Sequential Refined Partitioning for probabilistic dependence assessment

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Abstract

Modelling dependence probabilistically is crucial for many applications in risk assessment and decision making under uncertainty. Neglecting dependence between multivariate uncertainties can distort model output and prevent a proper understanding of the overall risk. Whenever relevant data for quantifying and modelling dependence between uncertain variables is lacking, expert judgement might be sought to assess a joint distribution. Key challenges for the use of expert judgement for dependence modelling are over- and underspecification. An expert can sometimes provide assessments which are not consistent with any probability distribution (overspecification), and on the other hand, without making very restrictive parametric assumptions an expert cannot fully define a full probability distribution (underspecification). The Sequential Refined Partitioning method addresses over- and underspecification whilst allowing for flexibility about which part of a joint distribution is assessed and its level of detail. Potential overspecification is avoided by ensuring low cognitive complexity for experts through eliciting single conditioning sets and by offering feasible assessment ranges. The feasible range of any (sequential) assessment can be derived by solving a linear programming problem. Underspecification is addressed by modelling the density of directly and indirectly assessed distribution parts as minimally informative given their constraints. Hence, our method allows for modelling the whole distribution feasibly and in accordance with experts' information. A non-parametric way of assessing and modelling dependence flexibly in such detail has not been presented in the expert judgement literature for probabilistic dependence models so far. We provide an example of assessing terrorism risk in insurance underwriting.

Keywords: Structured Expert Judgement, Dependence Modelling, Minimum Information, Terrorism Risk, Uncertainty Modelling, Risk Analysis

1 INTRODUCTION

In many risk and decision analysis problems, we need to quantify uncertainties and their dependence as otherwise a model for risk assessment and decision making might not be fit for purpose. Indeed, quantifying dependence for probabilistic modelling is listed repeatedly among the most significant topics which decision and risk analysis research faces [1, 2]. Therefore, modelling joint distributions in various ways and for several problem types is an active research area (e.g. Durante and Sempi [3], Hanea et al. [4], McNeil et al. [5], Joe [6], Genest et al. [7], Kurowicka and Cooke [8], Embrechts et al. [9]). A common challenge is a lack of relevant data for quantifying dependence models. In such cases, this information should be assessed through expert judgements. A *structured* expert judgement (SEJ) elicitation is the most sensible solution to missing historical data whenever a simplifying assumption, such as independence, is not applicable. Werner et al. [10] and Werner et al. [11] discuss expert judgement methods for dependence in more detail. The former outlines how it is used for several dependence models and reviews commonly elicited forms together with their implication on experts' cognitive burden. The latter presents the main steps of structured dependence elicitations and reviews the most prevalent cognitive fallacies for assessing dependence as their mitigation is a main aim of structured processes. Most applications discussed in these reviews are based on Cooke and Goossens [12, 13] which are among the first guides on SEJ procedures for dependence. Both guides are of further relevance for this paper as they consider in particular the elicitation of conditional exceedance probabilities, an elicited form we will address in more detail later. In this paper, we focus mainly on the process for quantitative elicitation, though we do discuss an approach to structuring experts' knowledge prior to elicitation in the illustrative example of section 4.

For us, dependence means that multiple uncertainties are present and obtaining information about one changes the uncertainty assessment of the other(s). More specifically, we consider the bivariate dependence between two random variables X and Y with joint distribution function $F_{X,Y}(x, y)$ and marginal distributions $F_X(x)$ and $F_Y(y)$. The variables are independent if the assessment of Y does not change when given information about X. Dependence is simply the absence of independence. It is a property of experts' knowledge (and beliefs) and its definition falls therefore into the subjective probability context as in line with De Finetti [14],Savage [15] and Ramsey [16].

We address the problem that experts can only ever assess certain aspects of a joint distribution whereas a decision-maker might desire these assessments to be made at a detailed level. The former implies that we have a partially unknown distribution for which various alternatives fit the given information. This is known as model *underspecification*. More specifically, we are only ever given the probability mass (or density) within some distribution parts, either through their direct assessment or (in parts which are never assessed) through the indirect result of these parts together with related assessed parts having to comply with the marginals. However, we can model these probability masses in various forms which all have the right amount (i.e. are feasible). Of course, we might elicit additional information from experts to distinguish between distributions, yet we need to acknowledge the impossibility of ever eliciting enough information to single out a unique distribution. This is unless adopting a low-dimensional parametric model early on in the modelling process[1]. Such parametric assumptions nevertheless restrict the obtained knowledge on dependencies and we might miss potentially important model aspects, such as random variables' behaviour in the extreme parts (tails) of a joint distribution. Hence, it is often desirable to avoid distributional assumptions which might exclude phenomena that the expert thinks are important.

Within a non-parametric setting, an elicitation should capture detailed distribution features, e.g. the probability mass within narrowly defined parts of the distribution, such as the tails to determine tail dependence, as they result in a more specific distribution, thus making the model more valuable for a decision-maker. Nevertheless, while detailed assessments might be desired by decision-makers, they increase the experts' cognitive burden, potentially resulting in inconsistent and infeasible assessments. This is termed *overspecification*, the second modelling challenge that we encounter[2].

As a non-parametric approach, addressing under- and overspecification, we present the *sequential refined partitioning* (SRP) method for assessments that can be made to any level of detail for any part of a joint distribution. In the SRP method, we address overspecification through an elicitation procedure which never increases the conditioning set to more than one condition and thus maintains a low cognitive complexity. Further, the procedure ensures consistent and feasible assessments through explicit guidance on assessments' feasibility ranges. Underspecifcation is dealt with by allowing the expert to specify as much detail as is desired and by then determining the density form of directly and indirectly assessed parts of the distribution through the unique copula distribution that is minimally informative with respect to the independent copula and that corresponds to the elicited information. This makes only the weak assumption that in the absence of any specific guidance from the expert we should make the copula as close as possible to the independent copula (in the sense of minimizing information). This ensures that the whole distribution is in agreement with the experts beliefs.

We note that there may be other situations where a joint distribution is to be defined but data is incomplete. For example there may be few data, and/or there may be few or no data in the tails. In these cases, we can apply SRP as part of a hybrid method for dependence assessment, so that it can also be applied for copula model selection more generally, i.e. in the research area of empirical

[1]Under low-dimensional parametric assumptions, it suffices to assess a chosen form's main parameters. E.g. eliciting the mean vector and the covariance matrix quantifies a multivariate Gaussian distribution sufficiently.

[2]Overspecification can also occur with parametric models, e.g. if assessed covariances jointly do not result in a positive definite matrix.

copula estimation.

The minimum information approach offers a recognised approach to incomplete knowledge[17]. Further, it allows us to stop the elicitation process at any time and still derive a unique distribution (in contrast to common probabilistic dependence models for which a full conditional probability table is required, e.g. Bayesian (Belief) nets (BNs) [18]). In the context of dependence elicitation, minimum information methods (and related approaches) have been used before, for instance in probabilistic inversion (PI) methods [19, 20, 21, 22], vine-copula quantification [23, 24], or as well joint distributions more generally within decision analysis contexts [25, 26, 27, 28]. However, these previous methods do not consider flexible nor detailed (e.g. tail) dependence assessments and their impact on potential overspecification of experts' judgements and on the minimum information solution to underspecification. For example, Bedford et al. [19] explicitly provide guidance on feasibility constraints. Yet, they consider dependence elicitation at a rather broad level, eliciting only a small number of assessments. This restricts the information to be obtained already early on in the modelling process and thus neglects focusing on specific parts of a distribution more exclusively. The SRP method's contribution is therefore that we provide an elicitation procedure to assess any part of a distribution to any desired level of detail while maintaining low cognitive complexity and avoiding infeasible expert judgements. As such, it also contributes to expert judgement methods for dependence in which increasing conditioning sets pose a concern (see Werner et al. [10] for a discussion). Similarly, the SRP method's approach to underspecification is more detailed than in previous research.

These contributions emphasise the applicability of our method in higher dimensions more generally. While in this paper we focus on assessing bivariate dependence, it should be noted that any d-dimensional copula density can be built through $d(d-1)/2$ so-called pair (bivariate) copulas through a vine structure [24]. This method of modelling dependence, in conjunction with appropriate simplifying assumptions, can avoid the *curse of dimensionality* (see Nagler and Czado [29] for using this approach in the context of kernel estimation). In that way, our method can be extended to higher dimensions of dependence and be used more generally in the area of multivariate density estimation. As such, the SRP method can contribute to more traditional methods of copula estimation for tail dependence assessment through experts when data on extremes are rare.

Figure 1 illustrates the method's modelling context schematically.

In the upper part, we observe that incomplete knowledge leads inevitably to an underspecified model. This is solved by a minimum information approach. In order to derive a model that is valuable for a decision-maker, the modelling process deviates along the dashed lines to the lower part. Here, the constraints of the minimum information problem determined by the experts' judgements

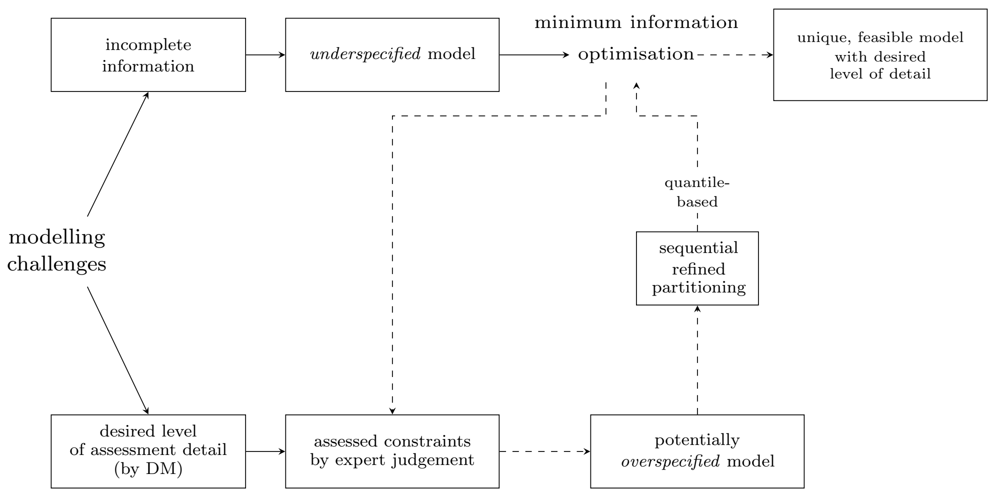


Figure 1: Modelling context of the SRP method.

are assessed as detailed as desired. As these might be overspecified, we the use an elicitation process that leads to feasible assessments. In the remainder of this paper, this is presented in section 2, introducing the elicitation procedure, and section 3, outlining the optimisation problem. Section 4 shows how our method has been used in an insurance underwriting risk assessment of political violence/terrorism in which a detailed and flexible method is of particular interest for stress-testing a model. Finally, section 5 concludes the paper.

2 ELICITING DETAILED DEPENDENCE INFORMATION FEASIBLY AND CONSISTENTLY THROUGH *SEQUENTIAL REFINED PARTITIONING*

In this section, we introduce our sequential elicitation procedure which addresses the potential issue of overspecification by providing explicit guidance on making feasible and consistent assessments. In the expert judgement literature, several approaches to ensuring feasibility and consistency are proposed, each with different implications on the robustness of the final assessment result. As such, some methods (always) allow for an assessment within the elicited forms' standard ranges (for correlation coefficients $\in [-1, 1]$ and for conditional and joint probabilities $\in [0, 1]$). However, this might jeopardise experts' commitment and confidence in the elicitation method if assessments are adjusted afterwards (for ensuring feasibility). While other methods do not modify assessments, they might increase experts' cognitive complexity. For instance, by limiting assessment ranges (away from the aforementioned standard ones), or by imposing unrealistic assumptions onto experts' understanding of elicited forms, e.g. when eliciting conditional judgements with large conditioning sets. For the latter, we might expect an expert to include and equally consider all the information given by a large conditioning set so that common cognitive fallacies, such as the conjunction fallacy and its conditional version (see Werner et al. [11] for an overview on heuristic and biases in dependence assessment), should be (ideally) avoided and hence feasibility is given. Yet, this might not be guaranteed.

In our method, we do not impose such unrealistic assumptions on experts' cognitive capabilities, nor do we modify assessments after they have been given. Rather, we only ever elicit single conditioning sets and give guidance on possible feasible assessment ranges. This includes not only providing the corresponding upper and lower bounds but also explaining their interpretation.

Mathematically, the feasibility range for any sequential assessment procedure is derived by solving a linear programming (LP) problem (see Vanderbei [30] for an introduction to LP). The number of constraints is restricted to a maximum of nine, irrespective of the number of elicitations. In the remainder of this section, we first present the general set-up together with the relevant proofs before we outline some specific elicitation sequences, which we regard as of interest for several practical applications.

2.1 General set-up of sequentially refined partitioning

We shall start by introducing some definitions. The unit square is here defined as the product of $(0,1] \times (0,1]$. Given values $u_0 = 0 < u_1 < \cdots < u_n < 1 = u_{n+1}$, and $v_0 = 0 < v_1 < \cdots < v_m < 1 = v_{m+1}$, we define the associated *quantile partition* of the unit square as the set of rectangles of the form $(u_i, u_{i+1}] \times (v_i, v_{i+1}]$. We call this set of rectangles $QP(u, v)$.

Given (p, q) with p different to the u_i and q different to the v_j, the (p, q)-refinement of $QP(u, v)$, denoted $QP(u, v; p, q)$, is the quantile partition obtained by including p and q in the values for u and v respectively. All rectangles in the old partition are either in the new partition or are a union of two or four rectangles of the old partition. Figure 2 shows two partitioned example distributions which result from any number of previously elicited quantiles (solid lines) in addition to new ones (dashed lines).

A probability distribution on a quantile partition $QP(u, v)$ simply assigns a probability value to each rectangle of the quantile partition. A (p, q)-*refinement* of such a probability distribution is a probability distribution on $QP(u, v; p, q)$ such that the probability of a rectangle in $QP(u, v)$ is either the same as it is in the (p, q)-refinement of $QP(u, v)$, or it equals the sum of the probabilities of the rectangles that make it up.

A *merging* of a quantile partition $QP(u; v)$ is obtained by merging together some of the partition rectangles in such a way that we still have a quantile partition. This can also be obtained by taking

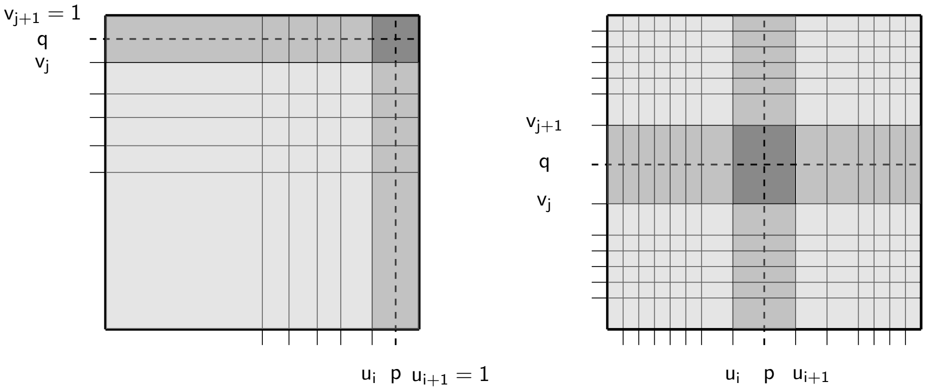


Figure 2: Partition example $QP(\tilde{u}, \tilde{v}; p, q)$ with solid lines for previously elicited quantiles and dashed lines for new ones.

a subsequence of the u's and v's and building the corresponding quantile partition. A *merged probability distribution* on the refined quantile partition is obtained by adding together the probabilities of the rectangles in each refined rectangle.

We always work with *discrete copula distributions*, which are probability distributions on a quantile partition that have the additional property that (for any k) the sum of probabilities of rectangles $(u_i, u_{i+1}] \times (v_i, v_{i+1}]$ with $u_{i+1} \leq u_k$ is equal to u_k, and similarly, the sum of all probabilities of rectangles $(u_i, u_{i+1}] \times (v_i, v_{i+1}]$ with $v_{i+1} \leq v_k$ is equal to v_k. For a general introduction to copula theory, see Nelsen [31],Joe [6] and Durante and Sempi [3]. However, note that most theory is on continuous copulas with marginals being continuous uniform distributions. For an overview on elicitation methods for copulas, see Werner et al. [10].

Proposition 1. *Suppose we are given values $u_0 = 0 < u_1 < \cdots < u_n < 1 = u_{n+1}$, and $v_0 = 0 < v_1 < \cdots < v_m < 1 = v_{m+1}$ (where $n, m > 0$), $0 < p, q < 1$, with p different to the u_i and q different to the v_j. Then a copula distribution on $QP(u, v)$ can be refined to a copula distribution on $QP(u, v; p, q)$.*

The proof of proposition 1 is found in the Appendix.

Having shown that we can always refine a copula distribution as above, we now wish to establish the possible range of values that can be taken by the rectangle $(p, 1] \times (q, 1]$ in a refined copula distribution. That is, we depart from the specific copula refinement defined in the Proof of Proposition 1, and ask what range of values can be allocated as the probability of $(p, 1] \times (q, 1]$ in some copula refinement.

Suppose that i and j are chosen such that u_i is the largest of the u-quantiles that is smaller than p, and v_j is the largest of the v-quantiles that is smaller than q (this includes the possibility that u_i or v_j is 0, or that u_{i+1} or v_{j+1} is 1). Define $\tilde{u}_1 = u_i$, $\tilde{u}_2 = u_{i+1}$, $\tilde{v}_1 = v_j$ and $\tilde{v}_2 = v_{j+1}$. The quantile

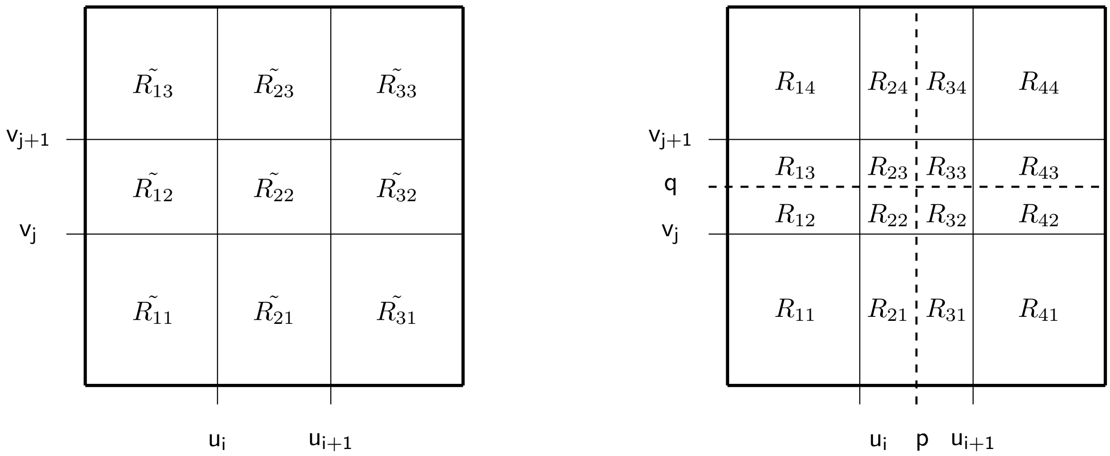


Figure 3: Maximum case of 16 partitions (right) resulting from partitioning 9 rectangles (left).

partition $QP(\tilde{u}, \tilde{v})$ is a merging of $QP(u, v)$, and we can merge the copula distribution on $QP(u, v)$ to get one on $QP(\tilde{u}, \tilde{v})$.

Furthermore $QP(\tilde{u}, \tilde{v}; p, q)$ is a merging of $QP(u, v; p, q)$. Note that $QP(\tilde{u}, \tilde{v})$ has at most 9 rectangles and that $QP(\tilde{u}, \tilde{v}; p, q)$ has at most 16 rectangles - see Figure 3.

For convenience we shall now consider only the case of 16 rectangles, which occurs when $u_i, v_j \neq 0$ and $u_{i+1}, v_{j+1} \neq 1$, as shown on the right of Figure 3. Other cases are simplifications of the one we consider here and can be dealt with in the same way.

We label the 16 rectangles of $QP(\tilde{u}, \tilde{v}; p, q)$ as $R_{11}, \ldots, R_{44}$ as shown in the right hand of Figure 3.

The 9 rectangles of $QP(\tilde{u}, \tilde{v})$ are labelled as $\tilde{R}_{11}, \ldots, \tilde{R}_{3,3}$ as shown in the left hand of Figure 3. Clearly $R_{11}, \ldots, R_{4,4}$ are each unions of rectangles in $QP(u, v)$, and furthermore,

$$R_{12} \cup R_{13} = \tilde{R}_{12}$$

$$R_{42} \cup R_{43} = \tilde{R}_{32}$$

$$R_{21} \cup R_{31} = \tilde{R}_{21}$$

$$R_{24} \cup R_{34} = \tilde{R}_{23}$$

$$R_{22} \cup R_{23} \cup R_{33} \cup R_{32} = \tilde{R}_{22}.$$

Suppose we are given a copula distribution on $QP(\tilde{u}, \tilde{v})$, for which $\tilde{p}_{st}$ is the probability of $\tilde{R}_{st}$ $(s, t = 1, 2, 3)$. We wish to assign copula probabilities p_{st} to the rectangles R_{st} $(s, t = 1, 2, 3, 4)$ so that the new distribution merges to p on $QP(\tilde{u}, \tilde{v})$.

For the merging we simply require,

- for the corner rectangles of $QP(\tilde{u}, \tilde{v})$: $p_{11} = \tilde{p}_{11}$, $p_{14} = \tilde{p}_{13}$, $p_{41} = \tilde{p}_{31}$, $p_{44} = \tilde{p}_{33}$, 187 - for the central rectangle in $QP(\tilde{u}, \tilde{v})$: $p_{22} + p_{32} + p_{23} + p_{33} = \tilde{p}_{22}$,

- for the remaining rectangles

$$p_{12} + p_{13} = \tilde{p}_{12}$$
$$p_{42} + p_{43} = \tilde{p}_{32}$$
$$p_{21} + p_{31} = \tilde{p}_{21}$$
$$p_{24} + p_{34} = \tilde{p}_{23}.$$

To ensure that the new distribution is a copula we also need to impose two constraints corresponding to a row and a column:

$$p_{21} + p_{22} + p_{23} + p_{24} = p - \tilde{u}_1$$
$$p_{12} + p_{22} + p_{32} + p_{42} = q - \tilde{v}_1.$$

(Note that these constraints correspond to row 2 and column 2 of the right hand of Figure 3. We could also have specified similar constraints on row 3 and column 3, but it straightforward to see that these are redundant).

Now define,

$$f(p_{11}, ..., p_{44}) = p_{33} + p_{43} + p_{34} + p_{44}$$

188 to be the total probability in the square $(p, 1] \times (q, 1]$. This in a linear function of the p_{st} and we are
189 free to choose it to take any value subject to the constraints listed above. As all these are linear,
190 we immediately see that we have the form of a linear programming problem, and so the range of
191 allowable values is an interval whose maximum and minimum values can be found be solving 2 LP
192 problems. The cases in which $QP(\tilde{u}, \tilde{v}; p, q)$ has fewer than 16 rectangles work similarly. The above
193 discussion (with minor adaptations to the other cases by removing further redundant constraints)
194 can be summarized in the following Proposition:

Proposition 2. *The range of feasible values for the probability of $(p, 1] \times (q, 1]$ in any copula refinement of the copula distribution on $QP(\tilde{u}, \tilde{v})$ is given by the interval:*

$$[\min f, \max f],$$

195 *given the corresponding constraint sets.*

196 *We can obtain* $\min f$ *and* $\max f$ *by solving feasible LP problems with at most* 12 *variables and* 9
197 *constraints.*

This now allows us to construct an algorithm for assessing copulas with expert judgements for quantile exceedance probabilities of the form:

$$P(Y > y_q | X > x_p)$$

where x_p and y_q index the p^{th} and q^{th} quantile for X and Y accordingly. For example, $p = 0.5$ and $q = 0.5$ correspond to the medians of X and Y. Other distribution areas can then be derived. Given a number of such coherent elicitations at quantile pairs $(u_1, v_1), \ldots, (u_n, v_n)$ we can calculate the copula distribution on the copula partition $QP(u, v)$.

For a new quantile pair (p, q), we then solve the LP problem to obtain the exact feasible range for the probability of $(p, 1] \times (q, 1]$. Note that this does not fully specify the distribution on all elements of the refined partition $QP(u, v; p, q)$. To achieve this, either

(a) we can carry out further elicitations at corner points in $QP(u, v; p, q)$ using proposition 2 repeatedly for obtaining feasible ranges from the expert; or

(b) we can make assumptions, such as minimally informative probabilities to restrict the number of elicitations required.

In the next section, we give a simple example of making assessments in the tail of the distribution along the lines of (a) but carried out in a slightly different order as there are few constraints in this case.

2.2 Commonly assessed quantile partition sequences

After having presented the mathematical set-up of refined partitioning generally, we now discuss some partitions that might be commonly assessed in practice.

One recurrent way of refining a joint distribution's assessments is by sequentially choosing a quantile for p and/or q that is always either higher or lower than any previously assessed value. Then, we elicit the corresponding area above a previously elicited quantile for a new maximum or below it for a new minimum. Such sequences assess in particular the distribution tails more explicitly.

Alternatively, it is (also) possible in our method to elicit probabilities of specific values, e.g. for $1, 10, \ldots, 100$ (units of elicited variable) rather than common quantiles, such as the median, if this can increase intuitiveness in particular for more extreme parts in the distribution tails. This relates to the choice of whether to frame the elicitation question in terms of quantiles or values. Both have been suggested (as P- and V-methods) since the pioneering probability elicitations by the Stanford Research Institute in the 1970s [32]. While a more recent discussion on this choice is given in [33], we consider in the following the elicitation of quantiles.

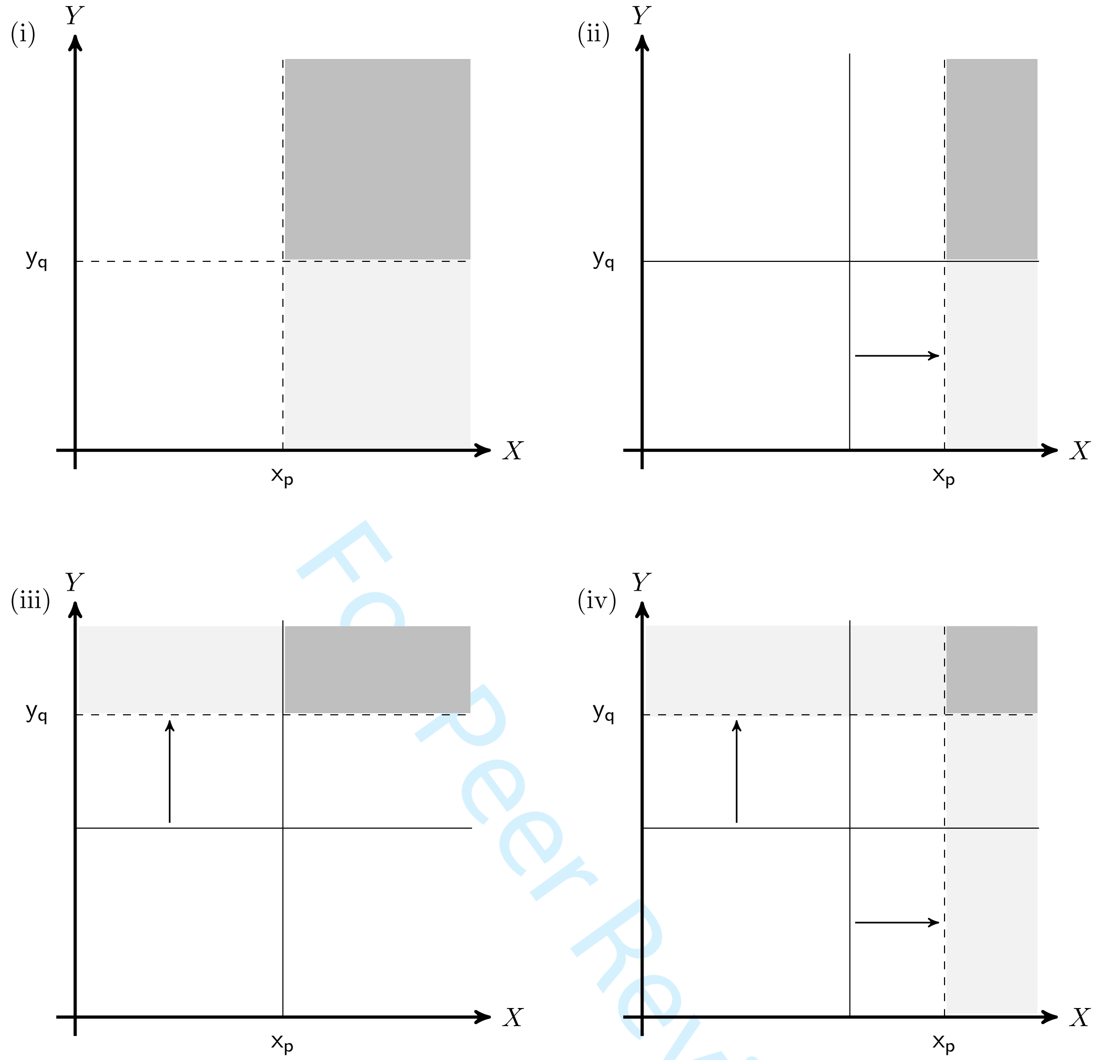


Figure 4: Example of a quantile partition for assessing the upper tail.

Figure 4 illustrates a sequence of quantile partitions on the upper tail constructed through setting new quantile maxima in (ii) to (iv) following an initial assessment (i) (note that this carries out the option (a) described in the previous section). We consider the procedure of Figure 4, i.e. further partitioning that probability mass which has been assessed directly in step (i) as most intuitive and practically useful. Nevertheless, the initial assessment also determines the probability mass in areas of the joint distribution which are not assessed further, $P(Y > y_q|X \leq x_p)$, $P(Y \leq y_q|X > x_p)$ and $P(Y \leq y_q|X \leq x_p)$, meaning we can also use a similar procedure to refine these.

First (in (i)), we elicit an overall probability mass and then subsequently refine the assessment. Suppose we first elicit $P(Y > y_{0.5}|X > x_{0.5})$.

Following (i), we elicit a refined quantile partition as determined by a new x_p in (ii). A common choice here might be the 90^{th} or 95^{th} quantile in order to assess the probability mass in the joint

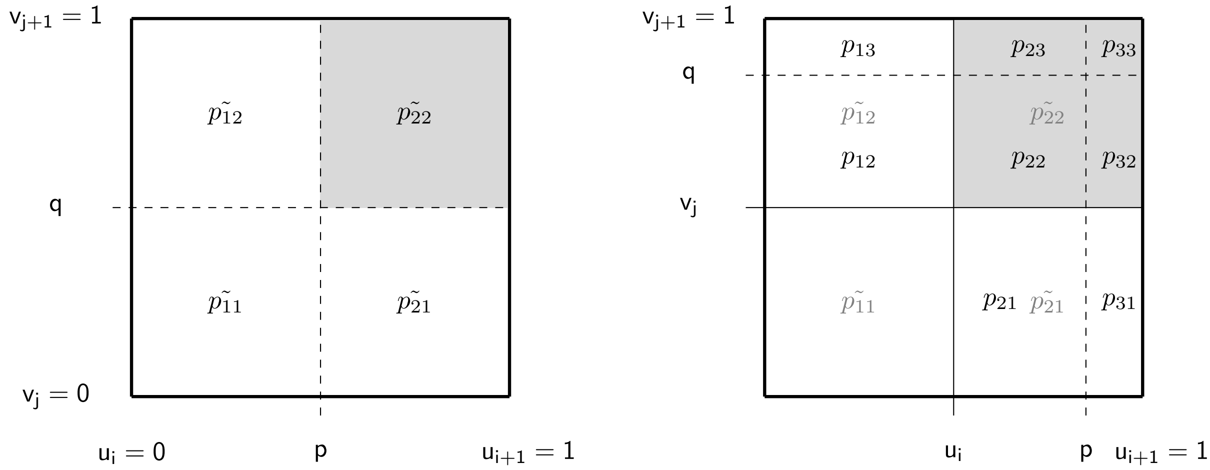


Figure 5: Quantile partition of the joint distribution from (i) to (iv).

distribution's extreme (tail) region. Thus, we elicit for instance $P(Y > y_{0.5}|X > x_{0.95})$. In the illustrative case-study of Section 4, we use a scenario mapping method [34] prior to the elicitations in order to gauge experts' familiarity with such tail judgements and decide on a quantile for which experts are comfortable to make assessments.

In (iii), we condition on Y and the new y_q is chosen to assess the tail region. With x_p being the median, we thus elicit $P(X > x_{0.5}|Y > y_{0.95})$. Depending on the underlying meaning of the variables and knowledge about causal or probabilistic relationships (see e.g. Rottman and Hastie [35], Werner et al. [11]), the expert might find it easier to condition on one variable rather than the other. Our method is flexible enough to allow for this.

In the last step of this quantile partition sequence, experts assess either $P(Y > y_{0.95}|X > x_{0.95})$ or $P(X > x_{0.95}|Y > y_{0.95})$, depending on case-specific interest, whereas p and q are the ones from the previous two rounds. Thus we further explore the joint tail region. Figure 5 displays the refinement in the quantile partition from the first to the latest assessment.

The assessments' feasibility ranges are as follows. The assessment in (i) is unrestricted, meaning experts can assess any value between $[0, 1]$. If the expert believes the variables are independent, the assessment is equal to $P(Y > y_q)$, that is learning about X does not change experts' belief. For negative dependence, the assessment is between $[0, P(Y > y_q))$ and for positive dependence, it is within $(P(Y > y_q), 1]$.

All following assessments on the other hand are restricted and only feasible if the assessed value falls within the range which is determined by solving the LP problem of minimising and maximising the possible values of the assessed area subject to the constraints that any new partition simply adds up to their previous assessments (see medium and dark grey areas $\tilde{P}_k$ in Figure 2) while areas which have not been newly partitioned do not change (see light grey areas $\tilde{P}_k$ in Figure 2). Consider for example the assessment in (iv). It is only feasible within the range that is determined by solving the following LP problem (with regards to Figure 4 on the right):

$$\left.\begin{matrix}\min\\ \\ \max\end{matrix}\right\} p_{33} \tag{2.1.1}$$

subject to

$$p_{13} + p_{12} = \tilde{p_{12}} \tag{2.1.2}$$

$$p_{23} + p_{22} + p_{33} + p_{32} = \tilde{p_{22}} \tag{2.1.3}$$

$$p_{11} = \tilde{p_{11}} \tag{2.1.4}$$

and

$$p_{21} + p_{31} = \tilde{p_{21}} \tag{2.1.5}$$

Experts express negative dependence again through a judgement close or equal to the lower bound, positive dependence is expressed by judgements close or equal to the upper bound and independence is assessed as before. As the upper and/or lower bounds deviate from the standard range of $[0, 1]$, it is necessary to communicate these restricted feasibility bounds to an expert and explain their interpretation.

The procedure for assessments (ii) to (iv) is repeated as often as necessary (with appropriate modifications) to obtain a desired level of detail (see assessments (v) to (vii) in Figure 6 for the next round of three assessments). Having assessed previously the 90^{th} or the 95^{th} quantile of X and Y, we now might consider the 99^{th} quantile. This allows for "zooming in" on the joint distribution's tail even further.

The resulting quantile partitions are illustrated in Figure 7.

While this section presents an example with a focus on refining the upper distribution tail, remember that the generality of the method (as introduced in Section 2.1) allows for any further refinement of the distribution, such as for instance shown in Figure 2 (on the right).

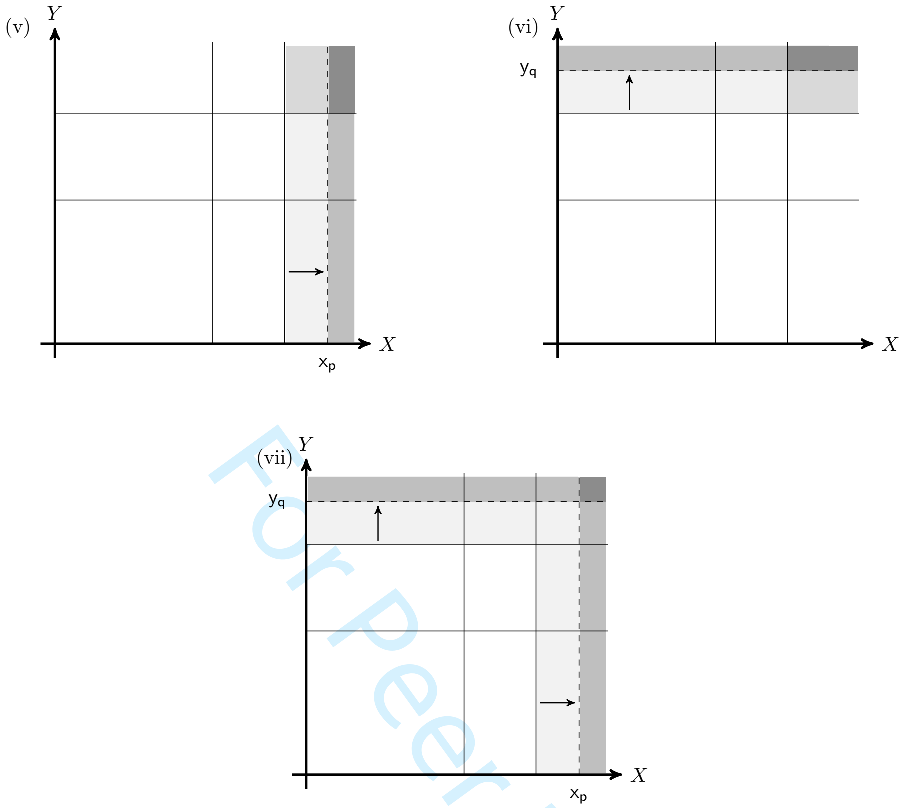


Figure 6: Further refining the assessment on the joint upper distribution tail.

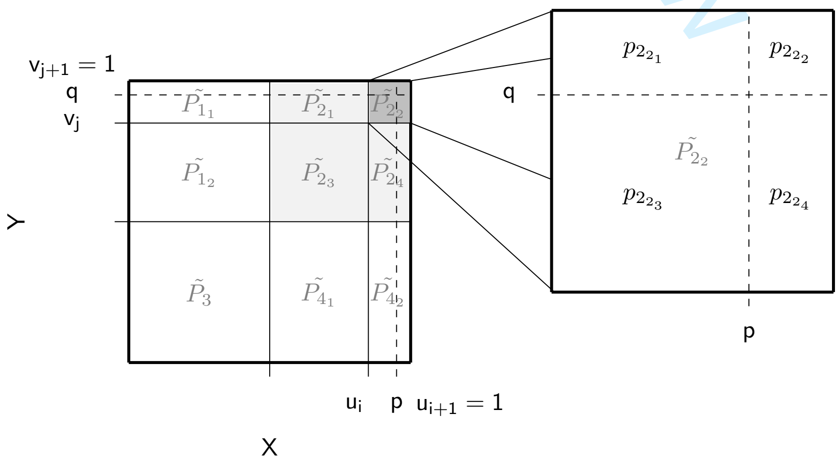


Figure 7: Resulting quantile partitions after further refining the previous assessments.

3 MODELLING THE FORM OF DIRECTLY AND INDIRECTLY ASSESSED PROBABILITY MASSES THROUGH MINIMUM INFORMATION

After having presented the elicitation procedure, which allows for feasibly assessing the probability mass within any part of the joint distribution, in this section we outline how we model the form of directly and indirectly assessed parts as minimally informative.

The reason for a minimum information approach is to address the modelling issue of underspecification. We do not have enough information for choosing a distribution that fits the experts' assessments uniquely but we wish to find the simplest distribution that matches them. This approach allows us to derive a unique distribution regardless the quantile partition's level of detail. As such, it does not restrict the flexibility of the assessment procedure from section 2.

Formally, we aim for modelling dependence through that copula which is chosen to have minimum information (also called *Kullback-Leibler divergence* [36]) with respect to the uniform copula given the quantile constraints. The resulting distribution is considered the most independent copula satisfying the constraints.

Consider the joint distribution $g(x, y)$ with marginal densities $g_1(x)$ and $g_2(y)$. Whenever g_1 and g_2 are not independent, i.e. $g(x, y) \neq g_1(x)g_2(y)$, we need to model the dependence between them. To do so, we introduce the concept of relative information $I(g; h)$ which is a measure of similarity between the two distributions and it is defined for $g(x)$ with respect to $h(x)$ as:

$$I(g; h) = \int g(x) \log \left(\frac{g(x)}{h(x)} \right) \mathsf{d}x$$

Whenever $g(x) = h(x)$, it follows that $I(g; h) = 0$. A higher value of $I(g_1; g_2)$ corresponds to less similarity. We consider $h(x)$ a background distribution, commonly chosen as uniform or log-uniform. Alternatively, we use sensitivity analysis for selecting an appropriate form [19]. Together with the constraints, this choice determines the form of $g(x)$ in absence of further information [23].

Information is invariant under monotone transformations. Therefore, if c_g and c_h are copula densities associated with the previous densities g and h, we have $I(c_g; c_h) = I(g; h)$. In particular if h is the joint independent distribution with the same marginals as g (g_1 and g_2), so that $h = g_1 g_2$ then $I(g; g_1 g_2) = I(c; \textbf{uniform})$ where h is the uniform copula. This gives the interpretation of our minimum information copula as the most independent copula given the constraints.

See Bedford and Wilson [37] for a detailed derivation on how a minimum information distribution can be approximated by the equivalent distribution of maximum entropy[3] [39].

For an extensive discussion on obtaining a minimum information copula through the convex optimisation problem, we refer to [23, 37, 19]. Here, it suffices to say that the conditional density within each rectangle is uniform. As discussed in Section 2, when we stop eliciting information from experts, some rectangles' density has been directly assessed by an expert while for other rectangles the mass is given indirectly through related assessment and the marginals. In order to obtain a unique solution for the whole distribution, we hence need to solve the minimisation problem of equation for directly and indirectly assessed parts.

We refer to Bedford and Wilson [37] and Meeuwissen and Bedford [21] for the corresponding proofs that such a minimum information distribution exists and is unique. Furthermore, Bedford et al. [19] and Bickel and Smith [28] discuss and apply a Lagrangian dual for a minimum information problem to show a way for obtaining more insight on the optimal solution.

4 AN ILLUSTRATIVE CASE-STUDY: ASSESSING SPATIAL DEPENDENCE OF POLITICAL VIOLENCE/ TERRORISM RISK IN INSURANCE UNDERWRITING

Given the flexibility and detail that the SRP method allows for when modelling dependence, we regard it as of particular interest for application areas in which common simplifying assumptions, such as bivariate normality, are not justified. Rather, different kinds of tail dependencies which potentially induce extreme impact scenarios are prevalent. For these, we often assess and model upper and lower tail dependence exclusively (similarly to testing the goodness of fit for asymmetric, Archimedean copulas to historical data when available) given that e.g. joint large losses are typically not observed together with joint large gains[40, 41].

As such, we consider (re-)insurance as an industry in which rigorous dependence modelling approaches are of particular interest. Due to the increasing complexity of (re-)insurance products, new (holistic) modelling approaches, such as *dynamic financial analysis* (DFA) (a Monte Carlo simulation-based method to model risks jointly), have become popular among actuaries to better understand the risks an insurer underwrites [9]. For these new approaches, flexible and detailed assessments of dependencies under a specific probability model are required. Exemplary for a DFA application, Eling and Toplek [42] present how various parametric copulas can be used for stress-testing an insurer's risk management strategies together with the implication on stakeholders, such

[3]In the context of expert judgement, an invariance approach to encoding information probabilistically is considered a main justification for maximum entropy methods North [38].

as regulators and rating agencies. The DFA model inputs, the perils (or risks) covered by an insurer, are informed by a *catastrophe model*. The components of catastrophe models are a hazard, inventory, vulnerability and loss estimation module. The loss estimation output is usually an exceedance probability curve specifying probabilistically the severity levels of a certain hazard in a region. Capturing relevant dependencies between severity levels is crucial for a more robust output. See Grossi and Kunreuther [43] for an introduction to catastrophe models.

We have already established that a common challenge is lacking relevant historical data for quantifying dependence relationships serving as model input. In actuarial risk assessment, *non-life* insurance underwriting is particularly challenged. So called *low frequency-high severity* perils, natural and man-made, are by definition not frequently observed but cannot be ignored. Therefore, we require structured expert judgement to model their uncertainty. In this illustrative case-study, we apply the SRP method to elicit and model the spatial dependence of the man-made peril of terrorism. Terrorism attacks are not only often low frequency-high severity catastrophes but pose an additional challenge due to intelligent adversaries which further inhibit the use of historical data. Better understanding the dependence between terrorism attacks' frequencies in different regions globally is nevertheless key for an insurer to quantify and price this peril's risk when managing a portfolio of (global) clients[4].

4.1 Pricing terrorism risk in insurance

Traditionally, pricing of terrorism risk in insurance has not been evaluated from actuarial principles, but rather covered by the balance of supply and demand in the insurance market together with some less formal risk selection from site surveys [44]. Terrorism coverage (e.g. in the United States) had been included in standard commercial insurance policies as an unnamed peril on all-risk commercial and home owners coverages for property and contents [45]. More recent loss developments though have highlighted the necessity of treating its risk assessment more rigorously. A major turning point for dealing with terrorism risk in insurance was the attacks of September 11^{th}, 2001 (9/11) on the United States. The attacks incurred an estimated monetary loss up to 60 billion US dollars, distributed among various lines of business, such as property insurance, business interruption insurance and workers' compensation [46]. Globally, the worst 15 terrorist attacks in terms of casualty numbers have occurred since 1982 with many more near-miss events [45]. Mathematically, the relationship between the frequency of more recent attacks and their severity can be described by a power law, i.e. attack severities that are orders of magnitude larger than the mean can be common

[4] As Woo [44] emphasises, we must not confuse quantifying terrorism risk with predicting a next attack. This is similar to natural catastrophes, such as earthquakes, for which we cannot determine the time, location and severity of the next event, but the aim is rather to evaluate the annual exceedance probability of loss, for instance to inform a property insurance portfolio.

[47]. The changing nature of its risk through an increasing number of frequencies and severities in multiple regions globally underlines the urgent need for improved assessment.

4.2 Expert judgement for adversarial risk

A specific aspect of assessing terrorism risk is the role of intelligent adversaries. Their impact is thus included in recent discussions on risk definitions [48, 49, 50, 51]. In fact, 9/11 led many researchers to propose modified risk definitions [52]. For instance, the triplet definition by Kaplan and Garrick [53] is extended to include adversaries in Garrick et al. [54] and Garrick [55] by considering the likelihood of a hazard as the conditional probability of a successful attack given that an attack is planned.

Models addressing adversarial risk are typically of game-theoretic nature [56, 57, 58] whereas the area of *adversarial risk analysis* comprises decision-analytic approaches combining traditional *probabilistic risk analysis* (PRA) methods with game theory [59, 60, 61, 62]. Nevertheless, there is some debate on (traditional) PRA's effectiveness for adversarial problems (see Ezell et al. [56] defending its usefulness and Brown and Cox Jr [63] and Cox Jr [64] arguing against it). A main argument against PRA approaches for adversaries is the dynamic attacker's decision rule for choosing a target as this choice might be based on the anticipated defender's assessment of targets' likelihoods. In other words, a defender's PRA might inform the attacker's choice and hence override its purpose as the previously most likely target has now zero probability of being attacked (closely related in terrorism risk analysis are decision on allocating defensive resources Bier [65]). Experts quantifying adversarial risk should therefore decompose their judgement in accordance with adversarial risk definitions, so that we understand experts' beliefs about attackers' choices. When doing so, assessments of an attack choice might be based on attackers' motivations, resources and capabilities together with defenders' vulnerabilities. In that way, expert judgement is used in the Probabilistic Terrorism Model by Risk Management Solutions Inc. (RMS[5]) for assessing likelihoods on target selection, capabilities of attack modes and an attack's overall likelihood. However, dependence between targets is neglected [66]. In other approaches, event trees are used to reason from an attacker's capabilities through a defender's countermeasures [67, 68]. In addition, several qualitative approaches for structuring the available knowledge on terrorists' objectives and motivations exist in the risk and decision analysis literature [69, 70, 54].

[5]RMS, founded at Stanford University in 1989, provides services in the area of catastrophe modelling for (re-)insurers.

4.3 Expert judgement for spatial dependence of terrorism attacks

Knowledge and beliefs on terrorists' motivations, resources and capabilities together with defender's vulnerabilities inform experts directly about the spatial dependence between attack frequencies. Terrorist groups, such as the Irish Republican Army (IRA), Basque Separatist Group (ETA) or as well Hamas and the Palestine Liberation Organization (PLO), had and have specific geographical foci with a politically motivated attack purpose. Their goals are formulated and self-proclaimed as separatism or liberation. The attacks' geographical impact is identified straightforwardly. Based on the number of active terrorist groups per region plus their resources and capabilities relative to counter-measures, an expert assesses either positive or negative dependence. While positive dependence might not seem intuitive at first due to different local foci and typically a lack of collaboration between these groups, learning and encouragement by another groups' successes can still occur. Woo [71] regards learning of optimal behaviours beyond the own organisation as a main strength of some well-known terrorist groups. Other scenarios for positive dependence can be due to defenders' collaboration, joint counter-terrorism activities and sharing of intelligence resources.

In contrast to terrorists motivated by self-proclaimed liberatism and separatism, other groups derive their goals from religious ideology. These groups are often globally active. Their members are organised as multiple independent hubs with satellite cells. Al-Qaeda and the Islamic State of Iraq and Syria/the Levant (ISIS/ISIL) are typical examples of such network-based organisations [44, 72]. Models from swarm intelligence and statistical network analyses are used to evaluate the effectiveness of counter-terrorism measures and understand the attackers' capabilities. It is understood that organisations like Al-Qaeda and ISIS/ISIL are more resilient and capable of more severe attacks than (hierarchical) army-like structured groups [71]. For dependence assessments, understanding the global presence of members and sympathisers (potentially future recruits) together with the functioning of the network structure is crucial. For instance, scenarios of positive dependence can occur when a terrorist group obtains more power and resources to extend globally or when new attack types are used for which little intelligence or counter-measures exists. Scenarios of negative dependence might describe attackers' scarce resources, e.g. lacking financial support for regional hubs, so the target focus shifts towards a certain region. The latter also depends on vulnerabilities of target countries, desired attention through media or as well a planned revenge, e.g. for a country's military actions.

While these are only brief considerations for scenarios that can influence the assessment of dependence between the number of terrorist attacks in different regions (see Woo [71] for a more extensive discussion on regional and global terrorism), it shows the complexity of factors to be thought of. In this illustrative case-study, we focus on the geographical regions of Central Asia (CA) and Western

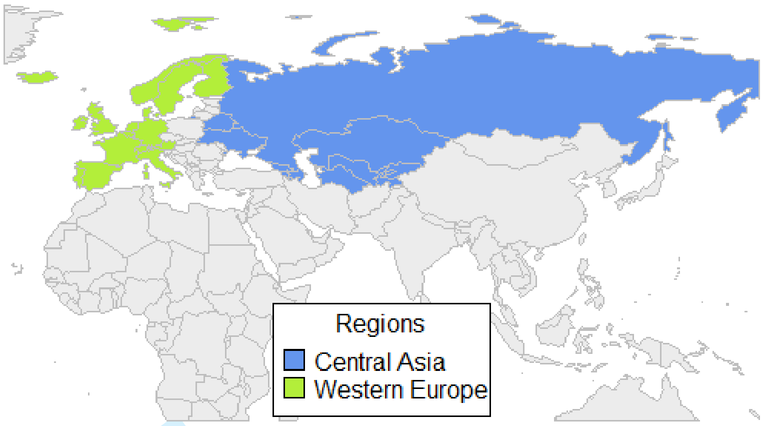


Figure 8: Regions of interest for dependence assessment.

Europe (WE) which are shown in Figure 8 (see the Appendix for a full list of the countries included per region).

4.4 Eliciting the marginal probabilities

Before eliciting dependence assessments from experts, we need to specify the marginal distributions for the variables of interest. Otherwise, the experts condition their judgements on different marginal probabilities and their assessments cannot be sensibly aggregated. The specification is done either through historical data (if available) or another, prior elicitation with a structured expert judgement method for univariate uncertainty, such as Quigley et al. [73], Gosling [74], Hanea et al. [75]. A structured elicitation for the marginal distributions is also encouraged when eliciting dependence only from one expert, i.e. without aggregation, as this mitigates potential biases of the marginals and ensures transparency[11].

In our case-study, the marginal distributions have been assessed by 16 experts[6]. The experts are involved in analysing and pricing the peril of terrorism and other armed conflict categories. They work for different (re-)insurers, catastrophe modellers and related service providers. The elicitation session was organised as part of the European Cooperation in Science and Technology, COST Action IS1304 - Expert Judgement Network, which aims at stimulating the emergence and spread of high quality evidence-based decision support approaches through structured expert judgement methods. The marginal distributions $F_X(x)$ and $F_Y(y)$ are defined as the number of terrorist attacks in Central Asia (x) and in Western Europe (y), both in 2017. We define a terrorist attack in accordance with common global data-bases on the topic (see START [76]). Thus, for an attack to be recorded as

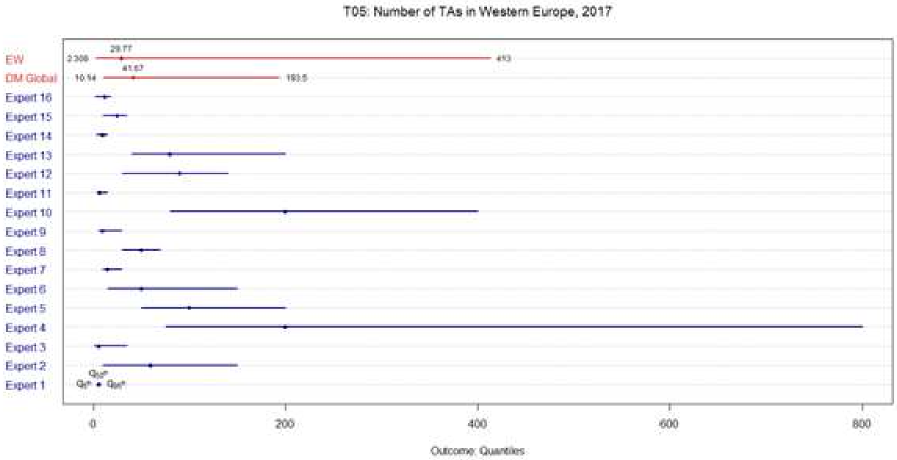
[6]The 16 experts are from a first elicitation round from a currently ongoing study that aims to include more experts.

such there must be evidence of an intention to coerce, intimidate, or convey some other message to a larger audience (or audiences) than the immediate victims. In this regard, any perpetrator group, any weapon type (e.g. biological, chemical, explosive, firearms etc.), any attack type (e.g. armed assault, bombing, facility/infrastructure attack, hostage taking etc.), any target apart from private persons (i.e. business, infrastructure, military, educational/religious institutions etc.) is included. We elicited $F_X(x)$ and $F_Y(y)$ through the so called *Classical Model* [73, 77]. Experts provide various quantile assessments for a continuous quantity rather than point estimates. Usually (and in our case), we elicit the 5^{th}, 50^{th} and 95^{th} quantile. The experts answer two types of questions. The first questions are about so called seed or calibration variables. For these, the true value is known to the analyst but not the experts at the time of the elicitation (or they will be known later and within the time frame of the study). The second question type is about the actual target value or variable of interest, i.e. the uncertainties we intend to include in the model. Based on each expert's assessments of the seed variables, the experts are aggregated. For that, two performance measures are derived, the calibration and information score. Loosely, the calibration score measures the statistical accuracy of the experts whose assessments are treated as statistical hypotheses. The information score measures the assessments' concentrations relative to a background distribution. Good expertise is shown by a high calibration and information score (see Cooke [77] for a more detailed introduction). Figure 9 shows each experts' individual assessment for the target variables' marginal distributions together with the aggregated assessments of equal weighting (EW) and the classical method (DM global).

We observe in Figure 9 that the marginal distribution assessments are similar for both regions whereas most of the experts provide narrow uncertainty bounds. The experts who are more uncertain are so for both assessments. Hence, the performance-based and the equally weighted combination show no major difference for either region. As commonly observed with the classical method, the performance-based aggregation is more informative even if both combinations lead to similar median assessments. The full documentation, the elicitation protocol together with results and raw data for the above elicitation can be found in Werner [78].

4.5 Applying the SRP method for quantifying spatial dependence of terrorism risk

Once the marginal distributions had been elicited, we proceeded with eliciting and modelling dependence through the SRP method. This elicitation was done with a single expert who is a professional in the area of terrorism catastrophe modelling within (re-)insurance (as well) and who subscribed to the aggregated results for the marginal distributions. In total, we elicited six dependence judgements



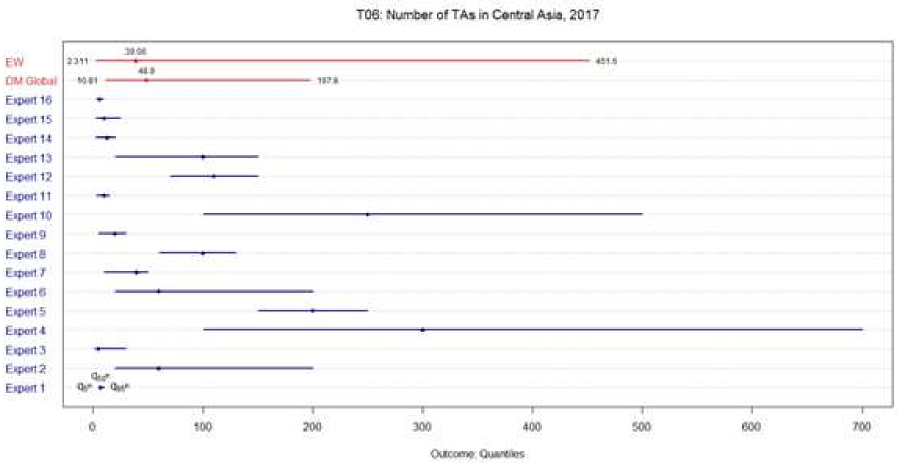


Figure 9: Outcome of eliciting the marginal distribution for each region.

Table I: Overview of dependence elicitation procedure and results.

	Framing "Given that we observe [...]"	**Conditional Probability**	**Assessment**
(i)	"[...] more than **73** terrorist attacks in CA, what is your probability that we observe more than **62** terrorist attacks in WE?"	$P(Y > y_{0.5} \mid X > x_{0.5})$	**0.5**
(ii)	"[...] more than **199** terrorist attacks in CA, what is your probability that we observe more than **62** terrorist attacks in WE?"	$P(Y > y_{0.5} \mid X > x_{0.95})$	**0.03**
(iii)	"[...] more than **197** terrorist attacks in WE, what is your probability that we observe more than **73** terrorist attacks in CA?"	$P(X > x_{0.5} \mid Y > y_{0.95})$	**0.045**
(iv)	"[...] more than **199** terrorist attacks in CA, what is your probability that we observe more than **197** terrorist attacks in WE?"	$P(Y > y_{0.95} \mid X > x_{0.95})$	**0.025**
(v)	"[...] more than **199** terrorist attacks in CA, what is your probability that we observe more than **225** terrorist attacks in WE?"	$P(Y > y_{0.99} \mid X > x_{0.95})$	**0.04**
(vi)	"[...] more than **225** terrorist attacks in WE, what is your probability that we observe more than **199** terrorist attacks in CA?"	$P(X > x_{0.95} \mid Y > y_{0.99})$	**0.01**

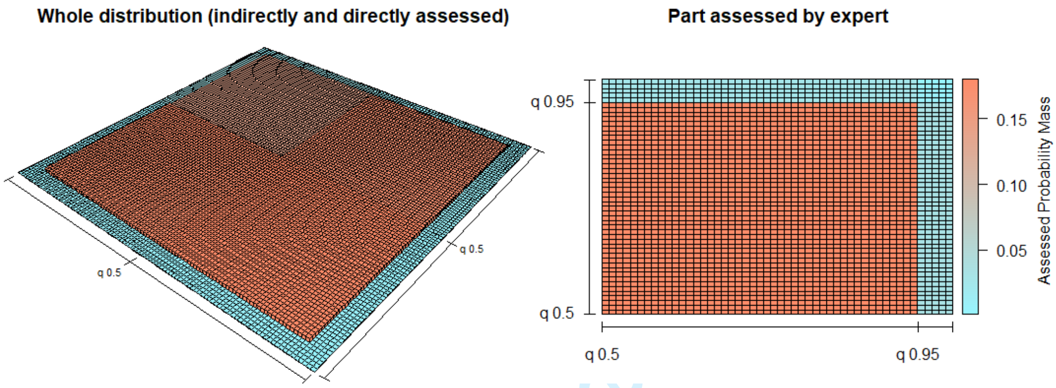


Figure 10: The experts joint distribution: overall (left) and assessed upper quadrant (right).

in addition to one further marginal assessment. The latter was required as we had not considered the 99^{th} quantiles previously. As outlined in the initial exemplary procedure of section 2.2, we started by first eliciting an overall probability mass which was later partitioned to further explore the joint upper distribution tail. The first elicitation is therefore on the probability of the terrorist attack frequency in Western Europe (y) being above its 50^{th} quantile, 62 attacks, given that we observe more than 73 attacks in Central Asia (x) (again the corresponding 50^{th} quantile), both in the year 2017. All judgements were conditional probabilities given the expert's familiarity with its interpretation. Table I summarises the dependence assessments by showing the results together with the framing of the questions.

As second part of the SRP method, we then modelled the overall joint distribution for the spatial dependence through solving the minimum information minimisation problem (section 3) based on the above assessments. The result can be seen in Figure 10.

We see that the expert's distribution indicates a slight negative dependence relationship between the spatial terrorism risk of both regions which is however close to independence. This is particularly driven by the first assessment being equal to 0.5 which indicates independence for a broad area of the joint distribution. In more detail, the difference between assessment *ii.*) and *iii.*) shows that in the joint tail, the expert assesses that an extreme year in terms of number of attacks for WE affects CA more than vice versa. The slight negative dependence (close to independence) corresponds to the expert's rationale which has been formally facilitated in order to support the expert with structuring his/her knowledge about the spatial dependence between both regions. For that, we used a conditional scenario mapping method [34]. In addition to mitigating some prevalent cognitive fallacies of assessing dependence, such as the *confusion of the inverse* or *confusing joint and conditional probabilities*(see also Werner et al. [11] for an overview), this method allows for considering and reflecting explicitly which scenarios affect the probability spaces of both regions (in a conditional sense). Scenarios are defined as "sequences that link triggering events to specified consequences (or final states) through intermediate conditions" [34]. For the example shown in Figure 11, the expert first reasoned through backwards logic, i.e. starting from the specified consequence, about observing more than 199 in Central Asia until the end of 2017 (α_5). Then, based on the initiating events that might cause Central Asia to experience more than 199 attacks and which are (at least partly) observable today, the expert reasoned (in forward logic) how these same initiating events affect the development of the number of terrorist attacks in Western Europe until the end of 2017. Based on the the number and plausibility of these conditional scenarios causing more than 255 attacks (again α_5), the expert could then make a dependence assessment in a more informed and confident manner. Werner et al. [34] presents the structured process of generating such conditional scenarios in more detail.

As can be seen in Figure 11, the expert considers both regions to be slightly negatively dependent (close to independence) due to the consideration that the active terrorist groups in both regions are different. In Central Asia, local separatists have political and regional motivations while in Western Europe mainly islamist groups are prevalent despite e.g. Russia's military involvement in the Middle East. Furthermore, the expert considers both regions to be different with regards to their vulnerability given not only the types of active terrorist groups but also the varying counter-terrorism and intelligence capabilities which drive the negative dependence relationship.

Before concluding this illustrative example, a first remark is that for quantifying the spatial dependence of terrorism attacks the definition used in this example is rather broad by including all attack types. Thus, the consideration of specific attack types might have very particular effects on the geographical interdependencies. As such, of growing interest in the adversarial risk literature have been biological attacks [56] and cyber attacks [79]. For these, it can be informative to assess the

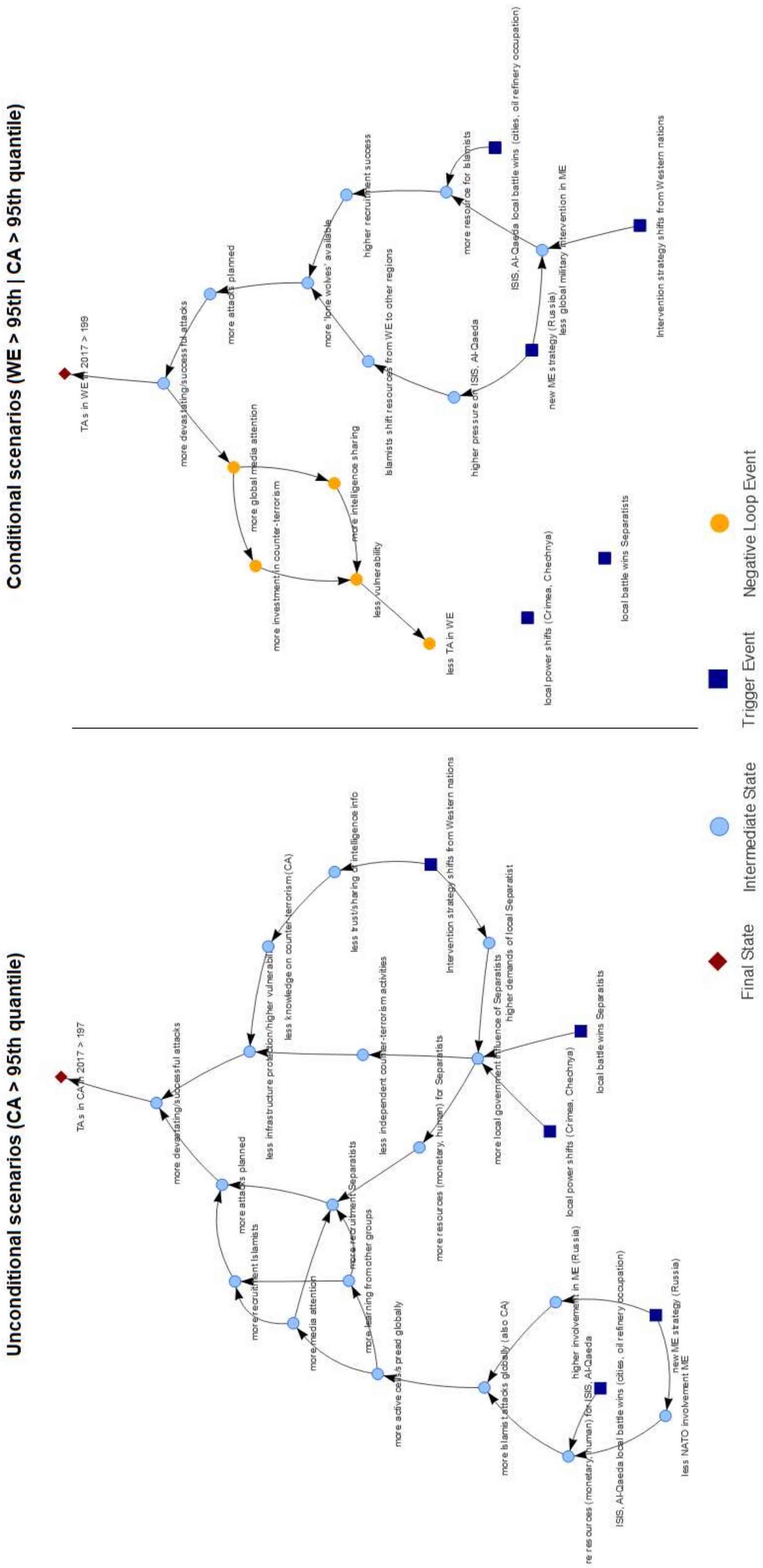


Figure 11: Unconditional and conditional scenarios for assessments.

dependence between variables of interest, such as casualties or monetary losses.

Further, we understand that an elicitation considering more explicitly the geographical interdependencies of critical infrastructure can be informative for insurers, for instance when offering business interruption coverage. Our method could hence build upon some modelling approaches that have ranked the susceptibility of critical infrastructures targeted by attacks [80].

Lastly, we acknowledge the inherent difficulties particular when considering attacks, such as 9/11, which some might title ”black swans”. For dependencies, the term ”perfect storms” appeared (see Paté-Cornell [81] for a discussion on the use of these terms in risk analysis and management). However, even for such events, structured assessment through experts can be informative and it is interesting that e.g. Zelikow (as director of the 9/11 Commission) called the misreading of precursors to these events as ”failure of imagination” given that air-planes had been used before as weapon and the World Trade Center in New York had been targeted already in 1993 [81].

5 DISCUSSION AND CONCLUSIONS

When using expert judgement for assessing dependence, there is a trade-off between easing the assessment burden for experts and sufficiently capturing a real-world phenomenon of interest in our model [10]. Therefore, we have presented an elicitation method that aims to satisfy a decision-maker's desired level of detail for a model whereas the procedure for eliciting dependence from experts provides an intuitive way of assessing even detailed dependence information (such as extreme parts of a joint distribution) while avoiding infeasible and inconsistent assessments. We argued that for the decision-maker a non-parametric setting of modelling multivariate uncertainties is more desirable and therefore we addressed the potential assessment issues of under- and overspecification. Concluding on the application shown in this paper, we note that in future research more applications are desirable to explore how the SRP method performs and obtain insights on potential modifications like alternative ways of framing the judgements, the implication of restricted feasibility ranges, or the elicitation of different forms (other than conditional probabilities). For example, as an alternative to eliciting quantile-based assessments, we can elicit conditional expectations. This follows from the discussion of Werner et al. [10] on modelling and elicitation strategies that are determined by the choice of considering influencing factors of dependence relationships explicitly or implicitly. The latter is similar to PI methods which aim at satisfying reasonable conditions of a model output due to its easier understanding and quantification. This is of particular interest when we cannot observe (and hence elicit) our variables of interest directly. Bedford et al. [19] show an elicitation procedure and minimum information modelling method for expectations on the whole joint distribution. Hence, considering its elaboration based on our method could allow for a more detailed specification of multivariate uncertainty for non-observable model input parameters. In the actuarial context of section 4, we might ask experts to assess the conditional expectation for a risk measure, such as *probable maximum loss* (see Grossi and Kunreuther [43]), which can be used (partly) as model output, whereas we assess dependence through PI on the function generating it. Similarly, our method can be used, either through quantile-based assessments or modifications, in other sectors in which understanding and quantifying tail risk is becoming of growing interest, such as financial decision-making on asset allocation [82].

ACKNOWLEDGEMENTS

APPENDIX

Proof for Proposition 1:

Suppose we are given values $u_0 = 0 < u_1 < \cdots < u_n < 1 = u_{n+1}$, and $v_0 = 0 < v_1 < \cdots < v_m < 1 = v_{m+1}$ (where $n, m > 0$), $0 < p, q < 1$, with p different to the u_i and q different to the v_j. Then a copula distribution on $QP(u, v)$ can be refined to a copula distribution on $QP(u, v; p, q)$.

Proof. In order to prove proposition 1, we divide the set $QP(u, v)$ into four subsets:

1. $A(p, q)$ has a single element which is the rectangle of $QP(u, v)$ containing the point (p, q).
2. $U(p, q)$ is the set of rectangles in $QP(u, v)$ that overlap the line $v = q$, except the one in $A(p, q)$.
3. $V(p, q)$ is the set of rectangles in $QP(u, v)$ that overlap the line $u = p$, except the one in $A(p, q)$.
4. $B(p, q)$ is the set of all rectangles in $QP(u, v)$ that are not in $A(p, q)$, $B(p, q)$, or $V(p, q)$.

Define also $A^*(p, q)$ to be the rectangles in $QP(u, v; p, q)$ which are sub-rectangles of $A(p, q)$, and define U^*, V^* and B^* similarly.

Note that $B^*(p, q) = B(p, q)$, that is, the rectangles in $B(u, v)$ do not get subdivided by the lines $u = p$, $v = q$. Rectangles in U^* are obtained by dividing rectangles in U by the line $v = q$, and rectangles in V^* are obtained by dividing rectangles in V by the line $u = p$.

We now define the refined copula distribution on $QP(u, v; p, q)$.

Let $\alpha = (p - u_i)/(u_{i+1} - u_i)$, and $\beta = (q - v_j)/(v_{j+1} - v_j)$. We specify how to define the refined copula distribution as follows:

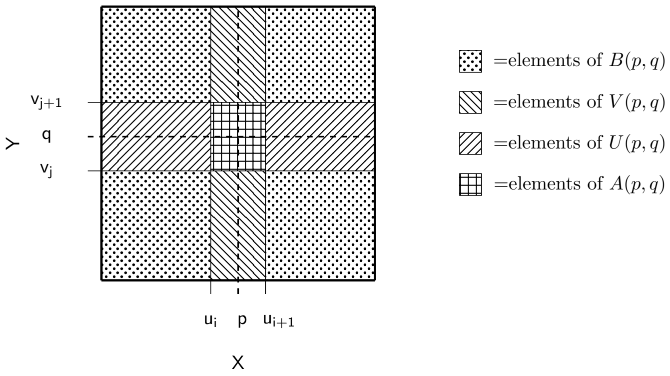


Figure 12: Different set of rectangles in $QP(u, v)$.

1. For the rectangles in A^* , the lower left sub-rectangle is allocated $\alpha\beta$ of the mass of A, the lower right one gets proportion $(1-\alpha)\beta$, the upper left one gets proportion $\alpha(1-\beta)$, and the upper right one gets proportion $(1-\alpha)(1-\beta)$.

2. Each rectangle in U is subdivided into two sub-rectangles in U^* by the line $v = q$, and the lower sub-rectangle is allocated proportion β of its mass and the upper sub-rectangle is allocated proportion $(1-\beta)$ of the mass.

3. Each rectangle in V is subdivided into two sub-rectangles in V^* by the line $u = p$, whereas the left sub-rectangle is allocated proportion α of its mass and the upper sub-rectangle is allocated proportion $(1-\alpha)$ of its mass.

4. Any rectangle in $B^*(p,q) = B(p,q)$ is assigned the same probability as it was in in the copula distribution on $QP(u, v)$.

This allocation of probabilities to the rectangles of $QP(u, v; p, q)$ adds to 1, while it is straightforward to check that it is a copula distribution. □

Regions of interest in illustrative case-study (section 4):

- Central Asia: Armenia, Azerbaijan, Belarus, Georgia, Kazakhstan, Kyrgyzstan, Russia, Tajikistan, Turkmenistan, Ukraine, Uzbekistan.

- Western Europe: Austria, Belgium, Denmark, Finland, France, Germany, Iceland, Ireland, Italy, Luxembourg, Malta, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom.

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