NONCOMMUTATIVE SPHERICALLY SYMMETRIC SPACETIMES AT SEMICLASSICAL ORDER

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Abstract. Working within the recent formalism of Poisson-Riemannian geometry, we completely solve the case of generic spherically symmetric metric and spherically symmetric Poisson-bracket to find a unique answer for the quantum differential calculus, quantum metric and quantum Levi-Civita connection at semiclassical order $O(\lambda)$. Here λ is the deformation parameter, plausibly the Planck scale. We find that $r, t, \mathrm{d}r, \mathrm{d}t$ are all forced to be central, i.e. undeformed at order λ, while for each value of r, t we are forced to have a fuzzy sphere of radius r with a unique differential calculus which is necessarily nonassociative at order λ^2. We give the spherically symmetric quantisation of the FLRW cosmology in detail and also recover a previous analysis for the Schwarzschild black hole, now showing that the quantum Ricci tensor for the latter vanishes at order λ. The quantum Laplace-Beltrami operator for spherically symmetric models turns out to be undeformed at order λ while more generally in Poisson-Riemannian geometry we show that it deforms to

$$\Box f + \frac{\lambda}{2}\omega^{\alpha\beta}(\mathrm{Ric}^{\gamma}{}_{\alpha} - S^{\gamma}{}_{;\alpha})(\widehat{\nabla}_{\beta}\mathrm{d}f)_{\gamma} + O(\lambda^2)$$

in terms of the classical Levi-Civita connection $\widehat{\nabla}$, the contorsion tensor S, the Poisson-bivector ω and the Ricci curvature of the Poisson-connection that controls the quantum differential structure. The Majid-Ruegg spacetime $[x, t] = \lambda x$ with its standard calculus and unique quantum metric provides an example with nontrivial correction to the Laplacian at order λ.

1. Introduction

In recent years it has come to be fairly widely accepted that quantum gravity effects could render spacetime better modelled as a 'quantum' geometry than a classical one, with coordinates x^μ now generating a noncommutative coordinate algebra. We refer to [24, 14, 26, 12, 21] for some early works, as well as [34] from the 1940s although this did not propose a closed coordinate algebra as such. A further ingredient to such a *quantum spacetime hypothesis* was to include differential forms $\mathrm{d}x^\mu$ such as in [31, 1, 33], while in recent years one also has quantum metrics and quantum bimodule connections within a systematic framework of 'noncommutative Riemannian geometry' [16, 29, 9, 28]. The latter links to spectral triples or 'Dirac operators' in the general approach to noncommutative geometry of Connes[13] as well as to quantum group frame bundles in 2+1 quantum gravity[27]. It may also

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relate to other ideas for 'quantum geometry' from spin foams and loop quantum cosmology, see for example[18, 5, 4].

In the present paper we continue recent work [10] which explores the content of such noncommutative Riemannian geometry at the Poisson level of first order in a deformation or 'quantisation' parameter λ. This is obviously useful to understand issues at order λ before attempting the full theory, but it also turns out to be surprisingly rich with compatibility conditions between the Poisson bivector $\omega^{\mu\nu}$ that controls the quantum spacetime relations $[x^\mu, x^\nu] = \lambda\omega^{\mu\nu} + O(\lambda^2)$ and the classical Riemannian metric $g_{\mu\nu}$ that we also want to quantise. This emergence of a well-defined order λ 'Poisson-Riemannian geometry' in [10] implies a specific paradigm of physics governing first order corrections and coming out of the quantum spacetime hypothesis in much the same way as classical mechanics emerges from quantum mechanics at first order in $\hbar$. In our case λ is plausibly the Planck scale so, although this is a Poisson-level theory, it includes quantum gravity effects and could be called 'semi-quantum gravity'[10] or 'classical-quantum gravity'.

The key further ingredient in this theory is a type of connection ∇ which controls commutators such as

$$[x^\mu, \mathrm{d}x^\nu] = -\lambda\omega^{\mu\rho}\Gamma^\nu_{\rho\sigma}\mathrm{d}x^\sigma + O(\lambda^2)$$

where Γ are the Christoffel symbols of ∇. It is only the combination $\omega^{\mu\rho}\Gamma^\nu_{\rho\sigma}$ which we actually need here and which can be seen as the structure constants of a Lie-Rinehart or 'contravariant' connection known to be relevant to quantising vector bundles [20, 22, 11, 17, 6]. One can also think of these as covariant derivatives partially defined just along hamiltonian vector fields. In our case we follow [10] and suppose a full ordinary covariant derivative ∇ of which only the hamiltonian vector field directions are relevant to the commutation relations. This is physically reasonable given that covariant derivatives already arise extensively in General Relativity but does mean that our covariant derivatives have extra directions that do not play an immediate role for the quantisation (but which could couple to physical fields later on). The field equations for this connection ∇ are [10]:

(1) Poisson compatibility $\nabla_\gamma\omega^{\alpha\beta} + T^\alpha{}_{\delta\gamma}\omega^{\delta\beta} + T^\beta{}_{\delta\gamma}\omega^{\alpha\delta} = 0$ where T is the torsion of ∇;
(2) Metric compatibility $\nabla_\gamma g_{\mu\nu} = 0$;
(3) A condition on the curvature and torsion of ∇ (see (2.14) in Section 2).

It was shown in [10] that (1) allows for the entire classical exterior algebra to quantise uniquely at lowest order, now with a quantum wedge product $\wedge_1$; (2) allows for the metric similarly to quantise to a quantum metric g_1 and (3) for the classical Levi-Civita connection $\widehat{\nabla}$ to quantise to a quantum Levi-Civita ∇_1. In fact the formulae for ∇_1 in [10] give a unique 'closest to quantum Levi-Civita' connection at order λ even when (3) does not hold but in that case $\nabla_1 g_1$ has an order λ correction. Our first main goal of the present work is to describe these results more explicitly using tensor calculus methods as in classical General Relativity (starting with Lemma 2.2) and also to extend them to cover the quantum Laplacian and quantum Ricci tensor in Theorem 2.3 and Section 2.2. This takes considerable work and occupies our 'formalism' Section 2.

We believe that these Poisson-Riemannian field equations deserve further study as an extension of classical General Relativity. In this respect our second main goal is a full analysis of their content in the spherically symmetric case. This includes the example of the Schwarzschild black hole already covered in [10] but now taken further and also, which is new, the FLRW or big-bang cosmological model. In our class of quantisations we assume that both the metric and the Poisson tensor are spherically symmetric and find generically that t must be central. The radius variable r and the differentials $\mathrm{d}t, \mathrm{d}r$ are then also central as an outcome of our analysis. This means that the only quantisation that can take place is on the spheres at each fixed r, t and we find that these are necessarily the 'nonassociative fuzzy sphere' quantisation of S^2 and calculus at order λ obtained in [7] as a cochain twist and later in [10] within Poisson-Riemannian geometry. This result is both positive and negative. It is positive because our analysis says that this simple form of quantisation is unique under our assumptions at order λ, it is negative because it is hard to extract physical predictions in this model and we show in particular that more obvious sources such as corrections to the quantum scaler curvature and quantum Laplace-Beltrami operator vanish at order λ, in line with cochain twist as a kind of 'gauge transformation'. We do still have changes to the form of the quantum metric (and quantum Ricci tensor) and more subtle effects such as nonassociativity of the differential calculus at order λ^2.

To explain this latter point in more detail, one can see[6, 10] that the Jacobi identity in the form $0 = [x^\mu, [\mathrm{d}x^\nu, x^\rho]]$+cyclic at order λ^2 amounts to vanishing of the curvature of ∇ after contraction with ω. Thus, usual associative noncommutative geometry[13] where the quantum differential forms define a differential graded algebra corresponds essentially to ∇ a flat connection (this being precisely true in the symplectic case). In general, the existence of a flat connection respecting a Lie symmetry can have a topological obstruction (it is governed by the relevant Atiyah class) and this goes some way towards understanding why some noncommutative algebras[9, 28] admit few covariant noncommutative geometries. At the semiclassical level we can see this as fixing ω and finding only very restricted solutions for ∇, g in the presence of rotational symmetry. Our new result in Theorem 4.1 is a similar rigidity where we fix g and find no flat ∇ and ω with rotational symmetry. We are not limited to flat ∇ in Poisson-Riemannian geometry as the nonassociativity shows up at order λ^2 not order λ and indeed from a General Relativity point of view if assuming a flat connection is too restrictive then it is reasonable to accept that we need a curved one. It is also worth remembering that noncommutative geometry was only meant to be an effective description and λ is so small that λ^2 is not relevant in practice away from singular situations that blow up its effective value. Therefore we have no real evidence that the world is in fact 'flat' in this respect. It is therefore one of the notable outcomes of our analysis that spherical symmetry generically requires such nonassociativity of differentials at order λ^2.

It is worth noting that a primary reason for wanting associative algebras is a practical one that these are much easier and more familiar to work with. In modern thinking, however, there is a class of *quasiassociative algebras*, shown in the 1990s to include the octonions, where the breakdown of associativity is nevertheless strictly controlled by a certain 3-cocycle 'associator'. In formal terms the algebra is associative in a monoidal category, where a coherence theorem of Mac Lane[23] says that one can work *as if* the algebra is strictly associative; one can put in brackets as

needed for compositions to make sense (this involves inserting the 3-cocycle) and different ways to do this give the same final result. Such categories are familiar in topological quantum field theory and in quantum group theory, where they are induced by a 'Drinfeld cochain twist'[15] of an underlying symmetry. The quantum group $\mathbb{C}_q[SU_2]$, for example, has no bicovariant differential calculus quantising the classical one but does have a nonassociative one where the exterior algebra is a super co-quasi-quantum group[6]. Our nonassociative fuzzy spheres have similarly been conjectured in [7] to extend to all orders in λ as quasiassociative cochain twists for a certain action of the Lorentz group. This is recalled in Section 2.5. In general there is a considerable amount of current interest in nonassociative twists in various contexts[8, 3, 30] including in relation to contravariant connections[2].

Finally on this topic, although not exactly the same as far as we know, there is a similarity here with quantum anomalies in physics where symmetries do not survive quantisation due to curvature obstructions. In that context it is sometime possible to cancel anomalies by introducing extra dimensions and in quantum group examples one can often do something similar (thus $\mathbb{C}_q[SU_2]$ does have a bicovariant associative differential calculus[35], but it is 4-dimensional). It is not known if we can do the same for the nonassociative fuzzy sphere to make it associative or by implication for spherically symmetric mildly nonassociative spacetimes in our analysis, but if so it may link up with the associative noncommutative Schwarzschild black hole with a 5-dimensional differential calculus in [25]. This is outside our current scope since it leaves our Poisson-Riemannian deformation theory setting, but could provide an alternative extra dimensions 'consequence' of our analysis.

Other possible effects include the form of the quantum metric g_1 and its inverse $(\ ,\ )_1$. Here a natural way to write its coefficients is as $g_1 = \mathrm{d}x^\mu \bullet \tilde{g}_{\mu\nu} \otimes_1 \mathrm{d}x^\nu$ where $\bullet$ is the quantum product, which is arranged so that $\tilde{g}_{\mu\nu}$ is inverse to the equally natural matrix $\tilde{g}^{\mu\nu} = (\mathrm{d}x^\mu, \mathrm{d}x^\nu)_1$. Then we find in the general analysis that

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{\lambda}{2} h_{\mu\nu}$$

at order λ where $h_{\mu\nu}$ is a certain antisymmetric tensor (or 2-form) built from the classical data in (2.12). The physical interpretation of this is not clear but *if* we suppose that the $\tilde{g}_{\mu\nu}$ are the observed 'effective metric' then we see that this acquires an anti-symmetric or spin 1 component, making contact with other scenarios where non-symmetric metrics have been studied. On the other hand, $h_{\mu\nu}$ is not tensorial i.e. transforms in a more complicated way if we change coordinates, albeit in such a way that when proper account is taken of the quantum tensor product $\otimes_1$, our constructions themselves are coordinate invariant. We look at this closely on one of the models in Section 4.3. The same applies for the quantum Ricci tensor. Theorem 2.3 also shows that the quantum Laplacian $\Box_1 = (\ ,\ )_1 \nabla_1 \mathrm{d}$ gets generically an order λ correction given by the Ricci curvature of ∇ and the covariant derivative of the contorsion tensor of ∇. In both cases Poisson-Riemannian geometry leads in principle to calculable effects due to our standing assumption that quantum field are identified with quantum ones just with modified operations. The precise *physical* significance of these effects, however, is much more involved question due to the necessity of working on a curved background but frequency dependence of the speed of light and of gravitational redshift could both be expected features based on limited ad-hoc experience from other models[1, 25]. The difference is that

Poisson-Riemannian geometry now offers the possibility of a systematic geometric treatment of such phenomena as an important direction for further work.

A third main goal of the paper is the detailed computation of some examples so as to explore some of the theory and issues above. Here the 2-dimensional Majid-Ruegg spacetime $[x,t]=\lambda x$ is explored at the Poisson level in Section 2.3 and has the merit that its full noncommutative geometry is known already by algebraic means[9] (our results are reconciled with that work in the Appendix). The classical metric

$$g=\mathrm{d}x^2+bv^2;\quad v=x\mathrm{d}t-t\mathrm{d}x$$

describes either a strong gravitational source with a Ricci singularity at $x=0$ or an expanding universe, depending on the sign of the parameter b. We find now that there *are* order λ corrections in the quantum wave equation with plane waves at first order now provided by Kummer M and U functions. One of the surprising outcomes of the paper is that such cases are relatively rare and for example in Section 2.4 we find no order λ correction to the quantum Laplacian for the Bertotti-Robinson metric quantisation of [28] on the same coordinate algebra (this has the same ω but a different ∇,g). The other model that we look at in particular detail, in Section 3, is the rotationally invariant quantisation of the classical spatially-flat FLRW metric

$$g=-\mathrm{d}t^2+a^2(t)\sum_i(\mathrm{d}x^i)^2.$$

We find that everything works in the sense that, as for the black hole, there do exist ω,∇ solving our field equations. Here ∇ pulls back to a unique contravariant connection, so there is a unique noncommutative geometry at order λ. We further find that $h_{\mu\nu}=0$ when computed in this section so that the quantum metric, and also the quantum Levi-Civita connection, look remarkably undeformed at order λ. This model is a warm up to the general analysis but because it is done in Cartesian and not the polar coordinates used in Section 4, it provides a good illustration of the subtle issues concerning changes of coordinates as reconciled in Section 4.3. The paper ends with some further discussion in Section 6.

2. Formalism

Throughout this paper by 'quantum' we mean extended to noncommutative geometry to order λ. There is a physical assumption that quantities will extend further to all orders according to axioms yet to be determined but we do not consider the details of that yet (the idea is to proceed order by order strictly as necessary). This is for convenience and one could more precisely say 'semiquantum' as in [10]. We use $\bullet$ for the deformed product and ; for the covariant derivative with respect to the Poisson compatible 'quantising' connection ∇. This usually has torsion and should not be confused with the Levi-Civita connection.

2.1. Poisson-Riemannian geometry and the quantum Laplacian.

We start with a short recap of results we need from [10] but in a more explicit tensorial form, along with some new general results in the same spirit. We let M be a smooth manifold with exterior algebra Ω, equipped with a metric tensor g and torsion free metric compatible Levi-Civita connection $\widehat{\nabla}$ on Ω^1 and Christoffel symbols $\widehat{\Gamma}$. We

let $\nabla : \Omega^1 \to \Omega^1 \otimes \Omega^1$ be another connection on Ω^1 with similarly defined 'Christoffel symbols' Γ, so that

$$\widehat{\nabla}_\beta \mathrm{d}x^\alpha = -\widehat{\Gamma}^\alpha{}_{\beta\gamma} \mathrm{d}x^\gamma, \quad \nabla_\beta \mathrm{d}x^\alpha = -\Gamma^\alpha{}_{\beta\gamma} \mathrm{d}x^\gamma$$

respectively for the two connections. The tensor product here is over $C^\infty(M)$ and we view the covariant derivative abstractly as a map or in practice with its first output against $\partial/\partial x^\beta$ to define ∇_β. Its action on the component tensor of a 1-form $\eta = \eta_\alpha \mathrm{d}x^\alpha$ is $\nabla_\beta \eta_\alpha = (\nabla_\beta \eta)_\alpha = \partial_\beta \eta_\alpha - \eta_\gamma \Gamma^\gamma{}_{\beta\alpha}$, which fixes its extension to other tensors. The torsion and curvature tensors of ∇ are

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}, \quad R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\delta\beta,\gamma} - \Gamma^\alpha{}_{\gamma\beta,\delta} + \Gamma^\kappa{}_{\delta\beta}\Gamma^\alpha{}_{\gamma\kappa} - \Gamma^\kappa{}_{\gamma\beta}\Gamma^\alpha{}_{\delta\kappa}.$$

in the conventions of [10]. In the presence of a metric we have a contorsion tensor S defined by

$$\Gamma^\alpha{}_{\beta\gamma} = \widehat{\Gamma}^\alpha{}_{\beta\gamma} + S^\alpha{}_{\beta\gamma}$$

where metric compatibility $\nabla g = 0$ is equivalent to the first of

$$S^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(T_{\delta\beta\gamma} + T_{\beta\delta\gamma} + T_{\gamma\delta\beta}), \quad T^\alpha{}_{\beta\gamma} = S^\alpha{}_{\beta\gamma} - S^\alpha{}_{\gamma\beta}$$

and also implies that the lowered $R_{\alpha\beta\gamma\delta}$ is antisymmetric in the first pair of indices (as well as the second). Also note that the lowered index contorsion here obeys $S_{\alpha\beta\gamma} = -S_{\gamma\beta\alpha}$. In this setup, torsion becomes the relevant field which determines metric compatible ∇ via S. This set up is slightly more than we strictly needed for the noncommutative geometry itself as explained in the introduction.

We let $\omega^{\alpha\beta}$ define the Poisson bracket $\{f, h\} = \omega^{\alpha\beta}(\partial_\alpha f)\partial_\beta h$. This is a bivector field and it is shown in [10] that ∇ in our full sense is Poisson compatible if and only if

$$\nabla_\gamma \omega^{\alpha\beta} + T^\alpha{}_{\delta\gamma}\omega^{\delta\beta} + T^\beta{}_{\delta\gamma}\omega^{\alpha\delta} = 0 \tag{2.1}$$

or equivalently in the Riemannian case that

$$\widehat{\nabla}_\gamma \omega^{\alpha\beta} + S^\alpha{}_{\delta\gamma}\omega^{\delta\beta} + S^\beta{}_{\delta\gamma}\omega^{\alpha\delta} = 0. \tag{2.2}$$

We also want ω to be a Poisson tensor even though this is not strictly needed at order λ,

$$\sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \omega^{\alpha\mu}\omega^{\beta\gamma}{}_{,\mu} = 0 \tag{2.3}$$

which given Poisson-compatibility is equivalent [10] to

$$\sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \omega^{\alpha\mu}\omega^{\beta\nu}T^\gamma{}_{\mu\nu} = 0. \tag{2.4}$$

Given the Poisson tensor and (2.1) respectively we quantize the product of functions with each other and 1-forms by functions,

$$f \bullet h = fh + \frac{\lambda}{2}\{f, h\}, \quad f \bullet \eta = f\eta + \frac{\lambda}{2}\omega^{ab} f_{,a} \nabla_b \eta, \quad \eta \bullet f = \eta f - \frac{\lambda}{2}\omega^{ab} f_{,a} \nabla_b \eta$$

to order λ, so that

$$[x^\alpha, \eta] = \lambda \omega^{\alpha\beta} \nabla_\beta \eta \tag{2.5}$$

to order λ in the quantum algebra. It is shown in [10] that we also can quantize the wedge product of 1-forms and higher,

$$\mathrm{d}x^\alpha \wedge_1 \mathrm{d}x^\beta = \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta + \frac{\lambda}{2}\omega^{\gamma\delta}\nabla_\gamma \mathrm{d}x^\alpha \wedge \nabla_\delta \mathrm{d}x^\beta + \lambda H^{\alpha\beta} \tag{2.6}$$

to order λ. This gives anticommutation relations

$$\{\mathrm{d}x^\alpha, \mathrm{d}x^\beta\}_1 = \lambda\omega^{\gamma\delta}\Gamma^\alpha{}_{\gamma\mu}\Gamma^\beta{}_{\delta\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + 2\lambda H^{\alpha\beta}. \tag{2.7}$$

Here the extra 'non-functorial' term needed is given by a family of 2-forms

$$H^{\alpha\beta} = \frac{1}{4}\omega^{\alpha\gamma}(\nabla_\gamma T^\beta{}_{\nu\mu} - 2R^\beta{}_{\nu\mu\gamma})\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

The exterior derivative d is taken as undeformed in the underlying vector spaces. Note that because the products by functions is modified, the quantum tensor product $\otimes_1$, i.e. over the quantum algebra is not the usual tensor product. It is characterised by

$$\eta \otimes_1 f \bullet \zeta = \eta \bullet f \otimes_1 \zeta$$

for all functions f and any η, ζ. If we denote by A the vector space $C^\infty(M)$ with this modified product, which can always be taken to be associative, and if Ω^1 with $\bullet$ is separately a left and right action of A (even if they do not associate) then $\otimes_1$ is just the usual tensor product $\otimes_A$ over A. Note that $\Omega^1 \otimes_1 \Omega^1 \otimes_1 \Omega^1$ in the case of nonassociative differentials will still have ambiguities at order λ^2. The quantum and classical tensor products are in fact identified by a natural transformation q to order λ, as explained in [10]. The fact that everything works and is consistent at order λ is a nontrivial part of that work, where we work associatively modulo λ^2.

Next, although not strictly necessary, it is useful to optionally require that the ∇_Q is geometrically well behaved in noncommutative geometry in that its associated quantum torsion commutes with the quantum product by functions (a 'bimodule map'). This is a quantum tensoriality property in the same spirit as requiring centrality of the quantum metric. We say that such a ∇ is *regular*, amounting to

$$\omega^{\alpha\beta}\nabla_\beta T^\gamma{}_{\mu\nu} = 0 \tag{2.8}$$

With or without this simplifying assumption, there is a quantum metric $g_1 \in \Omega^1 \otimes_1 \Omega^1$ to order λ given by[10]

$$g_1 = g_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2}\omega^{\alpha\beta}\Gamma_{\mu\alpha\kappa}\Gamma^\kappa{}_{\beta\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2}\mathcal{R}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu \tag{2.9}$$

and obeying $\wedge_1(g_1) = 0$ as well as a 'reality' property $\mathrm{flip}(* \otimes *)g_1 = g_1$. Note the quantum Ω^1 as a vector space is identified with the classical Ω^1 and the above formula specifies an element of $\Omega^1 \otimes_1 \Omega^1$ by giving the classical 1-forms for each factor in each term. This should not be confused with $\tilde{g}_{\mu\nu}$ which we will introduce later as coefficients with respect to the $\bullet$ product. In our case $x^{\mu *} = x^\mu$ since our classical manifold has real coordinates and also acts trivially on all classical (real) tensor components, while $\lambda^* = -\lambda$. The action of $*$ on a $\bullet$-product reverse orders while on a $\wedge_1$-product it reverse order with sign according to the degrees. For the most part this $*$-operation takes care of itself given that our classical tensors are real, so we will not emphasise it. The first two terms in (2.9) are the functorial part g_Q and the last term is a correction. Here

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}g_{\alpha\beta}\omega^{\alpha\gamma}(\nabla_\gamma T^\beta{}_{\mu\nu} - R^\beta{}_{\mu\nu\gamma} + R^\beta{}_{\nu\mu\gamma}) \tag{2.10}$$

is antisymmetric and can be viewed as the *generalised Ricci 2-form*

$$\mathcal{R} = \frac{1}{2}\mathcal{R}_{\mu\nu}\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = g_{\mu\nu}H^{\mu\nu}$$

(note the sign and factor in our conventions for 2-form components). Next we let $(\ ,\ )_1 : \Omega^1 \otimes_1 \Omega^1 \to A$ be the inverse metric as a bimodule map. We define A-valued coefficients $g_{1\mu\nu}, \tilde{g}_{\mu\nu}$ by

$$g_1 = g_{1\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu = \mathrm{d}x^\mu \bullet \tilde{g}_{\mu\nu} \otimes_1 \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes_1 \tilde{g}_{\mu\nu} \bullet \mathrm{d}x^\nu$$

so that

$$\tilde{g}_{\mu\nu} = g_{1\mu\nu} + \frac{\lambda}{2}\omega^{\alpha\beta}\Gamma^\gamma{}_{\alpha\mu}g_{\gamma\nu,\beta} = g_{\mu\nu} + \frac{\lambda}{2}h_{\mu\nu} \tag{2.11}$$

to order λ, where we also write $h_{\mu\nu}$ for the leading order correction in $\tilde{g}_{\mu\nu}$. Here $g_{1\mu\nu}$ is read off from (2.9) as the quantum metric coefficients when we choose to use the undeformed product and $\tilde{g}_{\mu\nu}$ are the coefficients when we choose to reorder and use the deformed product as stated (we can also place the $\tilde{g}_{\mu\nu}$ with the second factor since $\otimes_1$ behaves well with respect to the $\bullet$ product as we explained above). The two sets of coefficients are related by (2.11) but in different calculations one or the other may be easier to work depending on the context (the same remark will apply to all our other quantum tensors). From (2.9) and (2.11) we find

$$h_{\mu\nu} = \mathcal{R}_{\mu\nu} + \omega^{\alpha\beta}(\Gamma_{\mu\alpha\kappa}\Gamma^\kappa{}_{\beta\nu} + \Gamma^\gamma{}_{\alpha\mu}g_{\gamma\nu,\beta}) = -h_{\nu\mu} \tag{2.12}$$

where we use metric compatibility of ∇ in the form $g_{\gamma\nu,\beta} = \Gamma_{\gamma\beta\nu} + \Gamma_{\nu\beta\gamma}$ to replace the second term to more easily verify antisymmetry. We let $\tilde{g}^{\mu\nu}$ be the A-valued matrix inverse so that $\tilde{g}_{\mu\nu} \bullet \tilde{g}^{\nu\gamma} = \delta^\gamma_\mu = \tilde{g}^{\gamma\nu} \bullet \tilde{g}_{\nu\mu}$ and define

$$(\mathrm{d}x^\mu, \mathrm{d}x^\nu)_1 = \tilde{g}^{\mu\nu} = g^{\mu\nu} - \frac{\lambda}{2}\tilde{h}^{\mu\nu} \tag{2.13}$$

which we extend by $(f \bullet \mathrm{d}x^\mu, \mathrm{d}x^\nu \bullet \tilde{f})_1 = f \bullet (\mathrm{d}x^\mu, \mathrm{d}x^\nu)_1 \bullet \tilde{f}$ for any functions $f, \tilde{f}$. This gives us a bimodule map $(\ ,\ )_1 : \Omega^1 \otimes_1 \Omega^1 \to A$ inverse to g_1 in the usual sense of noncommutative geometry [9], namely

$$((\ ,\ )_1 \otimes \mathrm{id})(\eta \otimes_1 g_1) = \eta = (\mathrm{id} \otimes (\ ,\ )_1)(g_1 \otimes_1 \eta)$$

for all $\eta \in \Omega^1$, except that we only claim these facts to order λ. From the above,

$$\begin{aligned}\tilde{h}^{\mu\nu} &= g^{\mu\alpha}g^{\nu\beta}h_{\alpha\beta} + g^{\mu\alpha}\{g_{\alpha\beta}, g^{\beta\nu}\} = \mathcal{R}^{\mu\nu} + \omega^{\alpha\beta}(\Gamma^\mu{}_{\alpha\kappa}\Gamma^\kappa{}_{\beta\gamma}g^{\nu\gamma} + \Gamma^\mu{}_{\alpha\gamma}g^{\nu\gamma}{}_{,\beta}) \\ &= \mathcal{R}^{\mu\nu} - \omega^{\alpha\beta}g^{\eta\zeta}\Gamma^\mu{}_{\alpha\eta}\Gamma^\nu{}_{\beta\zeta} = -\tilde{h}^{\nu\mu}\end{aligned}$$

and $\mathcal{R}$ has indices raised by g. As an application, in bimodule noncommutative geometry there is a quantum dimension[9] which we can now compute.

Proposition 2.1. *In the setting above, the 'quantum dimension' to order λ is*

$$\dim_1 := (\ ,\ )_1(g_1) = \dim(M) + \frac{\lambda}{2}\{g_{\mu\nu}, g^{\mu\nu}\}.$$

Proof. Given the above results, we have

$$\begin{aligned}\dim_1 &= (\mathrm{d}x^\mu \bullet \tilde{g}_{\mu\nu}, \mathrm{d}x^\nu)_1 = \tilde{g}_{\mu\nu} \bullet \tilde{g}^{\mu\nu} + ([\mathrm{d}x^\mu, g_{\mu\nu}], \mathrm{d}x^\nu) \\ &= \dim(M) + \frac{\lambda}{2}(h_{\mu\nu} - h_{\nu\mu})g^{\mu\nu} + \lambda\omega^{\alpha\beta}g_{\mu\nu,\alpha}\Gamma^\mu{}_{\beta\gamma}g^{\gamma\nu} = \dim(M) - \lambda\omega^{\alpha\beta}g^{\mu\nu}{}_{,\alpha}\Gamma_{\nu\beta\mu}\end{aligned}$$

where the middle term vanishes as $g^{\mu\nu}$ is symmetric and we transferred to the derivative to the inverse metric. We can now use metric compatibility in the form $\Gamma_{\mu\beta\nu}+\Gamma_{\nu\beta\mu}=g_{\mu\nu,\beta}$ to obtain the answer. □

Finally, the theory in [10] says that there is a quantum torsion free quantum metric compatible (or *quantum Levi-Civita*) connection $\nabla_1:\Omega^1\to\Omega^1\otimes_1\Omega^1$ to order λ if and only if

$$\widehat{\nabla}\mathcal{R}+\omega^{\alpha\beta}\,g_{\rho\sigma}\,S^{\sigma}{}_{\beta\nu}(R^{\rho}{}_{\mu\gamma\alpha}+\nabla_\alpha S^{\rho}{}_{\gamma\mu})\,\mathrm{d}x^\gamma\otimes\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=0 \tag{2.14}$$

In fact the theory always gives a unique 'best possible' ∇_1 at this order for which the symmetric part of $\nabla_1 g_1$ vanishes. This leaves open that $\nabla_1 g_1=O(\lambda)$, namely proportional to the left hand side of (2.14). The construction of ∇_1 takes the form

$$\nabla_1=\nabla_Q+q^{-1}Q(S)+\lambda K$$

where the first two terms are functorial and the last term is a further correction. Translating the formulae in [10] into indices and combining, one has

Lemma 2.2. *cf*[10] *Writing* $\nabla_1\mathrm{d}x^\iota=-\Gamma_1{}^\iota{}_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu$, *the construction of* [10] *can be written explicitly as*

$$\Gamma_1{}^\iota{}_{\mu\nu}=\widehat{\Gamma}^\iota{}_{\mu\nu}+\frac{\lambda}{2}\omega^{\alpha\beta}\left(\widehat{\Gamma}^\iota{}_{\mu\kappa,\alpha}\Gamma^\kappa{}_{\beta\nu}-\widehat{\Gamma}^\iota{}_{\kappa\tau}\Gamma^\kappa{}_{\alpha\mu}\Gamma^\tau{}_{\beta\nu}+\widehat{\Gamma}^\iota{}_{\alpha\kappa}(R^\kappa{}_{\nu\mu\beta}+\nabla_\beta S^\kappa{}_{\mu\nu})\right).$$

Proof. It is already stated in [10] that

$$\nabla_Q(\mathrm{d}x^\iota)=-\left(\Gamma^\iota{}_{\mu\nu}+\frac{\lambda}{2}\omega^{\alpha\beta}(\Gamma^\iota{}_{\mu\kappa,\alpha}\Gamma^\kappa{}_{\beta\nu}-\Gamma^\iota{}_{\kappa\tau}\Gamma^\kappa{}_{\alpha\mu}\Gamma^\tau{}_{\beta\nu}-\Gamma^\iota{}_{\beta\kappa}R^\kappa{}_{\nu\mu\alpha})\right)\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu$$

Next, we carefully we write the term $\omega^{ij}\nabla_i\circ\nabla_j(S)$ in $Q(S)$ in [10, Lemma 3.2] as curvature plus an extra term involving ∇S and Γ, to give

$$q^{-1}Q(S)(\mathrm{d}x^\iota)=\Big(S^\iota{}_{\mu\nu}+\frac{\lambda}{2}\omega^{\alpha\beta}(S^\iota{}_{\mu\kappa,\alpha}\Gamma^\kappa{}_{\beta\nu}-S^\iota{}_{\kappa\tau}\Gamma^\kappa{}_{\alpha\mu}\Gamma^\tau{}_{\beta\nu})+\frac{\lambda}{4}R_\omega(S)^\iota{}_{\mu\nu}$$
$$-\frac{\lambda}{2}\omega^{\alpha\beta}\Gamma^\iota{}_{\alpha\kappa}\nabla_\beta S^\kappa{}_{\mu\nu}\Big)\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu$$

where

$$R_\omega(S)^\iota{}_{\mu\nu}=\omega^{\alpha\beta}\left(R^\iota{}_{\kappa\alpha\beta}S^\kappa{}_{\mu\nu}-R^\kappa{}_{\mu\alpha\beta}S^\iota{}_{\kappa\nu}-R^\kappa{}_{\nu\alpha\beta}S^\iota{}_{\mu\kappa}\right)$$

is the curvature of ∇ evaluated on the Poisson bivector and acting on the contorsion tensor S. Finally, we take K given explicitly in [10, Corollary 5.9],

$$K(\mathrm{d}x^\iota)=\left(\frac{1}{2}\omega^{\alpha\beta}(S^\iota{}_{\alpha\kappa}\nabla_\beta S^\kappa{}_{\mu\nu}-S^\iota{}_{\beta\kappa}R^\kappa{}_{\nu\mu\alpha})-\frac{1}{4}R_\omega(S)^\iota{}_{\mu\nu}\right)\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu$$

and combine all the terms to give the compact formula stated. □

As a bimodule connection there is also a generalised braiding $\sigma_1:\Omega^1\otimes_1\Omega^1\to\Omega^1\otimes_1\Omega^1$ that expresses the right-handed Liebniz rule for a bimodule left connection,

$$\sigma_1(\mathrm{d}x^\alpha\otimes_1\mathrm{d}x^\beta)=\sigma_Q(\mathrm{d}x^\alpha\otimes_1\mathrm{d}x^\beta)+\lambda\omega^{\beta\mu}(\nabla_\mu S)(\mathrm{d}x^\alpha) \tag{2.15}$$

which comes out as

$$\sigma_1(\mathrm{d}x^\alpha\otimes_1\mathrm{d}x^\beta)=\mathrm{d}x^\beta\otimes_1\mathrm{d}x^\alpha+\lambda\left(\omega^{\mu\nu}\Gamma^\alpha{}_{\mu\gamma}\Gamma^\beta{}_{\nu\delta}-\omega^{\mu\beta}(R^\alpha{}_{\gamma\delta\mu}+S^\alpha{}_{\delta\gamma;\mu})\right)\mathrm{d}x^\delta\otimes_1\mathrm{d}x^\gamma \tag{2.16}$$

The bimodule noncommutative geometry also has a natural definition of quantum Laplacian [9] and we can now compute this

Theorem 2.3. *In Poisson-Riemannian geometry the quantum Laplacian to order λ is*

$$\Box_1 f := (\ ,\ )_1 \nabla_1 \mathrm{d}f = \Box f + \tfrac{\lambda}{2}\omega^{\alpha\beta}(\mathrm{Ric}^\gamma{}_\alpha - S^\gamma{}_{;\alpha})(\widehat{\nabla}_\beta \mathrm{d}f)_\gamma$$

Proof. Here $\mathrm{Ric}^\gamma{}_\alpha = g^{\gamma\nu}R^\beta{}_{\nu\beta\alpha} = -R^\gamma{}_{\nu\beta\alpha}g^{\nu\beta}$ and $(\widehat{\nabla}_\alpha \mathrm{d}f)_\gamma = f_{,\alpha\gamma} - \widehat{\Gamma}^\iota{}_{\alpha\gamma}f_{,\iota}$ as usual. Let us also note that d is not deformed but can look different, namely write $\mathrm{d}f = (\tilde{\partial}_\alpha f)\bullet \mathrm{d}x^\alpha$ so that

$$\tilde{\partial}_\mu = \partial_\mu + \frac{\lambda}{2}\omega^{\alpha\beta}\Gamma^\nu{}_{\beta\mu}\partial_\alpha\partial_\nu$$

and we similarly write $\nabla_1 \mathrm{d}x^\iota = -\tilde{\Gamma}_1{}^\iota{}_{\mu\nu}\bullet \mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu$ so that

$$\tilde{\Gamma}_1{}^\iota{}_{\mu\nu} = \Gamma_1{}^\iota{}_{\mu\nu} + \frac{\lambda}{2}\omega^{\alpha\beta}\widehat{\Gamma}^\iota{}_{\nu\kappa,\alpha}\Gamma^\kappa{}_{\beta\mu} = \widehat{\Gamma}^\iota{}_{\mu\nu} + \frac{\lambda}{2}\gamma^\iota{}_{\mu\nu},$$

say, using symmetry of the last two indices of $\widehat{\Gamma}$. Then by the bimodule and derivation properties at the quantum level, we deduce

$$\Box_1 f = (\ ,\ )_1(\mathrm{d}\tilde{\partial}_\nu f \otimes_1 \mathrm{d}x^\nu + (\tilde{\partial}_\alpha f)\bullet \nabla_1 \mathrm{d}x^\alpha) = \big(\tilde{\partial}_\mu\tilde{\partial}_\nu f - (\tilde{\partial}_\alpha f)\bullet \tilde{\Gamma}_1{}^\alpha{}_{\mu\nu}\big)\bullet \tilde{g}^{\mu\nu}$$

We then expand this out to obtain as the classical $\Box f$ and five corrections time $\lambda/2$ as follows:

(i) From the deformed product with $\tilde{g}^{\mu\nu}$ we obtain

$$\{\partial_\mu\partial_\nu f - (\partial_\alpha f)\widehat{\Gamma}^\alpha{}_{\mu\nu}, g^{\mu\nu}\}$$

(ii) From the deformation in $\tilde{g}^{\mu\nu}$ we obtain

$$-(\partial_\mu\partial_\nu f - (\partial_\alpha f)\widehat{\Gamma}^\alpha{}_{\mu\nu})\tilde{h}^{\mu\nu} = 0$$

by the antisymmetry of $\tilde{h}^{\mu\nu}$ compared to symmetry of $\widehat{\Gamma}^\alpha{}_{\mu\nu}$ and of $\partial_\mu\partial_\nu f$. So there is no contribution from this aspect at order λ.

(iii) From the deformation in $\tilde{\partial}_\mu\tilde{\partial}_\nu f$ we obtain

$$\begin{aligned}&\omega^{\alpha\beta}\Gamma^\gamma{}_{\beta\mu}g^{\mu\nu}\partial_\alpha\partial_\gamma\partial_\nu f + g^{\mu\nu}\partial_\mu(\omega^{\alpha\beta}\Gamma^\gamma{}_{\beta\nu}\partial_\alpha\partial_\gamma f)\\ &\quad = 2\omega^{\alpha\beta}\Gamma^\gamma{}_{\beta\mu}g^{\mu\nu}\partial_\alpha\partial_\gamma\partial_\nu f + g^{\mu\nu}(\partial_\alpha\partial_\gamma f)\partial_\mu(\omega^{\alpha\beta}\Gamma^\gamma{}_{\beta\nu})\end{aligned}$$

(iv) From the deformation in $-\tilde{\partial}_\alpha f\bullet\widehat{\Gamma}^\alpha{}_{\mu\nu}$ we obtain

$$-\omega^{\alpha\beta}\Gamma^\gamma{}_{\beta\kappa}g^{\mu\nu}\widehat{\Gamma}^\kappa{}_{\mu\nu}\partial_\alpha\partial_\gamma f - \{\partial_\alpha f, \widehat{\Gamma}^\alpha{}_{\mu\nu}\}g^{\mu\nu}$$

(v) From the deformation in $\tilde{\Gamma}^1$ and our above formulae for that, we obtain

$$-\gamma^\iota{}_{\mu\nu}g^{\mu\nu}\partial_\iota f = -(\partial_\iota f)\omega^{\alpha\beta}\left(2\widehat{\Gamma}^\iota{}_{\mu\kappa,\alpha}\Gamma^\kappa{}_{\beta\nu}g^{\mu\nu} + \widehat{\Gamma}^\iota{}_{\beta\kappa}\mathrm{Ric}^\kappa{}_\alpha + \widehat{\Gamma}^\iota{}_{\alpha\kappa}S^\kappa{}_{;\beta}\right)$$

where $S^\kappa = S^\kappa{}_{\mu\nu}g^{\mu\nu}$ is the 'contorsion vector field' and ; is with respect to ∇.

Now, comparing we see that the cubic derivatives of f in (i) and (iii) cancel using metric compatibility to write a derivative of the metric in terms of Γ. Similarly the 1-derivative term from (i) is $-\partial_\iota f$ times

$$\begin{aligned}\{\widehat{\Gamma}^\iota{}_{\mu\nu}, g^{\mu\nu}\} &= \omega^{\alpha\beta}\widehat{\Gamma}^\iota{}_{\mu\kappa,\alpha}g^{\mu\nu}{}_{,\beta}g_{\eta\nu}g^{\kappa\eta} = -\omega^{\alpha\beta}\widehat{\Gamma}^\iota{}_{\mu\kappa,\alpha}g^{\mu\nu}(\Gamma_{\eta\beta\nu} + \Gamma_{\nu\beta\eta})g^{\kappa\eta}\\ &= -2\omega^{\alpha\beta}\widehat{\Gamma}^\iota{}_{\mu\kappa,\alpha}g^{\mu\nu}\Gamma^\kappa{}_{\beta\nu}\end{aligned}$$

where we inserted $g_{\eta\nu}$, turned ∂_β onto this and used metric compatibility of ∇. In the last step we used that $\widehat{\Gamma}$ is torsion free so symmetric in the last two indices. The result exactly cancels with a term in (v) giving

$$\Box_1 f = \Box f + \frac{\lambda}{2}(\partial_\iota f)\omega^{\alpha\beta}\widehat{\Gamma}^\iota{}_{\alpha\gamma}\left(\mathrm{Ric}^\gamma{}_\beta - \mathcal{S}^\gamma{}_{;\beta}\right) + O(\partial^2)f$$

where we have not yet analysed corrections with quadratic derivatives of f. Turning to these, the remainder of (i) and (iv) contribute

$$\begin{aligned}
&-\{\partial_\gamma f, \widehat{\Gamma}^\gamma\} - (\partial_\alpha\partial_\gamma f)\omega^{\alpha\beta}\Gamma^\gamma{}_{\beta\kappa}\widehat{\Gamma}^\kappa = -\omega^{\alpha\beta}(\partial_\alpha\partial_\gamma f)\widehat{\Gamma}^\gamma{}_{;\beta}\\
&= \omega^{\alpha\beta}(\partial_\alpha\partial_\gamma f)S^\gamma{}_{;\beta} - (\partial_\alpha\partial_\gamma f)g^{\mu\nu}\omega^{\alpha\beta}\Gamma^\gamma{}_{\mu\nu;\beta}\\
&= \omega^{\alpha\beta}(\partial_\alpha\partial_\gamma f)S^\gamma{}_{;\beta} - (\partial_\alpha\partial_\gamma f)g^{\mu\nu}\omega^{\alpha\beta}\left(\Gamma^\gamma{}_{\mu\nu,\beta} + \Gamma^\gamma{}_{\beta\kappa}\Gamma^\kappa{}_{\mu\nu} - \Gamma^\gamma{}_{\kappa\nu}\Gamma^\kappa{}_{\beta\mu} - \Gamma^\gamma{}_{\mu\kappa}\Gamma^\kappa{}_{\beta\nu}\right).
\end{aligned}$$

Meanwhile in (iii), we use poisson-compatibility in the direct form[10]

$$\omega^{\alpha\beta}{}_{,\mu} = \omega^{\beta\eta}\Gamma^\alpha{}_{\eta\mu} + \omega^{\eta\alpha}\Gamma^\beta{}_{\eta\mu}$$

to obtain

$$(\partial_\alpha\partial_\gamma f)g^{\mu\nu}\left(\omega^{\alpha\beta}\Gamma^\gamma{}_{\beta\nu,\mu} + \omega^{\beta\eta}\Gamma^\alpha{}_{\eta\mu}\Gamma^\gamma{}_{\beta\nu} - \omega^{\alpha\beta}\Gamma^\kappa{}_{\beta\nu}\Gamma^\gamma{}_{\kappa\mu}\right)$$

using $g^{\mu\nu}$ symmetric to massage the last term. The middle term vanishes as it is antisymmetric in α, γ and the remaining two terms together with the above terms from $\Gamma^\gamma{}_{\mu\nu;\beta}$ combine to give $(\partial_\alpha\partial_\gamma f)g^{\mu\nu}\omega^{\alpha\beta}R^\gamma{}_{\nu\mu\beta}$. This gives our 2-derivative corrections at order λ as

$$\frac{\lambda}{2}(\partial_\alpha\partial_\gamma f)\omega^{\alpha\beta}(S^\gamma{}_{;\beta} - \mathrm{Ric}^\gamma{}_\beta).$$

We then combine our results to the expression stated. □

2.2. **Quantum Riemann and Ricci curvatures.** The quantum Riemann curvature in noncommutative geometry is defined by

$$\mathrm{Riem}_1 = (\mathrm{d}\otimes_1 \mathrm{id} - (\wedge_1\otimes_1 \mathrm{id})(\mathrm{id}\otimes_1\nabla_1))\nabla_1 \tag{2.17}$$

and we start by obtaining an expression for it to semiclassical order in terms of tensors. It will be convenient to define components by

$$\mathrm{Riem}_1(\mathrm{d}x^\alpha) := -\frac{1}{2}\widehat{R}_1{}^\alpha{}_{\beta\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\beta := -\frac{1}{2}\widetilde{R_1}{}^\alpha{}_{\beta\mu\nu}\bullet(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu)\otimes_1\mathrm{d}x^\beta$$

$$\widetilde{R_1}{}^\alpha{}_{\beta\mu\nu} = \widehat{R}_1{}^\alpha{}_{\beta\mu\nu} + \frac{\lambda}{2}\omega^{\delta\gamma}\left(\widehat{R}^\alpha{}_{\beta\eta\nu,\delta}\Gamma^\eta{}_{\gamma\mu} + \widehat{R}^\alpha{}_{\beta\mu\eta,\delta}\Gamma^\eta{}_{\gamma\nu}\right)$$

depending on how the coefficients enter. If we write $\Gamma_1 = \widehat{\Gamma} + \frac{\lambda}{2}\widehat{\gamma}$ then

$$\begin{aligned}
\mathrm{Riem}_1(\mathrm{d}x^\alpha) &= (\mathrm{d}\otimes_1 \mathrm{id} - (\wedge_1\otimes_1\mathrm{id})(\mathrm{id}\otimes_1\nabla_1))\nabla_1(\mathrm{d}x^\alpha)\\
&= -(\mathrm{d}\otimes_1 \mathrm{id} - (\wedge_1\otimes_1\mathrm{id})(\mathrm{id}\otimes_1\nabla_1))\Gamma_1{}^\alpha{}_{\mu\beta}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\beta\\
&= -\left(\Gamma_1{}^\alpha{}_{\mu\beta,\nu}\mathrm{d}x^\nu\wedge\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\beta + (\Gamma_1{}^\alpha{}_{\mu\gamma}\mathrm{d}x^\mu)\wedge_1(\Gamma_1{}^\gamma{}_{\nu\beta}\mathrm{d}x^\nu)\otimes_1\mathrm{d}x^\beta\right)\\
&= -\left(\widehat{\Gamma}^\alpha{}_{\nu\beta,\mu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\beta + (\widehat{\Gamma}^\alpha{}_{\mu\gamma}\mathrm{d}x^\mu)\wedge_1(\widehat{\Gamma}^\gamma{}_{\nu\beta}\mathrm{d}x^\nu)\otimes_1\mathrm{d}x^\beta\right)\\
&\quad+\frac{\lambda}{2}\left(\widehat{\gamma}^\alpha{}_{\mu\beta,\nu} + \widehat{\Gamma}^\alpha{}_{\nu\gamma}\widehat{\gamma}^\gamma{}_{\mu\beta} - \widehat{\gamma}^\alpha{}_{\mu\gamma}\widehat{\Gamma}^\gamma{}_{\nu\beta}\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\beta\\
&= -\frac{1}{2}\widehat{R}^\alpha{}_{\beta\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\beta + \frac{\lambda}{2}\widehat{\gamma}^\alpha{}_{\mu\beta;\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\beta\\
&\quad-\frac{\lambda}{2}\omega^{\eta\zeta}\nabla_\eta\left(\widehat{\Gamma}^\alpha{}_{\mu\gamma}\mathrm{d}x^\mu\right)\wedge\nabla_\zeta\left(\widehat{\Gamma}^\gamma{}_{\nu\beta}\mathrm{d}x^\nu\right)\otimes_1\mathrm{d}x^\beta\\
&\quad-\lambda\widehat{\Gamma}^\alpha{}_{\mu\gamma}\widehat{\Gamma}^\gamma{}_{\nu\beta}H^{\mu\nu}\otimes_1\mathrm{d}x^\beta
\end{aligned}\tag{2.18}$$

to $O(\lambda^2)$, where $\hat{;}$ is with respect to the Levi-Civita connection. The term $\widehat{\gamma}^\alpha{}_{\gamma\beta}\widehat{\Gamma}^\gamma{}_{\nu\mu}$ does not contribute due to the antisymmetry of the wedge product. This implies for the components

$$\widehat{R}_1{}^\alpha{}_{\beta\mu\nu} = \widehat{R}^\alpha{}_{\beta\mu\nu} + \lambda\Big(\frac{1}{2}(\widehat{\gamma}^\alpha{}_{\nu\beta\hat{;}\mu} - \widehat{\gamma}^\alpha{}_{\mu\beta\hat{;}\nu}) + \omega^{\eta\zeta}(\widehat{\Gamma}^\alpha_{\mu\gamma,\eta} - \widehat{\Gamma}^\alpha{}_{\kappa\gamma}\Gamma^\kappa{}_{\eta\mu})(\widehat{\Gamma}^\gamma_{\nu\beta,\zeta} - \widehat{\Gamma}^\gamma{}_{\kappa\beta}\Gamma^\kappa{}_{\zeta\nu})$$
$$+\frac{\omega^{\eta\kappa}}{2}\widehat{\Gamma}^\alpha{}_{\eta\gamma}\widehat{\Gamma}^\gamma{}_{\zeta\beta}(R^\zeta{}_{\mu\nu\kappa} - R^\zeta{}_{\nu\mu\kappa} - T^\zeta{}_{\mu\nu;\kappa})\Big)$$

where we inserted a previous formula for H in terms of the curvature and torsion of ∇. One can similarly read off $\widehat{\gamma}$ from the quantum Levi-Civita connection in Lemma 2.2.

Next, following [9], we consider the classical map $i:\Omega^2\to\Omega^1\otimes\Omega^1$ that sends a 2-form to an antisymmetric 1-1 form in the obvious way.

Proposition 2.4. *The map i quantises to a bimodule map such that $\wedge_1 i_1 = \mathrm{id}$ to $O(\lambda^2)$ by*

$$i_1(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu) = \frac{1}{2}(\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu - \mathrm{d}x^\nu\otimes_1\mathrm{d}x^\mu) + \lambda I(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu)$$

for any tensor map $I(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu) = I^{\mu\nu}{}_{\alpha\beta}\mathrm{d}x^\alpha\otimes\mathrm{d}x^\beta$ where the tensor I is antisymmetric in μ,ν and symmetric in α,β. The functorial choice here is

$$I^{\mu\nu}{}_{\alpha\beta} = -\frac{1}{4}\omega^{\kappa\tau}(\Gamma^\mu{}_{\kappa\alpha}\Gamma^\nu{}_{\tau\beta} + \Gamma^\mu{}_{\tau\alpha}\Gamma^\nu{}_{\kappa\beta}).$$

Proof. The functorial construction in [10] gives $i_Q:\Omega^2\to\Omega^1\otimes_1\Omega^1$ necessarily obeying $\wedge_Q i_Q = \mathrm{id}$. Here $\nabla(i) = 0$ since the connection on Ω^2 is descended from the connection on $\Omega^1\otimes\Omega^1$ so that $i_Q = q^{-1}i$ on identifying the vector spaces. This gives the expression stated for the canonical I and this also works for $\wedge_1$ since this on $\frac{1}{2}(\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu - \mathrm{d}x^\nu\otimes_1\mathrm{d}x^\mu)$ differs from $\wedge_Q$ by $\frac{\lambda}{2}(H^{\mu\nu} - H^{\nu\mu}) = 0$. Finally, if we change the canonical I to any other tensor with the same symmetries then its wedge is not changed and we preserve all required properties. Note that canonical choice can also be written as

$$i_1(\mathrm{d}x^\mu\wedge_1\mathrm{d}x^\nu) = \frac{1}{2}(\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu - \mathrm{d}x^\nu\otimes_1\mathrm{d}x^\mu) - \frac{\lambda}{2}\omega^{\alpha\beta}\Gamma^\mu{}_{\alpha\kappa}\Gamma^\nu{}_{\beta\tau}\mathrm{d}x^\tau\otimes_1\mathrm{d}x^\kappa + \lambda i(H^{\mu\nu})\tag{2.19}$$

when we allow for the relations of $\wedge_1$. □

Now we can follow [9] and use i_1 to lift the first output of Riem_1 and take a trace of this to compute the quantum Ricci tensor. To take the trace it is convenient, but not necessary, to use the quantum metric and its inverse, so

$$\mathrm{Ricci}_1 = ((\ ,\ )_1 \otimes_1 \mathrm{id} \otimes_1 \otimes \mathrm{id})(\mathrm{id} \otimes_1 i_1 \otimes_1 \mathrm{id})(\mathrm{id} \otimes_1 \mathrm{Riem}_1)(g_1) \tag{2.20}$$

We now calculate Ricci_1 from (2.20) taking first the 'classical' antisymmetric lift

$$i_1(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = \frac{1}{2}(\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\mu)$$

corresponding to $I = 0$. Then using the second form of the components of Riem_1,

$$\begin{aligned}
&\mathrm{Ricci}_1 = ((\ ,\ )_1 \otimes_1 \mathrm{id} \otimes_1 \otimes \mathrm{id})(\mathrm{id} \otimes_1 i_1 \otimes_1 \mathrm{id})(\mathrm{id} \otimes_1 \mathrm{Riem}_1)(g_1)\\
&= -\frac{1}{2}((\ ,\ )_1 \otimes_1 \mathrm{id} \otimes_1 \otimes \mathrm{id})(\mathrm{id} \otimes_1 i_1 \otimes_1 \mathrm{id})(\mathrm{d}x^\alpha \bullet \tilde{g}_{\alpha\beta} \otimes_1 \widetilde{R_1}^\beta{}_{\gamma\mu\nu} \bullet (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \otimes_1 \mathrm{d}x^\gamma)\\
&= -\frac{1}{2}((\ ,\ )_1 \otimes_1 \mathrm{id} \otimes_1 \otimes \mathrm{id})(\mathrm{d}x^\alpha \otimes_1 \tilde{g}_{\alpha\beta} \bullet \widetilde{R_1}^\beta{}_{\gamma\mu\nu} \bullet i_1(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \otimes_1 \mathrm{d}x^\gamma)\\
&= -\frac{1}{2}(\mathrm{d}x^\alpha, \tilde{g}_{\alpha\beta} \bullet \widetilde{R_1}^\beta{}_{\gamma\mu\nu} \bullet \mathrm{d}x^\mu)_1 \bullet \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\gamma\\
&= -\frac{1}{2}\tilde{g}^{\alpha\mu} \bullet \tilde{g}_{\alpha\beta} \bullet \widetilde{R_1}^\beta{}_{\gamma\mu\nu} \bullet \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\gamma - \frac{1}{2}(\mathrm{d}x^\alpha, [\widehat{R}_{\alpha\gamma\mu\nu}, \mathrm{d}x^\mu])\mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\gamma
\end{aligned}$$

In the fourth line we used the fact that the Riemann tensor is already antisymmetric in μ and ν. Note that $\tilde{g}^{\alpha\mu} \bullet \tilde{g}_{\alpha\beta} = (\tilde{g}^{\mu\alpha} + \lambda\tilde{h}^{\mu\alpha}) \bullet \tilde{g}_{\alpha\beta} = \delta^\mu{}_\beta + \lambda\tilde{h}^\mu{}_\beta$ to $O(\lambda^2)$ where we lower an index using the classical metric. Meanwhile, putting in general I adds a term

$$-\frac{\lambda}{2}\widehat{R}_{\alpha\gamma\mu\nu}((\ ,\ ) \otimes \mathrm{id} \otimes \otimes \mathrm{id})(\mathrm{d}x^\alpha \otimes_1 I(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \otimes_1 \mathrm{d}x^\gamma)$$

and we therefore obtain

$$\begin{aligned}
\mathrm{Ricci}_1 &= -\frac{1}{2}\widetilde{R_1}^\mu{}_{\gamma\mu\nu} \bullet \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\gamma - \frac{\lambda}{2}\left(\tilde{h}^{\mu\beta}\widehat{R}_{\beta\gamma\mu\nu} - \omega^{\eta\zeta}g^{\alpha\beta}\widehat{R}_{\alpha\gamma\mu\nu,\eta}\Gamma^\mu{}_{\zeta\beta}\right)\mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\gamma\\
&\quad -\frac{\lambda}{2}\widehat{R}^\alpha{}_{\gamma\eta\zeta}I^{\eta\zeta}{}_{\alpha\nu}\mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\gamma
\end{aligned} \tag{2.21}$$

The idea of [9] is then to use the freedom in I to arrange that

$$\wedge_1(\mathrm{Ricci}_1) = 0, \quad \mathrm{flip}(* \otimes *)\mathrm{Ricci}_1 = \mathrm{Ricci}_1$$

to order λ so that Ricci_1 enjoys the same quantum symmetry and 'reality' properties (to order λ) as g_1. (A further 'reality' condition on the map i_1 in [9] just amounts in our case to the entries of the tensor I being real.) If we write components

$$\mathrm{Ricci}_1 := -\frac{1}{2}\widetilde{R_1}_{\mu\nu} \bullet \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\mu = -\frac{1}{2}\mathrm{d}x^\nu \bullet \widetilde{\widetilde{R_1}}_{\nu\mu} \otimes_1 \mathrm{d}x^\mu$$

then (2.21) is equivalent to

$$\widetilde{R_1}_{\mu\nu} = \widetilde{R_1}^\alpha{}_{\mu\alpha\nu} + \lambda\left(\tilde{h}^{\alpha\beta}\widehat{R}_{\beta\mu\alpha\nu} - \omega^{\eta\zeta}g^{\alpha\beta}\widehat{R}_{\alpha\mu\kappa\nu,\eta}\Gamma^\kappa{}_{\zeta\beta} + \widehat{R}^\alpha{}_{\mu\eta\zeta}I^{\eta\zeta}{}_{\alpha\nu}\right) \tag{2.22}$$

and

$$\widetilde{\widetilde{R_1}}_{\nu\mu} = \widetilde{R_1}_{\mu\nu} - \lambda\omega^{\alpha\beta}\widehat{R}_{\mu\delta,\alpha}\Gamma^\delta{}_{\beta\nu}$$

respectively, where $\widehat{R}_{\mu\nu}$ is the classical Ricci tensor. This second version is useful for the quantum reality condition, which says that if we write $\widetilde{\widetilde{R_1}}_{\mu\nu} = \widehat{R}_{\mu\nu} + \lambda\rho_{\mu\nu}$

then the quantum correction $\rho_{\mu\nu}$ is required to be antisymmetric. Remember that this will have contributions from $\widetilde{R_1}$ as well as the terms directly visible in (2.22).

We then define the quantum Ricci scaler as

$$S_1 = (\ ,\ )_1\mathrm{Ricci}_1 = -\frac{1}{2}\widetilde{R_1}_{\mu\nu} \bullet \tilde{g}^{\nu\mu} \tag{2.23}$$

which does not depend on the lifting tensor I due to the antisymmetry of the first two indices of $\widehat{R}$. There does not appear to be a completely canonical choice of Ricci in noncommutative geometry as it depends on the choice of lifting for which we have not done a general analysis, but this constructive approach allows us to begin to explore it. The reader should note that the natural conventions in our context reduce in the classical limit to $-\frac{1}{2}$ of the usual Riemann and Ricci curvatures, which we have handled by putting this factor into the definition of the tensor components so that these all have limits that match standard conventions.

2.3. **Laplacian in the bicrossproduct model.** We apply the above formalism to the bicrossproduct model quantum spacetime[26]. Much of the quantum geometry (but not the Laplacian) was already solved to all orders by algebraic methods in [9] and the Appendix carefully checks that our new tensor calculus formulae agree with that to order λ (this is not easy and provides a critical check).

The 2D version here has coordinates t, r with r invertible and Poisson bracket $\{r,t\} = r$ or $\omega^{10} = r$ in the coordinate basis. The work [9] used r rather than x as this is also the radial geometry of a higher-dimensional model. The Poisson-compatible 'quantising' connection is given by $\Gamma^0{}_{01} = -r^{-1}, \Gamma^0{}_{10} = r^{-1}$ or in abstract terms $\nabla \mathrm{d}r = 0$ and $\nabla \mathrm{d}t = r^{-1}(\mathrm{d}t \otimes \mathrm{d}r - \mathrm{d}r \otimes \mathrm{d}t)$. Letting $v = r\mathrm{d}t - t\mathrm{d}r$, we have $\nabla \mathrm{d}r = \nabla v = 0$ so a pair of central 1-forms $v, \mathrm{d}r$ at least at first order. This model has trivial curvature of ∇ (and is indeed associative) but in other respects is a good test of our formulae with nontrivial torsion and contorsion and curvature of the Levi-Civita connection.

Next we take classical metric $g = \mathrm{d}r \otimes \mathrm{d}r + bv \otimes v$ where b is a nonzero real parameter which clearly has inverse $(\mathrm{d}r, \mathrm{d}r) = 1$, $(v, v) = b^{-1}$, $(\mathrm{d}r, v) = (v, \mathrm{d}r) = 0$, as the unique form of classical metric for which $\nabla g = 0$ for the above Poisson-compatible connection. This was shown in [9] where it was also shown that the classical Riemannian geometry is that of either a strongly gravitating particle or an expanding universe according to the sign of b. The Levi-Civita connection for g is

$$\widehat{\nabla} v = -\frac{2v}{r} \otimes \mathrm{d}r, \quad \widehat{\nabla}\mathrm{d}r = \frac{2bv}{r} \otimes v.$$

In tensor terms, now in the coordinate basis $x^0 = t$ and $x^1 = r$, the metric tensor and Levi-Civita connection are cf[9, Appendix]

$$g_{\mu\nu} = \begin{pmatrix} br^2 & -brt \\ -brt & 1 + bt^2 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \frac{1+bt^2}{br^2} & \frac{t}{r} \\ \frac{t}{r} & 1 \end{pmatrix}$$

$$\widehat{\Gamma}^0_{\mu\nu} = \begin{pmatrix} -2bt & r^{-1}(1+2bt^2) \\ r^{-1}(1+2bt^2) & -2r^{-2}t(1+bt^2) \end{pmatrix}, \quad \widehat{\Gamma}^1_{\mu\nu} = \begin{pmatrix} -2br & 2bt \\ 2bt & -2br^{-1}t^2 \end{pmatrix}$$

The contorsion tensor can be written[9]

$$S^\kappa{}_{\alpha\beta} = 2b\epsilon_{\alpha\mu}x^\mu\epsilon_{\beta\nu}g^{\kappa\nu}, \quad S^\mu = 2\frac{x^\mu}{r^2}$$

where $\epsilon_{01}=1$ is antisymmetric. Then formula (2.10) gives $\mathcal{R}_{10}=br$ or $\mathcal{R}_{\mu\nu}=-br\epsilon_{\mu\nu}$ and hence $\mathcal{R}=\mathcal{R}_{10}\mathrm{d}t\wedge\mathrm{d}r=bv\wedge\mathrm{d}r$ as in [10, Sec 7.1]

We also write

$$\mathrm{d}f=f_{,r}\mathrm{d}r+f_{,t}\mathrm{d}t=(\partial_r f)\mathrm{d}r+(\partial_v f)v;\quad \partial_v f=\frac{1}{r}f_{,t},\quad \partial_r f=f_{,r}+\frac{t}{r}f_{,t}$$

Then

$$\begin{aligned}\Box f&=(\ ,\ )\widehat{\nabla}\mathrm{d}f=(\ ,\ )((\partial_r f)\widehat{\nabla}\mathrm{d}r+(\partial_v f)\widehat{\nabla}v+\mathrm{d}\partial_r f\otimes\mathrm{d}r+\mathrm{d}\partial_v f\otimes v)\\&=\frac{2}{r}(\partial_r f)+\partial_r^2 f+b^{-1}\partial_v^2 f\end{aligned}$$

is the classical Laplacian for g. When $b<0$ the interpretation of the classical geometry is that of a strong gravitational source and curvature singularity at $r=0$. Being conformaly flat after a change of variables to $r'=1/r, t'=t/r$ the massless waves or zero eigenfunctions of the classical Laplacian are plane waves in t',r' space of the form $e^{\imath\omega t'}e^{\pm\frac{\imath\omega r'}{\sqrt{-b}}}$ or

$$\psi_\omega^\pm(t,r)=e^{\imath\frac{\omega t}{r}}e^{\pm\imath\frac{\omega}{r\sqrt{-b}}}$$

while the massive modes are harder to describe due to the conformal factor. One can similarly solve the expanding universe case where $b>0$ and the interpretation of the r,t variables is swapped. This completes the classical data.

Next, the quantum metric at semiclassical order from (2.9) is

$$\begin{aligned}g_1&=g_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu+\frac{\lambda}{2}\mathcal{R}_{01}(\mathrm{d}t\otimes_1\mathrm{d}r-\mathrm{d}r\otimes_1\mathrm{d}t)+\frac{\lambda}{2}\omega^{10}g_{\mu 0}\Gamma^0{}_{10}\Gamma^0{}_{01}\mathrm{d}x^\mu\otimes_1\mathrm{d}r\\&=g_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu+\frac{\lambda}{2}bt\mathrm{d}r\otimes_1\mathrm{d}r+\frac{\lambda}{2}br\mathrm{d}r\otimes_1\mathrm{d}t-\lambda br\mathrm{d}t\otimes_1\mathrm{d}r\end{aligned}$$

Also, from the formula (2.12), we have

$$h_{01}=\mathcal{R}_{01}+g_{00}\omega^{10}\Gamma^0{}_{10}\Gamma^0{}_{01}+\omega^{10}\Gamma^0{}_{10}g_{01,0}=-3br$$

and similarly $h_{10}=3br$, so that

$$\tilde{g}_{\mu\nu}=g_{\mu\nu}+\frac{\lambda}{2}\begin{pmatrix}0&-3br\\3br&0\end{pmatrix}=g_{\mu\nu}-\frac{\lambda 3br}{2}\epsilon_{\mu\nu}.$$

For the correction in ${\Gamma_1}^1{}_{\mu\nu}$ for the quantum Levi-Civita connection in Lemma 2.2 we have

$$\begin{aligned}&\frac{\lambda}{2}\omega^{\alpha\beta}\widehat{\Gamma}^1{}_{\mu 0,\alpha}\Gamma^0{}_{\beta\nu}-\frac{\lambda}{2}\omega^{\alpha\beta}\widehat{\Gamma}^1{}_{00}\Gamma^0{}_{\alpha\mu}\Gamma^0{}_{\beta\nu}-\frac{\lambda}{2}\omega^{01}S^1{}_{0\kappa}\nabla_1 S^\kappa{}_{\mu\nu}\\&=-\frac{\lambda}{2}\widehat{\Gamma}^1{}_{\mu 0,\nu}-\lambda b\epsilon_{\mu\nu}-\frac{\lambda}{2}r\widehat{\Gamma}^1{}_{0\kappa}\nabla_1 S^\kappa{}_{\mu\nu}=2\lambda b\begin{pmatrix}0&1\\0&-r^{-1}t\end{pmatrix}\end{aligned}$$

and

$$\nabla_1 S^0{}_{\mu\nu}=\begin{pmatrix}-\frac{2bt}{r}&\frac{2(1+bt^2)}{r^2}\\\frac{2bt^2}{r^2}&-\frac{2t(1+bt^2)}{r^3}\end{pmatrix},\quad \nabla_1 S^1{}_{\mu\nu}=\begin{pmatrix}-2b&\frac{2bt}{r}\\\frac{2bt}{r}&-\frac{2bt^2}{r^2}\end{pmatrix}$$

where ∇_1 in this context means with respect to r. Similarly, the correction to ${\Gamma_1}^0{}_{\mu\nu}$ in Lemma 2.2 is

$$\frac{\lambda}{2}\omega^{\alpha\beta}\widehat{\Gamma}^0{}_{\mu 0,\alpha}{\Gamma^0}_{\beta\nu} - \frac{\lambda}{2}\omega^{\alpha\beta}\widehat{\Gamma}^0{}_{00}{\Gamma^0}_{\alpha\mu}{\Gamma^0}_{\beta\nu} - \frac{\lambda}{2}\omega^{01}{S^0}_{0\kappa}\nabla_1{S^\kappa}_{\mu\nu}$$
$$= -\frac{\lambda}{2}\widehat{\Gamma}^0{}_{\mu 0,\nu} - \lambda b r^{-1} t\epsilon_{\mu\nu} - \frac{\lambda}{2} r\widehat{\Gamma}^0{}_{0\kappa}\nabla_1{S^\kappa}_{\mu\nu} = \lambda\begin{pmatrix} 2b & 0 \\ -\frac{2bt}{r} & \frac{1}{2r^2}\end{pmatrix}$$

giving

$$\nabla_1 \mathrm{d}r = -\widehat{\Gamma}^1_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - 2\lambda b(\mathrm{d}t - r^{-1}t\mathrm{d}r)\otimes_1 \mathrm{d}r$$
$$\nabla_1 \mathrm{d}t = -\widehat{\Gamma}^0{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - \frac{\lambda}{2}r^{-2}\mathrm{d}r\otimes_1 \mathrm{d}r - 2\lambda b(\mathrm{d}t - r^{-1}t\mathrm{d}r)\otimes_1 \mathrm{d}t.$$

One can check that the condition (2.14) holds so that this is the quantum Levi-Civita connection at order λ.

The quantum Riemann tensor by direct computation (using Maple) from (2.18) comes out as

$$\mathrm{Riem}_1(\mathrm{d}x^\alpha) = -\frac{1}{2}\widehat{R}^\alpha{}_{\beta\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\beta + \frac{5b\lambda}{r}\mathrm{d}t\wedge\mathrm{d}r\otimes_1\mathrm{d}x^\alpha \tag{2.24}$$

For the quantum Ricci tensor we need a lift map i_1 and we take

$$i_1(\mathrm{d}t\wedge\mathrm{d}r) = \frac{1}{2}(\mathrm{d}t\otimes_1\mathrm{d}r - \mathrm{d}r\otimes_1\mathrm{d}t) + \frac{7\lambda}{4r}g$$

where $\mathrm{d}t\wedge_1\mathrm{d}r = \mathrm{d}t\wedge\mathrm{d}r$ since $\nabla\mathrm{d}r = 0$ and only H^{00} is non-zero. The first term is the functorial term and the second term is $\lambda I(\mathrm{d}t\wedge\mathrm{d}r)$. Then (2.21) gives us $\mathrm{Ricci}_1 = g_1/r^2$ to order λ in agreement with the algebraic result in [9]. This means that the quantum Ricci scaler is undeformed.

Finally, the contracted contorsion tensor obeys

$${S^\mu}_{;0} = 0, \quad {S^\mu}_{;1} = -2\frac{x^\mu}{r^3}$$

while the curvature of ∇ vanishes. Hence the Laplacian in Theorem 2.3 is

$$\Box_1 f = \Box f + \frac{\lambda}{2}\omega^{10}(-{S^\mu}_{;1})(\widehat{\nabla}_0\mathrm{d}f)_\mu = \Box f + \frac{\lambda}{r^2}x^\mu(\widehat{\nabla}_0\mathrm{d}f)_\mu$$

which can be further expanded out using the values of $\Gamma^0_{\mu\nu}$. We see that there *is* an order λ correction. It is not so clear how to immediately read off physical predictions from this but one thing we can still do in the deformed case is make the conformal change of variables as classically and separate off $\psi = e^{\imath\omega t'}f$, to give an equation for null modes

$$\left(\frac{\partial^2}{\partial r'^2} + \lambda_P\omega\left(\frac{2}{r'} + \frac{\partial}{\partial r'}\right) - \frac{\omega^2}{b}\right)f = 0$$

where $\lambda = \imath\lambda_P$. This is solved by

$$f = r' e^{-\frac{1}{2}\omega r'\left(\lambda_P + \frac{\imath\sqrt{b\lambda_P^2+4}}{\sqrt{-b}}\right)}\left(AM(1 - \frac{\imath\sqrt{-b}\lambda_P}{\sqrt{b\lambda_P^2+4}}, 2, \imath\sqrt{b\lambda_P^2+4}\frac{\omega r'}{\sqrt{-b}})\right.$$
$$\left. + BU(1 - \frac{\imath\sqrt{-b}\lambda_P}{\sqrt{b\lambda_P^2+4}}, 2, \imath\sqrt{b\lambda_P^2+4}\frac{\omega r'}{\sqrt{-b}})\right)$$

for constants A and B. $M(a,b,z)$ and $U(a,b,z)$ denote the Kummer M and U functions (or hypergeometric ${}_1F_1$, U respectively in Mathematica). In the limit $\lambda_P \to 0$, this becomes

$$f = \frac{\imath}{2}\frac{\sqrt{-b}A}{\omega}e^{\imath\frac{\omega r'}{\sqrt{-b}}} + \frac{\imath}{2}\frac{\sqrt{-b}(B-A)}{\omega}e^{-\imath\frac{\omega r'}{\sqrt{-b}}}$$

which means we recover our two independent solutions $\psi^{\pm}_{\omega}$ as a check. Bearing in mind that our equations are only justified to order λ, we can equally well write

$$\psi_\omega(t,r) = \frac{e^{\imath\frac{\omega t}{r}}}{r}e^{-\imath\frac{\omega}{r\sqrt{-b}}\left(1-\frac{\imath\sqrt{-b}\lambda_P}{2}\right)}$$
$$\left(AM(1-\frac{\imath\sqrt{-b}\lambda_P}{2},2,\imath\frac{2\omega}{r\sqrt{-b}}) + BU(1-\frac{\imath\sqrt{-b}\lambda_P}{2},2,\imath\frac{2\omega}{r\sqrt{-b}})\right)$$

and proceed to analyse the behaviour for small λ_P in terms of integral formulae. Thus

$$M(1-a,2,z) = \frac{1}{\Gamma(1-a)\Gamma(1+a)}\int_0^1 e^{zu}(\frac{1-u}{u})^a\mathrm{d}u = \int_0^1 e^{zu}(1+a\ln(\frac{1-u}{u}))\mathrm{d}u + O(a^2)$$

which we evaluate for $z = 2\imath s$ and s real in terms of the function

$$\tau_M(s) = \imath\frac{M^{(1,0,0)}(1,2,2\imath s)}{M(1,2,2\imath s)} = \frac{\int_0^1 e^{2\imath su}\ln(\frac{1-u}{u})\mathrm{d}u}{\imath\int_0^1 e^{2\imath su}\mathrm{d}u}$$

shown in Figure 1. This function in the principal region (containing $s = 0$) is qualitatively identical to the trig function $-2\tan(s/2)$ but blows up slightly more slowly as $s \to \pm\pi$. This gives us $M(1-a,2,2\imath s) = \frac{1}{2\imath s}(e^{2\imath s}-1)(1+a\imath\tau_M(s)) + O(a^2)$ and hence with $a = \imath\frac{\sqrt{-b}\lambda_P}{2}$ and $s = \frac{\omega}{r\sqrt{-b}}$, we have up to normalisation

$$\psi^M_\omega(t,r) = e^{\imath\frac{\omega t}{r}}\sin(\frac{\omega}{r\sqrt{-b}})e^{-\frac{\omega}{2r}\lambda_P\left(1+\frac{r}{\omega}\sqrt{-b}\tau_M(\frac{\omega}{r\sqrt{-b}})\right)} + O(\lambda_P^2), \quad |r| > \frac{|\omega|}{\pi\sqrt{-b}}$$

as one of our independent solutions. Notice that for $\lambda_P \neq 0$ our solution blows up and our approximations break down as r approaches a certain minimum distance as shown to the classical Ricci singularity at $r = 0$, depending on the frequency. This is a geometric 'horizon' of some sort (with scale controlled by $\sqrt{-b}$) but frequency dependent, and very different effect from the usual Planck scale bound $|r| >> |\omega|\lambda_P$ needed in any case for our general analysis. Meanwhile for large $|r|$, the effective λ_P is suppressed as $\tau'_M(0) = -1$.

For the other mode, the similar integral

$$U(1-a,2,z) = \frac{1}{\Gamma(1-a)}\int_0^\infty e^{-zu}(\frac{1+u}{u})^a\mathrm{d}u$$

is not directly applicable as it is not valid on the imaginary axis but we can still proceed in a similar way for the other mode by defining

$$\tau_U(s) = \imath\frac{U^{(1,0,0)}(1,2,2\imath s)}{U(1,2,2\imath s)} = T(s) + \imath S(s)$$

where the real function $T(s)$ resembles $\frac{\pi}{2}\tanh(s)$ (but is vertical at the origin) and $S(s)$ resembles $-\ln(e^{-\gamma}+2|s|)$ as also shown in Figure 1, where $\gamma \approx 0.577$ is

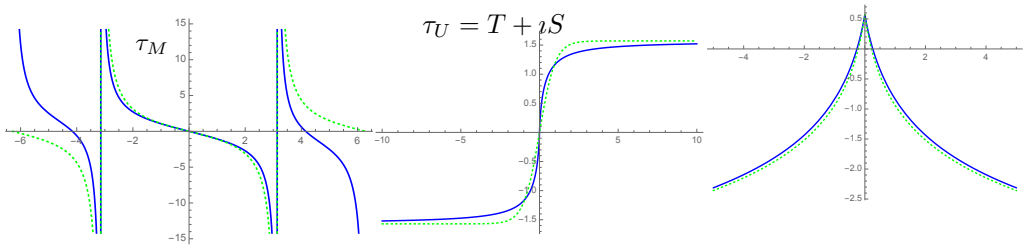

FIGURE 1. Functions τ_M and τ_U related to differentials of Kummer $M(1-a,2,2\imath s)$ and $U(1-a,2,2\imath s)$ at $a=0$ and similar to tan, tanh and a shifted ln (shown dashed for reference)

the Euler constant. Then $U(1-a,2,2\imath s) = \frac{1}{2\imath s}(1 + a\imath\tau_U(s)) + O(a^2)$ giving up to normalisation

$$\psi^U_\omega(t,r) = e^{\imath\frac{\omega t}{r}} e^{-\imath\frac{\omega}{r\sqrt{-b}}} e^{-\frac{\omega}{2r}\lambda_P\left(1+\frac{r}{\omega}\sqrt{-b}\tau_U(\frac{\omega}{r\sqrt{-b}})\right)} + O(\lambda_P^2)$$

as a second solution. This still has our general Planck scale lower bound needed for the general analysis but no specific geometric bound at finite radius as τ_U does not blow up and moreover has only a mild log divergence as $s\to\infty$ or $r\to 0$. There is no particular suppression of λ_P as $s\to 0$ or $r\to\infty$ and indeed τ_U tends to a constant nonzero imaginary value (the meaning of which is unclear as it can be absorbed in a normalisation).

Both of our solutions have been exhibited as deviations from the classical solutions and consequently they can reasonably be expected to lead to physical predictions, such as a change of the group velocity along the lines of [1] and of gravitational redshift along the lines of [25]. However, doing this in a convincing way in a GR setting requires rather more analysis and is beyond our scope here.

2.4. **Laplacian in the 2D Bertotti-Robinson model.** By way of contrast we note that the bicrossproduct spacetime algebra has an alternative differential structure for which the full quantum geometry was also already solved, in [28]. We have the same Poisson bracket as above but this time the zero curvature 'quantising' connection $\nabla \mathrm{d}r = \frac{1}{r}\mathrm{d}r\otimes\mathrm{d}r$, $\quad \nabla \mathrm{d}t = -\frac{\alpha}{r}\mathrm{d}t$ or non-zero Christoffel sysmbols $\Gamma^1{}_{11} = -r^{-1}$ and $\Gamma^0{}_{10} = \alpha r^{-1}$ and the de Sitter metric in the form

$$g = ar^{-2}\mathrm{d}r\otimes\mathrm{d}r + br^{\alpha-1}(\mathrm{d}r\otimes\mathrm{d}t + \mathrm{d}r\otimes\mathrm{d}r) + cr^{2\alpha}\mathrm{d}t\otimes\mathrm{d}t$$

where only the nonzero combination $\bar\delta = c\alpha^2/(b^2-ac)$ of parameters is relevant up coordinate transformations. One can easily compute the classical Levi-Civita connection in these coordinates as

$$\widehat\Gamma^0{}_{00} = -\frac{bc\alpha r^\alpha}{b^2-ac},\quad \widehat\Gamma^0{}_{10} = -\frac{ac\alpha}{r(b^2-ac)},\quad \widehat\Gamma^0{}_{11} = -\frac{ba\alpha r^{-\alpha-2}}{b^2-ac}$$

$$\widehat\Gamma^1{}_{00} = \frac{c^2\alpha r^{1+2\alpha}}{b^2-ac},\quad \widehat\Gamma^1{}_{10} = \frac{bc\alpha r^\alpha}{b^2-ac},\quad \widehat\Gamma^1{}_{11} = \frac{-b^2(1-\alpha)+ac}{r(b^2-ac)}$$

Combing this with the 'quantising' connection yields the contorsion tensor

$$S^0{}_{00} = -S^1_{01} = -S^1{}_{10} = \frac{bc\alpha r^\alpha}{b^2-ac},\quad S^0{}_{10} = -S^1{}_{11} = S^0{}_{01} + \frac{\alpha}{r} = \frac{b^2\alpha}{b^2-ac}$$

$$S^0{}_{11} = \frac{ba\alpha r^{-\alpha-2}}{b^2-ac}, \quad S^1{}_{00} = -\frac{c^2\alpha r^{1+2\alpha}}{b^2-ac}$$

From here we compute $S^\mu = \frac{\alpha}{b^2-ac}(br^{-\alpha}, -cr)$ for the t, r components giving $\nabla_\nu S^\mu = 0$ so that in conjunction with flatness of ∇, Theorem 2.3 shows that there is no order λ correction to the Laplace operator.

We can also find the geometric quantum Laplacian to all orders directly from the full quantum geometry at least after a convenient but non-algebraic coordinate transformation in [28]. If we allow this then the model has generators R, T with the only non-zero commutation relations $[T, R] = \lambda'$, $[R, \mathrm{d}R] = \lambda'\sqrt{\bar{\delta}}\mathrm{d}R$ where $\lambda' = \lambda\sqrt{b^2-ac}$ and the quantum metric and quantum Levi-Civita connection[28]

$$g_1 = \mathrm{d}R \bullet e^{2T\sqrt{\bar{\delta}}} \otimes_1 \mathrm{d}R - \mathrm{d}T \otimes_1 \mathrm{d}T,$$

$$\nabla_1 \mathrm{d}T = -\sqrt{\bar{\delta}} e^{2T\sqrt{\bar{\delta}}} \bullet \mathrm{d}R \otimes_1 \mathrm{d}R, \quad \nabla_1 \mathrm{d}R = -\sqrt{\bar{\delta}}(\mathrm{d}R \otimes_1 \mathrm{d}T + \mathrm{d}T \otimes_1 \mathrm{d}R),$$

which immediately gives us

$$(\mathrm{d}R, \mathrm{d}R)_1 = e^{-2T\sqrt{\bar{\delta}}}, \quad (\mathrm{d}T, \mathrm{d}T)_1 = -1, \quad \Box_1 T = (\ ,\ )\nabla_1 \mathrm{d}T = -\sqrt{\bar{\delta}}, \quad \Box_1 R = 0.$$

Finally, for a general normal-ordered function $f(T, R)$ with $T's$ to the left, we have

$$\mathrm{d}f = \frac{\partial f}{\partial T} \bullet \mathrm{d}T + \partial^1 f \bullet \mathrm{d}R; \quad \partial^1 f(R) = \frac{f(R) - f(R - \lambda'\sqrt{\bar{\delta}})}{\lambda'\sqrt{\bar{\delta}}}$$

due to the standard form of the commutation relations. With these ingredients and following exactly the same method as above, we have

$$\Box_1 f = (\ ,\ )_1 \nabla_1(\mathrm{d}f) = -\sqrt{\bar{\delta}}\frac{\partial f}{\partial T} - \frac{\partial^2 f}{\partial T^2} + (\partial^1)^2 f \bullet e^{-2T\sqrt{\bar{\delta}}} = \Box f + O(\lambda^2)$$

when we expand $\partial^1 = \frac{\partial}{\partial R} - \frac{\lambda'\sqrt{\bar{\delta}}}{2}\frac{\partial^2}{\partial R^2} + O(\lambda^2)$ and write the bullet as classical plus Poisson bracket. This confirms what we found from Theorem 2.3. We can also use identities from quantum mechanics applied to R, T in our case to further write

$$\Box_1 f = -\sqrt{\bar{\delta}}\frac{\partial f}{\partial T} - \frac{\partial^2 f}{\partial T^2} + e^{-2T\sqrt{\bar{\delta}}} \bullet \Delta_1 f$$

where

$$\Delta_1 f(R) = \frac{f(R + 2\lambda'\sqrt{\bar{\delta}}) - 2f(R - \lambda'\sqrt{\bar{\delta}}) + f(R)}{(\lambda'\sqrt{\bar{\delta}})^2}.$$

We see that the quantum Laplacian working in the quantum algebra with normal-ordered quantum wave functions has the classical form except that the derivative in the R direction is a finite difference one. It is also clear that we have eigenfunctions $\psi(T, R) = e^{\imath\omega T} e^{\imath k R}$. This is an identical situation to the standard Minkowski spacetime bicrossproduct model in [1] except that there time became a finite difference and there was no actual quantum geometry. Like there, one *could* claim that there is an order λ correction provided classical fields are identified with normal ordered ones, but from the point of view of Poisson-Riemannian geometry this is an artefact of such an assumption (the Poisson geometry being closer to Weyl ordering). We have focussed on the 2D case but the same conclusion holds for the Bertotti-Robinson quantum metric on $S^{n-1} \times dS_2$ in [28] keeping the angular coordinates to the left along with T; then only the double R-derivative deforms namely to Δ_1 on normal-ordered functions. [28] already obtained the quantum Ricci and scaler curvatures in the same form as classically (normal ordered in the former case).

2.5. **Fuzzy nonassociative sphere revisited.** The case of the sphere in Poisson-Riemannian geometry is covered in [10] mainly in very explicit cartesian coordinates where we broke the rotational symmetry. However, the results are fully rotationally invariant as is more evident if we work with z^i, $i = 1, 2, 3$ and the relation $\sum_i z^{i2} = 1$. We took $\nabla = \widehat{\nabla}$ (the Levi-Civita connection) so $S = 0$, and ω the inverse of the canonical volume 2-form on the unit sphere. Then the results of [10] give us a particular 'fuzzy sphere' differential calculus

$$[z^i, z^j]_\bullet = \lambda \epsilon^{ij}{}_k z^k, \quad [z^i, \mathrm{d}z^j]_\bullet = \lambda z^j \epsilon^i{}_{mn} z^m \mathrm{d}z^n.$$

to order λ. These are initially valid for $i = 1, 2$ but must hold in this form for $i = 1, 2, 3$ by rotational symmetry of both the Poisson bracket and the Levi-Civita connection. One also finds from the algebra that $z^m \bullet \mathrm{d}z^m = 0$ (sum over $m = 1, 2, 3$) at order λ on differentiating the radius 1 relation. Here Ω^1 is a projective module with $\mathrm{d}z^i$ as a redundant set of generators and a relation. We also have

$$\{\mathrm{d}z^i, \mathrm{d}z^j\}_\bullet = \lambda(3z^i z^j - \delta_{ij})\mathrm{Vol}$$

to order λ as derived in [10] for $i = 1, 2$ and which then holds for $i = 1, 2, 3$. This can also be derived by applying d to the bimodule relations and using $\mathrm{d}z^i \wedge \mathrm{d}z^j = \epsilon^{ij}{}_k z^k \mathrm{Vol}$ at the classical level on the unit sphere. We will also use the antisymmetric lift $\widetilde{\mathrm{Vol}} = \frac{1}{2}(z^3)^{-1}(\mathrm{d}z^1 \otimes \mathrm{d}z^2 - \mathrm{d}z^2 \otimes \mathrm{d}z^1)$ at the classical level. The classical sphere metric $g_{\mu\nu}$ is given in [10] in the z^1, z^2 coordinates but we can also write it as

$$g = \sum_{i=1}^{3} \mathrm{d}z^i \otimes \mathrm{d}z^i$$

Similarly, the inverse metric and metric inner product are

$$g^{\mu\nu} = \delta_{\mu\nu} - z^\mu z^\nu, \quad (\mathrm{d}z^i, \mathrm{d}z^j) = \delta_{ij} - z^i z^j$$

for $\mu, \nu = 1, 2$, which extends as the second equality for $i, j = 1, 2, 3$. The sphere is 2-dimensional so only two of the z^i are independent in any coordinate patch but the expressions themselves are rotationally invariant in terms of all three.

The work [10] also computes the quantum metric and quantum Levi-Civita connection at order λ. We have

$$\begin{aligned} g_1 =& g_{\mu\nu}\mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\nu - \frac{\lambda}{2(z^3)^2}\mathrm{d}z^3 \otimes_1 \epsilon_{3ij} z^i \mathrm{d}z^j + \lambda\widetilde{\mathrm{Vol}} \\ =& g_{\mu\nu}\mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\nu + \frac{\lambda}{2(z^3)^2}\epsilon_{3ij}\left(z^3 \mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i \mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right) \end{aligned}$$

$$\begin{aligned} \nabla_1 \mathrm{d}z^\mu &= -z^\mu \bullet g_1 = -\widehat{\Gamma}^\mu{}_{\alpha\beta}\mathrm{d}z^\alpha \otimes_1 \mathrm{d}z^\beta - \lambda z^\mu \widetilde{\mathrm{Vol}} + \frac{\lambda}{2}\left(\mathrm{d}z^3 \otimes_1 (\epsilon^{\mu\beta} g_{\beta\gamma} + \frac{z^\mu z^\beta}{(z^3)^2}\epsilon_{\beta\gamma})\mathrm{d}z^\gamma\right) \\ &= -\widehat{\Gamma}^\mu{}_{\alpha\beta}\mathrm{d}z^\alpha \otimes_1 \mathrm{d}z^\beta - \frac{\lambda}{2(z^3)^2}\left(\epsilon_{3ij} z^\mu z^3 \mathrm{d}z^i \otimes_1 \mathrm{d}z^j - \epsilon^\mu{}_{\nu 3}\mathrm{d}z^3 \otimes_1 \mathrm{d}z^\nu\right) \end{aligned}$$

where we massaged the formulae in [10]. The classical Christoffel symbols are $\widehat{\Gamma}^\mu{}_{\alpha\beta} = z^\mu g_{\alpha\beta}$.

If we work with coefficients $\tilde{g}_{ij}$ in the middle for the metric then the given quantum metric corresponds to the correction term

$$h = \frac{(2 - (z^3)^2)}{(z^3)^3}\epsilon_{3ij}\mathrm{d}z^i \otimes_1 \mathrm{d}z^j = \frac{2(2 - (z^3)^2)}{(z^3)^2}\widetilde{\mathrm{Vol}}$$

which we see is antisymmetric. For the inverse metric we have from (2.13) that

$$(\mathrm{d}z^i, \mathrm{d}z^j)_1 = g^{ij} + \frac{\lambda}{2}\epsilon_{ijk}z^k$$

to order λ when $i,j = 1,2$ but which extends to $i,j = 1,2,3$ with $g^{ij} = \delta_{ij} - z^i z^j$. For the connection it is a nice check that the formula in Lemma 2.2 gives the same answer for ∇_1. Then we can calculate the quantum Riemann tensor from (2.17) or directly from the above formulae for ∇_1.

$$\begin{aligned}\mathrm{Riem}_1(\mathrm{d}z^\alpha) &= (\mathrm{d}\otimes_1 \mathrm{id} - (\wedge_1\otimes_1 \mathrm{id})(\mathrm{id}\otimes_1 \nabla_1))\nabla_1(\mathrm{d}z^\alpha)\\ &= -\left(\mathrm{d}({\Gamma_1}^\alpha{}_{\mu\beta})\wedge \mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\beta + {\Gamma_1}^\alpha{}_{\mu\gamma}\mathrm{d}z^\mu \wedge_1 {\Gamma_1}^\gamma{}_{\nu\beta}\mathrm{d}z^\nu \otimes_1 \mathrm{d}z^\beta\right)\end{aligned}$$

which can be broken down into three terms as follows

(i) The first term gives

$$\begin{aligned}\mathrm{d}({\Gamma_1}^\alpha{}_{\mu\beta})\wedge \mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\beta &= -\widehat{\Gamma}^\alpha{}_{\mu\beta,\nu}\mathrm{d}z^\nu \wedge \mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\beta - \frac{\lambda}{2}\partial_\mu\left(\frac{z^\alpha}{z^3}\right)\epsilon_{3\nu\beta}\mathrm{d}z^\mu \wedge \mathrm{d}z^\nu \otimes_1 \mathrm{d}z^\beta\\ &= -\widehat{\Gamma}^\alpha{}_{\mu\beta,\nu}\mathrm{d}z^\nu \wedge \mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\beta - \frac{\lambda}{2(z^3)^2}\epsilon_{3\nu\beta}\left(z^3\mathrm{d}z^\alpha - z^\alpha \mathrm{d}z^3\right)\wedge \mathrm{d}z^\nu \otimes_1 \mathrm{d}z^\beta\\ &= -\widehat{\Gamma}^\alpha{}_{\mu\beta,\nu}\mathrm{d}z^\nu \wedge \mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\beta - \frac{\lambda}{2z^3}\left(z^3\mathrm{Vol}\otimes_1 \mathrm{d}z^\alpha - z^\alpha \mathrm{Vol}\otimes_1 \mathrm{d}z^3\right)\end{aligned}$$

The last step comes from expanding the expression in the previous line and simplifying, this will prove useful in comparing to the other terms.

(ii) Expanding $\wedge_1$ gives a further two terms at $\mathcal{O}(\lambda)$. But first, using the formula for the classical Christoffel symbols and metric compatibility note that

$$\nabla_\alpha \widehat{\Gamma}^\iota{}_{\mu\nu}\mathrm{d}z^\nu = \nabla_\alpha z^\iota g_{\mu\nu}\mathrm{d}z^\nu = (\delta^\iota{}_\alpha g_{\mu\nu} + z^\iota z_\nu g_{\alpha\mu})\mathrm{d}z^\nu$$

Now consider

$$\begin{aligned}&\omega^{\eta\zeta}\nabla_\eta \widehat{\Gamma}^\alpha{}_{\mu\gamma}\mathrm{d}z^\mu \wedge \nabla_\zeta \widehat{\Gamma}^\gamma{}_{\nu\beta}\mathrm{d}z^\nu \otimes_1 \mathrm{d}z^\beta\\ &\qquad = \omega^{\eta\zeta}(\delta^\alpha{}_\eta g_{\mu\gamma} + z^\alpha z_\gamma g_{\eta\mu})\mathrm{d}z^\mu \wedge (\delta^\gamma{}_\zeta g_{\nu\beta} + z^\gamma z_\beta g_{\zeta\nu})\mathrm{d}z^\nu \otimes_1 \mathrm{d}z^\beta\\ &\qquad = \omega^{\alpha\gamma}g_{\mu\gamma}g_{\nu\beta}\mathrm{d}z^\mu \wedge \mathrm{d}z^\nu \otimes_1 \mathrm{d}z^\beta\\ &\qquad = \frac{1}{z^3}(\epsilon^\alpha{}_{\mu 3} + \epsilon_{3\mu\gamma}z^\alpha z^\gamma)g_{\nu\beta}\mathrm{d}z^\mu \wedge z^\nu \otimes_1 \mathrm{d}z^\beta = \mathrm{Vol}\otimes_1 \mathrm{d}z^\alpha\end{aligned}$$

where the cancellations in the second line result from the antisymmetry of μ,ν and η,ζ. For the second term use

$$H^{\mu\nu} = \frac{1}{2}(z^\mu z^\nu - \delta^{\mu\nu})\mathrm{Vol}$$

giving

$$\begin{aligned}\widehat{\Gamma}^\alpha{}_{\mu\gamma}\widehat{\Gamma}^\gamma{}_{\nu\beta}H^{\mu\nu}\otimes_1 \mathrm{d}z^\beta &= \frac{1}{2}z^\alpha z^\gamma g_{\mu\gamma}g_{\nu\beta}(z^\mu z^\nu - \delta^{\mu\nu})\mathrm{Vol}\otimes_1 \mathrm{d}z^\beta\\ &= \frac{1}{2}z^\alpha z^\nu\left(\frac{(z^1)^2 + (z^2)^2}{(z^3)^4} - \frac{1}{(z^3)^4}\right)\delta_{\nu\beta}\mathrm{Vol}\otimes_1 \mathrm{d}z^\beta\\ &= -\frac{z^\alpha}{2z^3}\mathrm{Vol}\otimes_1 \mathrm{d}z^3\end{aligned}$$

Combining these two (remembering to add an overall 1/2 to the first) results in

$$\widehat{\Gamma}^{\alpha}{}_{\mu\gamma}\mathrm{d}z^{\mu}\wedge_1\widehat{\Gamma}^{\gamma}{}_{\nu\beta}\mathrm{d}z^{\nu}\otimes_1\mathrm{d}z^{\beta}$$
$$=\widehat{\Gamma}^{\alpha}{}_{\mu\gamma}\widehat{\Gamma}^{\gamma}{}_{\nu\beta}\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\nu}\otimes_1\mathrm{d}z^{\beta}+\frac{\lambda}{2z^3}\left(z^3\mathrm{Vol}\otimes_1\mathrm{d}z^{\alpha}-z^{\alpha}\mathrm{Vol}\otimes_1\mathrm{d}z^3\right)$$

(iii) The last term involves the $\mathcal{O}(\lambda)$ of ${\Gamma_1}^{\alpha}{}_{\mu\gamma}{\Gamma_1}^{\gamma}{}_{\nu\beta}\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\nu}\otimes_1\mathrm{d}z^{\beta}$ and is

$$z^{\gamma}g_{\nu\beta}(z^{\alpha}z^3\epsilon_{3\mu\gamma}\mathrm{d}z^{\mu}-\epsilon^{\alpha}{}_{\gamma 3}\mathrm{d}z^3)\wedge\mathrm{d}z^{\nu}\otimes_1\mathrm{d}z^{\beta}+z^{\alpha}g_{\mu\gamma}\mathrm{d}z^{\mu}\wedge(z^{\gamma}z^3\epsilon_{3\nu\beta}\mathrm{d}z^{\nu}-\epsilon^{\gamma}{}_{\beta 3}\mathrm{d}z^3)\otimes_1\mathrm{d}z^{\beta}$$

The second term, which given in components is

$$z^{\alpha}g_{\mu\gamma}(z^{\gamma}z^3\epsilon_{3\nu\beta}+\frac{1}{z^3}\epsilon^{\gamma}{}_{\beta}\delta_{\delta\nu}z^{\delta})$$

can be shown to be symmetric in μ,ν and therefore vanishes, whereas the first can be expanded and simplified to give

$$z^{\gamma}g_{\nu\beta}(z^{\alpha}z^3\epsilon_{3\mu\gamma}\mathrm{d}z^{\mu}-\epsilon^{\alpha}{}_{\gamma 3}\mathrm{d}z^3)\wedge\mathrm{d}z^{\nu}\otimes_1\mathrm{d}z^{\beta}=-\frac{1}{(z^3)^2}(1-(z^3)^2)\mathrm{Vol}\otimes_1\mathrm{d}z^{\alpha}-2\frac{z^{\alpha}}{z^3}\mathrm{Vol}\otimes_1\mathrm{d}z^3$$

Now, taking together the above terms gives the semiclassical Riemann tensor as

$$\mathrm{Riem}_1(\mathrm{d}z^{\alpha})=-\frac{1}{2}\widehat{R}^{\alpha}{}_{\beta\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\otimes_1\mathrm{d}x^{\beta}+\frac{\lambda}{2(z^3)^2}(1+(z^3)^2)\mathrm{Vol}\otimes_1\mathrm{d}z^{\alpha}$$

Where the classical Riemann tensor is $R^{\iota}{}_{\gamma\mu\nu}\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\nu}\otimes_1\mathrm{d}z^{\gamma}=\mathrm{d}z^{\iota}\wedge g$. This is the same result as the general tensorial calculation using (2.18), as a useful check.

For the Ricci tensor, the form of the quantum lift from Proposition 2.4 is

$$i_1(\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\nu})=\frac{1}{2}\left(\mathrm{d}z^{\mu}\otimes_1\mathrm{d}z^{\nu}-\mathrm{d}z^{\nu}\otimes_1 z^{\mu}\right)+\lambda I(\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\nu})$$

The functorial choice here comes out as $I(\mathrm{d}z^{\mu}\wedge\mathrm{d}z^{\nu})=0$, but we leave this general. In 2D the lift map has three independent components which, in tensor notation, we parametrize as $\alpha:=I^{12}{}_{11}$, $\beta:=I^{12}{}_{22}$ and $\gamma:=I^{12}{}_{12}$, with the remaining components being related by symmetry. Then the tensorial formula (2.21) gives us

$$\mathrm{Ricci}_1=-\frac{1}{2}g_1-\frac{3\lambda}{2}\widetilde{\mathrm{Vol}}$$
$$-\frac{\lambda}{(z^3)^2}\left((\alpha z^1z^2+\gamma((z^2)^2-1))\mathrm{d}z^1\otimes_1\mathrm{d}z^1-(\beta z^1z^2+\gamma((z^1)^2-1))\mathrm{d}z^2\otimes_1\mathrm{d}z^2\right.$$
$$\left.+(\gamma z^1z^2+\alpha((z^1)^2-1))\mathrm{d}z^1\otimes_1\mathrm{d}z^2-(\gamma z^1z^2+\beta((z^2)^2-1))\mathrm{d}z^2\otimes_1\mathrm{d}z^1\right)$$

Next, following our general method, we require I to be such that $\wedge_1\mathrm{Ricci}_1=0$, i.e. quantum symmetric. This results in the constraint

$$\gamma=-\frac{1}{4z^1z^2}\left(3z^3+2\alpha((z^1)^2-1)+2\beta((z^2)^2-1)\right)$$

with α and β undetermined. We also want Ricci_1 to be hermitian or 'real' in the sense $\mathrm{flip}(*\otimes *)\mathrm{Ricci}_1=\mathrm{Ricci}_1$ which already holds for $-\frac{1}{2}g_1$. Since λ is imaginary this requires the matrix of coefficents in the order λ terms displayed above to be antisymmetric as all tensors are real. This imposes three more constraints which are fortunately not independent and give us a unique suitable lift, namely with

$$\alpha=\frac{3}{4z^3}(1-(z^2)^2),\quad \beta=\frac{3}{4z^3}(1-(z^1)^2),\quad \gamma=\frac{3}{4}\frac{z^1z^2}{z^3}.$$

The result (and similarly in any rotated coordinate chart) is

$$i_1(\mathrm{d}z^1\wedge\mathrm{d}z^2)=\frac{1}{2}\left(\mathrm{d}z^1\otimes_1\mathrm{d}z^2-\mathrm{d}z^2\otimes_1 z^1\right)-\frac{3\lambda}{4z^3}g$$

$$\mathrm{Ricci}_1=-\frac{1}{2}g_1$$

where the latter in our conventions is analogous to the classical case. And from this or from (2.23) we get the quantum scalar curvature

$$S_1=-\frac{1}{2}\widehat{S},\quad \widehat{S}=\widehat{R}_{\mu\nu}g^{\mu\nu}=2$$

the same as classically in our conventions, so this has no corrections at order λ. As remarked in the general theory, the quantum Ricci scalar is independent of the choice of lift I.

We also find no correction to the Laplacian at order λ since the classical Ricci tensor is proportional to the metric hence the contraction in Theorem 2.3 gives $\omega^{\alpha\beta}(\widehat{\nabla}_\beta\mathrm{d}f)_\alpha$ which factors through $\widehat{\nabla}\wedge\mathrm{d}f=0$ due to zero torsion of the Levi-Civita connection.

We close with some other comments about the model. In fact the parameter λ in this model is dimensionless and if we want to have the usual finite-dimensional 'spin j' representations of our algebra then we need

$$\lambda=\imath/\sqrt{j(j+1)}$$

for some natural number j as a quantisation condition on the parameter. Our reality conventions require λ imaginary. It is also known from [7] that this differential algebra arises from twisting by a cochain at least to order λ^2 *but* in such a way that the twisting also induces the correct differential structure at order λ, i.e. as given by the Levi-Civita connection. We take $U(so_{1,3})$ with generators and relations

$$[M_i,M_j]=\epsilon_{ijk}M_k,\quad [M_i,N_j]=\epsilon_{ijk}N_k,\quad [N_i,N_j]=-\epsilon_{ijk}M_k$$

acting on the classical z^i (i.e. converting [7] to the coordinate algebra) as,

$$M_i\triangleright z^j=\epsilon_{ijk}z^k,\quad N_i\triangleright z^j=z^iz^j-\delta_{ij}.$$

This is the action of $so_{1,3}$ on the 'sphere at infinity'. The cochain we need is then[7]

$$F^{-1}=1+\lambda f+\frac{\lambda^2}{2}f^2+\cdots,\quad f=\frac{1}{2}M_i\otimes N_i$$

where the higher terms are conjectured to exist in such a way that the algebra remains associative at all orders (and gives the quantisation of S^2 as a quotient of $U(su_2)$). On the other hand cochain twisting extends the differential calculus to all orders as a graded quasi-algebra in the sense of [8]. Specifically, if we start with the classical algebra and exterior algebra on the sphere, the deformed products are

$$z^i\bullet z^j=(F^{-1}\triangleright z^i)(F^{-2}\triangleright z^j)=z^iz^j+\frac{\lambda}{2}\epsilon_{ijk}z^k$$

$$z^i\bullet\mathrm{d}z^j=(F^{-1}\triangleright z^i)\mathrm{d}F^{-2}\triangleright z^j=z^i\mathrm{d}z^j+\frac{\lambda}{2}z^j\epsilon_{imn}z^m\mathrm{d}z^n$$

$$\mathrm{d}z^j\bullet z^i=(F^{-1}\triangleright\mathrm{d}z^j)\mathrm{d}F^{-2}\triangleright z^i=(\mathrm{d}z^j)z^i-\frac{\lambda}{2}z^i\epsilon_{jmn}z^m\mathrm{d}z^n-\frac{\lambda}{2}\epsilon_{ijm}\mathrm{d}z^m$$

to order λ, giving relations

$$[z^i, \mathrm{d}z^j]_\bullet = \frac{\lambda}{2}((z^i\epsilon_{jmn} + z^j\epsilon_{imn})z^m\mathrm{d}z^n + \epsilon_{ijm}\mathrm{d}z^m) = \lambda z^j\epsilon_{jmn}z^m\mathrm{d}z^n$$

in agreement with the quantisation of the calculus by the Levi-Civita connecton. For the last step we let

$$w^i = \epsilon^i{}_{jk}z^j\mathrm{d}z^k.$$

and note that classically $z^iw^j\epsilon_{ijk} = -\mathrm{d}z^k$ using the differential of the sphere relation and hence $z^iw^j - z^jw^i = -\epsilon_{ijk}\mathrm{d}z^k$, which we use. This twisting result in [7] is in contrast to other cochain twist or deformation theory quantisations such as in [32], which consider only the coordinate algebra. It means that although the differential calculus is not associative at order λ^2, corresponding to the curvature of the sphere, different brackets are related via an associator and hence strictly controlled. One can then twist other aspects of the noncommutative geometry using the formalism of [8], see also more recently [3].

To get a sense of how these equations fit together even though nonassociative, we work now in the quantum algebra so *from now till the end of the section all products are deformed ones*. We have the commutation relations

$$[z^i, z^j] = \lambda\epsilon^{ij}{}_kz^k, \quad [z^i, \mathrm{d}z^j] = \lambda w^iz^j$$

to order λ. Then, if we apply d to the first relation we have

$$\lambda\epsilon_{ijk}\mathrm{d}z^k = [\mathrm{d}z^i, z^j] + [z^i, \mathrm{d}z^j] = \lambda(w^iz^j - w^jz^i) = \lambda\epsilon_{jik}\epsilon_{kmn}z^mw^n$$

$$= -\lambda\epsilon_{ijk}\epsilon_{kmn}z^m\epsilon_{nab}z^a\mathrm{d}z^b = -\lambda\epsilon_{ijk}(\delta^k_a\delta^m_b - \delta^k_b\delta^m_a)z^mz^a\mathrm{d}z^b = \lambda\epsilon_{ijk}\mathrm{d}z^k - \lambda z^m\epsilon_{ijk}z^k\mathrm{d}z^m$$

which confirms that $\sum z^m\mathrm{d}z^m = O(\lambda)$ (which is to be expected since it is zero classically). In fact we only need the commutation relations for $i, j = 1, 2$ to arrive at this deduction. Moreover,

$$0 = \mathrm{d}(\sum_m z^mz^m) = 2z^m\mathrm{d}z^m - \lambda w^mz^m + [\mathrm{d}z^3, z^3] + \lambda w^3z^3$$

and $w^mz^m = O(\lambda)$ since zero classically, which tells us that

$$[z^3, \mathrm{d}z^3] = \lambda w^3z^3 + 2z^m\mathrm{d}z^m.$$

Hence $z^m\mathrm{d}z^m = 0$ at order λ if the z^3 commutation relations hold as claimed. In fact assuming only the $i, j = 1, 2$ commutation relations one can deduce (so long as z^3 is invertible) that $[z^3, \mathrm{d}z^j] = \lambda w^3z^j$ for $j = 1, 2$ by looking at $[(z^3)^2, \mathrm{d}z^i] = 2z^3[z^3, \mathrm{d}z^i]$ on the one hand and using the radius relation on the other hand. From this and $\lambda\epsilon_{i3k}\mathrm{d}z^k = [\mathrm{d}z^i, z^3] + [z^i, \mathrm{d}z^3]$ we deduce that $[z^i, \mathrm{d}z^3] = \lambda w^iz^3$ as claimed. Then by the same calculation as for the $[z^3, \mathrm{d}z^i]$ relation we can deduce $[z^3, \mathrm{d}z^3] = \lambda w^3z^3$ as well. Thus, we have internal consistency of the quantum algebra relations even if we do not have associativity of the relations involving the $\mathrm{d}z^i$.

3. Semiquantum FLRW model

We will use both Cartesian and spatially polar coordinates t, r, θ, ϕ whereby $\mathrm{d}^2\Omega = \sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi$ is the unit sphere metric. It is already known from [10] that for a bivector ω to be rotationally invariant leads in polars to

$$\omega^{23} = \frac{f(t,r)}{\sin\theta} = -\omega^{32}, \quad \omega^{01} = g(t,r) = -\omega^{10} \tag{3.1}$$

for some functions f, g and other components zero. Our approach is to solve (2.2) for S using the above form of ω and $\widehat{\nabla}$ for the chosen metric, which in the present section is the spatial flat FLRW one

$$g = -\mathrm{d}t \otimes \mathrm{d}t + a(t)^2(\mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}^2\Omega)$$

with

$$\begin{aligned}
&\widehat{\Gamma}^0_{11} = \dot{a}a, \quad \widehat{\Gamma}^0_{22} = \dot{a}ar^2, \quad \widehat{\Gamma}^0_{33} = \dot{a}ar^2\sin^2(\theta),\\
&\widehat{\Gamma}^1_{01} = \frac{\dot{a}}{a}, \quad \widehat{\Gamma}^1_{22} = -r, \qquad \widehat{\Gamma}^1_{33} = -r\sin^2(\theta)\\
&\widehat{\Gamma}^2_{02} = \frac{\dot{a}}{a}, \quad \widehat{\Gamma}^2_{21} = \tfrac{1}{r}, \qquad \widehat{\Gamma}^2_{33} = -\sin(\theta)\cos(\theta)\\
&\widehat{\Gamma}^3_{03} = \frac{\dot{a}}{a}, \quad \widehat{\Gamma}^3_{31} = \tfrac{1}{r}, \qquad \widehat{\Gamma}^3_{23} = \cot(\theta)
\end{aligned}$$

Remarkably, if ω is generic in the sense that the functions a, f, g are algebraically independent and invertible then it turns out that one can next solve the Poisson-compatibility condition (2.2) for S *uniquely* using computer algebra. This is relevant if we drop the requirement (2.3) that ω obeys the Jacobi identity which is to say if we allow the coordinate algebra to be nonassociative at order λ^2 and if we drop (2.14) which is to say we allow a possible quantum effect where $\nabla_1 g_1 = O(\lambda)$ in its antisymmetric part. Such a theory appears quite natural for this reason, but for the present purposes we do want to go further and impose (2.3) as well as the condition (2.14) for the existence of a fully quantum Levi-Civita connection.

Proposition 3.1. *In the FLRW spacetime with spherically symmetric Poisson tensor, a Poisson-compatible connection obeying* (2.14) *and* (2.3) *requires up to normalisation that* $g(r,t) = 0$ *and* $f(r,t) = 1$. *The contorsion tensor in this case is*

$$S_{022} = a\dot{a}r^2, \quad S_{122} = a^2 r, \quad S_{033} = a\dot{a}r^2\sin^2(\theta), \quad S_{133} = a^2 r\sin^2(\theta)$$

$$S_{120} = S_{123} = S_{223} = S_{320} = S_{130} = S_{132} = S_{230} = S_{233} = 0$$

up to the outer antisymmetry of $S_{\mu\nu\gamma}$. *The remaining components* $S_{\mu 0\nu}$, $S_{\mu 1\nu}$ *are undetermined but are irrelevant to the combination* $\omega^{\alpha\beta}\nabla_\beta$ *(the contravariant connection), which is uniquely determined.*

Proof. As already noted in [10] for ω of the rotationally invariant form (3.1) to obey (2.3) comes down to

$$g\partial_t f = g\partial_r f = 0 \tag{3.2}$$

which tells us that either $f = k$ a constant or $g = 0$. We examine the former case, then the Poisson compatibility condition (2.2) becomes

$$S_{201} = 0, \quad S_{301} = 0, \quad S_{001}g - \partial_r g = 0, \quad S_{314} = 0, \quad S_{233} = 0, \quad S_{322} = 0$$

$$S_{031}kr^2a^2 + S_{112}g\sin(\theta) = 0, \quad r^2a^2k\sin(\theta)S_{021} + S_{311}g = 0, \quad a\dot{a}\sin(\theta)g + kS_{231} = 0$$

$$S_{203}g - r^2\sin(\theta)(ra^2 - S_{231}) = 0, \quad S_{002}a^2g + S_{112}g = 0, \quad \sin(\theta)g + S_{230}kr = 0$$

$$kr^4a^2\sin(\theta)a\dot{a} + S_{312}g + kS_{022} = 0, \quad a^2r^3k\sin^2(\theta) + S_{203}g + kS_{331} = 0$$

$$r^4a^3\sin^2(\theta)S_{033} - S_{312} = 0, \quad \sin(\theta)g - rkS_{320} = 0, \quad kr^2S_{031} - S_{002}g\sin(\theta) = 0$$

$$a^2S_{003} - S_{322} = 0, \quad S_{021}r^2\sin(\theta) + S_{003} = 0, \quad 2ka\dot{a}r^2\sin(\theta) - S_{022}\sin(\theta) - kS_{033} = 0$$

$$ga\dot{a}\sin(\theta) - kS_{321} = 0, \quad r^2a^2\partial_t g + r^2a\dot{a}g + S_{012} = 0, \quad (2ra^2 - S_{122})k\sin^2(\theta) + S_{331} = 0.$$

This is not enough to determine all the components of S and hence ∇ but determines enough of them for the Ricci 2-form to be uniquely determined for $k \neq 0$, as

$$\mathcal{R}_{01} = \frac{1}{r^2}\left(5r^2a\dot{a}\partial_t g - \dot{a}^2 + gr^2a\dot{a} + \partial_t^2 g r^2 a\dot{a} - \partial_r^2 g r^2 - 2r\partial_r g + 6g\right)$$

$$\mathcal{R}_{23} = \frac{1}{kr^2}\sin(\theta)\left(k^2r^4a^2 + ga^2r^2 - g^2\right).$$

We can now impose the Levi-Civita condition, using the *Physics* package in *Maple*, to expand (2.14) and solve for g simultaneously with the above requirements of (2.2) (details omitted). This results in $g = 0$ as the only unique solution permitting Poisson compatibility and a quantum Levi-Civita connection for f a (nonzero) constant. The case $f = 0$ also has this conclusion and we exclude this so as to exclude the unquantized case $\omega = 0$ in our analysis. We now go back and examine the second case, setting $g = 0$ and leaving f arbitrary. Now (2.2) includes

$$\partial_r f = 0, \quad \partial_t f = 0$$

independently of the contorsion tensor. Hence this takes us back to $g = 0$ and f constant again. We can absorb the latter constant in λ, i.e. we take $k = 1$ up to the overall normalisation of ω. Then the above-listed content of (2.2) setting $k = 1$ and $g = 0$ gives us the values of S and 12 undetermined components as stated.

In the process above we also solved (2.14) so this holds for the stated S with $f = 1$ and $g = 0$. As this depended on a Maple solution, we check it analytically, setting

$$Q_{\gamma\mu\nu} = \omega^{\alpha\beta}\, g_{\rho\sigma}\, S^{\sigma}{}_{\beta\nu}(R^{\rho}{}_{\mu\gamma\alpha} + \nabla_\alpha S^{\rho}{}_{\gamma\mu}) - \omega^{\alpha\beta}\, g_{\rho\sigma}\, S^{\sigma}{}_{\beta\mu}(R^{\rho}{}_{\nu\gamma\alpha} + \nabla_\alpha S^{\rho}{}_{\gamma\nu}) \tag{3.3}$$

while from the above with $k = 1, g = 0$, the Ricci two-form for our solution is

$$\mathcal{R} = -\frac{1}{2}ka^2r^2\sin(\theta)\mathrm{d}\theta\wedge\mathrm{d}\phi \tag{3.4}$$

and is independent of the undetermined components. This allows us to compute $\widehat{\nabla}_0\mathcal{R}_{\mu\nu} = \widehat{\nabla}_1\mathcal{R}_{\mu\nu} = 0$ as well as

$$\widehat{\nabla}_2\mathcal{R}_{\mu\nu} = ark\sin(\theta)\begin{pmatrix} 0 & 0 & -\dot{a}r & 0\\ 0 & 0 & -a & 0\\ \dot{a}r & a & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad \widehat{\nabla}_3\mathcal{R}_{\mu\nu} = ark\sin(\theta)\begin{pmatrix} 0 & 0 & \dot{a}r & 0\\ 0 & 0 & a & 0\\ -\dot{a}r & -a & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Further calculation yields $Q_{0\mu\nu} = Q_{1\mu\nu} = 0$ and

$$Q_{2\mu\nu} = ark\sin(\theta)\begin{pmatrix} 0 & 0 & -\dot{a}r & 0\\ 0 & 0 & -a & 0\\ \dot{a}r & a & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad Q_{3\mu\nu} = ark\sin(\theta)\begin{pmatrix} 0 & 0 & \dot{a}r & 0\\ 0 & 0 & a & 0\\ -\dot{a}r & -a & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Substituting into (2.14) we see that this holds in the form $\frac{1}{2}Q_{\gamma\alpha\beta} - \frac{1}{2}\widehat{\nabla}_\gamma\mathcal{R}_{\mu\nu} = 0$. □

Thus we see that if we want the Poisson bracket to obey the Jacobi identity so as to keep an associative coordinate algebra and if we want a full quantum Levi-Civita connection without on $O(\lambda)$ correction to the antisymmetric part of the quantum metric compatibility tensor $\nabla_1 g_1$, then rotational invariance forces us to a model in which time is central and in which the other commutation relations are also determined uniquely from $\omega^{\alpha\beta}\nabla_\beta$. To work these out it is convenient (though not essential) to work with the angular variables in terms of $z^i = x^i/r$ as redundant

unit sphere variables at each r,t, with $\mathrm{d}z^i = \frac{1}{r}\mathrm{d}x^i - \frac{x^i}{r^2}\mathrm{d}r$. Now, using the contorsion tensor above Christoffel symbols of the 'quantising' connection come out as

(3.5)
$$\Gamma^0_{\mu\nu} = \begin{pmatrix} 0 & -S^0{}_{01} & -S^0{}_{02} & -S^0{}_{03} \\ 0 & a\dot{a} - S^0{}_{11} & -S^0{}_{12} & -S^0{}_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma^1_{\mu\nu} = \begin{pmatrix} S^1{}_{00} & \frac{\dot{a}}{a} S^1{}_{02} & S^1{}_{03} \\ \frac{\dot{a}}{a} + S^1{}_{10} & 0 & S^1{}_{12} & S^1{}_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} S^2{}_{00} & S^2{}_{01} & \frac{\dot{a}}{a} & S^2{}_{03} \\ S^2{}_{10} & S^2{}_{11} & r^{-1} & S^2{}_{12} \\ 0 & 0 & \frac{z^1(1-(z^2)^2)}{(z^3)^2} & \frac{(z^1)^2 z^2}{(z^3)^2} \\ 0 & 0 & \frac{(z^1)^2 z^2}{(z^3)^2} & \frac{z^1(1-(z^1)^2)}{(z^3)^2} \end{pmatrix}, \Gamma^3_{\mu\nu} = \begin{pmatrix} S^3{}_{00} & S^3{}_{01} & S^3{}_{03} & \frac{\dot{a}}{a} \\ S^3{}_{10} & S^3{}_{11} & S^3{}_{12} & r^{-1} \\ 0 & 0 & \frac{z^2(1-(z^2)^2)}{(z^3)^2} & \frac{z^1(z^2)^2}{(z^3)^2} \\ 0 & 0 & \frac{z^1(z^2)^2}{(z^3)^2} & \frac{z^2(1-(z^1)^2)}{(z^3)^2} \end{pmatrix}$$

while the bimodule relations are independent of the undetermined components of S and come out as

$$[z^i, z^j] = \lambda \epsilon^{ij}{}_k z^k, \quad [z^i, \mathrm{d}z^j] = \lambda z^j \epsilon^i{}_{mn} z^m \mathrm{d}z^n.$$

Our quantum algebra at order λ is thus classical in the r,t directions and a standard fuzzy sphere as in Section 2.5 in the angular ones. We also have

$$[r, x^i] = 0, \quad [r, \mathrm{d}x^i] = 0, \quad [x^i, \mathrm{d}r] = 0$$

so that $r, t, \mathrm{d}t, \mathrm{d}r$ are all central. The undetermined contorsion components do not enter these relations from (2.5) because only ω^{23} is nonzero so contraction with the Christoffel symbols selects only the $\Gamma^\mu{}_{2\nu}$ and $\Gamma^\mu{}_{3\nu}$ components which depend on only the corresponding S components.

For the rest of this section for the sake of brevity, we shall concentrate on the case where the undetermined and irrelevant S components are all set to zero, returning later when analyzing general spherically symmetric metrics to see what happens when these are included. For the record, changing to Cartesians, the nonzero bimodule relations are

$$[x^i, x^j] = \lambda r \epsilon^{ij}{}_k x^k, \quad [x^i, \Omega^j] = \frac{\lambda}{r} x^j \epsilon^i{}_{mn} x^m \Omega^n$$

by letting $\mathrm{d}z^i = \Omega^i / r$ while our choice of the undetermined contorsion tensor components allows us to write down a nice expression for the 'quantising' connection

$$\Gamma^i{}_{jk} = -\frac{x^m}{r^2} \epsilon^i{}_{kn} \epsilon^n{}_{jm}, \quad \Gamma^i{}_{0j} = \frac{\dot{a}}{a} \delta^i{}_j \quad \Gamma^i{}_{j0} = 0$$

The torsion comes out as

$$T^i{}_{jk} = \frac{x^m}{r^2} \epsilon^i{}_{mn} \epsilon^n{}_{jk}, \quad T^i{}_{0j} = \frac{\dot{a}}{a} \delta^i{}_j$$

and the Riemann, Ricci and scaler curvatures are

$$R^i{}_{jkl} = \frac{1}{r^2} \epsilon^i{}_{jm} \epsilon^m{}_{kl} + \frac{1}{r^4} \left(x_j x^m \epsilon^i{}_{mn} \epsilon^n{}_{kl} + x^i x^m \epsilon_{jmn} \epsilon^n{}_{kl} \right) \tag{3.6}$$

$$R_{ij} = \frac{1}{r^4} (\delta_{ij} r^2 - x_i x_j), \quad S = \frac{2}{a^2 r^2} \tag{3.7}$$

and it should be noted that $R^0{}_{jkl} = R^i{}_{0kl} = R^i{}_{j0l} = 0$ and $R_{i0} = R_{0i} = 0$.

3.1. **Construction of quantum metric and quantum Levi-Civita connection.** Having solved for a Poisson bracket and Poisson compatible metric-compatible connection we are in a position to read off, according to the theory in [10], the full exterior algebra and the quantum metric to lowest order. First compute

$$H^{ij} = -\frac{1}{2r^3}\left(\epsilon^i{}_{nk}x_m x^j x^k - r^2\epsilon^i{}_{nk}\delta^j{}_m x^k\right)\mathrm{d}x^n\wedge\mathrm{d}x^m \tag{3.8}$$

from which we get

$$\mathcal{R}_{mn} = \frac{a^2}{r}\epsilon_{mnk}x^k,\quad \mathcal{R} = \frac{a^2}{2r}\epsilon_{mnk}x^k\mathrm{d}x^n\wedge\mathrm{d}x^m \tag{3.9}$$

As with the curvature, all time components are equal to zero. From Γ and H^{ij} we have

$$\mathrm{d}x^i\wedge_1\mathrm{d}x^j = \mathrm{d}x^i\wedge\mathrm{d}x^j + \frac{\lambda}{2r^3}\left(r^2\epsilon^i{}_{nm}x^j + r^2\epsilon^{ij}{}_m x_n + r^2\epsilon^i{}_{nk}\delta^j{}_m x^k + \epsilon^i{}_{nk}x^k x^j x_m\right)\mathrm{d}x^m\wedge\mathrm{d}x^n$$

$$\mathrm{d}r\wedge_1\mathrm{d}x^i = \mathrm{d}r\wedge\mathrm{d}x^i,\quad \mathrm{d}t\wedge_1\mathrm{d}x^i = \mathrm{d}t\wedge\mathrm{d}x^i,\quad \mathrm{d}x^i\wedge_1\mathrm{d}t = \mathrm{d}x^i\wedge\mathrm{d}t \tag{3.10}$$

$$\{\mathrm{d}x^i,\mathrm{d}x^j\}_1 = \frac{\lambda}{r^3}\left(r^2\epsilon^i{}_{nm}x^j + r^2\epsilon^{ij}{}_m x_n + r^2\epsilon^i{}_{nk}\delta^j{}_m x^k + \epsilon^i{}_{nk}x^k x^j x_m\right)\mathrm{d}x^m\wedge\mathrm{d}x^n$$

Similary, from Γ and $\mathcal{R}$ we compute g_1 from (2.9). Remarkably, the correction term $\frac{\lambda}{2}\omega^{\alpha\beta}g_{\mu\rho}\Gamma^\rho{}_{\alpha\kappa}\Gamma^\kappa{}_{\beta\nu}$ exactly cancels the $\lambda\mathcal{R}_{\mu\nu}$ so that $g_{1\mu\nu} = g_{\mu\nu}$ and

$$g_1 = g_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu \tag{3.11}$$

Moreover since the components $g_{\mu\nu}$ depend only on time, we also have that $g_{1\mu\nu} = \tilde{g}_{1\mu\nu}$. It is a nice check to verify that $\wedge_1(g_1) = 0$ is satisfied as it must from our general theory. The second version of the metric is subtly different and equality depends on the form of the FLRW metric. One can also compute

$$\omega^{\mu\nu}[\nabla_\mu,\nabla_\nu]T^\alpha{}_{\beta\gamma} = 0,\quad \omega^{\mu\nu}[\nabla_\mu,\nabla_\nu]S^\alpha{}_{\beta\gamma} = 0$$

$$\widehat{\nabla}_i\mathcal{R}_{mn} = -\frac{a^2}{r^3}\left(\epsilon_{mnk}x^k x_i - r^2\epsilon_{mni}\right) - \frac{a\dot{a}}{r}\left(\epsilon_{ink}x^k - \epsilon_{imk}x^k\right)$$

and see once again that (2.14) holds as it must by construction in Proposition 3.1.

Hence a quantum Levi-Civita connection for g_1 exists by the theory from [10] and from Lemma 2.2 we find it to be

$$\nabla_1(\mathrm{d}x^\iota) = \widehat{\Gamma}^\iota_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu$$

which, like the quantum metric earlier, keeps its undeformed coefficients in the coordinate basis *if* we keep all coefficients to the left and use $\otimes_1$. The theory in [10] ensures that this is quantum torsion free and quantum metric compatible as a bimodule connection with generalised braiding σ_1 from (2.16) which computes as

(3.12)
$$\begin{aligned}\sigma_1(\mathrm{d}x^a\otimes_1\mathrm{d}x^b) &= \mathrm{d}x^b\otimes_1\mathrm{d}x^a + \tfrac{\lambda}{r^3}\left(\epsilon_k{}^{bk}x^k x_m x_n + \epsilon_{kn}{}^b x^k x^a x_m + 2r^2\epsilon_k^{ab}x_k\delta_{mn}\right.\\ &\quad \left. + 2r^2\epsilon^b{}_{mk}x_k\delta^a{}_n - r^2\epsilon^a{}_{mk}x^k\delta^b{}_m + r^2\epsilon_{bck}x^k\delta^{ab}\right)\mathrm{d}x^n\otimes_1\mathrm{d}x^m\\ \sigma_1(\mathrm{d}t\otimes_1\mathrm{d}x^a) &= \mathrm{d}x^a\otimes_1\mathrm{d}t\\ \sigma_1(\mathrm{d}x^a\otimes_1\mathrm{d}t) &= \mathrm{d}t\otimes_1\mathrm{d}x^a\end{aligned}$$

It is a reassuring but rather nontrivial check to verify directly from our results for ∇_1, σ_1, g_1 that $\nabla_1 g_1 = 0$ as implied by the general theory in [10]. Lastly, we compute the quantum lift map from Proposition 2.4 as

$$\begin{aligned} i_1(\mathrm{d}x^a \wedge \mathrm{d}x^b) &= \frac{1}{2}\left(\mathrm{d}x^a \otimes_1 \mathrm{d}x^b - \mathrm{d}x^b \otimes_1 \mathrm{d}x^a\right) \\ &\quad - \frac{\lambda}{4r}\epsilon^{ab}{}_m x_n \left(\mathrm{d}x^m \otimes_1 \mathrm{d}x^n + \mathrm{d}x^n \otimes_1 \mathrm{d}x^m\right) \\ i_1(\mathrm{d}t \wedge \mathrm{d}x^\alpha) &= \frac{1}{2}\left(\mathrm{d}t \otimes_1 \mathrm{d}x^\alpha - \mathrm{d}x^\alpha \otimes_1 \mathrm{d}t\right) \\ i_1(\mathrm{d}x^\alpha \wedge \mathrm{d}t) &= \frac{1}{2}\left(\mathrm{d}x^\alpha \otimes_1 \mathrm{d}t - \mathrm{d}t \otimes_1 \mathrm{d}x^\alpha\right) \end{aligned} \tag{3.13}$$

with we have taken the functorial choice $I = 0$.

3.2. **Laplace operator and Curvature Tensors.** We first observe that $[\mathrm{d}x^m, g_{mn}] = 0$ for the FLRW metric since either the coefficients g_{mn} depend only on t or are constant in our basis. Hence the inverse metric is simply $(\mathrm{d}x^a, \mathrm{d}x^b)_1 = g^{ab}$ undeformed similarly to the coefficients of g_1, since then

$$\begin{aligned} (f \bullet \mathrm{d}x^\alpha, g_{\mu\nu} \bullet \mathrm{d}x^\mu)_1 \bullet \mathrm{d}x^\nu &= (f \bullet \mathrm{d}x^a, \mathrm{d}x^\mu \bullet g_{\mu\nu})_1 \bullet \mathrm{d}x^\nu - (f \bullet \mathrm{d}x^\alpha, [\mathrm{d}x^\mu, g_{\mu\nu}])_1 \bullet \mathrm{d}x^\nu \\ &= f \bullet (\mathrm{d}x^\alpha, \mathrm{d}x^\mu)_1 \bullet g_{\mu\nu} \bullet \mathrm{d}x^\nu = f \bullet \mathrm{d}x^\alpha \end{aligned}$$

as required, where we also need that $\{g^{am}, g_{mn}\} = 0$ which holds for the FLRW metric. Similarly on the other side. It follows that the quantum dimension is the same as the classical dimension, namely 4, in our model. Similarly, because $\nabla_1, g_1, (\ ,\ )_1$ also have their classical form, from Theorem 2.3 we get that

$$\Box_1 f = g^{\alpha\beta}\left(f_{,\alpha\beta} + f_{,\gamma}\widehat{\Gamma}^\gamma{}_{\alpha\beta}\right)$$

is also undeformed on the underlying vector space. We used that ∇_1 is a left connection. We can also calculate Riemann tensor using (2.17) from which we see that corrections come from $\wedge_1$.

$$\begin{aligned} \mathrm{Riem}_1(\mathrm{d}x^i) &= (\mathrm{d}\otimes \mathrm{id})\nabla_1 \mathrm{d}x^i - \widehat{\Gamma}^i{}_{\mu\gamma}\mathrm{d}x^\mu \wedge_1 \widehat{\Gamma}^\gamma{}_{\nu\alpha}\mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\alpha \\ &= -\frac{1}{2}\widehat{R}^i{}_{\alpha\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\alpha - \frac{\lambda\dot{a}^2}{2r^3}\epsilon^i{}_{nk}x^k x_j x_m \mathrm{d}x^m \wedge \mathrm{d}x^n \otimes_1 \mathrm{d}x^j \\ &\quad - \frac{\lambda\dot{a}^2}{2r}\left(\epsilon^i{}_{nm}x_j + \epsilon^i{}_{jm}x_n + \epsilon^i{}_{nk}\delta_{jm}x^k\right)\mathrm{d}x^m \wedge \mathrm{d}x^n \otimes_1 \mathrm{d}x^j \\ \mathrm{Riem}_1(\mathrm{d}t) &= -\frac{1}{2}\widehat{R}^0{}_{\alpha\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\alpha \end{aligned}$$

Next step is to calculate Ricci_1 which comes out as

$$\mathrm{Ricci}_1 = -\frac{1}{2}\widehat{R}_{\alpha\beta}\mathrm{d}x^\beta \otimes_1 \mathrm{d}x^\alpha$$

with no corrections to the coefficients in this form. The classical Ricci tensor for the Levi-Civita connection in our conventions is

$$\widehat{\mathrm{Ricci}} = -\frac{1}{2}\left(\left(2\dot{a}^2 + a\ddot{a}\right)\delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j - 3\frac{\ddot{a}}{a}\mathrm{d}t \otimes \mathrm{d}t\right)$$

and Ricci_1 has the same form just with $\otimes_1$. The components again depend only on time, hence are central, which means that $\rho = 0$ as well. It remains to verify that $\wedge_1(\mathrm{Ricci}_1) = 0$ as it should have the same quantum symmetry as g_1. So using

(3.10), we first see that $\mathrm{d}t\wedge_1\mathrm{d}t=0$ leaving (since the coefficients are time dependent they can be neglected here)

$$\delta_{ij}\mathrm{d}x^i\wedge_1\mathrm{d}x^j=\frac{\lambda}{2r^3}\delta_{ij}\left(r^2\epsilon^i{}_{nm}x^j+r^2\epsilon^{ij}{}_m x_n+r^2\epsilon^i{}_{nk}\delta^j{}_m x^k+\epsilon^i{}_{nk}x^k x^j x_m\right)\mathrm{d}x^m\wedge\mathrm{d}x^n=0$$

From (2.23) we calculate the scalar curvature. Since neither the quantum metric or Ricci tensor have any semiclassical correction, it is straightforward to see that the same is true of the Ricci scalar, i.e.

$$S_1=-\frac{1}{2}\widehat{S},\quad \widehat{S}=\widehat{R}_{\mu\nu}g^{\mu\nu}=\frac{6}{a^2}\left(\ddot{a}a+\dot{a}^2\right) \tag{3.14}$$

From Proposition 2.1, the quantum dimension of this model comes out as

$$\dim(M)_1=\dim(M)-\lambda\omega^{\alpha\beta}g^{\mu\nu}{}_{,\alpha}\Gamma_{\nu\beta\mu}=4$$

since the metric depends only on t the $\mathcal{O}(\lambda)$ term vanishes.

4. Semiquantisation of spherically symmetric metrics

4.1. **General analysis for the spherical case.** In the previous section we saw that for a spherically symmetric Poisson tensor, demanding a compatible connection that also satisfied (2.14) results in a unique quantisation at order λ of the FLRW metric. Something similar for the Schwarzschild black hole in [10] suggests a general phenomenon for the spherically symmetric case. We prove in the present section that this is generically true. For the metric we choose a diagonal form

$$g=a^2(r,t)\mathrm{d}t\otimes\mathrm{d}t+b^2(r,t)\mathrm{d}r\otimes\mathrm{d}r+c^2(r,t)(\mathrm{d}\theta\otimes\mathrm{d}\theta+\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi)$$

where a,b,c are arbitrary functional parameters. The Poisson tensor is taken to be the same as in Section 3, once again parameterized by

$$\omega^{23}=\frac{f(t,r)}{\sin\theta}=-\omega^{32},\quad \omega^{01}=g(t,r)=-\omega^{10}$$

The Christoffel symbols for the above metric are

$$\begin{aligned}
&\widehat{\Gamma}^0_{00}=\frac{\partial_t a}{a},\quad \widehat{\Gamma}^0_{01}=\frac{\partial_r a}{a},\widehat{\Gamma}^0_{33}=-\frac{b\partial_t b\sin^2(\theta)}{a^2},\ \widehat{\Gamma}^0_{11}=-\frac{b\partial_t b}{a^2},\widehat{\Gamma}^0_{22}=-\frac{c\partial_t c}{a^2}\\
&\widehat{\Gamma}^1_{00}=-\frac{a\partial_r a}{b^2},\widehat{\Gamma}^1_{11}=\frac{\partial_r b}{b},\widehat{\Gamma}^1_{33}=-\frac{b\partial_r b\sin^2(\theta)}{b^2},\ \widehat{\Gamma}^1_{01}=\frac{\partial_t b}{b},\quad \widehat{\Gamma}^1_{22}=-\frac{c\partial_r c}{b^2}\\
&\widehat{\Gamma}^2_{02}=\frac{\partial_t c}{c},\quad \widehat{\Gamma}^2_{21}=\frac{\partial_r c}{c},\widehat{\Gamma}^2_{33}=-\sin(\theta)\cos(\theta)\\
&\widehat{\Gamma}^3_{03}=\frac{\partial_t c}{c},\quad \widehat{\Gamma}^3_{31}=\frac{\partial_r c}{c},\widehat{\Gamma}^3_{23}=\cot(\theta)
\end{aligned} \tag{4.1}$$

Now for the quantum Levi-Civita connection.

Theorem 4.1. *For a generic spherically symmetric metric with functional parameters a,b,c and spherically symmetric Poisson tensor, the Poisson-compatibility* (2.2) *and the quantum Levi-Civita condition* (2.14) *require up to normalisation that $g(r,t)=0$ and $f(r,t)=1$ and the contorsion tensor components*

$$S_{022}=c\partial_t c,\quad S_{122}=c\partial_r c,\quad S_{033}=c\partial_t c\sin^2(\theta),\quad S_{133}=c\partial_r c\sin^2(\theta)$$

$$S_{120}=S_{123}=S_{223}=S_{320}=S_{130}=S_{132}=S_{230}=S_{233}=0$$

up to the outer antisymmetry of $S_{\mu\nu\gamma}$. The remaining components $S_{\mu 0\nu}$, $S_{\mu 1\nu}$ remain undetermined but do not affect $\omega^{\alpha\beta}\nabla_\beta$, which is unique. The relations of the quantum algebra are uniquely determined to $O(\lambda)$ as those of the fuzzy sphere

$$[z^i, z^j] = \lambda\epsilon^{ij}{}_k z^k, \quad [z^i, \mathrm{d}z^j] = \lambda z^j \epsilon^i{}_{mn} z^m \mathrm{d}z^n$$

as in Section 2.5 and

$$[t, x^\mu] = [r, x^\mu] = 0, \quad [x^\mu, \mathrm{d}t] = [x^\mu, \mathrm{d}r] = 0$$

so that t, r, dt, dr are central at order λ.

Proof. The first part is very similar to the proof of Proposition 3.1 but with more complicated expressions. We once again require that either $f = k$ or $g = 0$ for ω to be Poisson. Taking first $f = k$ and leaving g arbitrary gives the Poisson compatibility condition (2.2) as

$$S^1{}_{02} = 0, \quad S^3{}_{01} = 0, \quad S^3{}_{22} = 0, \quad S^3{}_{10} = 0, \quad S^0{}_{12} = 0, \quad S^3{}_{32} = 0,$$

$$gabS^0{}_{01} + ab\partial_r g + ga\partial_r b + gb\partial_r a = 0, \quad ab^2\partial_t g + abg\partial_t b + b^2 g\partial_t a + a^3 g S^0{}_{11} = 0$$

$$c^2 S^3{}_{31} - b^2 S^1{}_{22} + 2c\partial_r c = 0, \quad kc^2 S^0{}_{31} + gb^2\sin(\theta)S^1{}_{12} = 0, \quad g\partial_t c\sin(\theta) - cS^1{}_{32} = 0,$$

$$a^2 g\sin(\theta)S^3{}_{12} - ka^2 S^0{}_{22} - kc\partial_t c = 0, \quad gb^2 S^3{}_{02}\sin(\theta) + kb^2 S^1{}_{22}\sin(\theta) - kc\partial_r c = 0,$$

$$S^1{}_{12} + S^0{}_{02} = 0, \quad kcS^0{}_{32} + g\partial_r c\sin(\theta) = 0, \quad g\sin(\theta)S^3{}_{11} + kS^0{}_{21} = 0,$$

$$kc\partial_r c + gb^2\sin(\theta)S^3{}_{02} + kc^2 S^3{}_{31} = 0, \quad kd^3\sin(\theta)S^3{}_{21} - gb^2\partial_t c = 0,$$

$$c^2 S^3{}_{30} + a^2 S^0{}_{22} a^2 + 2c\partial_r c = 0, \quad gS^3{}_{00} b^2\sin(\theta) + ka^2 S^0{}_{21} = 0, \quad kc^2 S^0{}_{31} - gb^2 S^0{}_{02} = 0$$

$$kc^3\sin(\theta)S^3{}_{20} - a^2 g\partial_r c = 0, \quad kc\partial_t c - ga^2 S^3{}_{12}\sin(\theta) + kc^2 S^3{}_{30}, \quad a^2 S^3{}_{11} - b^2 S^3{}_{00} = 0$$

Once again, the above is enough to determine $\mathcal{R}$ and can then be solved for g simultaneously with (2.14) using computer algebra (details omitted) assuming that a, b, c are generic in the sense of invertible and not enjoying any particular relations. The only solution is $g(r, t) = 0$ as in the FLRW case. Now, starting over with $g = 0$ and f arbitrary, Poisson compatibility (2.2) gives a number of constraints including

$$\partial_r f = 0, \quad \partial_t f = 0$$

which again forces us back to $g = 0$, $f = k$ (which we set to be 1). Our above reduction of (2.2) setting $g = 0$ and $k = 1$ then gives the contorsion tensor is as stated and by construction we also solved (2.14).

Now we now check (2.14) for this solution directly and independently of the computer algebra (which then does not require a, b, c generic). For this, the generalised Ricci two-form comes out as

$$\mathcal{R} = -\frac{1}{2}c^2\sin(\theta)\mathrm{d}\theta\wedge\mathrm{d}\phi$$

giving us $\widehat{\nabla}_0\mathcal{R}_{\mu\nu} = \widehat{\nabla}_1\mathcal{R}_{\mu\nu} = 0$ as well as

$$\widehat{\nabla}_2\mathcal{R}_{\mu\nu} = c\sin(\theta)\begin{pmatrix} 0 & 0 & -\partial_t c & 0 \\ 0 & 0 & -\partial_r c & 0 \\ \partial_t c & \partial_r c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{\nabla}_3\mathcal{R}_{\mu\nu} = c\sin(\theta)\begin{pmatrix} 0 & 0 & -\partial_t c & 0 \\ 0 & 0 & -\partial_r c & 0 \\ \partial_t c & \partial_r c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Further calculation yields $Q_{0\mu\nu} = Q_{1\mu\nu} = 0$ and

$$Q_{2\mu\nu} = c\sin(\theta)\begin{pmatrix} 0 & 0 & -\partial_t c & 0 \\ 0 & 0 & -\partial_r c & 0 \\ \partial_t c & \partial_r c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_{3\mu\nu} = c\sin(\theta)\begin{pmatrix} 0 & 0 & -\partial_t c & 0 \\ 0 & 0 & -\partial_r c & 0 \\ \partial_t c & \partial_r c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Substituting, see see that (2.14) holds in the form $\frac{1}{2}Q_{\gamma\alpha\beta} - \frac{1}{2}\nabla_\gamma \mathcal{R}_{\mu\nu} = 0$, where Q is the expression (3.3). This we have solved for the contorsion tensor obeying (2.2) and (2.14) for any a, b, c and this gives us $\omega^{\alpha\beta}\nabla_\beta$ uniquely if these are generic.

Next we take the last two local coordinates z^1 and z^2 while identifying $(z^3)^2 = 1 - (z^2)^2 - (z^1)^2$. Then the Poisson tensor becomes

$$\omega = z^3\left(\frac{\partial}{\partial z^1} \otimes \frac{\partial}{\partial z^2} - \frac{\partial}{\partial z^2} \otimes \frac{\partial}{\partial z^1}\right)$$

giving the coordinate algebra as stated. Since only $\omega^{23} = -\omega^{32}$ is nonzero, we also have $\{t, x^\mu\} = \{r, x^\mu\} = 0$. The 'quantising' connection is

$$\nabla \mathrm{d}t = -\frac{\partial_t a}{a}\mathrm{d}t\otimes\mathrm{d}t - \frac{b\partial_t b}{a^2}\mathrm{d}r\otimes\mathrm{d}r - \frac{\partial_r a}{a}(\mathrm{d}r\otimes\mathrm{d}t + \mathrm{d}t\otimes\mathrm{d}r) - S^0{}_{0\mu}\mathrm{d}t\otimes\mathrm{d}x^\mu - S^0{}_{1\mu}\mathrm{d}r\otimes\mathrm{d}x^\mu$$

$$\nabla \mathrm{d}r = -\frac{a\partial_r a}{b^2}\mathrm{d}t\otimes\mathrm{d}t - \frac{\partial_t b}{b}\mathrm{d}r\otimes\mathrm{d}r - \frac{\partial_t b}{b}(\mathrm{d}r\otimes\mathrm{d}t + \mathrm{d}t\otimes\mathrm{d}r) - S^1{}_{0\mu}\mathrm{d}t\otimes\mathrm{d}x^\mu - S^1{}_{1\mu}\mathrm{d}r\otimes\mathrm{d}x^\mu$$

$$\nabla \mathrm{d}z^i = -\frac{\partial_r c}{c}\mathrm{d}r\otimes\mathrm{d}z^i - \frac{\partial_t c}{c}\mathrm{d}t\otimes\mathrm{d}z^i - \delta_{ab}z^i\mathrm{d}z^a\otimes\mathrm{d}z^b - S^i{}_{0\mu}\mathrm{d}t\otimes\mathrm{d}x^\mu - S^i{}_{1\mu}\mathrm{d}r\otimes\mathrm{d}x^\mu$$

due to the Christoffel symbols

$$\Gamma^0_{\mu\nu} = \begin{pmatrix} \frac{\partial_t a}{a} & \frac{\partial_r a}{a} S^0{}_{01} & S^0{}_{02} & S^0{}_{03} \\ \frac{\partial_r a}{a} & \frac{b\partial_t b}{a^2} + S^0{}_{11} & S^0{}_{12} & S^0{}_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma^1_{\mu\nu} = \begin{pmatrix} S^1{}_{00} + \frac{a\partial_r a}{b^2} & \frac{\partial_t b}{b} & S^1{}_{02} & S^1{}_{03} \\ \frac{\partial_t b}{b} + S^1{}_{10} & \frac{\partial_r b}{b} & S^1{}_{12} & S^1{}_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} S^2{}_{00} & S^2{}_{01} & \frac{\partial_t c}{c} & S^2{}_{03} \\ S^2{}_{10} & S^2{}_{11} & \frac{\partial_r c}{c} & S^2{}_{12} \\ 0 & 0 & \frac{z^1(1-(z^2)^2)}{(z^3)^2} & \frac{(z^1)^2 z^2}{(z^3)^2} \\ 0 & 0 & \frac{(z^1)^2 z^2}{(z^3)^2} & \frac{z^1(1-(z^1)^2)}{(z^3)^2} \end{pmatrix}, \Gamma^3_{\mu\nu} = \begin{pmatrix} S^3{}_{00} & S^3{}_{01} & S^3{}_{03} & \frac{\partial_t c}{c} \\ S^3{}_{10} & S^3{}_{11} & S^3{}_{12} & \frac{\partial_r c}{c} \\ 0 & 0 & \frac{z^2(1-(z^2)^2)}{(z^3)^2} & \frac{z^1(z^2)^2}{(z^3)^2} \\ 0 & 0 & \frac{z^1(z^2)^2}{(z^3)^2} & \frac{z^2(1-(z^1)^2)}{(z^3)^2} \end{pmatrix}$$

From (2.5) we immediately see that $[t, \mathrm{d}x^\mu] = [r, \mathrm{d}x^\mu] = 0$. Furthermore,

$$[z^1, \mathrm{d}x^\mu] = -z^3\Gamma^\mu{}_{3\beta}\mathrm{d}x^\beta, \quad [z^2, \mathrm{d}x^\mu] = z^3\Gamma^\mu{}_{2\beta}\mathrm{d}x^\beta$$

so we can read of from the Christoffel symbols that $[x^\mu, \mathrm{d}t] = [x^\mu, \mathrm{d}r] = 0$. Evaluating the nonzero terms gives

$$[z^1, \mathrm{d}z^1] = -\frac{z^1}{z^3}(z^1z^2\mathrm{d}z^1 - ((z^1)^2 - 1)\mathrm{d}z^2), \quad [z^2, \mathrm{d}z^2] = \frac{z^2}{z^3}(z^1z^2\mathrm{d}z^2 - ((z^2)^2 - 1)\mathrm{d}z^1)$$

$$[z^2, \mathrm{d}z^1] = -\frac{z^2}{z^3}(z^1z^2\mathrm{d}z^1 - ((z^1)^2 - 1)\mathrm{d}z^2), \quad [z^1, \mathrm{d}z^2] = \frac{z^1}{z^3}(z^1z^2\mathrm{d}z^2 - ((z^2)^2 - 1)\mathrm{d}z^1)$$

which upon using $\sum_i z^i\mathrm{d}z^i = 0$ becomes $[z^i, \mathrm{d}z^j] = \lambda z^j\epsilon^i{}_{mn}z^m\mathrm{d}z^n$. □

So we see that in generalizing the analysis, we recover the same bimodule structure as in the FLRW case and by extension, that of the fuzzy sphere in Section 2.5. The noncommutativity is purely spacial and confined to spatial 'spherical shells'; the surfaces of fuzzy spheres at each time and each classical radius r. We have

checked directly in the proof of the theorem that this is a solution for all a, b, c while for generic a, b, c we showed that it is the only solution, i.e. we are forced into this form from our assumptions and spherical symmetry. There do in fact exist particular combinations of these metric functional parameters which permit alternative solutions for f and g. In fact we already saw an example in Section 2.4 with the Bertotti-Robinson metric which had $f = 0$ and $g = -r$. To see why this was allowed, we take a brief look at the Poisson compatibility condition (2.2) again, now with arbitrary f and g and note the particular constraint

$$S^0{}_{32} f c + g\partial_r c \sin(\theta) = 0, \quad -S^1{}_{32} f c + g\partial_r c \sin(\theta) = 0$$

It is clear that with c arbitrary, we cannot have $f = 0$ without also having $g = 0$. However, allowing c =constant means we can also take $f = 0$ and g nonzero, as is the case with the Bertotti-Robinson metric. This leads to a different contorsion tensor with a flat ∇ and in fact this exceptional model was solved using algebraic methods in [28] including the quantum Levi-Civita connection to all orders in λ.

Proceeding with our generic spherically symmetric metric, for brevity we define

$$F_1 = \frac{1}{a^3 b}\left(a^2 b\partial_r^2 a - ab^2\partial_t^2 b + b^2\partial_t b\partial_t a - a^2\partial_r b\partial_r a\right), \quad F_2 = \frac{a^2}{b^2}F_1$$

$$F_3 = \frac{c}{a^3 b}\left(a\partial_t b\partial_r c - ba\partial_t\partial_r c + b\partial_t c\partial_r a\right), \quad F_4 = \frac{c}{a^3 b^2}\left(b^2\partial_t a\partial_t c + a^2\partial_r c\partial_r a - b^2 a\partial_t^2 c\right)$$

$$F_5 = \frac{c}{b^3 a^2}\left(a^2\partial_r b\partial_r c + b^2\partial_t c\partial_t b - a^2 b\partial_r^2 c\right), \quad F_6 = \frac{1}{b^2 a^2}\left(b^2 a^2 + b^2(\partial_t c)^2 - a^2(\partial_r c)^2\right)$$

$$F_7 = \frac{1}{a^2 bc}\left(a^2 b\partial_r^2 - a^2\partial_r b\partial_r c - b^2\partial_t b\partial_t c\right), \quad F_8 = \frac{1}{ab^2 c}\left(-b^2 a\partial_t^2 + a^2\partial_r a\partial_r c + b^2\partial_t a\partial_t c\right)$$

$$F_9 = \frac{1}{abc}\left(b\partial_r a\partial_t c + a\partial_t b\partial_r c - ab\partial_r\partial_t c\right)$$

in which terms the Riemann tensor for the Poisson-compatible 'quantising' connection comes out as

$$\mathrm{Riem}(\mathrm{d}t) = -F_1 \mathrm{d}r\wedge\mathrm{d}t\otimes\mathrm{d}r + C(\mathrm{d}t), \quad \mathrm{Riem}(dr) = F_2\mathrm{d}r\wedge\mathrm{d}t\otimes\mathrm{d}t + C(\mathrm{d}r),$$

$$\mathrm{Riem}(dz^i) = \delta_{ab}\mathrm{d}z^i\wedge\mathrm{d}z^a\otimes\mathrm{d}z^b + C(\mathrm{d}z^i),$$

where we have collected in the tensor C all contributions coming from the undetermined components of the contorison tensor, namely

$$\begin{aligned} C(\mathrm{d}x^\iota) = {} & \nabla_\mu S^\iota{}_{0\alpha}\mathrm{d}x^\mu\wedge\mathrm{d}t\otimes\mathrm{d}x^\alpha + \nabla_\mu S^\iota{}_{1\alpha}\mathrm{d}x^\mu\wedge\mathrm{d}r\otimes\mathrm{d}x^\alpha \\ & + \left(S^\iota{}_{0\kappa}S^\kappa{}_{\nu\alpha} + S^\iota{}_{0\alpha}S^0{}_{0\nu} + S^\iota{}_{1\alpha}S^1{}_{0\nu}\right)\mathrm{d}t\wedge\mathrm{d}x^\nu\otimes\mathrm{d}x^\alpha \\ & + \left(S^\iota{}_{1\kappa}S^\kappa{}_{\nu\alpha} + S^\iota{}_{0\alpha}S^0{}_{1\nu} + S^\iota{}_{1\alpha}S^1{}_{1\nu}\right)\mathrm{d}r\wedge\mathrm{d}x^\nu\otimes\mathrm{d}x^\alpha \\ & + \left(S^\iota{}_{0\alpha}S^0{}_{i\nu} + S^\iota{}_{1\alpha}S^1{}_{i\nu}\right)\mathrm{d}z^i\wedge\mathrm{d}x^\nu\otimes\mathrm{d}x^\alpha \end{aligned}$$

We also have the classical Ricci tensor for the Levi-Civita connection

$$\begin{aligned} \widehat{\mathrm{Ricci}} = -\frac{1}{2}\Big(& (F_2 + 2F_8)\mathrm{d}t\otimes\mathrm{d}t - (F_1 + 2F_7)\mathrm{d}r\otimes\mathrm{d}r + F_9(\mathrm{d}t\otimes\mathrm{d}r + \mathrm{d}r\otimes\mathrm{d}t) \\ & - \frac{1}{(z^3)^2}(F_6 + F_5 - F_4)\delta_{ij}\mathrm{d}z^i\otimes\mathrm{d}z^j\Big) \end{aligned}$$

and, for later reference, the Einstein tensor

$$\widehat{\mathrm{G}} = -\frac{1}{2}\left(\frac{a^2}{c^2}(F_5+2F_6)\mathrm{d}t\otimes\mathrm{d}t - \frac{a^2}{c^2}(F_6-2F_4)\mathrm{d}r\otimes\mathrm{d}r + F_9(\mathrm{d}t\otimes\mathrm{d}r+\mathrm{d}r\otimes\mathrm{d}t)\right.$$
$$\left. -\frac{1}{(z^3)^2}(F_5-F_4-\frac{c^2}{b^2}F_1)\delta_{ij}\mathrm{d}z^i\otimes\mathrm{d}z^j\right)$$

Before continuing, we turn briefly to the quantity $\omega^{\beta\alpha}(R^{\iota}{}_{\nu\mu\alpha}+S^{\iota}{}_{\mu\nu;\alpha})$ which appears in several formulas in Section 2, most importantly, the quantum Levi-Civita connection condition (2.14). In particular, we note that it is surprising simple with the only nonzero components

$$\omega^{2\alpha}(R^2{}_{22\alpha}+S^2{}_{22;\alpha}) = \frac{z^1z^2}{z^3},\quad \omega^{2\alpha}(R^3{}_{22\alpha}+S^3{}_{22;\alpha}) = \frac{(z^2)^2-1}{z^3},$$
$$\omega^{3\alpha}(R^3{}_{23\alpha}+S^3{}_{32;\alpha}) = \frac{z^2z^1}{z^3},\quad \omega^{3\alpha}(R^3{}_{23\alpha}+S^3{}_{32;\alpha}) = \frac{(z^2)^2-1}{z^3},$$
$$\omega^{2\alpha}(R^2{}_{32\alpha}+S^2{}_{32;\alpha}) = \frac{1-(z^1)^2}{z^3},\quad \omega^{2\alpha}(R^3{}_{32\alpha}+S^3{}_{32;\alpha}) = -\frac{z^1z^2}{z^3},$$
$$\omega^{3\alpha}(R^2{}_{33\alpha}+S^2{}_{33;\alpha}) = \frac{1-(z^1)^2}{z^3},\quad \omega^{3\alpha}(R^3{}_{33\alpha}+S^3{}_{33;\alpha}) = -\frac{z^1z^2}{z^3}$$

The undetermined components of S do not contribute. In general the 'quantising' connection always enters in combination with the Poisson tensor e.g. $\omega^{\alpha\beta}\Gamma^{\iota}{}_{\beta\gamma}$ so the same argument as in the proof of Theorem 4.1 applies and we the undetermined components $S_{\mu1\nu}$ or $S_{\mu2\nu}$ in geometrically relevant expressions, as demonstrated by the generalized Ricci 2-form which now is

$$\mathcal{R} = -\frac{1}{2}c^2\epsilon_{mnk}z^k\mathrm{d}z^m\wedge\mathrm{d}z^n = -\frac{c^2}{z^3}\mathrm{d}z^1\wedge\mathrm{d}z^2$$

From this we have the quantum wedge product

$$\mathrm{d}t\wedge_1\mathrm{d}x^\mu = \mathrm{d}t\wedge\mathrm{d}x^\mu,\quad \mathrm{d}x^\mu\wedge_1\mathrm{d}t = \mathrm{d}x^\mu\wedge\mathrm{d}t,\quad \mathrm{d}r\wedge_1\mathrm{d}x^\mu = \mathrm{d}r\wedge\mathrm{d}x^\mu,\quad \mathrm{d}x^\mu\wedge_1\mathrm{d}r = \mathrm{d}x^\mu\wedge\mathrm{d}r$$
$$\mathrm{d}z^i\wedge_1\mathrm{d}z^j = \mathrm{d}z^i\wedge\mathrm{d}z^j + \frac{\lambda}{2}\left(3z^iz^j-\delta^{ij}\right)\mathrm{d}z^1\wedge\mathrm{d}z^2$$
$$\{\mathrm{d}z^i,\mathrm{d}z^j\}_1 = \lambda\left(3z^iz^j-\delta^{ij}\right)\mathrm{d}z^1\wedge\mathrm{d}z^2$$

Our next step is to calculate the quantum metric.

$$g_1 = g_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu + \frac{\lambda c^2}{2(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i\otimes_1\mathrm{d}z^j - z^i\mathrm{d}z^3\otimes_1\mathrm{d}z^j\right)$$

Working with metric components $\tilde{g}_{ij}$ (in the middle) we get

$$h = \frac{c^2(2-(z^3)^2)}{(z^3)^3}\epsilon_{3ij}\mathrm{d}z^i\otimes_1\mathrm{d}z^j$$

Meanwhile, for the inverse metric with components $\tilde{g}^{ij}$ we get

$$(\mathrm{d}z^1,\mathrm{d}z^2)_1 = g^{23}+\frac{\lambda}{2}\frac{z^3}{c^2},\quad (\mathrm{d}z^2,\mathrm{d}z^1)_1 = g^{32}-\frac{\lambda}{2}\frac{z^3}{c^2}$$
$$(\mathrm{d}t,\mathrm{d}t)_1 = g^{00},\quad (\mathrm{d}r,\mathrm{d}r)_1 = g^{11},\quad (\mathrm{d}z^1,\mathrm{d}z^1)_1 = g^{22},\quad (\mathrm{d}z^2,\mathrm{d}z^2)_1 = g^{33}$$

Now, Lemma 2.2 gives the quantum connection as

$$\nabla_1(\mathrm{d}t) = -\widehat{\Gamma}^0{}_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu - \frac{\lambda}{2(z^3)^2}\frac{c\partial_tc}{a^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i\otimes_1\mathrm{d}z^j - z^i\mathrm{d}z^3\otimes_1\mathrm{d}z^j\right)$$

$$\nabla_1(\mathrm{d}r) = -\widehat{\Gamma}^1{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2(z^3)^2}\frac{c\partial_r c}{b^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right) \tag{4.2}$$

$$\nabla_1(\mathrm{d}z^a) = -\widehat{\Gamma}^i{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2}\left(\epsilon_{ijk}z^k z^a \mathrm{d}z^i \otimes_1 \mathrm{d}z^j - \frac{1}{(z^3)^2}\epsilon^a{}_{i3}\mathrm{d}z^3 \otimes_1 \mathrm{d}z^i\right)$$

Lastly, we calculate the associated braiding. Its contributions at order λ are

$$\sigma_1(\mathrm{d}z^i \otimes_1 \mathrm{d}z^j) = \mathrm{d}z^j \otimes_1 \mathrm{d}z^i + \lambda\left(\epsilon_{abc}z^c z^i z^j + \epsilon^j{}_{ac}z^c\delta^i{}_b + \epsilon^{ij}{}_c z^c\delta_{ab}\right)\mathrm{d}z^a \otimes_1 \mathrm{d}z^b$$

when calculated using (2.16). Meanwhile, from Proposition 2.4 quantum antisymmetric lift is

$$i_1(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = \frac{1}{2}\left(\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\mu\right) + \lambda I(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \tag{4.3}$$

The functorial choice gives $I(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = 0$, but we leave this general.

4.2. **Laplace operator and Curvature Tensor.** Following from the previous section, we first calculate the Laplace operator. From Theorem 2.3 we get that

$$\Box_1 f = g^{\alpha\beta}\left(f_{,\alpha\beta} + f_{,\gamma}\widehat{\Gamma}^\gamma{}_{\alpha\beta}\right)$$

as with the flat FLRW metric, is undeformed in the underlying algebra. Then, (2.18) gives the quantum Riemann tensor as

$$\begin{aligned}\mathrm{Riem}_1(\mathrm{d}t) = &-\frac{1}{2}\widehat{R}^0{}_{\alpha\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\alpha - \frac{\lambda}{2(z^3)^2}\left(\epsilon_{3ij}z^3(F_3\mathrm{d}t - F_4\mathrm{d}r) \wedge \mathrm{d}z^i \otimes_1 \mathrm{d}z^j\right.\\ &\left.+\epsilon_{3ij}z^i(F_3\mathrm{d}t - F_4\mathrm{d}r) \wedge \mathrm{d}z^j \otimes_1 \mathrm{d}z^3\right)\end{aligned}$$

$$\begin{aligned}\mathrm{Riem}_1(\mathrm{d}r) = &-\frac{1}{2}\widehat{R}^1{}_{\alpha\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\alpha + \frac{\lambda}{2(z^3)^2}\left(\epsilon_{3ij}z^3(F_5\mathrm{d}t - F_3\mathrm{d}r) \wedge \mathrm{d}z^i \otimes_1 \mathrm{d}z^j\right.\\ &\left.+\epsilon_{3ij}z^i(F_5\mathrm{d}t - F_3\mathrm{d}r) \wedge \mathrm{d}z^j \otimes_1 \mathrm{d}z^3\right)\end{aligned}$$

$$\mathrm{Riem}_1(\mathrm{d}z^\mu) = -\frac{1}{2}\widehat{R}^\mu{}_{\alpha\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\alpha + \frac{\lambda F_6}{2(z^3)^2}(1 + (z^3)^3)\mathrm{d}z^1 \wedge \mathrm{d}z^2 \otimes_1 \mathrm{d}z^\mu$$

Using the lift map (4.3) and the tensor formula (2.21) we get the quantum Ricci tensor as

$$\begin{aligned}\mathrm{Ricci}_1 = &-\frac{1}{2}\widehat{R}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - \frac{\lambda}{4(z^3)^2}(F_6 + F_5 - F_4)\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right)\\ &-\frac{3\lambda}{4z^3}F_6\epsilon_{3ij}\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - \frac{\lambda}{2}\widehat{R}^\alpha{}_{\gamma\eta\zeta}I^{\eta\zeta}{}_{\alpha\nu}\mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\gamma\end{aligned}$$

Lastly, we fix I so that we have both $\wedge_1(\mathrm{Ricci}_1) = 0$ and $\mathrm{flip}(*\otimes *)\mathrm{Ricci}_1 = \mathrm{Ricci}_1$. For the latter, it is easiest to consider the quantum Ricci tensor with components in the middle so that from Section 2.2 we have

$$\begin{aligned}\rho = &-\frac{1}{4(z^3)^3}\left((F_5 - F_4)(2 - (z^3)^2) + 2F_6(1 + (z^3)^2)\right)\epsilon_{3ij}\mathrm{d}z^i \otimes_1 \mathrm{d}z^j\\ &-\frac{1}{2}\widehat{R}^\alpha{}_{\gamma\eta\zeta}I^{\eta\zeta}{}_{\alpha\nu}\mathrm{d}x^\nu \otimes_1 \mathrm{d}x^\gamma\end{aligned}$$

where, for comparison, $\rho = -\frac{1}{2}\rho_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu$. The reality condition, since the coefficients are real and λ imaginary, requires this to be antisymmetric. Also,

$$\wedge_1(\mathrm{Ricci}_1) = -\frac{3\lambda}{2z^3}F_6\mathrm{d}z^1 \wedge \mathrm{d}z^2 - \frac{\lambda}{2}\widehat{R}^\alpha{}_{\gamma\eta\zeta}I^{\eta\zeta}{}_{\alpha\nu}\mathrm{d}x^\nu \wedge \mathrm{d}x^\gamma$$

Putting this together results in

$$\widehat{R}^{\alpha}{}_{\gamma\eta\zeta}I^{\eta\zeta}{}_{\alpha\nu}\mathrm{d}x^{\nu}\otimes_1\mathrm{d}x^{\gamma}=-\frac{3}{2z^3}F_6\epsilon_{3ij}\mathrm{d}z^i\otimes_1\mathrm{d}z^j$$

This answer for the contraction of the lift map with the Riemann tensor is unique, but the same is not true of the lift map itself and we are left with a large moduli of possible solutions with most components of I undetermined. We examine the simplest possible form by setting these to zero, leaving us with

$$i_1(\mathrm{d}z^1\wedge\mathrm{d}z^2)=\frac{1}{2}\left(\mathrm{d}z^1\otimes_1\mathrm{d}z^2-\mathrm{d}z^2\otimes_1\mathrm{d}z^1\right)-\frac{3\lambda}{4z^3}\delta_{ij}\mathrm{d}z^i\otimes_1\mathrm{d}z^j$$

as the only part with an $\mathcal{O}(\lambda)$ contribution and which is the same as for the fuzzy sphere seen previously. This results in

$$\rho=-\frac{1}{4(z^3)^3}(F_6+F_5-F_4)(2-(z^3)^2)\epsilon_{3ij}\mathrm{d}z^i\otimes_1\mathrm{d}z^j$$

which we note has the same structure as h for the quantum metric, but with different coefficients. The quantum Ricci tensor (with components on the left) is now

$$\mathrm{Ricci}_1=-\frac{1}{2}\widehat{R}_{\mu\nu}\mathrm{d}x^{\mu}\otimes_1\mathrm{d}x^{\nu}-\frac{\lambda}{4(z^3)^2}(F_6+F_5-F_4)\epsilon_{3ij}\left(z^3\mathrm{d}z^i\otimes_1\mathrm{d}z^j-z^i\mathrm{d}z^3\otimes_1\mathrm{d}z^j\right)$$

The scalar curvature, using (2.23), has no corrections and comes out as

$$S_1=-\frac{1}{2}\widehat{S},\quad \widehat{S}=\widehat{R}_{\mu\nu}g^{\mu\nu}=\frac{2}{c^2}(F_6+2F_5-2F_4)-\frac{2}{b^2}F_1$$

Note that it depends only on t and r and is therefore central in the algebra. From Proposition 2.1, the quantum dimension comes out as

$$\dim(M)_1=\dim(M)-\lambda\omega^{\alpha\beta}g^{\mu\nu}{}_{,\alpha}\Gamma_{\nu\beta\mu}=4$$

It might also be of interest to think about a quantum Einstein tensor. While a general theorem has not been established, we could consider a 'naive' construction by analogy to the classical expression. Since the quantum and classical dimensions are the same, we could take for example

$$G_1=\mathrm{Ricci}_1-\frac{1}{2}\mathrm{S}_1\mathrm{g}_1$$

which has the same form as the classical case. This can be written as

$$G_1=-\frac{1}{2}G_{1\mu\nu}\mathrm{d}x^{\mu}\otimes_1\mathrm{d}x^{\nu}=-\frac{1}{2}\mathrm{d}x^{\mu}\bullet\widetilde{G}_{1\mu\nu}\otimes_1\mathrm{d}x^{\nu}$$

where $\widetilde{G}_{1\mu\nu}=G_{1\mu\nu}-\lambda\omega^{\alpha\beta}\widehat{G}_{\gamma\nu,\alpha}\Gamma^{\gamma}{}_{\beta\mu}$ and as previously the hat denotes that this is for the Levi-Civita connection. Now since in our case S_1 is purely classical and central, this can be expressed in component form as

$$\widetilde{G}_{1\mu\nu}=\widetilde{\widehat{R}}_{1\mu\nu}-\frac{1}{2}\widehat{S}\widetilde{g}_{\mu\nu}$$

following the same pattern as how the components of Ricci_1 and g_1 are written. If we write $\widetilde{G}_{1\mu\nu}=\widehat{G}_{\mu\nu}+\lambda\Sigma_{\mu\nu}$ then

$$\Sigma=\rho-\frac{1}{4}\widehat{S}h=\frac{1}{4(z^3)^3}(F_5-F_4-\frac{c^2}{b^2}F_1)(2-(z^3)^2)\epsilon_{3ij}\mathrm{d}z^i\otimes_1\mathrm{d}z^j$$

where $\Sigma=-\frac{1}{2}\Sigma_{\mu\nu}\mathrm{d}x^{\mu}\otimes_1\mathrm{d}x^{\nu}$ and is manifestly antisymmetric corresponding to $\mathrm{flip}(*\otimes *)G_1=G_1$. Indeed, G_1 is both quantum symmetric and obeys the reality condition since Ricci_1, g_1 and $\hat{S}$ is real and central.

With the results of this section, we can calculate the quantum geometry for all metrics of the form (4.1) simply by choosing appropriate parameters for a,b and c.

4.3. **FLRW metric case.** Comparing the above results with those of the FLRW metric in Section 3, there is a disparity that previously the metric appeared undeformed while now it has a quantum correction. We resolve this here. We first specialise the general theory above to the FLRW metric

$$g = -\mathrm{d}t\otimes\mathrm{d}t + a^2(t)(\mathrm{d}r\otimes\mathrm{d}r + r^2\delta_{ij}\mathrm{d}z^i\otimes\mathrm{d}z^j)$$

where we identify the parameters

$$a(r,t)=1,\quad b(r,t)=a(t),\quad c(r,t)=ra(t).$$

This gives us the 'quantising' connection up to undetermined but irrelevant contorsion tensor components (which are set to zero for simplicity)

$$\nabla(\mathrm{d}t) = -a\dot{a}\mathrm{d}r\otimes\mathrm{d}r,\quad \nabla(\mathrm{d}r) = -a\dot{a}(\mathrm{d}r\otimes\mathrm{d}t + \mathrm{d}t\otimes\mathrm{d}r)$$

$$\nabla(\mathrm{d}z^i) = -\frac{1}{r}\mathrm{d}r\otimes\mathrm{d}z^i - \frac{\dot{a}}{a}\mathrm{d}t\otimes\mathrm{d}z^i - \delta_{ab}z^i\mathrm{d}z^a\otimes\mathrm{d}z^b$$

Meanwhile, for the classical Ricci tensor of the Levi-Civita connection we have

$$\widehat{\mathrm{Ricci}} = -\frac{1}{2}\left(-3\frac{\ddot{a}}{a}\mathrm{d}t\otimes\mathrm{d}t + \left(2\dot{a}^2+a\ddot{a}\right)(\mathrm{d}r\otimes\mathrm{d}r + r^2\delta_{ij}\mathrm{d}z^i\otimes\mathrm{d}z^j)\right)$$

and the curvature scalar is as in (3.14). Now, the quantum metric comes out as

$$g_1 = g_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu + \frac{\lambda r^2}{2(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i\otimes_1\mathrm{d}z^j - \mathrm{d}z^3\otimes_1 z^i\mathrm{d}z^j\right) \tag{4.4}$$

where x^μ refers to coordinates t, r, z^1, z^2 as we used polar coordinates. Equivalently, $\tilde{g}_{ij}$ (where the components are in the middle) has quantum correction

$$h = \frac{a(t)^2r^2(2-(z^3)^2)}{(z^3)^3}\epsilon_{3ij}\mathrm{d}z^i\otimes_1\mathrm{d}z^j \tag{4.5}$$

For the inverse metric with components $\tilde{g}^{ij}$ we obtain

$$(\mathrm{d}z^1,\mathrm{d}z^2)_1 = g^{23} + \frac{\lambda}{2}\frac{z^3}{r^2},\quad (\mathrm{d}z^2,\mathrm{d}z^1)_1 = g^{32} - \frac{\lambda}{2}\frac{z^3}{r^2}$$

$$(\mathrm{d}t,\mathrm{d}t)_1 = g^{00},\quad (\mathrm{d}r,\mathrm{d}r)_1 = g^{11},\quad (\mathrm{d}z^1,\mathrm{d}z^1)_1 = g^{22},\quad (\mathrm{d}z^2,\mathrm{d}z^2)_1 = g^{33}$$

The quantum connection is

$$\nabla_1(\mathrm{d}t) = -\widehat{\Gamma}^0{}_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu - \frac{\lambda}{2(z^3)^2}a\dot{a}r^2\epsilon_{3ij}\left(z^3\mathrm{d}z^i\otimes_1\mathrm{d}z^j - z^i\mathrm{d}z^3\otimes_1\mathrm{d}z^j\right)$$

$$\nabla_1(\mathrm{d}r) = -\widehat{\Gamma}^1{}_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu + \frac{\lambda}{2(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i\otimes_1\mathrm{d}z^j - z^i\mathrm{d}z^3\otimes_1\mathrm{d}z^j\right)$$

$$\nabla_1(\mathrm{d}z^a) = -\widehat{\Gamma}^i{}_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu + \frac{\lambda}{2}\left(\epsilon_{ijk}z^kz^a\mathrm{d}z^i\otimes_1\mathrm{d}z^j - \frac{1}{(z^3)^2}\epsilon^a{}_{i3}\mathrm{d}z^3\otimes_1\mathrm{d}z^i\right)$$

Then, by computing $(F_6+F_5-F_4) = r^2(2\dot{a}^2+a\ddot{a})$ the quantum Ricci tensor is

$$\mathrm{Ricci}_1 = \widehat{R}_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu - \frac{\lambda r^2}{4(z^3)^2}(2\dot{a}^2+a\ddot{a})\epsilon_{3ij}\left(z^3\mathrm{d}z^i\otimes_1\mathrm{d}z^j - z^i\mathrm{d}z^3\otimes_1\mathrm{d}z^j\right)$$

With components in the middle, this comes out as

$$\rho = -\frac{1}{4(z^3)^3} r^2(2\dot{a}^2 + a\ddot{a})(2 - (z^3)^2)\epsilon_{3ij} \mathrm{d}z^i \otimes_1 \mathrm{d}z^j$$

and in either case $S_1 = -\frac{1}{2}\widehat{S}$ in our conventions.

Now these results appear at first sight to be at odds with Section 3 since there the quantum metric from (3.11) looks the same as classical when written in Cartesian coordinates. We first write it terms of $\mathrm{d}z^i$ by writing $\mathrm{d}x^i = r\mathrm{d}z^i - z^i \mathrm{d}r$ and note that since $z^i \mathrm{d}r = z^i \bullet \mathrm{d}r$, we can take such z^i terms to the other side of $\otimes_1$. Since also $\mathrm{d}z^i \bullet z^i = O(\lambda^2)$ (sum over i), we find

$$g_1 = -\mathrm{d}t \otimes_1 \mathrm{d}t + a^2(t)(\mathrm{d}r \otimes_1 \mathrm{d}r + r^2 \delta_{ij} \mathrm{d}z^i \otimes_1 \mathrm{d}z^j)$$

This begins to look like (4.4) but note that only z^1, z^2 (say) are coordinates with z^3 a function of them. In particular,

$$\mathrm{d}z^3 = -(z^3)^{-1} \bullet (z^1 \bullet \mathrm{d}z^1 + z^2 \bullet \mathrm{d}z^2) = -(z^3)^{-1}(z^1 \mathrm{d}z^1 + z^2 \mathrm{d}z^2) - \frac{\lambda}{2}\epsilon_{3ij} z^i \mathrm{d}z^j$$

would be needed to reduce to the form of (4.4) where the first term has only $\mathrm{d}z^a \otimes_1 \mathrm{d}z^b$ for $a, b = 1, 2$. Equivalently, we show that we have the same $\tilde{g}_{\mu\nu}$. Considering only the angular part $\delta_{ij}\mathrm{d}z^i \otimes_1 \mathrm{d}z^j = \mathrm{d}z^1 \otimes_1 \mathrm{d}z^1 + \mathrm{d}z^2 \otimes_1 \mathrm{d}z^2 + \mathrm{d}z^3 \otimes_1 \mathrm{d}z^3$ and examine the last term more closely (sum over repeated indices understood)

$$\begin{aligned}
\mathrm{d}z^3 \otimes_1 \mathrm{d}z^3 &= (z^3)^{-1} \bullet z^a \bullet \mathrm{d}z^a \otimes_1 (z^3)^{-1} \bullet z^b \bullet \mathrm{d}z^b \\
&= \left(\frac{z^a}{z^3} + \frac{\lambda}{2}\epsilon_{ac} z^c\right) \bullet \mathrm{d}z^a \otimes_1 \left(\frac{z^b}{z^3} + \frac{\lambda}{2}\epsilon_{bd} z^d\right) \bullet \mathrm{d}z^b \\
&= \mathrm{d}z^a \bullet \left(\frac{z^a}{z^3} + \frac{\lambda}{2}\epsilon_{ac} z^c\right) \bullet \left(\frac{z^b}{z^3} + \frac{\lambda}{2}\epsilon_{bd} z^d\right) \otimes_1 \mathrm{d}z^b + [\frac{z^a}{z^3}, \mathrm{d}z^a] \bullet \frac{z^b}{z^3} \otimes_1 \mathrm{d}z^b \\
&= \mathrm{d}z^a \bullet \frac{z^a z^b}{(z^3)^2} \otimes_1 \mathrm{d}z^b + \frac{\lambda}{2}\left(\mathrm{d}z^a \frac{1}{(z^3)^3}(\epsilon_{ab} + z^3 \epsilon_{bd} z^a z^d + z^3 \epsilon_{ac} z^b z^c) \otimes_1 \mathrm{d}z^b \right. \\
&\quad \left. -\mathrm{d}z^a \frac{1}{(z^3)^3} \epsilon_{cb} z^c z^a \otimes_1 \mathrm{d}z^b\right) \\
&= \mathrm{d}z^a \bullet \frac{z^a z^b}{(z^3)^2} \otimes_1 \mathrm{d}z^b + \frac{\lambda}{2}\mathrm{d}z^a \frac{(2 - (z^3)^2)}{(z^3)^3} \epsilon_{ab} \otimes_1 \mathrm{d}z^b
\end{aligned}$$

The $\bullet$ in the first term is left unevaluated so as to obtain $\tilde{g}_{ij}$ and we clearly see that we now have the same semiclassical correction h as in (4.5). We can perform a similar calculation for the quantum Ricci tensor in Section (3.2), making the same coordinate transformation as for the metric

$$\mathrm{Ricci}_1 = -\frac{1}{2}\left(-3\frac{\ddot{a}}{a}\mathrm{d}t \otimes_1 \mathrm{d}t + \left(2\dot{a}^2 + a\ddot{a}\right)(\mathrm{d}r \otimes_1 \mathrm{d}r + r^2 \delta_{ij} \mathrm{d}z^i \otimes_1 \mathrm{d}z^j)\right)$$

Indeed, since t and r are central in the algebra, the procedure is simply a repeat of that for the metric and clearly results in the same ρ as above. Thus we obtain the same results as in Section 3 but only after allowing for the change of variables in the noncommutative algebra and $\otimes_1$.

4.4. **Schwarzschild metric.** We now look at some examples of well known metrics that fit the above analysis. For the first, we reexamine the Schwarzschild metric case in [10, Sec. 7.2]. There it was found that (as we would now expect), the quantum Levi-Civita condition is satisfied for a spherically symmetric Poisson tensor. A difference however, is that in [10] the torsion tensor was restricted to being rotationally invariant. By contrast, no such assumption is made here yet we are still led to a unique (con)torsion from Theorem 4.1 up to undetermined components which we show do not enter into the quantum metric, quantum connection etc. Here

$$g = -\left(1-\frac{r_S}{r}\right)\mathrm{d}t\otimes\mathrm{d}t+\left(1-\frac{r_S}{r}\right)^{-1}\mathrm{d}r\otimes\mathrm{d}r+r^2\delta_{ij}\mathrm{d}z^i\otimes\mathrm{d}z^j$$

so our three functional parameters are

$$a(r,t)=\left(1-\frac{r_S}{r}\right)^{\frac{1}{2}},\quad b(r,t)=\left(1-\frac{r_S}{r}\right)^{-\frac{1}{2}},\quad c(r,t)=r$$

giving the 'quantising' connection up to undetermined but irrelevant contorsion tensor components (which are set to zero for simplicity)

$$\nabla(\mathrm{d}t)=-\left(1-\frac{r_S}{r}\right)^{-\frac{1}{2}}\frac{r_S}{r^2}(\mathrm{d}r\otimes\mathrm{d}t+\mathrm{d}t\otimes\mathrm{d}r)$$

$$\nabla(\mathrm{d}r)=-\left(1-\frac{r_S}{r}\right)^{\frac{3}{2}}\frac{r_S}{r^2}\mathrm{d}t\otimes\mathrm{d}t+\left(1-\frac{r_S}{r}\right)^{-\frac{1}{2}}\frac{r_S}{r^2}\mathrm{d}r\otimes\mathrm{d}r$$

$$\nabla(\mathrm{d}z^i)=-\frac{1}{r}\mathrm{d}r\otimes\mathrm{d}z^i-\delta_{ab}z^i\mathrm{d}z^a\otimes\mathrm{d}z^b$$

As a check, by transforming into spherical polars and likewise neglecting the irrelevant components, we can recover the 'quantising' connection in [10]. In particular

$$\nabla(\mathrm{d}\theta)=-\frac{1}{r}\mathrm{d}r\otimes\mathrm{d}\theta+\cos(\theta)\sin(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi$$

$$\nabla(\mathrm{d}\phi)=-\frac{1}{r}\mathrm{d}r\otimes\mathrm{d}\phi-\cot(\theta)(\mathrm{d}\theta\otimes\mathrm{d}\phi+\mathrm{d}\phi\otimes\mathrm{d}\theta)$$

which agrees with [10]. Obviously, the classical Ricci tensor for the Levi-Civita connection vanishes for the Schwarzschild metric, likewise for the curvature scalar.

Now, the quantum metric comes out as

$$g_1=g_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu+\frac{\lambda r^2}{2(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i\otimes_1\mathrm{d}z^j-\mathrm{d}z^3\otimes_1 z^i\mathrm{d}z^j\right)$$

While $\tilde{g}_{ij}$ (components in the middle) is

$$h=\frac{r^2(2-(z^3)^2)}{(z^3)^3}\epsilon_{3ij}\mathrm{d}z^i\otimes_1\mathrm{d}z^j$$

For the inverse metric with components $\tilde{g}^{ij}$ we get

$$(\mathrm{d}z^1,\mathrm{d}z^2)_1=g^{23}+\frac{\lambda}{2}\frac{z^3}{r^2},\quad(\mathrm{d}z^2,\mathrm{d}z^1)_1=g^{32}-\frac{\lambda}{2}\frac{z^3}{r^2}$$

$$(\mathrm{d}t,\mathrm{d}t)_1=g^{00},\quad(\mathrm{d}r,\mathrm{d}r)_1=g^{11},\quad(\mathrm{d}z^1,\mathrm{d}z^1)_1=g^{22},\quad(\mathrm{d}z^2,\mathrm{d}z^2)_1=g^{33}$$

The quantum connection is

$$\nabla_1(\mathrm{d}t)=-\widehat{\Gamma}^0{}_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu$$

$$\nabla_1(\mathrm{d}r) = -\widehat{\Gamma}^1{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda r}{2(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right)$$

$$\nabla_1(\mathrm{d}z^a) = -\widehat{\Gamma}^i{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2}\left(\epsilon_{ijk}z^k z^a \mathrm{d}z^i \otimes_1 \mathrm{d}z^j - \frac{1}{(z^3)^2}\epsilon^a{}_{i3}\mathrm{d}z^3 \otimes_1 \mathrm{d}z^i\right)$$

Meanwhile, from calculating the parameter $F_6 + F_5 - F_4 = 0$, we see that analogous to the classical case, the quantum Ricci tensor also vanishes

$$\mathrm{Ricci}_1 = 0, \quad \rho = 0, \quad S_1 = 0.$$

4.5. **Bertotti-Robinson metric with fuzzy spheres.** Another interesting example is the Bertotti-Robinson metric, discussed in the context of a different differential algebra in Section 2.4. In order to draw a comparison between this case and the previous one, we define our metric as

$$g = -a^2 r^{2\alpha}\mathrm{d}t \otimes \mathrm{d}t + b^2 r^{-2}\mathrm{d}r \otimes \mathrm{d}r + c^2\delta_{ij}\mathrm{d}z^i \otimes \mathrm{d}z^j$$

To chime with the conventions in this section, we relabel the constant terms and compared to the metric in Section 2.4, the off diagonal component is zero (either by diagonalising or setting the corresponding coefficient to zero). So our three functional parameters are

$$a(r,t) = ar^\alpha, \quad b(r,t) = br^{-1}, \quad c(r,t) = c$$

As explained after Theorem 4.1, the theorem in this case does not give a unique quantum geometry but does give one. Dropping the undetermined and irrelevant contorsion components, Poisson-connection come out as

$$\nabla(\mathrm{d}t) = -\frac{\alpha}{r}(\mathrm{d}t \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}t), \quad \nabla(\mathrm{d}r) = -\alpha r^{3\alpha}\frac{a^2}{c^2}\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{r}\mathrm{d}r \otimes \mathrm{d}r$$

$$\nabla(\mathrm{d}z^i) = -\delta_{ab}z^i\mathrm{d}z^a \otimes \mathrm{d}z^b$$

This is markedly different from that in Section 2.4 (apart from the different choice of coordinates), in particular with regard to the bimodule relations since previously t was not central. We also have the Ricci tensor for the Levi-Civita connection

$$\widehat{\mathrm{Ricci}} = -\frac{1}{2}\left(\alpha^2 r^{2\alpha}\frac{a^2}{b^2}\mathrm{d}t \otimes \mathrm{d}t - \frac{\alpha^2}{r^2}\mathrm{d}r \otimes \mathrm{d}r + \delta_{ij}\mathrm{d}z^i \otimes \mathrm{d}z^j\right)$$

with the corresponding scalar curvature

$$\widehat{S} = \widehat{R}_{\mu\nu}g^{\mu\nu} = \frac{2}{c^2} - \frac{2\alpha^2}{b^2}$$

The quantum metric is

$$g_1 = g_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda c^2}{2(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right)$$

While $\tilde{g}_{ij}$ (components in the middle) has the deformation term

$$h = \frac{c^2(2-(z^3)^2)}{(z^3)^3}\epsilon_{3ij}\mathrm{d}z^i \otimes_1 \mathrm{d}z^j$$

For the inverse metric with components $\tilde{g}^{ij}$ we get

$$(\mathrm{d}z^1, \mathrm{d}z^2)_1 = g^{23} + \frac{\lambda}{2}\frac{z^3}{c^2}, \quad (\mathrm{d}z^2, \mathrm{d}z^1)_1 = g^{32} - \frac{\lambda}{2}\frac{z^3}{c^2}$$

$$(\mathrm{d}t, \mathrm{d}t)_1 = g^{00}, \quad (\mathrm{d}r, \mathrm{d}r)_1 = g^{11}, \quad (\mathrm{d}z^1, \mathrm{d}z^1)_1 = g^{22}, \quad (\mathrm{d}z^2, \mathrm{d}z^2)_1 = g^{33}$$

Now, the quantum connection is

$$\nabla_1(\mathrm{d}t) = -\widehat{\Gamma}^0{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu, \quad \nabla_1(\mathrm{d}r) = -\widehat{\Gamma}^1{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu$$

$$\nabla_1(\mathrm{d}z^a) = -\widehat{\Gamma}^i{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2}\left(\epsilon_{ijk}z^k z^a \mathrm{d}z^i \otimes_1 \mathrm{d}z^j - \frac{1}{(z^3)^2}\epsilon^a{}_{i3}\mathrm{d}z^3 \otimes_1 \mathrm{d}z^i\right)$$

Again, calculating the parameter $F_6 + F_5 - F_4 = 1$, the quantum Ricci tensor is

$$\mathrm{Ricci}_1 = -\frac{1}{2}\widehat{R}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - \frac{\lambda}{4(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i\mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right)$$

With components in the middle, this comes out as

$$\rho = -\frac{1}{4(z^3)^3}(2 - (z^3)^2)\epsilon_{3ij}\mathrm{d}z^i \otimes_1 \mathrm{d}z^j$$

and in either case $S_1 = -\frac{1}{2}\widehat{S}$ in our conventions.

5. Conclusions

In this paper we simplified and extended the study of Poisson-Riemannian geometry introduced in [10] to include a formula for the quantum Laplace-Beltrami operator at semiclassical order (Theorem 2.3) and we also looked at the lifting map needed to define a reasonable Ricci tensor in a constructive approach to that. Our second main piece of analysis was Theorem 4.1 for spherically symmetric Poisson tensors on spherically symmetric spacetimes. We found that if the metric components are sufficiently generic (in particular the coefficient of the angular part of the metric is not constant) then any quantisation has to have $t, \mathrm{d}t, r, \mathrm{d}r$ central and nonassociative fuzzy spheres[7] at each value of time and radius. We also found that the Laplace-Beltrami operator has no corrections at order λ. Key to the startling rigidity here was condition (2.14) from [10] needed for the existence of a quantum Levi-Civita connection ∇_1. Hence if one wanted to avoid this conclusion then [10] says that we *can* drop (2.14) and still have a canonical ∇_1 and now with a larger range of spherically symmetric models but with a new physical effect of $\nabla_1 g_1 = O(\lambda)$. One can also drop our other assumption in the analysis that ω obeys (2.3) for the Jacobi identity. In that generic (nonassociative algebra) context we noted that spherical symmetry and Poisson-compatibility leads to a unique contorsion tensor, while imposing the Jacobi identity leads to half the modes of S being undetermined but in such a way that the contravariant connection $\omega^{\alpha\beta}\nabla_\beta$ more relevant to the quantum geometry is still unique. This suggests an interesting direction for the general theory.

The paper also included detailed calculations of the quantum metric, quantum Levi-Civita connection and quantum Laplacian for a number of models, some of them, such as the FLRW, Schwarzschild and the time-central Bertotti-Robinson model being covered by Theorem 4.1. The important case of the FLRW model was first solved directly in Cartesian coordinates both as a warm up and as an independent check of the main theorem (the needed quantum change of coordinates was provided in Section 4.3). Two models not covered by our analysis of spherical symmetry are the 2D bicrossproduct model for which most of the algebraic side of the quantum geometry but not the quantum Laplacian was already found in [9], and the non-time central but spherically symmetric Bertotti-Robinson model for which the full quantum geometry was already found in [28] (this case is not excluded by Theorem 4.1

since the coefficient of the angular metric is constant). In both cases the quantum spacetime algebra is the much-studied Majid-Ruegg spacetime $[x_i, t] = \lambda x_i$ in [26]. The non-time central Bertotti-Robinson model quantises $S^{n-1} \times dS_2$ and the quantum Laplacian in Section 2.4 is quite similar to the old 'Minkowski spacetime' Laplacian for this spacetime algebra which has previously led to variable speed of light[1] in that provided wave functions are normal ordered, one of the double-differentials becomes a finite-difference (the main difference from[1] is that this time there is an actual quantum geometry forcing the classical metric not to be flat[28]). However, when we analysed this within Poisson-Riemannian geometry we found no order λ correction to the quantum Laplacian. We traced this to the formula for the bullet product in Poisson-Riemannian geometry in [10] being realised on the classical space by an antisymmetric deformation, which is analogous to Weyl-ordered rather than left or right normal ordered functions in the noncommutative algebra being identified with classical ones. Our conclusion then is that order λ predictions from such models[1] were an artefact of the hypothesised normal ordering assumption and that Theorem 2.3 is a more stringent test within the paradigm of Poisson-Riemannian geometry. We should not then be too surprised that order λ corrections are more rare than one might naively have expected from the formula in Theorem 2.3. The 2D bicrossproduct model in Section 2.3 *does* however have an order λ deformation to the Laplacian even within Poisson-Riemannian geometry and we were able to solve the deformed massless wave equation at order λ using Kummer functions (i.e. it is effectively the Kummer equation). This behaviour is reminiscent of the minimally coupled black hole in the wave operator approach of [25] without yet having a general framework for the physical interpretation of the order λ deformations obtained from Poisson-Riemannian geometry.

It is even less clear at the present time how to draw physical conclusions from our formulae for the quantum metric g_1 and quantum Ricci tensor Ricci_1. In the FLRW model for example we found that g_1 looks identical to the classical metric but of course as an element of the quantum tensor product $\Omega^1 \otimes_1 \Omega^1$. The physical understanding of how quantum tensors relate to classical ones is suggested here as a topic of further work. Another topic on which we made only a tentative comment at the end of Section 4.2, is what should be the quantum Einstein tensor. Its deformation could perhaps be reinterpreted as an effective change to the stress energy tensor. This is another direction for further work.

Appendix A. Match up with the algebraic bicrossproduct model

This is a supplement to Section 2.3 in which we will verify that the semiclassical theory obtained by our tensor calculus formulae agrees with the order λ part of the full quantum geometry found for this model by algebraic means in [9]. This provide a completely independent check of the main formulae in Section 2.1.

We let $\nu = r \bullet \mathrm{d}t - t \bullet \mathrm{d}r = v + \frac{\lambda}{2}\mathrm{d}r$ and the (full) quantum metric, inverse quantum metric and quantum Levi-Civita connection in [9] are

$$g_1 = (1 + b\lambda^2)\mathrm{d}r \otimes_1 \mathrm{d}r + b\nu \otimes_1 \nu - b\lambda\nu \otimes_1 \mathrm{d}r$$

$$(\nu, \nu)_1 = b^{-1}, \quad (\mathrm{d}r, \nu)_1 = 0, \quad (\nu, \mathrm{d}r)_1 = \frac{\lambda}{1 + b\lambda^2}, \quad (\mathrm{d}r, \mathrm{d}r)_1 = \frac{1}{1 + b\lambda^2}$$

$$\nabla_1 \mathrm{d}r = \frac{8b}{r(4+7b\lambda^2)} v \otimes_1 \nu - \frac{12b\lambda}{r(4+7b\lambda^2)} v \otimes_1 \mathrm{d}r$$

$$\nabla_1 \nu = -\frac{4b\lambda}{r(4+7b\lambda^2)} v \otimes_1 \nu - \frac{8(1+b\lambda^2))}{r(4+7b\lambda^2)} v \otimes_1 \mathrm{d}r$$

(there is a typo in the coefficient of β' in [9]).

We note immediately by expanding g_1 to $O(\lambda)$ and changing from ν to v that

$$\begin{aligned} g_1 &= \mathrm{d}r \otimes_1 \mathrm{d}r + bv \otimes_1 v + b\frac{\lambda}{2}(\mathrm{d}r \otimes_1 v - v \otimes_1 \mathrm{d}r) \\ &= g_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + b\frac{\lambda}{2}\mathrm{d}r \otimes_1 v - \lambda b v \otimes_1 \mathrm{d}r \\ &= g_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu + \frac{\lambda}{2}bt\mathrm{d}r \otimes_1 \mathrm{d}r + \frac{\lambda}{2}br\mathrm{d}r \otimes_1 \mathrm{d}t - \lambda br \mathrm{d}t \otimes_1 \mathrm{d}r \end{aligned}$$

the same as we obtained from (2.9). We used $r\mathrm{d}t = r\bullet \mathrm{d}t - \frac{\lambda}{2}\mathrm{d}r$ to move all coefficients to the left through $\otimes_1$ in order make this comparison. We can do a similar trick with $r\mathrm{d}t = (\mathrm{d}t)\bullet r + \frac{\lambda}{2}\mathrm{d}r$ to put the coefficients in the middle, giving

$$\begin{aligned} g_1 =& \mathrm{d}r \otimes_1 \mathrm{d}r + b(\mathrm{d}t)\bullet r^2 \otimes_1 \mathrm{d}t + b(\mathrm{d}r)\bullet t^2 \otimes_1 \mathrm{d}r - b(\mathrm{d}r)\bullet t \bullet r \otimes_1 \mathrm{d}t \\ & - b(\mathrm{d}t)\bullet r \bullet t \otimes_1 \mathrm{d}r + b\lambda(\mathrm{d}r \otimes_1 v - v \otimes_1 \mathrm{d}r) \end{aligned}$$

so that

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{\lambda}{2}\begin{pmatrix} 0 & -3br \\ 3br & 0 \end{pmatrix}.$$

which agrees with $\tilde{g}_{\mu\nu}$ implied by $h_{\mu\nu} = -3br\epsilon_{\mu\nu}$ as computed from (2.12).

Similarly, working to $O(\lambda)$, the quantum connection from [10, Prop 7.1] is

$$\begin{aligned} \nabla_1 \mathrm{d}r &= \frac{2b}{r}(v \otimes_1 v - \lambda v \otimes_1 \mathrm{d}r) \\ &= \frac{2b}{r} v \otimes_1 (r \bullet \mathrm{d}t - t \bullet \mathrm{d}r - \frac{\lambda}{2}\mathrm{d}r) - \lambda\frac{2b}{r} v \otimes_1 \mathrm{d}r \\ &= 2bv \otimes_1 \mathrm{d}t - 2bv(r^{-1}\bullet t) \otimes_1 \mathrm{d}r - \lambda\frac{3b}{r} v \otimes_1 \mathrm{d}r \\ &= -\widehat{\Gamma}^1_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - 2\lambda b(\mathrm{d}t - r^{-1}t\mathrm{d}r) \otimes_1 \mathrm{d}r \end{aligned}$$

in the same manner as the computation of $v \otimes_1 v$ for the metric, which agrees with the correction in ${\Gamma_1}^1{}_{\mu\nu}$ of

$$\begin{aligned} &\frac{\lambda}{2}\omega^{\alpha\beta}\widehat{\Gamma}^1{}_{\mu 0,\alpha}{\Gamma^0}_{\beta\nu} - \frac{\lambda}{2}\omega^{\alpha\beta}\widehat{\Gamma}^1{}_{00}{\Gamma^0}_{\alpha\mu}{\Gamma^0}_{\beta\nu} - \frac{\lambda}{2}\omega^{01}{S^1}_{0\kappa}\nabla_1 {S^\kappa}_{\mu\nu} \\ &\quad = -\frac{\lambda}{2}\widehat{\Gamma}^1{}_{\mu 0,\nu} - \lambda b\epsilon_{\mu\nu} - \frac{\lambda}{2} r\widehat{\Gamma}^1{}_{0\kappa}\nabla_1 {S^\kappa}_{\mu\nu} = 2\lambda b\begin{pmatrix} 0 & 1 \\ 0 & -r^{-1}t \end{pmatrix} \end{aligned}$$

in Lemma 2.2. Here

$$\nabla_1 {S^0}_{\mu\nu} = \begin{pmatrix} -\frac{2bt}{r} & \frac{2(1+bt^2)}{r^2} \\ \frac{2bt^2}{r^2} & -\frac{2t(1+bt^2)}{r^3} \end{pmatrix}, \quad \nabla_1 {S^1}_{\mu\nu} = \begin{pmatrix} -2b & \frac{2bt}{r} \\ \frac{2bt}{r} & -\frac{2bt^2}{r^2} \end{pmatrix}$$

where ∇_1 in this context means with respect to r. Similarly, the semiquantum connection $\nabla_1 v = -\frac{2}{r}(v \otimes_1 \mathrm{d}r + b\lambda v \otimes_1 v)$ implies

$$\nabla_1 \mathrm{d}t = \nabla_1(r^{-1} \bullet v + (r^{-1}t) \bullet \mathrm{d}r) = \mathrm{d}r^{-1} \otimes_1 v + \mathrm{d}(r^{-1}t) \otimes_1 \mathrm{d}r + r^{-1} \bullet \nabla_1 v + (r^{-1}t) \bullet \nabla_1 \mathrm{d}r$$
$$= -r^{-2}\mathrm{d}r \otimes_1 (r \bullet \mathrm{d}t - t \bullet \mathrm{d}r - \frac{\lambda}{2}\mathrm{d}r) - r^{-2}t\mathrm{d}r \otimes_1 \mathrm{d}r + r^{-1}\mathrm{d}t \otimes_1 \mathrm{d}r$$
$$- 2r^{-2}(v \otimes_1 \mathrm{d}r + b\lambda v \otimes_1 v) + 2b(r^{-1}t) \bullet r^{-1}(v \otimes_1 v - \lambda v \otimes_1 \mathrm{d}r)$$
$$= -r^{-1}\mathrm{d}r \otimes_1 \mathrm{d}t + (r^{-2} \bullet t)\mathrm{d}r \otimes_1 \mathrm{d}r + \frac{\lambda}{2}r^{-2}\mathrm{d}r \otimes_1 \mathrm{d}r - r^{-2}t\mathrm{d}r \otimes_1 \mathrm{d}r + r^{-1}\mathrm{d}t \otimes_1 \mathrm{d}r$$
$$- 2r^{-2}v \otimes_1 \mathrm{d}r - \lambda b r^{-2} v \otimes_1 v + 2br^{-2}tv \otimes_1 (r \bullet \mathrm{d}t - t \bullet \mathrm{d}r - \frac{\lambda}{2}\mathrm{d}r) - \lambda 2br^{-2}tv \otimes_1 \mathrm{d}r$$
$$= -r^{-1}\mathrm{d}r \otimes_1 \mathrm{d}t + r^{-2}t\mathrm{d}r \otimes_1 \mathrm{d}r - \frac{\lambda}{2}r^{-2}\mathrm{d}r \otimes_1 \mathrm{d}r - r^{-2}t\mathrm{d}r \otimes_1 \mathrm{d}r + r^{-1}\mathrm{d}t \otimes_1 \mathrm{d}r$$
$$- 2r^{-2}v \otimes_1 \mathrm{d}r - \lambda b r^{-2} v \otimes_1 v + 2b(r^{-2}t) \bullet rv \otimes_1 \mathrm{d}t - 2b(r^{-2}t) \bullet tv \otimes_1 \mathrm{d}r - \lambda 3br^{-2}tv \otimes_1 \mathrm{d}r$$
$$= -\widehat{\Gamma}^0{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - \frac{\lambda}{2}r^{-2}\mathrm{d}r \otimes_1 \mathrm{d}r - \lambda b r^{-2} v \otimes_1 v - \lambda b r^{-1} v \otimes_1 \mathrm{d}t - \lambda b r^{-2} tv \otimes_1 \mathrm{d}r$$
$$= -\widehat{\Gamma}^0{}_{\mu\nu}\mathrm{d}x^\mu \otimes_1 \mathrm{d}x^\nu - \frac{\lambda}{2}r^{-2}\mathrm{d}r \otimes_1 \mathrm{d}r - 2\lambda b(\mathrm{d}t - r^{-1}t\mathrm{d}r) \otimes_1 \mathrm{d}t$$

which agrees with the correction to ${\Gamma_1}^0{}_{\mu\nu}$ of

$$\frac{\lambda}{2}\omega^{\alpha\beta}\widehat{\Gamma}^0{}_{\mu 0,\alpha}{\Gamma^0}_{\beta\nu} - \frac{\lambda}{2}\omega^{\alpha\beta}\widehat{\Gamma}^0{}_{00}{\Gamma^0}_{\alpha\mu}{\Gamma^0}_{\beta\nu} - \frac{\lambda}{2}\omega^{01}{S^0}_{0\kappa}\nabla_1 {S^\kappa}_{\mu\nu}$$
$$= -\frac{\lambda}{2}\widehat{\Gamma}^0{}_{\mu 0,\nu} - \lambda b r^{-1} t\epsilon_{\mu\nu} - \frac{\lambda}{2}r\widehat{\Gamma}^0{}_{0\kappa}\nabla_1 {S^\kappa}_{\mu\nu} = \lambda \begin{pmatrix} 2b & 0 \\ -\frac{2bt}{r} & \frac{1}{2r^2} \end{pmatrix}$$

in Lemma 2.2.

Next note that because $\nabla v = \nabla \mathrm{d}r = 0$, we do not have any corrections to products with these basic 1-forms and this allows us to equally well write

$$\mathrm{d}f = (\partial_r f) \bullet \mathrm{d}r + (\partial_v f) \bullet v$$

with the classical derivatives if we use this basis. Then working to $O(\lambda)$,

$$\Box_1 f = (\ ,\ )_1 \nabla_1 \mathrm{d}f = (\ ,\ )_1((\partial_r f) \bullet \nabla_1 \mathrm{d}r + (\partial_v f) \bullet \nabla_1 v + \mathrm{d}\partial_r f \otimes_1 \mathrm{d}r + \mathrm{d}\partial_v f \otimes_1 v)$$
$$= (\ ,\ )_1((\partial_r f) \bullet \frac{2b}{r} v \otimes_1 v - (\partial_r f) \bullet \frac{2b\lambda}{r} v \otimes_1 \mathrm{d}r - (\partial_v f) \bullet \frac{2}{r} v \otimes_1 \mathrm{d}r - (\partial_v f) \bullet \frac{2b\lambda}{r} v \otimes_1 v$$
$$+ (\partial_r^2 f)\mathrm{d}r \otimes_1 \mathrm{d}r + (\partial_v \partial_r f) v \otimes_1 \mathrm{d}r + (\partial_r \partial_v f)\mathrm{d}r \otimes_1 v + (\partial_v^2 f) v \otimes_1 v)$$
$$= (\partial_r f)\frac{2b}{r} b^{-1} + \frac{\lambda}{2}\partial_v \partial_r f 2b b^{-1} - (\partial_v f)\frac{2}{r}\frac{\lambda}{2} - (\partial_v f)\frac{2b\lambda}{r} b^{-1}$$
$$+ (\partial_r^2 f) + (\partial_v \partial_r f)\frac{\lambda}{2} - (\partial_r \partial_v f)\frac{\lambda}{2} + (\partial_v^2 f) b^{-1}$$
$$= \Box f + \lambda(-\frac{3}{r}\partial_v f + \frac{1}{2}[\partial_v, \partial_r] f + \partial_v \partial_r f)$$
$$= \Box f + \lambda\left(\frac{1}{r}\frac{\partial}{\partial t}\frac{\partial}{\partial r} f + \frac{t}{r^2}\frac{\partial^2}{\partial t^2} f - \frac{1}{r^2}\frac{\partial}{\partial t} f\right)$$

where we used $h \bullet r^{-1} = hr^{-1} + \frac{\lambda}{2}\partial_v h$ for any function h and, to $O(\lambda)$,

$$(v, v)_1 = b^{-1}, \quad (\mathrm{d}r, v)_1 = -\frac{\lambda}{2}, \quad (v, \mathrm{d}r)_1 = \frac{\lambda}{2}, \quad (\mathrm{d}r, \mathrm{d}r)_1 = 1.$$

This agrees with the quantum Laplacian to order λ obtained in Section 2.3.

In this model we can in fact write down the full quantum Laplacian in noncommutative geometry in the setting of [9] just as easily and we do this now as it was not done in that work. We again write

$$\mathrm{d}f = (\partial_r f)\bullet \mathrm{d}r + (\partial_v f)\bullet \nu$$

where ∂_r, ∂_v are now quantised versions of the ones before and are derivations of the noncommutative algebra since $\mathrm{d}r, v$ are central. They obey

$$\partial_v f(r) = 0, \quad \partial_v f(t) = r^{-1}\bullet \partial_0 f, \quad \partial_r f(r) = f', \quad \partial_r f(t) = -r^{-1}\bullet t\partial_0 f$$

as easily found using the derivation rule, the values on r, t and the relations $t\bullet r^{-1} = r^{-1}\bullet(t+\lambda)$ in the algebra. Here $\partial_0 f(t) = \lambda^{-1}(f(t+\lambda) - f(t))$ is a finite difference. Then for any f in the algebra, one can compute $(\ ,\ )_1\nabla_1\mathrm{d}f$ using the full expressions above to obtain

$$\Box_1 f = \frac{(8-6b\lambda^2)\partial_r f - \lambda(8+4b\lambda^2)\partial_v f}{4+7b\lambda^2}\bullet r^{-1} + \frac{\partial_r^2 f + \lambda\partial_v\partial_r f}{1+b\lambda^2} + b^{-1}\partial_v^2 f$$

for ∇_1 the quantum Levi-Civita connection stated above for this model.

Finally, the work [9] already contained the full quantum Ricci as proportional to the quantum metric. The first ingredient for this is the quantum Riemann tensor in [9] and expanding this gives

$$\begin{aligned}
\mathrm{Riem}_1(\mathrm{d}r) &= -2b\frac{1}{r^2}\bullet\nu\wedge_1\mathrm{d}r\otimes_1(r\bullet\mathrm{d}t - t\bullet\mathrm{d}r) + \frac{7b\lambda}{r}\mathrm{d}t\wedge\mathrm{d}r\otimes\mathrm{d}r\\
&= -2b\mathrm{d}t\wedge_1\mathrm{d}r\otimes_1\mathrm{d}t + 2b\frac{1}{r^2}\bullet t\bullet r\bullet(\mathrm{d}t\wedge_1\mathrm{d}r)\otimes_1\mathrm{d}r + \frac{7b\lambda}{r}\mathrm{d}t\wedge\mathrm{d}r\otimes\mathrm{d}r\\
&= -2b\mathrm{d}t\wedge\mathrm{d}r\otimes_1\mathrm{d}t + 2b\left(\frac{t}{r} + \frac{\lambda}{2r^2}\{t,r\} + \frac{\lambda}{2}\{\frac{1}{r^2},t\}r\right)\bullet(\mathrm{d}t\wedge\mathrm{d}r)\otimes_1\mathrm{d}r\\
&\quad + \frac{7b\lambda}{r}\mathrm{d}t\wedge\mathrm{d}r\otimes\mathrm{d}r\\
&= -2b\mathrm{d}t\wedge\mathrm{d}r\otimes_1\mathrm{d}t + 2b(\frac{t}{r})\bullet(\mathrm{d}t\wedge\mathrm{d}r)\otimes_1\mathrm{d}r + \frac{4b\lambda}{r}\mathrm{d}t\wedge\mathrm{d}r\otimes\mathrm{d}r\\
&= -\frac{1}{2}\widehat{R}^1{}_{\beta\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes_1\mathrm{d}x^\beta + 5\lambda\frac{b}{r}\mathrm{d}t\wedge\mathrm{d}r\otimes_1\mathrm{d}r + O(\lambda^2).
\end{aligned}$$

where we use $\nu, \mathrm{d}r$ central for the second equality, then $\mathrm{d}t\wedge_1\mathrm{d}r = \mathrm{d}t\wedge\mathrm{d}r$ and $\nabla_1(\mathrm{d}t\wedge\mathrm{d}r) = -r^{-1}\mathrm{d}t\wedge\mathrm{d}r$. There is a similar formula for $\mathrm{Riem}_1(\mathrm{d}t)$ obtained from $\mathrm{Riem}_1(\nu) = r\bullet\mathrm{Riem}_1(\mathrm{d}t) - t\bullet\mathrm{Riem}_1(\mathrm{d}r)$ given in [9] and expanding. Thus the curvature agrees with (2.24) obtained from our tensor formulae.

Next the lifting map i_1 was given by the method in [9] uniquely (by the time the reality property is included) as,

$$i_1(\nu\wedge_1\mathrm{d}r) = \frac{1}{2}(\nu\otimes_1\mathrm{d}r - \mathrm{d}r\otimes_1\nu) + \frac{7\lambda}{4}g_1 + O(\lambda^2)$$

according to the order λ part of the full calculation in [9, Sec 6.2.1] (the 9/4 in [9, eqn (5.21)] was an error and should be 7/4). It was then shown in [9] that $\mathrm{Ricci}_1 = g_1/r^2$. Expanding the quantum metric from [9] as recalled above, the

quantum Ricci is to $O(\lambda^2)$,

$$\begin{aligned}\mathrm{Ricci}_1 &= \frac{1}{r^2}\bullet(\mathrm{d}r\otimes_1\mathrm{d}r + br\bullet\nu\otimes_1\mathrm{d}t - bt\bullet\nu\otimes_1\mathrm{d}r - b\lambda\nu\otimes_1\mathrm{d}r)\\ &= \frac{1}{r^2}\mathrm{d}r\otimes_1\mathrm{d}r + \frac{b}{r}(v+\frac{\lambda}{2}\mathrm{d}r)\otimes_1\mathrm{d}t - (\frac{b}{r^2}\bullet t)(v+\frac{\lambda}{2}\mathrm{d}r)\otimes_1\mathrm{d}r - \frac{b}{r^2}\lambda v\otimes_1\mathrm{d}r\\ &= -\frac{1}{2}\widehat{R}_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu - \frac{\lambda}{2}\frac{bt}{r^2}\mathrm{d}r\otimes_1\mathrm{d}r + \frac{\lambda}{2}\frac{b}{r}\mathrm{d}r\otimes_1\mathrm{d}t\\ &= -\frac{1}{2}\widehat{R}_{\mu\nu}\mathrm{d}x^\mu\otimes_1\mathrm{d}x^\nu + \frac{\lambda}{2}\frac{b}{r^2}\mathrm{d}r\otimes_1 v + O(\lambda^2).\end{aligned}$$

where for the second equality uses $\nabla\mathrm{d}r = \nabla v = 0$ so that bullet products with these are classical products. For the third equality we used $\{\frac{1}{r^2}, t\} = -\frac{2}{r^2}$ from the $\bullet$ which cancels the last term. We obtain exactly this answer by calculation from (2.21).

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