Are the Barriers that Inhibit Mathematical Models of a Cyclic Universe, which Admits Broken Symmetries, Dark Energy, and an Expanding Multiverse, Illusory?

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Abstract. We argue the thesis that if (1) a physical process is mathematically representable by a Cauchy sequence; and (2) we accept that there can be no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena; then (a) in the absence of an extraneous, *evidence-based*, proof of 'closure' which *determines* the behaviour of the physical process in the limit as *corresponding* to a 'Cauchy' limit; (b) the physical process *must* tend to a discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the physical process. We support our thesis by mathematical models of the putative behaviours of (i) a virus cluster; (ii) an elastic string; and (iii) a Universe that recycles from Big Bang to Ultimate Implosion, in which parity and local time reversal violation, and the existence of 'dark energy' in a multiverse, need not violate Einstein's equations and quantum theory. We suggest that the barriers to modelling such processes in a mathematical language that seeks unambiguous communication are illusory; they merely reflect an attempt to ask of the language chosen for such representation more than it is designed to deliver.

Keywords. Big Bang 1; black hole 2; broken symmetries 3; Cauchy limit 4; cosmological constant 5; cyclic universe 6; dark energy 7; Einstein's equations 8; expanding multiverse 9; gravitational field 10; Hawking radiation 11; phase change 12; repulsive gravitational field 13; Zeno's paradox 14

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1. Introduction

In this cross-disciplinary foundational investigation, we argue the thesis (Thesis 2) that if:

(a) a physical process is representable by a Cauchy sequence; and

(b) we accept that there can be no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena;

then:

(c) in the absence of an extraneous, *evidence-based*, proof of 'closure' which determines the behaviour of the physical process in the limit as corresponding to a 'Cauchy' limit;

(d) the physical process *must* tend to a discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the physical process.

We support our thesis by, *in principle verifiable*, mathematical models of the putative behaviours of:

(i) a virus cluster; and

(ii) an elastic string.

We then show how Thesis 2 admits an *essentially unverifiable* mathematical model of the behaviour of:

(iii) a Universe:

- that recycles endlessly from Big Bang to Ultimate Implosion;
- that admits broken symmetries and accelerated expansion locally in a multiverse;
- in which local parity violation and time reversal violation, as also the existence of 'dark energy', need not violate Einstein's equations, quantum theory, and our physical observations of the universe we inhabit.

We suggest that perceived barriers to modelling the behaviour of such a universe, in a mathematical language seeking unambiguous communication, could be illusory; they may arise out of an attempt to ask of the language selected for such representation more than the language is designed to deliver.

2. The significance of evidence-based reasoning for mathematically representing conceptual metaphors of physical phenomena

This cross-disciplinary foundational investigation is rooted in the *evidence-based* perspective towards 'provability' and 'truth' introduced in the paper [An16], '*The Truth Assignments That Differentiate Human Reasoning From Mechanistic Reasoning: The Evidence-Based Argument for Lucas' Gödelian Thesis*'.

The paper appeared in the December 2016 issue of *Cognitive Systems Research*, and addressed the philosophical challenge that arises when an intelligence—whether human or mechanistic—accepts mathematical propositions as true under an interpretation—either axiomatically or on the basis of subjective self-evidence—*without* any specified methodology for evidencing such acceptance.

For the purposes of the investigation we shall make an arbitrary distinction between (compare [Ma08]; see also [Fe99]):

- The ***natural scientist's hat***, whose wearer's responsibility is recording—as precisely and as objectively as possible—our sensory observations (corresponding to computer scientist David Gamez's 'Measurement' in [Gam18], Fig.5.2, p.79) and their associated perceptions of a 'common' external world (corresponding to Gamez's 'C-report' in [Gam18], Fig.5.2, p.79; and to what some cognitive scientists, such as Lakoff and Núñez in [LR00], term as 'conceptual metaphors');

- The ***philosopher's hat***, whose wearer's responsibility is abstracting a coherent—albeit informal and not necessarily objective—holistic perspective of the external world from our sensory observations and their associated perceptions (corresponding to Carnap's *explicandum* in [Ca62]; and to Gamez's 'C-theory' in [Gam18], F, p.79); and

- The ***mathematician's hat***, whose wearer's responsibility is providing the tools for adequately expressing such recordings and abstractions in a symbolic language of unambiguous communication (corresponding to Carnap's *explicatum* in [Ca62]; and to Gamez's 'P-description' and 'C-description' in [Gam18], Fig.5.2, p.79).

That this distinction may not reflect conventional wisdom is highlighted in §8., where we argue that:

— if mathematics is to *serve* as a lingua franca for the physical sciences,

— then it can only represent physical phenomena *unambiguously* by insistence upon *evidence-based* reasoning (in the sense of Chapter 5 in [An18])

— which, in some cases, may prohibit us from building a mathematical theory of a physical process

  - based on the assumption that the limiting behaviour of *every* physical process which can be described by a Cauchy sequence

  - *must* correspond to the behaviour of the classically defined Cauchy limit of the sequence.

The distinction seeks to crystalise Hermann Weyl's perspective that (see also Chapter 42 of [An18]):

> "...I believe the human mind can ascend toward mathematical concepts only by processing reality as it is given to us. So the applicability of our science is only a symptom of its rootedness, not a genuine measure of its value. It would be equally fatal for mathematics—this noble tree that spreads its wide crown freely in the ether, but draws its strength from the earth of real intuitions and perceptions (*Anschauungen und Vorstellungen*)—if it were cropped with the shears of a narrow-minded utilitarianism or were torn out of the soil from which it grew."
>
> *... Weyl: [We10], p.10.*

Without attempting to address the issue in its broader dimensions, we shall also argue from the perspective that:

(i) Mathematics is to be considered as a set of precise, symbolic, languages;

(ii) Any language of such a set seeks to express—in a finitary, unambiguous, and communicable manner—relations between elements that are external to the language;

(iii) Moreover, each such language is two-valued if we assume that a specific relation either holds or does not hold externally under any valid interpretation of the (symbolic) language.

From such a perspective, eliminating ambiguity in critical cases—such as communication between mechanical artefacts, or a putative communication between terrestrial and/or extra-terrestrial intelligences (whether mechanical or organic)—seems to be the very raison d'être of mathematical activity.

We would view such activity:

(1) First, as the construction of richer and richer mathematical languages that can symbolically express those of our abstract concepts (corresponding to Lakoff and Núñez's *conceptual metaphors*, and Carnap's *explicandum*) which can be subjectively addressed unambiguously.

Languages such as, for instance, the first-order Set Theory ZF, which can be well-defined formally but which have no constructively well-defined model that would admit *evidence-based* assignments of 'truth' values to set-theoretical propositions by a mechanical intelligence.

By 'subjectively address unambiguously' we intend in this context that there is essentially a subjective acceptance of identity by a human mind between:

- an abstract concept in the mind (corresponding to Lakoff and Núñez's 'conceptual metaphor' in [LR00], p.5) that the mind intended to express symbolically in a language; and
- the abstract concept created in the mind each time it subsequently attempts to understand the import of that symbolic expression (a process which can be viewed in engineering terms as analogous to formalising the specifications, i.e., *explicatum*, of a proposed structure from a prototype).

(2) Second, the study of the ability of a mathematical language to precisely express and objectively communicate the formal expression (corresponding to Carnap's *explicatum*) of *some* such concepts effectively.

A language such as, for instance, the first order Peano Arithmetic PA, which can not only be well-defined formally, but which has a finitary model (see Theorem 6.8 of [An16]) that admits *evidence-based* assignments of 'truth' values to arithmetical propositions by a mechanical intelligence, and which is categorical (albeit, with respect to algorithmic computability—Corollary 7.2 in [An16]).

By 'objectively communicate effectively' we intend in this context that there is essentially:

(a) first, an objective (i.e., on the basis of *evidence-based* reasoning in the sense of Chapter 5 in [An18]) acceptance of identity by another mind between:
   - the abstract concept created in the other mind when first attempting to understand the import of what some human mind has expressed symbolically in a language; and
   - the abstract concept created in the other mind each time it subsequently attempts to understand the import of that symbolic expression (a process which can also be viewed in engineering terms as analogous to confirming that the formal specifications, i.e., *explicatum*, of a proposed structure do succeed in uniquely identifying the prototype, i.e., *explicandum*);

(b) second, an objective acceptance of functional identity between abstract concepts that can be 'objectively communicated effectively' based on the *evidence* provided by a commonly accepted doctrine such as, for instance, the view that a simple functional language can be used for specifying *evidence* for propositions in a constructive logic ([Mu91]).

Moreover, we argue that we need to recognise explicitly the limitations imposed by *evidence-based* reasoning on:

– the ability of highly expressive mathematical languages such as ZF to *effectively communicate* abstract concepts (Lakoff and Núñez's conceptual metaphors), such as, for instance, those involving putative limits of Cauchy sequences, Cantor's first limit ordinal ω, and other conceptual metaphors whose interpretations cannot be *evidence-based*; and

– the ability of effectively communicating mathematical languages such as PA to *adequately express* concepts whose interpretations cannot be *evidence-based* (see §20.7 of [An18]).

We suggest, further, that from an *evidence-based* perspective, the barriers that inhibit the mathematical modeling of a universe that admits dark energy and broken symmetries in an expanding multiverse, and which recycles endlessly without violating Einstein's equations, quantum theory, and physical observations of our universe, are illusory—they may arise out of a blurring of the preceding distinction, and reflect merely an attempt to ask of a language more than it is designed to deliver.

They would dissolve once we accept that the ontology of any interpretation of a language is determined not by the 'logic' of the language—which, contrary to conventional wisdom, we take as intended solely to assign unique 'truth' values to the declarative sentences of the language (in the sense of the proposed Definitions 21.3 to 21.7 of [An18])—but by the rules (see Theorem 11.8 of [An18]) that determine the 'terms' which can be admitted into the language without inviting contradiction.

3. When is the concept of a completed infinity consistent with its interpretation?

Which raises the question:

Query 1. *When is the concept of a completed infinity consistent with the interpretation of a formal language?*

Clearly, the consistency of the concept would follow immediately in any constructively well-defined interpretation of the axioms (and rules of inference) of a set theory such as the Zermelo-Fraenkel ([BF58]) first-order theory ZF (whether such an interpretation exists at all is, of course, another question).

In view of the perceived power of ZFC as an unsurpassed language of rich and adequate expression of mathematically expressible abstract concepts precisely (see [An18], Thesis 42.1, p.347), it is not surprising that many of the semantic and logical paradoxes (see Chapter 24 in [An18])—which create doubt in our minds as to our ability to define scientific 'truths unambiguously'—depend on the implicit assumption that the domain over which the paradox quantifies can always be treated as a well-defined mathematical object that can be formalised in ZFC, even if neither the domain nor the object are explicitly definable set-theoretically.

This assumption is rooted in the questionable[1] belief that ZF can express all mathematical 'truths' (see, for instance, [Ma18] and [Ma18a]).

From this it is but a short step to non-constructive perspectives—such as Gödel's Platonic interpretation of his own formal reasoning in his 1931 paper ([Go31])—which fallaciously argue (see [An18], §20.14, p.158) that the first-order Peano Arithmetic PA must have non-standard models.

However, it is our contention that both of the above foundational issues need to be reviewed carefully, and that we need to recognize explicitly the limitations on the ability of highly expressive mathematical languages such as ZF to unambiguously represent and effectively communicate abstract concepts that claim to be rooted in an external physical reality.

4. Asking more of a language than it is designed to deliver

For instance, consider the claim (e.g., [Bar88], p.37, Theorem 1) that fractal 'constructions'—such as the Cantor ternary set, which is defined classically as a 'putative' set-theoretical limit ([Ru53], p34; [Bar88], pp.44-45) of an iterative process in the 'putative' completion of a metric space—yield valid mathematical objects (sets) in the 'limit' (*presumably in some Platonic mathematical model*).

[1] 'Questionable' since, in [An18], Chapter 22 we show how—in the case of Goodstein's Theorem—such a belief leads to a curious conclusion ([An18], Theorem 22.3).

Now, the Cantor Set T_∞ is defined as the putative 'fractal' limit of the set of points obtained by taking the closed interval $T_0 = [0, 1]$):

- removing the open middle third to yield the set $T_1 = \{[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]\}$,
- then removing the middle third of each of the remaining closed intervals to yield the set $T_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$,
- and so on ad infinitum.

To see why such a limit needs to be treated as 'putative' (compare with Lakoff and Núñez's analysis of the similar 'length paradox' in [LR00], p.325-333) from an *evidence-based* perspective (as detailed in [An18], Chapter 5), consider the equilateral triangle BAC of height h and side s in Fig.1 (below):

- Divide the base BC in half and construct two isosceles triangles of height $h.d$ and base $s/2$ on BC, where $1 \geq d > 0$.
- Iterate the construction on each constructed triangle ad infinitum.
- Thus, the height of each of the 2^n triangles on the base BC at the n'th construction is $h.d^n$, and the base of each triangle $s/2^n$.
- Hence, the total area of all these triangles subtended by the base BC is $s.h.d^n/2$.
- Now, if $d = 1$, the total area of all the constructed triangles after each iteration remains constant at $s.h/2$, although the total length of all the sides opposing the base BC increases monotonically.
- However, if $1 > d > 0$, it would appear that, geometrically, the base BC of the original equilateral triangle will always be the 'limiting' configuration of the sides opposing the base BC.

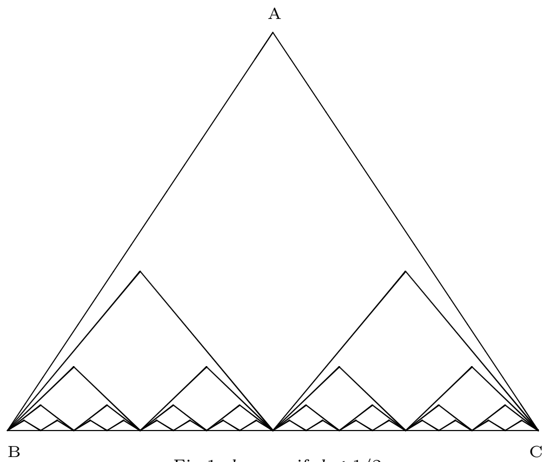

Fig.1: $l_n \rightarrow s$ if $d < 1/2$

This is indeed so if $0 < d < 1/2$ (Fig.1), since the total length of all the sides opposing the base BC at the n'th iteration—say l_n—yields a Cauchy sequence whose limiting value is, indeed, the length s of the base BC.

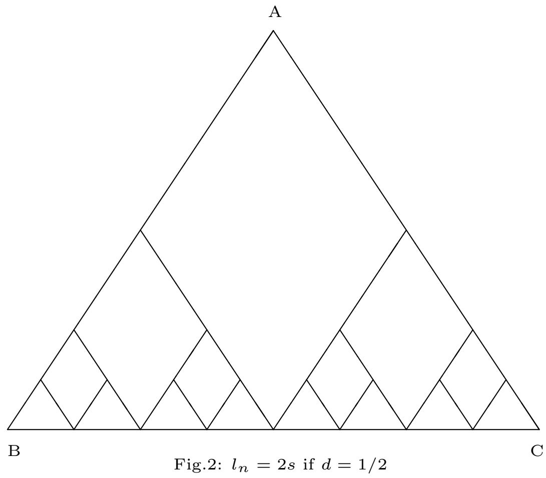

Fig.2: $l_n = 2s$ if $d = 1/2$

However, if $d = 1/2$ (Fig.2), the total length of all the sides opposing their base on BC is always $2s$; which, by definition, also yields a Cauchy sequence whose limiting value is $2s$!

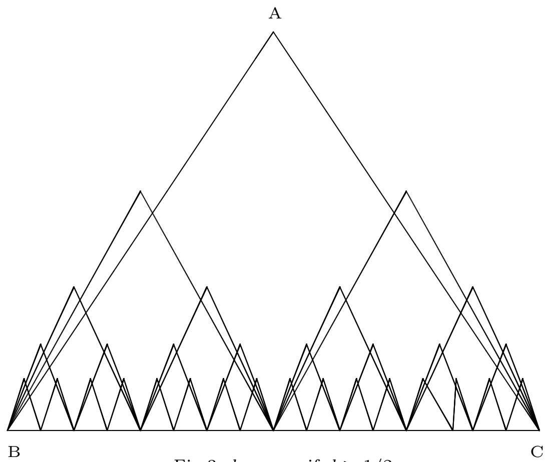

Fig.3: $l_n \rightarrow \infty$ if $d > 1/2$

Finally, if $d > 1/2$ (Fig. 3), the total length of all the sides opposing their base on BC is a monotonically increasing value.

Consider now:

5. Interpretation as a virus cluster

Case 1: Let the area BAC denote the population size of a virus cluster, where each virus cell has a 'virulence' measure h/s.

Let each triangle at the n'th iteration denote a virus cluster—with a virulence factor $h.d^n/(s/2^n)$—that reacts to the next generation anti-virus by splitting into two smaller clusters with inherited virulence $h.d^{n+1}/(s/2^{n+1})$.

We then have that:

(a) If $d < 1/2$, the effects of the virus can—in a sense—be contained and eventually 'eliminated', since both the total population of the virus, and its virulence in each cluster, decrease monotonically;

(b) If $d = 1/2$, the effects of the virus can be 'contained', but never 'eliminated' since, even though the total population of the virus decreases monotonically, its virulence in each cluster remains constant, albeit at a containable level, until the virus suffers a sudden, dinosaur-type, extinction at the 'limiting' point as $n \rightarrow \infty$;

(c) However, if $d > 1/2$, the effects of the virus can neither be 'contained' nor 'eliminated' since, even though the total population of the virus decreases monotonically, its virulence in each cluster resists containment by increasing monotonically until, again, the virus suffers a sudden, dinosaur-type, extinction at the 'limiting' point as $n \rightarrow \infty$.

6. Interpretation as an elastic string

Case 2: Let the base BC denote an elastic string, stretched iteratively into the above configurations. We then have that:

(a) If $d < 1/2$, the elastic will, in principle, eventually return to its original state;

(b) If $d > 1/2$, then the elastic must break at some point; in a 'phase change' that is apparently 'normal', and invites no untoward curiosity, since it forms part of our everyday experience;

(c) However, what if $d = 1/2$?

7. Phase change: Zeno's argument in 2-dimensions

We then arrive at a two-dimensional version of Zeno's arguments where—unlike in Zeno's classical, one-dimensional, 'arrow' argument ([Rus37], pp.347-353)—the extraneous, *evidence-based*, proof of 'closure' which *determines* the behaviour of the physical process at the limit does *not correspond* to the Cauchy limit of the sequence that seeks to mathematically represent the physical process.

One way of resolving the 'apparent' paradox is by admitting the possibility that, in this case, such an elastic 'length' undergoes a 'steam-to-water-like' phase change in the 'limit' (akin to the breaking of the elastic string in §6., Case 2(b)) that need not correspond (see [An18], §19.4) to the putative limit of its associated Cauchy sequence[2]!

We note that Theorem 19.4 in [An18] shows that Cauchy sequences which are defined as algorithmically *verifiable*, but not algorithmically *computable*, can correspond to 'essentially incompletable' real numbers (such as, for instance, the fundamental dimensionless constants considered in [An18], §27.6) whose Cauchy sequences cannot, in a sense, be known 'completely' even to Laplace's 'intellect'.

The above example now show further that—and why—the numerical values of some algorithmically computable Cauchy sequences may also need to be treated as formally specifiable, first-order, non-terminating processes:

- which are 'eternal work-in-progress' in the sense of Theorem 19.4 in [An18], and
- which cannot be uniquely identified by a putative 'Cauchy limit' without limiting the ability of such sequences to model phase-changing physical phenomena faithfully.

In view of Theorem 19.7 in [An18], the gedanken in §5. and §6. highlight the disquieting issue sought to be raised, for instance, by Krajewski in [Kr16][3] (see also §2.2 in [An18]), Lakoff and Núñez in [LR00] (p.325-333), and Simpson in [Sim88], which can be expressed as:

[2]We note that, by definition, the sequence $\{a_0, a_1, a_2, \ldots\}$ where $a_0 = 1$ and $a_i = 3$ for all $i \geq 1$, is a Cauchy sequence whose 'Cauchy' limit is 3.

[3]"Examples of possible theological influences upon the development of mathematics are indicated. The best known connection can be found in the realm of infinite sets treated by us as known or graspable, which constitutes a divine-like approach. Also the move to treat infinite processes as if they were one finished object that can be identified with its limits is routine in mathematicians, but refers to seemingly super-human power." . . . [Kr16].

Query 2. *Since the raison d'être of a mathematical language is—or ideally should be—to express our abstractions of natural phenomena precisely, and communicate them unequivocally, in what sense can we sensibly admit an interpretation of a mathematical language that constrains all the above cases by 'limiting' configurations in a putative, set-theoretical, 'completion' of Euclidean Space?*

8. The mythical completability of metric spaces

From the *evidence-based* perspective of this investigation, we can now hypothesise:

Thesis 1. *There are no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena.*

Thesis 2. *If:*

(a) a physical process is representable by a Cauchy sequence; and

(b) we accept that there can be no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena;

then:

(c) in the absence of an extraneous, evidence-based, proof of 'closure' which determines the behaviour of the physical process in the limit as corresponding to a 'Cauchy' limit;

(d) the physical process must tend to a discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the physical process.

The *significance* of such insistence on *evidence-based* reasoning for the physical sciences is that we may then be prohibited from claiming legitimacy for a mathematical theory which seeks to represent a physical process based on the assumption that the limiting behaviour of *every* physical process which can be described by a Cauchy sequence in the theory must correspond to—and so be constrained by—the behaviour of the Cauchy limit of the corresponding sequence.

For instance the existence of Hawking radiation in cosmology is posited on the assumption that '*the consistent extension of this local thermal bath has a finite temperature at infinity*':

> "Hawking radiation is required by the Unruh effect and the equivalence principle applied to black hole horizons. Close to the event horizon of a black hole, a local observer must accelerate to keep from falling in. An accelerating obsrver sees a thermal bath of particles that pop out of the local acceleration horizon, turn around, and free-fall back in. The condition of local thermal equilibrium implies that the consistent extension of of this local thermal bath has a finite temperature at infinity, which imples that some of these particles emitted by the horizon are not reabsorbed and become outgoing Hawking radiation."
>
> ... *https://en.wikipedia.org/wiki/Hawking_radiation.* (Accessed 04/06/2018, 08:00 IST.)

As we have demonstrated in Fig. 2 (§4.) and §6., Case 2(c), *the consistent extension of the state of a stretched elastic string—as defined in Fig. 2—does not have a limiting mathematical value at infinity* which can be taken to correspond to its putatively limiting physical state.

The gedanken in §9. further illustrates that a mathematical singularity need not constrain a physical theory from positing a well-definable value for a limiting state of a physical process; contrary to conventional wisdom, which apparently perceives such singularities as *unsurmountable* barriers in the limiting cases of Einstein's equations for General Relativity:

> "The Big Bang is probably the most famous feature of standard cosmology. But it is also an undesirable one. That's because the classical model of the universe, described by Einstein's equations, breaks down in the conditions of the Big Bang, which include an infinite density and temperature, or what physicists call a singularity."
>
> *. . . Padmanabhan: [Pd17].*

We shall argue there that introduction of a normally *weak* anti-gravitational field whose strength can, however, accept quantum states that cause a universe to explode and implode in a predictable way at their corresponding 'mathematical' singularities, yields a mathematical model of a time-reversal invariant universe:

- That recycles endlessly from Big Bang to Ultimate Implosion;
- In which the existence of 'dark energy' in a 'multiverse' is intuitively unobjectionable;
- In which individual sub-universes expand at an accelerated rate and need not be time-reversal invariant.

Whether or not such features can be made to apply to the physical universe we inhabit is a separate issue that lies beyond the focus of the *evidence-based* perspective of this investigation.

However, the mathematical representations of the putative behaviour of a putative virus cluster in §5., and more particularly that of an elastic string in §6., suggest that arguments which admit a premise such as:

> "Particle physics interactions can occur at zero distance—but Einstein's theory of gravity makes no sense at zero distance. . . . String interactions don't occur at one point but are spread out in a way that leads to more sensible quantum behaviour."
>
> *. . . Schwarz: http://superstringtheory.com/basics/basic3.html; accessed 21/06/2018, 18:30:00.*

may be vulnerable to conflating:

- the limitations imposed upon Einstein's equations by the mathematical language in which they are expressed; and
- the limitations that nature in fact imposes upon the physical phenomena that the equations seek to express and communicate in a mathematical language.

It is thus worthwhile noting some of the 'barriers' that mathematical 'singularities' are perceived as imposing upon our ability to faithfully comprehend, and mathematically represent, the laws of nature. For instance, as queried by Thanu Padmanabhan in [Pd17a]:

> "But what if there was no singularity? Since the 1960s, physicists have been working on describing the universe without a Big Bang by attempting to unify gravitational theory and quantum theory into something called quantum gravity. Physicists John Wheeler and Bryce deWitt were the first to apply these ideas to a hypothetical pre-geometric phase of the universe, in which notions of space and time have not yet-emerged from some as-yet unknown structure. This heralded the study of quantum cosmology, in which physicists attempted to describe the dynamics of simple toy models of the universe in quantum language. Needless to say, several different, but related, ideas for the description of the pre-geometric phase mushroomed over the decades. The unifying theme of these models is that the classical universe arises, without any singularity, through a transition from a pre-geometric phase to one in which spacetime is described by Einstein's equations. The main difficulty in constructing such a description is that we do not have a complete theory of quantum gravity, which would allow us to model the pre-geometric phase in detail."
>
> *. . . Padmanabhan: [Pd17].*

The issue is highlighted by his belief in [Pd17a] that 'meaningful theories must be nonsingular':

"I will now raise a question which, at the outset, may sound somewhat strange. Why does the universe expand and, thereby, give us an arrow of time? To appreciate the significance of this question, recall that Eq. (9) is invariant under time reversal $t \to -t$. (After all, Einstein's equations themselves are time reversal invariant.) To match the observations, we have to choose a solution with $\dot{a} > 0$ at some fiducial time $t = t_{fid} > 0$ (say, at the current epoch), thereby breaking the time-reversal invariance of the system. This, by itself, is not an issue for a laboratory system. We know that a particular solution to the dynamical equations describing the system need not respect all the symmetries of the equations. But, for the universe, this is indeed an issue.

To see why, let us first discuss the case of $(\rho + 3p) > 0$ for all t. The choice $\dot{a} > 0$, at any instant of time, implies that we are postulating that the universe is expanding at that instant. Then Eq. (9) tells us that the universe will expand at all times in the past and will have a singularity ($a = 0$) at some finite time in the past (which we can take to be $t = 0$ without loss of generality). The structure of Eq. (9) prevents us from specifying the initial conditions at $t = 0$. So, if you insist on specifying the initial conditions and integrating the equations forward in time, you are forced to take $\dot{a} > 0$ at some time $t = \epsilon > 0$, thereby breaking the time reversal symmetry. The universe expands at present 'because' we chose it to expand at some instant in the past. This expansion, in turn, gives us an arrow of time [where] either t or a can be used as a time coordinate. But why do we have to choose the solution with $\dot{a} > 0$ at some instant? This is the essence of the so called expansion problem [6]. An alternative way of posing the same question is the following: How come a cosmological arrow of time emerges from the equations of motion which are time-reversal invariant?

In a laboratory, we can usually take another copy of the system we are studying and explore it with a time-reversal choice of initial conditions, because the time can be specified by degrees of freedom external to the system. We cannot do it for the universe because we do not have extra copies of it handy and—equally importantly—there is nothing external to it to specify the time. So the problem, as described, is specific to cosmology.

So far we assumed that $(\rho + 3p) > 0$, thereby leading to a singularity. Since meaningful theories must be nonsingular, we certainly expect a future theory of gravity—possibly a model for quantum gravity—to eliminate the singularity [effectively leading to $(\rho + 3p) < 0$. Can such a theory solve the problem of the arrow of time? This seems unlikely. To see this, let us ask what kind of dynamics we would expect in such a 'final' theory. The classical dynamics will certainly get modified at the Planck epoch, to govern the evolution of an (effective) expansion factor. The solutions could, for example, have a contracting phase (followed by a bounce) or could start from a Planck-size universe at $t = -\infty$, just to give two nn-singular possibilities. While we do not know these equations or their solutions, we can be confident that they will still be time-reversal invariant because quantum theory, as we know it, is time-reversal invariant.

So except through a choice for initial conditions (now possibly at $t = -\infty$), we still cannot explain how the cosmological arrow of time emerges. Since quantum gravity is unlikely to produce an arrow of time, it is a worthwhile pursuit to try and understand this problem in the (semi)classical context."

. . . Padmanabhan: [Pd17a].

9. Interpretation as the confinement state of the total energy in a universe that recycles

"Both general relativity and Newtonian gravity appear to predict that negative mass would produce a repulsive gravitational field."

. . . Anti-gravity: https://en.wikipedia.org/wiki/Anti-gravity; accessed 08/06/2018, 10:13:00.

To illustrate why an *evidence-based* perspective towards interpreting the propositions of a mathematical model realistically would view such barriers as illusory, we consider the following gedanken.

Case 3: We can also treat Fig.2 as a mathematical representation of the 'confinement parameter' that determines the state of the total energy s, in a *finite* universe $\mathcal{U}$, which is subject to two constantly unequal and opposing, assumed additive, forces due to:

- A *strong* confinement field G (induced by matter), whose state is determined by a single discrete dimensionless constant, defined as an Einsteinian confinement, or gravitational strength, 'gravitational constant' (gsp), which is always $\frac{1}{2}$; and

- A *weak* anti-confinement field R (induced by anti-matter), whose state is determined by discrete dimensionless values, defined as the Einsteinian anti-confinement, or repulsive gravitational strength, 'cosmological constants' (asp), where:
  - $asp = 1 > gsp$ when $\mathcal{U}$ is in an *exploding* state *at* event e_0;
  - $asp = \frac{1}{3} + \frac{2}{3}(1 - \frac{1}{n+1}) > gsp$ when $\mathcal{U}$ is in an *imploding* state *at* event e_n for $n \geq 1$;
  - $asp = \frac{1}{3} < gsp$ when $\mathcal{U}$ is in a *steady* state:
    * during which events, denoted by $e'_n, e''_n, \ldots,$
    * occur *between* events e_n and e_{n+1};
    * where $e'_n < e_m$ is an abbreviation for 'event e'_n occurs causally before event e_m'.

and where the following are assumed to hold:

(a) Classical laws of nature which determine the nature and behaviour of all those properties of the physical world that are both *determinate* and *predictable*, and are therefore mathematically describable at any event $e(n)$ by *algorithmically computable* functions from a given initial state at event $e(0)$ (Thesis 27.2 in [An18]);

(b) Neo-classical (quantum) laws of nature which determine the nature and behaviour of those properties of the physical world that are *determinate* but *not predictable*, and are therefore mathematically describable at any event $e(n)$ only by functions that are *algorithmically verifiable* but not *algorithmically computable* from any given initial state at event $e(0)$ (Thesis 27.3 in [An18]);

(c) There can be no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena;

(d) All laws of nature are subject to *evidence-based* accountability as follows (Thesis 1):
  - If a physical process is representable by a Cauchy sequence (as in the above cases in §5. and §6.);
  - then, in the absence of an extraneous, *evidence-based*, proof of 'closure' which determines the behaviour of the physical process in the limit as corresponding to a 'Cauchy' limit;
  - the physical process *must* be taken to tend to a discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the physical process.

A: We then define:

(i) The total, say s, units of energy of the universe $\mathcal{U}$ is:
  - in an *exploding* state *at* event e_0;
  - in a *steady* state *between* events e_n and e_{n+1} for $n \geq 1$;
  - in an *imploding* state *at* events e_n for $n \geq 1$.

(ii) The state of the anti-confinement field in $\mathcal{U}$ at an event is defined with reference to Fig.2 as follows:

- Initially at the Big Bang event e_0, where the energy s is in an unstable *exploding* state, the anti-confinement field strength:
  * is determined by the ratio $asp = \frac{s}{s} = 1 > gsp$ of the absolute value of the total energy s of the universe, and the absolute value of a confinement parameter represented by the length BC where, for convenience, we define the length BC as s;
  * which corresponds to the limiting case of the confinement parameter as $n \rightarrow \infty$ in Fig.2.
- Between events e_n and e_{n+1} for $n > 0$, where the energy s is in a *steady* state, the anti-confinement field strength:
  * is determined by the ratio $asp = \frac{s}{l_n} = \frac{1}{3} < gsp$,
  * where the confinement parameter $l_n = 3s$ is represented by the cumulative perimeter lengths of all the triangles on their common base BC in Fig.2.
- At event e_n for $n \geq 1$, where the energy s is in an unstable *imploding* state, the anti-confinement field strength:
  * is determined by $asp = \frac{s}{l_n} + \frac{2}{3}(1 - \frac{1}{n+1}) > gsp > \frac{1}{3}$;
  * where $\frac{2}{3}(1 - \frac{1}{n+1}) > \frac{1}{3}$ is defined as the implosion constant at event e_n.

B: We further define:

(iii) At event e_0 the universe $\mathcal{U}$ explodes and expands 'instantaneously'—in a water-to-steam like phase change—to a *steady* state termed as event e'_0 where:

- The strength of the confinement field, $gsp = \frac{1}{2}$,

is now greater than:

- The strength of the anti-confinement field, $asp = \frac{s}{3s} = \frac{1}{3}$.

(iv) At any event e'_0 the total energy s of the universe $\mathcal{U}$—which we assume can neither be created nor destroyed—is subjected to a confinement field due to gravitational effects that gradually concentrates:

- some energy to form isolated matter;
- some isolated matter to form stars;
- some stars to form supernovas;
- some supernovas to form 'black holes';
- some 'black hole' to form the first 'critical black hole':
  * which we define as event e''_0 where $e''_0 \geq e_0$;
  * during which matter is gradually drawn into the 'black hole',
  * until, at event e_1, a 'critical' proportion of the total energy s of the parent universe corresponding to the state BAC has been drawn into the 'critical black hole':
    · which proportion, without loss of generality, we may take as $\frac{1}{2}$ in this example;

· where we treat event e_1 as a singularity corresponding to the mid-point of BC;

· such that this energy $(\frac{s}{2})$ has now been 'confined' into an imploding state with $asp = \frac{1}{3} + \frac{2}{3}(1 - \frac{1}{2}) = \frac{2}{3} > gsp$;

- and is extinguished in an 'instantaneous' implosion, defined as the event $e_1 \geq e''_0$,
- which forms an electromagnetically disconnected, independent, universe;
- which, without loss of generality, we treat as the splitting of the energy s of the parent universe $\mathcal{U}$ into two disconnected, isomorphic but not identical, twin sub-universes corresponding to the states $BAC_{1,1}$ and $BAC_{1,2}$ in Fig.2,
- that are situated in common, universal, confinement and anti-con- finement fields G and R;
- and which, without loss of generality, we assume obey identical laws of nature;
- where the total energy s is now divided equally between the twin states $BAC_{1,1}$ and $BAC_{1,2}$;
- where, without loss of generality, we may assume that the distribution of particles and their anti-particles between the twin states $BAC_{1,2}$ and $BAC_{1,1}$ is not necessarily symmetrical.

(v) Whence it follows that:

– The total of any Hawking—or other, similarly putative[4]—energy radiated back into the 'observable' universe $\mathcal{U}$ corresponding to the state BAC during the period, defined as event e''_0, between the creation of the 'critical black hole' and its eventual extinction at event e_1 (corresponding to the mid-point of BC):

∗ is *not* $s/2$ (as conventional wisdom would expect in such a model);

∗ but, if at all, only a tiny fraction of the total energy—which is now $s/2$—of each sub-universe;

∗ although each sub-universe:

· unaware of its isomorphic sibling,

· and under the illusion that it is still the entire parent universe,

· with merely 'black hole' concentrates of energy within it,

· which it believes will gradually extinguish once all the energy has seeped back into its domain as a result of a putative Hawking, or similar, radiation,

· continues to lay claim to the energy of its extinguished sibling as 'dark energy',

· by an 'unknowably' misapplied appeal to the law of preservation of the total energy s of the original universe corresponding to the state BAC;

– Although the universe $\mathcal{U}$ is time-reversal invariant, each of the twin (isomorphic but not identical) sub-universes corresponding to the states $BAC_{1,1}$ and $BAC_{1,2}$ need not be time-reversal invariant, since the ratio of particles to their anti-particles in each of the twin sub-universes may no longer be symmetrical;

[4]'Putative' since the existence of such energy may be only on the basis of the debatable—see §7. and §8.—mathematical assumption that the limit of the mathematical representations of a sequence of physical phenomena must necessarily correspond to the putative behaviour of the physical phenomena in the putative limiting state.

- Each sub-universe in turn forms the *next* 'critical black hole' singularity;
  - that implodes similarly at—assumed without loss of generality as a common—event e_2,
  - into two, isomorphic but electro-magnetically disconnected, twin sub-universes with equal, but asymmetrical, division of energy;
- The universe at event e_2 is a 'multiverse' of mutually disconnected 2^2 sub-universes corresponding to the states $\{BAC_{2,1}, BAC_{2,2}, BAC_{2,3}, BAC_{2,4}\}$;
  - and so on ad infinitum.

C: In other words, the n^{th} implosion at event e_n, for $n > 1$, is when the universe $\mathcal{U}$ is confined into the imploding state with a monotonically increased imploding anti-confinement strength $asp = \frac{1}{3} + \frac{2}{3}(1 - \frac{1}{n+1}) > \frac{1}{3}$; and its energy divides further—corresponding to each of the 2^n triangles $BAC_{n,i}$ on the base BC, where $1 \leq i \leq 2^n$, dividing further into two similar sub-triangles—where:

(vi) The total energy corresponding to each of the 2^n triangles *after* the event e_n is $s/2^{n-1}$ for $n > 0$;

(vii) The strength of the anti-confinement field within each sub-universe remains constant at $asp = 1/3$ between events e_n and e_{n+1}, which is below the minimum imploding $asp = \frac{2}{3}$ of event e_1.

D: We thus have a mathematical model of an exploding and then imploding universe:

(viii) That can be viewed as recycling endlessly in *either* direction of time;

(ix) Whose state—exploding, steady, or imploding—at any event e is determined by the strength of an anti-confinement field that—in the direction of time chosen in this example—regularly impels $\mathcal{U}$ to split itself into a monotonically increasing number of isomorphic, but electromagnetically disconnected, sub-universes, all situated in a common confinement/anti-confinement field:

- where the laws of nature remain unchanged;
- where, for $n > 0$, the total energy within each sub-universe at event e_n has decreased monotonically to $s/2^{n-1}$ due to persisting imploding effects of assumed gravitational/anti-gravitational forces;
- that will further split each sub-universe into two at event e_{n+1} as illustrated in Fig.2 if the strength of the anti-confinement field is in the state $1 > asp > \frac{1}{3}$;

(x) Where the energy within each sub-universe during the steady state between events e_n and e_{n+1} appears as 'dark' to its siblings:

- since it is disconnected from, and disappears forever beyond, their event-horizon at an implosion;
- and because each sub-universe, unaware of its siblings, assumes that—since energy can neither be created nor destroyed—the total energy s of the universe must remain constant within their illusory 'universe', either as visible or as 'dark' energy;
- where the distribution of matter outside the critical black hole within each sub-universe may be perceived at any instant by an observer within the sub-universe as accelerating away from the observer in an apparently expanding 'universe';

- where any two, isomorphic but electro-magnetically disconnected, twin sub-universes have equal, but asymmetrical, division of energy;

(xi) Where each sub-universe during the steady state between events e_n and e_{n+1} is expanding at an accelerating rate since the 'cosmological constant' $asp = \frac{1}{3} > 0$;

(xii) The energy within each sub-universe at the limiting Zeno-type phase-change point— describable mathematically as '$n \rightarrow \infty$'—implodes finally to a 'dark point' in BC;

(xiii) Where the energy within the universe as a whole experiences a steam-to-water phase-changing collapse into the original Big Bang configuration represented by an exploding anti-gravitational state $asp = 1$ denoted by BC;

- thus triggering the next cycle of its rebirth (in the chosen time direction of this example);

10. Conclusion

In this investigation we have argued for the plausibility of the thesis (Thesis 2) that if:

(a) a physical process is representable by a Cauchy sequence; and

(b) we accept that there can be no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena;

then:

(c) in the absence of an extraneous, *evidence-based*, proof of 'closure' which determines the behaviour of the physical process in the limit as corresponding to a 'Cauchy' limit;

(d) the physical process *must* tend to a discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the physical process.

We have highlighted the practical significance of our thesis for the physical sciences by defining an, *in principle verifiable*, mathematical model in Fig.2 that can be interpreted as describing the putative behaviour under a well-defined iteration of:

(i) a virus cluster; and

(ii) an elastic string.

where the physical process in each case can be 'seen' to tend to an 'ultimate' discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the process.

We have then highlighted the theoretical significance of our thesis for a realistic philosophy of science by showing that Fig.2 can also be interpreted as representing the, *essentially unverifiable*, state of the total energy of:

(iii) a finite Universe $\mathcal{U}$:

- that recycles endlessly from Big Bang to Ultimate Implosion; and

– in which the existence of 'dark energy' is mathematically and intuitionistically unobjectionable.

Moreover, the only assumptions we have made are that $\mathcal{U}$ obeys Einstein's equations and classical quantum theory, and that:

Thesis 3. *The anti-matter in $\mathcal{U}$ produces a repulsive, anti-gravitational, field:*

- *that is consistent with both general relativity and Newtonian gravity;*
- *whose state at any instant is either exploding, steady, or imploding;*
- *whose 'energy anti-confinement' strength at any instant is determined by an anti-gravitational dimensionless 'cosmological constant' asp that can assume any of three values $asp = 1$ (exploding at the instant of the Big Bang), $asp = \frac{1}{3}$ (steady between an explosion and an implosion) or $asp = \frac{1}{3} + \frac{2}{3}(1 - \frac{1}{n+1})$ (imploding at the instant of the extinguishing of the n^{th} 'critical black hole' for all $n \geq 1$);*
- *which constantly opposes the 'energy confinement' strength of the Newtonian gravitational field whose state is determined at any instant by only one dimensionless gravitational constant*[5] *$gsp = \frac{1}{2}$.*

Since it is conventional wisdom (see [BCST], [Vi11], [Ch97], [NG91]) that the existence of anti-matter which could produce a repulsive, anti-gravitational, field is admitted by both general relativty and Newtonian gravity, we conclude from Theses 2 and 3 that the commonly perceived barriers to modelling the behaviour of such a universe $\mathcal{U}$ unambiguously in a mathematical language may be illusory, and reflect merely an attempt to ask of the language selected for such representation more than it is designed to deliver unequivocally.

More specifically, from the perspective of the *evidence-based* reasoning introduced in [An16]—some of whose consequences for mathematics, philosophy and the physical sciences are investigated in [An18]—it can reasonably be argued that the commonly perceived barriers to modelling the behaviour of such a universe $\mathcal{U}$ realistically in a mathematical language may reflect the fact that:

- since the real numbers are defined by conventional wisdom in set-theoretical terms as the postulated limits of Cauchy sequences in a second-order *dichotomous*[6] arithmetic such as ACA_0,
- the prevailing language of choice for representing physical phenomena and their associated abstractions (conceptual metaphors) mathematically is generally some language of Set Theory,
- which admits axioms—such as an axiom of infinity—whose veridicality cannot be *evidence-based* (in the sense of [An18], Chapter 6) under a well-defined interpretation,
- and in which the *dichotomy* highlighted in ACA_0 could admit a contradiction under any well-defined interpretation of the theory.

[5] Which could be viewed as corresponding to the gravitational constant, denoted by G, common to both Newton's law of universal gravitation and Einstein's general theory of relativity; whose value in Planck units is *defined* as 1, and whose measured value is expressed in the *International System of Units* as approximately $6.674\ x\ 10^{-11} N.kg^{-2}.m^{2}$.

[6] Since, in [An18], Chapter 22 we show how—in the case of Goodstein's Theorem—such a belief leads to a dichotomous conclusion ([An18], Theorem 22.3).

11. Further directions suggested by this investigation

We note that Fig.2 is not a unique model for the ‘confinement’ properties of the universe $\mathcal{U}$. For instance, we could have started essentially similar iterations with a square $ABCD$ of side s.

Moreover, it is not necessary that each ’black hole’ create isomorphic sub-universes; an assumption intended only to illustrate that an event such as an Ultimate Implosion is well-definable mathematically.

However, since the Ultimate Implosion is defined as corresponding to a mathematical limit as $n \rightarrow \infty$, and we postulate that there are no infinite processes in physical phenomena, it follows that the law determining such an Ultimate Implosion (as also the point of implosion of a ‘black hole’) may be of an essentially ‘unknowable’ quantum nature; in which case we cannot even assume in principle that a universe such as $\mathcal{U}$ can be shown to *actually* exist on the basis of *evidence-based* reasoning, nor whether or not it would recycle identically each time (in either direction).

It may thus be worth considering further, by the principle of Occam’s razor, whether the above simple mathematical model of the properties of a universe $\mathcal{U}$—which, defined as obeying Einstein’s equations and quantum theory, seems to fit our known experimental observations—can be taken to suggest that, as implicitly argued by physicist Sabine Hossenfelder, we may have reached the foundations of physics beyond which the laws of nature are essentially ‘unknowable’:

> “So you want to know what holds the world together, how the universe was made, and what rules our existence goes by? The closest you will get to an answer is following the trail of facts down into the basement of science. Folow it until facts get sparse and your onward journey is blocked by theoreticians arguing whose theory is prettier. That’s when you know you’ve reached the foundations.
>
> The foundations of physics are those ingredients of our theories that cannot, for all we presently know, be derived from anything simpler. At this bottommost level we presently have space, time, and twenty-five particles, together with the equations that encode their behaviour. . . .
>
> In the foundations of physics we deal only with particles that cannot be further decomposed; we call them “elementary particles.” For all we presently know, they have no substructure. But the elementary particles can combine to make up atoms, molecules, proteins—and thereby create the enormous variety of structures we see around us. It’s these twenty-five particles that you, I, and everything else in the universe are made of.
>
> But the particles themselves aren’t all that interesting. What is interesting are the relations between them, the principles that determine their interaction, the structure of the laws that gave birth to the universe and enabled our existence. In our game, it’s the rules we care about, not the pieces. And the most important llesson we have learned is that nature plays by the rules of mathematics.”
>
> *. . . Hossenfelder: [Hos18], p.6.*

From the broader, multi-disciplinary, *evidence-based* perspective of this investigation, we view Hossbender as essentially arguing in [Hos18] that committing intellectual and physical resources to seeking experimental verification for the putative existence of physical objects, or of a ‘Theory’, should:

- only follow if such putative objects, or the putative elements of the ‘Theory’, can be theoretically defined—even if only in principle—in a categorical mathematical language, such as the first-order Peano Arithmetic, which (see [An16]) has a finitary *evidence-based* interpretation, and admits unambiguous communication between any two intelligences—whether human or mechanistic;

- and not merely on the basis that they can be conceptualised metaphorically and represented in a set-theoretical language such as ZF which, even though first-order, has no *evidence-based* interpretation that would admit unambiguous communication.

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