

# ON AGMON METRICS AND EXPONENTIAL LOCALIZATION FOR QUANTUM GRAPHS

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**ABSTRACT.** We investigate the rate of decrease at infinity of eigenfunctions of quantum graphs by using Agmon's method to prove  $L^2$  and  $L^\infty$  bounds on the product of an eigenfunction with the exponential of a certain metric. A generic result applicable to all graphs is that the exponential rate of decay is controlled by an adaptation of the standard estimates for a line, which are of classical Liouville-Green (WKB) form. Examples reveal that this estimate can be the best possible, but that a more rapid rate of decay is typical when the graph has additional structure. In order to understand this fact, we present two alternative estimates under more restrictive assumptions on the graph structure that pertain to a more rapid decay. One of these depends on how the eigenfunction is distributed along a particular chosen path, while the other applies to an average of the eigenfunction over edges at a given distance from the root point.

## 1. INTRODUCTION

The goal of this article is to study the rate of decrease of eigenfunctions on infinite quantum graphs  $(\Gamma, H)$ , as the distance (arc length along edges) from a designated root point  $0$  goes to infinity. In fact our results apply more generally to any  $L^2 \cap AC^1$  solution of the eigenvalue equation

$$H\psi := -\psi'' + V(x)\psi = E\psi \quad (1)$$

on infinite subgraphs  $\Gamma_0 \subset \Gamma$ , which can be disconnected from  $\Gamma$  by the removal of a compact subset of the graph  $S$ , and on which the requisite conditions are satisfied at the vertices, without regard to what happens on other portions of the full graph  $\Gamma$ . (For us a subgraph  $\Gamma_0$  is taken to consist of a subset  $\mathcal{V}_0$  of the vertices  $\mathcal{V}(\Gamma)$  along with a subset of edges  $\mathcal{E}(\Gamma)$  connected to vertices in  $\mathcal{V}_0$ . A compact subset is closed and contained within the union of a finite set of vertices and edges, with finite total edge length.) For brevity we shall refer to solutions of (1) on infinite subgraphs as **exterior eigenfunctions**.

The philosophy introduced by Agmon [1] for proving exponential localization of (exterior) eigenfunctions of equations like  $(-\Delta + V(\mathbf{x}))\psi(\mathbf{x}) = E\psi(\mathbf{x})$  on  $\mathbb{R}^d$  is to introduce an **Agmon multiplier**  $F(\mathbf{x})$  in terms of which any solution  $\psi \in L^2$  on an exterior domain  $\mathcal{D}$  must additionally satisfy  $F\psi \in L^2(\mathcal{D})$ . We in fact go beyond this and show that  $F\psi$  is finite in Sobolev norm. The canonical case is when  $F = e^{\rho(0, \mathbf{x})}$ , or, typically,  $F = e^{(1-\epsilon)\rho(0, \mathbf{x})}$ , for an Agmon metric  $\rho(0, \mathbf{x})$  that tends to infinity as  $\mathbf{x} \rightarrow \infty$ . With some further effort such integrated bounds establish exponential decrease in the

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pointwise sense. A great success of the Agmon method was to extend to the case of PDEs some estimates which, in sharper one-dimensional versions, date from Liouville and Green and are more widely known as the WKB approximation in semiclassical analysis. We refer to [1, 7] for a full account of the Agmon method and to [11] for a definitive account of the Liouville-Green approximation with rigorous error control.

Quantum graphs offer an interesting middle ground between the one-dimensional and higher-dimensional situations, and our intent here is to explore how the Agmon philosophy can be adapted to prove exponential localization of eigensolutions of quantum graphs. We are unaware of previous treatments of this question, although localization on quantum graphs in different contexts has been considered in, e.g., [5, 6] and references therein.

We recall that a quantum graph consists of a metric graph  $\Gamma$  for which a one-dimensional Schrödinger operator

$$H = -\frac{d^2}{dx^2} + V(x)$$

is applied to functions on the edges  $e$ , and vertex conditions connecting the values on adjacent edges are imposed. The Hamiltonian  $H$  of the quantum graph is defined as a self-adjoint operator on

$$L^2(\Gamma) := \oplus_e L^2(e, dx)$$

by extending a symmetric quadratic form initially given on the Sobolev space  $H^1(\Gamma)$ , which is by definition the subspace of  $\oplus_e H^1(e)$  consisting of functions that are continuous at the vertices, cf. [2], Definition 1.3.6. The norm on  $H^1(\Gamma)$  is defined by  $\|\phi\|_{H^1}^2 = \sum_e \int (|\phi'|^2 + |\phi|^2) dx$ . Following [2, 8, 9], we shall assume a global lower bound  $\ell_{\min}$  to the length of the edges of  $\Gamma$  and choose the potential-energy function  $V(x)$  to be continuous and to satisfy

$$V(x) > E > -\infty \tag{2}$$

for some eigenparameter  $E$ , outside a finite part of the graph. (Throughout this article  $E$  will be treated as a fixed parameter.) The vertex conditions we impose will be of Kirchhoff type, *viz.*

**Definition 1.1.** *For any connected subgraph  $\Gamma_0 \subset \Gamma$ ,  $\mathcal{K}(\Gamma_0)$  will designate the set of functions  $f \in AC^1$  on all edges (considered as open sets) in  $\Gamma_0$ , such that at each vertex  $v$ ,  $f$  is continuous, and*

$$\sum_{e \sim v} f'_e(v^+) = 0. \tag{3}$$

*The notation in the summand indicates the limit along the edge  $e$  of  $f'(x)$  as  $x \rightarrow v$ , calculated in the outgoing orientation from  $v$ . We shall dispense with the explicit reference to  $\Gamma_0$  when the context is clear. We refer to the functions in  $\mathcal{K}(\Gamma_0)$  as satisfying Kirchhoff conditions.*

In our analysis, the edges of  $\Gamma_0$  will sometimes be oriented with increasing distance from a root point  $0$ , according to a metric  $\rho$  on the graph. This will be termed the **distance orientation** according to the metric  $\rho$ . If the graph contains cycles, the assignment of edge orientation will break down for any edge containing a  $x_c$  at the same distance from  $0$  by two or more distinct paths. We refer to any  $x_c$  connected to  $0$  by two or more distinct paths with

the same distance  $\rho(x_c, 0)$  as a *cut point*, and find it convenient to regard any edge containing an interior cut point as a pair of distinct oriented edges joined by a degree-2 vertex located at  $x_c$ . In particular, in Definition 1.4 we use **Euclidean distance** by which we mean the Euclidean length of a path. Furthermore, in our integrals  $dx$  e.g. in (16) and (18) the orientation plays no role and the integrals could be taken with respect to arclength  $ds$ . However,  $dx$  was chosen for ease of notation.

Given an edge orientation defined by a metric, the Kirchhoff condition at a vertex  $v$ , if not also a cut point, reads

$$\sum_{e \sim v, \rho(e) \geq v} f'_e(v^+) = f'_{\text{incoming } e}(v^-).$$

Thus each daughter edge  $e_\ell$  spawned at  $v$  carries forward a certain **fraction of the incoming derivative**

$$p_{v_\ell} := f'_{e_\ell}(v^+) / f'_{\text{incoming } e}(v^-) \quad (4)$$

with  $\sum_\ell p_{v_\ell} = 1$ . In this paper we will sometimes work with a path  $P$  from a root to  $\infty$ . In this case, only one edge spawned at a vertex will lie on the path  $P$  and the subscript  $\ell$  can be dropped from the notation for  $p_{v_\ell}$ . The quantities  $p_v$  will be crucial in Theorem 1.2.

We note that the Kirchhoff conditions are analogous to classical Neumann boundary conditions in that they are the boundary conditions associated by the Friedrichs extension with the quadratic form (1), *viz.*

$$\phi \rightarrow \sum_e \int_e (|\phi'|^2 + V(x)|\phi|^2) dx \quad (5)$$

for  $\phi \in H^1(\Gamma)$  According to [9], Theorem 9, when  $V = 0$  this extension is a nonnegative self-adjoint operator, since the vertex operator included there vanishes. By [12], §X.3, the same is true when we add a potential that is bounded from below. All quantum graphs considered in this article are defined by such Friedrichs extensions. We first prove several basic properties of exterior eigensolutions. We show that solutions exist for general graphs in Proposition 2.1, and we offer a partitioned uniqueness result for graphs with additional structure in Corollary 2.2. Furthermore, in Section 4.5 we offer an example demonstrating that such solutions are not unique in general.

Then, implementing an Agmon-style argument, we will show that  $L^2$  solutions multiplied by a certain growing function  $F$ , will still have a finite Sobolev norm. Our key technical estimate is the following:

**Proposition 1.1.** *Let  $\Gamma_0$  be a subgraph of  $\Gamma$  and assume that  $\liminf V(x, \psi) - E > 0$  on  $\Gamma_0 \setminus S_1$ , where  $S_1$  is compact. Suppose that  $\psi \in L^2(\Gamma_0)$  and that on each edge of  $\Gamma_0 \setminus S_1$ ,  $0 < \psi \in AC^1$  and  $\psi'' \geq (V(x, \psi) - E)\psi$ . Let a function  $F$  defined on each edge be such that  $0 < F \in AC^1$  and for some  $\delta > 0$ ,*

$$V(x, \psi) - E - \left( \frac{F'}{F} \right)^2 \geq \delta. \quad (6)$$

Then we have the following bound on the Sobolev norm of  $F\psi$ :

$$\|F\psi\|_{H^1(\Gamma_0)}^2 \leq C_1 \left( \|\psi\|_{H^1(S_2)}^2 + \sum_{v \in \Gamma_0 \setminus S_2} \sum_{e \in \Gamma_0 \setminus S_2, e \sim v} F^2 \psi_e \psi'_e(v+) \right), \quad (7)$$

where  $S_2$  is a compact set such that  $S_1 \subset S_2$  and  $S_2 \setminus S_1$  contains no vertices, and  $\psi'_e(v+)$  indicates the outward derivative.

We caution that although  $F\psi$  has a finite Sobolev norm, we have as yet said nothing about its continuity at the vertices, without which it may not belong to the space  $H^1(\Gamma)$ . We shall construct three different possibilities for  $F$ . In all of them, we ensure that the second term on the right side of (7) vanishes. Our aim is to construct an  $F$  that grows as rapidly as possible under the constraint (6) in order to obtain the best control on  $\psi$ .

In our first estimate, we show that exterior eigenfunctions exhibit at least as rapid exponential decay estimates as is the case for the line. Since one would expect, correctly, that the familiar one-dimensional Liouville-Green expressions will play a central role in extending Agmon's method to quantum graphs, we introduce notation for the metric that corresponds to the Liouville-Green approximation, extending the definition of the standard Agmon metric to the setting of graphs in the following way:

**Definition 1.2.** Let the *classical action metric* from 0 to  $x$  be given by

$$\rho_a(y, x; E) := \min_{\text{paths } P_{y \text{ to } x}} \int_P (V(t) - E)_+^{1/2} dt. \quad (8)$$

For simplicity we usually set  $y = 0$  for a designated root point and when clear from the context write  $\rho_a(x; E)$  or  $\rho_a(x)$  in place of  $\rho_a(0, x; E)$ .

Our first application of Proposition 1.1 shows that the classical action estimate applies universally to quantum graphs as a bound on the Sobolev norm. (In fact the improvement from an  $L^2$  estimate to a Sobolev estimate in the following also goes through in the classical cases [1, 7], but this is not widely remarked upon.)

**Theorem 1.1.** Suppose that  $\Gamma_0 \subset \Gamma$  is a connected, infinite subgraph on which  $\liminf(V(x) - E) > 0$ . If  $\psi \in L^2(\Gamma) \cap \mathcal{K}(\Gamma_0)$  satisfies

$$-\psi'' + V(x)\psi = E\psi$$

on the edges of  $\Gamma_0$ , then for any  $\delta < \liminf(V - E)$ ,

$$e^{\rho_a(x; E - \delta)} \psi \in H^1(\Gamma_0) \cap L^\infty(\Gamma_0). \quad (9)$$

Generically this decay estimate cannot be improved. In Section 4.5, we discuss the ladder graph which has an exterior eigenfunction that exhibits the decay in (9). However, other examples indicate that more rapid decay is in some circumstances typical. Our motivating example is the tree or, more generally, a graph which is composed of a union of trees outside of a set of compact support. The illustrative case of a regular tree with equal lengths and equal branching numbers is treated in Section 4.1. In particular, we see in Section 4.1 that for a regular tree the exterior eigenfunction is in  $L^1$  and not just  $L^2$ . We generalize this phenomenon in the next theorem, in which we make the crucial assumption that the exterior eigenfunction has a negative

derivative. We show in Corollary 2.1 that a graph which is a union of trees outside of a set of compact support satisfies this assumption. We introduce an Agmon metric adapted to a given path and a given eigenfunction, and this new metric exceeds  $\rho_a$  by an additional contribution from the vertices.

**Definition 1.3.** *Let  $P$  be a path from the root 0 to  $x$  and let  $\psi$  be an exterior eigensolution such that  $\psi' < 0$ . Suppose that at each vertex  $v \in P$ ,  $p_v$  as in (4) is the fraction of the derivative continuing down edge  $e \in P$  that is adjacent to the vertex  $v$ . Let the **Agmon path metric**  $\rho_P$  be given by*

$$\rho_P(x, E) = \int_P \left( (V(t) - E)_+^{1/2} + \frac{1}{2} \sum_{\{v \in P: p_v > 0\}} \delta_v(t) \log(1/p_v) \right) dt, \quad (10)$$

which yields an Agmon multiplier of

$$F_P(x, E) = \left( \prod_{\{v \in P: p_v > 0\}} \sqrt{\frac{1}{p_v}} \right) e^{\int_P (V(t) - E)_+^{1/2} dt}.$$

Using this new version of the Agmon metric we formulate the next theorem, which captures the additional decay particular to a given path.

**Theorem 1.2.** *Suppose that  $\Gamma_0 \subset \Gamma$  is a connected, infinite subgraph on which  $\liminf(V(x) - E) > 0$  and that  $\psi \in L^2(\Gamma) \cap \mathcal{K}(\Gamma_0)$  satisfies*

$$-\psi'' + V(x)\psi = E\psi$$

on the edges of  $\Gamma_0$  and  $\psi' < 0$  outside of a set of compact support. Consider any infinite path  $P \subset \Gamma_0$ , on which the fraction of the derivative exiting from a vertex  $v$  is designated  $p_v$ . Then for any  $\delta < \liminf(V - E)$ ,  $e^{\rho_P(x, E - \delta)}\psi \in L^2(P)$ . That is,

$$\sqrt{\prod_{v \in P} \frac{1}{p_v}} e^{\rho_a(x, E - \delta)}\psi \in L^2(P) \cap L^\infty(P).$$

At first sight the utility of this theorem could be questioned because the information it provides about  $\psi$  seems to depend on knowing  $\psi$ . However, we shall show in Section 4 that it is sometimes possible to determine the fractions  $p_v$  from the structure of the graph. Moreover, our final theorem, which is a consequence of Theorem 1.2, will eliminate the use of the  $p_{v_j}$  by averaging. Specifically, for a category of regular graphs, we shall show that an average of an exterior eigenfunction over the edges of the same generation always decreases more rapidly than the one-dimensional estimate of Theorem 1.1.

Recall that a rooted tree is regular in the sense of Naimark and Solomyak [10, 14] if the vertices and edges occur in generations at equal distances from the root 0, and for each  $j = 0, 1, \dots$ ,

- All vertices of the  $j$ -th generation have the same branching number  $b_j$ .
- The edges emanating onward from a vertex of the  $j$ -th generation have identical lengths.

By convention the root 0 corresponds to  $j = 0$ .

We extend this definition to include certain graphs that may contain cycles, and we allow non-uniformity in the potential energy in the following definition.

**Definition 1.4.** *Consider a rooted graph  $\Gamma$  with the distance orientation with respect to Euclidean distance  $y$ . Vertices are assumed to occur on all paths from 0 whenever  $y = v_j$ , where  $v_j$  is an increasing sequence with  $v_{j+1} - v_j \geq \epsilon$  for some  $\epsilon > 0$ . This implies that the edges of each generation have identical lengths, as in the case of regular trees. In addition, at each generation  $j$ ,*

- *Every vertex at  $v_j$  has the same ongoing branching number  $b_j \geq 2$ .*
- *Every vertex at  $v_j$  has the same arriving branching number  $a_j \geq 1$ .*

*Such a metric graph will be termed a **regular braided graph**; the case of a regular tree corresponds to  $a_j = 1$  for all  $j$ . We define a quantum graph on a regular  $\Gamma$  with the usual Kirchhoff conditions at the vertices and refer to it as having **regular topology**. On a graph of regular topology, given an  $L^2$  eigensolution  $\psi$ , we define an **averaged wave function**, depending on  $y \in \mathbb{R}^+$ , by*

$$\Psi(y) := \sum_{e: \exists x \in e: \text{dist}(0, x) = y} \left( \prod_{j: v_j < x} \left( \frac{a_j}{b_j} \right) \right) \psi_e(x), \quad (11)$$

We observe that  $\sum_{e: \exists x \in e: \text{dist}(0, x) = y} \left( \prod_{j: v_j < y} \left( \frac{a_j}{b_j} \right) \right) = 1$ , thus making our use of the term ‘‘averaged’’ justified. Indeed  $\Psi(y)$  is the average of  $\psi(x)$  over all points  $x$  at a given Euclidean distance  $y$  from the root.

**Theorem 1.3.** *Suppose that  $\Psi$  is the averaged eigenfunction on a quantum graph with regular topology corresponding to a solution  $\psi$  of (1), for which  $\psi \in L^2(\Gamma) \cap \mathcal{K}$ , and that for all  $x$  such that  $\text{dist}(0, x) = y$ ,  $V(x) \geq V_m(y)$ , where  $\liminf(V_m(y) - E) > 0$ . Define*

$$F_{\text{ave}}(y, E) := \left( \prod_{j: v_j < y} \sqrt{\frac{b_j}{a_j}} \right) e^{\int_0^y \sqrt{V_m(t) - E} dt}. \quad (12)$$

*Then for each  $0 < \delta < \liminf(V_m - E)$ ,*

$$F_{\text{ave}}(y, E - \delta)\Psi \in H^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+).$$

The rest of the paper is organized as follows. In Section 2, we prove several basic facts. We first show existence of eigensolutions in general. In the case where the graph is a finite union of trees outside of a set of compact support, we also prove a limited uniqueness and the crucial fact that  $\psi' < 0$ . In Section 3 we offer proofs of our theorems. The final section of this article (Section 4) contains case studies of illustrative examples, including regular trees and trees with a lesser degree of regularity involving two lengths (see 4.2). We also include a study of the ladder graph, which is not a union of trees outside of a set of compact support, and thus provides a good testing ground for the validity our theorems in general settings.

## 2. EXISTENCE, UNIQUENESS, AND BASIC PROPERTIES OF EIGENSOLUTIONS

In this section we collect some useful facts about eigensolutions of quantum graphs. Since we allow  $E$  to be an arbitrary real parameter, it might be asked whether  $L^2$  solutions exist and, if so, whether they can be characterized with a degree of uniqueness. We first tackle existence.

**Proposition 2.1.** *Consider a Hamiltonian on an infinite graph  $\Gamma$  satisfying the assumptions in the introduction, and fix a connected subset of positive measure  $i_0 \subset \Gamma$ . Then for any  $E < \inf_{\Gamma \setminus i_0}(V)$ , there exists a function  $\psi(x) \in L^2(G)$  that satisfies*

$$-\psi'' + V(x)\psi = E\psi \quad (13)$$

and the Kirchhoff conditions on  $\Gamma \setminus i_0$ .

*Proof.* As shown in [9], the assumptions guarantee that the Friedrichs extension of (5) defines a nonnegative self-adjoint operator. We now perturb this operator by adding a potential of the form

$$\alpha w(x),$$

where  $w \geq 0$  is a  $C^\infty$  function supported in a finite subinterval of  $i_0 \cap e$  for some edge  $e$  (and not identically 0). We first note that that  $\alpha w(x)$  is a relatively form compact perturbation of  $H$  and therefore leaves its essential spectrum unchanged, by the following fairly standard argument based on Problem XIII.39 of [13]: It suffices to show that multiplication by  $w$  is a compact mapping from  $H^1(\Gamma)$  to its dual space  $H^{-1}(\Gamma)$ . Now, multiplication by  $w$  is a bounded map  $H^1(\Gamma) \rightarrow H_0^1(I)$  for some compact interval  $I$ , and the latter space is compactly embeddable in  $H^{-1}(I)$ , which is in turn isomorphic to a subspace of  $H^{-1}(\Gamma)$ . It follows that  $\sigma_{\text{ess}}(H + \alpha w) = \sigma_{\text{ess}}(H)$ .

Therefore, if  $E < 0$  is in the spectrum of  $\sigma_{\text{ess}}(H + \alpha w)$ , it is an eigenvalue of finite multiplicity ([13], §XIII.4), implying that there exists an  $L^2$  eigenfunction  $\psi$  solving

$$(H + \alpha w)\psi = E\psi$$

on  $\Gamma$ . In particular (13) holds outside  $i_0$ .

Now, since multiplication by  $w$  is a bounded operator, the spectrum depends continuously on  $\alpha$  ([13], §XII.2). Thus consider a normalized test function  $\varphi$  supported in  $\text{supp } w$  and note that

$$\langle \varphi, (H + \alpha w)\varphi \rangle = \mathcal{E}(\varphi) + \alpha \int_{i_1} w|\varphi|^2 dx$$

tends continuously to  $-\infty$  as  $\alpha \rightarrow -\infty$ . The Rayleigh-Ritz inequality states that

$$\inf \sigma(H + \alpha w) \leq \langle \varphi, (H + \alpha w)\varphi \rangle,$$

so by continuity, for any given  $E < 0$ , there exists a value of  $\alpha$  for which

$$E \in \sigma(H + \alpha w),$$

which finishes the proof of existence.  $\square$

As a variant of standard fact, to be found for example in [2], we note that exterior eigensolutions are in  $H^1(\Gamma)$ :

**Proposition 2.2.** *If  $\psi$  is an  $L^2$  eigensolution of (1) on a subgraph  $\Gamma_0$  where  $V - E \geq 0$ , and  $\Gamma_0$  can be disconnected from  $S := \{x : V - E < 0\}$  by the removal of a finite number of points  $x_k$ ,  $k = 1, \dots, k_{\max}$  then  $\psi' \in L^2(\Gamma_0)$  and  $\sqrt{V - E}\psi \in L^2(\Gamma_0)$ .*

*Proof.* By integrating by parts and invoking the vertex conditions, (1) implies that

$$\int_{\Gamma_0} (|\psi'|^2 + (V - E)\psi^2) dx = \sum_{k=1}^{k_{\max}} (\pm\psi(x_k)\psi'(x_k)),$$

showing that a finite quantity is the sum of the squared norms of  $\psi'$  and  $\sqrt{V - E}\psi$ .  $\square$

In Sturmian theory, the characterization of the solution set is related to unique continuation. In particular, if  $\liminf V(x) > 0$ , then there can only be a finite number of nodes for any solution of  $-\psi'' + V(x)\psi = E\psi$  on a finite or infinite interval. In contrast, there are examples of quantum graphs for which the zero set of an eigenfunction contains intervals or an infinite number of discrete nodes (cf. the ladder example in Section 4.5). However, the following proposition allows a generalization of the classical statement about the finite number of nodes:

**Proposition 2.3.** *Assume that a connected infinite quantum graph has a minimal edge length and that outside a compact subset  $S$ ,  $V(x) - E > 0$ . Assume that  $x_0 \notin S$  is a boundary point of the zero set  $Z(\psi) := \{x : \psi(x) = 0\}$  of an exterior  $L^2$  eigenfunction  $\psi$ . (The point  $x_0$  is either an isolated node of  $\psi$  or else a vertex that abuts an edge on which  $\psi$  vanishes identically.) Then there are at least two oriented paths beginning at  $x_0$ , along each of which  $|\psi(x)|$  strictly increases until the path enters  $S$ .*

*Proof.* As mentioned,  $x_0$  could either be a node in the interior of an edge or a vertex, and if a vertex there are two possibilities, *viz.*, it may touch an edge on which the eigenfunction is identically zero, or else it touches at least two other edges on which the eigenfunction is not identically zero. In any of these cases, by the existence-uniqueness theorem for ODEs there must be at least one edge  $\mathcal{E}_0$  leaving  $x_0$  on which  $\psi'(x_0^+) > 0$ , and, due to the Kirchhoff condition, one edge where  $\psi'(x_0^+) < 0$ . (In the case where  $x_0$  is a node we regard it as a vertex of degree 2 and consider the edge where it is located as two distinct edges.) We discuss only the case  $\psi'(x_0^+) > 0$ , as the argument for the case where  $\psi'(x_0^+) < 0$  is the same with a systematic sign difference. Therefore there is an interval in  $\mathcal{E}_0$  of the form  $(0, \epsilon)$  in the variable which = distance from  $x_0$ , for which  $\psi(x) > 0$  and  $\psi'(x) > 0$ . Since  $V - E > 0$  in  $S^c$ ,  $\psi'' > 0$ , which means that  $\psi(x)$  and  $\psi'(x)$  must increase on all of  $\mathcal{E}_0$ . Call the vertex at which  $\mathcal{E}_0$  terminates  $v_1$ . Because of the Kirchhoff condition, there is at least one edge  $\mathcal{E}_1 \neq \mathcal{E}_0$  emanating from  $v_1$  such that  $\psi'(v_1^+) > 0$ . Repeating the argument,  $\psi(x)$  and  $\psi'(x)$  increase on all of  $\mathcal{E}_1$  and on a continuing chain of edges  $\mathcal{E}_k, k = 1, \dots$ . Recall that the length of each edge is bounded from below. Such a chain can therefore not include an infinite number of vertices, because in that case  $\psi \notin L^2(\Gamma)$ . The remaining possibility is that the chain on which  $\psi(x)$  and  $\psi'(x)$  increase enters  $S$ .  $\square$



**Corollary 2.1.** *If  $\Gamma$  has only a finite number of cycles, and  $\psi$  is an exterior eigenfunction, then the null set  $Z(\psi)$  has at most a finite number of connected components. Furthermore, there exists a compact set  $S$  such that on  $\Gamma \setminus S$ ,  $\psi$  is monotonically decreasing as a function of the distance from  $S$ .*

The proof is a straightforward consequence of the observation that at most a finite number of paths can enter  $S$  and thus only a finite number of paths as described in Proposition 2.3 are possible given that the number of cycles is bounded. We shall henceforth refer to such functions  $\psi$  as **monotonic exterior eigenfunctions**.

It is not excluded that  $\psi$  may vanish identically on certain maximal connected infinite subgraphs, but there can be at most finitely many such subgraphs  $\Gamma_Z$ , and any such  $\Gamma_Z$  connects to  $\Gamma \setminus \Gamma_Z$  at only a finite set of vertices. Thus, for graphs with  $V - E > 0$  and treelike structures outside a compact set, there is no loss of generality in assuming that the exterior eigenfunctions are positive, decreasing, and convex.

A further corollary of Proposition 2.3 is a partitioned uniqueness theorem for  $L^2$  solutions of  $H\psi = E\psi$  on trees.

**Corollary 2.2.** *Assume that a graph  $\Gamma$  contains only a finite number of cycles and that  $V - E \geq 0$  outside a compact set. Then there is a (possibly different) compact set  $S$  such that  $\Gamma \setminus S$  can be partitioned into a finite number of maximal connected subgraphs  $\{\mathcal{T}_k\}$ , intersecting only at vertices, such that any exterior eigenfunction  $\psi$  is a linear combination of functions supported on exactly one of the  $\{\mathcal{T}_k\}$ . Moreover, the solution set supported in each  $\{\mathcal{T}_k\}$  is one-dimensional.*

*Proof.* Suppose that there were two linearly independent  $L^2$  solutions of the eigenvalue equation,  $\psi_{1,2}$ , and that their supports contain an interval in common. Since they are linearly independent, some linear combination

$$\psi_3 = a\psi_1 - b\psi_2$$

must change sign on  $\mathcal{T}$ . However, this contradicts Proposition 2.3, by which no solution that changes signs on  $\mathcal{T}$  can belong to  $L^2$ .

□

### 3. AGMON ESTIMATES FOR QUANTUM GRAPHS

In this section we prove decay estimates for exterior eigenfunctions on quantum graphs. In some regards we follow the line of reasoning laid out in the book by Hislop and Sigal [7], which contains a treatment of the Agmon method in the standard case. However, we not only adapt their argument to graphs, but generalize it in some ways, in particular by providing Sobolev estimates in addition to  $L^2$  estimates. The adaptation of Agmon's method to graphs begins with analogues of two simple integration-by-parts lemmas from [7]. As a matter of convenience, we state our results in the case of real functions defined on a quantum graph. The extension to complex solutions is immediate, since the real and imaginary parts of a complex eigensolution are real eigensolutions.

The first lemma, replacing [7], Lemma 3.6, is an elementary identity:

**Lemma 3.1.** *Suppose that  $\phi$  and  $F > 0$  are real-valued functions on the metric graph  $\Gamma$  such that  $\phi \in AC^1$  and  $F \in AC$ . Then for any  $x$  in an edge of  $\Gamma$ ,*

$$(F\phi)' \left( \frac{\phi}{F} \right)' = (\phi')^2 - \left( \frac{F'}{F} \right)^2 \phi^2. \quad (14)$$

Moreover, on any subgraph  $\Gamma_0 \subset \Gamma$

$$\begin{aligned} \sum_{e \in \Gamma_0} \int_e F\phi \left( -\frac{d^2}{dx^2} + V(x) - E \right) \frac{1}{F} \phi dx = & - \sum_{v \in \Gamma_0} \sum_{e \in \Gamma_0, e \sim v} F\phi \frac{d}{dx} \left[ \frac{1}{F} \phi \right] (v^+) \\ & + \sum_{e \in \Gamma_0} \int_e |\phi'|^2 + \left( V - E - \left| \frac{F'}{F} \right|^2 \right) |\phi|^2 dx. \end{aligned} \quad (15)$$

The notation in (15) is meant to convey that the derivatives are taken in the outward sense at the vertex.

**Remark 3.1.** *In [7] Lemma 3.6,  $F$  is assumed to be bounded. By keeping track of boundary terms in this lemma we are able to eliminate the need for this assumption.*

*Proof.* The first identity (14) is an easy calculation. The other form follows by integration by parts, the result of which is that

$$\begin{aligned} \sum_{e \in \Gamma_0} \int_e F\phi \left( -\frac{d^2}{dx^2} + V(x) - E \right) \frac{1}{F} \phi dx = & - \sum_{e \in \Gamma_0} F\phi \frac{d}{dx} \left( \frac{1}{F} \phi \right) \Big|_{ei}^{ef} \\ & + \sum_{e \in \Gamma_0} \int_e \left( |\phi'|^2 + \left( V - E - \left| \frac{F'}{F} \right|^2 \right) |\phi|^2 \right) dx, \end{aligned} \quad (16)$$

where  $ei$  is the initial vertex of the edge  $e$  and  $ef$  the final vertex. The dependence on the edge orientation in this expression is only apparent, however: At each vertex, all derivatives in the integrated term are summed with an inward orientation. Thus when the integrated terms are collected at each vertex, the result is the expression (15) which does not depend on how the edges are oriented.  $\square$

Our second lemma replaces [7] Lemma 3.7.

**Lemma 3.2.** *Let  $S_1 \subset S_2$  be two compact subsets of a subgraph  $\Gamma_0$ , such that  $S_2 \setminus S_1$  contains no vertices. Let  $\eta \geq 0$  be a smooth function supported in  $\Gamma_0 \setminus S_1$  such that  $\eta(x) = 1$  on  $\Gamma_0 \setminus S_2$ . Furthermore, let  $\psi > 0$  satisfy  $\psi'' \geq (V(x, \psi) - E)\psi$  on each edge. Then for each  $x \in \Gamma_0 \setminus S_1$ ,*

$$F^2 \eta \psi \left( -\frac{d^2}{dx^2} + V(x) - E \right) \eta \psi \leq C_0 \chi_{\text{supp } \eta}(x) (\psi)^2(x) + G'(x), \quad (17)$$

where  $C_0$  is a finite constant and  $G := -\frac{1}{2}((F\psi)^2(\eta^2)')$ .

**Remark 3.2.** *The left side of (17) is the integrand in (14) after setting  $\phi = F\eta\psi$ , as we shall do in the proof of Proposition 1.1. Furthermore, we note that  $G$  as above is 0 on all vertices since  $\eta$  is chosen in such a way that  $\text{supp } \eta'$  contains no vertices.*

*Proof.* Expanding the derivatives and using that  $\psi$  is a subsolution, we get that

$$\begin{aligned}
 F^2\eta\psi\left(-\frac{d^2}{dx^2} + V(x) - E\right)\eta\psi &= -F^2\eta\psi(\eta''\psi + 2\eta'\psi' + \eta\psi'') + (V - E)(F\eta\psi)^2(x) \\
 &\leq -F^2\eta\psi(\eta''\psi + 2\eta'\psi') \\
 &= -(F\psi)^2\eta\eta'' - \frac{1}{2}F^2(\eta^2)'(\psi^2)' \\
 &= -(F\psi)^2\eta\eta'' - \frac{1}{2}((F\psi)^2(\eta^2)')' + \frac{1}{2}(F^2(\eta^2)')'\psi^2 \\
 &= \psi^2\left(\frac{1}{2}(F^2(\eta^2)')' - F^2\eta\eta''\right) - \frac{1}{2}((F\psi)^2(\eta^2)')'.
 \end{aligned}$$

Since the first term is supported within  $\text{supp } \eta'$  and  $\text{supp } \eta'$  contains no vertices, it is dominated by  $C_0\chi_{\text{supp } \eta}(x)\psi^2(x)$  as claimed, establishing (17).  $\square$

With these two lemmas in hand we are ready to prove our theorems, following the philosophy of Agmon.

*Proof of Proposition 1.1.* We let  $\eta$  be a smoothed characteristic function such that  $\eta = 0$  on  $S_1$  and 1 outside  $S_2$ , and set  $\phi = F\eta\psi$ . Using (15) and (6) we get

$$\begin{aligned}
 \sum_{e \in \Gamma_0} \int_e F^2\eta\psi\left(-\frac{d^2}{dx^2} + V(x) - E\right)\eta\psi \, dx &\geq \sum_{e \in \Gamma_0} \int_e ((F\eta\psi)')^2 + \delta(F\eta\psi)^2 \, dx \\
 &= \sum_{e \in \Gamma_0} \int_e \eta^2 \left[ ((F\psi)')^2 + \delta(F\psi)^2 \right] \, dx + \sum_{e \in \Gamma_0} \int_e (\eta')^2 (F\psi)^2 + 2\eta\eta'((F\psi)')^2 \\
 &= \sum_{e \in \Gamma_0} \int_e \eta^2 \left[ ((F\psi)')^2 + \delta(F\psi)^2 \right] \, dx + \sum_{e \in \Gamma_0} \int_e (\eta')^2 (F\psi)^2 - (\eta\eta')'(F\psi)^2 \\
 &= \sum_{e \in \Gamma_0} \int_e \eta^2 \left[ ((F\psi)')^2 + \delta(F\psi)^2 \right] \, dx - \sum_{e \in \Gamma_0} \int_e \eta\eta''(F\psi)^2.
 \end{aligned} \tag{18}$$

To establish an upper bound we integrate (17) over  $\Gamma_0$  to get

$$\sum_{e \in \Gamma_0} \int_e \eta^2 \left[ ((F\psi)')^2 + \delta(F\psi)^2 \right] \, dx \leq \sum_{e \in \Gamma_0} \int_e \eta\eta''(F\psi)^2 \, dx + C_0\chi_{\text{supp } \eta}(x)(\psi)^2(x) + G'(x). \tag{19}$$

Thus,

$$\sum_{e \in \Gamma_0} \int_e \eta^2 \left[ ((F\psi)')^2 + (F\psi)^2 \right] \, dx \leq C_2 \|\psi\|_{L^2(\text{supp}(\eta'))}^2,$$

where in the last line we used that  $G$  is 0 on all vertices to handle the last term in (19), and  $C_2$  is a constant large enough to incorporate  $C_0$ , the finite maximum value of  $\eta\eta''F^2$  on the compact set  $\text{supp}(\eta')$ , and the effect of making the coefficients 1 and  $\delta$  uniform on the left side.

To complete the proof of Proposition 1.1 on both sides we add

$$\sum_{e \in \Gamma_0} \int_e (1 - \eta^2) \left[ ((F\psi)')^2 + (F\psi)^2 \right] \, dx,$$

which is dominated by a constant times  $\|\psi\|_{L^2(S_2)}^2 + \|\psi'\|_{L^2(S_2)}^2$ , because the support of  $1 - \eta^2$  is contained in the compact set  $S_2$ .  $\square$

*Proof of Theorem 1.1, Step 1.* In Step 1 we establish the finite Sobolev norm of  $F(x, E - \delta)\psi(x)$  with  $F(x, E) := e^{\rho_a(x, E)}$ . Since by assumption  $\psi$  is an exterior eigenfunction and  $F(x, E - \delta)$  satisfies assumption (6) of Proposition 1.1, (7) follows. Since  $F$  is continuous and  $\psi$  satisfies Kirchhoff conditions at the vertices, the vertex contributions to (7) vanish, establishing that  $e^{\rho_a(x; E - \delta)}\psi \in H^1(\Gamma_0)$ .  $\square$

Step 2 is to show that the  $H^1$  bound that has been established above implies a pointwise bound on  $F\psi$ . This is immediate from the following lemma, choosing  $\phi = F\psi$ .

**Lemma 3.3.** *Suppose that  $\phi \in AC^1$  on the edges of  $\Gamma_0$ , and that*

$$\|\phi\|_{H^1}^2 \leq \infty.$$

*Then  $\phi \in L^\infty(\Gamma_0)$ .*

*Proof.* Although  $\phi$  is not assumed continuous at the vertices, being in  $H^1$  on the edges implies that  $\phi$  has well-defined finite limits as  $x$  tends to a vertex along any given edge.

We now fix  $x_0 \in \Gamma_0$ , and choose a function  $\chi$  supported in  $\{x : \text{dist}(x, x_0) \leq \frac{\ell_{\min}}{2}\}$ ,  $\chi \in C^1$  on all edges intersecting this set, and continuous and equal to 1 at  $x_0$ . (The circumlocution is only necessary in case  $x_0$  is a vertex. Here if  $x_0$  is a vertex, we interpret  $\phi(x_0)$  as the limiting value along any given edge.) The function  $\chi$  is to be chosen so that its  $C^1$  norm does not depend on  $x_0$ .

Because of the assumption that there is a minimum edge length, we can write  $\chi\phi(x)$  as the integral of its derivative over an interval, which we may assume without loss of generality, by choosing an orientation for  $x$ , to be of the form  $I = (x_0 - \frac{\ell_{\min}}{2}, x_0)$ , finding

$$\begin{aligned} |\phi(x_0)| &= |\chi(x_0)\phi(x_0)| = \left| \int_{x_0 - \frac{\ell_{\min}}{2}}^{x_0} (\chi(y)\phi(y))' dy \right| \\ &= \left| \int_{x_0 - \frac{\ell_{\min}}{2}}^{x_0} (\chi'(y)\phi(y) + \chi(y)\phi'(y)) dy \right| \\ &\leq \frac{1}{2} \int_{x_0 - \frac{\ell_{\min}}{2}}^{x_0} ((\chi')^2(y) + (\phi(y))^2 + (\chi(y))^2 + (\phi'(y))^2) dy, \end{aligned} \tag{20}$$

which is bounded independently of  $x_0$  by the assumptions of the lemma.  $\square$

*Proof of Theorem 1.2.* We again apply Proposition 1.1. The Agmon multiplier  $e^{\rho_P(x, E - \delta)}$  has been chosen so that the boundary terms in (7) vanish when  $\Gamma_0$  is identified with  $P$ . This establishes that  $e^{\rho_P(x, E - \delta)}\psi$  has finite  $H^1$  norm and therefore finite  $L^2$  norm. (It fails, however, to have the continuity necessary to belong to the space  $H^1(P)$ .)

The  $L^\infty$  bound follows as before by an application of Lemma 3.3.

□

We now turn to the proof of Theorem 1.3, showing that in the case of regular braided graphs, when the number of vertices in generation  $j$  increases without bound, the exterior eigenfunctions as defined in Definition 1.4 decrease on average more rapidly than the one-dimensional upper bound of Theorem 1.1. As a consequence of Proposition 2.3 we may assume that each  $\psi_e(x) > 0$ , and consequently that the averaged wave function  $\Psi(y) > 0$ .

**Proposition 3.1.** *Let  $\psi$  be an exterior eigenfunction on a regular braided graph and suppose  $V(x) \geq V_m(y)$  where  $\liminf(V_m(y) - E) > 0$ . Then the averaged wave function  $\Psi(y)$ , as defined in (11), enjoys the following properties:*

- (1)  $\Psi$  is continuous and decreasing in magnitude.
- (2) Except at the positions of the vertices  $y = v_j$ ,  $\Psi$  satisfies

$$\Psi'' \geq (V_m(y) - E) \Psi \geq 0.$$

- (3) The derivative  $\Psi'$  is discontinuous at  $y = v_j$ , decreasing in magnitude by a factor  $p_j = \frac{a_j}{b_j}$ .

*Proof.* 1.  $\Psi$  must decrease in magnitude as a consequence of Proposition 2.3. To see that  $\Psi$  is continuous, first, for any  $y \in \Gamma$ , define  $W_t := \prod_{\substack{\text{vertices} \\ \text{leading to } t}} \frac{a_v}{b_v}$ , and observe that  $\sum_{t: \text{dist}(0,t)=y} W_t = 1$ . Since  $\Psi(y) = \sum_{t: \text{dist}(0,t)=y} W_t \psi(t)$ , each time a  $y$  passes a value  $v_j$ , a contribution of  $\left(\prod_{k < j} \frac{a_k}{b_k}\right) \sum_{\ell=1}^{a_j} \psi_{j-1,\ell}(v^-)$  to  $\Psi(y)$  is replaced by

$$\left(\prod_{k < j} \frac{a_k}{b_k}\right) \cdot \frac{a_j}{b_j} \sum_{n=1}^{b_j} \psi_{j,n}(v_k^+) = \left(\prod_{k < j} \frac{a_k}{b_k}\right) \sum_{\ell=1}^{a_j} \psi_{j-1,\ell}(v^-)$$

by the continuity of  $\psi$ .

2. This is clear by linearity.

3. When  $y$  passes a value  $v_j$ ,  $a_j$  summands of the form  $\psi'_{e'}(v_j^-)$  are replaced by  $\sum_{e > v_j} \frac{a_j}{b_j} \psi'_{e'}(v_j^+) = \sum_{e' < v_j} \psi'_{e'}(v_j^-)$ , according to the Kirchhoff condition. □

*Proof of Theorem 1.3.* We note that the averaged wave function can be considered as an exterior eigenfunction on a path as in Theorem 1.2, where  $p_j = \left(\frac{a_j}{b_j}\right)$ . Although it is possible that some of the  $p_j > 1$ , this does not affect the proof of the theorem, and convergence is not at issue, because of the following argument:

The number of vertices at the  $j$ -th generation is  $\prod_{\ell \leq j} \left(\frac{b_\ell}{a_\ell}\right)$ , which shows that the factor  $\prod_{\ell \leq j} \left(\frac{a_\ell}{b_\ell}\right) \leq 1$  and implies that if the number of vertices at the  $j$ -th generation is bounded below by a function of  $j$  that tends monotonically to  $+\infty$ , the Agmon multiplier for  $\Psi$  is exponentially smaller than  $e^{-\rho_a(x)}$ .

Thus Theorem 1.3 is a special case of Theorem 1.2. Since  $\Psi$  is continuous at the vertices as observed in Proposition 3.1, we know that  $F_{\text{ave}}(y, E-\delta)\Psi \in H^1$  and not merely in  $L^2$  with finite  $H^1$  norm.  $\square$

#### 4. CASE STUDIES

In this section we develop several illuminating examples. We begin by reviewing the case of the most regular tree.

**4.1. The regular tree with equal lengths  $L$  and branching numbers  $b$ .** We consider a tree rooted at  $v_0$  which starts with one edge and splits into  $b$  edges at each vertex henceforth. We are able to construct an explicit exterior eigenfunction on such a tree.

We will work with transfer matrices. Suppose edge  $e_j$  and  $e_{j+1}$  are adjacent at a vertex  $v$ . If  $\psi_1, \psi_2$  is a basis of the solution space and on an edge  $e_j$  the solution is  $A_j\psi_1 + B_j\psi_2$  and the solution on edge  $e_{j+1}$  is given by  $A_{j+1}\psi_1 + B_{j+1}\psi_2$ . Then a **transfer matrix  $T$**  is a matrix such that

$$(A_{j+1} \ B_{j+1})^t = T(A_j \ B_j)^t.$$

We will usually take  $(\psi_1(x), \psi_2(x)) = (\cosh(kx), \sinh(kx))$  on an edge with  $k = \sqrt{E}$ .

We take the transfer matrix at each vertex to be

$$T = \begin{pmatrix} \cosh kL & \sinh kL \\ \frac{1}{b} \sinh kL & \frac{1}{b} \cosh kL \end{pmatrix} \quad (21)$$

Since all the transfer matrices are equal by construction, it suffices for the purpose of characterizing the  $L^2$  eigenfunction to find the eigenvalues of  $T$ . Since  $\det T = 1/b$  and  $\text{tr}T = (1 + \frac{1}{b}) \cosh kL > 2/\sqrt{b}$  both eigenvalues are real. Then solving for the eigenvector will give us the initial conditions that yield the decay corresponding to  $\lambda_1^n$  where  $\lambda_1$  is the smaller of the two eigenvalues and  $n$  is the number of vertices away from the root. We want to compare how the decay on the tree compares to the decay on the line.

The eigenvalue  $\lambda_1$  is given by

$$\begin{aligned} \lambda_1 &= \left(\frac{1}{2} + \frac{1}{2b}\right) \cosh kL - \sqrt{\left(\left(\frac{1}{2} + \frac{1}{2b}\right) \cosh kL\right)^2 - 1/b} \\ &= \frac{1}{\left(\frac{b}{2} + \frac{1}{2}\right) \cosh kL + \sqrt{\left(\left(\frac{b}{2} + \frac{1}{2}\right) \cosh kL\right)^2 - b}} \\ &< \frac{1}{b \cosh kl} \end{aligned} \quad (22)$$

where the last inequality follows from the fact that

$$\left(\left(\frac{b}{2} + \frac{1}{2}\right) \cosh kL\right)^2 - b > \left(\left(\frac{b}{2} - \frac{1}{2}\right) \cosh kL\right)^2.$$

This implies that the solution we have constructed is in  $L^2$  for the tree since

$$\int_{\Gamma} |\phi|^2 = C \sum_n b^n \lambda_1^{2n}.$$

If  $\lambda_1 < \alpha/\sqrt{b}$  for  $\alpha < 1$  then the above sum converges.

To compare this to the case of the line, we immediately see that the factor of  $\frac{1}{b^n}$  makes the pointwise decay faster than the case of the line, where the decay is just  $e^{-kx}$ . On the other hand, if a solution is to be in  $L^2$  for a tree the  $1/\sqrt{b}$  factor is required for convergence. However, we have a factor of  $1/b$  instead, which means that even if we consider partial integrals, the decay on the tree will be faster than on the line.

**4.2. The 2-lengths tree.** In this subsection we find an exterior eigenfunction for a certain tree, which is more sophisticated than the regular tree but still can be solved explicitly and exhibits more rapid decrease of than the general result with the classical action. The key to this and other examples is that if one approaches exponential decay through transfer matrices, and parameters can be adjusted so that all the matrices in a product share a common eigenvector, then the the growth properties of the full solution built upon that eigenvector will be determined by the product of the associated eigenvalues of the transfer matrices.

**Definition 4.1.** *Let the 2-lengths tree be a rooted tree which at each vertex splits into two edges with lengths  $L_1$  and  $L_2$ .*

When  $L_1 = L_2$ , we recover the regular tree with branching number 2. Similar to the regular tree, the transfer matrix at a vertex  $v$  assuming that the edge terminating at  $v$  has length  $L_j$  will be

$$T_j = \begin{pmatrix} \cosh kL_j & \sinh kL_j \\ p_j \sinh kL_j & p_j \cosh kL_j \end{pmatrix} \quad (23)$$

We will seek weights for the derivative fraction  $p_1, p_2$  at the vertex  $v$ , so that the eigenvector corresponding to the smaller eigenvalue of  $T_1$  is the same as the eigenvector corresponding to the smaller eigenvalue of  $T_2$ .

We introduce the following notation. Let  $c_j = \cosh kL_j$ ,  $s_j = \sinh kL_j$ ,  $\lambda$  be the smaller eigenvalue of  $T_1$ ,  $\mu$  be the smaller eigenvalue of  $T_2$ , and  $(1, w)$  be the eigenvector common to  $T_1$  and  $T_2$  associated to  $\lambda$  and  $\mu$  respectively. Then from the eigenvalue equations we get that

$$\begin{aligned} \lambda &= \frac{1}{2} \left( c_1 + p_1 c_1 - \sqrt{(c_1 + p_1 c_1)^2 - 4p_1} \right) \\ \mu &= \frac{1}{2} \left( c_2 + (1 - p_1)c_2 - \sqrt{(c_2 + (1 - p_1)c_2)^2 - 4(1 - p_1)} \right). \end{aligned}$$

From the two eigenvector equations we get that

$$w = \frac{\lambda - c_1}{s_1} = \frac{\mu - c_2}{s_2}.$$

Putting these together we obtain the following equation for  $p_1$ :

$$\frac{p_1 c_1 - c_1 - \sqrt{(c_1 + p_1 c_1)^2 - 4p_1}}{2s_1} = \frac{(1 - p_1)c_2 - c_2 - \sqrt{(c_2 + (1 - p_1)c_2)^2 - 4(1 - p_1)}}{2s_2}. \quad (24)$$

We will consider the two sides of the above equation separately and look for an intersection point in  $(0, 1)$ . We notice that when  $p_1 = 0$  the left side is  $-c_1/s_1 < -1$  and the right side is

$$\frac{-\sqrt{c_2^2 - 1}}{s_2} = -1,$$

while symmetrically the right is  $-1$  at  $p_1 = 1$  and  $-c_2/s_2 < -1$  for  $p_1 = 0$  which implies by the intermediate-value theorem that there is an intersection point, yielding a solution for some value of  $p_1 \in (0, 1)$ .

**4.3. A regular tree in the sense of Naimark and Solomyak.** Consider next a tree with equal lengths and branching numbers at each generation as in Definition 1.4. Let  $b_n$  be the branching number at generation  $n$ . It is clear by the uniqueness of the exterior eigenfunction that at each vertex for  $j$  between 0 and  $b_n$ ,  $p_j = 1/b_n$ . Suppose that  $\psi$  is an exterior eigenfunction and at some generation for some  $j_0$ ,  $p_{j_0} \neq 1/b_n$ . Then there exists  $j_1$  such that  $p_{j_0} \neq p_{j_1}$ . However the tree is self-similar under permutation of the branches, so a composition of  $\psi$  with the isometry that maps  $j_0$ th branch to the  $j_1$ th branch will yield a second distinct eigenfunction, which would be a contradiction. From this we have a complete characterization of the eigenfunction, and thus by Theorem 1.2 and Corollary 2.2 we obtain that for any directed path  $P$  which includes  $n$  vertices

$$\psi(x) \leq C e^{(1-\delta)\rho_a(x)} \prod_{k=1}^n \frac{1}{\sqrt{b_k}}.$$

**4.4. The millipede.** Consider a graph consisting of the half axis  $[0, \infty)$  with additional half axes attached at each even integer position. On the main half-axis, called the “body,” we posit  $V(x) = 0$ , whereas on each “leg” of the millipede emanating from position  $x = k$  we posit a potential  $V(x) = -1 + \delta^2$  for  $\delta > 0$ . The position 0 on each leg corresponds to the vertex. We set  $E = -1$  so that the eigenfunction satisfies  $\psi'' = \psi$  on the body between integer vertices. On the legs it satisfies  $\psi'' = \delta^2 \psi$ . The  $L^2$  solutions are thus proportional to  $e^{-\delta x}$  on the legs, and the solutions on the body are determined by a transfer matrix which, after an elementary calculation, has the form

$$\begin{pmatrix} \cosh 2 & \sinh 2 \\ \sinh 2 + \delta \cosh 2 & \cosh 2 + \delta \sinh 2 \end{pmatrix}.$$

The smaller eigenvalue of the transfer matrix is

$$\begin{aligned} \cosh 2 + \frac{\delta \sinh 2}{2} - \sqrt{\left(\cosh 2 + \frac{\delta \sinh 2}{2}\right)^2 - 1} &= e^{-2} \left(1 - \frac{\delta}{2}\right) + 0(\delta^2) \\ &= e^{-2 - \frac{\delta}{2} + 0(\delta^2)}. \end{aligned} \quad (25)$$

This implies that the  $L^2$  solution along the body is of the form  $e^{(-1 - \frac{\delta}{4} + 0(\delta^2))x}$  times a periodic function.

**4.5. The ladder.** In closing we present an analysis of a “ladder” graph (see Figure 1), which fits within the analysis of the generic Theorem 1.1, but not some of the other results, being infinitely multiply connected. In particular, partitioned uniqueness (Corollary 2.2) no longer applies. Here we construct two possible eigensolutions.



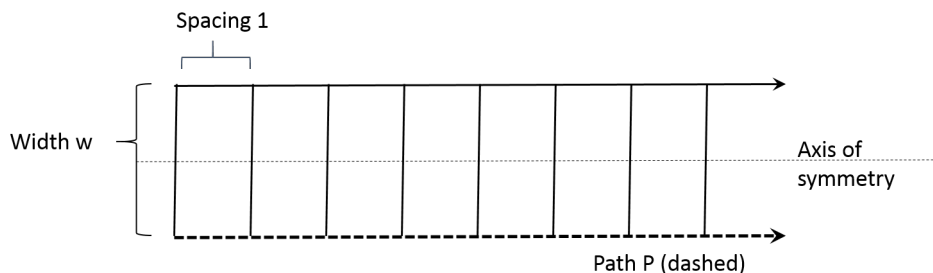


FIGURE 1. The Ladder Graph

4.5.1. *Symmetric:* Let us take two copies of the half-line with a decaying solution. Then let us connect them at the integers with edges (rungs of the ladder) on which the solution is a constant. The result is indeed an eigensolution with the Kirchoff boundary condition at the vertices and it decays exactly at the same rate as solutions on the half-line.

4.5.2. *Antisymmetric:* One can construct a different solution on the ladder that decays faster than the above. The ladder is symmetric under reflection across a line connecting the midpoints of the rungs (the axis of symmetry is as marked in Figure 1). We assume that the rungs have length  $w$  and are located at the integers. We want to construct an exterior eigenfunction which is odd under this reflection. We assume work with  $V = 0$  and  $E = -1$ . This yields that if we parametrize each rung by  $t$  with  $t = 0$  in the middle, then the solution on the rung should be  $\sinh t$  so that it is 0 at 0. Choosing a path along the bottom of the ladder, we compute the transfer matrix to be

$$T = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 + \cosh 1 \coth \frac{w}{2} & \cosh 1 + \sinh 1 \coth \frac{w}{2} \end{pmatrix}.$$

We find that  $\det T = 1$  and  $\text{tr} T = 2 \cosh 1 + \gamma$  where  $\gamma = \coth \frac{w}{2} \sinh 1 \in (\sinh 1, \infty)$ . The characteristic equation is

$$\lambda^2 - \text{tr} T \lambda + 1 = 0.$$

If we let  $\text{tr} T = t$  be a parameter and differentiate the characteristic equation in  $t$  we get

$$2\lambda\lambda' - t\lambda' - \lambda = 0$$

yielding that

$$\lambda' = \frac{\lambda}{2(\lambda - \text{tr} T/2)}$$

which is negative for  $\lambda_-$  since  $\lambda_- < \text{tr} T/2$ . This shows that  $\lambda_-$  is monotonically decreasing in  $\gamma$ . One can see that  $\lim_{\gamma \rightarrow \infty} \lambda_- = 0$  and  $\gamma = 0$  if and only if  $\lambda_- = e^{-1}$  (the 1D value). Therefore,  $\lambda_- < e^{-1}$  and thus the solution satisfies a bound of the form  $ge^{-|\ln \lambda_-|x}$  where  $g$  is periodic and  $|\ln \lambda_-| > 1$ .

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