Queen Mary University of London

Centre for Research in String Theory
School of Physics and Astronomy

Thesis submitted for the degree of Doctor of Philosophy

CHY Formulae and Soft Theorems in $\mathcal{N} = 4$ Super Yang-Mills Theory

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March 2017

This thesis is dedicated to my family,
for their unconditional love, continual support and infinite imagination.

The most fatal illusion is the settled point of view.

— Brooks Atkinson, theatre critic

Abstract

The study of scattering amplitudes in quantum field theories (QFTs) is equally important for high energy phenomenology and for theoretical understanding of fundamental physics. Over the last 15 years there has been an explosion of new techniques, inspired by Witten's celebrated twistor string theory [1]. The $\mathcal{N} = 4$ super Yang-Mills theory (SYM) provides a playground for applying and extending these methods, heavily constrained by spacetime, internal and hidden symmetries.

Recently, Cachazo, He and Yuan proposed an algebraic construction of scattering amplitudes at tree level in various QFTs, based on the solution of certain scattering equations [2]. This formula was later extended to tree-level form factors of $\mathrm{Tr}(F_{\mathrm{SD}}^2)$ in four dimensional Yang-Mills theory [3]. In this thesis we show how this result may be naturally supersymmetrised, and derived from a dual connected formulation. Moreover, we relate our results to a geometric construction of form factors via the Grassmannian [4]. Finally, we argue that ambitwistor string theory provides a natural way to lift the result to arbitrary dimensions, paving the way for loop-level results.

In complementary work, it was shown that the subleading soft behaviour of tree-level amplitudes in gauge theory and gravity is universal [5–7]. This unexpected property is related to extended symmetries of the theory acting at null infinity. Moreover, the hidden structure provides additional information relevant for resummation of physical observables. In this thesis, we extend the known results to one-loop level in $\mathcal{N} = 4$ SYM, arguing that IR divergences introduce anomaly terms through finite order in the regulator. We constrain these terms using dual superconformal symmetry, and derive explicit formulae in the MHV and NMHV sectors.

This thesis contains documentation for two Mathematica packages, illustrating the original calculations we have performed.

Acknowledgements

First and foremost, I owe a huge debt of gratitude to my co-supervisors Andreas Brandhuber and Gabriele Travaglini, who have nurtured, encouraged and motivated me throughout my doctoral studies. It has been a great pleasure and privilege to learn from their considerable advice and expertise, both technical and practical. I wish to thank them wholeheartedly for their unwavering enthusiasm and friendship, a large positive constant amidst the fluctuations of research life.

I must also thank my collaborators Bill Spence and Rodolfo Panerai for their invaluable contributions to research discussions, excellent sense of humour and forensic ability to spot my mistakes. Our research meetings will remain among my happiest memories from the last few years. Moreover, I thank all the academics and administrative staff associated with the Centre for Research in String Theory, who provide such a stimulating, challenging and exciting environment for research.

My fellow Ph.D. students at QMUL have been a constant source of support. I am extremely grateful for the many debates, jokes and puzzles we have shared, and I shall truly miss their inspirational company in the office. In particular I thank Joel Berkeley, David Garner, Joe Hayling, Zac Kenton, Arnau Koemans, Martyna Kostacińska, Zoltán Laczkó, Chris Lewis-Brown, Paolo Mattioli, James McGrane, Rob Mooney, Emanuele Moscato, Ray Otsuki, Matteo Parisi, Brenda Penante, Sam Playle and Felix Rudolph.

The direction and success of my research has been shaped by a number of very fruitful interactions at meetings and conferences. In particular, I wish to thank Nima Arkani-Hamed, Rutger Boels, Scott Davies, Francesca Day, Joe Farrow, Yvonne Geyer, Arthur Lipstein, Lorenzo Magnea, Lionel Mason, Josh Nohle, Jan Plefka, Kai Roehrig, Matt von Hippel, Matthias Wilhelm and Chris White for interesting conversations. I am especially grateful to Paolo Mattioli, Katy Clough and Christian Käding, who have enabled many discussions through the Collabor8 conference series.

The completion of my Ph.D. would have been impossible if not for my wonderful friends outside of QMUL. They have provided a melodious counterpoint to my research, and wonderful companionship through thick and thin. I am particularly grateful to Jessie Acton, Victoria Bullard-Smith, Hugh Collins, James Corbett, Alex Davan Wetton, Francesca Day, Hannah Fiddy, Jo Hale, Kate Massey, Alfie Monkey, Stefan Ward-Caddle and Travis Winstanley.

Finally, I wish to thank my family, for their steadfast support and belief in me. I thank my partner, Sophie Hopkins, for her boundless good humour and inventiveness. I thank my brother and sister-in-law, Alistair and Svetli Hughes, for keeping me open-minded. I am enormously grateful to my parents, Jeremy and Caroline Hughes, for instilling in me the ambition to discover the secrets of the universe. And I am greatly indebted to my grandmother, Jacqueline Chappell, who first showed me the joy of learning, setting me on the quest for knowledge. This thesis is dedicated to you all.

The research contained in this thesis was supported by an STFC studentship. Some of the figures were produced with the aid of JaxoDraw [8].

Declaration

I, Edward Fauchon Hughes, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material.

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Edward Hughes

Date: September 3, 2017

Details of collaboration and publications:

This thesis describes research carried out with my supervisors Andreas Brandhuber and Gabriele Travaglini which was published in [9, 10]. It also contains some unpublished material. We collaborated with Bill Spence on [9, 10] and additionally with Rodolfo Panerai on [10]. Where other sources have been used, they are cited in the bibliography.

Contents

Chapter 1

Introduction

The past 15 years have seen a flourishing of so-called on-shell methods for computing scattering amplitudes. In many ways the motivation for this research is identical to the analytic S-matrix programme of the 1960s [11]. A principal aim is to construct compact representations of amplitudes based on central tenets of quantum field theory, such as locality, unitarity, and gauge and global symmetries. One might hope that such formulae expose interesting algebro-geometrical structures, shedding light on hidden properties of QFT. Moreover, they should be designed to bypass the factorial complexity associated with Feynman graph calculations.

There are several reasons why on-shell methods have succeeded where the analytic S-matrix programme failed. Chief among these is the discovery of $\mathcal{N} = 4$ SYM [12], a mathematical playground nowadays widely regarded as the simplest renormalisable QFT [13]. A related factor is the maturity of string theory, and its relation to $\mathcal{N} = 4$ SYM via the AdS/CFT correspondence [14, 15]. Twistor theory has also played a vital role, ever since Witten's seminal paper [1]. At present the crowning achievements can be pleasingly categorised as algebraic and geometric. The former is based on the solution of scattering equations [16], while the latter involves measurements on Grassmannians [17].

Of course, the S-matrix does not encode all the information in a QFT, expect perhaps in quantum gravity [18]. Therefore it is important to ask whether on-shell methods extend beyond merely constructing scattering amplitudes. Most obviously, one could consider a matrix element between asymptotic states and an operator inserted at a spacetime point, defining a form factor. More subtly, one could study the infrared behaviour of amplitudes in various on-shell guises, crucial for generating accurate inclusive cross-sections.

index	type
$\mu, \nu, \ldots$	$SO(1,3)$ Lorentz index
$\alpha, \beta, \ldots$	$SL(2;\mathbb{C})$ chiral spinor index
$\dot{\alpha}, \dot{\beta}, \ldots$	$SL(2;\mathbb{C})$ antichiral spinor index
$a, b, i, j, \ldots$	label external particles
$J \in \mathsf{m}$, $i \in \mathsf{p}$	label subsets of particles
$A, B, \ldots$	$SU(4)$ R-symmetry index
$a, a', \ldots$	$SU(2)$ R-symmetry index
$\mathsf{A}, \mathsf{B}, \ldots$	$PSL(4\|4;\mathbb{C})$ supertwistor index
$I, J, \ldots$	$PSL(4;\mathbb{C})$ twistor index

Table 1: An inexhaustive summary of our index conventions.

In this thesis we shall expand the literature in both these cases, demonstrating that the on-shell approach may be profitably employed. Our form factor results owe an intellectual debt to [19], arguably the first attempt to understand scattering with operator insertions in a modern framework. Our work on soft limits is largely inspired by [6], in which the theoretical importance of subleading behaviour was first articulated.

We shall now review several topics of central revelance to our original arguments in Chapters 2 and 3. In Section 1.1 we formally introduce $\mathcal{N} = 4$ SYM, and comment on several appealing properties. We then define scattering amplitudes and form factors, and provide a detailed exposition of several on-shell methods. Section 1.3 fills a gap in the literature for a pedagogical review of the scattering equations and related twistor theories. We have found the following textbooks and papers particularly useful as reference material, and cite them exactly once: [20–28].

Throughout this thesis we work in four dimensions and employ a mostly minus $(+---)$ metric signature, unless explicitly stated otherwise. Table 1 outlines our most common index conventions for the convenience of the reader. Where there are conflicts, the correct interpretation should be clear from the context.

1.1 $\mathcal{N} = 4$ Super Yang-Mills Theory

We begin by introducing the sole theory of study in this thesis, $\mathcal{N} = 4$ SYM. In a sense this is a souped-up version of QCD, incorporating extra quark and scalar fields.

However, $\mathcal{N}=4$ SYM is a much simpler theory, bound by the strictures of supersymmetry. In fact, it is the unique renormalisable maximally supersymmetric theory in four dimensions that can be described with a Lagrangian. The requirement of renormalisability prohibits particles of spin >1. Since massless supermultiplets contain $\mathcal{N}/2$ states, the maximal allowed $\mathcal{N}$ is 4, providing 16 real supercharges.[1] Uniqueness follows from applying the additional constraints of extended supersymmetry to the well-known restrictions on a $\mathcal{N}=1$ action, detailed in for example [29].

More explicitly, the unique massless supermultiplet in $\mathcal{N}=4$ is

$$(A_\mu, \lambda_{\alpha A}, \tilde{\lambda}^A_{\dot{\alpha}}, \phi^{AB}) \,, \tag{1.1.1}$$

comprising a gauge field A_μ, fermions $\lambda_{\alpha A}$ and $\tilde{\lambda}^A_{\dot{\alpha}}$ for $A=1,\dots 4$, and complex scalars ϕ^{AB} for $A,B=1,\dots 4$. All fields are in the adjoint representation of a gauge group $SU(N)$. The indices A,B indicate non-trivial representations of R-symmetry, the $SU(4)$ rotation of supersymmetry generators among themselves. Clearly the gauge field transforms in the singlet representation, while the fermions transform in the (anti-)fundamental representation. The scalars transform in the adjoint representation, hence represent 6 real degrees of freedom[2]. In particular we have

$$\phi^{AB} = -\phi^{BA} \,, \quad \phi_{AB} = \overline{\phi}^{AB} \,. \tag{1.1.2}$$

The Lagrangian density for $\mathcal{N}=4$ SYM is, using the conventions in (1.2.5),

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Bigg(&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\tilde{\lambda}^A_{\dot{\alpha}}\sigma^{\alpha\dot{\alpha}}_\mu D^\mu \lambda_{\alpha A} - D_\mu\phi_{AB}D^\mu\phi^{AB} \\ &-\frac{1}{2}[\phi_{AB},\phi_{CD}][\phi^{AB},\phi^{CD}] - \frac{i}{2}\lambda^\alpha_A[\phi^{AB},\lambda_{\alpha B}] - \frac{i}{2}\tilde{\lambda}^A_{\dot{\alpha}}[\phi_{AB},\tilde{\lambda}^{\dot{\alpha}B}]\Bigg) \,,\end{aligned} \tag{1.1.3}$$

where we have absorbed all dependence on the coupling constant g into our definition of the fields. The kinetic terms for gauge field and fermions and the minimal coupling interaction between them are familiar features from QCD, while the third and fourth term describe scalar electrodynamics.

The supersymmetry and R-symmetry we have discussed thus far are merely part of the spacetime symmetry exhibited by $\mathcal{N}=4$ SYM. At the classical level, the theory is superconformal invariant. In addition to the translations, boosts and rotations of Poincaré symmetry, the supersymmetry generators $Q^{A\alpha}$ and $\tilde{Q}^{\dot{\alpha}}_A$ and R-symmetry, the

[1] Of course, $\mathcal{N}=3$ supersymmetry is identical to $\mathcal{N}=4$ on-shell since CPT conjugation naturally produces all supermultiplets in the latter from those in the former.

[2] The real scalars are commonly denoted ϕ^i for $i=1,\dots 6$ and are related to the complex scalars in the following way [30]. Let $\Upsilon^i=\phi^i$ and $\Xi^i=\phi^{i+3}$ for $i=1,2,3$. Define 4×4 antisymmetric matrices $\Upsilon_{ij}=-\epsilon_{ijk}\Upsilon_k$, $\Upsilon_{i4}=-\Upsilon_i$ and $\Xi_{ij}=\epsilon_{ijk}\Xi_j$, $\Xi_{i4}=\Xi_i$. Then $\phi_{AB}=\Upsilon_{AB}+i\Xi_{AB}$.

action is invariant under dilatations, conformal boosts and conformal supersymmetries. Together the generators define the graded Lie algebra $\mathfrak{su}(2,2|4)$.

We shall review relevant consequences of this large symmetry group in more detail in Section 1.3. For now we confine ourselves to stating that the superconformal symmetry survives in the full quantum theory. Indeed, the beta function of $\mathcal{N}=4$ SYM vanishes to all orders of the coupling [31, 32], so the theory is ultraviolet (UV) finite. Hence conformal symmetry is not broken by anomalies. Therefore, we shall have no need of the renormalisation toolbox when calculating loop amplitudes. Note that form factors of unprotected operators do require renormalisation in general – see for example [33]. We shall not encounter such complications.

In Section 1.2 we shall define our main objects of interest in $\mathcal{N}=4$ SYM, scattering amplitudes and form factors. It is important to note that our definitions below aren't strictly meaningful for $\mathcal{N}=4$ SYM. Indeed, since the theory is conformal, the notion of an asymptotic state is ill-defined. This problem manifests itself as infrared (IR) divergences in observables. We may rectify this by introducing an IR regulator, when required. Most commonly, one chooses to work in $4-\epsilon$ dimensions, manifestly breaking the conformal symmetry. This entails introducing a dimensionful parameter μ called the regularisation scale, and requiring that all physical quantities are independent of this parameter. For simplicity of notation, we set $\mu=1$ throughout this thesis, arguing here that one may include it a posteriori via dimensional analysis. These subtleties do not affect tree level calculations, but have an important role to play at one loop, as we shall see in Chapter 3.

1.2 Scattering Amplitudes and Form Factors

The most general observable in a four dimensional quantum field theory (QFT) with fields $\{\phi,\ldots,\psi\}$ and Langrangian density $\mathcal{L}(\phi,\ldots,\psi)$ is the correlation function,[3]

$$\begin{aligned}&\langle 0|T\phi(x_1)\cdots\phi(x_n)\cdots\psi(y_1)\cdots\psi(y_m)|0\rangle\\ &\quad=\lim_{T\to\infty(1-i\epsilon)}\frac{\int\mathcal{D}\phi\cdots\mathcal{D}\psi\ \phi(x_1)\cdots\phi(x_n)\cdots\psi(y_1)\cdots\psi(y_m)\ \exp\left(i\int_{-T}^{T}\mathrm{d}^4x\ \mathcal{L}\right)}{\int\mathcal{D}\phi\cdots\mathcal{D}\psi\ \exp\left(i\int_{-T}^{T}\mathrm{d}^4x\ \mathcal{L}\right)}\,.\end{aligned}\tag{1.2.1}$$

[3]Confusingly also referred to as a Green's function in much of the literature. In general, only the two-point correlation function of a free field theory is a right-inverse for a meaningful differential operator, namely the Hamiltonian.

computing the expected value of a time-ordered product of operators in the (true interacting) vacuum state $|0\rangle$. Mathematically, these are regarded as fundamental quantities by virtue of the Wightman reconstruction theorem [34], which states that the correlation functions of a QFT satisfying the Wightman axioms determine the theory up to unitary equivalence. More pragmatically, correlation functions describe how quantum fields co-vary, with measurable consequences for fields from cosmology to condensed matter – see for example [35, 36].

Of course, the most important observable in high-energy physics is the cross-section presented by the scattering of particles. Theoretical predictions are typically determined in silico by libraries such as Sherpa [37] which perform phase space integration over a scattering amplitude. The scattering amplitude is defined as the overlap between incoming and outgoing field quanta of definite momenta in the asympotic past and future respectively. These are related to correlation functions by virtue of the celebrated LSZ reduction formula [38], quoted below for illustration in the case of a scalar theory.

$$
\begin{aligned}
{}_{\mathrm{out}}\langle p_1 \dots p_n | k_1 \dots k_m \rangle_{\mathrm{in}} = \prod_{i=1}^{n} \mathrm{d}^4 x_i e^{ip_i \cdot x_i} (m^2 - \partial_{x_i}) \prod_{j=1}^{m} \mathrm{d}^4 y_j e^{ik_j \cdot y_j} (m^2 - \partial_{y_j}) \\
\times \langle 0 | T\phi(x_1) \cdots \phi(x_n) \phi(y_1) \cdots \phi(y_n) | 0 \rangle \,. \quad (1.2.2)
\end{aligned}
$$

In the amplitude each of the momenta p_i and k_j are constrained to be on-shell[4], that is satifying the constraints $p_i^2 = m^2$ and $k_j^2 = m^2$ where m is the mass of the field ϕ. Henceforth we drop the subscripts on the asymptotic states.

For the remainder of this thesis, all fields we consider will be massless, with this property protected by symmetry and so unaffected by quantum corrections. Our reasons for this are threefold. Firstly, it is a convenient mathematical simplification, and indeed one embodied by the toy model $\mathcal{N} = 4$ SYM which we introduced in Section 1.1. Secondly, it is physically reasonable to neglect mass for high-energy processes, at least for certain spin-$\frac{1}{2}$ particles. Indeed the mass term enters scattering amplitudes through a propagator factor $(K^2 - m^2)^{-1}$ where K is a sum of momenta. For sufficiently high energy scattering, the K^2 term generically dominates. In QCD processes at the LHC, the gluons are already massless and the light quarks (up, down, charm and strange) have negligable mass ($\lesssim$ 1GeV) compared with the centre of mass energy of a typical collision ($\gtrsim$ 1TeV). Thirdly, with the discovery of the Higgs, it is clear that mass

[4]The nomenclature arises from picturing the hyperboloid described by the condition in three dimensions. More specifically the asymptotic particles are assumed to satisfy the free-field equations of motion, which necessarily imply the Klein-Gordon equation $p^2 = m^2$ in Fourier space for relativistic theories.

is an emergent property, and not of fundamental importance to the blueprints of our universe.

A fundamental property of scattering amplitudes in relativistic field theories is their adherence to crossing symmetry. Explicitly, the amplitude for a process is unchanged upon replacing an incoming particle of momentum p with an outgoing antiparticle of momentum $-p$. This is easily derived from the Feynman rules or directly from the action of symmetry generators [39], with appropriate conventions for the analytic continuation of polarisation spinors. For massless particles, this entails flipping the helicity. Without loss of generality, we may therefore restrict our attention to amplitudes with all particles outgoing, viz.

$$A_n(1, \dots n) = \langle p_1 \dots p_n | 0 \rangle \, . \tag{1.2.3}$$

Form factors[5] provide a bridge between completely on-shell amplitudes and completely off-shell correlation functions. Intuitively, they describe the scattering of particles off a quantum field localised in space. Mathematically we write

$$F_n^{\mathcal{O}}(1, \dots, n, q) = \int \mathrm{d}^4 x \, e^{iq\cdot x} \langle p_1 \dots p_n | \mathcal{O}(x) | 0 \rangle \, , \tag{1.2.4}$$

with $\mathcal{O}$ some product of operators in the theory. Such objects are phenomenologically important as approximations to terms in on-shell scattering amplitudes. For example, to calculate $H \to ggg$ cross-section, one may approximate the scattering amplitude by a form factor of $\mathrm{Tr}(F^2)$ in the limit of large top quark mass [42]. Furthermore one can even recover part of this QCD form factor from a calculation in $\mathcal{N} = 4$ at two-loops [43]. Other processes for which form factors are typically employed include electron-positron collision [44] and deep inelastic scattering [45]. Since crossing symmetry is essentially a property of the asymptotic states, it remains valid for form factors.

When the couplings are small, it is convenient to calculate scattering amplitudes and form factors perturbatively. In this thesis, we work exclusively in such a regime. The traditional diagrammatic method is due to Feynman [46], and naturally expresses results order-by-order in couplings and $\hbar$. Each factor of $\hbar$ is associated with the appearence of a loop in a diagram, comprising an off-shell particle whose phase space must be integrated over. By invoking Euler's formula, one may show that for a fixed number of external particles, amplitudes and form factors scale uniformly in $\hbar$ and couplings. Therefore we shall not include any such explicit factors in the sequel; rather

[5]This arcane term appears to have originated in forestry [40] to describe the shape of a tree, more precisely giving the ratio between the volume of a tree and that of a cylinder of the same diameter and height. In the mid-twentieth century it was appropriated by physicists to denote the effective size and shape of a target in a scattering experiment, as reviewed in [41].

we shall distinguish between contributions by stating the numbers of external particles and loops. Moreover, we shall refer to the leading order 0-loop term as tree level, as is customary in the literature.

1.2.1 Colour Ordering

In this thesis we will focus exclusively on amplitudes and form factors in $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. All particles live in the adjoint representation of the gauge group, hence their amplitudes are functions of traceless Hermitian generators T^a and structure constants f_{abc} for $a, b, c = 1, \ldots N^2-1$. For concreteness, we choose a basis in which

$$\mathrm{Tr}(T^a T^b) = \delta^{ab}\,, \qquad f_{abc} = -\frac{i}{\sqrt{2}}\mathrm{Tr}(T^a[T^b, T^c])\,, \tag{1.2.5}$$

and recall the Fierz identity,[6]

$$(T^a)_i^j (T^a)_k^l = \delta_i^l \delta_k^j - \frac{1}{N}\delta_i^j \delta_k^l\,. \tag{1.2.6}$$

Combining these formulae, it is possible to organise amplitudes and form factors as a sum of kinematic terms dressed with products of traces $\mathrm{Tr}(T^a \ldots T^b)$, thus disentangling the colour dependence. An argument of 'tHooft [47] establishes that each trace comes with an additional factor of $1/N$. Thus amplitudes and form factors admit a natural decomposition into colour-ordered kinematic pieces, labelled by a trace structure and a cyclic ordering of the external adjoint particles.

Henceforth we shall examine only the single-trace colour-ordered partial amplitudes and form factors. Abusing notation, we refer to these as A_n and $F_n^{\mathcal{O}}$ respectively. From a phenomenological viewpoint, we are making a leading order in N approximation. Mathematically speaking, we are working in the $N \to \infty$ limit, which corresponds to considering only planar Feynman diagrams. This simplification brings us in touching distance of string theory by virtue of the AdS/CFT correspondence [14], and reveals unexpected hidden symmetries which are particularly important in Chapter 3.

In this thesis, the restriction to single-trace terms is without loss of generality. Indeed, we only consider amplitudes in $\mathcal{N} = 4$ up to one loop, for which the full result can be extracted from the single-trace contribution alone [48]. Moreover, our operator insertions for form factors will be colour singlets comprising the $\mathrm{Tr}(\phi_{12}^2)$ supermultiplet. Therefore the same conclusion holds for the tree-level form factors we encounter.

[6] Also known as a completeness relation, since it may be derived from the fact that the Lie algebra $\mathfrak{su}(N)$, augmented with the identity matrix, spans the vector space of $N \times N$ anti-Hermitian complex matrices.

1.2.2 Spinor-Helicity Variables

There has been much progress in our understanding of amplitudes and form factors in recent years, particularly in planar $\mathcal{N} = 4$ SYM theory. The search for convenient variables with which to describe scattering processes has been a guiding principle. In particular, such variables should transform simply under relevant symmetries, and transparently yield compact results where possible. The first step on this journey is appropriate for any Lorentz invariant theory of massless particles, namely the adoption of spinor-helicity variables.[7]

It is well-known that the Lie algebra of the Lorentz group is isomorphic to $\mathfrak{sl}(2;\mathbb{C})$. One may therefore put real representations of the Lorentz algebra in bijection with complex representations of $\mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$. The simplest non-trivial such representations involve taking the direct sum of fundamental representation and a trivial representation of $\mathfrak{sl}(2;\mathbb{C})$. We call the vector space elements Weyl spinors and write

$$\lambda^\alpha \equiv |\lambda\rangle^\alpha \in \left(\frac{1}{2}, 0\right), \qquad \tilde{\lambda}_{\dot{\alpha}} \equiv |\tilde{\lambda}]_{\dot{\alpha}} \in \left(0, \frac{1}{2}\right). \tag{1.2.7}$$

for left-handed and right-handed spinors respectively. It is convenient to define lowering and raising of indices as

$$\lambda_\alpha \equiv \langle\lambda|_\alpha = \epsilon_{\alpha\beta}|\lambda\rangle^\beta, \qquad \tilde{\lambda}^{\dot{\alpha}} \equiv [\tilde{\lambda}|^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}|\tilde{\lambda}]_{\dot{\beta}}, \tag{1.2.8}$$

where the antisymmetric ϵ tensors satisfy

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon^{\alpha\beta} = -\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.2.9}$$

$$\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \epsilon^{\beta\gamma}\epsilon_{\gamma\alpha} = \delta^\beta_\alpha, \quad \epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\beta}} = \epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\gamma}\dot{\alpha}} = \delta^{\dot{\beta}}_{\dot{\alpha}}. \tag{1.2.10}$$

Hence we find the complementary raising and lowering rules,

$$|\lambda\rangle^\alpha = \epsilon^{\alpha\beta}\langle\lambda|_\beta, \qquad |\tilde{\lambda}]_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}[\tilde{\lambda}|^{\dot{\beta}}. \tag{1.2.11}$$

The vector representation of the Lorentz group is isomorphic to the $(\frac{1}{2}, \frac{1}{2})$ tensor representation. We may express this unitary equivalence explicitly in terms of our favourite basis of $\mathfrak{u}(2)$ namely the generalised Pauli matrices,

$$\sigma_\mu^{\alpha\dot{\alpha}} = (1, \sigma_i), \tag{1.2.12}$$

[7] We shall work in four dimensions, but similar constructions can be made in arbitrary dimension by constraining spinors appropriately [49].

where σ_i are the usual Pauli matrices. We define

$$q^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}} q^\mu \,, \tag{1.2.13}$$

with lowering performed according to

$$q_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} q^{\beta\dot{\beta}} \,. \tag{1.2.14}$$

We may then observe that

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} p^{\alpha\dot{\alpha}} q^{\beta\dot{\beta}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \sigma_\mu^{\alpha\dot{\alpha}} \sigma_\nu^{\beta\dot{\beta}} p^\mu q^\nu = 2p\cdot q \,, \tag{1.2.15}$$

and in particular, by the combinatorial definition of the determinant,

$$\det q = q \cdot q \,. \tag{1.2.16}$$

Therefore if p is a null vector, we may write

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \equiv |p\rangle^\alpha [p|^{\dot{\alpha}} \,, \tag{1.2.17}$$

for some Weyl spinors λ^α and $\tilde{\lambda}^{\dot{\alpha}}$ which must satisfy $(\lambda^\alpha)^* = \pm\tilde{\lambda}^{\dot{\alpha}}$ to ensure that p is real.[8] There is some freedom in choosing spinors satisfying these conditions, namely

$$\lambda^\alpha \to e^{-i\theta}\lambda^\alpha \,, \quad \tilde{\lambda}^{\dot{\alpha}} \to e^{i\theta}\tilde{\lambda}^{\dot{\alpha}} \,. \tag{1.2.18}$$

This corresponds to the representation of the little group $SE(2)$ on physically reasonable states.[9] The associated conserved charge is known as helicity, and is defined by

$$h = -\frac{1}{2}\left(\lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} - \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial\tilde{\lambda}^{\dot{\alpha}}}\right) \equiv -\frac{1}{2}\left(|\lambda\rangle \cdot \frac{\partial}{\partial|\lambda\rangle} + |\tilde{\lambda}] \cdot \frac{\partial}{\partial|\tilde{\lambda}]}\right) \,, \tag{1.2.19}$$

where spinor differentiation obeys

$$\frac{\partial}{\partial|\lambda\rangle^\alpha}|\lambda\rangle^\beta = \delta_\alpha^\beta \,, \quad \frac{\partial}{\partial|\tilde{\lambda}]_{\dot{\alpha}}}|\tilde{\lambda}]_{\dot{\beta}} = -\delta_{\dot{\beta}}^{\dot{\alpha}} \,. \tag{1.2.20}$$

We conclude that angle (undotted) spinors carry helicity $-\frac{1}{2}$ while square (dotted) spinors carry helicity $+\frac{1}{2}$. By convention, every spinor carries mass dimension $\frac{1}{2}$.

[8] This requirement motivates the nomenclature that λ^α is holomorphic and $\tilde{\lambda}^{\dot{\alpha}}$ is antiholomorphic.

[9] The little group [50] is the subgroup of Lorentz transformations that leave a vector p invariant, namely translations and rotations. Naïvely for a massless quantum state of null momentum p, the little group acts to continuously deform the helicity. Since we don't observe massless particles with continuous spin, we enforce additional constraints on physical states, so that the only freedom that remains is (1.2.18).

It is often convenient to employ complex momenta, allowing us to apply powerful techniques from complex analysis. In this setting, we may remove the reality condition below (1.2.17), so that the positive and negative helicity spinors are truly independent. The little group phase shift (1.2.18) then naturally extends to a scaling symmetry $\lambda \to t^{-1}\lambda, \tilde{\lambda} \to t\tilde{\lambda}$ for $t \in \mathbb{C}$. We shall assume complex momenta henceforth.

To illustrate the notation we rewrite (1.2.15) more compactly, for p null and q is arbitrary:

$$2p \cdot q = \langle p|q|p] \,. \tag{1.2.21}$$

It is surprisingly useful to observe that three arbitrary Weyl spinors must be linearly dependent, leading to the Schouten identities,

$$\langle i\ j\rangle\langle k| + \langle j\ k\rangle\langle i| + \langle k\ i\rangle\langle j| = 0\,, \quad [i\ j][k| + [j\ k][i| + [k\ i][j| = 0\,. \tag{1.2.22}$$

We have motivated spinor-helicity variables as natural and fundamental quantities with which to parameterise Lorentz invariant processes. They are also of great practical benefit, since they seamlessly subsume the polarisation vectors of massless external states in scattering amplitudes and form factors. For fermions, the relationship is obvious. Recall that the Weyl equations in momentum space take the form,

$$\sigma^{\mu}_{\alpha\dot{\alpha}} p_{\mu} \psi^{\alpha} = 0\,, \quad \sigma^{\alpha\dot{\alpha}}_{\mu} p^{\mu} \tilde{\psi}_{\dot{\alpha}} = 0\,. \tag{1.2.23}$$

By construction, the spinors $\psi^{\alpha} = |p\rangle^{\alpha}$ and $\tilde{\psi}_{\dot{\alpha}} = |p]_{\dot{\alpha}}$ satisfy (1.2.23), and thus comprise the polarisation spinors of on-shell massless fermions.

For vector bosons we require arbitrary reference spinors μ^{α} and $\tilde{\mu}^{\dot{\alpha}}$ such that $[\tilde{\mu}\ p] \neq 0$ and $\langle \mu\ p\rangle \neq 0$. This freedom reflects gauge invariance as embodied by the Ward identity $p^{\mu} A_{\mu} = 0$ where A_{μ} is the amplitude stripped of the polarisation vector $\epsilon^{\mu}(p)$. The correct identification turns out to be

$$\epsilon_{+}^{\alpha\dot{\alpha}}(p) = \sqrt{2}\frac{|\mu\rangle^{\alpha}[p|^{\dot{\alpha}}}{\langle \mu p\rangle}\,, \quad \epsilon_{-}^{\alpha\dot{\alpha}}(p) = \sqrt{2}\frac{|p\rangle^{\alpha}[\mu|^{\dot{\alpha}}}{[p\mu]}\,. \tag{1.2.24}$$

To conclude, in spinor-helicity notation we may express any colour-ordered partial scattering amplitude of massless particles $i = 1, \dots n$ using only the data $(\lambda_i, \tilde{\lambda}_i, h_i)$. Moreover, we may use parity to determine the amplitudes with flipped helicity by exchanging angle and square spinors.

1.2.3 Superamplitudes and Super Form Factors

Just as Lorentz symmetry led us to an elegant choice of variables, so too does the enormous supersymmetry of $\mathcal{N}=4$ SYM help us to compute scattering amplitudes and form factors. Unlike for $\mathcal{N}=1$ supersymmetric theories, $\mathcal{N}=4$ SYM does not admit a finite-dimensional off-shell superspace. Nevertheless, one can construct an on-shell superspace [51] and a harmonic off-shell superspace incorporating certain operators [52–54]. This trivialises the action of the supersymmetry generators $Q^{A\alpha}$ and $\tilde{Q}^{\dot{\alpha}}_A$, leading to important simplifying constraints.

On-shell, the massless states of $\mathcal{N}=4$ SYM form a single supermultiplet, consisting of 16 states. Each state may be labelled by the $SU(4)$ R-symmetry indices A, B and so forth. By introducing auxiliary Grassmann variables η_A we may combine the states into a single superfield, viz.

$$\begin{aligned}\Phi(p,\eta) = g^+(p) + \eta^A\lambda_A(p) + \frac{1}{2}\eta^A\eta^B\phi_{AB}(p)\\ + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\tilde{\lambda}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}g^-(p)\,. \qquad (1.2.25)\end{aligned}$$

Any amplitude invariant under supersymmetry necessarily requires external particles to appear in the combination (1.2.25). We term such an object a superamplitude $A_n(\Phi_i)$, where we have introduced an additional index $i = 1, \ldots n$ which labels the particles.[10] For comparison to less supersymmetric theories and ease of notation, it is often convenient to extract the component amplitudes associated with a given particle content. This may be done by differentiating (or integrating) the superamplitude with respect to the Grassmann variables associated with the desired states. For example:

$$A_n(g_1^+ \ldots g_{n-2}^+ g_{n-1}^- g_n^-) = \prod_{A=1}^{4}\frac{\partial}{\partial\eta_{n-1}^A}\prod_{B=1}^{4}\frac{\partial}{\partial\eta_n^B}A_n(\Phi_1,\ldots\Phi_n)|_{\eta_{kC}=0}\,. \qquad (1.2.26)$$

We may immediately constrain the form of a superamplitude by requiring that it be annihilated by the symmetry generators of $\mathcal{N}=4$ SYM. Invariance under $SU(4)$ R-symmetry immediately requires that the superamplitude is a polynomial in the Grassmann terms $\epsilon_{ABCD}\eta_i^A\eta_j^B\eta_k^C\eta_l^D$. Therefore the superamplitude naturally decomposes into terms of Grassmann order $4(k+2)$ for integers $-2 \leq k \leq n-2$, in principle.

Before considering the action of the supersymmetry generators, it is instructive to recall how translation invariances manifests itself at the level of amplitudes. In spinor-helicity

[10] We do not use different notation to distinguish superamplitudes from their components – this should always be clear from the context.

notation, the generator of global translations is the momentum operator,

$$P^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} |i\rangle^{\alpha}[i|^{\dot{\alpha}}\,, \tag{1.2.27}$$

which acts on amplitudes multiplicatively. In particular then, it must be solved by imposing a delta function of momentum conservation as part of the amplitude, writing

$$A_n = \delta^{(4)}\left(\sum_{i=1}^{n} |i\rangle^{\alpha}[i|^{\dot{\alpha}}\right) \times \cdots\,. \tag{1.2.28}$$

A similar result holds for form factors, where one must now include the additional off-shell momentum associated with the operator insertion [54]. Given the universal nature of this simplification, we shall freely omit the delta function of momentum conservation in writing amplitudes where it is not absolutely necessary for logical clarity.

In on-shell superspace, the supersymmetry generators are represented as

$$Q^{A\alpha} = \sum_{i=1}^{n} |i\rangle^{\alpha}\eta_i^A\,, \quad \tilde{Q}_A^{\dot{\alpha}} = [i|^{\dot{\alpha}}\frac{\partial}{\partial\eta_i^A}\,. \tag{1.2.29}$$

The requirement that these annihilate the superamplitude is precisely the Ward identity arising from supersymmetry of the path integral. The action of the holomorphic generator $Q^{A\alpha}$ is multiplicative, just like the momentum generator (1.2.27). By analogy, we therefore impose a supermomentum conserving delta function,[11]

$$A_n = \delta^{(4)}\left(\sum_{i=1}^{n} |i\rangle^{\alpha}[i|^{\dot{\alpha}}\right)\delta^{(8)}\left(\sum_{i=1}^{n} |i\rangle^{\alpha}\eta_i^A\right) \times \cdots\,, \tag{1.2.30}$$

where $n > 3$. By virtue of the equivalence of integration and differentiation for Grassmann variables, the Grassmann delta function is merely a product of its arguments for $A = 1, \ldots 4$ and $\alpha = 1, 2$. Hence, we observe that a superamplitude has minimal Grassmann degree 8, corresponding to $k = 0$ in our decomposition above.

The sector with $k = 0$ is termed maximally helicity violating (MHV), because its purely gluonic component amplitudes have the form $g^+ \ldots g^+ g^- g^-$. After applying crossing symmetry, this corresponds to a process $g^- g^- \to g^+ \ldots g^+ g^- g^-$ which represents the largest proportion of g^+ which can be produced from two g^- with non-zero amplitude

[11] There is an additional subtlety when $n = 3$. Three particle special kinematics implies that either $[i\ j] = 0$ or $\langle i\ j\rangle = 0$ for all $i, j = 1, 2, 3$. In the latter case, the $Q^{A\alpha}$ constraint becomes vacuous, and the $\tilde{Q}_A^{\dot{\alpha}}$ constraint may be solved by $\delta^{(4)}([1\ 2]\eta_3 + [2\ 3]\eta_1 + [3\ 1]\eta_2)$ courtesy of a Schouten identity.

for a given total number of particles.[12] This motivates a natural naming convention for the degree $4(k{+}2)$ terms in the superamplitude – N^{k}MHV amplitudes, where N stands for next-to.

It is convenient to modify the definition of the helicity operator (1.2.19) such that superamplitudes have uniform helicity 1 in all external legs. The appropriate alteration is

$$h = -\frac{1}{2}\left(|\lambda\rangle \cdot \frac{\partial}{\partial|\lambda\rangle} + |\tilde{\lambda}] \cdot \frac{\partial}{\partial|\tilde{\lambda}]} - \eta^A \frac{\partial}{\partial \eta_A}\right) . \tag{1.2.31}$$

As a concrete example of an amplitude in $\mathcal{N} = 4$ SYM, we quote the supersymmetric version of the famous Parke-Taylor MHV amplitude at tree level [55]. This may be easily derived by induction using the BCFW method reviewed in Section 1.2.5, starting with the 3-point amplitude which is fixed by consistency conditions [56].

$$A_n^{\text{tree,MHV}} = \frac{\delta^{(4)}(P)\delta^{(8)}(Q)}{\langle 1\ 2\rangle\langle 2\ 3\rangle \cdots \langle n\ 1\rangle} . \tag{1.2.32}$$

We now desire to supersymmetrise form factors in $\mathcal{N} = 4$. This requires us to promote operators to superfields, at least in some limited sector. Recall that the standard procedure for constructing superfields involves introducing new Grassmann coordinates θ^A_α and $\bar{\theta}_{\dot{\alpha}A}$ for $A = 1, \ldots 4$, extending spacetime to superspace. One then seeks to combine operators as a polynomial in the Grassmann variables, such that the superalgebra closes off-shell.

Given our success with the on-shell $\mathcal{N} = 4$ supermultiplet, we are tempted to extend this off-shell by adding further auxiliary variables. However, this naïve procedure turns out to require infinitely many variables in general [57], and moreover there is no known formulation of the resulting superfield. To proceed, we may restrict our attention to supercurrents, which naturally admit an off-shell extension [58]. In particular we shall consider the supermultiplet $\mathcal{T}$ containing the (improved) stress-tensor.

To construct this multiplet explicitly, we must first express the on-shell vector multiplet as a superfield in the operators (1.1.1). Note that the Nair superfield (1.2.25) does not suffice for our purposes, since it only encodes the creation operators of $\mathcal{N} = 4$ SYM. The resulting Grassmann field W^{AB}, a function of superspace variables, is constrained by Bianchi identities, reviewed in [59]. Compactly, the defining constraints take the form,

$$\mathcal{D}^\alpha_C W^{AB} = -\frac{2}{3}\delta^{[A}_C \mathcal{D}^\alpha_D W^{B]D} , \quad \mathcal{D}^{\dot{\alpha}(C} W^{A)B} = 0 , \tag{1.2.33}$$

[12] At tree level, this result carries over to pure Yang-Mills. Indeed the tree-level gluon amplitudes in pure Yang-Mills are identical to those in $\mathcal{N} = 4$ SYM since the gluon couples to the gluinos and scalars quadratically in (1.1.3), so amplitudes with only gluon external states can only involve gluinos and scalars running in loops.

where the supercovariant derivatives are defined by

$$\mathcal{D}^{\alpha}_{A} = \frac{\partial}{\partial \theta^{A}_{\alpha}} + i\bar{\theta}_{A\dot{\alpha}}\frac{\partial}{\partial x_{\alpha\dot{\alpha}}} + ig\Gamma^{\alpha}_{A}\,, \quad \bar{\mathcal{D}}^{\dot{\alpha}A} = -\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}A}} - i\theta^{A}_{\alpha}\frac{\partial}{\partial x_{\alpha\dot{\alpha}}} + ig\bar{\Gamma}^{\dot{\alpha}A}\,. \tag{1.2.34}$$

with Γ^{α}_{A} and $\bar{\Gamma}^{\dot{\alpha}A}$ spinor superconnections, required for gauge invariance. For the remainder of the derivation we shall set $g = 0$. Indeed, one may argue that the gauge dependent terms that arise for non-zero g don't affect our conclusions, so long as the operator we consider is gauge invariant [53].

In [60], it was shown that the Bianchi identities (1.2.33) imply the equations of motion for $\mathcal{N} = 4$ SYM. Therefore, to deviate off-shell we consider only the chiral half of the theory by letting $\bar{\theta}^{A}_{\dot{\alpha}} = 0$. The surviving fields are the self-dual field strength $F^{\alpha\beta}_{\text{SD}} = \sigma^{\alpha\dot{\alpha}}_{\mu}\sigma^{\beta}_{\nu\dot{\alpha}}F^{\mu\nu}$, the chiral gluinos λ^{A}_{α} and the scalars ϕ_{AB}. This additional restriction breaks the second constraint, but allows us to solve the first one explicitly in harmonic superspace [61].

The solution requires projecting the $SU(4)$ R-symmetry onto a subgroup $SU(2) \times SU(2) \times U(1)$, first argued for in [62]. We do this in a democratic manner by introducing two $SU(2)$ matrices u^{+a}_{A} and $u^{-a'}_{A}$, oppositely charged under $U(1)$, where $a, a' = 1, 2$. Explicitly the projection is

$$\theta^{+a}_{\alpha} := u^{+a}_{A}\theta^{A}_{\alpha}\,, \quad \theta^{-a'}_{\alpha} := u^{-a'}_{A}\theta^{A}_{\alpha}\,. \tag{1.2.35}$$

After projecting the superspace constraint, the only non-trivial equation is

$$\frac{\partial}{\partial \theta^{-a'}_{\alpha}} W^{++} = 0\,, \tag{1.2.36}$$

where we define

$$\epsilon^{ab}W^{++} \equiv W^{+a+b} := u^{+a}_{A}u^{+b}_{A}W^{AB}\,. \tag{1.2.37}$$

We see that W^{++} is independent of the $\theta^{-a'}_{\alpha}$ variables and thus constitutes a short multiplet. In fact, it is annihilated by the chiral half of the supersymmetry generators, hence is $\frac{1}{2}$-BPS. This property ensures that any function of W^{++} is immune from UV divergences [63].[13] By contrast, generic multiplets can develop anomalies, the archetypal example being the Konishi multiplet [64].

Now we have constructed W^{++} we may determine the chiral part of the stress-tensor multiplet as $\mathrm{Tr}(W^{++}W^{++})$, according to [58]. The explicit field content of this multiplet was worked out in the abelian case by [65], by acting with the supersymmetry

[13] Strictly speaking, we must restrict to gauge invariant functions in the interacting theory, courtesy of the discuss below (1.2.34).

generators on supercurrents.[14] It is useful to exhibit the lowest and highest order Grassmann components,

$$\mathrm{Tr}(W^{++}W^{++}) = \mathrm{Tr}(\phi^{++}\phi^{++}) + \cdots + \frac{1}{3}(\theta^{+})^4 \mathcal{L}_{\text{on-shell,chiral}} \,. \tag{1.2.38}$$

We now have all the ingredients to define the super form factor of study in this thesis, namely

$$F_n^{\mathcal{T}_{\text{chiral}}}(1, \dots n, q) = \int \mathrm{d}^4x \, \mathrm{d}^4\theta^+ \, e^{-iq\cdot x - i\gamma^{\alpha}_{+a}\theta^{+a}_{\alpha}} \langle p_1 \dots p_n | \mathcal{T}_{\text{chiral}}(x)|0\rangle \,, \tag{1.2.39}$$

where γ^{α}_{+a} is the Fourier conjugate to θ^{+a}_{α}. Observe that if we specify purely gluon external states, the highest θ^+ (lowest γ_+) component of the super form factor at tree level computes the phenomenologically interesting $\langle g^+ \dots g^+ g^- \dots g^- | \mathrm{Tr} F^2_{\mathrm{SD}}(x)|0\rangle$. Henceforth we shall drop the superscript identifying the operator.

As for superamplitudes, we may obtain universal simplifications by examining the action of the unbroken supersymmetry generators. These are represented as

$$Q^{\alpha}_{+a} = \gamma^{\alpha}_{+a} - \sum_{i=1}^{n} |i\rangle^{\alpha}\eta_{+a,i} \,, \quad Q^{\alpha}_{-a} = \sum_{i=1}^{n} |i\rangle^{\alpha}\eta_{-a,i} \,, \tag{1.2.40}$$

where we perform projection of lower indices with the conjugate $SU(2)$ matrices $\bar{u}^{A}_{+a}$ and $\bar{u}^{A}_{-a}$. Since both act multiplicatively, we may immediately conclude that they must be enforced by delta functions, and write

$$F_n = \delta^{(4)}\left(q - \sum_{i=1}^{n} |i\rangle[i|\right) \delta^{(4)}\left(\gamma_+ - \sum_{i=1}^{n} |i\rangle\eta_{+i}\right) \delta^{(4)}\left(\sum_{i=1}^{n} |i\rangle\eta_{-i}\right) \times \cdots \,. \tag{1.2.41}$$

Like superamplitudes, super form factors admit an MHV classification, where N^{k}MHV super form factors have Grassmann degree $4(k{+}2)$. For amplitudes, we saw below (1.2.24) that parity may be used to directly obtain N^{n-k-2}MHV from N^{k}MHV behaviour. However, our restriction to a chiral operator breaks CPT invariance, so the same is not true for the super form factors we consider. Indeed, generic super form factors are non-zero up to and including the maximally non-MHV $k = n-2$ level.[15]

Finally, we exhibit the formulae for tree level MHV and maximally non-MHV super

[14] A more modern construction might proceed via the representation theory of the superconformal algebra, along the lines of [66].

[15] The parity conjugate denominators of (1.2.42) and (1.2.43) are not quite a coincidence. In fact, one may relate MHV and maximally non-MHV super form factors via a Grassmann Fourier transform [54].

form factors, first derived in [67]:

$$F_n^{\text{tree,MHV}} = \frac{\delta^{(4)}(P)\delta^{(4)}(Q_+)\delta^{(4)}(Q_-)}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,, \tag{1.2.42}$$

$$F_n^{\text{tree,N}^{\max}\text{MHV}} = \delta^{(4)}(P)\delta^{(4)}(Q_+)\int\left(\prod_{i=1}^{n}\mathrm{d}^4\tilde{\eta}_i\, e^{i\eta_{iA}\tilde{\eta}_i^A}\right)\frac{\delta^{(4)}\left(\sum_{j=1}^{n}\tilde{\lambda}_j\tilde{\eta}_j^+\right)}{[1\,2][2\,3]\cdots[n\,1]}\,, \tag{1.2.43}$$

where P now encodes the total momentum for form factors, as in the first term of (1.2.41). In particular, the maximally non-MHV form factor of $\mathrm{Tr}F_{\mathrm{SD}}^2$ is given by the $(\gamma_+)^0$ component,

$$\eta_1^4\eta_2^4\cdots\eta_n^4\frac{q^4}{[1\,2][2\,3]\cdots[n\,1]}\,. \tag{1.2.44}$$

These will be important ingredients in the BCFW and MHV diagram examples we compute in Section 1.2.5 and 1.2.7.

1.2.4 Dual Superconformal Symmetry

In this section we shall focus exclusively on amplitudes in $\mathcal{N}=4$, although similar ideas have been applied to form factors, for example in [67–69].

In Section 1.2.2, we used spinor-helicity to trivialise the null condition $p^2=0$. In this spirit it is natural to ask whether there exist variables that trivialise other constraints, such as supermomentum conservation $\delta^{(4)}(P)\delta^{(8)}(Q)$. To answer this question, we picture momenta as vectors concatenated nose to tail. Momentum conservation then ensures that these vectors define a closed polygon. The vertices of this polygon are exactly the convenient dual variables we seek. Explicitly, the dual variables (x_i,θ_i) are defined by[16] [70]

$$(x_i - x_{i+1})^{\dot{\alpha}\alpha} = |i\rangle^{\alpha}[i|^{\dot{\alpha}}\,, \quad (\theta_i-\theta_{i+1})^{\alpha A} = |i\rangle^{\alpha}\eta_i^A\,, \tag{1.2.45}$$

where supermomentum conservation provides the identifications $x_{n+1}=x_1$ and $\theta_{n+1}=\theta_1$. The true power of these variables is hardly apparent at first glance. To exhibit it, we recall that gauge theories contain an observable which is associated with closed polygons (and more general contours) by definition – the Wilson loop. It is a remarkable fact that superamplitudes in $\mathcal{N}=4$ SYM may be computed as the expectation values of supersymmetric Wilson loops, schematically

$$\mathcal{W} = \frac{1}{N}\mathrm{Tr}\left[\mathcal{P}\exp\left(\oint\mathcal{A}\right)\right]\,, \tag{1.2.46}$$

[16]These variables are certainly not the same as the spacetime x and superfield θ we encountered above. In the main text we disambiguate this notation where it is not already clear from the context.

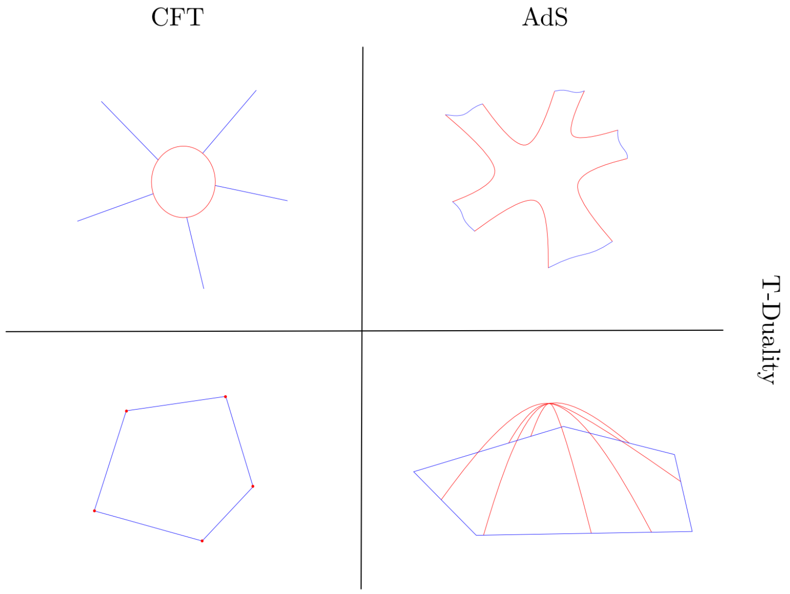

Figure 1: A pictorial representation of Alday-Maldecena duality, to be read clockwise from top left. Starting with a scattering amplitude in planar $\mathcal{N} = 4$ SYM at strong 't Hooft coupling g^2N, we may equivalently calculate the scattering of open strings in AdS space. After a T-duality, this is equal to the area of a minimal surface ending on a boundary polygon. Back on the CFT side, this may be computed as the expectation value of the Wilson loop (1.2.46).

where $\mathcal{A}$ is a certain superconnection, and the contour is a polygon [71,72]. Furthermore the identification of variables in this duality is precisely given by (1.2.45). Surprisingly, this result was first motivated at strong coupling via string theory [73]. The argument follows from the AdS/CFT correspondence and T-duality as shown in Figure 1. Subsequently the duality was established perturbatively in [74, 75].

We may now exhibit a key benefit of the dual coordinates (x, θ). The Wilson loop transforms covariantly under a superconformal symmetry acting in dual space. By virtue of the Alday-Maldecena duality, the same is true for superamplitudes. Moreover, it may be shown that this symmetry is distinct from the standard spacetime superconformal symmetry.[17] Therefore the dual variables manifest a hidden dual superconformal symmetry, which we may use to simplify calculations.

In Chapter 3, we will be particularly interested in one particular operator, namely the dual conformal boost,

$$K_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \left(x_{i\dot{\alpha}}^{\beta} \langle i|_{\alpha} \frac{\partial}{\partial |i\rangle^{\beta}} + x_{i+1\alpha\dot{\beta}} |i]_{\dot{\alpha}} \frac{\partial}{\partial |i]_{\dot{\beta}}} + \theta_{i+1\alpha}^{A} |i]_{\dot{\alpha}} \frac{\partial}{\partial \eta_i^A} \right) . \tag{1.2.47}$$

[17] For example, tree level pure Yang-Mills amplitudes are conformally invariant, but do not transform with uniform weight for each external particle under dual conformal inversion.

Under this transformation, tree amplitudes behave covariantly [76],

$$K_{\alpha\dot{\alpha}}A_n^{\text{tree}} = -\Big(\sum_{i=1}^{n} x_{i\alpha\dot{\alpha}}\Big)A_n^{\text{tree}}\,. \tag{1.2.48}$$

At loop level the symmetry is anomalous, owing to divergences in the Wilson loop [77]. In particular the IR divergences of loop amplitudes correspond to the UV cusp divergences of Wilson loops [78]. This provides us with a useful means to visualise the entanglement between subleading soft behaviour and IR divergences at loop level, which we shall discuss in Chapter 3.

The anomalous Ward identities calculated in [79, 80] suffice to explain the BDS ansatz for all-loop MHV amplitudes [81], which is correct up to a function of dual conformal invariant cross-ratios. The explicit form of the one-loop anomaly was proved in [82], giving

$$K_{\alpha\dot{\alpha}}A_n^{\text{1-loop}} \;=\; \frac{2}{\epsilon}c_\Gamma A_n^{\text{tree}}\sum_{i=1}^{n} x_{i\alpha\dot{\alpha}}\big[-(i-1\;i)\big]^{-\epsilon} \;-\; A_n^{\text{1-loop}}\sum_{i=1}^{n} x_{i\alpha\dot{\alpha}}\,, \tag{1.2.49}$$

valid through $\mathcal{O}(\epsilon^0)$, where $(i\;j) := 2p_i\cdot p_j$, ϵ is an IR regulator, and

$$c_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)}\,. \tag{1.2.50}$$

The fundamental conformally invariant quantities are the cross-ratios,

$$u_{ijkl} = \frac{x_{ik}^2 x_{jl}^2}{x_{il}^2 x_{jk}^2}\,, \tag{1.2.51}$$

where we define $x_{ij} = x_i - x_j$. One may derive this fact by considering the action of the special conformal generators on ratios of distances – see for example [83]. We will find it convenient to single out a particular class of cross-ratios which appear frequently in $\mathcal{N}=4$ superamplitudes, namely

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}\,. \tag{1.2.52}$$

1.2.5 BCFW Recursion

The BCFW method for amplitudes [84, 85] and form factors [54] is a convenient means of constructing higher point and MHV degree quantities recursively. We shall review the procedure at tree level without detailing the proof. For brevity we refer only

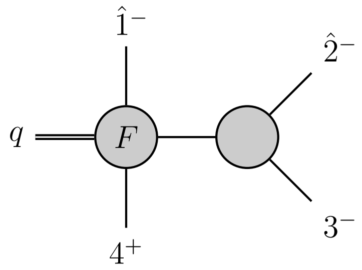

Figure 2: The vanishing BCFW diagram in the calculation of $F^{1^-2^-3^-4^+}$.

to amplitudes, but the method works identically for form factors of the stress-tensor multiplet.

1. Perform an $[i,j\rangle$ shift by defining

$$[\hat{i}| = [i| + z[j|\,, \quad |\hat{j}\rangle = |j\rangle - z|i\rangle\,. \tag{1.2.53}$$

2. Prove that the shifted amplitude $\hat{A}_n(z)$ does not have a pole at $z = \infty$.

3. Draw all possible colour-ordered diagrams (indexed by I) involving two shifted subamplitudes joined by a single shifted propagator of momentum $\hat{P}_I$.

4. For each diagram, solve for z_I by setting $\hat{P}_I^2 = 0$, and hence evaluate the left and right shifted subamplitudes.

5. Determine the full amplitude as a sum,

$$A_n^{\text{tree}} = \sum_I \hat{A}_{\text{left}}(z_I)\frac{1}{P_I^2}\hat{A}_{\text{right}}(z_I)\,. \tag{1.2.54}$$

The supersymmetric version of this procedure [13,86] naturally produces dual conformal invariant expressions called R-invariants [76]. We shall encounter such terms for NMHV amplitudes in Section 1.3.4.

Instead of reviewing this well-known calculation, we illustrate the method with an example recently computed in [10], which we shall require in Chapter 2. Explicitly, we compute the $n = 4, k = 1$ form factor of $\text{Tr}F_{\text{SD}}^2$ at tree level with purely gluon external states. We assign helicities as $1^-2^-3^-4^+$ and perform a $[1,2\rangle$ shift. It was proven in [54] that the shifted amplitude is regular at infinity. In principle one can draw four BCFW diagrams. However one of these immediately vanishes, depicted in Figure 2.

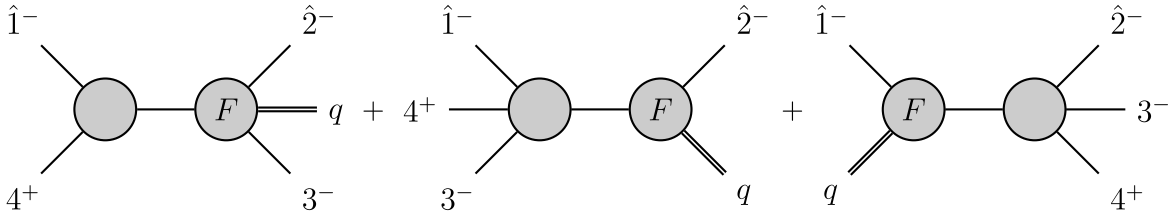

Figure 3: The non-vanishing BCFW diagrams in the calculation of $F^{1^-2^-3^-4^+}$.

Indeed the diagram contains a subamplitude,

$$\frac{\langle \hat{2}\ 3\rangle^3}{\langle \hat{P}\ 2\rangle\langle 3\ \hat{P}\rangle}\,, \tag{1.2.55}$$

which must be evaluated on the support of the on-shell condition $\langle \hat{2}\ 3\rangle[3\ 2] = 0$. For generic kinematics $[3\ 2] \neq 0$, so the diagram gives zero contribution. The remaining diagrams are shown in Figure 3 and term-by-term yield the result,[18]

$$F^{1^-2^-3^-4^+} = -\frac{\langle 1\ 3\rangle^4 q^4}{s_{134}\langle 1\ 4\rangle\langle 3\ 4\rangle\langle 3|q|2]\langle 1|q|2]} - \frac{\langle 3|q|4]^3}{s_{124}[1\ 2][1\ 4]\langle 3|q|2]} - \frac{\langle 1|q|4]^3}{s_{324}[3\ 2][3\ 4]\langle 1|q|2]}\,. \tag{1.2.56}$$

For completeness we explicitly calculate the first term. The sub form factor is maximally non-MHV, thus evaluates to

$$\frac{q^4}{[2\ \hat{P}]^2}\,, \tag{1.2.57}$$

while the subamplitude is MHV,

$$\frac{\langle 1\ 3\rangle^4}{\langle \hat{P}\ 3\rangle\langle 3\ 4\rangle\langle 4\ 1\rangle\langle 1\ \hat{P}\rangle}\,. \tag{1.2.58}$$

Notice that the only shifted quantity appearing is the propagator $\hat{P}$. We eliminate this by virtue of momentum conservation,

$$[2\ \hat{P}]\langle P\ 3\rangle\langle 1\ \hat{P}\rangle[P\ 2] = [2|q|3\rangle\langle 1|q|2]\,, \tag{1.2.59}$$

producing the desired result.

[18] Note that the result of [10] suffers from a typographical error in the numerator of the first term.

1.2.6 Grassmannian and Link Representations

Choosing different valid shifts provides different BCFW representations of amplitudes and form factors. Moreover, typically BCFW constructions generate terms with unphysical poles,[19] which cancel in the sum. The search for a unifying principle underpinning these properties leads to expressions for scattering amplitudes and form factors as contour integrals over auxiliary Grassmannians [17, 87]. In such constructions, the residues picked up at poles yield BCFW terms, with the equivalence of different representations explained by the global residue theorem, which we review in Section 1.3.2. The amplitude case was proven using on-shell diagrams in [88], while the form factor expression remains a conjecture.

Explicitly the N^{k-2}MHV tree-level superamplitude may be written as an integral over the Grassmannian $G(k,n)$ of k-dimensional linear subspaces of $\mathbb{C}^n$. This is a compact smooth manifold, with coordinates given by full-rank $k \times n$ matrices c_{Ja} modulo $GL(k;\mathbb{C})$ gauge transformations. The natural $GL(k;\mathbb{C})$-invariant expressions are $k \times k$ minors $(a_1 \cdots a_k)$ obtained by taking the determinant of the submatrix formed by columns $a_1, \dots a_k$. For an N^{k-2}MHV tree amplitude we write

$$A_{n,k} = \int \frac{\mathrm{d}^{k\times n} c_{Ja}\; \mathrm{d}^{2k}\rho_J}{|GL(k;\mathbb{C})|} \frac{\delta^{(2n)}(\rho_J c_{Ja} - \lambda_a)\delta^{(2k)}(c_{Ja}\tilde{\lambda}_a)}{(1\,2\cdots k)(2\,3\cdots k+1)\cdots(n\,1\cdots k-1)}\,. \tag{1.2.60}$$

For practical calculations, it is convenient to gauge fix by forcing an $k \times k$ submatrix to be the identity. This naturally partitions the set $\{1, \dots n\}$ into indices J and i corresponding to fixed and unfixed columns respectively. Then the Grassmannian is described by link variables c_{Ji} where J labels the rows via the canonical order-preserving bijection. In the sequel, we will often consider purely gluonic external states, where it is natural to let i range over the positive helicity particles and J the negative helicity ones. For example, to calculate the split helicity $1^+2^+3^-4^-$ amplitude, we could use coordinates,

$$\begin{pmatrix} 1 & 0 & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \end{pmatrix}\,. \tag{1.2.61}$$

We shall denote the set of i by p and the set of J by m with $\mathsf{p} \cup \mathsf{m} = \{1, \dots n\}$. We may now derive the link representation of N^kMHV tree-level superamplitudes,[20]

[19] Amplitudes are expected to have singular behaviour when sums of kinematic variables go on-shell, reflecting the locality of the theory. Indeed this property is manifested by the local interaction and propagator structure of Feynman diagrams. The appearance of poles outside such singular kinematic regions is said to be unphysical.

[20] In this gauge fixing $c_{JK} = \delta_{JK}$. Therefore that $\prod_J \delta^{(2)}(c_{Ja}\tilde{\lambda}_a) = \prod_J \delta^{(2)}(c_{JK}\tilde{\lambda}_K + c_{Ji}\tilde{\lambda}_i) = \prod_J \delta^{(2)}(\tilde{\lambda}_J + c_{Ji}\tilde{\lambda}_i)$. The procedure works similarly for the holomorphic delta functions, upon integrating out the ρ_J spinors.

$$A_{n,k} = \oint \frac{\mathrm{d}^{k\times(n-k)} c_{Ji}}{(1\,2\cdots k)(2\,3\cdots k{+}1)\cdots(n\,1\cdots k{-}1)}$$
$$\times \prod_{i\in\mathsf{p}} \delta^{(2)}\left(\lambda_i - \sum_{J\in\mathsf{m}} c_{Ji}\lambda_J\right) \prod_{J\in\mathsf{m}} \delta^{(2)}\left(\tilde{\lambda}_J + \sum_{i\in\mathsf{p}} c_{Ji}\tilde{\lambda}_i\right) \delta^{(4)}\left(\eta_J + \sum_{i\in\mathsf{p}} c_{Ji}\eta_i\right), \tag{1.2.62}$$

where the contour is taken to enclose a subset of the poles corresponding to the desired BCFW terms. Recently, there have been efforts to derive the contour from deeper principles, for instance by interpreting the amplitude as the volume of a certain generalised polytope called the amplituhedron – see for example [89, 90].

To adapt these results to form factors, one introduces two auxiliary on-shell legs $n{+}1$ and $n{+}2$ with momenta summing to q. The Grassmannian representation for the N^{k-2}MHV tree-level super form factor of the chiral stress tensor multiplet is then an integral over $G(k, n{+}2)$:

$$\langle n{+}1\ n{+}2\rangle^2 \int \frac{\mathrm{d}^{k\times(n+2)} C_{Ja}\ \mathrm{d}^{2k}\rho_J}{|GL(k;\mathbb{C})|}$$
$$\times \sum_{\text{ins}} \frac{\Omega_{n,k}(C)\delta^{2(n+2)}(\rho_J c_{Ja} - \lambda_a)\delta^{(2k)}(c_{Ja}\tilde{\lambda}_a)\delta^{(4k)}(c_{Ja}\eta_a)}{(1\cdots k)\cdots(n+2\cdots k-1)}, \tag{1.2.63}$$

where we define $\bar{\mathsf{p}} = \mathsf{p} \cup \{n{+}1, n{+}2\}$, and the numerator factor is

$$\Omega_{n,k}(C) = \frac{Y}{1-Y}, \quad Y = \frac{(n{+}2{-}k\cdots n\,n{+}1)(n{+}2\,1\cdots k{-}1)}{(n{+}2{-}k\cdots n\,n{+}2)(n{+}1\,1\cdots k{-}1)}. \tag{1.2.64}$$

The sum is over certain insertions of $\{n{+}1, n{+}2\}$ into $\{1, \ldots n\}$. After gauge fixing to the link representation, the contour required may be different for each term of the sum. We shall see a consequence of this subtlety in Section 2.3. A conjectured off-shell version of the amplituhedron [91] appeared during the preparation of this thesis, and may provide an useful new perspective on such issues.

1.2.7 MHV Diagrams

MHV diagrams [92] provide an alternative recursive method for constructing amplitudes and form factors, which can be computationally advantageous. As above, we review the procedure at tree level without proof.[21] For brevity we refer only to amplitudes,

[21] In the original paper, the rules were derived from twistor string theory, reviewed in Section 1.3. Later, Risager found a more direct argument, based on similar ideas to BCFW [93].

but the method works identically for form factors of any operator.[22]

1. Draw all possible colour-ordered diagrams (indexed by I) involving two MHV subamplitudes joined by a single propagator of momentum P_I.

2. Compute the MHV subamplitudes as if P_I were on-shell.

3. Make the replacement $|P_I\rangle \to P_I|\xi]$ in each diagram, where $|\xi]$ is an arbitrary reference spinor.[23]

4. Determine the full amplitude as a sum,

$$A_n^{\text{tree}} = \sum_I A_{\text{left}}^{\text{MHV}}(|\xi])\frac{1}{P_I^2}A_{\text{right}}^{\text{MHV}}(|\xi])\,. \tag{1.2.65}$$

As an example, we compute the $n = 5, k = 1$ tree level form factor of $\text{Tr}(F_{SD}^2)$ with the helicity assignment $1^-2^-3^-4^+5^+$. The MHV diagrams required are depicted in Figure 4. The first one yields the term,

$$-\frac{\langle 2\ 3\rangle^3\langle 1|q|\xi]^2}{s_{2345}\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5|1+q|\xi]\langle 3|1+q|\xi]}\,. \tag{1.2.66}$$

We have evaluated the full result using Mathematica, and employ it for numerical checks in Section 2.5.

1.2.8 Unitarity and The Symbol

In the previous sections, we detailed methods for recursively computing tree-level amplitudes and form factors. It is both phenomenologically and theoretically important to consider loop corrections. Both BCFW [94–96] and MHV diagrams [97] admit loop-level generalisations, particularly useful for constructing loop integrands. However, some properties of a QFT (such as anomalies) typically only appear after regularisation and integration. Here we review two methods for constructing and manipulating such integrated quantities. In this section we shall consider only superamplitudes.

A well-known consequence of the unitarity of the S-matrix is the optical theorem. In the Cutkosky formulation [98], this relates discontinuities across branch cuts in loop amplitudes to products of lower loop amplitudes produced by taking pairs of

[22] The validity of MHV diagrams for higher dimensional operators is particularly useful, since one might expect singular behaviour for $z \to \infty$ under BCFW shifts.

[23] The only restriction is that $|\xi]$ may not be proportional to the antiholomorphic spinor of a negative helicity external leg.

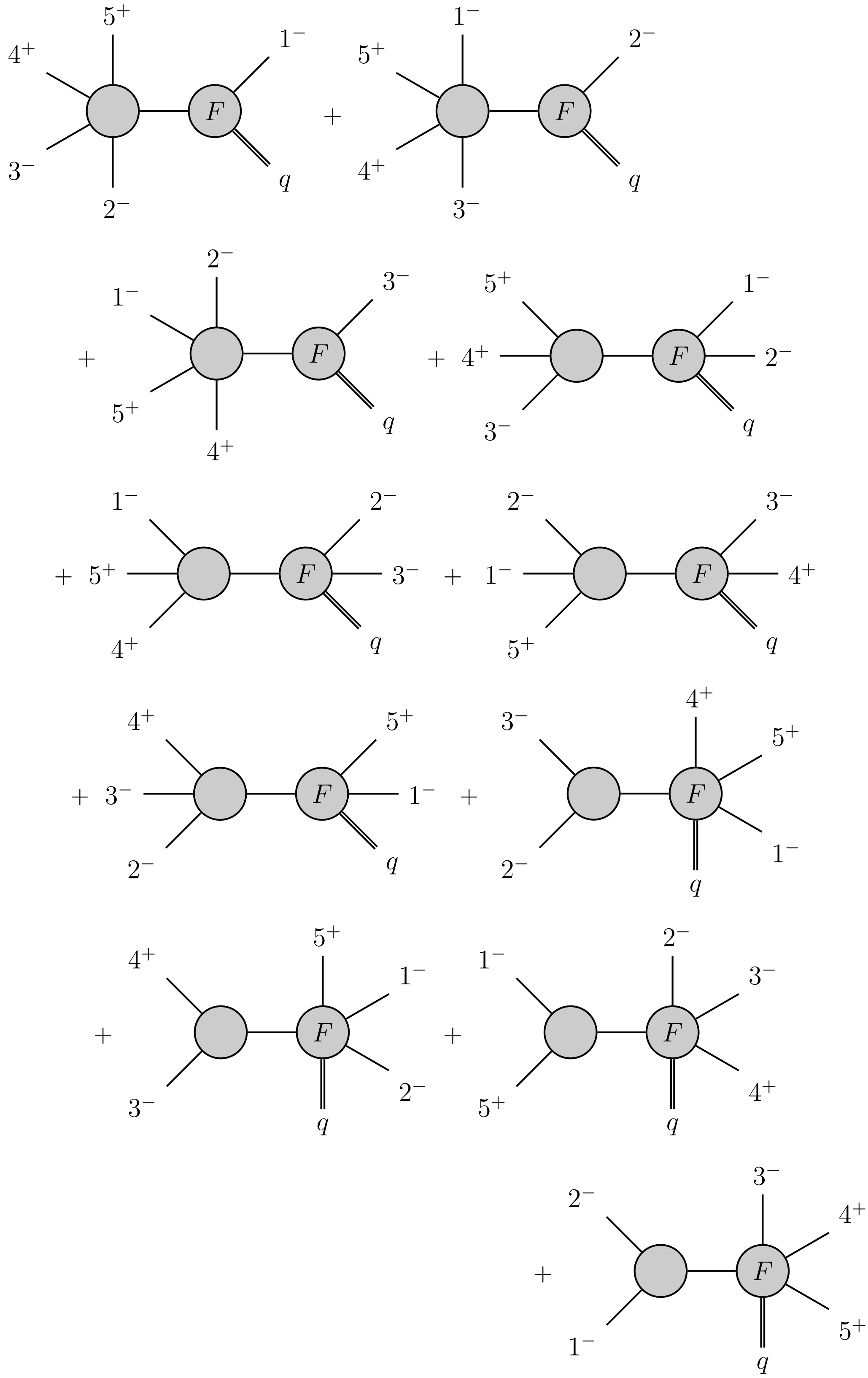

Figure 4: The eleven MHV diagrams required to calculate $F^{1^-2^-3^-4^+5^+}$.

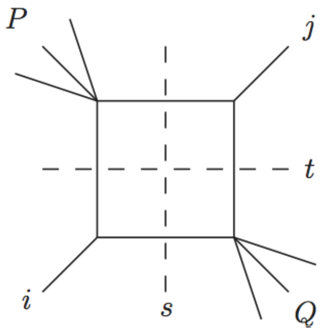

Figure 5: The Feynman diagram defining a generic two-mass easy box function.

propagators on-shell, a procedure known as cutting. In principle, given a sufficiently small basis of scalar integrals, the unitarity cuts suffice to determine the full loop amplitude by comparing coefficients. In a landmark paper [48] this technique was shown to work for one-loop MHV amplitudes in $\mathcal{N} = 4$ SYM. Explicitly the result is[24]

$$\mathcal{A}_n^{\text{1-loop}} = \mathcal{A}_n^{\text{tree}} \sum_{\text{channels}} F^{\text{2me}}\,, \tag{1.2.67}$$

where F^{2me} are generically scalar integrals called two-mass easy box functions, defined by the ϕ^k theory Feynman diagram depicted in Figure 5, where k can be any integer greater than 2.

As we hinted at in Section 1.1, amplitudes involving the propagation of massless particles suffer from IR divergences at loop level. To evaluate the integrals in (1.2.67), we must regularise these divergences, typically by moving to $4 - 2\epsilon$ dimensions. We view ϵ as an IR regulator. Upon integrating, one arrives at

$$\begin{aligned} F^{\text{2me}}(K, L) &= -\frac{1}{\epsilon^2}\left[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-K^2)^{-\epsilon} - (-L^2)^{-\epsilon}\right] + \mathrm{Li}_2\left(1 - \frac{K^2}{s}\right) \\ &+ \mathrm{Li}_2\left(1 - \frac{K^2}{t}\right) + \mathrm{Li}_2\left(1 - \frac{L^2}{s}\right) + \mathrm{Li}_2\left(1 - \frac{L^2}{t}\right) - \mathrm{Li}_2\left(1 - \frac{K^2 L^2}{st}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right)\,, \end{aligned} \tag{1.2.68}$$

where P and Q denote the momenta of massive corners and (s, t) are defined by $(\text{vertical}, \text{horizontal})$ cuts respectively. We must also include degenerate cases in the sum, where one or both of the massive corners become massless. These are given

[24] Here and elsewhere in this thesis we neglect a factor of c_Γ as defined in (1.2.50).

by

$$F^{0\mathrm{m}} = -\frac{1}{\epsilon^2}\left[(-s)^{-\epsilon} + (-t)^{-\epsilon}\right] + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \frac{\pi^2}{2}\,,$$
$$F^{1\mathrm{m}}(K) = -\frac{1}{\epsilon^2}\left[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-K^2)^{-\epsilon}\right] + \mathrm{Li}_2\left(1 - \frac{K^2}{s}\right) + \mathrm{Li}_2\left(1 - \frac{K^2}{t}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \frac{\pi^2}{6}\,. \tag{1.2.69}$$

The technique was extended to construct one-loop NMHV superamplitudes in [99], demonstrating that these may be written in terms of dual superconformal R-invariants and V-functions.[25] These quantities are most conveniently expressed in momentum twistor variables, so we shall delay providing explicit examples until Section 1.3.

In physical processes, IR divergences arising from loop integrations cancel against soft and collinear divergences from lower loop amplitudes upon computing the cross-section [103, 104].[26] This cancellation suggests that such divergences take a universal form, independent of the number of particles and helicity configuration. For example, the IR divergent terms for one-loop amplitudes in $\mathcal{N} = 4$ SYM take the form [105],

$$-\frac{1}{\epsilon^2}\sum_{i=1}^{n}\big(-(i\; i+1)\big)^{-\epsilon}\,, \tag{1.2.70}$$

up to a factor of A_n^{tree}, where $(i\; j) := (p_i + p_j)^2$.

Having constructed loop-level superamplitudes, we now consider a method for their manipulation and simplification. The symbol [106] is a powerful way to represent transcendental functions, converting functional relations to algebraic identities. More specifically, it is a map taking transcendental functions to tensor products of their rational arguments. In particular,

$$\mathrm{Sym}\left[\log(R_a)\log(R_b)\right] = R_a \otimes R_b + R_b \otimes R_a\,, \tag{1.2.71}$$
$$\mathrm{Sym}\left[\mathrm{Li}_2(1 - R_a)\right] = -R_a \otimes (1 - R_a)\,. \tag{1.2.72}$$

The target space is defined modulo the identifications,

$$R_aR_b \otimes R_cR_d = R_a \otimes R_c + R_b \otimes R_c + R_a \otimes R_d + R_b \otimes R_d\,, \tag{1.2.73}$$
$$\text{constant} \otimes R_a = R_a \otimes \text{constant} = 0\,, \tag{1.2.74}$$
$$R_a \otimes (R_b)^{-1} = (R_a)^{-1} \otimes R_b = -R_a \otimes R_b\,. \tag{1.2.75}$$

[25] A general formula for one-loop NMHV gluon amplitudes was already known [100]. The proof of dual conformality for arbitrary n was completed in [101, 102].

[26] Each external carries a factor of $\hbar$ by virtue of the commutation relation $[a, a^\dagger] = \hbar$, so the power counting agrees.

As an example, we use the symbol to compute a version of the one-loop MHV superamplitude first derived in [9], inspired by partial results in [107,108]. The finite part[27] of a two-mass easy box may compactly be defined as [97]

$$F^{\text{2me,fin}}(P,Q,s,t) = \mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at)\,, \tag{1.2.76}$$

where

$$a = \frac{P^2+Q^2-s-t}{P^2Q^2-st}\,. \tag{1.2.77}$$

It is convenient to write the momentum invariants in terms of differences of dual momenta,

$$P^2 = x_{i+1j}^2\,, \quad Q^2 = x_{ij+1}^2\,, \quad s = x_{ij}^2\,, \quad t = x_{i+1j+1}^2\,. \tag{1.2.78}$$

We evaluate the symbols,

$$\begin{aligned}
\mathrm{Sym}\left[\mathrm{Li}_2(1-aP^2)\right] &= a\otimes(P^2Q^2-st) + P^2\otimes(P^2Q^2-st)\\
&\quad - a\otimes(s-P^2)(P^2-t) - P^2\otimes(s-P^2)(P^2-t)\,,\\
\mathrm{Sym}\left[\mathrm{Li}_2(1-aQ^2)\right] &= a\otimes(P^2Q^2-st) + Q^2\otimes(P^2Q^2-st)\\
&\quad - a\otimes(s-Q^2)(Q^2-t) - Q^2\otimes(s-Q^2)(Q^2-t)\,,\\
\mathrm{Sym}\left[\mathrm{Li}_2(1-as)\right] &= a\otimes(P^2Q^2-st) + s\otimes(P^2Q^2-st)\\
&\quad - a\otimes(P^2-s)(Q^2-s) - s\otimes(P^2-s)(Q^2-s)\,,\\
\mathrm{Sym}\left[\mathrm{Li}_2(1-as)\right] &= a\otimes(P^2Q^2-st) + t\otimes(P^2Q^2-st)\\
&\quad - a\otimes(P^2-t)(Q^2-t) - t\otimes(P^2-t)(Q^2-t)\,.
\end{aligned} \tag{1.2.79}$$

The first and third terms in each symbol cancel in the sum defining the symbol of $F^{\text{2me,fin}}$. The second terms combine to yield

$$\frac{P^2Q^2}{st}\otimes(P^2Q^2-st) = \frac{x_{ij+1}^2x_{i+1j}^2}{x_{ij}^2x_{i+1j+1}^2}\otimes(x_{i+1j}^2x_{ij+1}^2 - x_{ij}^2x_{i+1j+1}^2)\,. \tag{1.2.80}$$

It is convenient to write the fourth terms in dual variables,

$$P^2\otimes(s-P^2)(P^2-t) = x_{i+1j}^2\otimes(x_{ij}^2-x_{i+1j}^2) + x_{i+1j}^2\otimes(x_{i+1j}^2-x_{i+1j+1}^2)\,, \tag{1.2.81}$$

$$Q^2\otimes(s-Q^2)(Q^2-t) = x_{ij+1}^2\otimes(x_{ij}^2-x_{ij+1}^2) + x_{ij+1}^2\otimes(x_{ij+1}^2-x_{i+1j+1}^2)\,, \tag{1.2.82}$$

$$s\otimes(P^2-s)(Q^2-s) = x_{ij}^2\otimes(x_{i+1j}^2-x_{ij}^2) + x_{ij}^2\otimes(x_{ij+1}^2-x_{ij}^2)\,, \tag{1.2.83}$$

$$t\otimes(P^2-t)(Q^2-t) = x_{i+1j+1}^2\otimes(x_{i+1j}^2-x_{i+1j+1}^2) + x_{i+1j+1}^2\otimes(x_{ij+1}^2-x_{i+1j+1}^2)\,. \tag{1.2.84}$$

To produce the complete finite part of the amplitude we must sum over all distinct

[27]The finite part of the box function is defined to be the part of (1.2.68) without explicit dependence on ϵ. Note that this does not capture the full $\mathcal{O}(\epsilon^0)$ behaviour.

boxes. This corresponds to summing over all i and j not adjacent and dividing by a factor of 2. We now apply this procedure to the symbols (1.2.80)–(1.2.84) to exhibit hidden cancellations.

Consider for fixed i the telescoping sum,

$$\sum_{j \notin \{i-1,i,i+1\}} A_{ij+1} - A_{ij} = A_{ii-1} - A_{ii+2}\,. \tag{1.2.85}$$

We may employ this formula to find the contribution of (1.2.81)–(1.2.84) to the full symbol. The resulting term is

$$\sum_i \mathrm{Sym}\left[\log^2(x^2_{ii-2})\right]\,. \tag{1.2.86}$$

We now massage (1.2.80) into a form we can integrate, writing

$$\frac{x^2_{ij+1}x^2_{i+1j}}{x^2_{ij}x^2_{i+1j+1}} \otimes (x^2_{i+1j}x^2_{ij+1} - x^2_{ij}x^2_{i+1j+1}) \;=\; u_{ij} \otimes (1-u_{ij}) + u_{ij} \otimes x^2_{ij}x^2_{i+1j+1}\,. \tag{1.2.87}$$

We immediately identify the first term as the symbol of $-\mathrm{Li}_2(1-u_{ij})$. The second term expands to give neatly paired contributions,

$$\begin{aligned} &x^2_{ij+1} \otimes x^2_{ij} + x^2_{i+1j} \otimes x^2_{i+1j+1} + x^2_{ij+1} \otimes x^2_{i+1j+1} + x^2_{i+1j} \otimes x^2_{ij} \\ &- x^2_{ij} \otimes x^2_{ij} - x^2_{i+1j+1} \otimes x^2_{i+1j+1} - x^2_{ij} \otimes x^2_{i+1j+1} - x^2_{i+1j+1} \otimes x^2_{ij}\,. \end{aligned} \tag{1.2.88}$$

Performing the sum over non-adjacent i and j we find that

$$\begin{aligned} &\sum_i \sum_{j \notin \{i-2,i-1,i,i+1\}} \mathrm{Sym}\left[\log(x^2_{ij})\log(x^2_{ij+1})\right] + \sum_i \sum_{j \notin \{i-1,i,i+1,i+2\}} \mathrm{Sym}\left[\log(x^2_{ij})\log(x^2_{i+1j})\right] \\ &- \sum_i \sum_{j \notin \{i-1,i,i+1\}} \mathrm{Sym}\left[\log(x^2_{ij})\log(x^2_{i+1j+1})\right] - \sum_i \sum_{j \notin \{i-1,i,i+1\}} \mathrm{Sym}\left[\log(x^2_{ij})\log(x^2_{ij})\right]\,. \end{aligned} \tag{1.2.89}$$

Combining the terms (1.2.86) and (1.2.89), integrating the symbol[28] and dividing by 2 yields

$$\sum_i \left[\frac{1}{2}\sum_{j \notin \{i-2,i-1,i,i+1,i+2\}} \log(x^2_{ij})\log(u_{ij}) + \log(x^2_{ii-2})\log\left(\frac{x^2_{i+1i-2}}{x^2_{i+1i-1}}\right)\right]\,. \tag{1.2.90}$$

[28] For the uninitiated, this is a colloquial term for inverting the map which projects transcendental functions onto their symbols. The inverse is unique up to rational terms, which one usually fixes either by symmetry arguments or by considering explicit examples.

We finally split our expression for the finite part of the amplitude into generic terms,

$$\frac{1}{2}\sum_{i}\sum_{j\notin\{i-2,i-1,i,i+1,i+2\}}\left(-\mathrm{Li}_2(1-u_{ij})+\log x_{ij}^2\log u_{ij}\right)\,, \tag{1.2.91}$$

and edge cases,

$$\sum_{i}\log(x_{ii-2}^2)\log\left(\frac{x_{i+1i-2}^2}{x_{i+1i-1}^2}\right)\,. \tag{1.2.92}$$

Note that the edge cases comprise the full finite part of the amplitude at five-point, which we have verified by comparison with [48].

Finally, we may naturally incorporate the universal IR divergent terms (1.2.70), yielding an expression for the full MHV superamplitude,

$$\begin{aligned}\frac{A_n^{\text{1-loop}}}{A_n^{\text{tree}}}=&\frac{1}{2}\sum_{i}\sum_{j\notin\{i-2,i-1,i,i+1,i+2\}}\left(-\mathrm{Li}_2(1-u_{ij})+\log x_{ij}^2\log u_{ij}\right)\\&+\sum_{i}\log(x_{ii-2}^2)\log\left(\frac{x_{i+1i-2}^2}{x_{i+1i-1}^2\sqrt{x_{ii-2}^2}}\right)\,.\end{aligned} \tag{1.2.93}$$

We shall make use of this formula in Chapter 3.

1.2.9 Soft Limits

In the previous sections, we have concentrated entirely on constructing convenient representations for superamplitudes and super form factors. However, it is also useful to study particular properties of such quantities, especially if the behaviour happens to be universal, in the sense we described above (1.2.70). The soft limits of amplitudes are a good example of this phenomenon.

Leading soft divergences were first studied at tree level in QED [109], and subsequently determined in gravity [110] and Yang-Mills theories [111]. The resulting factorisation was shown to be universal, in accordance with our expectation that IR singularities should cancel in inclusive cross-sections. It is natural to ask whether similar properties hold at subleading soft order. This question was answered in the affirmative long ago for QED [112, 113]. However, the analogous results in gravity and Yang-Mills theories were derived only recently [5–7].

There are both theoretical and practical reasons underpinning the current interest in subleading soft theorems. Naïvely one might argue that the subleading soft behaviour is mathematically unimportant, since it is not divergent.[29] However, this is phenomenologically misleading. Indeed, a central problem in collider physics is to manage the large logarithms arising from the cancellation of IR divergences, which threaten to invalidate perturbation theory. Typically, one generates accurate predictions by resumming the large terms, making use of their universal properties. In this context, information about next-to-soft terms can reduce the theoretical error of certain observables – see for example [114].

From a theoretical perspective, subleading soft theorems may be viewed as Ward identities for symmetries acting at null infinity [115–117]. In gauge theory, the symmetry is locally a Kac-Moody algebra, while in gravity it is described by the extended BMS group [118]. This has inspired two dimensional holographic CFTs reproducing single soft [119, 120] and multi-soft [121] factorisation from current correlators, and related string models [122,123]. Moreover, Hawking et al. used similar ideas to propose a novel solution to the black hole information paradox [124].

It is clearly important to ask whether universal behaviour prevails at one-loop level. It has been known since the early days that leading soft behaviour does not get renormalised in gravity [110]. This good IR behaviour intuitively arises from the dimensionful coupling in the Einstein-Hilbert action [125]. Leading soft behaviour is universally renormalised in QED [126] and QCD [127, 128]. Much less is known about the subleading soft theorems at loop level, particularly in Yang-Mills theory. In Chapter 3, we extend the partial results [129–131] through finite order in the IR regulator, and to higher point amplitudes, finding evidence for a limited form of universality.

Henceforth, we shall focus exclusively on soft gluon limits in $\mathcal{N} = 4$ SYM.[30] We briefly review the standard results, in preparation for further calculations in Chapter 3. Consider the holomorphic soft limit of a positive-helicity gluon n^+ in an n-particle amplitude,

$$|n\rangle \to \delta|n\rangle \ , \quad |n] \to |n] \ , \quad p_n \to \delta p_n \, . \tag{1.2.94}$$

Clearly this is related by the little group transformation (1.2.18) to the democratic soft scaling of holomorphic and antiholomorphic spinors. In common with much of the recent literature, we find it convenient to use the holomorphic soft scaling, since then the subleading behaviour appears as a divergent term in δ.

In this section, and in Chapter 3, it will be important to distinguish between amplitudes written with and without the momentum conservation delta function. We let

[29]This argument is not quite watertight. Indeed, we shall see in Chapter 3 that the subleading soft terms include divergent $\log \delta$ contributions at loop level.

[30]Note that soft theorems can also be formulated for gluinos and scalars [132], but the subleading behaviour for these particles is suppressed relative to gluons. Therefore, we do not consider such cases in this thesis.

$\mathcal{A}_n$ denote an n-particle colour-ordered superamplitude, including the aforementioned delta function. Then expanding in δ one has, at tree level [7],

$$\mathcal{A}_n^{\text{tree}} \to \left(\frac{1}{\delta^2} S^{(0)} + \frac{1}{\delta} S^{(1)}\right) \mathcal{A}_{n-1}^{\text{tree}} , \tag{1.2.95}$$

where $S^{(0)}$ and $S^{(1)}$ are given by

$$S^{(0)} = \frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle\langle n\ 1\rangle} , \tag{1.2.96}$$

$$S^{(1)} = \frac{|n]}{\langle n-1\ n\rangle} \cdot \frac{\partial}{\partial |n-1]} + \frac{|n]}{\langle n\ 1\rangle} \cdot \frac{\partial}{\partial |1]} . \tag{1.2.97}$$

Note that these operators are antisymmetric about particle n. This is enforced, since colour-ordered n-point amplitudes obey a reflection symmetry in particle labels up to a factor of $(-1)^n$.

To perform practical calculations it is convenient to work with stripped amplitudes A_n, where

$$\mathcal{A}_n = A_n\, \delta^{(4)}(P_n) , \tag{1.2.98}$$

with $P_n := \sum_{i=1}^n p_i$. Note that a stripped amplitude is ambiguous without a momentum conservation prescription. One means of resolving this is by eliminating two antiholomorphic spinors $|a]$ and $|b]$ [6]. We may define such an elimination for any function f of external kinematics as

$$f^{(ab),n} = \int \mathrm{d}|a]\mathrm{d}|b]\ |\langle a\ b\rangle|\, \delta^{(4)}(P_n)\, f , \tag{1.2.99}$$

so that an unambiguous stripped amplitude may be written as $A_n^{(ab),n}$. Clearly it is useful to have an explicit prescription for performing the integral in (1.2.99). We impose the equalities,

$$|a] = \frac{1}{\langle a\ b\rangle} \sum_{i\neq a}^{n} \langle b\ i\rangle |i], \qquad |b] = \frac{1}{\langle b\ a\rangle} \sum_{i\neq b}^{n} \langle a\ i\rangle |i] . \tag{1.2.100}$$

These relations are especially important when considering the soft behaviour at one loop, which turns out to depend on the choice of $|a]$ and $|b]$.[31]

Taking the integrals through the derivatives in (1.2.95) proves the result for stripped amplitudes,

$$A_n^{\text{tree}(ab),n} \to \left(\frac{1}{\delta^2} S^{(0)} + \frac{1}{\delta} S^{(1)}\right) A_{n-1}^{\text{tree}(ab),n-1} , \tag{1.2.101}$$

[31] In other words, it depends on how one implements momentum conservation, in a way similar to stripped amplitudes.

as found in [6] in the case of gravity.

In [129] Bern, Nohle and Davies argued for a statement equivalent to (1.2.95), namely

$$\mathcal{A}_n^{\text{tree}} \;\to\; \delta^{(4)}(P_n)\left(\frac{1}{\delta^2}S^{(0)}+\frac{1}{\delta}S^{(1)}\right)\mathcal{A}_{n-1}^{\text{tree}}\,, \qquad (1.2.102)$$

where the momentum conservation delta function sits in front of the soft operator. The distinguishing property of this expression is that it features n-point momentum conservation on both sides. Many explicit examples have been calculated in the literature demonstrating the equivalence of (1.2.95) and (1.2.102) and the issue was discussed formally in [133]. One may verify this equivalence by Taylor expanding $\delta^{(4)}(P_n)$ and applying the chain rule to $S^{(1)}\delta^{(4)}(P_{n-1})$.

We may easily write down the stripped amplitude version of (1.2.102); that is

$$A_n^{\text{tree}(ab),n} \to \left[\left(\frac{1}{\delta^2}S^{(0)}+\frac{1}{\delta}S^{(1)}\right)A_{n-1}^{\text{tree}}\right]^{(ab),n}\,. \qquad (1.2.103)$$

This formulation has an advantage over (1.2.101) because it allows one to adopt the following two step strategy to verify soft theorems:

1. Choose arbitrary forms for A_n and A_{n-1} and determine

$$A_n - \frac{1}{\delta^2}S^{(0)}A_{n-1} - \frac{1}{\delta}S^{(1)}A_{n-1}\,. \qquad (1.2.104)$$

2. Apply n-point momentum conservation and expand in δ, then one finds zero up to $\mathcal{O}(\delta^0)$.

We emphasise that this approach leads to so-called feed-down terms from Taylor-expanding the term

$$\left[-\frac{1}{\delta^2}S^{(0)}A_{n-1}^{\text{tree}}\right]^{(ab),n}\,, \qquad (1.2.105)$$

in (1.2.104), evaluated using (δ-dependent) n-point momentum conservation. In this thesis we shall consider soft theorems only in the language of (1.2.102) and (1.2.103), which is better suited to a loop-level generalisation.

At one-loop level, the leading soft behaviour is well-known [127, 128, 134]. Subleading soft theorems for the infrared-divergent part of generic one-loop amplitudes were found in [129]. Based on this, one may conjecture the one-loop extension to the subleading

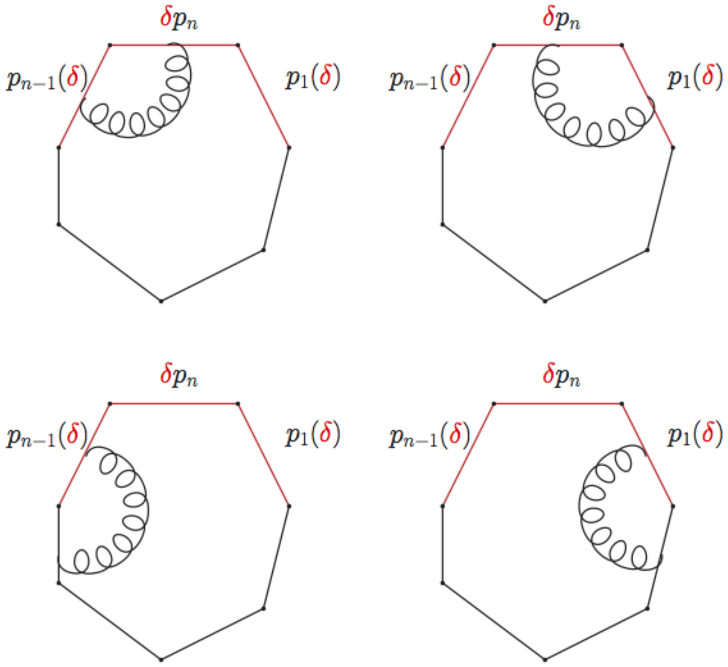

Figure 6: *The four diagrams contributing to the infrared-divergent terms in the soft theorem at one loop.*

soft theorem,

$$A_n^{\text{1-loop}} \to \frac{1}{\delta^2}\left(S^{(0)}A_{n-1}^{\text{1-loop}} + S^{(0)\text{1-loop}}A_{n-1}^{\text{tree}}\right) + \frac{1}{\delta}\left(S^{(1)}A_{n-1}^{\text{1-loop}} + S^{(1)\text{1-loop}}A_{n-1}^{\text{tree}}\right), \tag{1.2.106}$$

where the leading soft factor is [127, 128],

$$S^{(0)\text{1-loop}} = S^{(0)}F^{(0)}, \qquad F^{(0)} = \left(\frac{c_\Gamma}{\epsilon^2}\frac{\pi\epsilon}{\sin(\pi\epsilon)}\right)\left(-\frac{1}{\delta^2}\frac{(n-1\ 1)}{(n-1\ n)(n\ 1)}\right)^{\epsilon}, \tag{1.2.107}$$

the infrared-divergent part of the subleading soft operator is [129]

$$\begin{aligned} S^{(1)\text{1-loop}}\Big|_{\text{div.}} &= \frac{c_\Gamma}{\epsilon^2}\left[1+\epsilon\log\left(-\frac{1}{\delta^2}\frac{(n-1\ 1)}{(n-1\ n)(n\ 1)}\right)\right]S^{(1)\text{tree}} \\ &+ \frac{c_\Gamma}{\epsilon}\left[\frac{[n-1\ n]}{[n-1\ 1]\langle 1\ n\rangle} + \frac{[2\ n]}{[2\ 1]\langle 1\ n\rangle} - \frac{[1\ n]}{[1\ n-1]\langle n-1\ n\rangle} - \frac{[n-2\ n]}{[n-2\ n-1]\langle n-1\ n\rangle}\right], \end{aligned} \tag{1.2.108}$$

and the notation is defined around (1.2.50). More generally we conjecture that the subleading soft operator takes the form,

$$S^{(1)\text{1-loop}} = F^{(1)}S^{(1)} + c_\Gamma Z\,. \tag{1.2.109}$$

Here $F^{(1)}$ and Z are functions of external kinematics. We shall refer to Z as the sub-

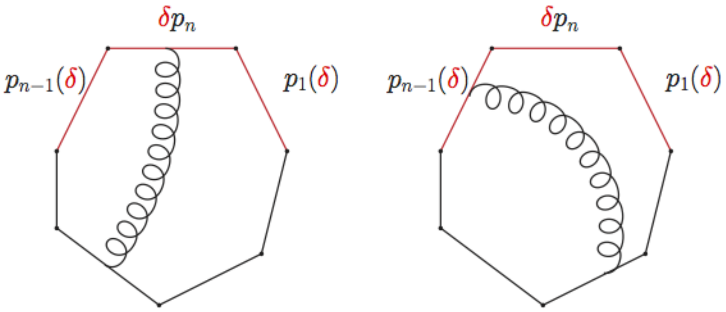

Figure 7: *Two of the* $3n-10$ *diagrams contributing to the finite terms in the soft theorem at one loop.*

leading soft anomaly. Note that Z is only defined up to a momentum conservation prescription – it is frame dependent. Nevertheless it remains a useful and practical quantity. Indeed we may immediately transform Z between frames using the elimination (1.2.100). In Section 3.2 we will fix $F^{(1)}$ and derive a differential constraint on Z. Section 3.3 then provides explicit computations of Z for amplitudes in the MHV and NMHV sectors. All of our results will be valid through finite order in ϵ.

From a Wilson loop perspective, one-loop amplitudes decompose into a sum of diagrams with one internal gluon. Evaluating each diagram requires a ultraviolet regulator ϵ which corresponds exactly to the infrared regulator of the loop amplitude. Only diagrams in which a gluon attaches to a δ-dependent external momentum will contribute to the one-loop soft anomaly.

It is useful to distinguish the diagrams in which the internal gluon connects adjacent edges of the polygon. These have a ultraviolet cusp divergence, and in fact capture all infrared-divergent terms in the amplitude [78]. This restriction limits the number of diagrams required to analyse the infrared-divergent soft anomaly. In fact, choosing a symmetric momentum conservation prescription eliminating $(|n-1], |1])$ we see that the four diagrams in Figure 6 suffice.

The remaining diagrams generate the finite parts of box functions [71,72,75]. Examples are displayed in Figure 7. It is important to note a conceptual subtlety: although the terms from these diagrams are independent of ϵ they still contribute to the subleading soft anomaly. The large number of contributing diagrams makes finite order analysis significantly harder; nevertheless in Section 3.3 we shall see surprising cancellations leading to compact formulae.

1.3 Twistors, Strings and Scattering Equations

In the previous section, we used Lorentz symmetry to motivate spinor-helicity variables, supersymmetry to constrain amplitudes and form factors, and translation symmetry to inspire dual coordinates. However, we have failed to exploit the two superconformal symmetries enjoyed by $\mathcal{N} = 4$ SYM. We address this omission by introducing supertwistors and momentum supertwistors, which transform in the fundamental representation of superconformal and dual superconformal symmetry respectively.

The complexified[32] superconformal group is locally isomorphic to $PSL(4|4, \mathbb{C})$. Therefore it is natural to define supertwistor space as $\mathbb{P}^3$ equipped with four additional Grassmann coordinates. We denote a twistor[33] in homogeneous coordinates by

$$Z^{\mathsf{A}} = (\lambda_\alpha, \tilde{\mu}^{\dot{\alpha}}, \chi^A) \, . \tag{1.3.1}$$

Penrose and Ferber [136, 137] determined the correct way to identify twistors with complex chiral Minkowsi superspace variables, namely via the incidence relations,

$$\tilde{\mu}^{\dot{\alpha}} = i x^{\alpha\dot{\alpha}} \lambda_\alpha \, , \quad \chi^A = \theta^A_\alpha \lambda^\alpha \, . \tag{1.3.2}$$

The bosonic relations have a beautiful geometric interpretation, summarised in Figure 8. In particular the conformal structure of spacetime, determined by its null cones, is mapped onto the complex structure of twistor space, determined by its projective lines. A priori it is not clear how to express functions on spacetime (such as amplitudes) in terms of twistor variables. The dictionary is provided by the Penrose transform, which we now review without supersymmetry.

1.3.1 Penrose Transform

The field configurations associated with on-shell massless fields naturally biject with (Čech or Dolbeault) cohomology classes of functions on twistor space according to the Penrose transform [138]. While the Čech representation is naturally associated with topology, the Dolbeault version is closely tied to holomorphicity. The latter property is more obviously advantageous for the computation of scattering amplitudes, so we shall review the Dolbeault approach.

Recall that a complex manifold $\mathcal{M}$ possesses an almost complex structure, allowing us to split the complexified tangent bundle into holomorphic and antiholomorphic parts.

[32] The restriction to real spacetime coordinates requires additional conditions on twistor space, comprehensively reviewed in [135]. We sidestep this issue by using complex variables throughout.

[33] Henceforth we refer to supertwistors as twistors for brevity.

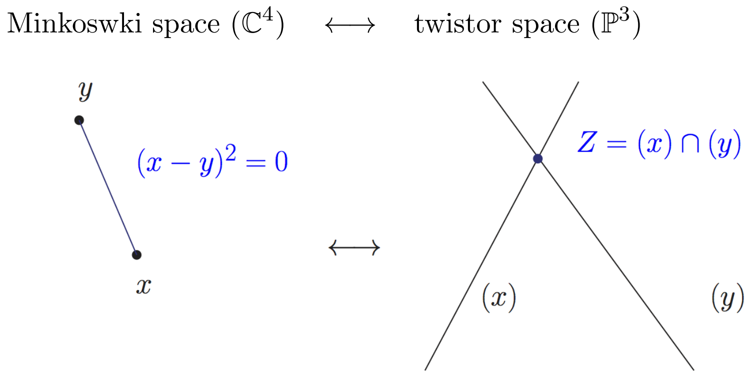

Figure 8: The geometry of the bosonic twistor correspondence.

An (r, s)-form is defined to be a complexified $(r+s)$-form ω satisfying $\omega(V_1, \dots V_{r+s}) = 0$ unless r of the V_i are holomorphic and s are antiholomorphic. We note immediately that the exterior dervative of an (r, s) form is the sum of an $(r+1, s)$-form and an $(r, s+1)$-form, yielding the decomposition, $d = \partial + \bar{\partial}$. We may now define the Dolbeault cohomology classes,[34]

$$H^{r,s}(\mathcal{M}) = \{\omega : \bar{\partial}\omega = 0\}/\{\omega : \omega = \bar{\partial}\eta\}\,, \tag{1.3.3}$$

where ω denotes an (r, s)-form on $\mathcal{M}$. The bijection provided by the Penrose transform may then be written:[35]

$$\begin{aligned}&\{\text{on-shell massless fields of helicity } \frac{h}{2} \text{ on } \mathbb{C}^4\}\\ &\qquad\qquad\longleftrightarrow \{f \in H^{0,1}(\mathbb{P}^3) : f \text{ homogeneous of degree } h-2\}\,,\end{aligned} \tag{1.3.4}$$

where we view f as a function of the bosonic part of the supertwistor homogeneous coordinates Z^{A}. Explicitly the transform may be performed via an integration,

$$\begin{aligned}\phi^{\alpha_1\dots\alpha_h}(x) &= \frac{1}{2\pi i}\int_{\mathbb{P}^1} \lambda^{\alpha_1}\dots\lambda^{\alpha_h} f(ix^{\beta\dot{\beta}}\lambda_\beta, \lambda_\beta) \wedge \lambda^\gamma d\lambda_\gamma\,,\\ \phi^{\dot{\alpha}_1\dots\dot{\alpha}_h}(x) &= \frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{\partial}{\partial\tilde{\mu}_{\dot{\alpha}_1}}\dots\frac{\partial}{\partial\tilde{\mu}_{\dot{\alpha}_h}} f(ix^{\beta\dot{\beta}}\lambda_\beta, \lambda_\beta) \wedge \lambda^\gamma d\lambda_\gamma\,,\end{aligned} \tag{1.3.5}$$

over the projective line incident with the spacetime point x. Indeed it is then simple

[34] The selection of $\bar{\partial}$ is convenientional, and without loss of generality, since complex conjugation provides an isomorphism between the different definitions.

[35] Strictly speaking the transform is defined on suitably chosen open sets, rather than globally. Such subtleties shall not be important for our purposes.

to verify that the following field equations are satisfied,

$$
\begin{aligned}
\partial_{\alpha_1\dot{\alpha}}\phi^{\alpha_1\ldots\alpha_h}(x) &= 0 \quad \text{for helicity } +h/2\,, \\
\partial_{\alpha\dot{\alpha}_1}\phi^{\dot{\alpha}_1\ldots\dot{\alpha}_h}(x) &= 0 \quad \text{for helicity } -h/2\,, \\
\partial_{\alpha\dot{\alpha}}\partial^{\alpha\dot{\alpha}}\phi(x) &= 0 \quad \text{for helicity } 0\,,
\end{aligned}
\tag{1.3.6}
$$

by realising that $\partial_{\alpha\dot{\alpha}} = i\lambda_\alpha \frac{\partial}{\partial\bar{\mu}^{\dot{\alpha}}}$, by virtue of the incidence relations (1.3.2). In fact, this observation may be taken as a starting point for the twistor correspondence, re-expressing it as a half-Fourier transform [1]. Similarly the fermionic coordinates η_a^A for on-shell amplitudes are Fourier conjugates of the supertwistor components χ_a^A representing an asymptotic state a.[36] This perspective provides the most natural means for translating superamplitudes in spinor-helicity notation into twistor language. Strictly speaking, the proof of equivalence is valid in split signature [139], but again complex momenta render this subtlety irrelevant.

It is instructive to compute an explicit example, to illustrate the integration procedure, taking inspiration from [140,141]. For simplicity we shall consider a scalar field, encoded by the holomorphic twistor space $(0,1)$-form of homogeneity -2,

$$
f(Z) = \frac{\bar{Z}_{\mathsf{A}} d\bar{Z}^{\mathsf{A}}}{\left(Z^{\mathsf{B}}\bar{Z}_{\mathsf{B}}\right)^2}\,. \tag{1.3.7}
$$

Upon restricting to the projective line defined by x, the integral (1.3.5) gives

$$
\phi(x) = \frac{1}{\pi}\int \frac{\langle\lambda d\lambda\rangle\langle\bar{\lambda}d\bar{\lambda}\rangle}{x^2\langle\lambda\bar{\lambda}\rangle^2}\,, \tag{1.3.8}
$$

where $\bar{\lambda}_\alpha = (\bar{\lambda}_2, -\bar{\lambda}_1)$. The measure is nothing but the Fubini-Study metric on $\mathbb{P}^1$ in homogeneous coordinates. The integration hence gives the volume of a 2-sphere of radius $\frac{1}{2}$, yielding

$$
\phi(x) = \frac{1}{x^2}\,. \tag{1.3.9}
$$

which satisfies the wave equation away from $x = 0$. Of course, to describe external particles in scattering processes we shall be interested in a particular class of on-shell massless fields, namely the momentum eigenstates. By Fourier transforming the field equations (1.3.6) we immediately see that these take the form,

$$
\phi_a^{\alpha_1\ldots\alpha_h}(x) = \lambda_a^{\alpha_1}\ldots\lambda_a^{\alpha_h}e^{ip\cdot x}\,, \quad \phi_a^{\dot{\alpha}_1\ldots\dot{\alpha}_h}(x) = \tilde{\lambda}_a^{\dot{\alpha}_1}\ldots\tilde{\lambda}_a^{\dot{\alpha}_h}e^{ip\cdot x}\,, \tag{1.3.10}
$$

in agreement with our formulae in (1.2.23) and (1.2.24), where a labels the particles and

[36] It is common to abuse terminology and define the fermionic components of an external state twistor to be η_a^A. We use such a definition in Sections 1.3.2 and 2.1.

$p_a = \lambda_a \tilde{\lambda}_a$. In twistor space, we define equivalent cohomology representatives,

$$V_a(\lambda, \tilde{\mu}) = \int_{\mathbb{C}} \frac{ds_a}{s_a^{h-1}}\, e^{s_a[\tilde{\mu}\tilde{\lambda}_a]}\, \bar{\delta}^{(2)}(\lambda_a - s_a\lambda)\,, \tag{1.3.11}$$

for particles of helicity $h/2$, where $\bar{\delta}^{(2)}$ is a $(0,1)$-form ensuring that V_a is supported only when λ projectively coincides with λ_a. Upon substitution into the integral formulae (1.3.5), and integrating out s_a to effect the constraint, the exponent automatically becomes $ix{\cdot}p$ and the prefactor is trivially the polarisation data encoded in (1.3.10).

1.3.2 From Twistor String Theory to Scattering Equations

In [142], Witten proposed that Yang-Mills amplitudes could be calculated using a topological string theory. More precisely, he demonstrated that certain $\mathcal{N} = 4$ SYM amplitudes localise on curves in supertwistor space $\mathbb{P}^{3|4}$ and conjectured that such a property holds at all loops. In this formalism, to calculate an N^{k-2}MHV amplitude one must integrate over the moduli space of curves of degree $k-1$. In a sense this work was the generalisation of Nair's observation that tree-level MHV amplitudes in Yang-Mills theory may be obtained from a two-dimensional Wess-Zumino-Witten model [51].

We motivate the theory by considering the twistor space representation of MHV amplitudes. We work in split signature $(--++)$ where bosonic twistor space may be regarded as the real manifold $\mathbb{RP}^3$. In this case the Penrose transform reduces to the Witten half-Fourier transform, as we mentioned in Section 1.3.1. Therefore, to move from momentum space to twistor space, we simply Fourier transform with respect to the antiholomorphic spinors $|j]$. For MHV amplitudes all dependence on the antiholomorphic spinors lies in the momentum conserving delta function so we may write

$$A_n^{\mathrm{MHV}}(Z) = g(|i\rangle) \int \left(\prod_{j=1}^{n} \mathrm{d}^2|j] e^{i[j\mu_j]} \right) \delta^4(P)\,. \tag{1.3.12}$$

The delta function may be expressed as an ordinary Fourier integral,

$$\delta^4(P) = \int \mathrm{d}^4 x\, e^{-ix_{\alpha\dot{\alpha}} \sum_j |j\rangle^{\alpha} [j|^{\dot{\alpha}}}\,. \tag{1.3.13}$$

Substituting and exchanging the order of integration we find

$$A_n^{\mathrm{MHV}}(Z) = g(|i\rangle) \int \mathrm{d}^4 x \left(\prod_{j=1}^{n} \delta^2(|\mu_j]_{\dot{\alpha}} + x_{\alpha\dot{\alpha}}|j\rangle^{\alpha}) \right)\,. \tag{1.3.14}$$

Hence in twistor space the kinematic variables are constrained to lie on a line. Under the twistor correspondence a line defines a point in Minkowski space. So we could view MHV amplitudes as fundamental local interactions. This perspective leads directly to the MHV diagram method we reviewed in Section 1.2.7.

These results are suggestive of an underlying string picture. According to Witten the correct theory reproducing an $SU(N)$ gauge theory is the topological B-model on $\mathbb{P}^{3|4}$ with N $D5$-branes enriched by Euclidean $D1$-brane instantons wrapping all holomorphic curves $\mathcal{C}$. Witten showed that this configuration yields the right classical equations of motion and gives an $\mathcal{N}=4$ SYM multiplet upon quantisation.

To extract scattering amplitudes we must examine the low energy effective action of $D1$-$D5$ and $D5$-$D1$ strings in the presence of the background from $D5$-$D5$ strings. Quantising the zero modes of the $D1$-$D5$ and $D5$-$D1$ strings yields fermion fields α and β living on the $D1$. The zero modes of the $D5$-$D5$ strings contribute a gauge field E and from string field theory one may derive the action,

$$S = \int_{\mathcal{C}} \mathrm{d}z\, \beta(\overline{\partial} + E)\alpha\,. \tag{1.3.15}$$

The $\mathcal{N}=4$ SYM scattering amplitudes may now be computed as correlation functions of fermionic currents $J(z) = \beta\alpha\,\mathrm{d}z$ coupled to the gauge field E. We must also integrate over all possible $D1$-instantons or equivalently over the moduli space of curves $\mathcal{C}$. Following [143] we write down the explicit formula,

$$A_n = \int \mathrm{d}\mathcal{M} \left\langle \int_{\mathcal{C}} J_1 E_1 \ldots \int_{\mathcal{C}} J_n E_n \right\rangle\,, \tag{1.3.16}$$

where $\mathrm{d}\mathcal{M}$ is the holomorphic measure on the space of holomorphic curves of genus 0 and degree $k-1$ for tree level N^{k-2}MHV processes. In [144], this quantity was computed directly, providing a conjecture for the full tree level S-matrix of $\mathcal{N}=4$ SYM. This has become known as the connected formulation and determines the N^{k-2}MHV superamplitude in supertwistor variables as

$$A_{n,k} = \int \frac{\mathrm{d}^{4k|4k}\mathcal{A}\; \mathrm{d}^n\sigma\; \mathrm{d}^n\xi}{|GL(2;\mathbb{C})|} \prod_{a=1}^{n} \frac{\delta^{(4|4)}(Z_a - \xi_a \mathcal{P}(\sigma_a))}{\xi_a(\sigma_a - \sigma_{a+1})}\,. \tag{1.3.17}$$

where $\mathcal{P} : \mathbb{P}^1 \to \mathbb{P}^{3|4}$ is a holomorphic curve of degree $k-1$ with supermoduli $\mathcal{A}_r$ describing the embedding of the string into twistor space, explicitly

$$\mathcal{P}(\sigma) = \sum_{J=1}^{k} \mathcal{A}_J\, \sigma^{J-1}\,, \tag{1.3.18}$$

and the ξ_a are auxiliary variables. Equation (1.3.17) is closely related to the Grassmannian formula (1.2.60). We may see this in two complementary ways, depending on whether we gauge fix the Grassmannian. Firstly, we may rewrite (1.3.17) as [145]

$$A_{n,k} = \int \mathrm{d}^{k\times(n-k)} c_{Ji}\, U(c_{Ji}) \prod_{i\in\mathsf{p}} \delta^{(2)}(\lambda_i - c_{Ji}\lambda_J) \prod_{J\in\mathsf{m}} \delta^{(2|4)}(\tilde{\lambda}_J + c_{Ji}\tilde{\lambda}_i, \eta_J + c_{Ji}\eta_i)\,, \tag{1.3.19}$$

where we have employed the summation convention, and we define

$$U(c_{Ji}) = \int \frac{\mathrm{d}^n\sigma\, \mathrm{d}^n s}{|GL(2;\mathbb{C})|} \prod_{a=1}^{n} \frac{1}{s_a(\sigma_a - \sigma_{a+1})} \prod_{i\in\mathsf{p},J\in\mathsf{m}} \delta\left(c_{Ji} - \frac{s_J s_i}{\sigma_J - \sigma_i}\right)\,. \tag{1.3.20}$$

This provides an expression for the amplitude precisely in terms of the c_{Ji} link variables appearing in the gauge fixed formula (1.2.62). We shall fully explain a generalisation of the derivation, appropriate for form factors, in Chapter 2. Secondly, we may transform all supertwistors in (1.3.17) back to momentum space, yielding

$$\begin{aligned} A_{n,k} = \int & \frac{\mathrm{d}^n\sigma\, \mathrm{d}^n\xi\, \mathrm{d}^{2k}\rho}{|GL(2;\mathbb{C})|} \prod_{a=1}^{n} \frac{1}{\xi_a(\sigma_a - \sigma_{a+1})} \\ & \times \prod_{J=1}^{k} \delta^{(2)}\left(\sum_{a=1}^{n} \xi_a \sigma_a^{J-1} \tilde{\lambda}_a\right) \delta^{(4)}\left(\sum_{a=1}^{n} \xi_a \sigma_a^{J-1} \eta_a\right) \prod_{a=1}^{n} \delta^{(2)}\Big(\xi_a \lambda(\sigma_a) - \lambda_a\Big)\,, \end{aligned} \tag{1.3.21}$$

where we define the rational map,

$$\lambda(\sigma) = \sum_{J=1}^{k} \rho_J \sigma^{J-1}\,. \tag{1.3.22}$$

Starting with the manifestly gauge invariant Grassmannian integral (1.2.60), we may immediately obtain (1.3.21) by applying the Veronese map,

$$C_{Ja} = \xi_a\, \sigma_a^{J-1}\,, \tag{1.3.23}$$

which embeds $G(2,n)$ into $G(k,n)$.

We may now reinterpret the connected formulation as an integral over rational maps from the Riemann sphere to momentum space, producing the celebrated CHY formulation of $\mathcal{N}=4$ SYM amplitudes. First we express the integral (1.3.21) in a manifestly parity symmetric form [144, 146, 147],

$$\begin{aligned} A_{n,k} = \int & \frac{\mathrm{d}^n\xi\, \mathrm{d}^n\tilde{\xi}\, \mathrm{d}^n\sigma\, \mathrm{d}^{2k}\rho\, \mathrm{d}^{2\tilde{k}}\tilde{\rho}\, \mathrm{d}^{2k}\chi\, \mathrm{d}^{2\tilde{k}}\tilde{\chi}}{|GL(2;\mathbb{C})|} \\ & \times \prod_{a=1}^{n} \left[\delta^{(1)}\left(\xi_a\tilde{\xi}_a - \frac{1}{\prod_{b\neq a}(\sigma_a - \sigma_b)}\right) \frac{1}{\sigma_a - \sigma_{a+1}} \right. \end{aligned}$$

$$\times\, \delta^{(2)}\left(\tilde{\lambda}_a - \tilde{\xi}_a \tilde{\lambda}(\sigma_a)\right) \delta^{(2)}\left(\tilde{\eta}^R_a - \tilde{\xi}_a \tilde{\eta}(\sigma_a)\right) \delta^{(2)}\left(\lambda_a - \xi_a \lambda(\sigma_a)\right) \delta^{(2)}\left(\eta^L_a - \xi_a \eta(\sigma_a)\right)\Bigg] \,, \tag{1.3.24}$$

where $\tilde{k} = n - k$, the χ and $\tilde{\chi}$ are two-component Grassmann moduli, $\eta_a = (\eta^L_a, \eta^R_a)$ and the $\tilde{\eta}_a$ are the Fourier conjugates. The rational maps are defined by

$$\begin{aligned} \lambda(\sigma) &= \sum_{J=1}^{k} \rho_J \sigma^{J-1}\,, \quad \tilde{\lambda}(\sigma) = \sum_{\tilde{J}=1}^{\tilde{k}} \tilde{\rho}_{\tilde{J}} \sigma^{\tilde{J}-1}\,, \\ \eta(\sigma) &= \sum_{J=1}^{k} \chi_J \sigma^{J-1}\,, \quad \tilde{\eta}(\sigma) = \sum_{\tilde{J}=1}^{\tilde{k}} \tilde{\chi}_{\tilde{J}} \sigma^{\tilde{J}-1}\,. \end{aligned} \tag{1.3.25}$$

In [148], Cachazo, He and Yuan suggested rewriting the bosonic delta functions as a single term,

$$\prod_{a=1}^{n} \delta^{(4)}\left(k_a - \oint_a \frac{k(\sigma)\mathrm{d}\sigma}{\prod_{b=1}^{n}(\sigma - \sigma_b)}\right)\,, \tag{1.3.26}$$

where the contour encloses only the pole $\sigma = \sigma_a$ and the rational map $k(\sigma)$ is defined by

$$k(\sigma) = \lambda(\sigma) \otimes \tilde{\lambda}(\sigma)\,. \tag{1.3.27}$$

In this language (1.3.24) manifestly expresses the scattering amplitude as an integral over the moduli space of rational maps from the Riemann sphere to momentum space. Moreover, since $k(\sigma)$ is a simple tensor we have $k(\sigma)^2 = 0$ so $\mathbb{P}^1$ maps to the null cone in momentum space. This suggests another underlying twistor string theory, which we review in Section 1.3.3.

The delta function (1.3.26) implies a set of constraints known as the scattering equations, which we may readily derive. Trivially we have for $a = 1, \dots n$

$$k(\sigma_a) = k_a \prod_{b \neq a} (\sigma_a - \sigma_b)\,. \tag{1.3.28}$$

By the definitions (1.3.25) and (1.3.27) we see that $k(\sigma)^2$ is a non-monic polynomial of degree $2n-4$, so its vanishing constrains $2n-3$ variables. These are most profitably expressed as conditions at the marked points σ_a, namely

$$k^2(\sigma_a) = 0 \quad \text{and} \quad k(\sigma_a) \cdot k'(\sigma_a) = 0\,. \tag{1.3.29}$$

The former yields n on-shell conditions for the external particles, $k_a^2 = 0$. The latter generates $n-3$ independent scattering equations,

$$f_a(\sigma, k) = \sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0\,, \tag{1.3.30}$$

where the counting arises by observing that (1.3.30) is invariant under the action of a $PGL(2;\mathbb{C})$ gauge symmetry on the worldsheet.[37] The scattering amplitude is then naturally expressed as a sum over the $(n-3)!$ solutions to these equations, providing an purely algebraic mechanism for calculating amplitudes.[38]

The integrand in (1.3.24) is rather unwieldy. In principle we could obtain a simpler formula by integrating out the moduli and auxiliary variables, then specialising to gluon amplitudes. Perhaps counterintuitively, it is also convenient to sum over N^{k-2}MHV levels. This computation yields an ansatz of the form,[39]

$$A_n = \oint \frac{\mathrm{d}^n\sigma}{\mathrm{d}\omega} \mathcal{J} \prod_{a=1}^{n} \frac{1}{\sigma_a - \sigma_{a+1}} \cancel{\prod_{a=1}^{n}} \frac{1}{f_a(\sigma, k)}\,. \tag{1.3.31}$$

where $\mathcal{J}$ is a Jacobian factor and we have defined the slashed product,[40]

$$\cancel{\prod_{a=1}^{n}} = (z_i - z_j)(z_j - z_k)(z_k - z_i) \prod_{a \neq i,j,k}\,, \tag{1.3.32}$$

and the invariant measure on the Möbius group $PGL(2;\mathbb{C})$,

$$d\omega = \frac{dz_r dz_s dz_t}{(z_r - z_s)(z_s - z_t)(z_t - z_r)}\,. \tag{1.3.33}$$

By virtue of Möbius symmetry, the result is independent of the choices i, j, k, r, s, t. Henceforth we shall make the choice $\{i, j, k\} = \{r, s, t\}$ elementwise, allowing us to fix σ_i, σ_j and σ_k to arbitrary distinct values. In [2], the Jacobian was determined explicitly based on the requirements of gauge invariance, permutation invariance and appropriate Möbius transformation. Explicitly $\mathcal{J}$ is given by the reduced Pfaffian $\cancel{\mathrm{Pf}}(\Psi_n)$ of the

[37] This symmetry is exactly the induced action of the $GL(2;\mathbb{C})$ redundancy in (1.3.24) on the projectivisation of $\mathbb{C}^2$.

[38] The number of solutions is perhaps most transparent from the Dolan-Goddard form of the scattering equations [149], comprising equations e_i of degree i for $1 \leq i \leq n-3$ with generically distinct solutions.

[39] We have followed the notation of [150] and expressed the delta functions in the form of a contour integral. The contour should be taken to enclosed exactly the poles associated with the simultaneous vanishing of the f_a.

[40] This is alternatively notated with a prime symbol in the literature. We prefer the slash, which carries appropriate connotations of cancellation.

$2n \times 2n$ antisymmetric matrix,

$$\Psi_n(z, k, \epsilon) = \begin{pmatrix} A & B \\ -B^\top & D \end{pmatrix}, \tag{1.3.34}$$

with entries,

$$A_{ab} = \frac{k_a \cdot k_b}{z_a - z_b}, \quad D_{ab} = \frac{\epsilon_a \cdot \epsilon_b}{z_a - z_b}, \quad B_{ab} = \frac{\epsilon_a \cdot k_b}{z_a - z_b}, \quad B_{aa} = -\sum_{c \neq a} \frac{\epsilon_a \cdot k_c}{z_a - z_c}, \tag{1.3.35}$$

where $a \neq b$ and $1 \leq a, b \leq n$. The reduced Pfaffian is defined by

$$\text{Pf}' \, \Psi_n = 2 \frac{(-1)^{a+b}}{z_a - z_b} \text{Pf} \, \Psi_n^{\not{a}, \not{b}}, \tag{1.3.36}$$

where $\Psi_n^{\not{a}, \not{b}}$ denotes the $(2n-2) \times (2n-2)$ submatrix of Ψ_n obtained by deleting the a^{th} and b^{th} rows and columns. For completeness we observe that the Pfaffian obeys $\det \Psi_n = (\text{Pf} \, \Psi_n)^2$. In Appendix A we document a Mathematica package which may be used to verify the formula (1.3.31) for low values of n by explicitly solving the scattering equations in an equivalent polynomial form due to Dolan and Goddard [149].

Remarkably, scattering amplitudes in many theories besides Yang-Mills admit representations as sums over solutions to the scattering equations – see for example [151]. The simplest possibility is to consider double partial amplitudes in ϕ^3 theory with gauge group $U(N) \times U(\tilde{N})$, as first observed in [152]. In this case the Pfaffian factor is absent. We conclude this section by explicitly computing $A_5(1\,2\,3\,4\,5 \,|\, 1\,2\,3\,4\,5)$ in this theory, to illustrate how a global residue theorem may be used to avoid solving the scattering equations directly. We shall apply this technique again in Chapter 2.

We expect our answer to correspond to the sum of planar tree diagrams with momentum ordering $(k_1, \ldots k_5)$. From the Feynman rules this is seen to be

$$A_5 = \frac{1}{s_{23}s_{234}} + \frac{1}{s_{34}s_{234}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{345}s_{34}} + \frac{1}{s_{345}s_{45}}, \tag{1.3.37}$$

where $s_{i_1 \ldots i_r} = (p_{i_1} + \cdots + p_{i_r})^2$. We choose $(i, j, k) = (r, s, t) = (1, 2, 5)$ and evaluate the CHY formula,

$$A_5 = \frac{1}{4}(z_2 - z_5)^2 \oint \frac{1}{f_3(z,k) f_4(z,k)} \frac{dz_3 dz_4}{(z_2 - z_3)^2 (z_3 - z_4)^2 (z_4 - z_5)^2}. \tag{1.3.38}$$

Using the Möbius symmetry we may choose the gauge $z_1 = \infty$, $z_2 = 1$, $z_5 = 0$ and the amplitude reduces to

$$A_5 = \frac{1}{4} \oint \frac{1}{f_3(z,k) f_4(z,k)} \frac{dz_3 dz_4}{(1-z_3)^2 (z_3 - z_4)^2 z_4^2} \,. \tag{1.3.39}$$

To manifest the singularity structure it is convenient to define

$$g_a(z) = f_a(z,k) \prod_{c \neq 1,a} (z_a - z_c) = \sum_{b \neq 1,a} k_a \cdot k_b \prod_{c \neq 1,a,b} (z_a - z_c) \,, \tag{1.3.40}$$

and substituting into (1.3.39) yields

$$A_5 = -\frac{1}{4} \oint \frac{z_3 (1 - z_4) dz_3 dz_4}{g_3 g_4 z_4 (1 - z_3)} \,. \tag{1.3.41}$$

Abusing notation we write $z = z_3$ and $w = z_4$ to arrive at the integral,

$$A_5 = -\frac{1}{4} \oint \frac{z(1-w) dz dw}{g_3 g_4 w (1-z)} \,, \tag{1.3.42}$$

where g_3 and g_4 are explicitly

$$g_3(z,w) = k_3 \cdot k_2 z (z - w) + k_3 \cdot k_4 z (z - 1) + k_3 \cdot k_5 (z - 1)(z - w) \,, \tag{1.3.43}$$

$$g_4(z,w) = k_4 \cdot k_2 w (w - z) + k_4 \cdot k_3 w (w - 1) + k_4 \cdot k_5 (w - 1)(w - z) \,. \tag{1.3.44}$$

It only remains to evaluate the integral. Recall that the contour is defined to enclose only the poles a_i arising when $g_3 = g_4 = 0$. Naïve complex analysis tells us to sum the residues of the integrand at each a_i. By a generalisation of the residue theorem it suffices to sum the residues at poles b_i not arising from $g_3 = g_4 = 0$. This turns out to be a computationally easier task.

We first review the calculation procedure for multivariate residues, following [153]. Consider a holomorphic differential form,

$$\omega = \frac{h(z) dz_1 \wedge \cdots \wedge dz_n}{t_1(z) \ldots t_n(z)} \,, \tag{1.3.45}$$

and suppose a is a simultaneous zero of all the t_i with $h(a) \neq 0$. We define the residue of ω at a by

$$\mathrm{res}(\omega)_a = \left(\frac{1}{2\pi i} \right)^n \oint \omega \,, \tag{1.3.46}$$

where the contour encloses a and no other singularities. We say that a residue is non-degenerate if the Jacobian at a is non-zero, viz.

$$J(a) = \det \left(\frac{\partial t_i}{\partial z_j} \right) \Bigg|_{z=a} \neq 0 \,. \tag{1.3.47}$$

This may be viewed as the multivariate version of a simple pole. Analogously to the single variable case we evaluate a non-degenerate residue as

$$\mathrm{res}(\omega)_a = \frac{s(a)}{J(a)}\,. \tag{1.3.48}$$

For degenerate residues we can always introduce regulators which separate the multiple poles. We are then free to apply (1.3.48). One can also tackle the non-degenerate case head on using Gröbner bases [154], but we shall not have need of such technology.

We may now state the global residue theorem precisely. Suppose that the simultaneous zeroes of the t_i form a discrete set Z. If $\deg(h)+n < \deg(t_1)+\cdots+\deg(t_n)$ then

$$\sum_{a\in Z}\mathrm{res}(\omega)_a = 0\,. \tag{1.3.49}$$

We may apply these results to our integral (1.3.42). Indeed the integrand is of the form (1.3.45) with $t_1 = wg_3$, $t_2 = (1-z)g_4$ and $s = z(1-w)$. Let us denote by $\mathcal{R}(\alpha,\beta)$ the sum of residues when $\alpha=\beta=0$. Clearly the bound needed for the global residue theorem is satisfied so we may write

$$A_5 = \mathcal{R}(g_3,g_4) = -\mathcal{R}(g_3,1-z) - \mathcal{R}(w,g_4) - \mathcal{R}(w,1-z)\,. \tag{1.3.50}$$

We start with the obviously non-degenerate case $\mathcal{R}(w,1-z)$. Applying (1.3.48) gives

$$\mathcal{R}(w,1-z) = -\frac{1}{4k_3\cdot k_2k_4\cdot k_5} = -\frac{1}{s_{23}s_{45}}. \tag{1.3.51}$$

A quick computation shows that $\mathcal{R}(w,f_4)$ is degenerate. We regulate the problem by modifying the integrand to yield

$$\oint \frac{w(1-z)dzdw}{g_3g_4(w-\delta)(1-z)}\,. \tag{1.3.52}$$

The residue can now be evaluated in Mathematica and after taking the $\delta\to 0$ limit we obtain

$$\mathcal{R}(w,g_4) = -\frac{1}{s_{345}}\left(\frac{1}{s_{34}}+\frac{1}{s_{45}}\right)\,. \tag{1.3.53}$$

A similar procedure determines that

$$\mathcal{R}(g_3,1-z) = -\frac{1}{s_{234}}\left(\frac{1}{s_{23}}+\frac{1}{s_{34}}\right)\,. \tag{1.3.54}$$

Collating our results we find exact agreement with (1.3.37).

1.3.3 From Ambitwistor String Theory to Scattering Equations

The ambitwistor string [155] is a chiral infinite-tension version of the RNS string, living inside the space of complexified null geodesics, known as ambitwistor space. The scattering amplitudes of the ambitwistor string yield particularly compact forms of field theory amplitudes in various theories, depending on the matter content on the worldsheet. We swiftly review elements of the construction, relevant for arguments in Chapter 2. For our purposes, it suffices to focus on the bosonic construction, indicating appropriate supersymmetrisations where necessary.

Recall that the phase space action yielding Hamilton's equations may be written,

$$S[x,p] = \int p\ dx - H(x,p)\ dt\,. \tag{1.3.55}$$

where H denotes the Hamiltonian. Hence, we may write the phase space version of the worldline action for a massless particle as

$$S[x,p] = \int p_\mu dx^\mu - \frac{e}{2} p_\mu p^\mu d\tau\,, \tag{1.3.56}$$

where $e(\tau)$ is an auxiliary field, required for reparameterisation invariance, and $p_\mu = e^{-1}\dot{x}_\mu$ is the conjugate momentum. Note that p_μ coincides with the total momentum, since the theory is free. The gauge transformations are given by

$$\delta_\xi X^\mu = \xi P^\mu\,, \quad \delta_\xi P_\mu = 0\,, \quad \delta_\xi e = d\xi\,, \tag{1.3.57}$$

and e is easily seen to be a Lagrange multiplier enforcing the null condition $p^2 = 0$.

From (1.3.56) we may obtain the bosonic ambitwistor string action by complexifying both the worldsheet and the target space. Moreover, we require the model to be chiral, with the action involving only derivatives $\bar{\partial}$, similar to Witten's construction. Therefore we arrive at

$$S[X,P] = \frac{1}{2\pi} \int P_\mu \bar{\partial} X^\mu - \frac{e}{2} P_\mu P^\mu\,, \tag{1.3.58}$$

where we interpret X^μ as a map from the worldsheet to ambitwistor space, P_μ as a $(1,0)$-form, and e as a tangent-bundle-valued $(0,1)$-form[41]. This action may be appropriately supersymmetrised, and the resulting theory has critical dimension 10.

[41] For concreteness, recall that a (p,q) form may be written $\omega = f(z,\bar{z})(dz)^p(d\bar{z})^q$ in some coordinates $(z,\bar{z})$ on the worldsheet, where f is an arbitrary (not necessarily holomorphic or anti-holomorphic) smooth function.

We shall now restrict our attention to four dimensions, where quantum anomalies render the theory inconsistent.[42] Nevertheless, if we remain at tree level, ambitwistor string scattering amplitudes yield correct and compact expressions for field theory amplitudes [156]. As a first motivation, we note that ambitwistor space in four dimensions has a convenient parameterisation as a quadric inside the Cartesian product of twistor spaces $\mathbb{P}^3 \times (\mathbb{P}^3)^*$, where the second factor is the dual projective space. Thus we may immediately bring to bear the power of the Penrose transform.

More explicitly, let us coordinatise the Cartesian product by a pair of twistors (Z, W), then ambitwistor space is the submanifold given by $Z^{\mathsf{A}} W_{\mathsf{A}} = 0$. We may write the action (1.3.58) in terms of (Z, W) by using the incidence relations (1.3.2) and identifying $(\mathbb{P}^3)^*$ with $\overline{\mathbb{P}^3}$. The result is

$$S = \frac{1}{2\pi} \int W_{\mathsf{A}} \bar{\partial} Z^{\mathsf{A}} - Z^{\mathsf{A}} \bar{\partial} W_{\mathsf{A}} + a Z^{\mathsf{A}} W_{\mathsf{A}} \,. \tag{1.3.59}$$

where we interpret W and Z as $\left(\frac{1}{2}, 0\right)$-forms and a as a $(0,1)$-form enforcing the target space constraint. To calculate string scattering amplitudes, we require vertex operators representing the external states. These are given by the pullback of (1.3.11) onto the quadric defining ambitwistor space. There are two options, reflecting the two overlying twistor spaces, the dual being

$$\tilde{V}_a(\tilde{\lambda}, \mu) = \int_{\mathbb{C}} \frac{ds_a}{s_a^{h-1}} \; e^{s_a \langle \mu \lambda_a \rangle} \; \bar{\delta}^{(2)}(\tilde{\lambda}_a - s_a \tilde{\lambda}) \,. \tag{1.3.60}$$

Operators (1.3.11) and (1.3.60) turn out to represent helicity $\pm h/2$ particles respectively. For non-abelian gauge theories, we must additionally include a colour factor J for each particle. This takes the form of a Lie algebra valued current, permitted to flow on the worldsheet. More explicitly we may construct the current using N free complex fermions ψ^i and $SU(N)$ generators T^a, viz.

$$J^a(\sigma) = \frac{i}{2} T^a_{ij} : \psi^i(\sigma) \bar{\psi}^j(\sigma) : \,, \tag{1.3.61}$$

where i, j are fundamental representation indices and a is an adjoint representation index. To work in $\mathcal{N} = 4$ SYM, one may introduce an additive term of the form $\chi_A \eta_a^A$ in the exponent of each vertex operator, along the lines suggested by Witten [142].

We now have sufficient data to calculate the N^{k-2}MHV scattering amplitude,

$$A_{n,k} = \left\langle \int d\sigma_1 \dots d\sigma_n \; \tilde{V}_1 \dots \tilde{V}_k V_{k+1} \dots V_n \right\rangle , \tag{1.3.62}$$

[42] It is possible that one could add further matter content to the theory and reduce the critical dimension for loop-level consistency. This remains an open problem.

as a correlator of vertex operators which may be inserted anywhere on the worldsheet. We may express this as a path integral,

$$A_{n,k} = \int \frac{\mathcal{D}(\lambda,\tilde{\lambda},\mu,\tilde{\mu})}{|GL(2;\mathbb{C})|} \int \prod_{i=1}^{k} \frac{d\sigma_i ds_i}{s_i} \prod_{j=k+1}^{n} \frac{dz_j ds_j}{s_j}$$
$$\times \exp\left(-[\tilde{\mu}\ \bar{\partial}\tilde{\lambda}] - \langle \mu\ \bar{\partial}\lambda\rangle + \sum_{i=1}^{k} s_i \langle \mu\ \lambda_i\rangle \bar{\delta}(\sigma-\sigma_i) + \sum_{j=k+1}^{n} s_j [\tilde{\mu}\ \tilde{\lambda}_j] \bar{\delta}(\sigma-\sigma_j) \right)$$
$$\times\, \bar{\delta}^{(2)}(\tilde{\lambda}_i - s_i\tilde{\lambda}(\sigma_i))\, \bar{\delta}^{(2)}(\lambda_j - s_j\lambda(\sigma_j))\, \langle J_1^{a_1}\cdots J_n^{a_n}\rangle\,, \quad (1.3.63)$$

in the gauge $a=0$, where we've used integration by parts on the action. Observe that $(\mu,\tilde{\mu})$ only appear in the exponential, which is exactly linear in these variables. Hence upon integrating out $(\mu,\tilde{\mu})$ we obtain functional delta functions, enforcing

$$\bar{\partial}\lambda = \sum_{i=1}^{k} s_i\lambda_i\bar{\delta}(\sigma-\sigma_i)\,, \quad \bar{\partial}\tilde{\lambda} = \sum_{j=k+1}^{n} s_j\tilde{\lambda}_j\bar{\delta}(\sigma-\sigma_j)\,. \quad (1.3.64)$$

Now integrating out $(\lambda,\tilde{\lambda})$ amounts to solving these equations. To do so, observe that $\bar{\delta}(z)$ is rigorously defined as

$$\bar{\delta}(z) = \delta(x)\delta(y)d\bar{z} = \frac{1}{2\pi i}\bar{\partial}\frac{1}{z}\,, \quad (1.3.65)$$

where the second equality is a consequence of the two-dimensional Green's function for the Laplacian,

$$\delta(x)\delta(y) = \frac{1}{2\pi}\nabla^2\log(\sqrt{x^2+y^2})\,. \quad (1.3.66)$$

Therefore solving (1.3.64) is trivial, yielding

$$\lambda(\sigma) = \sum_{i=1}^{k}\frac{s_i\lambda_i}{\sigma-\sigma_i}\,, \quad \tilde{\lambda}(\sigma) = \sum_{j=k+1}^{n}\frac{s_j\tilde{\lambda}_j}{\sigma-\sigma_j}\,. \quad (1.3.67)$$

Finally we must deal with the current correlator. Recall that the only non-vanishing Wick contraction between complex fermions takes the form,

$$\langle\psi^i(\sigma_1)\bar{\psi}^j(\sigma_2)\rangle = \frac{\delta^{ij}}{\sigma_1-\sigma_2}\,. \quad (1.3.68)$$

Therefore the current correlator evaluates to

$$\langle J_1^{a_1}\cdots J_n^{a_n}\rangle = \frac{\mathrm{Tr}\,(T^{a_1}\ldots T^{a_n})}{(\sigma_1-\sigma_2)\cdots(\sigma_n-\sigma_1)} + \cdots\,, \quad (1.3.69)$$

where we have ignored multiple trace terms, since we work in the planar limit. Interestingly there exist ambitwistor string constructions which rigorously disallow multitrace contributions [157]. With these equalities, the amplitude becomes

$$A_{n,k} = \int \frac{1}{|GL(2;\mathbb{C})|} \prod_{a=1}^{n} \frac{d\sigma_a ds_a}{s_a(\sigma_a - \sigma_{a+1})} \prod_{i=1}^{k} \bar{\delta}^{(2)}(\tilde{\lambda}_i - s_i \tilde{\lambda}(\sigma_i)) \prod_{j=k+1}^{n} \bar{\delta}^{(2)}(\lambda_j - s_j \lambda(\sigma_j))\,. \tag{1.3.70}$$

To make this expression slightly neater we may change to homogeneous coordinates on the Riemann sphere, defining by a slight abuse of notation

$$\sigma^\alpha = \frac{1}{s}(1,\sigma)\,, \quad (i\ j) = \sigma_i^\alpha \epsilon_{\alpha\beta} \sigma_j^\beta\,. \tag{1.3.71}$$

Rewriting in terms of these variables, and including the contributions from supersymmetry sketched below (1.3.61) we arrive at[43]

$$\lambda(\sigma) = \sum_{i=1}^{k} \frac{\lambda_i}{(\sigma\ \sigma_i)}\,, \quad \tilde{\lambda}(\sigma) = \sum_{j=k+1}^{n} \frac{\tilde{\lambda}_j}{(\sigma\ \sigma_j)}\,, \quad \eta(\sigma) = \sum_{j=k+1}^{n} \frac{\eta_j}{(\sigma\ \sigma_j)}\,. \tag{1.3.72}$$

Hence, the superamplitude may be expressed as[44]

$$A_{n,k} = \int \frac{1}{|GL(2;\mathbb{C})|} \prod_{a=1}^{n} \frac{d^2\sigma_a}{(a\ a+1)} \prod_{i=1}^{k} \bar{\delta}^{(2|4)}(\tilde{\lambda}_j - \tilde{\lambda}(\sigma_j), \eta_j - \eta(\sigma_j)) \prod_{j=k+1}^{n} \bar{\delta}^{(2)}(\lambda_j - \lambda(\sigma_j))\,. \tag{1.3.73}$$

This represents the full tree-level S-matrix of Yang-Mills theory in a remarkably compact fashion, as a weighted sum over solutions to the rational scattering equations (also known as the refined scattering equations),

$$\begin{gathered} \sum_{i=1}^{k} \frac{\lambda_i}{(\sigma_j\ \sigma_i)} = \lambda_j \quad \text{for } j = k+1, \dots, n\,, \\ \sum_{j=k+1}^{n} \frac{\tilde{\lambda}_j}{(\sigma_i\ \sigma_j)} = \tilde{\lambda}_i\,, \quad \sum_{j=k+1}^{n} \frac{\eta_j}{(\sigma_i\ \sigma_j)} = \eta_i \quad \text{for } i = 1, \dots, k\,, \end{gathered} \tag{1.3.74}$$

where we have borrowed nomenclature from [159]. We note immediately that this formula may be derived directly from the linked connected formulation (1.3.19) by integrating out the variables c_{Ji} using the delta functions in the definition of U. Moreover, (1.3.74) may be viewed as a refinement of the scattering equations (1.3.30).[45] Indeed

[43] We have rescaled λ and $\tilde{\lambda}$ by a factor of $\frac{1}{s}$, permitted since they are only projectively meaningful.

[44] Our superamplitudes have η^0 for positive helicity and η^4 for negative helicity gluons, which is the opposite of the convention employed in [158].

[45] The number of solutions for given n,k is the Eulerian number $\left\langle {n-3 \atop k-2} \right\rangle$ [160].

applying partial fractions we find [159]

$$\left(\sum_{i=1}^{k}\frac{\lambda_i s_i}{\sigma-\sigma_i}\right)\left(\sum_{j=k+1}^{n}\frac{\tilde{\lambda}_j s_j}{\sigma-\sigma_j}\right)=\sum_{i=1}^{k}\frac{\lambda_i}{\sigma-\sigma_i}\sum_{j=k+1}^{n}\frac{\tilde{\lambda}_j s_i s_j}{\sigma_j-\sigma_i}+\sum_{j=k+1}^{n}\frac{\tilde{\lambda}_j}{\sigma-\sigma_j}\sum_{i=1}^{k}\frac{\lambda_i s_i s_j}{\sigma_i-\sigma_j}\,,$$
$$=\sum_{i=1}^{k}\frac{\lambda_i\tilde{\lambda}_i}{\sigma-\sigma_i}+\sum_{j=k+1}^{n}\frac{\lambda_j\tilde{\lambda}_j}{\sigma-\sigma_j}\,,\tag{1.3.75}$$

and defining

$$p^\mu(\sigma)=\prod_{b=1}^{n}(\sigma-\sigma_b)\sum_{c=1}^{n}\frac{p_c^\mu}{\sigma-\sigma_c}\,,\tag{1.3.76}$$

we see that $p^\mu(\sigma)$ is a simple tensor and $p^\mu(\sigma_a)=k_a\prod_{b\neq a}(\sigma_a-\sigma_b)$. These are exactly the statements from which we derived the scattering equations in Section 1.3.2.

We conclude this section by extracting the 3-point MHV amplitude from (1.3.73). We shall perform similar calculations for form factors in Chapter 2. We make the gauge choice,

$$\sigma_1=(1,0)\,,\quad \sigma_2=(0,1)\,,\quad \sigma_3=(\tau,\sigma)\,,\tag{1.3.77}$$

so that

$$(1\ 2)=1\,,\quad (2\ 3)=-\tau\,,\quad (3\ 1)=\sigma\,.\tag{1.3.78}$$

The delta functions reduce to

$$\bar{\delta}^{(2)}\left(\tilde{\lambda}_1+\frac{\tilde{\lambda}_3}{\sigma}\right)\bar{\delta}^{(2)}\left(\tilde{\lambda}_2+\frac{\tilde{\lambda}_3}{\tau}\right)\bar{\delta}^{(2)}\left(\lambda_3-\frac{\lambda_1}{\sigma}-\frac{\lambda_2}{\tau}\right)\,.\tag{1.3.79}$$

The third delta function fixes

$$\sigma=-\frac{\langle 1\ 2\rangle}{\langle 2\ 3\rangle}\,,\quad \tau=\frac{\langle 1\ 2\rangle}{\langle 1\ 3\rangle}\,.\tag{1.3.80}$$

also contributing a Jacobian factor,

$$\begin{vmatrix}\lambda_1/\sigma^2 & 0\\ 0 & \lambda_2/\tau^2\end{vmatrix}^{-1}=\frac{\sigma^2\tau^2}{\langle 1\ 2\rangle}\,.\tag{1.3.81}$$

The other two delta functions become

$$\bar{\delta}^{(2)}\left(\frac{\langle 2\ 1\rangle\tilde{\lambda}_1+\langle 2\ 3\rangle\tilde{\lambda}_3}{\langle 2\ 1\rangle}\right)\bar{\delta}^{(2)}\left(\frac{\langle 1\ 2\rangle\tilde{\lambda}_2+\langle 1\ 3\rangle\tilde{\lambda}_3}{\langle 1\ 2\rangle}\right)$$
$$=\bar{\delta}^{(2)}\left(\frac{\langle 2|P}{\langle 2\ 1\rangle}\right)\bar{\delta}^{(2)}\left(\frac{\langle 1|P}{\langle 1\ 2\rangle}\right)=\delta^{(4)}(P)\langle 1\ 2\rangle^2\,,\tag{1.3.82}$$

where P is the total momentum. Hence (1.3.73) evaluates to

$$\frac{1}{\sigma\tau}\frac{\sigma^2\tau^2}{\langle 1\ 2\rangle}\delta^{(4)}(P)\langle 1\ 2\rangle^2 = \delta^{(4)}(P)\frac{\langle 1\ 2\rangle^3}{\langle 2\ 3\rangle\langle 3\ 1\rangle}\,, \tag{1.3.83}$$

in agreement with (1.2.32).

1.3.4 Momentum Twistors

Thus far, we have used twistors as an inspiration for deriving formulae in momentum space. However, it is often convenient to work directly with twistor variables, yielding more compact expressions for conformally invariant quantities. In Chapter 3, we shall find it useful to introduce twistor variables in the fundamental representation of dual conformal symmetry. These are known as momentum twistors, and were first advocated by Hodges [89]. We define

$$Z_i^I = (\lambda_i^\alpha, \tilde{\mu}_i^{\dot{\alpha}}) = (\lambda_i^\alpha, x_i^{\dot{\alpha}\beta}\lambda_{i\beta})\,, \tag{1.3.84}$$

where $I = (\alpha, \dot{\alpha})$ and we have made explicit the twistor correspondence with dual space. Note that Z_i^I scales uniformly under the action of the little group, so momentum twistors are only projectively meaningful. The momentum twistors may be readily supersymmetrised, along the lines of (1.3.2). More precisely, we could augment (1.3.84) with new Grassmann variables $\chi_i^A = \eta_i^{\beta A}\lambda_{i\beta}$. In this thesis, we prefer to keep Z_i^I bosonic, and explicitly indicate dependence on χ_i^A where required. The on-shell variables $\tilde{\lambda}_i$ and η_i can be regained using the relations [161, 162],

$$\begin{aligned}
\tilde{\lambda}_i &= \frac{\tilde{\mu}_{i-1}\langle i\ i+1\rangle + \tilde{\mu}_i\langle i+1\ i-1\rangle + \tilde{\mu}_{i+1}\langle i-1\ i\rangle}{\langle i-1\ i\rangle\langle i\ i+1\rangle}\,,\\
\eta_i &= \frac{\chi_{i-1}\langle i\ i+1\rangle + \chi_i\langle i+1\ i-1\rangle + \chi_{i+1}\langle i-1\ i\rangle}{\langle i-1\ i\rangle\langle i\ i+1\rangle}\,.
\end{aligned} \tag{1.3.85}$$

The canonical dual conformal invariant quantity is the four-bracket,

$$\langle i\ j\ k\ l\rangle = \epsilon_{IJKL}Z_i^I Z_j^J Z_k^K Z_l^L\,. \tag{1.3.86}$$

Such objects obey various identities, which we employ liberally in Chapter 3. From the definition (1.3.84) it is immediate that

$$\langle i\ j-1\ j\ k\rangle = \langle j-1\ j\rangle\langle i|x_{ij}x_{jk}|k\rangle\,. \tag{1.3.87}$$

We trivially have a five term Schouten identity,

$$Z_a^I \langle b\ c\ d\ e\rangle + \text{cyclic} = 0\,, \tag{1.3.88}$$

which quickly yields a formula for computing intersections of projective lines and planes,

$$(i\ j) \cap (a\ b\ c) = Z_i \langle j\ a\ b\ c\rangle - Z_j \langle i\ a\ b\ c\rangle\,, \tag{1.3.89}$$

$$(i\ j\ k) \cap (a\ b\ c) = Z_i Z_j \langle k\ a\ b\ c\rangle + Z_j Z_k \langle i\ a\ b\ c\rangle + Z_k Z_i \langle j\ a\ b\ c\rangle\,, \tag{1.3.90}$$

where we've introduced the notation $(a\ b) = Z_a \wedge Z_b$. Finally we have the important relation,

$$\langle x\ y\ (i\ j\ k) \cap (a\ b\ c)\rangle = \langle (x\ y) \cap (a\ b\ c)\ i\ j\ k\rangle\,, \tag{1.3.91}$$

valid as a statement in homogeneous coordinates, where scale is important. To verify this, first observe that it has the correct vanishing behaviour in the case of linear dependence. Then it suffices to evaluate one non-vanishing example.

To illustrate the notation, we reproduce the expression for the NMHV tree amplitude as a sum over dual superconformal R-invariants [70, 76, 99],

$$A_n^{\text{NMHV,tree}} = A_n^{\text{MHV,tree}} \sum_{1<j-1<j<k-1} R_{1jk}\,, \tag{1.3.92}$$

which are most naturally defined in terms of momentum twistor variables [161], viz.

$$R_{ijk} = \frac{\delta^{(4)}\left(\langle j-1\ j\ k-1\ k\rangle \chi_i^A + \text{cyclic}\right)}{\langle i\ j-1\ j\ k-1\rangle\langle j-1\ j\ k-1\ k\rangle\langle j\ k-1\ k\ i\rangle\langle k-1\ k\ i\ j-1\rangle\langle k\ i\ j-1\ j\rangle}\,, \tag{1.3.93}$$

motivating the five-bracket notation,

$$R_{ijk} = [i\ j-1\ j\ k-1\ k]\,. \tag{1.3.94}$$

1.4 Outline of Thesis

The remainder of this thesis is structured in the following way.

In Chapter 2, we construct a number of CHY-inspired formulae for the form factor of $\text{Tr}F_{\text{SD}}^2$. In Section 2.1, we review the connected formulation for form factors presented in [10] and derive from it an equivalent expression as a sum over solutions to the rational scattering equations, first conjectured in [3]. We verify this formula with explicit computations using link variables in Section 2.2, noting an intriguing duality with BCFW terms. We then indicate how this formula is related to the Grassmannian representation and comment on a possible derivation from ambitwistor string theory. In Section 2.5, we present a previously unpublished CHY formula for the tree-level form factor of TrF^2 in pure Yang-Mills theory.

In Chapter 3, we consider the subleading soft gluon theorems at tree level and one loop. We describe how conformal symmetry may be used to derive the universal operators under mild assumptions, following Larkoski [163]. We then introduce a Larkoski method based on constraints from dual conformal symmetry, and demonstrate how this may be extended to loop level. In Section 3.3, we perform explicit calculations to determine the subleading soft anomaly in the MHV and NMHV sectors, restating a conjecture of [9] that universal breaks between helicity sectors, but holds within them. All our results are valid through finite order in the infrared regulator.

Chapter 4 provides a summary of our conclusions, and a few brief suggestions for future work. In Appendix A, we document two Mathematica packages we have developed, which may be used to verify some of the original calculations we have undertaken. Appendix B contains formulae required in Chapter 3.

Chapter 2

Scattering Equations and Form Factors

In this chapter we conjecture various formulae computing the super form factor of the chiral stress tensor supermultiplet, sometimes focussing on its lowest γ^+ component. Many of the results were previously presented in [10] and the expressions in Section 2.2 have some overlap with parallel work in [3,164]. The observations in 2.5 are incomplete unpublished work, and are therefore more speculative.

2.1 Connected Prescription in Four Dimensions

We propose the following connected formula computing the N^{k-2}MHV tree level super form factor of the $\mathrm{Tr}F_{\mathrm{SD}}^2$ supermultiplet (as defined in Section 1.2.3) in supertwistor variables:

$$\begin{aligned} F(Z_a) = \langle Z_x I Z_y \rangle^2 \int \frac{\mathrm{d}^{4k|4k}\mathcal{A}\ \mathrm{d}^{n+2}\sigma\ \mathrm{d}^{n+2}\xi}{|GL(2;\mathbb{C})|} \\ \times \frac{\prod_{a=x,y} \delta^{(4|4)}(Z_a - \xi_a \mathcal{P}(\sigma_a;\mathcal{A}))}{\xi_x\, \xi_y\, (\sigma_x - \sigma_y)^2} \prod_{a=1}^{n} \frac{\delta^{(4|4)}(Z_a - \xi_a \mathcal{P}(\sigma_a;\mathcal{A}))}{\xi_a\, (\sigma_a - \sigma_{a+1})}\,, \end{aligned} \tag{2.1.1}$$

where we describe the form factor insertion using two extra particles x and y, which are related to the form factor momentum and supermomentum via

$$q = \lambda_x \tilde{\lambda}_x + \lambda_y \tilde{\lambda}_y\,, \quad \gamma = \lambda_x \eta_x + \lambda_y \eta_y\,. \tag{2.1.2}$$

As in Section 1.3.2, we take $\mathcal{P}(\sigma)$ to be a degree $k-1$ curve in $\mathbb{P}^{3|4}$ with supermoduli $\mathcal{A}$. The prefactor is built using the infinity twistor I, and precisely evaluates to $\langle x\, y\rangle^2$.

All particle indices are defined modulo n, as usual. From (2.1.1) we may deduce a supersymmetric generalisation of the form factor result in [165], translating our expression into a sum over solutions to the rational scattering equations (1.3.74). The proof parallels closely that of [166, 167].

We begin by dividing the n physical particles into two sets p and m containing k and $n-k$ particles respectively, just as we did in Section 1.2.6. Furthermore, we augment the first set with the two auxiliary particles, defining $\bar{\mathsf{p}} = \mathsf{p} \cup \{x, y\}$. When working with gluon component amplitudes, it is convenient to assign gluons of positive (negative) helicity to $\bar{\mathsf{p}}$ (m). This parallels the assignments made in [165], where the two auxiliary particles are treated as positive helicity gluons. Henceforth we shall use the index notation $i \in \bar{\mathsf{p}}$ and $J \in \mathsf{m}$.

Clearly, to obtain a formula similar to (1.3.19) we desire to integrate over the moduli $\mathcal{A}$. The integral becomes trivial if we Fourier transform the supertwistors Z_i to obtain dual supertwistors W_i. In terms of these variables we have

$$F(W_i, Z_J) = \int \frac{\mathrm{d}^{4k|4k}\mathcal{A}\,\mathrm{d}^{n+2}\sigma\,\mathrm{d}^{n+2}\xi}{|GL(2;\mathbb{C})|} \frac{\prod_{J\in\mathsf{m}} \delta^{(4|4)}(Z_J - \xi_J \mathcal{P}(\sigma_J))}{\xi_x\,\xi_y\,(\sigma_x - \sigma_y)^2\,\prod_{a=1}^{n} \xi_a\,(\sigma_a - \sigma_{a+1})} \times \left\langle \frac{\partial}{\partial W_x}\, I\, \frac{\partial}{\partial W_y} \right\rangle^2 \prod_{i\in\bar{\mathsf{p}}} \exp\left(i\,\xi_i\,W_i \cdot \mathcal{P}(\sigma_i)\right), \quad (2.1.3)$$

and we are left with exactly as many delta functions as moduli. Performing the integral over $\mathcal{A}$ localises the degree $k-1$ curve onto

$$\mathcal{P}(\sigma) = \sum_{J\in\mathsf{m}} \frac{Z_J}{\xi_J} \prod_{K\neq J} \frac{\sigma_K - \sigma}{\sigma_K - \sigma_J}, \quad (2.1.4)$$

and the integral becomes

$$F(W_i, Z_J) = \int \frac{\mathrm{d}^{n+2}\sigma\,\mathrm{d}^{n+2}\xi}{|GL(2;\mathbb{C})|} \frac{1}{\xi_x\,\xi_y\,(\sigma_x - \sigma_y)^2} \prod_{a=1}^{n} \frac{1}{\xi_a\,(\sigma_a - \sigma_{a+1})} \times \left\langle \frac{\partial}{\partial W_x}\, I\, \frac{\partial}{\partial W_y} \right\rangle^2 \exp\left(i \sum_{i\in\bar{\mathsf{p}}, J\in\mathsf{m}} W_i \cdot Z_J\, \frac{\xi_i}{\xi_J} \prod_{K\neq J} \frac{\sigma_K - \sigma_i}{\sigma_K - \sigma_J} \right). \quad (2.1.5)$$

We may simplify the integral by introducing new variables,

$$s_i = \xi_i \prod_K (\sigma_K - \sigma_i), \quad s_J^{-1} = \xi_J \prod_{K\neq J} (\sigma_K - \sigma_J), \quad (2.1.6)$$

and spinor coordinates $\sigma_\alpha = s^{-1}(1, \sigma)$, defining the notation $(a\ b) = \epsilon_{\alpha\beta}\sigma_a^\alpha \sigma_b^\beta$. Our formula now reduces to

$$F(W_i, Z_J) = \int \frac{1}{|GL(2;\mathbb{C})|} \frac{\mathrm{d}^2\sigma_x \, \mathrm{d}^2\sigma_y}{(x\,y)^2} \\ \times \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \left\langle \frac{\partial}{\partial W_x} \, I \, \frac{\partial}{\partial W_y} \right\rangle^2 \exp\left(i \sum_{i\in\bar{\mathsf{p}}, J\in\mathsf{m}} \frac{W_i \cdot Z_J}{(i\ J)} \right). \quad (2.1.7)$$

Finally we may return to spinor variables by performing Witten's half-Fourier transform, recalling that in the language of footnote 36

$$Z = (\lambda_\alpha, \tilde{\mu}^{\dot{\alpha}}, \eta^A)\,, \quad W = (\mu^\alpha, \tilde{\lambda}_{\dot{\alpha}}, \tilde{\eta}_A)\,, \quad Z \cdot W = \langle \lambda \mu \rangle + [\tilde{\mu} \tilde{\lambda}] + \tilde{\eta}_A \eta^A\,. \quad (2.1.8)$$

The exponential factor may be written

$$\exp\left(i \sum_{i\in\bar{\mathsf{p}}, J\in\mathsf{m}} \frac{W_i \cdot Z_J}{(i\ J)} \right) = \prod_{i\in\bar{\mathsf{p}}} \exp\left(\sum_{J\in\mathsf{m}} \frac{\langle J\ \mu_i \rangle}{(i\ J)} \right) \prod_{J\in\mathsf{m}} \exp\left(\sum_{i\in\bar{\mathsf{p}}} \frac{[\tilde{\mu}_J\ i] + \tilde{\eta}_{iA}\eta_J^A}{(i\ J)} \right). \quad (2.1.9)$$

Following [168], we observe that

$$\prod_{i\in\bar{\mathsf{p}}} \exp\left(\sum_{J\in\mathsf{m}} \frac{\langle J\ \mu_i \rangle}{(i\ J)} \right) = \prod_{i\in\bar{\mathsf{p}}} \int \mathrm{d}^2\lambda_i \, e^{\langle i\ \mu_i \rangle} \, \delta^{(2)}(\lambda_i - \lambda(\sigma_i))\,, \quad (2.1.10)$$

where the scattering function $\lambda(\sigma)$ is defined by an appropriate generalisation of (1.3.72). One may obtain similar expressions for Fourier transforms with respect to $\tilde{\lambda}_{\dot{\alpha}}$ and $\tilde{\eta}_A$. Hence we find

$$F(\lambda_a, \tilde{\lambda}_a) = \langle x\,y \rangle^2 \int \frac{1}{|GL(2;\mathbb{C})|} \frac{\mathrm{d}^2\sigma_x \, \mathrm{d}^2\sigma_y}{(x\,y)^2} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \\ \times \prod_{i\in\bar{\mathsf{p}}} \delta^{(2)}(\lambda_i - \lambda(\sigma_i)) \prod_{J\in\mathsf{m}} \delta^{(2|4)}(\tilde{\lambda}_J - \tilde{\lambda}(\sigma_J), \eta_J - \eta(\sigma_J))\,, \quad (2.1.11)$$

This is a supersymmetric version of the form factor conjecture presented in [165]. By performing in reverse the same steps of this proof, one can of course derive the connected prescription for form factors (2.1.1) from the rational scattering equation formula.

2.2 Verifying the Rational Formula

In this section we provide explicit calculations verifying the formula (2.1.11). We first argue that (2.1.2) encodes the correct dependence on γ^+. Then, by explicitly solving the rational scattering equations, we demonstrate agreement with known results from [54] for all maximally non-MHV form factors. Next, we argue that calculations are more natural in link variables, using these to prove the all-point MHV case. Finally, we examine the simplest component NMHV form factor, employing the global residue theorem to expose a duality with BCFW recursion.

To make a connection with the supermomentum conservation delta functions, we contract the variables η with the harmonic matrices $\bar{u}_+$ and $\bar{u}_{-a}$ to yield fermionic contributions,

$$\prod_{J\in\mathsf{m}} \delta^{(2)}\left(\eta_{+J} - \sum_{i\in\bar{\mathsf{p}}} \frac{\eta_{+i}}{(J\ i)}\right) \prod_{J\in\mathsf{m}} \delta^{(2)}\left(\eta_{-J} - \sum_{i\in\bar{\mathsf{p}}} \frac{\eta_{-i}}{(J\ i)}\right) . \tag{2.2.1}$$

The first factor certainly implies the constraint,

$$\delta^{(4)}\left(\sum_{J\in\mathsf{m}} \eta_{+J}\lambda_J - \sum_{i\in\bar{\mathsf{p}}}\sum_{J\in\mathsf{m}} \frac{\eta_{+i}\lambda_J}{(J\ i)}\right) = \delta^{(4)}\left(\sum_{a=1}^{n} \eta_{+a}\lambda_a - \eta_{+x}\lambda_x - \eta_{+y}\lambda_y\right) , \tag{2.2.2}$$

on the support of the holomorphic delta functions. Now invoking (2.1.2) yields exactly the γ_+ dependent delta function implied by the supersymmetry Ward identities in (1.2.41). Similarly, one may extract the universal dependence on η_{-a} from the second factor, recalling that $\gamma_- = 0$. Note that our identification of γ_+ is identical to that found later in [164].

2.2.1 Maximally Non-MHV Sector

We explicitly evaluate the maximally non-MHV $n = k$ case. Solving the rational scattering equations is trivial here, since they are linear in the $(i\ J)^{-1}$ variables. The delta functions are

$$\prod_{J=1}^{n} \delta^{(2|4)}\left(\tilde{\lambda}_J - \frac{\tilde{\lambda}_x}{(J\ x)} - \frac{\tilde{\lambda}_y}{(J\ y)},\ \eta_J - \frac{\eta_x}{(J\ x)} - \frac{\eta_y}{(J\ y)}\right)$$
$$\delta^{(2)}\left(\lambda_x - \sum_{J=1}^{n} \frac{\lambda_J}{(x\ J)}\right) \delta^{(2)}\left(\lambda_y - \sum_{J=1}^{n} \frac{\lambda_J}{(y\ J)}\right) . \tag{2.2.3}$$

To proceed we must fix the $GL(2;\mathbb{C})$ redundancy. There is a canonical choice of gauge, namely

$$\sigma_x = (1,0)\,, \quad \sigma_y = (0,1)\,. \tag{2.2.4}$$

The antiholomorphic delta functions enforce

$$(J\ x) = \frac{[x\ y]}{[J\ y]}\,, \quad (J\ y) = \frac{[y\ x]}{[J\ x]}\,. \tag{2.2.5}$$

Note that the integration $\int \mathrm{d}^{n+2}\sigma$ may be equivalently performed over the combinations appearing in (2.2.5) in this gauge (up to a possible sign) because contraction with σ_{n+1} and σ_{n+2} exactly picks out the spinor components. In solving the δ functions as above, we gain the Jacobian factor,

$$\prod_{J=1}^{n} \frac{(J\ x)^2(J\ y)^2}{[x\ y]}\,. \tag{2.2.6}$$

To evaluate the integrand we need to write $(a\ a+1)$ in terms of combinations appearing in (2.2.5). We have, courtesy of Schouten identities,

$$(a\ a+1) = (a\ y)(x\ a+1) - (x\ a)(a+1\ y)\,. \tag{2.2.7}$$

Now we examine the two remaining bosonic delta functions, observing that they combine to yield momentum conservation, viz.

$$\delta^{(2)}\left(\frac{[y\ x]\lambda_x + \sum_{J=1}^{n}[y\ J]\lambda_J}{[y\ x]}\right)\delta^{(2)}\left(\frac{[x\ y]\lambda_y + \sum_{J=1}^{n}[x\ J]\lambda_J}{[x\ y]}\right) = [x\ y]^2\delta^{(4)}(P)\,, \tag{2.2.8}$$

where P denotes the total momentum. We now have all the ingredients required to evalute the super form factor. Substituting (2.2.5) and performing some light algebraic manipulations yields

$$F^{\text{max non-MHV}} = \frac{q^4}{[1\,2]\cdots[n\,1]}\prod_{J=1}^{n}\delta^{(4)}\Big(\eta_J + \frac{[J\,y]}{[y\,x]}\eta_x + \frac{[J\,x]}{[x\,y]}\eta_y\Big)\,\delta^{(4)}(P)\,. \tag{2.2.9}$$

At first glance, it is not at all obvious that this agrees with the known result (1.2.43). In [10], we verified that the $(\gamma^+)^0$ component evaluates to (1.2.44). Here, we go one step further, and check that the full supersymmetric formulae precisely agree. First observe that

$$\sum_{J=1}^{n}\lambda_J\eta_J = \sum_{J=1}^{n}\frac{\lambda_J[J\ y]}{[y\ x]}\eta_x + \frac{\lambda_J[J\ x]}{[x\ y]}\eta_y = \lambda_x\eta_x + \lambda_y\eta_y\,, \tag{2.2.10}$$

by virtue of momentum conservation, establishing the supermomentum conservation constraint we already knew would hold from (2.2.2). Hence it suffices to prove that

$$\begin{aligned}\int \mathrm{d}^4\gamma_+\,\frac{q^4}{[1\,2]\cdots[n\,1]}\prod_{j=1}^{n}\delta^{(4)}\Big(\eta_j + \frac{[j\,y]}{[y\,x]}\eta_x + \frac{[j\,x]}{[x\,y]}\eta_y\Big) \\ = \int\left(\prod_{i=1}^{n} d^4\tilde{\eta}_i\, e^{i\eta_{iA}\tilde{\eta}_i^A}\right)\frac{\delta^{(4)}\left(\sum_{j=1}^{n}\tilde{\lambda}_j\tilde{\eta}_j^+\right)}{[1\ 2][2\ 3]\cdots[n\ 1]}\,.\end{aligned} \tag{2.2.11}$$

Starting with the left hand side, we project with the harmonic matrices, and eliminate η_x and η_y in favour of γ_+, yielding

$$\int \mathrm{d}^4\gamma_+ \, \frac{q^4}{[1\,2]\cdots[n\,1]} \prod_{j=1}^{n} \delta^{(2)}\left(\eta_{-j}\right) \delta^{(2)} \left(\eta_{+j} - \frac{[j|q|\gamma_+\rangle}{q^2} \right) . \tag{2.2.12}$$

Moving to Fourier space the integral becomes

$$\int \mathrm{d}^4\gamma_+ \exp\left(\frac{q_{\alpha\dot{\alpha}}\gamma_+^{\alpha}}{q^2} \sum_{j=1}^{n} \tilde{\eta}_j^{+} \tilde{\lambda}_j^{\dot{\alpha}} \right) \frac{q^4}{[1\,2]\cdots[n\,1]} \,, \tag{2.2.13}$$

which is exactly the integrand on the right hand side of (2.2.11), upon performing the integral over γ_+ and picking up the Jacobian q^{-4}.

It is interesting to contrast the remarkable simplicity of this derivation with the original calculation presented in [169], which required a more significant amount of work. Likewise, the equivalent computation in the Grassmannian formulation [87] required the numerical evaluation of some complicated momentum twistor expressions. Nevertheless, we shall see in Section 2.3 that the Grassmannian integral trivially reduces to the formula (2.1.11) in this sector and beyond.

2.2.2 Link Variables and MHV Sector

In the previous section, we observed that the rational scattering equations were linear in $(i\ J)^{-1}$ for maximally non-MHV configurations. This motivates us to introduce the link variables,[46]

$$c_{iJ} = \frac{1}{(i\,J)} \,, \tag{2.2.14}$$

where the first and second index run over the sets $\bar{\mathsf{p}}$ and m, respectively. The identification is achieved by introducing $1 = \int \mathrm{d}c_{iJ}\, \delta\left(c_{iJ} - 1/(iJ)\right)$. Doing so, we can recast (2.1.11) as

$$F = \langle x\,y\rangle^2 \int \prod_{i\in\bar{\mathsf{p}},J\in\mathsf{m}} \mathrm{d}c_{iJ}\; U(c_{iJ}) \prod_{i\in\bar{\mathsf{p}}} \delta^{(2)}(\lambda_i - c_{iJ}\lambda_J) \prod_{J\in\mathsf{m}} \delta^{(2|4)}(\tilde{\lambda}_J + c_{iJ}\tilde{\lambda}_i, \eta_J + c_{iJ}\eta_i) \,, \tag{2.2.15}$$

where

$$U(c_{iJ}) = \int \frac{1}{|GL(2;\mathbb{C})|} \frac{\mathrm{d}^2\sigma_x \mathrm{d}^2\sigma_y}{(x\,y)^2} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \prod_{i\in\bar{\mathsf{p}},J\in\mathsf{m}} \delta\Big(c_{iJ} - \frac{1}{(i\,J)}\Big) . \tag{2.2.16}$$

[46] Our convention here is the transpose of that in Section 1.2.6, following the notation of [166].

There are several reasons why it is interesting to study the link representation form (2.2.15). Firstly, it has the advantage of linearising momentum conservation in terms of the c_{iJ} variables. Secondly, the quantity $U(c_{iJ})$ defined in (2.2.16) is easily computable for any k. Finally, it was shown in [166] that by using the global residue theorem, one can arrive at an alternative representation of the amplitudes which precisely matches BCFW diagrams, thus establishing a direct connection between the twistor-string representation of amplitudes and on-shell recursion relations. We will see that the same is also true for our representation of form factors in Section 2.2.3.

In performing calculations, we shall always use the canonical gauge-fixing (2.2.4), so that

$$U(c_{iJ}) = \int \prod_{a=1}^{n} \frac{\mathrm{d}(x\,a)\;\mathrm{d}(y\,a)}{(a\,a+1)} \prod_{i\in\bar{\mathsf{p}},J\in\mathsf{m}} \delta\Big(c_{iJ} - \frac{1}{(i\,J)}\Big)\,. \tag{2.2.17}$$

As above, we obtain all other required brackets via the Schouten identity,

$$(x\,y)(a\,b) = (x\,a)(y\,b) - (x\,b)(y\,a)\,. \tag{2.2.18}$$

In (2.2.17) we have $2n$ integration variables and $k(n+2-k)$ delta functions, which means that $U(c_{iJ})$ contains $(k-2)(n-k)$ delta functions after integration. In (2.2.15), four of the Grassmann-even delta functions enforce momentum conservation, leaving $2n$ delta functions and $k(n+2-k)$ variables c_{iJ} to integrate over. This leaves $(k-2)(n-k)$ integration variables, which we denote by τ. Thus (2.2.15) can be written as

$$F = J\,\langle x\,y\rangle^2\,\delta^{(4)}\Big(q - \sum_{a=1}^{n} p_a\Big) \int \mathrm{d}^{(k-2)(n-k)}\tau\;U(c_{iJ}) \prod_{J\in\mathsf{m}} \delta^{(4)}(\eta_J + c_{iJ}\eta_i)\,, \tag{2.2.19}$$

for some $c_{iJ}(\tau)$ linear in τ, and an appropriate Jacobian J.

We shall illustrate the link variable method by proving that (2.2.19) computes the MHV super form factor correctly. Abusing notation, we set $\mathsf{m} = \{J, K\}$. We start by writing the integrand as

$$\prod_{a=1}^{n} \frac{1}{(a\ a+1)} = \frac{1}{(J-1\ J)(J\ J+1)\ (K-1\ K)(K\ K+1)} \prod_{a\neq J-1,J,K-1,K}^{n} \frac{1}{(a\ a+1)}\,. \tag{2.2.20}$$

Now introducing factors of $(J\ K)$ and using the Schouten identity (2.2.18) gives

$$\frac{1}{(a\ a+1)} = \frac{(J\ K)}{(a\ J)(a+1\ K) - (a\ K)(a+1\ J)}\,. \tag{2.2.21}$$

On the support of the delta functions this becomes

$$\frac{c_{aJ}\, c_{a+1\,K}\, c_{aK}\, c_{a+1\,J}}{c_{xy;JK}\, c_{a\,a+1;JK}}\,, \tag{2.2.22}$$

where we have defined the notation,

$$c_{ab;JK} = c_{aJ}\, c_{bK} - c_{aK}\, c_{bJ}\,. \tag{2.2.23}$$

To solve the delta functions, we change variables from $(x\ a)$ and $(y\ a)$ to $(i\ J)$, incurring a Jacobian $(J\ K)^{n-2}$. The Jacobian factor from solving the delta functions is then

$$\begin{aligned} &\frac{c_{xJ}^{-2}\, c_{xK}^{-2}\, x_{yJ}^{-2}\, c_{yK}^{-2}}{c_{J-1\,J}\, c_{J\,J+1}\, c_{K-1\,K}\, c_{K\,K+1}\, c_{J-1\,K}\, c_{J+1\,K}\, c_{J\,K-1}\, c_{J\,K+1}} \\ &\qquad\qquad \times \prod_{a\neq J-1,J,K-1,K}^{n} \frac{1}{c_{aJ}\, c_{a+1\,J}\, c_{aK}\, c_{a+1\,K}} \end{aligned} \tag{2.2.24}$$

After significant cancellations, we are left with

$$U^{\mathrm{MHV}} = \frac{1}{(c_{xy;JK})^2\, c_{J-1\,K}\, c_{J+1\,K}\, c_{J\,K-1}\, c_{J\,K+1}} \prod_{a\neq J-1,J,K-1,K}^{n} \frac{1}{c_{a\,a+1;JK}}\,. \tag{2.2.25}$$

Now solving the chiral delta functions in (2.2.15), we find that

$$c_{iJ} = \frac{\langle i\ K\rangle}{\langle J\ K\rangle}\,, \quad c_{iK} = \frac{\langle i\ J\rangle}{\langle K\ J\rangle}\,, \tag{2.2.26}$$

for all $i \in \bar{\mathsf{p}}$. In performing this integration, we gain another Jacobian $\langle J\ K\rangle^{-n}$. As in the maximally non-MHV case, the antichiral delta functions combine to yield momentum conservation. On the support of (2.2.26), the fermionic delta functions become

$$\delta^{(4)}\left(\frac{\eta_J\langle J\ K\rangle + \eta_i\langle i\ K\rangle}{\langle J\ K\rangle}\right)\delta^{(4)}\left(\frac{\eta_K\langle K\ J\rangle + \eta_i\langle i\ J\rangle}{\langle K\ J\rangle}\right) = \langle J\ K\rangle^{-4}\delta^{(4)}(Q_+)\delta^{(4)}(Q_-)\,, \tag{2.2.27}$$

encoding supermomentum conservation. We finally arrive at

$$F^{\mathrm{MHV}} = \frac{1}{\langle 1\,2\rangle \cdots \langle n\,1\rangle}\ \delta^{(4)}(P)\delta^{(4)}(Q_+)\delta^{(4)}(Q_-)\,, \tag{2.2.28}$$

in agreement with (1.2.42).

2.2.3 $1^-2^-3^-4^+$ and Relation to BCFW

We now examine a component form factor, namely extracting the $(\gamma_+)^0$ behaviour by working with purely gluonic external states. More specifically, we shall perform an NMHV calculation, exposing an interesting parallel with BCFW recursion. We choose $I = \{1,2,3\}$ and $i = \{4,x,y\}$, whence (2.2.17) reads

$$U^{1^-2^-3^-4^+} = \int \prod_{a=1}^{4} \frac{\mathrm{d}(x\,a)\,\mathrm{d}(y\,a)}{(a\,a+1)} \prod_{J=1}^{3} \delta\Big(c_{xJ} - \frac{1}{(x\,J)}\Big)\,\delta\Big(c_{yJ} - \frac{1}{(y\,J)}\Big)\,\delta\Big(c_{4J} - \frac{1}{(4\,J)}\Big)\,. \tag{2.2.29}$$

With nine delta functions and eight integrations, there is one delta function remaining after all integrations are carried out. The integrations over $(x\,J)$ and $(y\,J)$ are straightforward, and one can then choose to solve the two delta functions involving $(4\,1)$ and $(4\,2)$, producing a Jacobian, and insert this solution into the remaining delta function for $(4\,3)$. Collecting all terms from this process, one finds that

$$U^{1^-2^-3^-4^+} = \frac{c_{x2}\,c_{y2}}{c_{42}\,c_{xy;21}\,c_{xy;23}}\,\delta(S_{123;4xy})\,, \tag{2.2.30}$$

where, following the notation introduced in [166], we define

$$S_{ijk;lmn} := c_{mi}\,c_{mj}\,c_{lk}\,c_{nk}\,c_{ln;ij} - c_{ni}\,c_{nj}\,c_{lk}\,c_{mk}\,c_{lm;ij} - c_{li}\,c_{lj}\,c_{mk}\,c_{nk}\,c_{mn;ij}\,. \tag{2.2.31}$$

Following (2.2.19), the form factor can be obtained by integrating out the remaining delta function. We have performed this calculation using Mathematica, and found that the result agrees numerically with (1.2.56). However, the resulting expression has a considerable number of terms, and so is not particularly enlightening for analytic study. Fortunately, there is a more efficient way to derive the final result which avoids solving the constraint of $\delta(S_{123;4xy})$ altogether. As in Section 1.3.2, we appeal to the global residue theorem, by reinterpreting $\int \mathrm{d}\tau$ as a contour integral.

More explicitly we may write

$$F^{1^-2^-3^-4^+} = \oint \mathrm{d}\tau \frac{c_{x2}\,c_{y2}}{c_{42}\,c_{xy;21}\,c_{xy;23}}\frac{1}{S_{123;4xy}}\,, \tag{2.2.32}$$

where the contour surrounds only the poles at $S_{123;4xy}(\tau) = 0$. By the global residue theorem, we may equivalently compute the result in terms of the other poles of the integrand, namely the simple zeros of $c_{42}(\tau)$, $c_{xy;21}(\tau)$ and $c_{xy;23}(\tau)$. The corresponding residues are

$$
\begin{aligned}
F_{42} &= -\frac{\langle 1\,3\rangle^4\, q^4}{s_{134}\,\langle 1\,4\rangle\,\langle 3\,4\rangle\,\langle 3|q|2]\,\langle 1|q|2]}\,,\\
F_{xy;21} &= -\frac{\langle 3|q|4]^3}{s_{124}\,[1\,2]\,[1\,4]\,\langle 3|q|2]}\,,\\
F_{xy;23} &= -\frac{\langle 1|q|4]^3}{s_{324}\,[3\,2]\,[3\,4]\,\langle 1|q|2]}\,,
\end{aligned}
\tag{2.2.33}
$$

and the complete result is obtained by adding the three terms,

$$
F^{1^-2^-3^-4^+} = F_{42} + F_{xy;21} + F_{xy;23}\,. \tag{2.2.34}
$$

It is notable that each term in (2.2.33) depends on p_x and p_y only through the combination $p_x + p_y = q$. Moreover, each term is a rational function of external kinematics. Interestingly, these two properties do not hold for the four terms arising from the solutions of the scattering equation $S_{123;4xy} = 0$, and are only recovered in the sum over the four solutions.

Perhaps more remarkably, each term in (2.2.33) corresponds to a BCFW diagram for a $[1\ 2\rangle$ shift, analogously to the amplitude case, as discussed in [166]. Specifically, observe that the sum in (2.2.34) corresponds, term by term, to the sum given by the BCFW expansion of the form factor in Figure 3.

2.3 Relation to the Grassmannian

In Section 1.2.6, we saw that the N^{k}MHV super form factor of the chiral stress tensor multiplet admits a representation as a Grassmannian integral,[47],

$$
\begin{aligned}
F_{n,k} = \langle n{+}1\ n{+}2\rangle^2 \int & \frac{\mathrm{d}^{k\times(n+2)}C_{Ja}\ \mathrm{d}^{2k}\rho_J}{|GL(k;\mathbb{C})|}\\
&\times \sum_{\text{ins}} \frac{\Omega_{n,k}(C)\delta^{(2(n+2))}(\rho_J c_{Ja} - \lambda_a)\delta^{(2k)}(c_{Ja}\tilde{\lambda}_a)\delta^{(4k)}(c_{Ja}\eta_a)}{(1\cdots k)\cdots(n+2\cdots k-1)}\,,
\end{aligned}
\tag{2.3.1}
$$

where the numerator factor is

$$
\Omega_{n,k}(C) = \frac{Y}{1-Y}\,, \quad Y = \frac{(n{+}2{-}k\cdots n\, n{+}1)(n{+}2\, 1\cdots k{-}1)}{(n{+}2{-}k\cdots n\, n{+}2)(n{+}1\, 1\cdots k{-}1)}\,. \tag{2.3.2}
$$

[47]In this section we will set $\mathsf{m} = \{1,\ldots,k\}$, $\mathsf{p} = \{k{+}1,\ldots,n\}$ and $\{x,y\} = \{n{+}1, n{+}2\}$ for convenience. Recall that the sum is over certain insertions of $\{n{+}1, n{+}2\}$ into $\{1,\ldots n\}$, specified for various cases in [4].

We are interested in connecting this conjecture with the result (2.1.11) expressing the form factor as a sum over solutions to the rational scattering equations. An obvious approach is suggested by comparison to [170], in which Grassmannian amplitude formulae are mapped to CHY-type formulae courtesy of the Veronese map (1.3.23). We can perform a partial integration of (2.3.1), reducing it to an integral over $G(2, n{+}2)$ in this embedding:[48]

$$F_{n,k} = \langle n{+}1\, n{+}2\rangle^2 \int \frac{\mathrm{d}^{n+2}\sigma\, \mathrm{d}^{n+2}\xi\, \mathrm{d}^{2k}\rho}{|GL(2;\mathbb{C})|} \sum_{\text{ins}} \frac{\Omega^V_{n,k}(\sigma_a, \xi_a)}{\prod_{a=1}^{n+2} \xi_a(\sigma_a - \sigma_{a+1})}$$
$$\times \prod_{a=1}^{n+2} \delta^{(2)}\Big(\lambda_a - \xi_a \sum_{J\in\mathsf{m}} \rho_J \sigma_a^{J-1}\Big) \prod_{J\in\mathsf{m}} \delta^{(2|4)}\Big(\sum_{a=1}^{n+2} \xi_a \sigma_a^{J-1} \{\tilde{\lambda}_a|\eta_a\}\Big) . \tag{2.3.3}$$

The delta functions enforce the polynomial scattering equations in the language of [171]. For practical purposes it it more convenient to partially gauge fix the $GL(k;\mathbb{C})$ symmetry before applying the Veronese map, enforcing the rational scattering equations of [158]. We also apply the change of variables (2.1.6) to yield

$$\langle n{+}1\, n{+}2\rangle^2 \int \frac{\mathrm{d}^{2(n+2)}\sigma}{GL(2)} \sum_{\text{ins}} \frac{\Omega^V_{n,k}(\sigma_a, t_a)}{(\sigma_1\sigma_2)\cdots(\sigma_{n+2}\sigma_1)}$$
$$\times \prod_{i\in\bar{\mathsf{p}}} \delta^{(2)}(\lambda_i - \lambda(\sigma_i)) \prod_{J\in\mathsf{m}} \delta^{(2|4)}(\tilde{\lambda}_J - \tilde{\lambda}(\sigma_J), \eta_J - \eta(\sigma_J)) , \tag{2.3.4}$$

where the functions defining the scattering equations are given by (1.3.72). Under the Veronese map, Y then becomes

$$Y^V(\sigma_a, \xi_a) = \prod_{j=n+2-k}^{n} \frac{\sigma_j - \sigma_{n+1}}{\sigma_j - \sigma_{n+2}} \prod_{i=1}^{k-1} \frac{\sigma_{n+2} - \sigma_i}{\sigma_{n+1} - \sigma_i} , \tag{2.3.5}$$

after using the Vandermonde determinant formula. Note immediately that this is independent of the ξ_a, thus the transition to the rational scattering equation version is simply the identity map. In terms of the homogeneous coordinates, (2.3.5) is

$$Y^V(\sigma_a) = \prod_{j=n+2-k}^{n} \frac{(j\, n+1)}{(j\, n+2)} \prod_{i=1}^{k-1} \frac{(n+2\, i)}{(n+1\, i)} . \tag{2.3.6}$$

The authors of [165] conjectured a simpler formula for the chiral stress tensor super form factor, namely (2.1.11). In the cases $k = 2$ and $k = n$ a short calculation shows

[48] Naively one might worry that there are four fewer integration variables than δ-functions. However the leftover constraints combine to form the δ-function of momentum conservation in the final answer, as required.

agreement with the formula (2.3.6) obtained from the Grassmannian. Indeed, these cases correspond to the MHV and maximally non-MHV form factor, where the sum in (2.3.1) consists of a single term. More generally, one must sum over terms arising from several top-cell forms constructed via on-shell diagrams. These correspond to particular cyclic shifts of the insertion point of the additional legs representing the form factor.

The first non-trivial case in which we wish to show agreement between (2.3.4) and (2.1.11) is $n = 4$, $k = 3$, which corresponds to the helicity assignment $1^-2^-3^-4^+$ in our chosen convention. For this case it was shown in [87] that the appropriate insertions are $\{1, 2, 3, 4, 5, 6\}$ and $\{1, 2, 5, 6, 3, 4\}$. A little algebra suffices to prove that

$$\begin{aligned}&\frac{Y_1}{1-Y_1}\frac{1}{(1\ 2)(2\ 3)(3\ 4)(4\ 5)(5\ 6)(6\ 1)}+\frac{Y_2}{1-Y_2}\frac{1}{(1\ 2)(2\ 5)(5\ 6)(6\ 3)(3\ 4)(4\ 1)}\\&=\frac{1}{(1\ 2)(2\ 3)(3\ 4)(4\ 1)(5\ 6)^2}\,,\end{aligned}\tag{2.3.7}$$

where

$$Y_1=\frac{(3\ 5)(4\ 5)(6\ 1)(6\ 2)}{(3\ 6)(4\ 6)(5\ 1)(5\ 2)}\,,\qquad\qquad Y_2=\frac{(1\ 5)(2\ 5)(6\ 3)(6\ 4)}{(1\ 6)(2\ 6)(5\ 3)(5\ 4)}\,.\tag{2.3.8}$$

Note that in (2.3.7) we have obtained the expected integrand, where the auxiliary particles associated to the form factor now only appear in the factor $(5\ 6)^2$.

The next non-trivial case is $n = 5$, $k = 3$. In this case, we have checked numerically that no combination of insertions reproduces the formula (2.1.11). This is not so surprising, since in this case different residues are required from each top-cell diagram, whereas the Veronese map treats terms democratically. In Section 2.5 we indirectly verify our formula in the 5-point NMHV sector, confirming that the tension with the Grassmannian integral representation is a interesting mathematical feature.

It would be interesting to determine whether there is an improved choice of top-cells compatible with a Veronese reduction. As a first step towards this goal, we consider uplifting our link representation formula (2.2.30) to a Grassmannian integral, following the procedure first outlined for amplitudes in [172]. In terms of 3×3 minors we find

$$\begin{aligned}F^{1^-2^-3^-4^+}=&\frac{(1\ 3\ 6)(1\ 3\ 5)}{(1\ 3\ 4)(1\ 5\ 6)(3\ 5\ 6)(1\ 2\ 3)}\\&\times\frac{1}{(1\ 2\ 3)(3\ 4\ 5)(1\ 5\ 6)(2\ 4\ 6)-(2\ 3\ 4)(4\ 5\ 6)(1\ 2\ 6)(1\ 3\ 5)}\,,\end{aligned}\tag{2.3.9}$$

where we have omitted the integration and measure for brevity. In deriving this formula, we employed the quadratic Plücker relations,

$$(a\ b\ c)(d\ i\ j) - (b\ c\ d)(a\ i\ j) + (c\ d\ a)(b\ i\ j) - (d\ a\ b)(c\ i\ j) = 0\,, \tag{2.3.10}$$

arising from the implicit non-linear embedding of the Grassmannian in the projective space of the k^{th} exterior power of $\mathbb{C}^n$. We also scaled the answer by factors of $(1\ 2\ 3)$ to ensure that the expression has uniform weight -3 in every leg. This is a permitted operation since $(1\ 2\ 3) = 1$ in our chosen gauge.

In (2.3.9) the square bracketed term emerges from the rational scattering equation delta function. Naïvely, we should evaluate the residue when this term vanishes. However, we saw in the previous section that it can be advantageous to consider the other poles, by virtue of the global residue theorem. Doing so will allow us to partially match our uplifted formula onto (2.3.1).

Both the residue at $(1\ 2\ 3) = 0$ and the residue at $(1\ 5\ 6) = 0$ may be equivalently obtained from the integrand,

$$\frac{1}{(1\ 2\ 3)(2\ 3\ 4)(3\ 4\ 5)(4\ 5\ 6)(5\ 6\ 1)(6\ 1\ 2)}\frac{(3\ 4\ 5)(6\ 1\ 2)}{(3\ 4\ 6)(5\ 1\ 2) - (3\ 4\ 5)(6\ 1\ 2)}\,, \tag{2.3.11}$$

as one may easily check with a little algebraic manipulation. Note that the second factor in (2.3.11) is exactly of the form $\frac{Y}{1-Y}$, as expected. The situation for the poles $(1\ 3\ 4) = 0$ and $(3\ 5\ 6) = 0$ is more mysterious, but may be resolved similarly by first applying the permutation identity derived in Section 3.2 of [173].

2.4 Ambitwistor String Theory

The result (2.1.11) bears a close resemblance to the formula (1.3.73) first derived in [158] from an ambitwistor-string model, describing the tree-level n-particle scattering in four-dimensional $\mathcal{N} = 4$ SYM. In this construction, the Parke-Taylor denominator of the measure emerges from a current algebra on the worldsheet, as we reviewed in Section 1.3.3.

We may construct the measure of formula (2.1.11) from ambitwistor strings in a similar way, at least up to an overall factor. We must include two additional vertex operators, corresponding to the punctures σ_{n+1} and σ_{n+2} on the Riemann sphere. These are dressed with additional currents defined as in (1.3.61). However, in order to obtain the chiral stress tensor super form factor, we now do not require the single trace term.

Rather we extract from Wick's theorem the double trace term displayed below,

$$\langle J^{a_1} \cdots J^{a_{n+2}} \rangle = \cdots + \frac{\mathrm{Tr}\,(T^{a_1} \cdots T^{a_n})}{(\sigma_1 - \sigma_2) \cdots (\sigma_n - \sigma_1)\,(\sigma_{n+1} - \sigma_{n+2})^2} \cdot \mathrm{Tr}\,(T^{a_{n+1}} T^{a_{n+2}}) + \cdots\,, \tag{2.4.1}$$

providing the appropriate denominator and colour factor for the on-shell state. It would be very interesting to have a complete derivation of (2.1.11) from ambitwistor strings, also explaining the $\langle x\,y\rangle^2$ prefactor. This may require a more complete description of the vertex operator corresponding to the operator insertion, perhaps inspired by recent work [174–177]. Such a construction may also be applicable for the $\mathrm{Tr}(\phi^k)$ form factors considered in [164].

Of course, the current algebra in the four-dimensional ambitwistor string construction is identical to that in the ten-dimensional formula of [178] which reproduces standard CHY formulae. We might thus recast the formula (2.1.11) as a sum over solutions to the standard scattering equations [2]. To do this would require an appropriate prescription for the polarisation vectors associated with the off-shell insertion. We detail such a proposal in Section 2.5.

Given that form factors emerge so naturally from an ambitwistor string construction, it is tempting to speculate that appropriate current algebra modifications might allow the construction of still more general objects, namely correlation functions. An obvious generalisation of the approach followed for form factors would be to include additional auxiliary particles to represent further operator insertions. The simplest example would be that of a two-point correlator of $\mathcal{O} = \mathrm{Tr}F^2_{\mathrm{SD}}$ and $\bar{\mathcal{O}} = \mathrm{Tr}F^2_{\mathrm{ASD}}$ with the vacuum as the external state. In order to contract the two operators, we choose the two pairs of auxiliary particles to have opposite helicity, (x^+, y^+) and (u^-, v^-). One might conjecture

$$\begin{aligned}\langle 0|\,\mathcal{O}(q)\,\bar{\mathcal{O}}(q')\,|0\rangle = \langle x\,y\rangle^2\,[u\,v]^2 \int \frac{1}{|GL(2;\mathbb{C})|} \frac{\mathrm{d}^2\sigma_x\,\mathrm{d}^2\sigma_y}{(x\,y)^2} \frac{\mathrm{d}^2\sigma_u\,\mathrm{d}^2\sigma_v}{(u\,v)^2} \\ \times \prod_{i=x,y} \delta^{(2)}(\lambda_i - \lambda(\sigma_i)) \prod_{J=u,v} \delta^{(2)}(\tilde{\lambda}_J - \tilde{\lambda}(\sigma_J))\,,\end{aligned} \tag{2.4.2}$$

where

$$\lambda(\sigma) = \sum_{J=u,v} \frac{\lambda_J}{(\sigma\;\sigma_J)}\,, \qquad \tilde{\lambda}(\sigma) = \sum_{i=x,y} \frac{\tilde{\lambda}_i}{(\sigma_i\;\sigma)}\,. \tag{2.4.3}$$

An explicit calculation shows that (2.4.2) is equal to

$$q^4\,\delta^{(4)}(p_x + p_y + p_u + p_v)\,, \tag{2.4.4}$$

with $q = p_x + p_y$. This is not quite the result one expects to find [179], namely

$$\langle 0|\, \mathcal{O}(q)\, \bar{\mathcal{O}}(q')\, |0\rangle \sim \delta^{(4)}(q+q')\, q^4 \log(q^2) \,+\, \text{analytic terms}\,. \tag{2.4.5}$$

In particular the $\log q^2$ term is absent. In order to be able to derive such terms one may need to understand scattering equations for off-shell quantities at loop level, along the lines of [180–182].

2.5 Towards a CHY Formula

In Section 2.2, we expressed the form factor of $\mathrm{Tr}F_{\mathrm{SD}}^2$ as a sum over solutions to the rational scattering equations (1.3.74). This formalism naturally exposes supersymmetry, but is closely tied to four spacetime dimensions. We may immediately contrast the CHY formula (1.3.31), which admits no known supersymmetrisation, but is valid in arbitrary dimensions. Nevertheless, there are at least two methods for translating between the formalisms. Firstly, one may take advantage of ambitwistor string theory, recalling from Section 1.3.3 that the rational scattering equations emerge from a four-dimensional specialisation of a general theory reproducing the full CHY result. Secondly, one can directly reduce from arbitrary dimensions to the four-dimensional refinement along the lines laid out by Zhang [183]. Importantly Zhang observed that the Pfaffian (1.3.36) exactly cancels a Jacobian factor upon performing the appropriate change of variables, explaining why (1.3.73) has unit numerator.

These methods provide a means of lifting the form factor (2.1.11) to a CHY version, expressed a sum over the scattering equations (1.3.30). More precisely, we specialise to purely gluonic external states, and hence conjecture the following formula for the tree level form factor of $\mathrm{Tr}F^2$ in pure Yang-Mills theory in four dimensions:

$$F_n = \sum_{\text{polarisations}} \frac{1}{2}(p_x{\cdot}p_y\ \bar{\epsilon}_x{\cdot}\bar{\epsilon}_y - \bar{\epsilon}_x{\cdot}p_y\ \bar{\epsilon}_y{\cdot}p_x) \oint \frac{\mathrm{d}^{n+2}\sigma}{\mathrm{d}\omega}\, \frac{\mathrm{Pf}'(\Psi_{n+2})}{(\sigma_x - \sigma_y)^2} \prod_{a=1}^{n} \frac{1}{\sigma_a - \sigma_{a+1}} \prod_{a=1}^{n+2}{}' \frac{1}{f_a(\sigma,k)}\,. \tag{2.5.1}$$

where the momenta of the two auxiliary particles sum to the off-shell momentum q and we sum over the polarisations ϵ_x and ϵ_y as in [184]. Note immediately that the integrand is Möbius invariant, since it differs from the amplitude case by a cross-ratio factor $\frac{(\sigma_n-\sigma_x)(\sigma_y-\sigma_1)}{(\sigma_x-\sigma_y)(\sigma_n-\sigma_1)}$. Leaving aside the gauge invariant prefactor for now, clearly the formula reduces to (2.1.11) on applying the Zhang rules and choosing x and y to have positive helicity. Moreover, it is trivially what we expect to obtain by extracting the trace structure (2.4.1) from the full Mason-Skinner ambitwistor string. It only remains to understand the agreement of the prefactors. In four dimensions we may represent the polarisation vectors by

$$\epsilon_x^+ = \sqrt{2}\frac{|x]\langle y|}{\langle x\ y\rangle}\,, \quad \epsilon_y^+ = \sqrt{2}\frac{|y]\langle x|}{\langle y\ x\rangle}\,, \quad \epsilon_x^- = \sqrt{2}\frac{|x\rangle[y|}{[x\ y]}\,, \quad \epsilon_y^- = \sqrt{2}\frac{|y\rangle[x|}{[y\ x]}\,. \tag{2.5.2}$$

Then we find that

$$\frac{1}{2}(p_x \cdot p_y\ \overline{\epsilon}_x^+ \cdot \overline{\epsilon}_y^+ - \overline{\epsilon}_x^+ \cdot p_y\ \overline{\epsilon}_y^+ \cdot p_x) = \langle x\ y\rangle^2\,, \tag{2.5.3}$$

as required for the self-dual part of the form factor, where we have implicitly restricted to real momenta by exchanging angle and square spinors upon complex conjugation. Similarly we obtain the prefactor $[x\ y]^2$ for the anti-self-dual case, where x and y are chosen to have negative helicity. The prefactor gives zero contribution when x and y have opposite helicities. Hence, we should expect (2.5.1) to compute the form factor of $\mathrm{Tr}F^2$ in four dimensions. We have verified that this is correct using the Mathematica package documented in Appendix A. Most importantly, we evaluated the $n = 5, k = 3$ case in which the Veronese reduction was shown to fail, demonstrating agreement with the sum of MHV diagrams in Figure 4. We did not compute this case in [10].

There are several potential extensions of these observations which we hope to report on in future work. Firstly, following Dolan and Goddard [185] one should be able to inductively prove the formula (2.5.1) in four dimensions by demonstrating the validity of an appropriate recurrence relation. Indeed, the appearance of BCFW terms in Section 2.2.3 lends weight to this proposal. Secondly, one might consider whether the formula (and any proof) still holds in arbitrary dimension. Indeed, even in the base case $n = 2, k = 2$ a naive evaluate of the Pfaffian produces a large number of terms, which must combine and cancel to produce the simple result demanded by Wick contractions,

$$F_2 = \frac{1}{4}\langle 0|\mathrm{Tr}\left[(q_\mu A_\nu - q_\nu A_\mu)(q^\mu A^\nu - q^\nu A^\mu)\right]|1,2\rangle = \frac{1}{2}(p_1 \cdot p_2\ \epsilon_1 \cdot \epsilon_2 - \epsilon_1 \cdot p_2\ \epsilon_2 \cdot p_1)\,. \tag{2.5.4}$$

A further advantage of the CHY formula over the rational version is the ability to extend the result to loop level.[49] Supposing that a five-dimensional formula could be verified, one might apply the arguments of [184] to obtain a loop integrand in the Q-cut representation [188], in particular reproducing the rational terms discovered by Davies [189]. Finally, one might attempt to find similar formulae for other operators in pure Yang-Mills, most obviously $\mathrm{Tr}F^3$. For this operator, a CHY formula in the soft

[49]At present the rational scattering equations at loop level are not well understood. The attempted construction [186] suffers from technical pathologies. For example, the s_a variables are set to zero for MHV amplitudes, rendering the formula meaningless. During the preparation of this manuscript, a more promising proposal appeared [187], based on the relations between CHY and Grassmannian formulae. It would be fascinating to extend these arguments to the form factor case.

limit $q \to 0$ was already determined in [190].

Chapter 3

Soft Gluon Theorems at Tree Level and One Loop

In this chapter, we constrain soft gluon theorems at tree level and one loop using dual conformal symmetry, inspired by [163]. We find that the tree level subleading soft theorem may be exactly reconstructed, under weak conditions. However, the one-loop constraint is insufficiently strong to provide a unique solution at subleading order. We then explicitly calculate the subleading soft anomaly in the MHV and NMHV sectors, as defined by (1.2.109), through finite order in the IR regulator. This extends the previously known IR divergent results of [129]. We find evidence for universality within, but not between, helicity sectors. The text follows very closely that of [9], in which the author contributed a large amount of the written content.

3.1 Constraints from Conformal Symmetry

In [163], conformal symmetry was used in order to determine the tree-level soft theorem (1.2.102). As a warm-up to our dual conformal calculations we shall briefly review this method. From now on we employ arbitrary forms of the stripped amplitudes, with the proviso that an n-point momentum conservation prescription should be applied afterwards.

First recall that the special conformal generator takes the form,[50]

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial|i\rangle\partial[i|}\,, \tag{3.1.1}$$

[50] Here, and elsewhere in this chapter, we leave some spinor indices implicit.

and upon expanding in the soft parameter δ,

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial|i\rangle\partial[i|} + \frac{1}{\delta}\frac{\partial^2}{\partial|n\rangle\partial[n|}\,. \tag{3.1.2}$$

Now note that k annihilates arbitrary forms of tree-level stripped superamplitudes, in particular order by order in δ. Applying k to (1.2.102) yields the constraint equations,

$$\frac{\partial^2}{\partial|n\rangle\partial[n|}\left(S^{(0)}A^{\text{tree}}_{n-1}\right) = 0\,, \tag{3.1.3}$$

$$\sum_{i=1}^{n-1}\frac{\partial^2}{\partial|i\rangle\partial[i|}\left(S^{(0)}A^{\text{tree}}_{n-1}\right) + \frac{\partial^2}{\partial|n\rangle\partial[n|}\left(S^{(1)}A^{\text{tree}}_{n-1}\right) = 0\,. \tag{3.1.4}$$

These equations allow us to determine the forms of the soft factors. In fact we shall require extra input from considerations of little group scaling, mass dimension and colour ordering. Firstly, the soft operators must have mass dimension -1. Furthermore, since we are taking a positive helicity particle soft, the soft operators must transform with weight -2 under the little group scaling,

$$|n\rangle \to t|n\rangle\ , \qquad |n] \to t^{-1}|n]\,, \tag{3.1.5}$$

and remain invariant under little group scaling for all other particles. Finally, since the amplitudes are colour ordered, the soft operators may only depend on particles $n-1$ and 1 adjacent to n, since only these share a colour line with n.[51] At one loop we will find that this simplifying assumption no longer holds since internal gluons carry colour dependence between arbitrary particles. Putting all this information together with the conformal Ward identities (3.1.3) and (3.1.4) suffices to determine $S^{(0)}$ and $S^{(1)}$ as written in (1.2.96) and (1.2.97).

It is difficult to generalise the method of [163] to loop level, because the conformal anomaly takes a complicated form. The current state of the art is restricted to MHV amplitudes and is rather intricate [191]. By contrast, the dual conformal anomaly (1.2.49) is very simple, which will allow us to make progress in the next section.

[51] At tree level in the planar sector, the only way that the color structure of particles can become entagled is if they are adjacent. The reader may swiftly verify this by attempting to find a counter-example in 't Hooft double line notation [47].

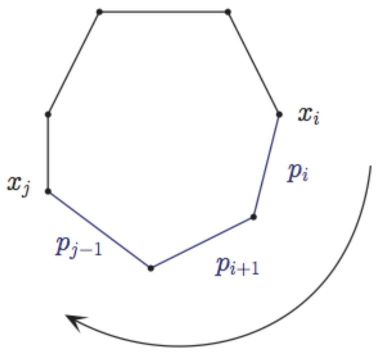

Figure 9: *Solving for $x_j(p)$ clockwise around the polygon from x_i.*

3.2 Constraints from Dual Conformal Symmetry

Our goal is to constrain soft factors using dual superconformal symmetry. Initially we work at tree level, then we extend the technique to one-loop amplitudes. To begin with, we collect a few results about soft limits in dual space.

In analogy with the previous section, we must expand the dual conformal boost generator (1.2.47) in powers of the soft parameter δ. This involves solving for dual momenta x_i in terms of momenta p_j. This procedure is ambiguous because of momentum conservation. In general we may freely fix any x_i allowing us to perform the change of variables $x \to p$. More precisely, we determine x_j as a sum of the p_k between x_i and x_j as indicated in Figure 9. The clockwise orientation is an arbitrary choice corresponding to taking $j > i$ cyclically.

In Section 1.2.9 we saw that momentum conservation is a subtle issue for subleading soft theorems. Therefore we must be careful regarding the ambiguity in base point x_i and orientation around the polygon when solving for $x(p)$. In the following we use a prescription that eliminates a pair of antiholomorphic spinors $(|a], |b])$ according to the substitution (1.2.100).

The simplest choice[52] is to fix $x_3 = 0$ and solve clockwise around the polygon, whence

$$x_{k\alpha\dot{\alpha}} = -\sum_{j=3}^{k-1} |j]\langle j| \,, \tag{3.2.1}$$

[52]In order to preserve momentum conservation and on-shell external momenta, a minimum of three momenta must acquire δ dependence, and hence a minimum of two dual momenta must be δ-dependent. Our choice (3.2.1) achieves this.

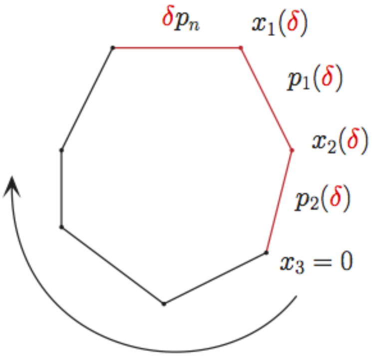

Figure 10: *Setting $x_3 = 0$ is compatible with eliminating $|1]$ and $|2]$.*

for $k \neq 3$. The solution (3.2.1) is compatible with eliminating $|1]$ and $|2]$. The only δ-dependent region momenta are then x_1 and x_2 as shown in Figure 10. Similarly, for the fermionic variables we set $\theta_3 = 0$ and write

$$\langle \theta_k^A | = -\sum_{j=3}^{k-1} \langle j | \eta_j^A \,, \tag{3.2.2}$$

for $k \neq 3$. We should view (3.2.1) and (3.2.2) as a frame choice well-adapted to the computations which follow. Of course, any result we derive in this frame may trivially be transformed to another using the substitution (1.2.100).

The soft expansion of the dual conformal boost generator is

$$\begin{aligned} K_{\alpha\dot\alpha} = &-\sum_{i\neq 3}\sum_{j=3}^{i-1}{}' |j]\langle i| \left(|j\rangle \cdot \frac{\partial}{\partial |i\rangle}\right) - \sum_{i\neq 2}\sum_{j=3}^{i}{}' |i]\langle j| \left(|j] \cdot \frac{\partial}{\partial |i]}\right) - \sum_{i\neq 2}\sum_{j=3}^{i}{}' |i]\langle j|\eta_j^A \frac{\partial}{\partial \eta_i^A} \\ &- \delta|n]\langle 2| \left(|n\rangle \cdot \frac{\partial}{\partial |2\rangle}\right) - \delta|n]\langle 1| \left(|n\rangle \cdot \frac{\partial}{\partial |1\rangle}\right) - \delta|n]\langle n| \left(|n] \cdot \frac{\partial}{\partial |n]}\right) \\ &- \delta|1]\langle n| \left(|n] \cdot \frac{\partial}{\partial |1]}\right) - \delta|n]\langle n|\eta_n^A \frac{\partial}{\partial \eta_n^A} - \delta|1]\langle n|\eta_n^A \frac{\partial}{\partial \eta_1^A} \,, \end{aligned} \tag{3.2.3}$$

where $\sum_j{}'$ indicates a sum over $j \neq n$. Similarly the statement of dual conformal covariance (1.2.48) yields

$$K_{\alpha\dot\alpha} A_n^{\text{tree}} = \Big(\sum_{i\neq 3}\sum_{j=3}^{i-1}{}' |j]\langle j| + 2\delta|n]\langle n| \Big) A_n^{\text{tree}} \,. \tag{3.2.4}$$

Paraphrasing (1.2.49), the dual conformal operator acts on one-loop amplitudes to give

$$K_{\alpha\dot\alpha} A_n^{\text{1-loop}} = (\text{anomaly})\, A_n^{\text{tree}} + (\text{covariance})\, A_n^{\text{1-loop}} \,. \tag{3.2.5}$$

For later convenience we reproduce the soft expansion of the covariance statement from (3.2.4),

$$(\text{covariance}) = \Big(\sum_{i\neq 3} \sum_{j=3}^{i-1}{}' |j]\langle j| + 2\delta |n]\langle n| \Big) . \tag{3.2.6}$$

In the frame choice (3.2.1) the soft expansion of the anomaly (1.2.49) is

$$\begin{aligned}(\text{anomaly}) = -\frac{2}{\epsilon} c_\Gamma \Big[\sum_{i\neq 1,3,n} \sum_{j=3}^{i-1}{}' |j]\langle j| \Big(-(i-1\ i) \Big)^{-\epsilon} + \sum_{j=3}^{n-1} |j]\langle j| \Big(-\delta(n-1\ n) \Big)^{-\epsilon} \\ + \sum_{j=3}^{n-1} |j]\langle j| \Big(-\delta(n\ 1) \Big)^{-\epsilon} + \delta |n]\langle n| \Big(-\delta(n\ 1) \Big)^{-\epsilon} + \delta |n]\langle n| \Big(-(1\ 2) \Big)^{-\epsilon} \Big] .\end{aligned} \tag{3.2.7}$$

3.2.1 Tree Level Constraints

By keeping the leading $1/\delta$ divergence in (3.2.3) we find the following constraint equation for the leading soft factor, in analogy with (3.1.3),

$$\big(K_{\alpha\dot\alpha} A_n^{\text{tree}}\big)_{\mathcal{O}(\delta^{-2})} = (K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)} \Big(S^{(0)} A_{n-1}^{\text{tree}} \Big) = \Big(\sum_{i\neq 3} \sum_{j=3}^{i-1}{}' |j]\langle j| \Big) S^{(0)} A_{n-1}^{\text{tree}} . \tag{3.2.8}$$

The covariance statement for $(n-1)$-point amplitudes gives

$$(K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)} A_{n-1}^{\text{tree}} = \Big(\sum_{i\neq 3,n} \sum_{j=3}^{i-1}{}' |j]\langle j| \Big) A_{n-1}^{\text{tree}} . \tag{3.2.9}$$

Hence (3.2.8) simplifies to

$$(K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)} S^{(0)} = \Big(\sum_{j=3}^{n-1} |j]\langle j| \Big) S^{(0)} . \tag{3.2.10}$$

This leading order behaviour can be checked explicitly using the known form of $S^{(0)}$ and the formulae in Appendix B. Conversely we may use (3.2.10) to determine the form of $S^{(0)}$, at least up to a constant factor. Since our amplitudes are colour ordered, we may assume

$$S^{(0)} = f(\langle a\ b\rangle, [a\ b]) , \tag{3.2.11}$$

where a, b can take values in $\{n-1, n, 1\}$. To obtain the dual conformal transformation (3.2.10) f must be proportional to

$$\frac{1}{\langle n\ 1\rangle} \text{ or } \frac{1}{[n-1\ n]}\,. \tag{3.2.12}$$

The constant of proportionality must have mass dimension 0. Moreover f must transform under little group scaling with weight $+2$ for particle n and 0 for all other particles. These constraints rule out the second option in (3.2.12), and lead us immediately to

$$f = k\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle\langle n\ 1\rangle} \tag{3.2.13}$$

Assuming universality, we may fix $k = 1$ by examining the simplest example, namely a four-point MHV amplitude.

At subleading order we employ the approach of [129], allowing the freedom to use arbitrary forms of the stripped amplitudes in our derivations. The dual conformal analogue of (3.1.4) is

$$\begin{aligned}\left(K_{\alpha\dot\alpha}A_n^{\text{tree}}\right)_{\mathcal{O}(\delta^{-1})} &= \left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^{1})}\left(S^{(0)}A_{n-1}^{\text{tree}}\right) + \left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^{0})}\left(S^{(1)}A_{n-1}^{\text{tree}}\right)\\ &= 2|n]\langle n|S^{(0)}A_{n-1}^{\text{tree}} + \Big(\sum_{i\neq 3}\sum_{j=3}^{i-1}{}'|j]\langle j|\Big)S^{(1)}A_{n-1}^{\text{tree}}\,.\end{aligned} \tag{3.2.14}$$

It is convenient to rewrite the first line of (3.2.14) to obtain

$$\begin{aligned}\left(K_{\alpha\dot\alpha}A_n^{\text{tree}}\right)_{\mathcal{O}(\delta^{-1})} = &-\frac{|n]\langle 1|}{\langle n\ 1\rangle}A_{n-1}^{\text{tree}} + S^{(0)}\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^{1})}A_{n-1}^{\text{tree}}\\ &+\left[\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^{0})}, S^{(1)}\right]A_{n-1}^{\text{tree}} + S^{(1)}\left[\Big(\sum_{i\neq 3,n}\sum_{j=3}^{i-1}{}'|j]\langle j|\Big)A_{n-1}^{\text{tree}}\right]\,.\end{aligned} \tag{3.2.15}$$

Using the covariance statement for $(n-1)$-point amplitudes we get

$$\begin{aligned}&-\frac{|n]\langle 1|}{\langle n\ 1\rangle}A_{n-1}^{\text{tree}} + S^{(0)}\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^{1})}A_{n-1}^{\text{tree}} + \left[\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^{0})}, S^{(1)}\right]A_{n-1}^{\text{tree}}\\ &+A_{n-1}^{\text{tree}}S^{(1)}\Big(\sum_{i\neq 3,n}\sum_{j=3}^{i-1}{}'|j]\langle j|\Big) = 2|n]\langle n|S^{(0)}A_{n-1}^{\text{tree}} + \Big(\sum_{j=3}^{n-1}|j]\langle j|\Big)S^{(1)}A_{n-1}^{\text{tree}}\,.\end{aligned} \tag{3.2.16}$$

We begin by verifying this using the known form of the subleading soft operator $S^{(1)}$. First note that

$$\frac{|n]\langle 1|}{\langle n\ 1\rangle} + S^{(1)}\Big(\sum_{i\neq 3,n}\sum_{j=3}^{i-1}{}'|j]\langle j|\Big) = 2|n]\langle n|S^{(0)}\,, \tag{3.2.17}$$

using a Schouten identity, whence (3.2.16) becomes

$$-2\frac{|n]\langle 1|}{\langle n\;1\rangle}A^{\text{tree}}_{n-1}+S^{(0)}\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^1)}A^{\text{tree}}_{n-1}+\Big[\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^0)},S^{(1)}\Big]A^{\text{tree}}_{n-1}=\Big(\sum_{j=3}^{n-1}|j]\langle j|\Big)S^{(1)}A^{\text{tree}}_{n-1}\,. \tag{3.2.18}$$

Using formulae from Appendix B and Schouten identities we evaluate

$$\begin{aligned}\Big[\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^0)},S^{(1)}\Big]&=\Big(\sum_{j=3}^{n-1}|j]\langle j|\Big)S^{(1)}+S^{(0)}|1]\langle n|\left(|n]\cdot\frac{\partial}{\partial|1]}\right)+S^{(0)}|n]\langle 1|\left(|n\rangle\cdot\frac{\partial}{\partial|1\rangle}\right)\\&\quad-\frac{|n]\langle 1|}{\langle n\;1\rangle}\left(|1\rangle\cdot\frac{\partial}{\partial|1\rangle}-|1]\cdot\frac{\partial}{\partial|1]}-\frac{\eta_1^A}{\langle n\;1\rangle}\frac{\partial}{\partial\eta_1^A}\right)\,,\end{aligned} \tag{3.2.19}$$

and observe that

$$S^{(0)}\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^1)}=-S^{(0)}|1]\langle n|\left(|n]\cdot\frac{\partial}{\partial|1]}\right)-S^{(0)}|n]\langle 1|\left(|n\rangle\cdot\frac{\partial}{\partial|1\rangle}\right)\,. \tag{3.2.20}$$

Hence (3.2.18) simplifies to

$$-\frac{|n]\langle 1|}{\langle n\;1\rangle}\left(|1\rangle\cdot\frac{\partial}{\partial|1\rangle}-|1]\cdot\frac{\partial}{\partial|1]}-\frac{\eta_1^A}{\langle n\;1\rangle}\frac{\partial}{\partial\eta_1^A}\right)A^{\text{tree}}_{n-1}=2\frac{|n]\langle 1|}{\langle n\;1\rangle}A^{\text{tree}}_{n}\,. \tag{3.2.21}$$

On the the left-hand side of (3.2.21) we immediately recognise the appearance of the helicity operator (1.2.31) for particle 1. Recalling that superamplitudes have unit helicity completes the verification.

Conversely, we can use (3.2.16) to derive the form of $S^{(1)}$ up to two constants. From Taylor series considerations it is natural to expect $S^{(1)}$ to be a derivative operator. We first split the constraint according to whether derivatives act, yielding

$$S^{(0)}\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^1)}+\Big[\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^0)},S^{(1)}\Big]=\Big(\sum_{j=3}^{n-1}|j]\langle j|\Big)S^{(1)}\,, \tag{3.2.22}$$

$$-\frac{|n]\langle 1|}{\langle n\;1\rangle}+S^{(1)}\Big(\sum_{i\neq 3,n}\sum_{j=3}^{i-1}{}'|j]\langle j|\Big)=2|n]\langle n|S^{(0)}\,. \tag{3.2.23}$$

Note that we might expect some mixing between the terms in each equation by virtue of the identity operator. The canonical representation of the identity under these circumstances is as a helicity operator. Hence we look for a form of $S^{(1)}$ which satisfies (3.2.22) up to additive helicity operators.

The key observation is found by studying the derivative structure of $S^{(0)}\left(K_{\alpha\dot\alpha}\right)_{\mathcal{O}(\delta^1)}$ in (3.2.20). After Schoutening, the second term on the right hand side yields a derivative structure which appears in $(K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)}$, namely

$$|n-1\rangle \cdot \frac{\partial}{\partial|1\rangle}\,, \tag{3.2.24}$$

and one which appears in the helicity operator for particle 1, namely

$$|1\rangle \cdot \frac{\partial}{\partial|1\rangle}\,. \tag{3.2.25}$$

Now (3.2.22) and Occam's razor suggest that the second term on the right hand side of (3.2.20) does not represent a contribution to $S^{(1)}$. On the contrary, the first term does not admit such a Schoutening since the derivative part involves antiholomorphic spinors. We thus propose that

$$S^{(1)} \propto |n] \cdot \frac{\partial}{\partial|1]}\,. \tag{3.2.26}$$

As in the leading case, the requirements of mass dimension, little group scaling and colour ordering constrain the constant of proportionality, leading to

$$S^{(1)} = \frac{k|n]}{\langle n\ 1\rangle} \cdot \frac{\partial}{\partial|1]}\,. \tag{3.2.27}$$

Of course, freedom to relabel the polygon in the opposite direction dictates the appearance of a similar term involving particle $(n-1)$. We finally arrive at

$$S^{(1)} = \frac{k|n]}{\langle n-1\ n\rangle} \cdot \frac{\partial}{\partial|n-1]} + \frac{l|n]}{\langle n\ 1\rangle} \cdot \frac{\partial}{\partial|1]}\,. \tag{3.2.28}$$

As above, k and l may be set to 1 by assuming universality and considering two non-trivial examples.

Similarly to [163] we can fix the leading and subleading soft operators at tree level, with mild assumptions. More importantly, we may now use the simple form of the one-loop anomaly of dual conformal symmetry to study soft factorisation at one-loop level, as we will see below. Note in this context that the conventional conformal anomaly is much more complicated and its general form is not known.

3.2.2 One Loop Constraints

In Section 1.2.9 we conjectured a form for the one-loop soft theorem

$$A_n^{\text{1-loop}} \to \frac{1}{\delta^2}\left(S^{(0)}A_{n-1}^{\text{1-loop}} + S^{(0)\text{1-loop}}A_{n-1}^{\text{tree}}\right) + \frac{1}{\delta}\left(S^{(1)\text{tree}}A_{n-1}^{\text{1-loop}} + S^{(1)\text{1-loop}}A_{n-1}^{\text{tree}}\right), \tag{3.2.29}$$

with previous results for $S^{(0)\text{1-loop}}$ and the infrared-divergent part of $S^{(1)\text{1-loop}}$ quoted in (1.2.107) and (1.2.108). We now derive dual conformal constraint equations on both $S^{(0)\text{1-loop}}$ and $S^{(1)\text{1-loop}}$ through $\mathcal{O}(\epsilon^0)$. These equations provide non-trivial checks on the known expressions. Furthermore, the one-loop subleading soft constraint suggests an ansatz for the hitherto unknown infrared-finite part of $S^{(1)\text{1-loop}}$.

The one-loop version of (3.2.8) is, by using (1.2.106),

$$\begin{aligned}\left(K_{\alpha\dot\alpha}A_n^{\text{1-loop}}\right)_{\mathcal{O}(\delta^{-2})} &= (K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)}\left(S^{(0)}F^{(0)}A_{n-1}^{\text{tree}} + S^{(0)}A_{n-1}^{\text{1-loop}}\right)\\ &= (\text{anomaly})_{\mathcal{O}(\delta^0)}\,S^{(0)}A_{n-1}^{\text{tree}} + (\text{covariance})_{\mathcal{O}(\delta^0)}\,S^{(0)}F^{(0)}A_{n-1}^{\text{tree}}\\ &\quad + (\text{covariance})_{\mathcal{O}(\delta^0)}\,S^{(0)}A_{n-1}^{\text{1-loop}}\,,\end{aligned}$$

where $F^{(0)}$ is defined in (1.2.107). This can be simplified significantly by recycling our tree-level knowledge; in fact, we can remove all terms involving the one-loop amplitude. Recall from (3.2.8) that

$$(K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)}\left(S^{(0)}A_{n-1}^{\text{tree}}\right) = (\text{covariance})_{\mathcal{O}(\delta^0)}\,S^{(0)}A_{n-1}^{\text{tree}}\,, \tag{3.2.30}$$

and hence we find that

$$(K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)}\left(S^{(0)}A_{n-1}^{\text{1-loop}}\right) = (\text{covariance})_{\mathcal{O}(\delta^0)}\,S^{(0)}A_{n-1}^{\text{1-loop}} + (\text{anomaly})_{n-1}\,S^{(0)}A_{n-1}^{\text{tree}}\,. \tag{3.2.31}$$

Using these results (3.2.30) simplifies to

$$(K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)}\,F^{(0)} + (\text{anomaly})_{n-1} = (\text{anomaly})_{\mathcal{O}(\delta^0)}\,, \tag{3.2.32}$$

or more explicitly,

$$(K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)}\,F^{(0)} = \frac{2}{\epsilon}c_\Gamma\Big(\sum_{j=3}^{n-1}|j]\langle j|\Big)\left[\Big(-\delta(n-1\;n)\Big)^{-\epsilon} + \Big(-\delta(n\;1)\Big)^{-\epsilon} - \Big(-(n-1\;1)\Big)^{-\epsilon}\right]. \tag{3.2.33}$$

Firstly we wish to verify that (3.2.33) holds using the known expression for $F^{(0)}$ in (1.2.107). It is easy to see that this is true at $\mathcal{O}(\epsilon^{-1})$. Using results from Appendix B we find that

$$(K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)}\left(\frac{(n-1\ 1)}{(n-1\ n)(n\ 1)}\right) = 2\Big(\sum_{j=3}^{n-1}|j]\langle j|\Big)\left(\frac{(n-1\ 1)}{(n-1\ n)(n\ 1)}\right), \tag{3.2.34}$$

whence at $\mathcal{O}(\epsilon^0)$ in (3.2.33) both sides evaluate to

$$2c_\Gamma\Big(\sum_{j=3}^{n-1}|j]\langle j|\Big)\log\left(-\frac{1}{\delta^2}\frac{(n-1\ 1)}{(n-1\ n)(n\ 1)}\right), \tag{3.2.35}$$

confirming the consistency of (3.2.33) at $\mathcal{O}(\epsilon^0)$ also.

Conversely we can use (3.2.33) as a constraint equation to determine $F^{(0)}$ up to and including ϵ^0 terms, provided that we assume that $F^{(0)}$ only depends on particles $n-1$, n and 1 and is a dimensionless, helicity-blind function. The derivation proceeds analogously to that in Section 3.2.1.

Naïvely, the restriction to particles neighbouring n seems unreasonable from the Wilson loop perspective. Indeed, we might expect contributions from diagrams where an internal gluon connects an arbitrary edge to p_n. However, the scalar boxes corresponding to the non-cusp diagrams do not contribute in the leading soft limit. This is perhaps most obvious from the perspective of MHV diagrams [134].

The one-loop version of (3.2.14) is

$$\begin{aligned}\Big(K_{\alpha\dot{\alpha}}A_n^{\text{1-loop}}\Big)_{\mathcal{O}(\delta^{-1})} &= (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)}\Big(S^{(0)}F^{(0)}A_{n-1}^{\text{tree}} + S^{(0)}A_{n-1}^{\text{1-loop}}\Big)\\ &\quad + (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)}\Big(S^{(1)}A_{n-1}^{\text{1-loop}} + F^{(1)}S^{(1)}A_{n-1}^{\text{tree}} + ZA_{n-1}^{\text{tree}}\Big)\\ &= (\text{anomaly})_{\mathcal{O}(\delta^1)}\,S^{(0)}A_{n-1}^{\text{tree}} + (\text{covariance})_{\mathcal{O}(\delta^1)}\,S^{(0)}F^{(0)}A_{n-1}^{\text{tree}}\\ &\quad + (\text{covariance})_{\mathcal{O}(\delta^1)}\,S^{(0)}A_{n-1}^{\text{1-loop}} + (\text{anomaly})_{\mathcal{O}(\delta^0)}\,S^{(1)}A_{n-1}^{\text{tree}}\\ &\quad + (\text{covariance})_{\mathcal{O}(\delta^0)}\,S^{(1)}A_{n-1}^{\text{1-loop}} + (\text{covariance})_{\mathcal{O}(\delta^0)}\,F^{(1)}S^{(1)}A_{n-1}^{\text{tree}}\\ &\quad + (\text{covariance})_{\mathcal{O}(\delta^0)}\,ZA_{n-1}^{\text{tree}},\end{aligned} \tag{3.2.36}$$

where $F^{(1)}$ and Z are defined in (1.2.109). Just as in the leading case, we can remove all terms involving $A_{n-1}^{\text{1-loop}}$ by recycling tree-level knowledge. Recall from (3.2.14) that

$$\begin{aligned}&(K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)}\Big(S^{(0)}A_{n-1}^{\text{tree}}\Big) + (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)}\Big(S^{(1)}A_{n-1}^{\text{tree}}\Big)\\ &= (\text{covariance})_{\mathcal{O}(\delta^1)}\,S^{(0)}A_{n-1}^{\text{tree}} + (\text{covariance})_{\mathcal{O}(\delta^0)}\;S^{(1)}A_{n-1}^{\text{tree}},\end{aligned} \tag{3.2.37}$$

and hence that

$$
\begin{aligned}
&(K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)} \left(S^{(0)} A_{n-1}^{\text{1-loop}}\right) + (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)} \left(\ S^{(1)} A_{n-1}^{\text{1-loop}}\right) \\
&= (\text{covariance})_{\mathcal{O}(\delta^1)} S^{(0)} A_{n-1}^{\text{1-loop}} + (\text{covariance})_{\mathcal{O}(\delta^0)}\ S^{(1)} A_{n-1}^{\text{1-loop}} + S^{(1)} \left[(\text{anomaly})_{n-1} A_{n-1}^{\text{tree}}\right] .
\end{aligned}
\tag{3.2.38}
$$

Applying these results to (3.2.36) we get

$$
\begin{aligned}
&S^{(0)} A_{n-1}^{\text{tree}} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)} F^{(0)} + (F^{(0)} - F^{(1)}) (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)} \left(S^{(0)} A_{n-1}^{\text{tree}}\right) \\
&+\ S^{(1)} A_{n-1}^{\text{tree}} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)} F^{(1)} + A_{n-1}^{\text{tree}} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)} Z \\
&= (F^{(0)} - F^{(1)}) (\text{covariance})_{\mathcal{O}(\delta^1)} S^{(0)} A_{n-1}^{\text{tree}} + (\text{anomaly})_{\mathcal{O}(\delta^1)} S^{(0)} A_{n-1}^{\text{tree}} \\
&+ (\text{anomaly})_{\mathcal{O}(\delta^0)}\ S^{(1)} A_{n-1}^{\text{tree}} + (\text{anomaly})_{n-1} S^{(1)} A_{n-1}^{\text{tree}} - S^{(1)} \left[(\text{anomaly})_{n-1}\right] A_{n-1}^{\text{tree}} \\
&+ \left[(\text{covariance})_{\mathcal{O}(\delta^0)} - (\text{covariance})_{n-1}\right] Z A_{n-1}^{\text{tree}} .
\end{aligned}
\tag{3.2.39}
$$

To proceed, we separate this result into two equations, depending on whether derivatives act on A_{n-1}^{tree}; of course there may be some cancellations between these equations via the appearance of helicity operators. With this separation we have derivative terms,

$$
\begin{aligned}
&(F^{(0)} - F^{(1)}) S^{(0)} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)} A_{n-1}^{\text{tree}} +\ S^{(1)} A_{n-1}^{\text{tree}} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)} F^{(1)} \\
&= (\text{anomaly})_{\mathcal{O}(\delta^0)}\ S^{(1)} A_{n-1}^{\text{tree}} - (\text{anomaly})_{n-1} S^{(1)} A_{n-1}^{\text{tree}} ,
\end{aligned}
\tag{3.2.40}
$$

and non-derivative terms,

$$
\begin{aligned}
&S^{(0)} A_{n-1}^{\text{tree}} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)} F^{(0)} + (F^{(0)} - F^{(1)}) A_{n-1}^{\text{tree}} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)} S^{(0)} + A_{n-1}^{\text{tree}} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)} Z \\
&= (F^{(0)} - F^{(1)}) (\text{covariance})_{\mathcal{O}(\delta^1)} S^{(0)} A_{n-1}^{\text{tree}} + (\text{anomaly})_{\mathcal{O}(\delta^1)} S^{(0)} A_{n-1}^{\text{tree}} \\
&- S^{(1)} \left[(\text{anomaly})_{n-1}\right] A_{n-1}^{\text{tree}} + \left[(\text{covariance})_{\mathcal{O}(\delta^0)} - (\text{covariance})_{n-1}\right] Z A_{n-1}^{\text{tree}} .
\end{aligned}
\tag{3.2.41}
$$

We focus first on equation (3.2.40). Note that the derivatives in $(K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)}$ and $S^{(1)}$ do not combine to yield a helicity operator. Therefore we may assume that this equation is truly decoupled from (3.2.41). Using (3.2.32) we see that (3.2.40) is satisfied if we choose $F^{(1)} = F^{(0)}$. This is consistent with the known infrared divergent behaviour of $F^{(1)}$ and extends it to finite order in ϵ.

With this choice, (3.2.41) simplifies to give

$$
\begin{aligned}
&S^{(0)} A_{n-1}^{\text{tree}} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^1)} F^{(0)} + A_{n-1}^{\text{tree}} (K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)} Z = (\text{anomaly})_{\mathcal{O}(\delta^1)} S^{(0)} A_{n-1}^{\text{tree}} \\
&- S^{(1)} \left[(\text{anomaly})_{n-1}\right] A_{n-1}^{\text{tree}} + \left[(\text{covariance})_{\mathcal{O}(\delta^0)} - (\text{covariance})_{n-1}\right] Z A_{n-1}^{\text{tree}} .
\end{aligned}
\tag{3.2.42}
$$

Thus we have arrived at a dual conformal constraint equation on the one-loop subleading soft anomaly.

Constraint on the Infrared-Divergent Anomaly

We now expand in ϵ to find constraint equations for Z at each order. We write

$$Z = \frac{1}{\epsilon^2} Z_{-2} + \frac{1}{\epsilon} Z_{-1} + Z_0 + \mathcal{O}(\epsilon)\,. \tag{3.2.43}$$

At leading order in ϵ the anomaly constraint (3.2.42) becomes

$$(K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)}\, Z_{-2} = \Big[(\text{covariance})_{\mathcal{O}(\delta^0)} - (\text{covariance})_{n-1}\Big]\, Z_{-2} = \Big(\sum_{j=3}^{n-1} |j]\langle j|\Big) Z_{-2}\,. \tag{3.2.44}$$

Clearly this is consistent with the choice $Z_{-2} = 0$ implicit in (1.2.108). For the converse argument, first note that (3.2.44) has exactly the same form as (3.2.10). We therefore employ logic similar to the leading tree-level case. Indeed, since we are dealing with infrared-divergent terms,

$$Z_{-2} = f(\langle a\ b\rangle, [a\ b])\,, \tag{3.2.45}$$

where a, b takes values in $\{n-2, n-1, n, 1, 2\}$ by the Wilson loop observations of Section 1.2.9. Following Section 3.2.1, we see that if $Z_{-2} \neq 0$ then we must have $Z_{-2} = S^{(0)}$. But reinserting factors of δ shows that $S^{(0)}$ can only appear as a leading soft divergence. Hence the constraint equation fixes

$$Z_{-2} = 0\,. \tag{3.2.46}$$

At subleading order in ϵ the anomaly constraint (3.2.42) becomes

$$\begin{aligned}
&\Big[\,(K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)} - \sum_{j=3}^{n-1} |j]\langle j|\Big] Z_{-1} \\
&= -|1]\langle 1| \frac{[n-1\ n]}{[n-1\ 1]\langle n\ 1\rangle} + \frac{|n]\langle n-1|}{\langle n-1\ n\rangle} - \frac{|n]\langle 1|}{\langle n\ 1\rangle} + |n-1]\langle n-1| \frac{[n\ 1]}{\langle n-1\ n\rangle [n-1\ 1]}\,.
\end{aligned} \tag{3.2.47}$$

Note that this is symmetric under relabelling the polygon anticlockwise, as expected. From (1.2.108) we have [129]

$$Z_{-1} = \frac{[n-1\ n]}{[n-1\ 1]\langle 1\ n\rangle} + \frac{[2\ n]}{[2\ 1]\langle 1\ n\rangle} - \frac{[1\ n]}{[1\ n-1]\langle n-1\ n\rangle} - \frac{[n-2\ n]}{[n-2\ n-1]\langle n-1\ n\rangle}\,. \tag{3.2.48}$$

The reader may verify that this satisfies the constraint equation, using formulae from Appendix B. Conversely we write an ansatz,

$$Z_{-1} = g(\langle a\ b\rangle, [a\ b])\,, \tag{3.2.49}$$

where a, b can take values in $\{n-2, n-1, n, 1, 2\}$ by the same logic as for Z_{-2}. We assume for simplicity that each term on the right-hand side of (3.2.47) emerges from a single term in Z_{-1}. Then (B.8)–(B.11) immediately suggest the result (3.2.48), which is clearly consistent with spinor weight and dimension constraints.

Constraint on the Infrared-Finite Anomaly

Finally we consider the $\mathcal{O}(\epsilon^0)$ terms. The anomaly constraint (3.2.42) becomes

$$\begin{aligned}
&\Big[(K_{\alpha\dot\alpha})_{\mathcal{O}(\delta^0)} - \sum_{j=3}^{n-1} |j]\langle j|\Big] Z_0 \\
&= 2\Big[Z_{-1}\sum_{j=3}^{n-1}|j]\langle j| - |n-1]\langle n-1|\frac{[n-2\ n]}{\langle n-1\ n\rangle[n-1\ n-2]} + |1]\langle 1|\frac{[2\ n]}{[2\ 1]\langle 1\ n\rangle}\Big] \\
&+\left[\frac{|n]\langle 1|}{\langle n\ 1\rangle} + 2\frac{|n]\langle n-1|}{\langle n-1\ n\rangle} - |1]\langle n|\frac{\langle n-1\ 1\rangle[n\ n-1]}{\langle n-1\ n\rangle\langle n\ 1\rangle[1\ n-1]}\right]\log\left(-\frac{(n-1\ 1)}{(n-1\ n)(n\ 1)}\right) \\
&- 2\frac{|n]\langle n-1|}{\langle n-1\ n\rangle}\log(-(n-1\ 1)) + 2|n]\langle n|\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle\langle n\ 1\rangle}\log(-(n\ 1))\,.
\end{aligned} \tag{3.2.50}$$

We now employ this formula to find plausible coefficients for the $\log\delta$ terms appearing in Z_0. The constraint equation (3.2.50) immediately suggests that these take the form,

$$A\log(-(n-1\ n)) + B\log(-(n\ 1))\,. \tag{3.2.51}$$

Relabelling symmetry ensures that it suffices to predict coefficient A. We make the ansatz,

$$A = S^{(0)}h(\langle a\ b\rangle, [a\ b])\,, \tag{3.2.52}$$

where a, b take values in $\{n-2, n-1, n, 1, 2\}$. Unlike the Z_{-2} and Z_{-1} cases considered above, there is no rigorous argument for this assumption, since the $\log\delta$ terms do not only emerge from cusp diagrams. Nevertheless it seems plausible to expect that such divergent terms only involve particles close to n. In Section 3.3 we shall see that this ansatz holds for MHV amplitudes, but not in the NMHV sector.

A general one-loop amplitude in $\mathcal{N}=4$ SYM theory involves functions of transcendentality 2. Therefore we expect the soft anomaly Z_0 to contain functions of transcendentality 1 and 0. We may hence deduce from (3.2.50) a constraint on h by examining the coefficient of $\log(-(n-1\;n))$, namely

$$(K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)}\, h = -|n-1]\langle n-1|\frac{(n\;1)}{(n-1\;1)} - |n]\langle n| + |1]\langle 1|\frac{(n-1\;n)}{(n-1\;1)}\,. \tag{3.2.53}$$

Equations (B.12) and (B.13) hence suggest that

$$Z_0|_{\log\delta} = \left(\frac{(n\;1)}{(n-1\;1)} + \frac{(n-2\;n)}{(n-2\;n-1)} - \frac{(n-2\;1)(n-1\;n)}{(n-2\;n-1)(n-1\;1)}\right) \times S^{(0)}\log(-(n-1\;n)) + (i\leftrightarrow n-i)\,. \tag{3.2.54}$$

We now proceed to verify this prediction by explicitly computing the subleading soft anomaly in the MHV and NMHV sectors. Beware that Z_0 itself does not suffice to reconstruct the subleading soft behaviour of an n-point amplitude; we must also remember feed-down terms, as discussed near (1.2.105). We consider this nicety in detail in Section 3.3.1.

3.3 Direct Calculation of the Subleading Soft Anomaly

In this section we determine the subleading soft contribution for n-point one-loop MHV amplitudes and for six-point and seven-point one-loop NMHV amplitudes. We first present the subleading soft behaviour of some low-point MHV cases, extracted via the unitarity method. We then use momentum twistor technology to derive a surprisingly compact expression for the subleading soft term at n-point modulo A_n^{tree}, namely[53]

$$\begin{aligned}&\frac{\langle n-1\;1\rangle}{\langle n-1\;n\rangle}\sum_{j=4}^{n-4}\log\left(\frac{y^2_{n-1j}}{y^2_{1j}}\right)\frac{\langle n-2\;n-1\;j-1\;j\rangle\langle n-2\;n-1\;n\;1\rangle}{\langle n-2\;n-1\;1\;j-1\rangle\langle n-2\;n-1\;1\;j\rangle}\\&+\frac{\langle n-1\;1\rangle}{\langle n\;1\rangle}\sum_{j=5}^{n-3}\log\left(\frac{y^2_{2j}}{y^2_{1j}}\right)\frac{\langle n-1\;n\;1\;2\rangle\langle j-1\;j\;1\;2\rangle}{\langle n-1\;1\;2\;j\rangle\langle n-1\;1\;2\;j-1\rangle}+\text{boundary terms}\,,\end{aligned} \tag{3.3.1}$$

The boundary terms have a universal form for all $n\geq 7$. In particular, the $\log\delta$ dependence is simply

$$\left(\frac{(n\;1)+(n\;2)}{(1\;2)} - \frac{s_{n-1,1,2}(n\;1)}{(n-1\;1)(1\;2)}\right)\log(-(n\;1)) + (i\leftrightarrow n-i)\,. \tag{3.3.2}$$

[53] Round brackets such as (6) appearing below represent the dual superconformal R-invariants, and are defined in (3.3.40).

where $s_{abc} := (p_a + p_b + p_c)^2$ denotes a three-particle invariant. We finally investigate the possibility of universality carrying over to NMHV amplitudes by identifying the subleading $\log\delta$ terms in low-point cases. Again intricate cancellations yield a remarkably simple result, but of a slightly different form to the MHV sector. Explicitly we find terms at six and seven points,

$$\frac{1}{2}\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle}(6)\frac{\langle 2\ 3\ 4\ 5\rangle\langle 4\ 5\ 6\ 1\rangle}{\langle 1\ 2\ 4\ 5\rangle\langle 3\ 4\ 5\ 1\rangle}\log(-(6\ 1)) + (i \leftrightarrow n-i)\,, \tag{3.3.3}$$

$$\frac{1}{2}\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle}\left[(5)\frac{\langle 2\ 3\ 4\ 6\rangle\langle 5\ 6\ 7\ 1\rangle}{\langle 3\ 4\ 6\ 1\rangle\langle 1\ 2\ 5\ 6\rangle} + (3)\frac{\langle 2\ 4\ 5\ 6\rangle\langle 5\ 6\ 7\ 1\rangle}{\langle 4\ 5\ 6\ 1\rangle\langle 1\ 2\ 5\ 6\rangle}\right]\log(-(7\ 1)) + (i \leftrightarrow n-i)\,, \tag{3.3.4}$$

respectively. We conjecture that the $\log\delta$ terms display universal behaviour for arbitrary n within each N^{k}MHV sector, but not between different sectors.

Throughout this section we employ the approach of [129], with a symmetric momentum conservation prescription eliminating $|n-1]$ and $|1]$. In particular this implies that the feed-down terms from Taylor-expanding $S^{(0)}A_{n-1}^{\text{1-loop}}$ in the soft parameter exactly cancel the contribution from $S^{(1)}A_{n-1}^{\text{1-loop}}$. Therefore the form of the lower-point amplitude becomes irrelevant to the calculation of the subleading soft anomaly.

3.3.1 MHV Sector

In Section 1.2.8, we saw that all one-loop MHV amplitudes in $\mathcal{N}=4$ SYM theory may be written as a sum over box functions. To calculate the subleading soft behaviour, we must in principle Taylor expand all box functions. Many may be immediately discarded, along the lines outlined in Section 1.2.9. Specifically the only nonzero terms emerge from boxes corresponding to Wilson loop diagrams in which the internal gluon ends on particle lines $n-1$, n or 1.

Five-Point Amplitude

The simplest non-trivial subleading soft behaviour appears at five points. In this case, the two-mass easy boxes degenerate to one-mass boxes. The subleading soft five-point infrared-finite term divided by A_5^{tree} is

$$-\frac{\langle 4\ 5\rangle[2\ 5]}{\langle 3\ 4\rangle[2\ 3]}\log\left(\frac{(4\ 5)}{(1\ 5)(2\ 3)}\right) - \frac{\langle 1\ 5\rangle[3\ 5]}{\langle 1\ 2\rangle[2\ 3]}\log\left(\frac{(1\ 5)}{(2\ 3)(4\ 5)}\right)\,, \tag{3.3.5}$$

For compactness we have implicitly recombined terms using four-point momentum conservation where appropriate. We have checked this result numerically using the Mathematica package `SubSoft.m` documented in Appendix A.

The simplifications required to reach (3.3.5) involve intricate cancellations between roughly 20 terms from different boxes. This suggests that box functions are poorly adapted to the calculation of subleading soft behaviour.

Six-Point Amplitude

At six points we discover new structure associated with the appearance of non-degenerate two-mass boxes. The subleading soft infrared-finite contribution modulo A_6^{tree} is

$$\begin{aligned}
&\frac{(3\ 4)(1\ 6)}{(1\ 2)(1\ 5)}\left[1+\log\left(\frac{(1\ 2)(1\ 5)}{(1\ 6)(3\ 4)}\right)\right]+\frac{(2\ 3)(5\ 6)}{(4\ 5)(1\ 5)}\left[1+\log\left(\frac{(1\ 5)(4\ 5)}{(2\ 3)(5\ 6)}\right)\right]\\
&-\frac{\langle 1\ 6\rangle[3\ 4](\langle 3\ 4\rangle[4\ 6]+\langle 3\ 5\rangle[5\ 6])}{(1\ 2)\langle 1\ 5\rangle[4\ 5]}\log(-(1\ 2))-\frac{\langle 5\ 6\rangle[2\ 3](\langle 1\ 3\rangle[1\ 6]+\langle 2\ 3\rangle[2\ 6])}{(4\ 5)\langle 1\ 5\rangle[1\ 2]}\log(-(4\ 5))\\
&+\frac{((1\ 6)+(2\ 6))}{(1\ 2)}\left[\log\left(-\frac{(1\ 6)(3\ 4)}{(1\ 5)}\right)-1\right]+\frac{((4\ 6)+(5\ 6))}{(4\ 5)}\left[\log\left(-\frac{(2\ 3)(5\ 6)}{(1\ 5)}\right)-1\right]\\
&+\frac{[2\ 6]\langle 5\ 6\rangle}{\langle 1\ 5\rangle[1\ 2]}\log\left(-\frac{(1\ 5)(2\ 3)}{(3\ 4)}\right)+\frac{[4\ 6]\langle 1\ 6\rangle}{\langle 1\ 5\rangle[4\ 5]}\log\left(-\frac{(1\ 5)(3\ 4)}{(2\ 3)}\right).
\end{aligned}\tag{3.3.6}$$

As in the five-point case, very non-trivial simplifications take place – the Taylor expansion initially produces some 200 terms. Mathematica numerics exactly confirm our concise formula.

It is instructive to perform a consistency check that the six-point result (3.3.6) reproduces the five-point result (3.3.5) when we make particle 3 soft. Taking the limit and relabelling appropriately we obtain

$$\begin{aligned}
&\left(\frac{\langle 4\ 5\rangle[2\ 5]}{\langle 1\ 4\rangle[1\ 2]}+\frac{\langle 1\ 5\rangle[3\ 5]}{\langle 1\ 4\rangle[3\ 4]}\right)\log(-(1\ 4))\\
&\qquad+\frac{(1\ 5)+(2\ 5)}{(1\ 2)}\log(-(1\ 5))+\frac{(3\ 5)+(4\ 5)}{(3\ 4)}\log(-(4\ 5)).
\end{aligned}\tag{3.3.7}$$

Naïvely it looks impossible to equate (3.3.7) and (3.3.5), however we only require them to match when a consistent momentum conservation prescription is applied to both. The relatively simple form of (3.3.5) is a consequence of the special four-point kinematics, $[1\ 2]=\langle 3\ 4\rangle[2\ 3]/\langle 1\ 4\rangle$.

$\log\delta$ Terms

To complete our analysis we concentrate on terms involving a $\log\delta$. Recent evidence [131] shows that these may be universal in QCD processes. Indeed these terms are truly infrared divergent, so intuitively one might expect enhanced universality to ensure such quantities cancel in any physical observable. At five points we have from (3.3.7)

$$\frac{(1\,5)+(2\,5)}{(1\,2)}\log(-(1\,5))+\frac{(3\,5)+(4\,5)}{(3\,4)}\log(-(4\,5))\,, \tag{3.3.8}$$

while at six points (3.3.6) yields

$$\begin{aligned}\frac{(1\,6)+(2\,6)}{(1\,2)}\log(-(1\,6))+\frac{(4\,6)+(5\,6)}{(4\,5)}\log(-(5\,6))&\\ -\frac{(3\,4)(1\,6)}{(1\,2)(1\,5)}\log(-(1\,6))-\frac{(2\,3)(5\,6)}{(1\,5)(4\,5)}\log(-(5\,6))\,.&\end{aligned} \tag{3.3.9}$$

The chances of a simple universal result look slim based on this evidence. In (3.3.9) new structures appear, in addition to a generalisation of (3.3.8). However, we shall see shortly that the complexity of $\log\delta$ terms does not grow with particle number in general. From the perspective of box functions, this is reasonable: a new type of box function enters at six points, after which no further new functions appear in the MHV sector.

n-point Formula via Momentum Twistors

We saw in Section 3.2.2 that two classes of Wilson loop diagrams contribute to the subleading soft behaviour. Cusp diagrams give rise to the infrared-divergent piece of any one-loop SYM amplitude,

$$-\frac{1}{\epsilon^2}\sum_{i=1}^{n}\big(-(i\;i+1)\big)^{-\epsilon}\quad . \tag{3.3.10}$$

Non-cusp diagrams with an internal gluon ending on at least one δ-dependent edge also feature. In the MHV sector these correspond to the finite parts of the two-mass easy box in Figure 5. Note that i and j must be separated by at least one intervening particle cyclically. For the symmetric momentum conservation prescription eliminating $(|n-1],|1])$, we can restrict to diagrams where i or j is in $\{n-1,n,1\}$.

We must sum over boxes to produce the full amplitude. This yields large cancellations, particularly between non-degenerate boxes in which i and j are separated by at least two intermediate particles. In Secton 1.2.8 we showed that the n-point one-loop MHV amplitude may be written at $\mathcal{O}(\epsilon^0)$ as

$$\begin{aligned}\frac{A_n^{\text{1-loop}}}{A_n^{\text{tree}}} = \frac{1}{2}\sum_i \sum_{j\notin\{i-2,i-1,i,i+1,i+2\}} &\left(-\mathrm{Li}_2(1-u_{ij}) + \log x_{ij}^2 \log u_{ij}\right) \\ + \sum_i \log(x_{ii-2}^2)\log &\left(\frac{x_{i+1i-2}^2}{x_{i+1i-1}^2\sqrt{x_{ii-2}^2}}\right),\end{aligned} \tag{3.3.11}$$

where u_{ij} denotes the dual conformal invariant cross-ratio,

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}, \tag{3.3.12}$$

and the square root arises from the infrared divergent pieces (3.3.10). The formula (3.3.11) phrases the MHV one-loop amplitude in a form which illuminates dual conformal properties. For example, it is particularly easy to verify that (1.2.49) holds. Presently, we shall see that this expression is especially convenient for extracting subleading soft behaviour. We anticipate that this form of the amplitude may find useful further applications.

We should point out that our description (3.3.11) is not entirely new; partial results of a similar flavour exist in the literature, confirming our observations. The double sum,

$$\frac{1}{2}\sum_i \sum_{j\notin\{i-2,i-1,i,i+1,i+2\}} \left(-\mathrm{Li}_2(1-u_{ij}) + \log x_{ij}^2 \log u_{ij}\right), \tag{3.3.13}$$

emerges from considering only non-degenerate two-mass easy-boxes, and accords with the results of [107]. The remaining sum,

$$\sum_i \log(x_{ii-2}^2)\log\left(\frac{x_{i+1i-2}^2}{x_{i+1i-1}^2\sqrt{x_{ii-2}^2}}\right), \tag{3.3.14}$$

comprises degenerate box and infrared divergent contributions, which were recognised but not calculated in [108].

Note that although (3.3.13) and (3.3.14) look symmetric under reversing the polygon labelling, this is not the case. On careful inspection, we see that this asymmetry is a consequence of our particular choices of telescoping cancellations in arriving at (1.2.89). Of course, the symmetry is restored when generic terms and edge cases are summed.

We first compute the subleading soft behaviour of the generic terms (3.3.13). Without loss of generality we consider only those terms in which $i \in \{n-1, n, 1\}$. It is convenient to use momentum twistor variables [89], which we reviewed in Section 1.3.4. In such variables the (super)soft limit may be expressed as [192]

$$Z_n \to \alpha Z_1 + \beta Z_{n-1} + \delta Z_n \, . \tag{3.3.15}$$

For generic α and β four spinors gain δ-dependence, namely $|n-1]$, $|n]$, $|n\rangle$ and $|1]$. Note that box functions transform with zero weight under little group scaling. Hence we may freely switch between holomorphic and antiholomorphic soft limits without affecting our results. We find it convenient to use the latter, explicitly

$$|n\rangle \to |n\rangle \, , \quad |n] \to \delta |n] \, , \quad \eta_n \to \delta \eta_n \, , \tag{3.3.16}$$

with the symmetric elimination of $|n-1]$ and $|1]$. By comparing (1.3.85) to (1.2.100) this stipulation forces

$$\alpha = \frac{\langle n-1 \; n\rangle}{\langle n-1 \; 1\rangle}(1-\delta) \quad \text{and} \quad \beta = \frac{\langle n \; 1\rangle}{\langle n-1 \; 1\rangle}(1-\delta) \, . \tag{3.3.17}$$

The δ-dependence of α and β is present to ensure that $|n\rangle$ remains fixed. The dual conformal cross-ratio u_{ij} may be expressed as a ratio of twistor four-brackets, namely

$$u_{ij} = \frac{\langle i-1 \; i \; j \; j+1\rangle\langle i \; i+1 \; j-1 \; j\rangle}{\langle i-1 \; i \; j-1 \; j\rangle\langle i \; i+1 \; j \; j+1\rangle} \, . \tag{3.3.18}$$

To evaluate the soft behaviour of relevant cross-ratios we will require the δ expansion of the four-brackets,

$$\langle n \; 1 \; j-1 \; j\rangle = \beta\langle n-1 \; 1 \; j-1 \; j\rangle + \delta\langle n \; 1 \; j-1 \; j\rangle \, , \tag{3.3.19}$$

$$\langle n-1 \; n \; j-1 \; j\rangle = \alpha\langle n-1 \; 1 \; j-1 \; j\rangle + \delta\langle n-1 \; n \; j-1 \; j\rangle \, . \tag{3.3.20}$$

Using twistor identities we can then derive simple forms for

$$u_{n-1j} = v_{n-1j}\left(1 - \frac{\delta\langle j-1 \; j \; j+1 \; n-1\rangle\langle n-1 \; n \; 1 \; j\rangle}{\alpha\langle n-1 \; 1 \; j-1 \; j\rangle\langle n-1 \; 1 \; j \; j+1\rangle}\right) \quad \text{for } 3 \leq j \leq n-4 \, , \tag{3.3.21}$$

$$u_{1j} = v_{1j}\left(1 - \frac{\delta\langle j-1 \; j \; j+1 \; 1\rangle\langle n-1 \; n \; 1 \; j\rangle}{\beta\langle n-1 \; 1 \; j-1 \; j\rangle\langle n-1 \; 1 \; j \; j+1\rangle}\right) \quad \text{for } 4 \leq j \leq n-3 \, , \tag{3.3.22}$$

$$u_{nj} = 1 + \frac{\delta\langle j-1 \; j+1 \; j \; n-1\rangle\langle n \; n-1 \; 1 \; j\rangle}{\alpha\langle n-1 \; 1 \; j-1 \; j\rangle\langle n-1 \; 1 \; j \; j+1\rangle} + \frac{\delta\langle j-1 \; j+1 \; 1 \; j\rangle\langle n-1 \; n \; 1 \; j\rangle}{\beta\langle n-1 \; 1 \; j-1 \; j\rangle\langle n-1 \; 1 \; j \; j+1\rangle}$$
$$\text{for } 3 \leq j \leq n-3 \, , \tag{3.3.23}$$

valid through subleading order in δ, where the v_{ij} are cross-ratios evaluated in $(n-1)$-

point kinematics. In addition we have special cases of cross-ratios that vanish in the soft limit

$$
\begin{aligned}
u_{1n-2} &= \frac{\delta\langle n\ 1\ n-2\ n-1\rangle\langle 1\ 2\ n-3\ n-2\rangle}{\beta\langle 1\ 2\ n-1\ n-2\rangle\langle n-1\ 1\ n-3\ n-2\rangle}\left(1-\frac{\delta\langle n\ 1\ n-3\ n-2\rangle}{\beta\langle n-1\ 1\ n-3\ n-2\rangle}\right),\\
u_{n-12} &= \frac{\delta\langle n-2\ n-1\ 2\ 3\rangle\langle n-1\ n\ 1\ 2\rangle}{\alpha\langle n-2\ n-1\ 1\ 2\rangle\langle n-1\ 1\ 2\ 3\rangle}\left(1-\frac{\delta\langle n-1\ n\ 2\ 3\rangle}{\alpha\langle n-1\ 1\ 2\ 3\rangle}\right).
\end{aligned}
\tag{3.3.24}
$$

Since these appear as arguments of logarithms we require these quantities through order δ^2 in order to extract terms subleading in δ. The multiparticle invariants x_{ij}^2 may be written as a ratio of a four-bracket to two holomorphic spinor brackets,

$$
x_{ij}^2 = \frac{\langle i-1\ i\ j-1\ j\rangle}{\langle i-1\ i\rangle\langle j-1\ j\rangle}\,, \tag{3.3.25}
$$

which breaks conformal symmetry due to the presence of the infinity twistor in the definition of the spinor brackets. The expansions of the only δ-dependent invariants are

$$
x_{nj}^2 = y_{1j}^2 + \delta\frac{\langle n-1\ n\ j-1\ j\rangle}{\langle n-1\ n\rangle\langle j-1\ j\rangle} \qquad \text{for } 2\le j\le n-2\,, \tag{3.3.26}
$$

$$
x_{1j}^2 = y_{1j}^2 + \delta\frac{\langle n\ 1\ j-1\ j\rangle}{\langle n\ 1\rangle\langle j-1\ j\rangle} \qquad \text{for } 3\le j\le n-1\,, \tag{3.3.27}
$$

through subleading order in δ, where the y_{ij}^2 are multiparticle invariants with $(n-1)$-point kinematics.

We have calculated the soft expansion of (3.3.13) by employing these formulae. There are significant telescopic cancellations, yielding bulk terms,

$$
\begin{aligned}
&\frac{1}{\alpha}\sum_{j=4}^{n-4}\log\left(\frac{y_{n-1j}^2}{y_{1j}^2}\right)\frac{\langle n-2\ n-1\ j-1\ j\rangle\langle n-2\ n-1\ n\ 1\rangle}{\langle n-2\ n-1\ 1\ j-1\rangle\langle n-2\ n-1\ 1\ j\rangle}\\
&+\frac{1}{\beta}\sum_{j=5}^{n-3}\log\left(\frac{y_{2j}^2}{y_{1j}^2}\right)\frac{\langle n-1\ n\ 1\ 2\rangle\langle j-1\ j\ 1\ 2\rangle}{\langle n-1\ 1\ 2\ j\rangle\langle n-1\ 1\ 2\ j-1\rangle}\,,
\end{aligned}
\tag{3.3.28}
$$

and boundary contributions,

$$
\begin{aligned}
&-\frac{(n-1\ 1\ 2)(n\ 1)}{(n-1\ 1)(1\ 2)}\left[\log\left(\frac{(n-1\ 1\ 2)(n\ 1)}{(n-1\ 1)(1\ 2)}\right)-1\right]+\left(\frac{(n\ 1)+(n\ 2)}{(1\ 2)}\right)\left[\log(-(n\ 1))-1\right]\\
&+\frac{(n-1\ 1\ 2)\langle n\ 1\rangle\left([n\ 1]\langle 1\ 3\rangle+[n\ 2]\langle 2\ 3\rangle\right)}{(1\ 2)\langle n-1\ 1\rangle\left([n-1\ 1]\langle 1\ 3\rangle+[n-1\ 2]\langle 2\ 3\rangle\right)}\log\left(\frac{(1\ 2)}{(n-1\ 1\ 2)}\right)\\
&-\frac{\langle n-1\ n\rangle[n\ 1]\langle 3\ 4\rangle[2\ 3]}{[1\ 2]\langle n-1\ 1\rangle([1\ 2]\langle 2\ 4\rangle+[1\ 3]\langle 3\ 4\rangle)}\log\left(\frac{(2\ 3)}{(1\ 2\ 3)}\right)\\
&+\left(\frac{(n\ 1)+(n\ 2)}{(1\ 2)}-\frac{\langle n-1\ n\rangle[n\ 2]}{\langle n-1\ 1\rangle[1\ 2]}\right)\log(-(n-1\ 1))+(i\leftrightarrow n-i)\,.
\end{aligned}
\tag{3.3.29}
$$

We have verified this result using box functions and Mathematica numerics in the case $n=7$. The computations are available as a package, documented in Appendix A.

Observe that the $\log\delta$ terms take a universal and simple form in the MHV sector for all n, namely

$$
\left(\frac{(n\ 1)+(n\ 2)}{(1\ 2)}-\frac{s_{n-1,1,2}(n\ 1)}{(n-1\ 1)(1\ 2)}\right)\log(-(n\ 1))+(i\leftrightarrow n-i)\,. \tag{3.3.30}
$$

where $s_{abc}:=(p_a+p_b+p_c)^2$ denotes a three-particle invariant. These structures were already visible at six points in (3.3.9). Note also that the purely rational terms have a similar universal behaviour.

We must now check that (3.3.30) is consistent with the coefficients of Z_0 predicted in (3.2.54). To see this, we first recall the ansatz (1.2.106), implicitly eliminating $(|n-1],|1])$ as discussed at the start of Section 3.3.

$$
A_n^{\text{1-loop}}\to\left(\frac{1}{\delta^2}S^{(0)\text{1-loop}}+\frac{1}{\delta}Z\right)A_{n-1}^{\text{tree}}\,. \tag{3.3.31}
$$

We focus exclusively on infrared finite $\delta^{-1}\log\delta$ terms. Then the left hand side is given by A_n^{tree} times (3.3.30). The right-hand side comprises the feed-down term[54]

$$
-\frac{(n\ 1)+(n-1\ n)}{(n-1\ 1)}S^{(0)}\log(-(n\ 1))+(i\leftrightarrow n-i)\,, \tag{3.3.32}
$$

and the soft anomaly

$$
Z_0|_{\log\delta}=\left(\frac{(n-1\ n)}{(n-1\ 1)}+\frac{(n\ 2)}{(1\ 2)}-\frac{(n-1\ 2)(n\ 1)}{(1\ 2)(n-1\ 1)}\right)S^{(0)}\log(-(n\ 1))+(i\leftrightarrow n-i)\,. \tag{3.3.33}
$$

[54]This is obtained by expanding $S^{(0)\text{1-loop}}$ to subleading order in δ, with the given momentum conservation prescription. For a full treatment of such subtleties, see Section 1.2.9 and in particular equation (1.2.105).

Summing (3.3.32) and (3.3.33) yields

$$\left(\frac{(n\ 1)+(n\ 2)}{(1\ 2)}-\frac{(n\ 1)}{(1\ 2)}-\frac{(n\ 1)}{(n-1\ 1)}-\frac{(n-1\ 2)(n\ 1)}{(1\ 2)(n-1\ 1)}\right)S^{(0)}\log(-(n\ 1))+(i\leftrightarrow n-i)\,, \tag{3.3.34}$$

which is identical to (3.3.30) up to $S^{(0)}$, as expected.

3.3.2 NMHV Sector

In the NMHV sector tree-level superamplitudes can be conveniently expressed in terms of dual superconformal R-invariants as we reviewed in Section 1.3.4. It is natural to ask whether the R-invariants have simple subleading soft behaviour, a study partially undertaken in [192]. There it was shown that in the (super)soft limit,

$$Z_n \to \alpha Z_1 + \beta Z_{n-1} + \delta Z_n\,, \tag{3.3.35}$$

the R-invariants R_{1jk} vanish at subleading order. Indeed when $k \neq n$, clearly R_{1jk} is independent of δ, so there is no subleading contribution. For $k = n$, the denominator becomes

$$\delta^2\langle 1\ j-1\ j\ n-1\rangle\langle j-1\ j\ n-1\ 1\rangle\langle j\ n-1\ n\ 1\rangle\langle n-1\ n\ 1\ j-1\rangle\langle n-1\ 1\ j-1\ j\rangle+\mathcal{O}(\delta^3)\,, \tag{3.3.36}$$

while the argument of the δ function is

$$\alpha\langle j-1\ j\ n-1\ 1\rangle\chi_1+\beta\langle n-1\ 1\ j-1\ j\rangle\chi_{n-1}+\langle 1\ j-1\ j\ n-1\rangle(\alpha\chi_1+\beta\chi_{n-1})+\mathcal{O}(\delta)\,. \tag{3.3.37}$$

Notice that the leading term in (3.3.37) exactly vanishes, hence the leading contribution of the numerator is $\mathcal{O}(\delta^4)$. Therefore R_{1jn} certainly vanishes at subleading order, as claimed.

Recall that for appropriate α and β the momentum conservation prescription associated with (3.3.35) is exactly the symmetric elimination of $(|n-1], |1])$. With this prescription the subleading term for tree amplitudes vanishes. Hence we conclude that each R_{1jk} individually obeys the amplitude soft theorem.

At one loop we may write a general planar NMHV amplitude in terms of dual conformal ratio functions $\mathcal{R}$ as

$$A_n^{\text{NMHV,1-loop}} = A_n^{\text{MHV,1-loop}}\mathcal{R}^{\text{tree}} + A_n^{\text{MHV,tree}}\mathcal{R}^{\text{1-loop}}\,. \tag{3.3.38}$$

$\mathcal{R}^{\text{tree}}$ is the sum of R-invariants appearing in (1.3.92). $\mathcal{R}^{\text{1-loop}}$ may be expressed in terms of general R-invariants and dual conformal combinations of box integrals called V-functions [70, 99, 102].

We now investigate the subleading soft behaviour of the R-invariants and V-functions appearing at one loop for six- and seven-point amplitudes, leaving general results to future work. More precisely we will focus on terms of order $\delta \log \delta$ in $\mathcal{R}^{\text{1-loop}}$ which, taking into account the $A_n^{\text{MHV,tree}}$ prefactor lead to terms of order $(1/\delta) \log \delta$. For illustration we outline the soft expansion of the various terms in (3.3.38),

$$\begin{aligned} A_n^{\text{MHV,1-loop}} &\sim \frac{1}{\delta^2} + \frac{1}{\delta} \log \delta + \frac{1}{\delta} \,, \qquad & \mathcal{R}^{\text{tree}} &\sim 1 + \delta^2 \,, \\ A_n^{\text{MHV,tree}} &\sim \frac{1}{\delta^2} \,, & \mathcal{R}^{\text{1-loop}} &\sim 1 + \delta \log \delta + \delta \,, \end{aligned} \tag{3.3.39}$$

where we employ a symmetric momentum conservation prescription that eliminates $|n-1]$ and $|1]$. The particular behaviour for $\mathcal{R}^{\text{tree}}$ was first observed in [192].

Six-Point Amplitude

At six points each five-bracket necessarily omits exactly one momentum twistor. This naturally provides a more concise notation by virtue of the cyclic symmetry of five-brackets. For example we write

$$(2) = [1\ 3\ 4\ 5\ 6] \,. \tag{3.3.40}$$

The six-point tree-level ratio function may then be written as

$$\mathcal{R}^{\text{tree}} = (1) + (3) + (5) = (2) + (4) + (6) \,, \tag{3.3.41}$$

which in the soft limit takes the form,

$$\mathcal{R}^{\text{tree}} = (6) + \mathcal{O}(\delta^2) \,, \tag{3.3.42}$$

noting that $(6) = [12345]$ has no δ dependence. The six-point one-loop ratio function is explicitly [193],

$$\mathcal{R}^{\text{1-loop}} = \frac{1}{2} \Big(\big[(1) + (4)\big] V_3 + \big[(2) + (5)\big] V_1 + \big[(3) + (6)\big] V_2 \Big) \,, \tag{3.3.43}$$

where the dual conformal V-functions are naturally expressed in terms of cross-ratios,

$$V_1 = -\log(u_{36}) \log(u_{25}) + X \,, \tag{3.3.44}$$

$$V_2 = -\log(u_{36})\log(u_{14}) + X\,, \tag{3.3.45}$$

$$V_3 = -\log(u_{14})\log(u_{25}) + X\,, \tag{3.3.46}$$

$$X = \frac{1}{2}\sum_{i=1}^{3}\Big(\log(u_{ii+3})\log(u_{i+1i+4}) + \mathrm{Li}_2(1-u_{ii+3})\Big) - 2\zeta_2\,, \tag{3.3.47}$$

and indices in X are implicitly modulo six. We know from tree-level reasoning that

$$(2) = \mathcal{O}(\delta^2), \quad (3) = \mathcal{O}(\delta^2), \quad (4) = \mathcal{O}(\delta^2)\,. \tag{3.3.48}$$

Therefore in the soft limit the ratio function (3.3.43) becomes

$$\begin{aligned}\mathcal{R}^{\text{1-loop}} = \frac{1}{2}\Bigg((6)\sum_{i=1}^{2}\mathrm{Li}_2(1-u_{ii+3}) + [(6)-(1)]\log(u_{36})\log(u_{25})\\ + [(6)-(5)]\log(u_{36})\log(u_{14})\Bigg) + \mathcal{O}(\delta^2)\,.\end{aligned} \tag{3.3.49}$$

From our calculations in the previous section observe that

$$\log(u_{36}) = \frac{\delta\langle 1\ 3\ 5\ 6\rangle}{\alpha\beta\langle 1\ 2\ 3\ 5\rangle\langle 1\ 3\ 4\ 5\rangle}\left(\beta\langle 2\ 3\ 4\ 5\rangle - \alpha\langle 1\ 2\ 3\ 4\rangle\right) + \mathcal{O}(\delta^2)\,. \tag{3.3.50}$$

Hence we need only expand the R-invariants to leading order, viz.

$$(1) \;=\; (6)\frac{\alpha\langle 1\ 2\ 3\ 4\rangle}{\alpha\langle 1\ 2\ 3\ 4\rangle - \beta\langle 2\ 3\ 4\ 5\rangle} + \mathcal{O}(\delta)\,, \tag{3.3.51}$$

$$(5) \;=\; (6)\frac{\beta\langle 2\ 3\ 4\ 5\rangle}{\beta\langle 2\ 3\ 4\ 5\rangle - \alpha\langle 1\ 2\ 3\ 4\rangle} + \mathcal{O}(\delta)\,. \tag{3.3.52}$$

Thus (3.3.49) reduces to

$$\begin{aligned}\mathcal{R}^{\text{1-loop}} = \frac{1}{2}(6)\Bigg(\sum_{i=1}^{2}\mathrm{Li}_2(1-u_{ii+3}) + \frac{\delta\langle 1\ 3\ 5\ 6\rangle\langle 2\ 3\ 4\ 5\rangle}{\alpha\langle 1\ 2\ 3\ 5\rangle\langle 1\ 3\ 4\ 5\rangle}\log(u_{25})\\ + \frac{\delta\langle 1\ 3\ 5\ 6\rangle\langle 1\ 2\ 3\ 4\rangle}{\beta\langle 1\ 2\ 3\ 5\rangle\langle 1\ 3\ 4\ 5\rangle}\log(u_{14})\Bigg) + \mathcal{O}(\delta^2)\,.\end{aligned} \tag{3.3.53}$$

It is instructive to extract the $\delta\log\delta$ terms, for these have the best hope of universal behaviour. Explicitly we find the contribution,

$$\frac{1}{2}\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle}(6)\frac{\langle 2\ 3\ 4\ 5\rangle\langle 4\ 5\ 6\ 1\rangle}{\langle 1\ 2\ 4\ 5\rangle\langle 3\ 4\ 5\ 1\rangle}\log(u_{25}) + (i \leftrightarrow n-i)\,. \tag{3.3.54}$$

Seven-Point Amplitude

At seven points we employ the formulae of [102], namely

$$\mathcal{R}^{\text{1-loop}} = \frac{1}{2}\left(\mathcal{R}^{\text{tree}}V^{\text{tot}} + R_{147}V_{147} + R_{157}V_{157} + \text{cyclic}\right)\,, \tag{3.3.55}$$

where the V-functions are defined by

$$7V^{\text{tot}} = -\text{Li}_2(1-u_{1246}^{-1}) + \frac{1}{2}\left[\text{Li}_2(1-u_{14}) + \text{Li}_2(1-u_{15})\right] - \log(u_{47})\log(u_{26}) + \text{cyclic}\,, \tag{3.3.56}$$

$$V_{147} = \text{Li}_2(1-u_{2476}) + \text{Li}_2(1-u_{2146}) + \log(u_{2476})\log(u_{2146}) - \zeta_2\,, \tag{3.3.57}$$

$$V_{157} = V_{147} + \text{Li}_2(1-u_{2745}) - \text{Li}_2(1-u_{1254}) - \log(u_{7145})\log\left(\frac{u_{1256}}{u_{2467}}\right)\,, \tag{3.3.58}$$

and general cross-ratios are written as

$$u_{ijkl} = \frac{x_{ik}^2 x_{jl}^2}{x_{il}^2 x_{jk}^2} = \frac{\langle i-1\ i\ k-1\ k\rangle\langle j-1\ j\ l-1\ l\rangle}{\langle i-1\ i\ l-1\ l\rangle\langle j-1\ j\ k-1\ k\rangle}\,. \tag{3.3.59}$$

We first examine the soft behaviour of the fourteen R-invariants explicitly entering (3.3.55). Eight of these have no $\mathcal{O}(\delta)$ term, namely

$$R_{147},\ R_{157},\ R_{261},\ R_{372}\ \sim\ \mathcal{O}(\delta^2)\,, \tag{3.3.60}$$

$$R_{362},\ R_{524},\ R_{625},\ R_{635}\ \ \text{independent of}\ \delta\,. \tag{3.3.61}$$

We obtain terms linear in δ from the remaining six, which are

$$R_{251},\ R_{514},\ R_{473},\ R_{413},\ R_{736},\ R_{746}\,. \tag{3.3.62}$$

At six points, we had no need to expand such R-invariants, courtesy of convenient behaviour of the V-functions. We must ask whether this property continues to hold at seven points. Hence we list the δ dependence of the relevant V-functions,

$$\begin{aligned} V_{251} &\sim \mathcal{O}(\delta)\,, & V_{736} &\sim \mathcal{O}(\delta)\,,\\ V_{514} &\sim \text{nonzero} + \mathcal{O}(\delta)\,, & V_{473} &\sim \text{nonzero} + \mathcal{O}(\delta)\,,\\ V_{746} - V_{736} &\sim \text{nonzero} + \mathcal{O}(\delta)\,, & V_{413} - V_{473} &\sim \mathcal{O}(\delta)\,. \end{aligned} \tag{3.3.63}$$

We also note that

$$V_{746} - V_{514}\ \sim\ \mathcal{O}(\delta)\,. \tag{3.3.64}$$

Therefore the only non-trivial R-invariants we must expand to $\mathcal{O}(\delta)$ are the combinations,

$$R_{413} + R_{473} \quad \text{and} \quad R_{514} + R_{746} \,. \tag{3.3.65}$$

Remarkably, through an intricate series of twistor bracket identities, both of these combinations have zero subleading soft dependence. Thus it only remains to expand the relevant V-functions explicitly. Henceforth we shall only look for $\delta \log \delta$ terms, these being the best candidates for universal behaviour.

It is convenient to express the V-functions only in terms of our earlier u_{ij} cross-ratios, defined by

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2} \,. \tag{3.3.66}$$

whose soft expansions were determined earlier. Observe that

$$u_{ii+1jj+2}^{-1} = u_{ij} u_{ij+1} \,, \tag{3.3.67}$$

and we trivially have relations,

$$u_{ijkl} = u_{ijlk}^{-1} = u_{klij} \,. \tag{3.3.68}$$

Thus we may write

$$7V^{\text{tot}} = -\text{Li}_2(1 - u_{14}u_{15}) + \frac{1}{2}\Big[\text{Li}_2(1 - u_{14}) + \text{Li}_2(1 - u_{15})\Big] - \log(u_{47})\log(u_{26}) + \text{cyclic}\,, \tag{3.3.69}$$

$$V_{147} = \text{Li}_2(1 - u_{62}u_{63}) + \text{Li}_2(1 - u_{14}u_{15}) + \log(u_{62}u_{63})\log(u_{14}u_{15}) - \zeta_2\,, \tag{3.3.70}$$

$$V_{157} = V_{147} + \text{Li}_2(1 - u_{47}u_{41}) - \text{Li}_2(1 - u_{14}) + \log(u_{74})\log\left(\frac{u_{62}u_{63}}{u_{15}}\right)\,. \tag{3.3.71}$$

We only obtain $\log \delta$ terms from the invariants u_{15} and u_{26}. By relabelling symmetry it suffices to determine only the $\log(u_{26})$ terms. The leading behaviour of the R-invariants involved is

$$R_{473} + R_{413} \to (5), \quad R_{625} \to (3), \quad R_{635} \to (1)\,, \tag{3.3.72}$$

$$R_{413} \to \frac{\beta\langle 2\ 3\ 4\ 6\rangle}{\beta\langle 2\ 3\ 4\ 6\rangle - \alpha\langle 1\ 2\ 3\ 4\rangle}(5)\,, \tag{3.3.73}$$

$$R_{251} \to \frac{\beta\langle 2\ 4\ 5\ 6\rangle}{\beta\langle 2\ 4\ 5\ 6\rangle - \alpha\langle 1\ 2\ 4\ 5\rangle}(3)\,. \tag{3.3.74}$$

On expanding the relevant V-functions many terms are produced. Quite unexpectedly, when multiplying by the respective R-invariants a highly non-trivial simplification takes place, yielding the expression,

$$R^{\text{tree}}\ \frac{\delta}{\alpha}\bigg(\frac{\langle 3\ 5\ 4\ 6\rangle\langle 7\ 6\ 1\ 4\rangle}{\langle 6\ 1\ 3\ 4\rangle\langle 6\ 1\ 4\ 5\rangle}+\frac{\langle 3\ 4\ 5\ 6\rangle\langle 1\ 2\ 6\ 7\rangle}{\langle 3\ 4\ 6\ 1\rangle\langle 1\ 2\ 5\ 6\rangle}+\frac{\langle 2\ 3\ 4\ 5\rangle\langle 1\ 2\ 6\ 7\rangle}{\langle 1\ 2\ 4\ 5\rangle\langle 2\ 3\ 6\ 1\rangle}$$
$$-\frac{\langle 1\ 2\ 6\ 7\rangle\langle 2\ 3\ 5\ 6\rangle}{\langle 1\ 2\ 5\ 6\rangle\langle 2\ 3\ 6\ 1\rangle}\bigg)+(5)\frac{\langle 2\ 3\ 4\ 6\rangle\langle 5\ 6\ 7\ 1\rangle}{\langle 3\ 4\ 6\ 1\rangle\langle 1\ 2\ 5\ 6\rangle}+(3)\frac{\langle 2\ 4\ 5\ 6\rangle\langle 5\ 6\ 7\ 1\rangle}{\langle 4\ 5\ 6\ 1\rangle\langle 1\ 2\ 5\ 6\rangle}\,. \quad (3.3.75)$$

All that remains is to extract the $\log(u_{26})$ pieces from V^{tot}. These come from

$$-\log(u_{47}u_{41})\log(u_{26})-\text{Li}_2(1-u_{25}u_{26})-\text{Li}_2(1-u_{62}u_{63})+\text{Li}_2(1-u_{26})\,, \quad (3.3.76)$$

yielding subleading soft terms,

$$-\frac{\delta}{\alpha}\bigg(\frac{\langle 3\ 5\ 4\ 6\rangle\langle 7\ 6\ 1\ 4\rangle}{\langle 6\ 1\ 3\ 4\rangle\langle 6\ 1\ 4\ 5\rangle}+\frac{\langle 2\ 3\ 4\ 5\rangle\langle 1\ 2\ 6\ 7\rangle}{\langle 1\ 2\ 4\ 5\rangle\langle 2\ 3\ 6\ 1\rangle}+\frac{\langle 3\ 4\ 5\ 6\rangle\langle 1\ 2\ 6\ 7\rangle}{\langle 1\ 2\ 5\ 6\rangle\langle 3\ 4\ 6\ 1\rangle}-\frac{\langle 1\ 2\ 6\ 7\rangle\langle 2\ 3\ 5\ 6\rangle}{\langle 1\ 2\ 5\ 6\rangle\langle 2\ 3\ 6\ 1\rangle}\bigg)\,. \quad (3.3.77)$$

Miraculously these terms exactly cancel terms in (3.3.75). Hence we arrive at the final expression for subleading $\log\delta$ contributions,

$$\frac{1}{2}\frac{\langle n-1\ 1\rangle}{\langle n-1\ n\rangle}\left[(5)\frac{\langle 2\ 3\ 4\ 6\rangle\langle 5\ 6\ 7\ 1\rangle}{\langle 3\ 4\ 6\ 1\rangle\langle 1\ 2\ 5\ 6\rangle}+(3)\frac{\langle 2\ 4\ 5\ 6\rangle\langle 5\ 6\ 7\ 1\rangle}{\langle 4\ 5\ 6\ 1\rangle\langle 1\ 2\ 5\ 6\rangle}\right]\log(u_{26})+(i\leftrightarrow n-i)\,. \quad (3.3.78)$$

Observe that these terms have the same overall structure as we found at six points in equation (3.3.54). Furthermore, one may immediately perform a consistency check that (3.3.78) reduces to (3.3.54) as we make particles 3 and 4 collinear. These terms seem amenable to an n-point generalisation, which we leave to future work.

We note finally that the coefficients in (3.3.78) were not predicted in Section 3.2.2; indeed they involve particles other than $\{n-2, n-1, n, 1, 2\}$ which were considered in deriving (3.2.54). It would be interesting to investigate whether the constraint equation (3.2.50), perhaps supplemented with further physical reasoning, is sufficently powerful to determine the NMHV one-loop subleading soft anomaly in general.

Chapter 4

Conclusions

In this thesis, we have presented two ways in which on-shell methods may be extended beyond evaluating the S-matrix. We shall now briefly review our main results, and outline various possible directions for future research.

In Chapter 2, we introduced a connected formula for the super form factor of the chiral stress tensor multiplet and showed precisely how it may be related to known results involving rational scattering equations, link representations, and Grassmannian integrals. Moreover, we verified our results both analytically and numerically in several non-trivial low-point cases. We then derived a CHY formula computing the form factor of $\mathrm{Tr}F^2$ in four dimensions. Interestingly, this prescription has no explicit dependence on dimensionality.

There are now several appealing avenues for future work. Most pressingly, it would be fascinating to determine what modifications are required to promote our CHY formula to arbitrary dimensions. To do so, one may need to refine the polarisation sum appropriately. This would provide an indirect means of generating Q-cut expressions for loop-level form factors, following [184]. Of course, simply evaluating our conjecture on the support of the one-loop scattering equations [194, 195] may suffice for this purpose.

Staying at tree level, one could ask whether other form factors admit scattering equation representations. Formulae for the $\mathrm{Tr}(\phi^k)$ case are already available [164]. Constructions for general operators may be possible by analogy with the programme of Koster et al. – see for example [196]. More abstractly, it is important to determine the exact obstruction to relating the Grassmannian integral [87] to our link representation. This may well uncover further valuable links between QFT and algebraic geometry. Finally, it would be interesting to determine whether Grassmannian expressions for gauge invariant off-shell amplitudes [197] can be reinterpreted as a sum over the rational scattering equations.

In Chapter 3, we derived dual conformal constraints on the soft behaviour of superamplitudes through subleading order. We demonstrated that these suffice to fix the universal soft operators at tree level. Our equations do not have a unique solution at one loop, but nonetheless provide non-trivial information about the interplay between IR divergences from different sources. We determined the full subleading soft anomaly in the MHV sector, arriving at a surprisingly simple all-point result. Analysis of two non-trivial NHMV examples showed that new features appear beyond the MHV sector. It seems natural to conjecture that universality is broken between helicity sectors but holds within each sector separately, at least for subleading $\log \delta$ terms.

Our results generate several intriguing questions. Firstly, it would be valuable to test our conjecture in different helicity sectors, providing additional insight into the surprising cancellations we discovered. This may lead to a soft-improved representation of one-loop amplitudes in $\mathcal{N} = 4$ SYM, along the lines of [198]. On a similar theme, suitable generalisations of our results may be applicable to the recent bootstrap programme [199,200]. One could also ask how our relatively compact results fit into a more general framework [201], bridging the gap between theory and phenomenology. From this perspective, it becomes important to also include collinear behaviour, perhaps building on the advances in [202].

There are also several open problems of a more formal nature. Given the emerging importance of asymptotic symmetry in physical models, one may ask how the subleading soft anomaly manifests itself from the perspective of symmetry breaking, and whether this can be tamed by renormalisation, as was recently argued for gravity [203]. It may also be worthwhile to investigate the interaction between soft limits, BCJ duality and the double copy, continuing the study of Oxburgh [204]. Indeed, the good IR behaviour of gravity may well be indirectly responsible for our unexpectedly simple subleading soft anomaly. Finally, one might consider extending our results to form factors, particularly in light of the recent Wilson loop duality [69].

Appendix A

Mathematica Packages

This appendix documents two Mathematica packages we have developed, which may be used to verify some of our computations in Chapters 2 and 3. The code for both packages is provided on the attached CD of illustrative material.

A.1 `SubSoft.m`

`SubSoft.m` is a Mathematica package for the automated calculation and verification of subleading soft theorems. A separate Mathematica file contains sample calculations, pertinent to our results in Section 3.3.1. The package extends Bourjaily's `bcfw.m` [205]. The files `SubSoft.m` and `bcfw.m` are required, and `SubSoft Examples.nb` is an optional walkthrough.

Setup

First ensure that both `SubSoft.m` and `bcfw.m` are saved to the same directory as the notebook you are writing. To initialise the package, simply call

```
In[1]:= SetDirectory[NotebookDirectory[]];

In[2]:= << SubSoft.m
```

SubSoft.m is a package for computing the subleading soft behaviour of amplitudes based on Bourjaily's bcfw.m.

from *Soft Theorems at 1-Loop via Dual Superconformal Symmetry* by Brandhuber, Hughes, Travaglini and Spence

Glossary

In Table 2, we collect descriptions of the most important expressions. The definitions of related expressions may be inferred, or determined by direct inspection of the source code. Strictly speaking, the verification functions compute the difference between subleading terms on the LHS and RHS of (1.2.103) or (1.2.106) at tree level or one loop respectively. Hence the subleading soft theorems are verified if the resulting quantity is within machine precision of zero.

A.2 CHY.m

`CHY.m` is a Mathematica package for the automated calculation and verification of CHY formulae. A separate Mathematica file contains sample calculations, pertinent to our results in Chapter 2. The package extends Bourjaily's `bcfw.m` [205]. The files `CHY.m` and `bcfw.m` are required, and `CHY Examples.nb` is an optional walkthrough.

Setup

First ensure that both `CHY.m` and `bcfw.m` are saved to the same directory as the notebook you are writing. To initialise the package, simply call

```
SetDirectory[NotebookDirectory[]];
```

```
<< CHY.m
```

CHY.m is a package for computing CHY formulae for amplitudes and form factors based on Bourjaily's bcfw.m.

from *CHY Formulae and Soft Theorems in N=4 Super Yang-Mills Theory* by Edward Hughes

Glossary

In Table 3, we collect descriptions of the most important expressions. The definitions of related expressions may be inferred, or determined by direct inspection of the source code. Note in particular that n is always taken to be the total number of external legs, including any used to define a form factor. This makes it easy to adapt the code in both amplitude and form factor settings, at the expense of introducing slight notational tension with Chapter 2.

Expression	**Type**	**Description**
`ab[i,j]`	object	represents the angle bracket $\langle i\ j\rangle$.
`sb[i,j]`	object	represents the square bracket $[i\ j]$.
`MHVTreeAmplitude[{i,j},n]`	function	returns the tree amplitude with helicity configuration $1^+ \cdots i^- \cdots j^- \cdots n^+$.
`F[i,n]`	function	returns the i^{th} box function for n-point kinematics.
`useRandomKinematics[n]`	function	sets up n-point random kinematics for numerical evalulation.
`NEvalute[expr]`	function	numerically evaluates an expression featuring `ab` and/or `sb`.
`deltaDependence[n]`	rule	introduces holomorphic δ dependence for particle n.
`momentumConservation WithDelta[n,a,b]`	rule	performs the substitution (1.2.100) assuming that particle n carries δ dependence.
`VerifySoftTheorem TreeLevel`$[\mathtt{A_n}, \mathtt{A_{n-1}}, \mathtt{n}, \mathtt{a}, \mathtt{b}]$	function	verifies the tree-level subleading soft theorem for given A_n, A_{n-1} with the elimination of $\|a]$ and $\|b]$.
`VerifySoftTheorem1LoopIRFinite Term`$[\mathtt{Alp_n}, \mathtt{Alp_{n-1}}, \mathtt{Atr_{n-1}}, \mathtt{n}, \mathtt{i}, \mathtt{j}]$	function	verifies the one-loop subleading soft term at finite order in ϵ for given $A_n^{\text{1-loop}}$, $A_{n-1}^{\text{1-loop}}$, A_{n-1}^{tree} with the elimination of $\|a]$ and $\|b]$.
`Z0[n]`	function	returns the predicted infrared-finite subleading soft anomaly, by default defined for MHV amplitudes with $n = 5, 6, 7$.

Table 2: Documentation for the `SubSoft.m` package.

Expression	Type	Description
`ab[i,j]`	object	represents the angle bracket $\langle i\ j\rangle$.
`sb[i,j]`	object	represents the square bracket $[i\ j]$.
`useRandomKinematics[n,1]`	function	sets up n-point random rational kinematics for numerical evalulation.
`kinematicSubstitution`	rule	numerically evaluates an expression using the random kinematics.
`PlusPolarizationVector[i,r]` `MinusPolarizationVector[i,r]`	function	spinor helicity form for $\epsilon_i^\pm$ with reference r in four dimensions
`IntegrandMatrix[n]`	function	provides the $2n\times 2n$ integrand matrix (1.3.34).
`SamplePolarizations[n,k]`	function	choice of 4-d polarisation vectors, k of which are $-$.
`ConvenientIntegrand[n]`	function	evaluates the integrand (2.5.1) in the gauge $\{z_{n-1}, z_n, z_1\} \to \{1, 59, 0\}$.
`ConvenientIntegrand[n,k]`	function	same as previous, but with 4-d sample polarisations.
`ScatteringEquation[i,n]`	function	returns the scattering equation for particle i of n.
`NumericalConvenientResult[n,k]`	function	sums integrand over solutions to scattering equations.
`NumericalConvenientDGResult[n,k]`	function	same as previous, but using the Dolan-Goddard scattering equations [149].

Table 3: Documentation for the `CHY.m` package.

Appendix B

Action of the Dual Conformal Boost Generator

We collect various formulae outlining the action of the dual conformal boost generator on spinors and multiparticle invariants used in Chapter 3. To adapt the formulae to $(K_{\alpha\dot{\alpha}})_{\mathcal{O}(\delta^0)}$ replace all $\sum_j$ by $\sum_j' = \sum_{j\neq n}$.

Suppose $a < b$ cyclically in $\{3, 4, \ldots 2\}$. Then we have

$$-K(\langle a\ b\rangle) = \sum_{j=3}^{a-1} |j]\langle j|\langle a\ b\rangle + \sum_{j=a+1}^{b-1} |j]\langle b|\langle a\ j\rangle\,, \tag{B.1}$$

$$-K([a\ b]) = \sum_{j=3}^{b} |j]\langle j|[a\ b] + \sum_{j=a+1}^{b-1} |a]\langle j|[b\ j]\,, \tag{B.2}$$

$$-K((a\ b)) = 2\sum_{j=3}^{a-1} |j]\langle j|(a\ b) + \sum_{j=a}^{b} |j]\langle j|(a\ b) + \sum_{j=a+1}^{b-1} |j]\langle b|\langle a\ j\rangle[b\ a] \tag{B.3}$$

$$- \sum_{j=a+1}^{b-1} |a]\langle j|[b\ j]\langle a\ b\rangle\,. \tag{B.4}$$

In particular if a and b are adjacent then

$$-K(\langle a\ b\rangle) = \sum_{j=3}^{a-1} |j]\langle j|\langle a\ b\rangle\,, \tag{B.5}$$

$$-K([a\ b]) = \sum_{j=3}^{b} |j]\langle j|[a\ b]\,, \tag{B.6}$$

$$-K((a\ b)) = 2\sum_{j=3}^{a-1}|j]\langle j|(a\ b) + \sum_{j=a,b}|j]\langle j|(a\ b)\,. \tag{B.7}$$

The following corollaries are of particular use in Section 3.2.2.

$$\Big[\big(K_{\alpha\dot\alpha}\big)_{\mathcal{O}(\delta^0)} - \sum_{j=3}^{n-1}|j]\langle j|\Big]\Big(\frac{[2\ n]}{[2\ 1]\langle 1\ n\rangle}\Big) \;=\; -\frac{|n]\langle 1|}{\langle n\ 1\rangle}\,, \tag{B.8}$$

$$\Big[\big(K_{\alpha\dot\alpha}\big)_{\mathcal{O}(\delta^0)} - \sum_{j=3}^{n-1}|j]\langle j|\Big]\Big(\frac{[n-2\ n]}{[n-2\ n-1]\langle n-1\ n\rangle}\Big) \;=\; -\frac{|n]\langle n-1|}{\langle n-1\ n\rangle}\,, \tag{B.9}$$

$$\Big[\big(K_{\alpha\dot\alpha}\big)_{\mathcal{O}(\delta^0)} - \sum_{j=3}^{n-1}|j]\langle j|\Big]\Big(\frac{[1\ n]}{[1\ n-1]\langle n-1\ n\rangle}\Big) \;=\; -|n-1]\langle n-1|\frac{[n\ 1]}{\langle n-1\ n\rangle[n-1\ 1]}\,, \tag{B.10}$$

$$\big[\big(K_{\alpha\dot\alpha}\big)_{\mathcal{O}(\delta^0)} - \sum_{j=3}^{n-1}|j]\langle j|\Big]\Big(\frac{[n-1\ n]}{[n-1\ 1]\langle 1\ n\rangle}\Big) \;=\; -|1]\langle 1|\frac{[n-1\ n]}{[n-1\ 1]\langle n\ 1\rangle}\,, \tag{B.11}$$

$$\big(K_{\alpha\dot\alpha}\big)_{\mathcal{O}(\delta^0)}\Big(\frac{(n\ 1)}{(n-1\ 1)}\Big) \;=\; -|n-1]\langle n-1|\frac{(n\ 1)}{(n-1\ 1)}\,, \tag{B.12}$$

$$\big(K_{\alpha\dot\alpha}\big)_{\mathcal{O}(\delta^0)}\Big(\frac{(n-2\ n)}{(n-2\ n-1)} - \frac{(n-2\ 1)(n-1\ n)}{(n-2\ n-1)(n-1\ 1)}\Big) \;=\; -|n]\langle n| + |1]\langle 1|\frac{(n-1\ n)}{(n-1\ 1)}\,. \tag{B.13}$$

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