Properties of Lipschitz Quotient Mappings on the Plane

by

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Abstract

In the present work, we are concerned with the relation between the Lipschitz and co-Lipschitz constants of a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the cardinality of the inverse image of a point under the mapping f, depending on the norm on $\mathbb{R}^2$.

In the paper *Lipschitz quotient mappings with good ratio of constants* (Mathematika, 2002), Maleva proves that there is a scale of real numbers $0 < \ldots < \rho_n < \ldots < \rho_1 < 1$ such that for any Lipschitz quotient mapping from the plane equipped with the Euclidean norm to itself, if the ratio between the co-Lipschitz and the Lipschitz constants of f is bigger than ρ_n, then the cardinality of any fibre of f is less than or equal to n. Furthermore, it is proven that for the Euclidean case the values of this scale are $\rho_n = {}^1\!/\!(n+1)$ for each $n \in \mathbb{N}$ and that these are sharp.

A natural question is: given a normed space $(\mathbb{R}^2, \|\cdot\|)$ whether it is possible to find the values of the scale $0 < \ldots < \rho_n^{\|\cdot\|} < \ldots < \rho_1^{\|\cdot\|} < 1$ such that for any Lipschitz quotient mapping from $(\mathbb{R}^2, \|\cdot\|)$ to itself, with Lipschitz and co-Lipschitz constants equal to L and c respectively, the relation $c/L > \rho_n^{\|\cdot\|}$ implies $\#f^{-1}(x) \leq n$ for all $x \in \mathbb{R}^2$.

We prove in Chapter 2 that the same "Euclidean scale", $\rho_n = {}^1\!/\!(n+1)$, works for any norm on the plane. Here we follow the general idea in *Point preimages under ball non-collapsing mappings* (GAFA, Lecture Notes in Math., 2003) by Maleva but verify details carefully. On the other hand, the question whether this scale is sharp leads to different conclusions. We show in Chapters 3 and 4 that for some non-Euclidean norms the "Euclidean scale" is not sharp, but there are also non-Euclidean norms for which a Lipschitz quotient exists satisfying $\max \#f^{-1}(x) = 2$ and $c/L = 1/2$.

To Julia

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Chapter 1
Introduction

We start this chapter in Section 1.1 with the introduction of the main subject of our study, Lipschitz quotient mappings. We give a brief survey of general properties of Lipschitz quotient mappings between normed spaces and state some open questions. We continue this section by explaining the question to which our research in this thesis is devoted and we describe the general structure of this work.

Section 1.2 will be devoted to the study of the Lipschitz and co-Lipschitz mappings between finite-dimensional and infinite-dimensional spaces. We will include the proof of some basic properties of these mappings there.

Once the problem has been explained and our main object of study has been presented, in the last section of this chapter we will be working on the development of the tools that we will be using to study the Lipschitz quotients on the plane in more depth. In particular, we will be concerned with measuring the length of a curve on the plane using non-Euclidean norms. In this section we include all the definitions and we state and prove all the general properties of the length that we will be using.

1.1 Motivation of the problem

The Lipschitz property has been widely used in different areas of mathematics and it has been of particular importance in geometric measure theory, nonlinear analysis and partial differential equations. The strengthening of the Lipschitz condition to reach stronger conclusions has led, in various contexts, to different notions of "well behaved Lipschitz mappings". Perhaps the better known of them is the bi-Lipschitz condition, but some other interesting weaker conditions —that do not require the mapping to be a homeomorphism— have been considered and studied. For instance, bounded length distortion mappings which are studied in [22], Lipschitz regular mappings studied in [9], Lipschitz ball noncollapsing mappings in [20] and Lipschitz quotient mappings which are the main object of study in this work. Lipschitz quotients are defined in the following way.

Definition 1.1.1. A map $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ between two normed spaces, is called a *Lipschitz mapping* if there exists a positive constant L such that

$$\|f(x) - f(y)\|_Y \leq L\|x - y\|_X \text{ for all } x, y \in X.$$

In other words, we require that there is a constant $L > 0$ such that for all $x \in X$ and all $r > 0$, we have $f(B_r^X(x)) \subseteq B_{Lr}^Y(f(x))$, where $B_r^W(x)$ denotes the open ball in $(W, \|\cdot\|_W)$ with radius r centred at x. The infimum of all such constants L is called the Lipschitz constant at the point x.

In a similar way, we say that f is a *co-Lipschitz mapping* if there exists a positive constant c such that

$$B_{cr}^Y(f(x)) \subseteq f(B_r^X(x)) \text{ for all } x \in X \text{ and } r > 0.$$

The co-Lipschitz constant of f is the supremum over all possible constants c.

Finally, if $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a Lipschitz and co-Lipschitz mapping, we say that f is a *Lipschitz quotient mapping.* We also say that Y is a *Lipschitz quotient* of X if there exists a Lipschitz quotient mapping from X to Y.

The definition of the co-Lipschitz condition appeared in [13] in the context of differential geometry right before the publication of the paper [1] in which Bates, Johnson, Lindenstrauss, Preiss and Schechtman reached very significant results concerning the structure of such mappings in the finite-dimensional case. In particular, they prove in [1] that for Lipschitz quotient mappings from the plane to itself the inverse image of any point under such mapping is finite. Furthermore, in a subsequent paper [15] they show that every Lipschitz quotient mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ can be viewed as a re-parametrization of a complex polynomial. In other words, there is a homeomorphism h on the plane and a polynomial P of one complex variable such that $f = P \circ h$.

Remark 1.1.2. The above implies that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a Lipschitz quotient mapping, then for any $x \in \mathbb{R}^2$, $f^{-1}(x)$ is a finite set and $\#f^{-1}(x) \leq \deg(P)$, where $\deg(P)$ denotes the degree of the polynomial P. Moreover, for all but finitely many points $x \in \mathbb{R}^2$ we have $\#f^{-1}(x) = \deg(P)$.

There is some resemblance between Lipschitz quotient mappings and the so-called quasiregular mappings, which are defined in the context of topological manifolds. (See [26] for a survey on quasiregular mapping theory). There is a result due to Reshetnyak (see [24] and [25]) that shows that the inverse image of a point under a quasiregular mapping is always discrete. This could suggest that the inverse image of a point of a Lipschitz quotient mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ is discrete. However, at this moment, this question has not yet been answered for $n > 2$.

It is easy to see that in the finite-dimensional case, the property of being Lipschitz quotient does not depend on a particular choice of norms (Proposition 1.2.3). Also, it is easy to see that every linear non degenerate transformation of $\mathbb{R}^n$ is a Lipschitz quotient mapping under any norm. However, the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x_1, x_2) = (x_1, |x_2|)$ is not a Lipschitz quotient mapping. In a sense, Lipschitz quotients can be thought of as a generalisation of linear quotient mappings (i.e.surjective linear mappings). Recall that linear quotients are "more than just open" in the sense that given such a mapping g, we can find a constant c such that $g(B_r^X(x)) \supseteq B_{cr}^Y(g(x))$ for all $r > 0$ and $x \in X$. It is then natural to ask for a Lipschitz mapping to have a similar openness property (e.g. the co-Lipschitz property) and to ask whether the existence of such mapping between Banach spaces yields some structure relations between these spaces.

A central question related to Lipschitz quotient mappings is: Given a Banach space X and a Lipschitz quotient $f : X \to Y$, under which conditions can we guarantee that Y is a linear quotient of X? In the case of linear quotients it is known that given a pair of Banach spaces X and Y, if there is a linear quotient mapping T from X onto Y, then there are relations between the structure of these two spaces (isomorphism theorems of Banach spaces). Regarding Lipschitz quotients there are some positive results in this direction. For instance, in [1] it is proven that if a Banach space Y is a Lipschitz quotient of L_p with $1 < p < \infty$ then Y is isomorphic to a linear quotient of L_p. However, although Lipschitz mappings carry strong continuity properties, there are examples of pairs of Banach spaces such that there is a Lipschitz quotient mapping between them but no linear quotient map exists between them. The first example of such a pair of Banach spaces was presented in [16].

Recently Lipschitz quotient mappings appear to have an interesting role in a more general setting, namely in a particular class of metric measure spaces with a form of differentiable structure where most of the major properties of these mappings, proved in [1], can be translated. See [10], and [4] for a more general reference on the structure of such metric measure spaces.

As we have mentioned there are strong results for Lipschitz mappings defined from the plane to itself, and we know that the fibers of points under Lipschitz quotients on the plane must be finite. However for the general case, $\mathbb{R}^n \to \mathbb{R}^m$, we have spread results. For instance, on the one hand, the inverse image of a point under a Lipschitz mapping could contain a set of co-dimension more than $m - n$. In [7] the author constructs an example of a Lipschitz quotient mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $f^{-1}(0)$ is a set containing a plane. On the other hand, in the particular case when $n = 2$ and $m = 1$, good progress has been made describing the structure of level sets of such mappings. In [23] it is proven that, for Lipschitz quotient mappings $f : \mathbb{R}^2 \to \mathbb{R}^1$, the inverse image of any point has a finite number of components and each component separates the plane. There is also a precise topological description of these sets, for instance, it is shown that they are hereditarily locally connected, locally compact and closed. Furthermore, in [21], the author gives an upper bound for the number of components of the level sets of a Lipschitz mapping from $\mathbb{R}^2$ to $\mathbb{R}$. This bound is given in terms of the Lipschitz and co-Lipschitz constants of the mapping. However the case $n = m > 2$ remains wide open and we do not even know if the inverse image of a point could be infinite. It is natural to think that adding assumptions on the difference between the Lipschitz and co-Lipschitz constants of a Lipschitz quotient mapping, could bring some positive results in that direction. See, for instance, [19, Theorem 1].

It is clear that much stronger results are known for the Lipschitz quotient mappings in the planar case, $n = m = 2$. Moreover, in this case we can say even more about the cardinality of the inverse image of a point under a Lipschitz quotient mapping. In [19] two questions were presented and answered for Lipschitz quotients from the plane to itself equipped with the Euclidean norm $|\cdot|$: Let $f : (\mathbb{R}^2, |\cdot|) \to (\mathbb{R}^2, |\cdot|)$, be an L-Lipschitz and c-co-Lipschitz mapping.

Question 1. Is it true that if the ratio between the Lipschitz and co-Lipschitz constants satisfies ${}^{c}/_{L} > {}^{1}/_{2}$, then f is a homeomorphism?

Question 2. Is there a scale $0 < \ldots < \rho_n < \ldots < \rho_1 < 1$ such that ${}^{c}/_{L} > \rho_n$ implies $\#f^{-1}(x) \leq n$ for any $x \in \mathbb{R}^2$?

The author solved both questions in the positive, by proving that in the Euclidean case the assumption ${}^{c}/_{L} > {}^{1}/_{(n+1)}$ implies $\#f^{-1}(x) \leq n$ for any $x \in \mathbb{R}^2$. Clearly, from the fact that any two norms on the plane are equivalent, the existence of such a scale ρ_n for the Euclidean case implies the existence of a scale for any norm on the plane, and we verify in Chapter 2 that the given scale $\rho_n = 1/(n+1)$ is in fact universal, in the sense that it does not depend on the norm. This also implies that for any norm on the plane, if $c/L > 1/2$ then f is a homeomorphism. The proof of this result (Theorem 2.7) is the main goal of Chapter 2. This is an expansion of the material presented in [20, Theorem 1].

However, as we shall first see in Chapter 3, when considering non-Euclidean norms the scale $\rho_n = 1/(n+1)$ works not in the same way as it does for the Euclidean norm. This shows that in fact the remark in Section 3 of [20] about the supremum norm, $\|\cdot\|_\infty$ is not correct. More precisely, for the Euclidean case we have examples of Lipschitz quotients with ratio of constants $c/L = 1/n$ and $\max \# f^{-1}(x) = n$. This means that the scale (ρ_n) is sharp for the Euclidean case. However, in Theorem 3.2.5 we prove that for the supremum norm on the plane, there does not exist a Lipschitz quotient with $c/L = 1/2$ and $\max \# f^{-1}(x) = 2$. This leads to three questions:

Question i. Is the Euclidean norm the only norm on the plane for which there exist Lipschitz quotients with $c/L = 1/n$ and $\max \# f^{-1}(x) = n$?

Question ii. Find a sharp scale for the supremum norm $0 < \ldots < \rho_n^\infty < \ldots < \rho_1^\infty < 1$ such that given any L-Lipschitz and c-co-Lipschitz mapping $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$, the condition $c/L > \rho_n^\infty$ implies $\# f^{-1}(x) \leq n$ for any $x \in \mathbb{R}^2$.

Question iii. For any norm on the plane, $\|\cdot\|$, find a sharp scale $0 < \ldots < \rho_n^{\|\cdot\|} < \ldots < \rho_1^{\|\cdot\|} < 1$ such that given any L-Lipschitz and c-co-Lipschitz mapping $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$, the condition $c/L > \rho_n^{\|\cdot\|}$ implies $\# f^{-1}(x) \leq n$ for any $x \in \mathbb{R}^2$.

In Chapter 4 we partially answer Question *i*. We give examples of non-Euclidean norms on the plane for which certain Lipschitz quotients do satisfy $c/L = 1/2$ and $\max \# f^{-1}(x) = 2$. We also include more examples of norms on the plane for which, as for the supremum norm, every Lipschitz quotient mapping with $\max \# f^{-1}(x) = n$ satisfies $c/L < 1/2$. Finally, in Chapter 5 we give partial results that indicate that for the supremum norm the second value of the scale ρ_n^∞ is equal to $1/3$, which leads to the conjecture $\rho_n^\infty = \rho_{n+1}$, where ρ_k denotes the sharp scale for the Euclidean norm.

1.2 Basic properties of Lipschitz quotient mappings

Our main object of study, Lipschitz quotients, has been defined in Definition 1.1.1 and now we are going to have a closer look at it. In this section we introduce some basic results regarding Lipschitz quotient mappings in general metric spaces. We will compare the Lipschitz quotients with the so called bi-Lipschitz mappings and we will study the local versions of these properties. We will show for example that, locally, the co-Lipschitz property, in the same way as the global Lipschitz condition, can be written directly in terms of the norm (see Corollary 1.2.12). However, it is worth noticing that even when, locally, this conditions seem to be of the very same kind, in order to achieve the global condition from the local one, these two properties do not behave exactly in the same way. For the Lipschitz condition the proof works for infinite-dimensional spaces (Proposition 1.2.6), whereas for the co-Lipschitz condition it does not work for infinite-dimensional spaces, see Proposition 1.2.7.

The following very basic statement establishes that the Lipschitz and co-Lipschitz constants of a mapping do satisfy the Lipschitz and co-Lipschitz conditions respectively.

Lemma 1.2.1. *Let $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ be a Lipschitz quotient mapping, with Lipschitz and co-Lipschitz constants equal to L_* and c_* respectively. Then for all $r > 0$ we have:*

$$B^Y_{c_* r}(f(x)) \subseteq f(B^X_r(x)) \subseteq B^Y_{L_* r}(f(x)).$$

Proof. First we deal with the Lipschitz constant. Let f be as in the hypothesis and take

$\varepsilon > 0$. Pick any two distinct points $x_1, x_2 \in X$. Since L_* is the infimum of the set

$$\mathcal{L} = \{L > 0 : \|f(x) - f(y)\|_Y \leq L\|x - y\|_X \text{ for all } x, y \in X\},$$

there is some $L \in \mathcal{L}$ such that $L - L_* < \varepsilon' = \frac{\varepsilon}{\|x_1 - x_2\|_X}$. Hence:

$$\begin{aligned}\|f(x_1) - f(x_2)\|_Y \leq L\|x_1 - x_2\|_X &< (\varepsilon' + L_*)\|x_1 - x_2\|_X \\ &= L_*\|x_1 - x_2\|_X + \varepsilon.\end{aligned}$$

Consequently, as the above estimate is satisfied for all $\varepsilon > 0$:

$$\|f(x_1) - f(x_2)\|_Y \leq L_*\|x_1 - x_2\|_X.$$

This means that $f(B_r^X(x)) \subseteq B_{L_* r}^Y(f(x))$ for all $x \in X$, $r > 0$.

Now, for the co-Lipschitz constant define the non-empty set

$$\mathcal{C} = \{c > 0 : B_{cr}^Y(f(x)) \subseteq f(B_r^X(x)) \text{ for all } r > 0\}.$$

Let $r > 0$ and $x_0 \in X$. Pick a point $y \in B_{c_* r}(f(x_0))$. Our aim is to prove that $y \in f(B_r(x_0))$. Since $\|y - f(x_0)\|_Y < c_* r$, we have $\frac{1}{r}\|y - f(x_0)\|_Y < c_* = \sup \mathcal{C}$, so there must be some $c \in \mathcal{C}$ such that $\|y - f(x_0)\|_Y < cr \leq c_* r$. This implies $y \in B_{cr}(f(x_0)) \subseteq f(B_r(x_0))$. Thus $B_{c_* r}(f(x_0)) \subseteq f(B_r(x_0))$. □

As the following lemma shows, for the finite-dimensional case, we can replace the open balls in the definition of the co-Lipschitz condition with closed balls, which is sometimes more convenient.

Lemma 1.2.2. *Let $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ be a continuous mapping with X finite-dimensional and let $c > 0$. The following conditions are equivalent:*

1. *For all $x \in X$ and for all $r > 0$,*

$$B^Y_{cr}(f(x)) \subseteq f(B^X_r(x)).$$

2. *For all $x \in X$ and for all $r > 0$,*

$$\overline{B^Y_{cr}}(f(x)) \subseteq f(\overline{B^X_r}(x)).$$

Proof. Assume first that c satisfies the condition *1*. Let $r > 0$ and $x_0 \in X$. Pick a point $y \in \overline{B^Y}_{cr}(f(x_0))$. We need to show that $y \in f(\overline{B^X}_r(x_0))$. Take a sequence (y_n) contained in $B^Y_{cr}(f(x_0))$ such that $y_n \to y$. Since f is a c-co-Lipschitz mapping, for every $n \in \mathbb{N}$ there is a point $x_n \in B^X_r(x_0)$ such that $f(x_n) = y_n$. By the finite-dimensionality of X we can assure that there is a convergent subsequence of x_n, say $x_{n_j} \to x \in \overline{B^X}_r(x_0)$. Hence, using the continuity of f, we have:

$$f(x) = f(\lim_{j\to\infty} x_{n_j}) = \lim_{j\to\infty} f(x_{n_j}) = \lim_{j\to\infty} y_{n_j} = y.$$

Thus $y \in f(\overline{B^X}_r(x_0))$. This shows that the first condition implies the second.

Now assume that the second condition is satisfied. Let $r > 0$ and $x_0 \in X$. Pick a point $y \in B^Y_{cr}(f(x_0))$, we need to show that $y \in f(B^X_r(x_0))$. Let $\delta := \|y - f(x_0)\|_Y$, from condition *2*, we have:

$$y \in \overline{B^Y_\delta}(f(x_0)) \subseteq f(\overline{B^X_{\delta/c}}(x_0)).$$

hence $y = f(x)$ for some $x \in \overline{B^X_{\delta/c}}(x_0)$. Now, since

$$\|x_0 - x\|_X \leq \tfrac{1}{c}\delta < \tfrac{1}{c}cr = r,$$

we know that $x \in B^X_r(x_0)$, therefore $y \in f(B^X_r(x))$.

Thus $B^Y_{cr}(f(x_0)) \subseteq f(B^X_r(x_0))$ and we conclude that conditions *1* and *2* are equivalent.

$\square$

We already mentioned some examples of Lipschitz quotient mappings: linear mappings, which clearly satisfy the Lipschitz quotient condition, and the mappings on the Euclidean plane $f_k : (\mathbb{R}^2, |\cdot|) \to (\mathbb{R}^2, |\cdot|)$, with $k \in \mathbb{N}$, defined as $f_k(re^{i\theta}) = re^{ik\theta}$. To see that the latter are in mappings are in fact Lipschitz quotients with Lipschitz constant equal to k and co-Lipschitz constant 1, notice that each point x is being mapped to the point that has k times the argument of x and the same norm as x. Hence f_k can separate points by at most a factor of k and at the same time f_k cannot shrink. Perhaps the best way to convince yourself about this fact is to look at the following picture, Figure 1.1, made for the case $k = 2$. In the picture we illustrate a ball centred at a point x with radius r and its image under the mapping f_k, both in yellow.

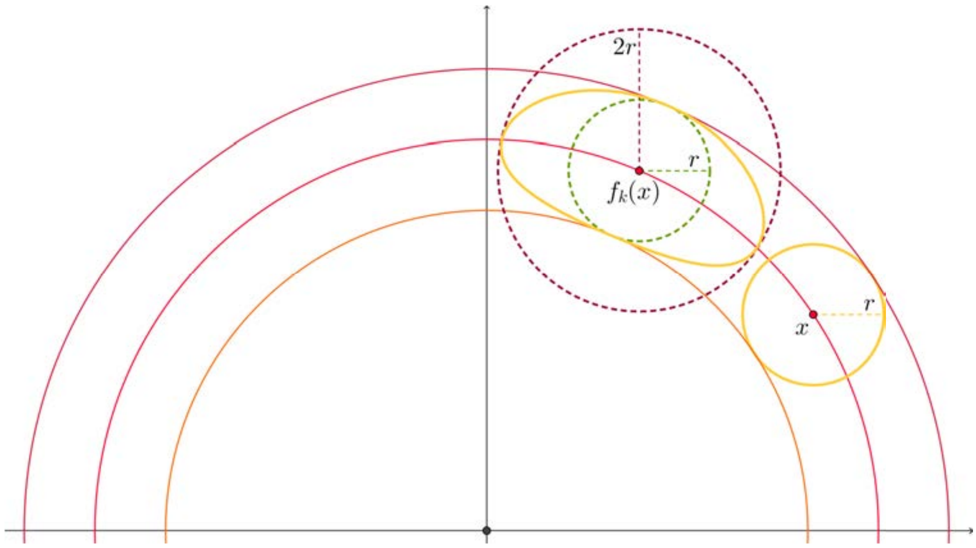


Figure 1.1

Notice that for each point $x \in \mathbb{R}^2 \setminus \{0\}$ we have $\# f_k^{-1}(x) = k$. As we shall see later in Theorem 2.6 this kind of examples are archetypal. So we now have more interesting examples of Lipschitz quotient mappings for the Euclidean plane, and hence for the plane in general. Indeed, as the next proposition shows, being a Lipschitz or co-Lipschitz mapping does not depend on the particular norm.

Proposition 1.2.3. *Let $f : (X, \|\cdot\|_X^1) \to (Y, \|\cdot\|_Y^1)$ be a Lipschitz quotient mapping. If $\|\cdot\|_X^2$ and $\|\cdot\|_Y^2$ are norms on X and Y equivalent to the norms $\|\cdot\|_X^1$ and $\|\cdot\|_X^1$ respectively, then the mapping f, considered as a map from $(X, \|\cdot\|_X^2)$ to $(Y, \|\cdot\|_Y^2)$, is a Lipschitz quotient mapping.*

In particular, if X and Y are finite-dimensional, the present lemma holds for any pair of norms defined on X and any pair of norms defined on Y.

Proof. Let L and c be the Lipschitz and co-Lipschitz constants of $f : (X, \|\cdot\|_X^1) \to (Y, \|\cdot\|_Y^1)$. Since X and Y are finite-dimensional, there exist constants k, k_*, s and s_* such that for all $x \in X$ and $y \in Y$:

$$\begin{aligned} k\|x\|_X^1 \leq \|x\|_X^2 \leq k_*\|x\|_X^1; \\ s\|y\|_Y^1 \leq \|y\|_Y^2 \leq s_*\|y\|_Y^1. \end{aligned} \tag{1.1}$$

Therefore, for any $x_1, x_2 \in X$ we have:

$$\|f(x_1) - f(x_2)\|_Y^2 \leq s_*\|f(x_1) - f(x_2)\|_Y^1 \leq Ls_*\|x_1 - x_2\|_X^1 \leq \tfrac{1}{k}Ls^*\|x_1 - x_2\|_X^2.$$

Hence, f is a Lipschitz mapping as a map from $(X, \|\cdot\|_X^2)$ to $(Y, \|\cdot\|_Y^2)$ with Lipschitz constant $L_2 \leq \frac{Ls^*}{k}$.

Now for the co-Lipschitz constant we work in a similar way. For $i = 1, 2$, let us denote by $B_X^i(x, r)$ the ball of radius r centred at x under the norm $\|\cdot\|_X^i$ and by $B_Y^i(y, r)$ the ball of radius r centred at y under the norm $\|\cdot\|_Y^i$. Take $x \in X$ and $r > 0$, from (1.1) and the fact that f is a co-Lipschitz mapping as a map from $(X, \|\cdot\|_X^1)$ to $(Y, \|\cdot\|_Y^1)$, we see that:

$$f(B_X^2(x,r)) \supseteq f(B_X^1(x,\tfrac{r}{k_*})) \supseteq B_Y^1(f(x),c\tfrac{r}{k_*}) \supseteq B_Y^2(f(x),sc\tfrac{r}{k_*}(x).$$

Since this holds for all $r > 0$, we conclude that $f : (X, \|\cdot\|_X^2) \to (Y, \|\cdot\|_Y^2)$ is a co-Lipschitz mapping with co-Lipschitz constant $c_2 \geq \frac{sc}{k_*}$.

Thus, f is a Lipschitz quotient mapping as a map from $(X, \|\cdot\|_X^2)$ to $(Y, \|\cdot\|_Y^2)$ and its Lipschitz and co-Lipschitz constants, L_2 and c_2 respectively, satisfy:

$$L_2 \leq (\tfrac{s_*}{k})L \text{ and } c_2 \geq (\tfrac{s}{k_*})c. \tag{1.2}$$

□

The Lipschitz quotient condition does not depend on the norm, but clearly the Lipschitz and co-Lipschitz constants do depend on the norm (see (1.2)). However, we can prove that if two norms on the plane are similar, in the sense that one can be obtained by scaling and rotating the other, then, given a Lipschitz quotient mapping f with certain constants under one of the norms, its appropriately scaled and rotated version will be a Lipschitz quotient mapping in the other norm and the constants will be preserved. This result will be useful later on in Chapter 4.

Proposition 1.2.4. *Let $\|\cdot\|$ and $\|\cdot\|_*$ be two norms on $\mathbb{R}^2$ and denote by $B_r(x)$ and $B_r^*(x)$ the ball of radius r centred at x under the norm $\|\cdot\|$ and $\|\cdot\|_*$ respectively. Assume there exist a rotation R and a constant $k > 0$ such that $k(R(B_1(0))) = B_1^*(0)$. If $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ is a Lipschitz quotient mapping with $\max_{x\in\mathbb{R}^2} \#f^{-1}(x) = n$, then the mapping $g : (\mathbb{R}^2, \|\cdot\|_*) \to (\mathbb{R}^2, \|\cdot\|_*)$ defined as $g = R \circ f \circ R^{-1}$ is a Lipschitz quotient with the same Lipschitz and co-Lipschitz constants as f and $\max_{x\in\mathbb{R}^2} \#g^{-1}(x) = n$.*

Proof. Let $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ be an L-Lipschitz and c-co-Lipschitz mapping. Let

R, k and g be as in the hypothesis.

It is clear that $\max_{x\in\mathbb{R}^2} \#g^{-1}(x) = n$, so we only need to show that L and c are the Lipschitz and co-Lipschitz constants of g.

Since $k(R(B_1(0))) = B_1^*(0)$ it is clear that for all $x \in \mathbb{R}^2$ we have

$$\|x\| = k\|R(x)\|_* \text{ and } \|x\|_* = \tfrac{1}{k}\|R^{-1}(x)\|. \tag{1.3}$$

For the Lipschitz constant, take $x, y \in \mathbb{R}^2$. From (1.3) we have

$$\begin{aligned}\|g(x) - g(y)\|_* = \|R(f(R^{-1}(x))) - R(f(R^{-1}(y)))\|_* = \tfrac{1}{k}\|f(R^{-1}(x)) - f(R^{-1}(y))\|\\ \leq \tfrac{1}{k}\left(L\|R^{-1}(x) - R^{-1}(y)\|\right) = L\|x - y\|_*.\end{aligned}$$

Therefore g is a Lipschitz mapping with Lipschitz constant $L_g \leq L$.

Now, for the co-Lipschitz constant take $x_0 \in \mathbb{R}^2$ and $r > 0$. We are going to show that $B_{cr}^*(g(x_0)) \subseteq g(B_r^*(x_0))$. Take $y \in B_{cr}^*(g(x_0))$ so that:

$$cr > \|y - g(x_0)\|_* = \|y - R(f(R^{-1}(x_0))\|_* = \tfrac{1}{k}\|R^{-1}(y) - f(R^{-1}(x_0))\|.$$

Therefore, $R^{-1}(y) \in B_{ckr}(f(R^{-1}(x_0)))$. Since c is the co-Lipschitz constant of f we know that $B_{ckr}(f(R^{-1}(x_0))) \subseteq f(B_{kr}(R^{-1}(x_0)))$, hence there exists $x \in B_{kr}(R^{-1}(x_0))$ such that $f(x) = R^{-1}(y)$, so we have:

$$kr > \|x - R^{-1}(x_0)\| = k\|R(x) - x_0\|_* \text{ and } R(f(x)) = y.$$

Therefore $R(x) \in B_r^*(x_0)$ and $y = R(f(x)) = R(f(R^{-1}(R(x)))) = g(R(x))$. Hence $y \in g(B_r^*(x_0))$ as we wanted to show. We conclude that g is a co-Lipschitz mapping with constant $c_g \geq c$.

So far, we proved that given an L-Lipschitz and c-co-Lipschitz mapping $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$, if $g = R \circ f \circ R^{-1}$, where R is as in the hypothesis, then g is an L_g-Lipschitz and c_g-co-Lipschitz mapping with $L_g \leq L$ and $c_g \geq c$. Since $f = R^{-1} \circ g \circ R$ and R^{-1} also satisfy the hypothesis of the present lemma, this result also shows that f is an L-Lipschitz and c-co-Lipschitz mapping with $L \leq L_g$ and $c \geq c_g$. Thus g is a Lipschitz quotient mapping with same Lipschitz and co-Lipschitz constants as f. $\square$

We will also consider local versions of Lipschitz, co-Lipschitz and Lipschitz quotient mappings which we define presently.

Definition 1.2.5. We say that a map $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is *locally Lipschitz* at the point $x \in X$ if there exist positive constants L and R such that, if $\|x - y\|_X < R$, then $\|f(x) - f(y)\|_Y \leq L\|x - y\|_X$, i.e. if for all $r \leq R$ we have $f(B_r^X(x)) \subseteq B_{Lr}^Y(f(x))$. The infimum of all such constants L, say L_x, is called the local Lipschitz constant of f at x. This is

$$L_x = \inf\{L > 0 : \exists R > 0 \text{ such that } \forall r < R, f(B_r(x)) \subseteq B_{L_r}^X(f(x))\}.$$

In a similar way, we say that a mapping f is *locally co-Lipschitz* at a point x if there exist positive constants R and c such that for all $r \leq R$ we have $B_{cr}^Y(f(x)) \subseteq f(B_r^X(x))$. The local co-Lipschitz constant of f at the point x is the supremum over all such possible constants c.

As the following results show, if for some map f the local Lipschitz condition, or the local co-Lipschitz condition, is satisfied at all points $x \in X$ with the same constant, then the map f satisfies the global Lipschitz, respectively co-Lipschitz, condition with the same constant. Here we, of course, assume that f is defined on the whole space X. The proof of the following two propositions are done following [7].

Proposition 1.2.6. *Let $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ be a map, where X and Y are normed vector spaces. If there is a constant L_* such that for all $x_0 \in X$ there exists $R_{x_0} > 0$ such that $\|f(x_0) - f(x)\|_Y \leq L_*\|x_0 - x\|_X$, whenever $\|x_0 - x\| < R_{x_0}$, then f is a Lipschitz mapping with Lipschitz constant less than or equal to L_*.*

Proof. Fix a point $x_0 \in X$, we shall prove that for all $x_1 \in X$ we have $\|f(x_0) - f(x_1)\|_Y \leq L_*\|x_0 - x_1\|_X$. Pick a point $x_1 \in X$ and consider the line segment $\mathcal{L}$ joining x_0 with x_1, i.e. $\mathcal{L} = (x_0, x_1]$. Define the set:

$$A = \{z \in \mathcal{L} : \|f(x_0) - f(x)\|_Y \leq L_*\|x_0 - x\|_X \quad \forall x \in (x_0, z]\}.$$

It is easy to see that the set A is a closed subset of $\mathcal{L}$. Indeed, take a point $z \in \bar{A} \cap \mathcal{L}$ and a sequence $z_n \subseteq A \cap \mathcal{L}$ such that $z_n \to z$. Let $\varepsilon > 0$. Take $N_1 \in \mathbb{N}$ such that

$$\|z - z_n\|_X < \tfrac{\varepsilon}{L_*+1} \text{ for all } n \geq N_1.$$

Since f is continuous, we can also find $N_2 \in \mathbb{N}$ such that

$$\|f(z) - f(z_n)\|_Y < \tfrac{\varepsilon}{L_*+1} \text{ for all } n \geq N_2.$$

Then, for any fixed $n \geq \max\{N_1, N_2\}$ we have:

$$\begin{aligned}
\|f(x_0) - f(z)\|_Y &\leq \|f(x_0) - f(z_n)\|_Y + \|f(z_n) - f(z)\|_Y \leq L_*\|x_0 - z_n\|_X + \tfrac{\varepsilon}{L_*+1} \\
&\leq L_*(\|x_0 - z\|_X + \|z - z_n\|_X) + \tfrac{\varepsilon}{L_*+1} \\
&< L_*\|x_0 - z\|_X + (L_* + 1)\tfrac{\varepsilon}{L_*+1} < L_*\|x_0 - z\|_X + \varepsilon.
\end{aligned}$$

Note that the left hand side and the right hand side values of the above estimate do not depend on n. As $\varepsilon > 0$ is arbitrary, we conclude that $\|f(x_0) - f(z)\|_Y \leq L_*\|x_0 - z\|$.

This shows the Lipschitz condition at z. We now show that $z \in A$. Indeed, if x is a point lying in the segment (x_0, z) then, since $z \in \overline{A} \cap \mathcal{L}$ we can take a point $z' \in A$ such that $(x_0, x) \subseteq (x_0, z')$, which means, by definition of A, that $\|f(x_0) - f(x)\|_Y \leq L_* \|x_0 - x\|_X$. Thus, for all $x \in (x_0, z]$ we have $\|f(x_0) - f(x)\|_Y \leq L_* \|x_0 - x\|_X$, i.e. $z \in A$. This proves that A is closed in $\mathcal{L}$.

Now we will prove that A is an open subset of $\mathcal{L}$. Pick a point $z \in A$, by the local Lipschitz property at the point z, there exists $R_z > 0$ such that $\|f(z) - f(x)\|_Y \leq L_* \|z - x\|_X$ whenever $\|z - x\|_X < R_z$. As we shall see this is enough to show that $B_{R_z}(z) \cap \mathcal{L} \subseteq A$. Take $x \in B_{R_z}(z) \cap \mathcal{L}$ and pick a point x' on the line segment $(x_0, x]$. If $x' \in (x_0, z]$ then, since $z \in A$, it is clear that x' satisfies $\|f(x_0) - f(x')\|_Y \leq L_* \|x_0 - x'\|_X$. Assume $x' \in (z, x]$, then $\|z - x'\|_X < R_z$, and since x_0, z and x' are collinear, we also have $\|x_0 - z\|_X + \|z - x'\|_X = \|x_0 - x'\|_X$, therefore:

$$\begin{aligned}\|f(x_0) - f(x')\|_Y &\leq \|f(x_0) - f(z)\|_Y + \|f(z) - f(x')\|_Y \\ &\leq L_* \|x_0 - z\|_X + L_* \|z - x'\|_X = L_* \|x_0 - x'\|_X.\end{aligned}$$

Hence we get $\|f(x_0) - f(x')\|_Y \leq L_* \|x_0 - x'\|_X$, whenever $x' \in (x_0, x]$. This means that $x \in A$, thus, we have shown that $B_{R_z}(z) \cap \mathcal{L} \subseteq A$ and we conclude that A is open in $\mathcal{L}$.

Therefore A is a closed and open subset of $\mathcal{L}$. Since $\mathcal{L}$ is connected we must have $A = \emptyset$ or $A = \mathcal{L}$, but it is clear that $A \neq \emptyset$. Indeed, using the local Lipschitz property of f at the point x_0 we may consider $R_0 > 0$ such that $\|f(x_0) - f(x)\|_Y \leq L_* \|x_0 - x\|_X$, whenever $\|x_0 - x\|_X < R_0$ and take $x \in B_{R_0}(x_0) \cap \mathcal{L}$. Then for all $x' \in (x_0, x]$ we have $\|x' - x_0\|_X < R_0$, so that $\|f(x_0) - f(x')\|_Y \leq L_* \|x_0 - x'\|_X$. This means that $B_{R_0}(x_0) \cap \mathcal{L} \subseteq A$. We conclude that $A = \mathcal{L}$, in particular $x_1 \in A$ and $\|f(x_0) - f(x_1)\|_Y \leq L_* \|x_0 - x_1\|_X$.

Thus f is a Lipschitz mapping, and the Lipschitz constant of f is less than or equal to L_*. $\square$

In a similar way, we prove in the next proposition that the co-Lipschitz condition satisfies the analogous property.

Proposition 1.2.7. *Let $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ be a continuous map, where X and Y are normed vector spaces, with X finite-dimensional. If there is a constant $c_* > 0$ such that for all $x \in X$ there exists a positive number R_x such that $B^Y_{c_*r}(f(x)) \subseteq f(B^X_r(x))$ for all $r \leq R_x$, then f is a co-Lipschitz mapping with co-Lipschitz constant greater than or equal to c_*.*

Proof. First fix $x \in X$ and let R be the radius of the ball centred at x in which the local co-Lipschitz property is satisfied. Pick a point $y \in B^Y_{c_*R}(f(x))$ and notice that this implies $y \in f(\overline{B}^X_{r_y}(x))$, where $r_y = \frac{1}{c_*}\|f(x) - y\|_Y$. To see this, observe that if $y \notin f(\overline{B}^X_{r_y}(x))$, then $f^{-1}(y)$ and $\overline{B}^X_{r_y}(x)$ are disjoint closed subsets of X, and, since X is a finite-dimensional space, $\overline{B}^X_{r_y}(x)$ is compact, therefore the distance between $f^{-1}(y)$ and $\overline{B}^X_{r_y}(x)$ is a positive number; hence there exists $r \in (r_y, R)$ such that $B^X_r(x) \cap f^{-1}(y) = \emptyset$. This is impossible because $y \in B^Y_{c_*r}(f(x)) \subseteq f(B^X_r(x))$ whenever $r \leq R$. Thus

$$B^Y_{c_*R}(f(x)) \subseteq f(\overline{B}^X_{r_y}(x)). \tag{1.4}$$

Now, let $r > 0$, $x_0 \in X$ and $y_0 \in B^Y_{c_*r}(f(x_0))$ be fixed. We need to show that there exists some $x' \in B^X_r(x_0)$ such that $f(x') = y_0$. Consider the line segment $\mathcal{L}$ contained in Y joining $f(x_0)$ with y_0, this is $\mathcal{L} = (f(x_0), y_0]$, and define the set:

$$A = \{z \in \mathcal{L} : y \in f(\overline{B}^X_{r_y}(x_0)) \;\forall\, y \in (f(x_0), z]\}, \text{ where } r_y = \tfrac{1}{c_*}\|f(x_0) - y\|_Y.$$

Let $R_0 > 0$ be such that $B^Y_{c_*r}(f(x_0)) \subseteq f(B^X_r(x_0))$, for all $r \leq R_0$; from (1.4), it is clear that $B^Y_{c_*R_0}(f(x_0)) \cap \mathcal{L} \subseteq A$, thus $A \neq \emptyset$.

Now we show that A is an open subset of $\mathcal{L}$. Consider a point $z \in A$, and let

$x' \in \overline{B}^X_{r_z}(x_0)$ be such that $f(x') = z$. Again, from the local co-Lipschitz property, there exists $R_{x'} > 0$ such that $B^Y_{c_* r}(f(x')) \subseteq f(B^X_r(x'))$, whenever $r \leq R_{x'}$. Pick any point $y \in B^Y_{c_* R_{x'}}(f(x')) \cap \mathcal{L}$. If $y \in (f(x_0), z]$ then from the definition of A we know that $y \in A$. Assume that $y \in [z, y_0]$. From (1.4) we gather that $y \in f(\overline{B}^X_{\delta_y}(x'))$, where $\delta_y = \frac{1}{c_*}\|f(x') - y\|_Y$. Hence, there is a point $x'' \in \overline{B}^Y_{\delta_y}(x') \cap f^{-1}(y)$, this leads to:

$$\begin{aligned}\|x'' - x_0\|_X \leq \|x'' - x'\|_X + \|x' - x_0\|_X &\leq \delta_y + r_z \\ &= \tfrac{1}{c_*}\|f(x') - y\|_Y + \tfrac{1}{c_*}\|f(x_0) - z\|_Y \\ &= \tfrac{1}{c_*}\left(\|z - y\|_Y + \|f(x_0) - z\|_Y\right) = \tfrac{1}{c_*}\|f(x_0) - y\|_Y.\end{aligned}$$

Therefore, $x'' \in \overline{B}^X_{r_y}(x_0)$, i.e. $y \in f(\overline{B}^X_{r_y}(x_0))$. So all points y belonging to $B^Y_{cR_{x'}}(z) \cap \mathcal{L}$ satisfy $y \in f(\overline{B}^X_{r_y}(x_0))$, hence the line segment $(f(x_0), z] \cup (B^Y_{c_* R_{x'}}(z) \cap \mathcal{L})$ is contained in A. Thus A is an open subset of $\mathcal{L}$.

Finally, pick $y \in \overline{A} \cap \mathcal{L}$, and consider a sequence y_n in $A \cap \mathcal{L}$ such that $y_n \to y$. Then, for every $n \in \mathbb{N}$ there is a point $x_n \in \overline{B}^X_{r_{y_n}}(x_0)$ such that $f(x_n) = y_n$. It is clear that, if for some y_n we have $\|f(x_0) - y_n\|_Y \geq \|f(x_0) - y\|_Y$ then $z \in A$. Let us assume that $\|f(x_0) - y_n\|_Y < \|f(x_0) - y\|_Y$ for all $n \in \mathbb{N}$. In this case, we have $(f(x_0), y) \subseteq A$, and since X is finite-dimensional, and $\{x_n\} \subseteq \overline{B}^X_{r_y}(x_0)$, we can take a convergent subsequence of x_n, say $x_{n_j} \to x_*$. Therefore, $x_* \in \overline{B}^X_{r_y}(x_0)$ and, by continuity we have:

$$f(x_*) = \lim_{j\to\infty} f(x_{n_j}) = \lim_{j\to\infty} y_{n_j} = y$$

Thus $y \in f(\overline{B}^X_{r_y}(x_0))$, i.e. $y \in A$. This shows that A is also closed in $\mathcal{L}$.

By the connectedness of $\mathcal{L}$, we must have $A = \mathcal{L}$. Thus $y_0 \in A$, but this means that there is a point $x' \in B^X_r(x_0)$ such that $f(x') = y_0$, because, since $y_0 \in A$, we can pick a point $x' \in f^{-1}(y_0) \cap \overline{B}^X_{r_{y_0}}(x_0)$, and then:

$$\|x' - x_0\|_X \leq \tfrac{1}{c_*}\|f(x)_0) - y_0\|_Y < \tfrac{1}{c_*}(c_* r) = r.$$

Thus $x' \in B^X_r(x_0)$, and this finishes the proof.

We conclude that f is a co-Lipschitz mapping with co-Lipschitz constant less than or equal to c_*. □

From the above results we gather that, for a function $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$, where X and Y are finite-dimensional normed vector spaces, it is enough for f to satisfy the Lipschitz and co-Lipschitz conditions locally —with the same constants for all $x \in X$— in order to achieve the "global" Lipschitz and co-Lipschitz conditions.

Another local property that would be useful to study is the local injectivity.

Definition 1.2.8. A function $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is locally injective at a point $x \in X$ if there exists $\varepsilon > 0$ such that f restricted to $B_\varepsilon(x)$ is an injective function.

Notice that if a Lipschitz quotient mapping $f : X \to Y$ is locally injective at a point x, then f is a local homeomorphism at x, i.e. there is an $\varepsilon > 0$ such that $\tilde{f} : B_\varepsilon(x) \to f(B_\varepsilon(x))$ is a homeomorphism.

It is known that for a continuous discrete open mapping f between n-dimensional topological manifolds the set in which f fails to be a local homeomorphism cannot be very big (it has dimension at most $n-2$).[1] This result was first proved in [5], [6] and a more accessible proof is given in [28]. From [1], we know that Lipschitz quotient mappings from the plane to itself are continuous, discrete and open, hence, from the above result it follows that every Lipschitz quotient mapping from the plane to itself is a local homeomorphism outside a discrete subset of $\mathbb{R}^2$.

[1]Here a mapping is said to be discrete if the inverse image of each point consists of isolated points and the dimension means the topological dimension as in [14].

The next proposition, similar to [20, Lemma 5], shows using the polynomial homeomorphism decomposition for Lipschitz quotient mappings on the plane that Lipschitz quotients are locally injective in all but at most a finite number of points, and thus a local homeomorphism at all but a finite number of points. Furthermore, in Corollary 1.2.12 we will show that this is also true if we ask for the local homeomorphism to be Lipschitz, see Remark 1.2.13.

Later on, in Proposition 2.4 and Corollary 2.5 we will prove stronger versions of these results for a particular type of Lipschitz quotients on the plane.

Proposition 1.2.9. *Let $f : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}^2, \|\cdot\|_2)$ be a Lipschitz quotient mapping. There exists a finite set $F \subseteq \mathbb{R}^2$ such that f is locally injective at x for all $x \in \mathbb{R}^2 \setminus F$.*

Proof. We assume without loss of generality that $\|\cdot\|_1 = \|\cdot\|_2$ is the Euclidean norm. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a Lipschitz quotient mapping, by [15] we know that there is a non-constant polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + b$ with $a_n \neq 0$ of one complex variable and a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$, such that $f = P \circ h$.

If the degree of P is equal to 1, there is nothing to prove. Assume $n > 1$ and let $R(P') = \{z \in \mathbb{R}^2 : P'(z) = 0\}$. Take $x_0 \in \mathbb{R}^2$ such that $h(x_0) \notin R(P')$ and let $y_0 := h(x_0)$. Since $y_0 \notin R(P')$, there exists $r_1 > 0$ such that $B_{r_1}(y_0) \cap R(P') = \emptyset$. Now, $h^{-1}(B_{r_1}(y_0))$ is an open set containing x_0, so there exists $r_2 > 0$ such that $B_{r_2}(x_0) \subseteq h^{-1}(B_{r_1}(y_0))$.

We now show that f is injective on $B_{r_2}(x_0)$. Take $x_1, x_2 \in B_{r_2}(x_0)$ such that $x_1 \neq x_2$ and let $y_i := h(x_i)$. We know that $P(y_2) - P(y_1) = P'(\xi)(y_2 - y_1)$ for some $\xi \in [y_1, y_2]$. Since $y_1, y_2 \in h(B_{r_2}(x_0)) \subseteq B_{r_1}(y_0)$, we have $\xi \in B_{r_1}(y_0) \subseteq \mathbb{R}^2 \setminus R(P')$. Hence:

$$|f(x_2) - f(x_1)| = |P(h(x_2)) - P(h(x_1))| = |P(y_2) - P(y_1)| = |P'(\xi)||y_2 - y_1| > 0.$$

Thus $f(x_2) \neq f(x_1)$ for all $x_1, x_2 \in B_{r_2}(x_0)$, $x_1 \neq x_2$.

Let $F = h^{-1}(R(P'))$, we have just shown that f is locally injective at every point

$x \in \mathbb{R}^2 \setminus F$. Since h is a homeomorphism, we know that $\#F \leq n-1$, thus f is locally injective at all but at most $n-1$ points. $\square$

The next proposition will be very useful to study the behaviour of Lipschitz quotients along boundaries of balls centred at the origin and some useful local properties of Lipschitz quotient mappings, see for instance Lemma 5.1.2, and Corollary 1.2.12. For this we will be using the "lifting of a curve" property of co-Lipschitz mappings, stated in [15]. The following is a restatement of [15, Lemma 2.2] and [1, Lemma 4.5].

Lemma 1.2.10. *Let X be a metric space and suppose that $f : \mathbb{R}^n \to X$ is a continuous c_f-co-Lipschitz mapping with $f(x) = y$. Suppose also that $\xi : [0, \infty) \to X$ is an L_ξ-Lipschitz curve with $\xi(0) = y$ and $L_\xi \leq c_f$. Then there is a curve $\phi : [0, \infty) \to \mathbb{R}^n$ with Lipschitz constant 1 such that $\phi(0) = x$ and $f(\phi(t)) = \xi(t)$ for all $t > 0$.*

We first show that Lipschitz quotient mappings are "well-behaved" with respect to boundaries of balls in the following sense.

Proposition 1.2.11. *Let $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ be a Lipschitz quotient mapping and assume that for some $x_0 \in \mathbb{R}^2$ and some $r > 0$ we have $f(\partial B_r(x_0)) = \varphi$ where φ is a simple closed curve with index one around $f(x_0)$. Then φ is in fact the boundary of $f(B_r(x_0))$, i.e. $\partial(f(B_r(x_0))) = f(\partial B_r(x_0))$.*

Proof. Let us denote by $E(\varphi)$ the exterior region of the curve φ i.e. the unbounded component of $\mathbb{R}^2 \setminus \varphi$. We will assume, without loss of generality that the co-Lipschitz constant of f, c_f, is equal to 1.

Let x_0 and r be as in the hypothesis and take $y_0 \in \partial(f(B_r(x_0)))$. Consider an open neighbourhood, U, of y_0, then we have:

$$U \cap f(B_r(x_0)) \neq \emptyset \neq U \cap (\mathbb{R}^2 \setminus f(B_r(x_0))).$$

This means that there is a $y' \in U$ such that, for some $x' \in B_r(x_0)$, we have $f(x') = y'$, and that there is a $y'' \in U$ such that $y'' \neq f(x)$ for all x in $B_r(x_0)$. Let $\xi : [0,1] \to \mathbb{R}^2$ be a 1-Lipschitz curve contained in U, joining y' and y''. Using Lemma 1.2.10, consider the 1-Lipschitz lifting ϕ of ξ under f with starting point at x', so that $\phi(0) = x'$ and $f(\phi(t)) = \xi(t)$ for all $t \in [0,1]$. Then, $\phi(0) = x' \in B_r(x_0)$ and, since $f(\phi(1)) = \xi(1) = y'' \notin f(B_r(x_0))$, we have $\phi(1) \in \mathbb{R}^2 \setminus B_r(x_0)$. Thus, ϕ goes from inside to outside $B_r(x_0)$, so it must intersect the curve $\partial B_r(x_0)$. Let $\phi(t_0)$ be a point in this intersection, then:

$$f(\phi(t_0)) = \xi(t_0) \in U \;\text{ and } f(\phi(t_0)) \in \; f(\partial B_r(x_0)).$$

Thus $\xi(t_0) \in U \cap f(\partial B_r(x_0))$. We have shown that for all basic neighbourhoods U of y_0 we have $U \cap f(\partial B_r(x_0)) \neq \emptyset$. Since $f(\partial B_r(x_0))$ is a closed set, we conclude that $y_0 \in f(\partial B_r(x_0))$. This shows that $\partial(f(B_r(x_0)) \subseteq f(\partial B_r(x_0))$.

We now prove the other inclusion. Let $y_0 \in f(\partial B_r(x_0)) =: \varphi$. Since f is continuous we know that any neighbourhood of y_0 intersects $f(B_r(x_0))$, so it only remains to show that any neighbourhood of y_0 also intersects $\mathbb{R}^2 \setminus f(B_r(x_0))$. Assume, for a contradiction, that there exists a basic neighbourhood U of y_0 such that $U \subseteq f(B_r(x_0))$. Since $y_0 \in f(\partial B_r(x_0))$, we can pick a point $y' \in E(\varphi) \cap U \subseteq f(B_r(x_0))$ and consider a point $x' \in B_r(x_0)$ such that $f(x') = y'$. Now, y' belongs to the unbounded component of $\mathbb{R}^2 \setminus \varphi$ so we can consider an unbounded 1-Lipschitz curve $\xi : [0,\infty) \to \mathbb{R}^2$ contained in $E(\varphi)$, and its 1-Lipschitz lifting $\phi : [0,\infty) \to \mathbb{R}^2$ with starting point at $x' \in B_r(x_0)$. By the Lipschitz property of f and ϕ, since ξ is unbounded so is ϕ, otherwise, if ϕ were bounded we can find $a \geq 0$, such that $\|\phi(t)\| \leq a$ for all $t \in [0,\infty)$, and we would have $\|f(\phi(t)) - f(0)\| \leq La$ $t \in [0,\infty)$, but $f(\phi(t)) = \xi(t)$ is unbounded. Therefore ϕ is unbounded and hence it must intersect $\partial B_r(x_0)$. Again, let $\phi(t_0)$ be a point belonging to this intersection. Then, $\xi(t_0) = f(\phi(t_0)) \in f(\partial B_r(x_0)) = \varphi$, which is impossible since $\phi \subseteq E(\varphi)$. Thus any neighbourhood of y_0 intersects $\mathbb{R}^2 \setminus f(B_r(x_0))$. We conclude that $y_0 \in \partial(f(B_r(x_0))$ and this finishes the proof. ◻

Corollary 1.2.12. *Let $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ be a Lipschitz quotient mapping. There exists a finite set F with such that for all $x \in \mathbb{R}^2 \setminus F$ there exists $\varepsilon_x > 0$ such that $\partial(f(B_r(x))) = f(\partial B_r(x))$ for all $r \leq \varepsilon_x$.*

Moreover, if c denotes the co-Lipschitz constant of f, then

$$\|f(x_1) - f(x_2)\| \geq c\|x_1 - x_2\|,$$

whenever $x_1, x_2 \in B_\delta(x)$, where $\delta = \frac{1}{4}\varepsilon_x$.

Proof. Let $x \in \mathbb{R}^2 \setminus F$, where F is as in Proposition 1.2.9. We know that F is finite and the same proposition allows us to consider $\varepsilon_x > 0$ such that f in injective in $B_{\varepsilon_x}(x)$. Take $r \in (0, \varepsilon_x)$, then the mapping f is injective along $\partial B_r(x)$, therefore $f(\partial B_r(x))$ is a simple closed curve with index one around $f(x)$. Hence, the first part of the statement follows from Proposition 1.2.11. To prove the second part, notice that if $x_1, x_2 \in B_\delta(x)$ with $\delta = \frac{1}{4}\varepsilon_x$, then $x_2 \in \partial B_r(x_1)$ with $r = \|x_1 - x_2\| < \frac{1}{2}\varepsilon_x$, therefore $\partial B_r(x_1) \subseteq B_{\varepsilon_x}(x)$. Hence f is injective along $\partial B_r(x_1)$, so from the first part of the statement of the present corollary we get:

$$f(x_2) \in f(\partial B_r(x_1)) = \partial(f(B_r(x_1))).$$

By the co-Lipschitz property we know that $f(B_r(x_1)) \supseteq B_{cr}(f(x_1))$, consequently

$$\|f(x_1) - f(x_2)\| \geq cr = c\|x_1 - x_2\|.$$

◻

Remark 1.2.13. Recall that a bi-Lipschitz mapping $f : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a bijective Lipschitz mapping whose inverse is also Lipschitz. In other words, we say that

f is bi-Lipschitz if there exist constants $c, L > 0$ such that for all $x_1, x_2 \in X$ we have

$$c\|x_1 - x_2\|_X \leq \|f(x_1) - f(x_2)\|_Y \leq L\|x_1 - x_2\|_X.$$

Notice that the Lipschitz constant of f^{-1} is equal to $1/c$. We can restate this property as a local property: We say that f is locally bi-Lipschitz at the point $x \in X$ if there exist $r_x > 0$ and constants $c_x, L_x > 0$ such that f is injective on $B_{r_x}(x)$ and

$$c_x\|x - x_1\|_X \leq \|f(x) - f(x_1)\|_Y \leq L_x\|x - x_1\|_X,$$

whenever $\|x - x_1\| < r_x$. From Corollary 1.2.12 it follows that if $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ is a Lipschitz quotient, then f is locally bi-Lipschitz at all but at most a finite number of points. Even more, if f is injective we can easily prove the following statement:

Corollary 1.2.14. *Let $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ be an injective mapping. Then f is an L-Lipschitz and c-co-Lipschitz mapping if and only if f is a bi-Lipschitz mapping and the Lipschitz constants of f and f^{-1} are equal to L and $1/c$, respectively.*

Proof. Assume f is a Lipschitz quotient mapping with Lipschitz and co-Lipschitz constants equal to L and c respectively. Since we are already assuming that f is injective on all of $\mathbb{R}^2$ (and hence bijective, since f is co-Lipschitz), we can follow the same argument used in the proof of Corollary 1.2.12 to show that for all $x_1, x_2 \in \mathbb{R}^2$ we have $\|f(x_1) - f(x_2)\| \geq c\|f(x_1) - f(x_2)\|$. Therefore for all $x_1, x_2 \in \mathbb{R}^2$, we have:

$$c\|x_1 - x_2\| \leq \|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|. \tag{1.5}$$

Taking $y_i = f(x_i)$, this gives $\|f^{-1}(y_1) - f^{-1}(y_2)\| \leq \frac{1}{c}\|y_1 - y_2\|$. Thus f^{-1} is a Lipschitz mapping with Lipschitz constant $L_* \leq 1/c$.

The other implication is clear since, from the bi-Lipschitz condition, it follows that if

$L_*, L > 0$ denote the Lipschitz constants of f and f^{-1}, respectively, then

$$\tfrac{1}{L_*}\|x_1 - x_2\| \leq \|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|.$$

Since f is bijective, for $y \in B_{r/L_*}(f(x_1))$ we can consider $x_2 = f^{-1}(y)$, and we get

$$\tfrac{1}{L_*}\|x_1 - x_2\| \leq \|f(x_1) - f(x_2)\| = \|f(x_1) - y\| < \tfrac{r}{L_*},$$

therefore $\|x_1 - x_2\| < r$ and we conclude that $y = f(x_2)$ and $x_2 \in B_r(x_1)$. Thus $B_{r/L_*}(f(x_1)) \subseteq f(B_r(x_1))$ and hence f is a co-Lipschitz mapping with co-Lipschitz constant $c \geq 1/L_*$.

We conclude that given an injective mapping f, f is a Lipschitz quotient mapping if and only if f is bi-Lipschitz and that the Lispchitz constant of f^{-1} is equal to the inverse of the co-Lipschitz constant of f. □

All these previous results are the main basic background of Lipschitz quotients that we will be using throughout this work. However, when studying Lipschitz quotients on the plane the use of curves turns out to be very useful, as we have already seen in Proposition 1.2.11. Hence, before starting the study of the relation between the Lipschitz and co-Lipschitz constants of a Lipschitz quotient mapping and the cardinality of the fibers of that mapping, we devote the next section to the study of some properties of curves on the plane.

1.3 Basic properties of the length of a curve on the plane

The lifting of curves (see Lemma 1.2.10) to study Lipschitz quotients on the plane is very useful. In fact, in [15], the proof that every Lipschitz quotient mapping on the plane can be written as a composition of a homeomorphism on the plane and a polynomial of one complex variable, is based on the lifting of a curve property mentioned in Lemma 1.2.10. So we will amply consider curves and their images under Lipschitz and co-Lipschitz mappings on the plane. Since we are concerned with the ratio between the Lipschitz and co-Lipschitz constants we will need to measure the length of a curve and compare it with the length of its image under a Lipschitz quotient. Hence, we will need to be able to measure the length of a curve under non-Euclidean norms. In this section we will first define the length of a curve using the Hausdorff measure and we will prove basic properties of this length, including the very basic: "the straight line is a shortest path between two points". This is stated and proved in Lemma 1.3.10 and Corollary 1.3.11.

To avoid any confusion, let us first clarify what we mean by a "curve".

Definition 1.3.1. Given a normed vector space X, we say that a set $\Phi \subseteq X$ is a *curve* if it is an image of a continuous function $\varphi : [a, b] \subseteq \mathbb{R} \to X$, where $a < b$ are real numbers. In this case we say that φ is a *parametrization* of Φ.

For instance, given a set of n points, $p_0, \ldots, p_n$ of X, we can consider the set Φ defined as the union of all the line segments $[p_i, p_{i-1}]$, $1 \leq i \leq n$. Clearly Φ is a curve. In this case we say that Φ is a *polygonal curve* and that the points p_i are the *vertices* of Φ.

Given a curve Φ, if there exists a parametrization $\varphi : [a, b] \to X$ of Φ which is injective on (a, b), then we say that Φ is a *simple curve* and φ is an *injective parametrization.* Finally, we say that Φ is a *simple closed curve* if it is a simple curve with $\varphi(a) = \varphi(b)$.

It is easy to see that a curve can always be parametrized by a continuous function whose domain is the interval $[0, 1]$.

Notice that, since the continuous image of a compact set is compact, a curve is always a compact set.

When the parametrization φ of a curve Φ is fixed we may use the same notation φ to refer to both, the parametrization and the set Φ.

As we shall see in Lemma 1.3.3, the boundary of a ball in any normed space $(\mathbb{R}^2, \|\cdot\|)$ is a curve. It is worth mentioning the well known correspondence between finite-dimensional Banach spaces and the symmetric convex bodies in $\mathbb{R}^n$, this is: *If $E \subseteq \mathbb{R}^n$ is a symmetric convex body centred at the origin, then E is a closed unit ball of some Banach space $(\mathbb{R}^n, \|\cdot\|)$. Conversely, the closed unit ball of any Banach space $(\mathbb{R}^n, \|\cdot\|)$ is a symmetric convex body in $\mathbb{R}^n$.* Recall that a convex body is defined in the following way.

Definition 1.3.2. Given any normed vector space X a set $E \subseteq X$ is a *convex body* if it is a compact, convex set with nonempty interior. Given a convex body $E \subseteq X$, we will say that a set $\Phi \subseteq X$ is *contained outside* E if $\Phi \cap \operatorname{Int}(E) = \emptyset$.

Recall that by Jordan's Theorem any simple closed curve in $\mathbb{R}^2$ divides the plane into two connected regions, one bounded and the other unbounded. We say that a set $P \subseteq X$ is a *polygon* if P is the bounded component of a simple closed polygonal curve, according to Definition 1.3.1.

Now we can prove the following statement.

Lemma 1.3.3. *Given any convex body $E \subseteq \mathbb{R}^2$ there is an injective continuous parametrization of the boundary of E. In other words, ∂E is a simple closed curve according to Definition 1.3.1.*

Proof. Let E be a convex body in a normed space $(\mathbb{R}^2, \|\cdot\|)$. Let us assume that the origin is an interior point of E. Notice that for every $\theta \in (-\pi, \pi]$ there is a unique point x_θ that belongs to ∂E and has argument equal to θ. More precisely, let ℓ_θ be the ray with starting point at the origin that forms an angle θ with the positive x-axis. Since ℓ_θ is not bounded, it must intersect ∂E at some point x_θ. Indeed, if $\ell : \mathbb{R}^+ \cup \{0\} \to \ell_\theta$ is given by $\ell(t) = mt$ then the set

$$A = \{t \in \mathbb{R}^+ \cup \{0\} : \ell(t) \in \operatorname{Int} E\}$$

is not empty because the origin is an interior point of E, furthermore since E is bounded, the set A is bounded as well, therefore we can consider $\beta = \sup A$. It is easy to see that

$\ell(\beta)$ belongs to ∂E. Hence $\ell(\beta) \in \ell_\theta \cap \partial E$.

Finally, we can show that this point is unique, for if $x' \in \ell_\theta \cap \partial E$, then x_θ and x' have the same argument. Without loss of generality assume that $\|x'\| < \|x_\theta\|$ and let θ_1, θ_2 be such that $\theta \in (\theta_1, \theta_2)$. Now, for $i = 1, 2$ take $x_{\theta_i} \in \ell_{\theta_i} \cap E$ such that $x_{\theta_i} \neq 0$. So that $x_\theta, x_{\theta_1}, x_{\theta_2}$ and the origin belong to E, therefore the quadrilateral Q whose vertices are these four points is contained in E. Since x' is an interior point of Q, it is also an interior point of E and this is a contradiction. Therefore $x' = x_\theta$.

Given that uniqueness, we can define the function $\varphi : [-\pi, \pi] \to \partial E$ as $\varphi(\theta) = x_\theta$ for $\theta \in (-\pi, \pi]$ and $\varphi(-\pi) = x_\pi$.

Clearly φ is onto, $\varphi(-\pi) = \varphi(\pi)$ and φ is injective along $(-\pi, \pi)$; so we only need to show that φ is continuous.

Fix some $\theta \in [-\pi, \pi]$ and consider a sequence t_n in $[-\pi, \pi]$ such that $t_n \to \theta$. Consider the sequence $\varphi(t_n)$ and any convergent subsequence $\varphi(t_{n_m})$ of $\varphi(t_n)$. Since ∂E is compact, the subsequence $\varphi(t_{n_m})$ converges to some point $x_0 \in \partial E$.

For $z \in \mathbb{R}^2$ let $\arg(z)$ denote the argument of z taking values in $(-\pi, \pi]$. Now, using that $\arg(\varphi(t_{n_m})) = t_{n_m}$ we get:

$$\arg(x_0) = \arg(\lim_{m\to\infty} \varphi(t_{n_m})) = \lim_{m\to\infty} \arg(\varphi(t_{n_m})) = \lim_{m\to\infty} t_{n_m} = \theta.$$

This implies that $x_0 = \varphi(\theta)$.

Therefore, every convergent subsequence of $\varphi(t_n)$ converges to $\varphi(\theta)$. Hence, $\varphi(t_n)$ converges to $\varphi(\theta)$.

We conclude that φ is continuous so ∂E is a curve. □

Remark 1.3.4. Note that from the proof of Lemma 1.3.3 it follows that the parametrization may be chosen in such a way that as the parameter increases, the point on the boundary "travels" in a counterclockwise direction with respect to the point in the interior of

E. We will return to this later, in Corollary 1.3.18.

Now that the concept of curve is clear, we define the *length* of a curve. First, let us recall the definition of the n-dimensional Hausdorff measure.

Definition 1.3.5. For a subset $A \subseteq \mathbb{R}^k$, we will use the notation $\mathcal{H}_n^{\|\cdot\|}(A)$ for the *n-dimensional Hausdorff measure of the set A*, under the norm $\|\cdot\|$, defined in the following way:

Given a subset $A \subseteq \mathbb{R}^k$, define for each fixed $\delta > 0$:

$$\mathcal{H}_{n,\delta}^{\|\cdot\|}(A) = \inf\left\{\sum_{j=1}^{\infty}(\operatorname{diam} C_j)^n : A \subseteq \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j \leq \delta\right\}, \tag{1.6}$$

where $\operatorname{diam} C_j$ is the diameter of the set C_j with respect to the norm $\|\cdot\|$.

It is clear that $\mathcal{H}_{n,\delta}^{\|\cdot\|}(A) \leq \mathcal{H}_{n,\varepsilon}^{\|\cdot\|}(A)$, whenever $\varepsilon \leq \delta$. We define the n-dimensional Hausdorff measure as

$$\mathcal{H}_n^{\|\cdot\|}(A) = \sup_{\delta>0} \mathcal{H}_{n,\delta}^{\|\cdot\|}(A) = \lim_{\delta\to 0} \mathcal{H}_{n,\delta}^{\|\cdot\|}(A).$$

Whenever the norm we are using is clear we may only write $\mathcal{H}_n(A)$.

Remark 1.3.6. Equivalently, it is not hard to prove that in (1.6) we might ask for the C_j's to be all open, or all closed. For a proof of this result see [29, Theorem 27.13]. For other general properties of the Hausdorff measures and dimensions see [18] and [2].

Given a curve $\Phi \subseteq \mathbb{R}^k$ and a parametrization $\varphi : [a,b] \to \Phi$ of Φ the 1-dimensional Hausdorff measure of Φ, relative to the norm $\|\cdot\|$ can be written in terms of the parametrization as:

$$\mathcal{H}_1^{\|\cdot\|}(\Phi) = \mathcal{H}_1^{\|\cdot\|}(\{\varphi(t) : t \in [a,b]\}).$$

Notice that this measure does not depend on the parametrization φ of Φ. This is the way we define the length of a curve when we think about it as a subset of $\mathbb{R}^k$. However, in this work we will also be concerned with the length of curves understood as parametrizations rather than as sets contained in $\mathbb{R}^k$. We will define the length of a locally injective curve —as a parametrization— using the 1-dimensional Hausdorff measure in the following way.

Definition 1.3.7. Let γ be a locally injective curve $\gamma : [a,b] \to \mathbb{R}^k$, and consider the points $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ in $[a,b]$ such that γ is injective along the interval (t_i, t_{i+1}), then the length of γ is defined as:

$$\operatorname{length}_{\|\cdot\|} \gamma = \sum_{i=1}^{n} \mathcal{H}_1^{\|\cdot\|}(\gamma_i), \tag{1.7}$$

where $\gamma_i = \gamma|_{[t_{i-1},t_i]}$. Again, whenever the norm we are working with is clear we may just write length(γ).

Remark 1.3.8. Notice that once the parametrization γ is fixed the length of the curve γ that we have just defined does not depend on the partition $\{a = t_0, \ldots, t_n = b\}$ in which γ is injective along each interval (t_i, t_{i+1}). Indeed, let $Q = \{a = t_0, \ldots, t_n = b\}$ be any such partition of $[a,b]$ and let $Q^* = \{a = \lambda_0, \ldots, \lambda_m = b\}$ be any refinement of Q. For $j \in \{0, \ldots, n\}$ let k_j be such that $\lambda_{k_j} = t_j$. We show that for all $j \in \{1, \ldots, n\}$ we have

$$\sum_{i=1}^{k_j} \mathcal{H}_1(\gamma|_{[\lambda_{i-1},\lambda_i]}) = \sum_{i=1}^{j} \mathcal{H}_1(\gamma|_{[t_{i-1},t_i]}). \tag{1.8}$$

We do this by induction over j. For $j = 1$ there are two options; if $k_1 = 1$ then there is nothing to do. If $k_1 > 1$ then since γ is injective along

$$[t_0, t_1) = \bigcup_{i=i}^{k_1} [\lambda_{i-1}, \lambda_i)$$

we have

$$\gamma([\lambda_{i-1}, \lambda_i)) \cap \gamma([\lambda_{j-1}, \lambda_j)) = \emptyset \ \text{ for all } \ i \neq j \ ; \ 1 \leq i, j \leq k_1.$$

Therefore,

$$\mathcal{H}_1(\gamma|_{[a,t_1]}) = \sum_{i=1}^{k_1} \mathcal{H}_1(\gamma|_{[\lambda_{i-1},\lambda_i]}).$$

Assume that (1.8) is true for some $j < n$. Again, for $j+1$ we have two options, if $k_{j+1} = k_j + 1$ there is nothing to do. If $k_{j+1} > k_j + 1$ then we can repeat the reasoning that we have just done to get:

$$\mathcal{H}_1(\gamma|_{[t_j,t_{j+1}]}) = \sum_{i=k_j}^{k_{j+1}} \mathcal{H}_1(\gamma|_{[\lambda_{i-1},\lambda_i]}).$$

This, together with the fact that (1.8) is true for j, implies that (1.8) is true for $j+1$. Thus the statement is true for all $j \in \{1, \dots, n\}$. In particular, for $j = n$ this is:

$$\sum_{i=1}^{m} \mathcal{H}_1(\gamma|_{[\lambda_{i-1},\lambda_i]}) = \sum_{i=1}^{n} \mathcal{H}_1(\gamma|_{[t_{i-1},t_i]}).$$

Finally, if $Q = \{a = t_0, \dots, t_n = b\}$ and $Q' = \{a = t'_0, \dots, t'_m = b\}$ are two partitions of $[a, b]$ such that γ is injective along each of the intervals (t_{i-1}, t_i), (t'_{i-1}, t'_i), then the partition $Q^* = Q \cup Q' = \{a = \lambda_0, \dots, \lambda_k = b\}$ is a refinement of Q and Q', hence we have:

$$\sum_{i=1}^{n} \mathcal{H}_1(\gamma|_{[t_{i-1},t_i]}) = \sum_{i=1}^{k} \mathcal{H}_1(\gamma|_{[\lambda_{i-1},\lambda_i]}) = \sum_{i=1}^{m} \mathcal{H}_1(\gamma|_{[t'_{i-1},t'_i]}).$$

This is what we wanted to show.

Notice also that whenever we consider injective parametrizations γ_1 and γ_2 of a curve $\Gamma \subseteq \mathbb{R}^2$ then

$$\mathcal{H}_1^{\|\cdot\|}(\Gamma) = \mathrm{length}_{\|\cdot\|}(\gamma_1) = \mathrm{length}_{\|\cdot\|}(\gamma_2).$$

Remark 1.3.9. If $f : (\mathbb{R}^m, \|\cdot\|_1) \to (\mathbb{R}^k, \|\cdot\|_2)$ is an L-Lipschitz mapping, then it is clear that for $E \subseteq \mathbb{R}^m$ we have $\mathrm{diam}_{\|\cdot\|_2}(f(E)) \leq L \,\mathrm{diam}_{\|\cdot\|_1}(E)$. Therefore, in view of the definition of the Hausdorff measure, we get:

$$\mathcal{H}_n^{\|\cdot\|_2}(f(E)) \leq L^n \mathcal{H}_n^{\|\cdot\|_1}(E) \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.9}$$

It follows that, if $\gamma : [a, b] \to \mathbb{R}^m$ is a locally injective L'-Lipschitz curve, and $f : \mathbb{R}^m \to \mathbb{R}^k$ is a locally injective L- Lipschitz mapping, then the length of the curve $f \circ \gamma$ is defined and $\text{length}_{\|\cdot\|_2} f \circ \gamma \leq (LL')(b - a)$. Moreover, in the particular case of the plane, say $f : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}^2, \|\cdot\|_2)$, from Proposition 1.2.9 we know that if f is a Lipschitz quotient mapping there exists a finite set F such that f is locally injective at every point $x \in \mathbb{R}^2 \setminus F$. Hence, if we choose $R_I > \sup\{\|x\|_1 : x \in F\}$, then for any curve γ contained outside $B_{R_I}(0)$ the length of the curve $f \circ \gamma$ is defined. Furthermore,

$$\text{length}_{\|\cdot\|_2}(f \circ \gamma) \leq L \, \text{length}_{\|\cdot\|_1}(\gamma). \tag{1.10}$$

Now that we have defined the length of a curve and we have understood how to measure it, we will state and prove some basic properties of curve length. Some of these properties sound quite obvious and intuitive. But let us just bear in mind that when working with non-Euclidean norms things are not that intuitive.

For example in Figure 1.2 we show a square and an octagon contained outside the interior of the square. Let us denote by γ the curve that describes the boundary of the octagon. If we agree that the side of the square is equal to 2, then the square is the unit ball under the supremum norm, $\|\cdot\|_\infty$, and we have: $\text{length}_\infty(\partial B_1^\infty) = 8$. On the other hand, since all the sides of the octagon are just a translation of some radii of the square ∂B_1^∞ we also have $\text{length}_\infty(\gamma) = 8$.

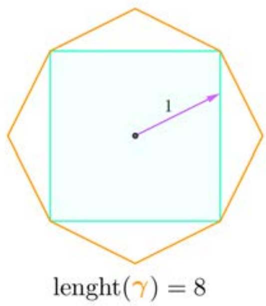


Figure 1.2

We start this survey of the basic properties of the length of a curve by showing that one of the shortest paths between two points is the straight line, which follows from the next Lemma 1.3.10 and Corollary 1.3.11.

Lemma 1.3.10. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^m$. Given any two points $x_0, y_0 \in \mathbb{R}^m$ the length of any locally injective curve joining these two points is at least $\|x_0 - y_0\|$.*

Proof. Let $\gamma : [0,1] \to \mathbb{R}^m$ be locally injective curve joining x_0 and y_0. Without loss of generality assume γ is injective. Consider any countable open cover $\mathcal{C}$ of γ, say

$$\gamma \subseteq \bigcup_{i=1}^{\infty} B_i.$$

Let $\mathcal{C}' = \{B_1, \ldots, B_n\}$ be a finite subcover of $\mathcal{C}$. We will pick some elements of $\mathcal{C}'$ and reorder them in the following way. Pick some $B_{i_1} \in \mathcal{C}'$ such that $x_0 \in B_{i_1}$. Let $t_1 := \sup\{t \in [0,1] : \gamma(t) \in B_{i_1}\}$. Clearly $t_1 > 0$. If $t_1 = 1$ then we set $\mathcal{C}^* = \{B_{i_1}\}$. If not, then since $\mathcal{C}'$ covers γ and $\gamma(t_1) \notin B_{i_1}$ there must be some $B_{i_2} \in \mathcal{C}' \setminus \{B_{i_1}\}$ such that $\gamma(t_1) \in B_{i_2}$. Now define $t_2 := \sup\{t \in [0,1] : \gamma(t) \in B_{i_2}\}$, so $t_1 < t_2$. If $t_2 = 1$ we define $\mathcal{C}^*$ as $\mathcal{C}^* = \{B_{i_1}, B_{i_2}\}$; if $t_2 < 1$ then, again since $\gamma(t_2) \notin (B_{i_1} \cup B_{i_2})$ we can find $B_{i_3} \in \mathcal{C}' \setminus \{B_{i_1}, B_{i_2}\}$ such that $\gamma(t_2) \in B_{i_3}$.

We continue this process till $t_k = 1$ for some k. This will certainly happen since $\gamma(1) = y_0$ belongs to some element of $\mathcal{C}'$ and $\mathcal{C}'$ has a finite number of elements.

So we have defined a subset $\mathcal{C}^* = \{B_{i_1}, \ldots, B_{i_k}\}$ of $\mathcal{C}' \subseteq \mathcal{C}$ (and $\mathcal{C}^*$ is not necessarily a cover of γ) and a sequence of points $0 = t_0 < t_1 < \ldots t_{k-1} < t_k = 1$ such that for all $j \in \{1, \ldots, k-1\}$:

$$\gamma(t_j) \in B_{i_{j+1}} \text{ and } \gamma(t_j) \notin \bigcup_{l=1}^{j} B_{i_l}.$$

This implies that the sets B_{i_j}, $j \in \{1, \ldots, k\}$ are all distinct. On the other hand, given $j \in \{1, \ldots, k\}$ we have $\gamma(t_{j-1}) \in B_{i_j}$ and $\gamma(t_j) \in \overline{B}_{i_j}$ therefore, $\operatorname{diam}(B_{i_j}) \geq \|\gamma(t_{j-1}) - \gamma(t_j)\|$

for all $j \in \{1, \dots, k\}$, and using the triangle inequality we have:

$$\sum_{i=1}^{\infty} \operatorname{diam}(B_i) \geq \sum_{B \in \mathcal{C}^*} \operatorname{diam}(B) \geq \sum_{i=1}^{k} \|\gamma(t_{i-1}) - \gamma(t_i)\| \geq \|\sum_{i=1}^{k} \gamma(t_{i-1}) - \gamma(t_i)\| = \|x_0 - y_0\|.$$

This shows that given any open cover of the curve γ the sum of the diameters of the elements of the cover is at least $\|x_0 - y_0\|$. Therefore, recalling Remark 1.3.6, we conclude that $\mathcal{H}_1(\gamma) \geq \|x_0 - y_0\|$. □

Corollary 1.3.11. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^m$. Given any two points $x_0, y_0 \in \mathbb{R}^m$ the $\|\cdot\|$-length of the straight line segment joining them is equal to $\|x_0 - y_0\|$.*

Proof. Let $x_0, y_0 \in \mathbb{R}^m$ and consider the line segment $\mathcal{L}$ joining them. Clearly, the parametrization of this line segment is injective whenever $x_0 \neq y_0$. We may assume for simplicity that $x_0 = 0$. From Lemma 1.3.10 we have $\mathcal{H}_1(\mathcal{L}) \geq \|y_0\|$.

Now, to get the opposite inequality it is enough to show that given any $\delta > 0$ there is some countable cover $\mathcal{C}$ of $\mathcal{L}$ such that for all $C \in \mathcal{C}$ we have $\operatorname{diam} C \leq \delta$ and $\sum_{C \in \mathcal{C}} \operatorname{diam}(C) \leq \|y_0\|$.

Take $n \in \mathbb{N}$ such that $\|y_0\|/n \leq \delta$ and for $i \in \{0, \dots, n-1\}$ let $z_i := \frac{2i+1}{2n} y_0$. Then, letting $r_\delta := \|y_0\|/2n$, the family $\{\overline{B}_{r_\delta}(z_{i-1}) : 0 \leq i \leq n-1\}$ is clearly a cover of $\mathcal{L}$ and

$$\sum_{i=0}^{n-1} \operatorname{diam}(\overline{B}_{r_\delta}(z_i)) = n(2r_\delta) = \|y_0\|.$$

Hence, for all $\delta > 0$ we have:

$$\inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(C_j) : \mathcal{L} \subseteq \cup_{i=1}^{\infty} C_j,\ \operatorname{diam}(C_j) \leq \delta\right\} \leq \sum_{i=0}^{n-1} \operatorname{diam}(\overline{B}_{r_\delta}(z_i)) = \|y_0\|.$$

Therefore $\mathcal{H}_1(\mathcal{L}) \leq \|y_0\|$ and we conclude that $\mathcal{H}_1(\mathcal{L}) = \|y_0\|$. □

Proposition 1.3.12. *Let φ be a locally injective curve on $\mathbb{R}^2$. Then the length of φ can be approximated by the lengths of polygonal curves. More precisely, there exists a family*

Ψ *of polygonal curves with all their vertices in* φ *such that:*

$$\text{length}(\varphi) = \sup\{\text{length}(\psi) : \psi \in \Psi\}.$$

Proof. Let $\varphi : [a,b] \to \mathbb{R}^2$ be a continuous parametrization of the given curve.

For every partition $Q = \{a = t_0, t_1, \ldots, t_n = b\}$ of the interval $[a,b]$ define the curve ψ_Q as the polygonal curve whose vertices are the points $\varphi(t_i)$, $i \in \{1, \ldots, n\}$ and let

$$\Psi := \{\psi_Q : Q \text{ is a partition of } [a,b]\}, \tag{1.11}$$
$$\beta := \sup\{\text{length}(\psi_Q)) : \psi_Q \in \Psi\}.$$

We show that $\beta = \text{length}(\varphi)$. From Lemma 1.3.10, it follows easily that $\text{length}(\varphi) \geq \text{length}(\psi_Q)$ for each partition Q. Therefore

$$\text{length}(\varphi) \geq \beta. \tag{1.12}$$

So we are left to prove the opposite inequality. Let us assume first that φ is injective, so that $\text{length}(\varphi) = \mathcal{H}_1(\varphi([a,b]))$. Notice that given any partition $Q = \{a = t_0, t_1, \ldots, t_n = b\}$ of $[a,b]$, if we denote by C_i the set $\varphi([t_{i-1}, t_i])$, then the family $\mathcal{C}_Q := \{C_i : 1 \leq i \leq n, t_i \in Q\}$ is a cover of φ. Furthermore, any cover constructed in this way must satisfy $\sum_{i=1}^n \text{diam}(C_i) \leq \beta$. Otherwise, if $\sum_{i=1}^n \text{diam}(C_i) > \beta$, we can define, as we will see now, a partition Q' of $[a,b]$ such that $\text{length}(\psi_{Q'}) > \beta$, which contradicts β being the supremum. For this, we pick $\delta > 0$ such that $\sum_{i=1}^n \text{diam}(C_i) > \beta + \delta$. Now for each $i \in \{1, \ldots, n\}$ choose $t_i', t_i'' \in (t_{i-1}, t_i)$ such that $t_i' < t_i''$ and $\|\varphi(t_i') - \varphi(t_i'')\| > \text{diam}(C_i) - \delta/n$. Let

$$Q' = \{a = t_0, t_1', t_1'', \ldots, t_{n-1}, t_{n-1}', t_{n-1}'', t_n = b\}$$

and consider the polygonal curve $\psi_{Q'}$ generated by Q', i.e. $\psi_{Q'}$ is the polygonal curve whose vertices are the points $\{\varphi(t_{i-1}), \varphi(t_i'), \varphi(t_i''), \varphi(t_i) : i = 1, \ldots, n\}$. Hence,

$$\begin{aligned}\beta \geq \text{length}(\psi_{Q'}) &= \sum_{i=1}^{n} \|\varphi(t_{i-1}) - \varphi(t_i')\| + \|\varphi(t_i') - \varphi(t_i'')\| + \|\varphi(t_i'') - \varphi(t_i)\| \\ &\geq \sum_{i=1}^{n} \|\varphi(t_i') - \varphi(t_i'')\| > \sum_{i=1}^{n} (\text{diam}(C_i) - \delta/n) > \beta + \delta - \delta = \beta.\end{aligned}$$

This is a contradiction. We conclude that for each cover $\mathcal{C}_Q$ of φ generated by a partition Q of $[a, b]$ we have:

$$\sum_{C \in \mathcal{C}_Q} \text{diam}(C) \leq \beta. \tag{1.13}$$

In fact, it is clear that given any $\delta > 0$ we can find a partition $Q = \{a = t_0, \ldots, t_n = b\}$ of $[a, b]$ such that the relevant cover $\mathcal{C}_Q = \{\varphi([t_{i-1}, t_i]) : t_i \in Q\}$ satisfies $\text{diam}(C) \leq \delta$ for all $C \in \mathcal{C}_Q$. Indeed, this follows easily from the uniform continuity of φ since we can find $\varepsilon > 0$ such that for all $t, t' \in [a, b]$ we have $\|\varphi(t) - \varphi(t')\| < \delta$ whenever $|t - t'| < \varepsilon$. Now we can take $N \in \mathbb{N}$ such that $(b - a)/N < \varepsilon$ and consider the partition $Q_\delta = \{a = t_0, \ldots, t_N = b\}$ of $[a, b]$, where $t_0 = a$ and $t_i = t_{i-1} + (b - a)/N$ for all $i = \{1, \ldots, N\}$. In this way it is clear that that the cover $\mathcal{C}_{Q_\delta} := \{\varphi([t_{i-1}, t_i]) : t_i \in Q\}$ satisfies $\text{diam}(C) \leq \delta$ for all $C \in \mathcal{C}_{Q_\delta}$.

Hence, from (1.13) it follows that for any $\delta > 0$:

$$\mathcal{H}_{1,\delta}(\varphi) := \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(C_i) : \varphi \subseteq \cup_{i=1}^{\infty} C_i, \text{diam}(C_i) < \delta \right\} \leq \sum_{C \in \mathcal{C}_{Q_\delta}} \text{diam}(C) \leq \beta.$$

Therefore,

$$\text{length}(\varphi) = \mathcal{H}_1(\varphi([a, b])) = \sup_{\delta > 0} \mathcal{H}_1^\delta(\varphi) \leq \beta.$$

Now, if φ is not injective, let $\varepsilon > 0$ and consider the points $a = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n = b$ such that φ is injective along $[\lambda_{i-1}, \lambda_i]$ and for $i \in \{1, \ldots n\}$ let $\varphi_i := \varphi|_{[\lambda_{i-1}, \lambda_i]}$. Now, for each φ_i let Q_i be a partition of $[\lambda_{i-1}, \lambda_i]$ such that the corresponding polygonal curve ψ_i satisfies $\mathcal{H}_1(\varphi_i) < \mathrm{length}(\psi_i) + \varepsilon/n$. Let $Q = \cup_{i=1}^n Q_i$ and let ψ be the polygonal curve relative to the partition Q. In this way, we have:

$$\mathrm{length}(\varphi) = \sum_{i=1}^n \mathcal{H}_1(\varphi_i) \leq \sum_{i=1}^n \mathrm{length}(\psi_i) + \varepsilon = \mathrm{length}(\psi) + \varepsilon.$$

Therefore, in this case we also have that for every ε there is a partition $Q = \{t_i : 0 \leq i \leq n\}$ of $[a, b]$ such that the polygonal curve whose vertices are the points $\varphi(t_i)$ satisfies $\mathrm{length}(\varphi) < \mathrm{length}(\psi) + \varepsilon$. Thus, also in this case we have

$$\mathrm{length}(\varphi) \leq \beta.$$

Together with (1.12) this gives $\mathrm{length}(\varphi) = \beta$. □

Lemma 1.3.13. *Let $(\mathbb{R}^2, \|\cdot\|)$ be a normed vector space. If γ is a locally injective closed curve contained outside a convex simple polygon $\mathcal{P}$ and $\mathrm{Ind}_p\, \gamma = 1$, with $p \in \mathrm{Int}\, \mathcal{P}$ then $\mathrm{length}(\gamma)$ is at least $\mathrm{length}(\partial\mathcal{P})$.*

Proof. Let γ be as in the hypothesis and denote by $A_0, \ldots, A_n$ the vertices of the simple convex polygon $\mathcal{P}$, placing the indices counterclockwise. Let $\mathcal{L}_1$ be the straight line that passes through A_0 and A_1. See Figure 1.3 for an illustration.

By Jordan's Theorem we can take $P_0, P_1 \in \gamma$ such that:

$$P_0 \in \gamma \cap \{\lambda \overrightarrow{A_1 A_0} : \lambda \geq 0\}, \tag{1.14}$$
$$P_1 \in \gamma \cap \{\lambda \overrightarrow{A_0 A_1} : \lambda \geq 0\};$$

Let us agree that the curve γ is parametrized by a continuous function $\gamma_0 : [0, 1] \to \mathbb{R}^2$ such that $\gamma_0(0) = P_0$ and goes along the curve counterclockwise. Pick a point $t_1 \in \gamma_0^{-1}(P_1)$

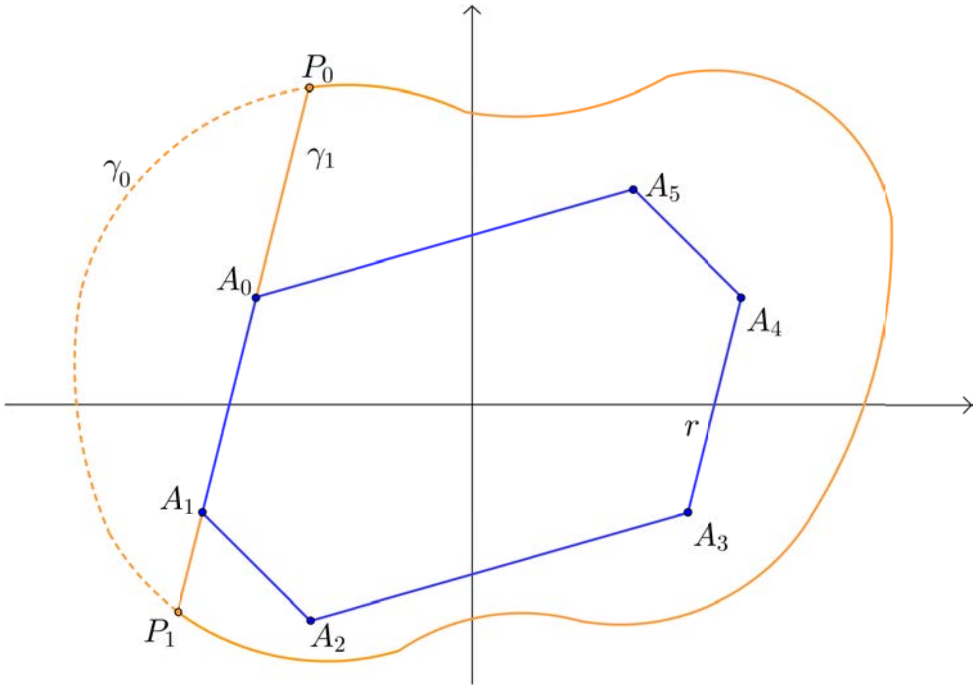


Figure 1.3

and define the curve γ_1 as

$$\gamma_1(t) = \begin{cases} (1 - \frac{t}{t_1})P_0 + \frac{t}{t_1}(P_1) & \text{if } \ 0 \leq t \leq t_1 \\ \gamma_0(t) & \text{if } \ t_1 \leq t \leq 1 \end{cases}$$

Denote by φ and φ' the restrictions of γ to the intervals $[0, t_1]$ and $[t_1, 1]$ respectively. It follows from Lemma 1.3.10 and Corollary 1.3.11 that:

$$\begin{aligned} \text{length}_{\|\cdot\|}(\gamma) &= \text{length}_{\|\cdot\|}(\varphi) + \text{length}_{\|\cdot\|}(\varphi') \geq \|P_0 - P_1\| + \text{length}_{\|\cdot\|}(\varphi') \\ &= \text{length}_{\|\cdot\|}([P_0, P_1]) + \text{length}_{\|\cdot\|}(\varphi') = \text{length}_{\|\cdot\|}(\gamma_1). \end{aligned}$$

Now, since γ_1 is a curve contained outside $\mathcal{P}$ we can repeat the same construction as before taking γ_1 instead of γ and A_1, A_2 instead of A_0, A_1. So now, in a similar way as we did in (1.14), we can consider two intersection points of the curve γ_1 with the line $\mathcal{L}_2$

that passes through A_1 and A_2, say

$$P_2 \in \gamma_1 \cap \{\lambda \overrightarrow{A_1 A_2} : \lambda \geq 0\},$$
$$P_2' \in \gamma_1 \cap \{\lambda \overrightarrow{A_2 A_1} : \lambda \geq 0\}.$$

Notice that in this case, we can choose P_2' to be precisely A_1. Take $t_2', t_2 \in (0,1)$ such that $t_2' \in \gamma_1^{-1}(A_1) = \gamma_1^{-1}(P_2')$ and $t_2 \in \gamma_1^{-1}(P_2)$ and define the curve:

$$\gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } \ 0 \leq t \leq t_2' \\ \left(1 - \frac{t-t_2'}{t_2-t_2'}\right) A_1 + \left(\frac{t-t_2'}{t_2-t_2'}\right) P_2 & \text{if } \ t_2' \leq t \leq t_2 \\ \gamma_1(t) & \text{if } \ t_2 \leq t \leq 1 \end{cases}$$

After repeating this process n times we will end up with a sequence, $\{\gamma, \gamma_1, \ldots, \gamma_n\}$, of curves contained outside $\mathcal{P}$, a sequence of points of the curve γ, $\{P_0, \ldots, P_n\} \subseteq \gamma$, and inverse image points $\{t_i \in \gamma_{i-1}^{-1}(P_i) : i = 1, \ldots, n\}$ and $\{t_i' \in \gamma_{i-1}^{-1}(A_{i-1}) : i = 1, \ldots, n\}$ such that for $i \in \{1, \ldots, n-1\}$ the curve γ_{i+1} is defined as:

$$\gamma_{i+1}(t) = \begin{cases} \gamma_i(t) & \text{if } 0 \leq t \leq t_{i+1}' \\ \left(1 - \frac{t-t_i'}{t_i-t_i'}\right) A_i + \left(\frac{t-t_i'}{t_i-t_i'}\right) P_i & \text{if } t_i' \leq t \leq t_i \\ \gamma_i(t) & \text{if } t_i \leq t \leq 1 \end{cases}$$

Again, by Lemma 1.3.10 and Corollary 1.3.11 it follows that for each $i \in \{1, \ldots, n\}$ we have:

$$\text{length}_{\|\cdot\|}(\gamma) \geq \text{length}_{\|\cdot\|}(\gamma_i) \geq \text{length}_{\|\cdot\|}(\gamma_{i+1}).$$

In particular for $i = n$, the curve γ_n is the curve defined as follows (writing down the definition of all the γ_i's that have been defined recursively):

$$\gamma_n(t) = \begin{cases} (1-\frac{t}{t_1})P_0 + \frac{t}{t_1}A_1 & \text{if } 0 \le t \le t_1 \\ \left(1-\frac{t-t_2'}{t_2-t_2'}\right)A_1 + \left(\frac{t-t_2'}{t_2-t_2'}\right)A_2 & \text{if } t_2' \le t \le t_2 \\ \left(1-\frac{t-t_2'}{t_2-t_2'}\right)A_2 + \left(\frac{t-t_2'}{t_2-t_2'}\right)A_3 & \text{if } t_2' \le t \le t_2 \\ \vdots & \vdots \\ \left(1-\frac{t-t_n'}{t_n-t_n'}\right)A_n + \left(\frac{t-t_n'}{t_n-t_n'}\right)P_n & \text{if } t_n' \le t \le t_n \\ \gamma(t) & \text{if } t_n \le t \le 1 \end{cases}$$

Repeating this construction once more we define the curve γ_{n+1}. In this case the line $\mathcal{L}_{n+1}$ that passes through A_n and A_0 intersects the curve γ_n at A_0 and A_n, since $A_0 = \gamma_n(t)$ for some $t \in [0, t_1]$ and $A_n = \gamma_n(t_n')$. Therefore $\gamma_{n+1} = \partial\mathcal{P}$.

Once again, from Lemma 1.3.10 and Corollary 1.3.11 it follows that $\text{length}_{\|\cdot\|}(\gamma_n) \ge \text{length}_{\|\cdot\|}(\gamma_{n+1}) = \mathcal{H}_1(\partial E)$ Hence:

$$\text{length}(\gamma) \ge \text{length}(\gamma_{n+1}) = \text{length}(\partial\mathcal{P}) = \sum_{j=1}^{n} \|A_{j-1} - A_j\| + \|A_n - A_0\|.$$

Thus the length of the curve γ is greater than or equal to the perimeter of the convex polygon $\mathcal{P}$. □

The next Corollary is a generalisation of [20, Lemma 3] in the case of *polygonal norms* (see Definition 4.1). However, together with Corollary 1.3.15, it will allow us to prove a full generalisation of [20, Lemma 3], see Corollary 1.3.16.

Lemma 1.3.14. *Given a norm $\|\cdot\|$ on $\mathbb{R}^2$, if $\gamma : [0,1] :\to \mathbb{R}^2$ is a locally injective closed curve contained outside a convex simple polygon $\mathcal{P}$ and $\operatorname{Ind}_p \gamma = k \ge 1$, with $p \in \operatorname{Int}\mathcal{P}$ then $\text{length}_{\|\cdot\|}(\gamma)$ is at least $k\,\text{length}_{\|\cdot\|}(\partial\mathcal{P})$.*

Proof. For $k = 1$ this is just Lemma 1.3.13. Let $k > 1$ and assume the statement is true for $k-1$. Let us denote the length of the curve γ under the norm $\|\cdot\|$ simply by $\text{length}(\gamma)$.

Since there exists $p \in \operatorname{Int}\mathcal{P}$ such that $\operatorname{Ind}_p \gamma = k$, we can find $x_1, x_2 \in [0,1]$ with $x_1 < x_2$ such that:

$$\operatorname{Ind}_p \gamma|_{[x_1,x_2]} = 1; \quad \gamma(x_1) = \gamma(x_2); \quad \operatorname{Ind}_p \gamma|_{[0,x_1]\cup[x_2,1]} = k - 1.$$

Given this partition of the curve γ, the statement follows easily from Lemma 1.3.13 as the curve $\gamma|_{[x_1,x_2]}$ is a closed curve contained outside the convex simple polygon $\mathcal{P}$ and $\operatorname{Ind}_p \gamma|_{[x_1,x_2]} = 1$. Therefore:

$$\begin{aligned}\operatorname{length}(\gamma) &= \operatorname{length}(\gamma|_{[x_1,x_2]}) + \operatorname{length}(\gamma|_{[0,x_1]\cup[x_2,1]}) \\ &\geq \operatorname{length}(\partial\mathcal{P}) + (k-1)\operatorname{length}(\partial\mathcal{P}) = k\operatorname{length}(\partial\mathcal{P}).\end{aligned}$$

Thus, the statement holds for all $k \in \mathbb{N}$. □

Corollary 1.3.15. *Let φ be a simple closed curve in $(\mathbb{R}^2, \|\cdot\|)$. If φ is a boundary of a convex body E, then the length of φ can be approximated by the perimeters of polygons inscribed in E. More precisely, there exists a family $\mathcal{P}$ of convex polygons whose vertices belong to ∂E such that:*

$$\operatorname{length}(\varphi) = \mathcal{H}_1^{\|\cdot\|}(\partial E) = \sup\{\mathcal{H}_1^{\|\cdot\|}(\partial P) : P \in \mathcal{P}\}.$$

Consequently,

$$\mathcal{H}_1^{\|\cdot\|}(\partial E) < \infty.$$

Proof. Let E be a convex body and let $\varphi : [-\pi,\pi] \to \mathbb{R}^2$ be the parametrization of ∂E constructed in Lemma 1.3.3. For each partition Q of $[-\pi,\pi]$, define ψ_Q to be the polygonal curve whose vertices are the points $\varphi(t_i)$, with $t_i \in Q$. From Proposition 1.3.12 we know that $\operatorname{length}(\varphi) = \sup\{\operatorname{length}(\psi_Q) : \psi_Q \in \Psi\}$ where $\Psi = \{\psi_Q :$

Q is a partition of $[-\pi, \pi]\}$. Hence, to show that the first equality of the statement is satisfied we need to show that each closed curve $\psi_Q \in \Psi$ is indeed the boundary of a convex polygon inscribed in $\partial(E)$.

For $\psi_Q \in \Psi$, let P_Q denote the bounded component of $\mathbb{R}^2 \setminus \psi_Q$. First notice that, given a partition Q of $[-\pi, \pi]$, the polygon P_Q is inscribed in E. Indeed, since E is convex all the straight lines $[\varphi(t_{i-1}), \varphi(t_i)]$, with $i \in \{1, \ldots, n\}$ are contained in E. Hence, the simple and closed polygonal curve

$$\psi_Q = \cup_{i=1}^{n} [\varphi(t_{i-1}), \varphi(t_i)]$$

is contained in E —recall that E being a convex body is closed. Hence the bounded component of $\mathbb{R}^2 \setminus \psi_Q$ is contained in E. By definition this bounded component is the polygon P_Q, thus $P_Q \subseteq E$.

Now we show that given a partition $Q = \{-\pi = t_0, \ldots, t_n = \pi\}$ of $[-\pi, \pi]$ the polygon P_Q is convex. We know that for any partition of the interval $[-\pi, \pi]$ with 2 or 3 elements the polygon P_Q is convex. So let us assume that $n \geq 3$. It is enough to show that all the internal angles of P_Q measure less than or equal to π radians. For, choose an internal angle of P_Q, say the angle at the vertex $\psi(t_k)$. See Figure 1.4 for an illustration.

Consider the triangle $\mathcal{T}$ whose vertices —modulo n— are $\psi(t_{k-1}), \psi(t_{k+1}), \psi(t_{k+2})$. Since E is convex and $\psi(t_i) \in \partial E \subseteq E$ for all $i \in \{0, \ldots, n\}$, $\mathcal{T} \subseteq E$ and we also have $\mathrm{Int}(\mathcal{T}) \subseteq \mathrm{Int}(E)$. Since $\psi(t_k) \in \partial E$, we must have $\psi(t_k) \notin \mathrm{Int}(\mathcal{T})$. This means that the quadrilateral with vertices $\psi(t_{k-1}), \psi(t_k), \psi(t_{k+1}), \psi(t_{k+2})$ is convex, so that the internal angle of this quadrilateral at $\psi(t_k)$, which is the internal angle at the vertex $\psi(t_k)$ of the polygon P_Q, is less than or equal to π radians, which is what we wanted. Thus P_Q is convex for any partition Q of $[-\pi, \pi]$.

Finally, it is clear that $\mathcal{H}_1(\partial E) < \infty$. Indeed, since E is bounded there exist points

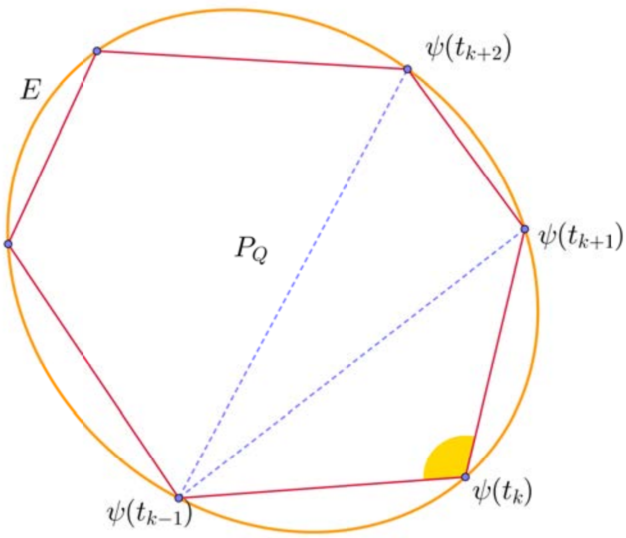


Figure 1.4

p_0, p_1, p_2 such that E is contained in the triangle T whose vertices are the points p_i, $i = 0, 1, 2$. Since the polygon P_Q is convex and the index of the curve ∂T is equal to 1 for any point in the interior of P_Q, from Lemma 1.3.13 it follows that:

$$\mathcal{H}_1(\partial P_Q) \le \mathcal{H}_1(\partial T) = \sum_{i=1}^{3} \|p_i - p_{i-1}\|,$$

for any partition Q of $[-\pi, \pi]$. Hence, $\mathcal{H}_1(\partial E) \le \sum_{i=1}^{3} \|p_i - p_{i-1}\| < \infty$. □

The next result generalises [20, Lemma 3] if we take E equal to a ball of radius r.

Corollary 1.3.16. *Consider a normed space $(\mathbb{R}^2, \|\cdot\|)$, a convex body $E \subseteq \mathbb{R}^2$ and a point $p \in \operatorname{Int} E$. The $\|\cdot\|$-length of a curve φ contained outside E with $\operatorname{Ind}_p \varphi = k$ is at least $k\mathcal{H}_1^{\|\cdot\|}(\partial E)$.*

In particular, if $\varphi \subseteq \mathbb{R}^2 \setminus B_r^{\|\cdot\|}(0)$ for some $r > 0$, then

$$\operatorname{length}_{\|\cdot\|}(\varphi) \ge k \operatorname{length}_{\|\cdot\|}(\partial B_r^{\|\cdot\|}(0)).$$

Proof. Let φ be a curve contained outside a convex body E with $\operatorname{Ind}_p \varphi = k$ for $p \in \operatorname{Int} E$

and take $\varepsilon > 0$. From Corollary 1.3.15 we can find a polygon $\mathcal{P}_Q$ inscribed in E such that

$$\mathcal{H}_1(\partial E) < \mathcal{H}_1(\partial \mathcal{P}_Q) + \varepsilon/k.$$

Since $\mathcal{P}_Q \subseteq E$, the curve φ is a curve contained outside $\mathcal{P}_Q$, and clearly $\operatorname{Ind}_p \varphi = k$ for $p \in \operatorname{Int} \mathcal{P}$. Therefore, from Lemma 1.3.14, we know that $\mathcal{H}_1(\varphi) \geq k\mathcal{H}_1(\partial \mathcal{P}_Q)$ so that:

$$\mathcal{H}_1(\varphi) > k\mathcal{H}_1(\partial E) - \varepsilon.$$

Since this is true for any $\varepsilon > 0$ we conclude that $\mathcal{H}_1(\varphi) \geq k\mathcal{H}_1(\partial E)$. □

The result in Corollary 1.3.16 will be of particular importance to prove Theorem 2.7, which is one of the main results in this work. In a sense, it could be understood that Corollary 1.3.16 was the motivation to develop this complete section. However, the importance of these results concerning lengths of curves "winding around other curves" its not limited to the proof of Theorem 2.7. We will also be concerned with Lipschitz parametrizations of curves. We finish this section showing that, further to Lemma 1.3.3, we can always define a 1-Lipschitz parametrization of the boundary of any convex body on the plane. As we shall see this result together with Corollary 1.3.16 will be present troughout the rest of this work. We first prove the following lemma.

Lemma 1.3.17. *Let $\gamma : [a, b] \to \mathbb{R}^m$ be a parametrization of a locally injective curve. Then the function $\mathscr{L}_\gamma : [a, b] \to \mathbb{R}$ defined as*

$$\mathscr{L}_\gamma(t) = \operatorname{length}(\gamma([a, t]))$$

is continuous.

Proof. Let $\varepsilon > 0$ and take $t_* \in [a, b)$. We show that the function $\mathcal{L}_\gamma$ is continuous at t_*.

By Lemma 1.3.12, we know that

$$\mathrm{length}(\gamma[a,b]) = \sup\{\mathrm{length}(\varphi_Q) : Q \text{ is a partition of } [a,b]\},$$

where φ_Q is the polygonal curve whose vertices are the points $\gamma(t_i)$, $t_i \in Q$. Therefore, we can consider a partition $Q_0 \supset \{a, t_*, b\}$ of $[a,b]$ such that

$$0 \le \mathrm{length}(\gamma[a,b]) - \mathrm{length}(\varphi_{Q_0}) < \tfrac{\varepsilon}{3}. \tag{1.15}$$

Now, take $\delta \in (0, b-t_*)$ such that $Q_0 \cap (t_*, t_*+\delta) = \emptyset$ and such that $\|\gamma(t) - \gamma(t_*)\| \le \varepsilon/3$, whenever $|t - t_*| \le \delta$. Define the following partitions of $[a, t_*]$, $[a, t_* + \delta]$ and $[a,b]$ respectively:

$$Q_* = Q_0 \cap [a, t_*]\,, \quad Q_*' = Q_* \cup \{t_* + \delta\}\,, \quad Q_0' = Q_0 \cup \{t_* + \delta\}\,.$$

Since $Q_0' \supseteq Q_0$ we have $\mathrm{length}(\varphi_{Q_0'}) \ge \mathrm{length}(\varphi_{Q_0})$, so from (1.15) we know that

$$0 \le \mathrm{length}(\gamma[a,b]) - \mathrm{length}(\varphi_{Q_0'}) < \tfrac{\varepsilon}{3}.$$

This implies that $0 \le \mathrm{length}(\gamma([a, t_*+\delta])) - \mathrm{length}(\varphi_{Q_*'}) < \frac{\varepsilon}{3}$, or equivalently,

$$0 \le \Big(\mathrm{length}(\gamma([a,t_*])) - \mathrm{length}(\varphi_{Q_*})\Big) + \Big(\mathrm{length}(\gamma([t_*, t_*+\delta])) - \|\gamma(t_*) - \gamma(t_*+\delta)\|\Big) < \tfrac{\varepsilon}{3}.$$

Since both terms of the sum in the left hand side of the inequality above are non-negative, we know that each term is smaller than $\varepsilon/3$. Using this fact we see that for $t \in (t_*, t_* + \delta)$:

$$\begin{aligned}
&\text{length}(\gamma([a,t])) - \text{length}(\varphi_{Q'_*}) \\
&= \text{length}(\gamma([a,t_*])) + \text{length}(\gamma([t_*,t])) - \Big(\text{length}(\varphi_{Q_*}) + \|\gamma(t_*) - \gamma(t_* + \delta)\|\Big) \\
&< \tfrac{\varepsilon}{3} + \text{length}(\gamma[t_*,t]) - \|\gamma(t_*) - \gamma(t_* + \delta)\| \\
&\leq \tfrac{\varepsilon}{3} + \text{length}(\gamma[t_*,t_* + \delta]) - \|\gamma(t_*) - \gamma(t_* + \delta)\| < \tfrac{2\varepsilon}{3}.
\end{aligned}$$

From this and recalling our choice of δ, we gather that:

$$\begin{aligned}
0 \leq \mathscr{L}_\gamma(t) - \mathscr{L}_\gamma(t_*) &= \text{length}(\gamma([a,t])) - \text{length}(\gamma([a,t_*])) \\
&= \text{length}(\gamma([a,t])) - \text{length}(\varphi_{Q'_*}) + \text{length}(\varphi_{Q'_*}) - \text{length}(\gamma([a,t_*])) \\
&< \tfrac{2\varepsilon}{3} + \text{length}(\varphi_{Q_*}) + \|\gamma(t_*) - \gamma(t_* + \delta)\| - \text{length}(\gamma([a,t_*])) \\
&\leq \tfrac{2\varepsilon}{3} - \Big(\text{length}(\gamma([a,t_*])) - \text{length}(\varphi_{Q_*})\Big) + \tfrac{\varepsilon}{3} \leq \varepsilon,
\end{aligned}$$

the last inequality holds since $\text{length}(\gamma([a,t_*])) - \text{length}(\varphi_{Q_*})$ is non-negative. Therefore, we have shown that for any locally injective curve γ defined on an interval $[a,b]$, the function $\mathscr{L}_\gamma$ is continuous from the right on $[a,b)$.

Now we show that this implies that it is also continuous from the left for any $t_* \in (a,b]$. For, consider the curve $\gamma_1 : [a,b] \to \mathbb{R}$ defined as $\gamma_1(t) = \gamma(b - t + a)$ and let $t'_* = a + b - t_*$. From what we have just shown, we know that there exists $\delta' > 0$ such that $0 \leq \mathscr{L}_{\gamma_1}(t') - \mathscr{L}_{\gamma_1}(t'_*) < \varepsilon$, whenever $t' \in (t'_*, t'_* + \delta')$.

Take $t \in (t_* - \delta', t_*)$ and let $t' = a + b - t$. Notice that $t' \in (t'_*, t'_* + \delta')$ and:

$$\begin{aligned}
\gamma_1[t'_*, t'] &= \{\gamma(b - u + a) : u \in [t'_*, t']\} = \{\gamma(u) : u \in [b - t' + a, b - t'_* + a]\} \\
&= \{\gamma(u) : u \in [t, t_*]\} = \gamma([t, t_*]).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
0 \leq \mathscr{L}_\gamma(t_*) - \mathscr{L}_\gamma(t) &= \text{length}(\gamma([t, t_*])) = \text{length}(\gamma_1([t'_*, t'])) \\
&= \text{length}(\gamma_1([a, t'])) - \text{length}(\gamma_1([a, t'_*])) = \mathscr{L}_{\gamma_1}(t') - \mathscr{L}_{\gamma_1}(t'_*) < \varepsilon,
\end{aligned}
$$

using $t' \in (t'_*, t'_* + \delta')$.

Thus γ is continuous at a and b and for $t_* \in (a, b)$, choosing $\delta_0 = \min\{\delta, \delta', t_* - a, b - t_*\}$, we have $|\mathscr{L}_\gamma(t) - \mathscr{L}_\gamma(t_*)| < \varepsilon$ for all $t \in (t_* - \delta_0, t_* + \delta_0)$.

We conclude that $\mathscr{L}_\gamma$ is a continuous function on $[a, b]$. □

We are now able to show that it is always possible to parametrize the boundary of any convex body in such a way that the parametrization is an injective 1-Lipschitz mapping. In particular, the boundary of any ball in a normed space $(\mathbb{R}^2, \|\cdot\|)$ can be parametrized in this way.

Corollary 1.3.18. *Let $\|\cdot\|$ be a norm in $\mathbb{R}^2$ and consider a convex body $E \subseteq \mathbb{R}^2$ and a point $x \in \partial E$. There exists a bijective 1-Lipschitz mapping $\varphi_* : [0, \mathcal{H}_1^{\|\cdot\|}(\partial E)] \to \partial E$ with starting point x and oriented counterclockwise.*

Proof. We will denote the 1-dimensional Hausdorff measure relative to the norm $\|\cdot\|$ simply by $\mathcal{H}_1$. Let E be a convex body on $(\mathbb{R}^2, \|\cdot\|)$. From Lemma 1.3.3 we know that there exists an injective continuous parametrization $\varphi : [a, b] \to \partial E$ of the boundary of E. Notice that we can assume that $\varphi(a) = x$ and that, by Remark 1.3.4, φ is oriented counterclockwise. Let $\mathcal{H}_E := \mathcal{H}_1^{\|\cdot\|}(\partial E)$ and define $\varphi_* : [0, \mathcal{H}_E] \to \partial E$ as:

$$
\varphi_*(t) = \varphi(\lambda) \ \text{ iff } \ \mathcal{H}_1(\varphi|_{[a,\lambda]}) = t. \tag{1.16}
$$

First we show that this function is well defined, i.e. we show that for all $t \in [0, \mathcal{H}_E]$ there exists a unique $\lambda \in [a, b]$ such that $\mathcal{H}_1(\varphi|_{[a,\lambda]}) = t$. This is equivalent to saying that the function $\mathscr{L} : [a,b] \to [0, \mathcal{H}_E]$ defined as $\mathscr{L}(\lambda) = \mathcal{H}_1(\varphi|_{[0,\lambda]})$ is a bijection on-to $[0, \mathcal{H}_E]$. From Lemma 1.3.17 we know that $\mathscr{L}$ is a continuous function. Moreover, it is strictly increasing, for if $t_1 < t_2$, then:

$$\begin{aligned}\mathscr{L}(t_2) = \mathcal{H}_1(\gamma[a,t_2]) &= \mathcal{H}_1(\gamma[a,t_1]) + \mathcal{H}_1(\gamma[t_1,t_2]) \\ &\geq \mathcal{H}_1(\gamma[a,t_1]) + \|\gamma(t_1) - \gamma(t_2)\| > \mathcal{H}_1(\gamma[a,t_1]) = \mathscr{L}(t_1).\end{aligned}$$

Since $\mathscr{L}(a) = 0$ and $\mathscr{L}(b) = \mathcal{H}_E$, we conclude that $\mathcal{L}$ is a bijection. Hence, φ_* is an injective parametrization of ∂E. Notice also that $\varphi_*(0) = \varphi(\lambda)$ where $\mathcal{H}_1(\varphi([a,\lambda]) = 0$, hence $\lambda = a$ and $\varphi_*(0) = \varphi(a) = x$. Thus, φ_* has starting point x, and it is clear that it is oriented in the same direction as φ.

It is easy to see that φ_* is an L-Lipschitz mapping with $L \leq 1$. Indeed, let $t_1, t_2 \in [0, \mathcal{H}_E]$ such that $t_1 < t_2$, and take $\lambda_1, \lambda_2 \in [a,b]$ such that $\varphi_*(t_i) = \varphi(\lambda_i)$ for $i = 1, 2$. This means that $\mathcal{H}_1(\varphi|_{[a,\lambda_i]}) = t_i$, therefore:

$$\begin{aligned}\|\varphi_*(t_2) - \varphi_*(t_1)\| = \|\varphi(\lambda_2) - \varphi(\lambda_1)\| &\leq \mathcal{H}_1(\varphi([\lambda_1, \lambda_2])) \\ &= \mathcal{H}_1(\varphi([a,\lambda_2]) - \mathcal{H}_1(\varphi([a,\lambda_1]) = |t_2 - t_1|,\end{aligned}$$

where the inequality uses Lemma 1.3.10. Thus φ_* is an L-Lipschitz mapping with $L \leq 1$. We are now left to show that $L = 1$.

Assume on the contrary that $L < 1$ and take $0 < \varepsilon < \mathcal{H}_E(1 - L)$. From Proposition 1.3.12, we know that there exists a partition $Q = \{0 = t_0, \ldots, t_n = \mathcal{H}_E\}$ of $[0, \mathcal{H}_E]$ and a polygonal curve ψ_Q with vertices $\{\varphi_*(t_i) : 0 \leq i \leq n\}$ such that:

$$\text{length}_{\|\cdot\|}(\varphi_*) < \text{length}_{\|\cdot\|}(\psi_Q) + \varepsilon.$$

Hence, since φ_* is an injective parametrization of ∂E, we have:

$$\begin{aligned}\mathcal{H}_1(\partial E) = \text{length}(\varphi_*) < \text{length}(\psi_Q) + \varepsilon &= \sum_{i=1}^{n} \|\varphi_*(t_i) - \varphi_*(t_{i-1})\| + \varepsilon \\ &\leq L \sum_{i=1}^{n} |t_i - t_{i-1}| + \varepsilon = L(t_n - t_0) + \varepsilon = L\mathcal{H}_E + \varepsilon \\ &< L\mathcal{H}_E + \mathcal{H}_E(1-L) = \mathcal{H}_E = \mathcal{H}_1(\partial E)\end{aligned}$$

This is a contradiction, therefore $L = 1$. □

Chapter 2
Bounds for the ratio of constants of an n-fold Lipschitz quotient on the plane

In this chapter we will study in more depth the behaviour of Lipschitz quotients from the plane to itself. As we have mentioned before, a remarkable property of Lipschitz quotient mappings $f : (X, \|\cdot\|_1) \to (Y, \|\cdot\|_2)$ with $X = Y = \mathbb{R}^2$ is that the preimage of each point is finite. Moreover, every such mapping can be written as a composition $P \circ h$, where P is a complex polynomial and h is a homeomorphism [15]. In this chapter we will see how the degree of P influences the ratio of the Lipschitz and co-Lipschitz constants of the mapping f, with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$.

It is proven in [19] that, for a Lipschitz quotient mapping from $\mathbb{R}^2$ to itself with the Euclidean norm, if the inverse image of every point consists of no more than n points, then $c/L \leq \rho_n := 1/n$, where c and L are the co-Lipschitz and Lipschitz constants of the mapping. Our aim is to generalise this result to any norm on the plane. Since every norm on the plane is equivalent to the Euclidean norm, we may use the tools developed in Section 1.3 and [19, Theorem 2] —which concerns the Euclidean case— to prove the more general statement Theorem 2.7. A generalisation of [19, Theorem 2] is also stated an proved in [20, Theorem 1].

First, we introduce some notation. In what follows, the ball of radius r under the norm $\|\cdot\|$, centred at the point $p \in \mathbb{R}^2$ will be denoted by $B_r^{\|\cdot\|}(p)$, or simply by $B_r(p)$ when it is clear with which norm we are working. The symbol $|z|$ will mean the modulus of z, if $z \in \mathbb{C}$, so $|\cdot|$ is the Euclidean norm when $\mathbb{C} \equiv \mathbb{R}$. Finally, recall that given a point

z of the plane, we use the symbol $\arg(z)$ to denote the argument of the complex number z taking values in $(-\pi, \pi]$.

For each norm $\|\cdot\|$ on $\mathbb{R}^2$ we will denote by $\mathscr{L}_{\|\cdot\|}$ the constant

$$\mathscr{L}_{\|\cdot\|} = \mathcal{H}_1^{\|\cdot\|}(\partial B_1(0)). \tag{2.1}$$

We introduce also the notion of an *n-fold mapping*:

Definition 2.1. We say that a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an *n-fold mapping* if

$$\max_{x\in\mathbb{R}^2} \#f^{-1}(x) = n.$$

Now that we have clarified the notation that we will be using, we can start the proof of the main result of this chapter. In order to do this we state and prove the following lemmas.

Lemma 2.2. *Assume $f : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}^2, \|\cdot\|_2)$ is a c-co-Lipschitz mapping, and $\#f^{-1}(0)$ is finite. Then there exists $M > 0$, defined by equation (2.2), such that*

$$\|f(p)\|_2 \geq c(\|p\|_1 - M) \ \ \textit{for all} \ \ p \in \mathbb{R}^2.$$

Consequently, for any $\varepsilon \in (0,1)$ there exists an $R_\varepsilon > 0$ such that for any p with $\|p\|_1 \geq R_\varepsilon$ we have $\|f(p)\|_2 > c(1-\varepsilon)\|p\|_1$.

Proof. For $i = 1, 2$, denote by $B_r^i(x)$ the ball of radius r centred at x under the norm $\|\cdot\|_i$. Let $p \in \mathbb{R}^2$ and set

$$M := \max\{\|p\|_1 : p \in f^{-1}(0)\}. \tag{2.2}$$

By Lemma 1.2.2 we have $\overline{B^2}_{\|f(p)\|_2}(f(p)) \subseteq f(\overline{B^1}_{\frac{1}{c}\|f(p)\|_2}(p))$, so there must exist a point

$p_0 \in \overline{B^1}_{\frac{1}{c}\|f(p)\|_2}(p)$ such that $f(p_0) = 0$. Hence:

$$\|f(p)\|_2 \geq c\|p - p_0\|_1 \geq c(\|p\|_1 - \|p_0\|_1) \geq c(\|p\|_1 - M).$$

The first inequality follows from $\|p - p_0\|_1 \leq \frac{1}{c}\|f(p)\|_2$.

To prove the second part of the statement, let $\varepsilon \in (0, 1)$ and $R_\varepsilon > M/\varepsilon$. Consider a point $p \in \mathbb{R}^2$ such that $\|p\|_1 \geq R_\varepsilon$. Now, using the above inequality we have:

$$\|f(p)\|_2 \geq c(\|p\|_1 - M) > c(\|p\|_1 - \varepsilon\|p\|_1) = c(1 - \varepsilon)\|p\|_1.$$

□

Lemma 2.3. *Let $f : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}^2, \|\cdot\|_2)$ be a Lipschitz quotient mapping. Then the mapping $f_1 = f - f(0)$ can be written as the composition $P_1 \circ h_1$ where P_1 is a polynomial of one complex variable with the leading coefficient equal to one, h_1 is a homeomorphism and $f_1(0) = P_1(0) = h_1(0) = 0$.*

Moreover for any $r > 0$ there exists an $r' > r$ such that

$$\|h_1(p)\|_1 > r \quad \textit{whenever} \quad \|p\|_1 \geq r'. \tag{2.3}$$

Proof. Let $f_1 := f - f(0)$, it is clear that f_1 is a Lipschitz quotient mapping with the same Lipschitz and co-Lipschitz constants as f and $f_1(0) = 0$. By [15] we know that there is a non-zero polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ (with $a_n \neq 0$) of one complex variable and a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$, such that $f_1 = P \circ h$. Let $h_1(z) := a_n^{1/n}(h(z) - h(0))$ and $P_1(z) = P(a_n^{-1/n} z + h(0))$, then it is clear that $h_1(0) = 0$, the leading coefficient of P_1 equals 1, and we also have:

$$\begin{aligned} P_1(h_1(z)) &= P_1\left(a_n^{1/n}(h(z) - h(0))\right) = P\Big(a_n^{-1/n}\left(a_n^{1/n}(h(z) - h(0)\right) + h(0)\Big) \\ &= P(h(z)) = f_1(z). \end{aligned}$$

Hence $P_1 \circ h_1 = f_1$. Finally notice that,

$$P_1(0) = P(h(0)) = f_1(0) = 0.$$

This finishes the first part of the statement, now we prove the second one. Take $\beta > 0$ and $\alpha \geq 1$ such that

$$\beta \|p\|_2 \leq |p| \leq \alpha \|p\|_1 \text{ for all } p \in \mathbb{R}^2,$$

where $|p|$ is the Euclidean norm of p.

Let us denote by b_k, $k = 1, \dots, n$, the coefficients of the polynomial P_1, so that $P_1(z) = b_n z^n + b_{n-1} z^{n-1} \dots + b_1 z$, with $b_n = 1$. Let $r > 0$, and M be as in Lemma 2.2 and pick any

$$r' > \max \left\{ r, \ \frac{\alpha^n}{c\beta} \sum_{k=1}^{n} |b_k| r^k + M \right\}. \tag{2.4}$$

Assume $\|p\|_1 \geq r'$, then we must have $\|h_1(p)\|_1 > r$. Indeed, if $\|h_1(p)\|_1 \leq r$ then $|h_1(p)| \leq \alpha r$, hence:

$$\begin{aligned}
\|f_1(p)\|_2 = \|P_1(h_1(p))\|_2 &\leq \sum_{k=1}^{n} \|b_k (h_1(p))^k\|_2 \leq \tfrac{1}{\beta} \sum_{k=1}^{n} |b_k| |h_1(p))^k| \\
&= \tfrac{1}{\beta} \sum_{k=1}^{n} |b_k| |h_1(p)|^k \leq \tfrac{1}{\beta} \sum_{k=1}^{n} |b_k| (\alpha r)^k \leq \tfrac{\alpha^n}{\beta} \sum_{k=1}^{n} |b_k| r^k \\
&< c(r' - M) \leq c(\|p\|_1 - M).
\end{aligned}$$

The latter is impossible since, by Lemma 2.2, we have $\|f_1(p)\|_2 \geq c(\|p\|_1 - M)$. Thus, $\|h_1(p)\|_1 > r$ whenever $\|p\|_1 \geq r'$. □

Actually, it will be convenient to revisit the local injectivity properties of Lipschitz quotients already mentioned in Section 1.2. As we will show now, Lemma 2.3 can be used to prove a somewhat stronger version of Proposition 1.2.9. As we will see in the next proposition, we can show that there exists a fixed constant $\varepsilon > 0$ such that f is injective in every neighbourhood of radius smaller than ε centred far enough from the origin. We prove this in the case of a 2-fold Lipschitz quotient mapping.

Proposition 2.4. *Let $f : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}^2, \|\cdot\|_2)$ be a 2-fold Lipschitz quotient mapping. There exist $N > 0$ and $\varepsilon > 0$ such that for every $x \in \mathbb{R}^2$ with $\|x\|_1 > N$ the mapping f is injective on $B_\varepsilon^{\|\cdot\|_1}(x)$.*

Proof. We may work out the proof in the Euclidean case since any other norm on the plane is equivalent to the Euclidean norm.

Let L and c be the Lipschitz and co-Lipschitz constants of f with respect to the Euclidean norm. Replacing f with

$$f_1(z) = \frac{f(z) - f(0)}{c}$$

we get that $f_1(0) = 0$, the Lipschitz constant of f_1 is equal to L/c and the co-Lipschitz constant is equal to 1. So assume that $f(0) = 0$ and that $c = 1$.

As in Lemma 2.3, we may consider the homeomorphism h with $h(0) = 0$ and the polynomial $P(z)$ with no constant term and leading coefficient equal to 1, such that $f = P \circ h$. Hence, from Remark 1.1.2, we see that in this case, since f is a 2-fold, we must have $P(z) = z^2 + az$; so that $f(p) = (h(p))^2 + ah(p)$. We will be using Lemma 2.2, so recall that the constant M was defined by (2.2) as $M = \max\{|p| : p \in f^{-1}(0)\}$, so in this case $M = |a|$. Also, from Proposition 1.2.9 we can consider a constant $M' > 0$ such that f is locally injective at x for all $x \in \mathbb{R}^2 \setminus B_{M'}(0)$.

Take $R > 0$ such that

$$\text{for all } z \in \mathbb{C} \text{ with } |z| > R \text{ we have } |\arg(z) - \arg(z + a)| < \tfrac{\pi}{4}. \tag{2.5}$$

From Lemma 2.3, we know that there is an $R' > R$ such that

$$|h(p)| > R \text{ whenever } |p| > R'. \tag{2.6}$$

Set the constants:

$$N' := \frac{\pi}{2}|a|^2; \quad \varepsilon := \frac{N'}{L} \quad \text{and} \quad N > \max\{|a|(2|a|+1) + N', R', M'\}.$$

Assume that $f(p_1) = f(p_2)$ for two different points $p_1, p_2 \in \mathbb{R}^2 \setminus \overline{B}_N(0)$. We show that $|p_1 - p_2| \geq \varepsilon$, so f is injective on $B_{\frac{\varepsilon}{2}}(x)$ for $|x| > N + \frac{\varepsilon}{2}$.

Let γ be the curve describing the straight line joining p_1 and p_2. Let us denote by γ_* the image of γ under h, so that $f(\gamma) = P(\gamma_*)$. Notice that $\gamma \subseteq \mathbb{R}^2 \setminus B_{M'}(0)$, so that f is locally injective at x, for all $x \in \gamma$.

First assume that there exists $w_0 \in \gamma_*$ such that $|w_0| \leq |a|$. In this case the curve $f(\gamma) = P(\gamma_*)$ contains the points $P(w_0)$ and $f(p_1)$. Recalling Lemma 2.2 and Lemma 1.3.10, we have:

$$\begin{aligned} \text{length}(f(\gamma)) &\geq |f(p_1)| - |P(w_0)| \geq c(|p_1| - |a|) - |w_0^2 + aw_0| \\ &> 1\,(|a|(2|a|+1) + N' - |a|) - (2|a|^2) = N'. \end{aligned}$$

Here length(γ) refers to the Euclidean length of the curve γ.

We will show that length$(f(\gamma)) \geq N'$ also in the case when there does not exist $w_0 \in \gamma_*$ such that $|w_0| \leq |a|$, i.e. when for all $w \in \gamma_*$ we have $|w| > |a|$. Indeed, let $z_1 := h(p_1)$ and $z_2 := h(p_2)$. Since $f(p_1) = f(p_2)$, $z_1^2 + az_1 = z_2^2 + az_2$. Therefore, $(z_1 - z_2)(z_1 + z_2 + a) = 0$. Also, since $p_1 \neq p_2$, then $z_1 = h(p_1) \neq h(p_2) = z_2$, so we must have $z_2 = -z_1 - a$.

Furthermore, by (2.6), since $|p_i| > N > R'$, $i = 1, 2$, we have $|z_i| > R$, so by (2.5):

$$|\arg(z_2) - \arg(z_1)| = |\arg(-(z_1 + a)) - \arg(z_1)| > \pi - \tfrac{\pi}{4}. \tag{2.7}$$

In order for a continuous curve γ_* to join the points z_1 and z_2 it must cover all the angles between $\arg(z_1)$ and $\arg(z_2)$, or between $\arg(z_2)$ and $\arg(z_1)$. In any case, from (2.7) we know that the argument range of γ_* between z_1 and z_2 is greater than $\pi/2$ and all points of γ_* are contained outside the circle of radius $|a|$. Hence,

$$\operatorname{length}(\gamma_*) \geq |a|\tfrac{\pi}{2}. \tag{2.8}$$

Now we will show that $\operatorname{length}(f(\gamma)) \geq |a| \operatorname{length}(\gamma_*)$. Since $\partial B_{|a|}(0)$ is a compact set and it does not intersect γ_* we can pick δ such that $0 < \delta \leq \operatorname{dist}(\gamma_*, \partial B_{|a|}(0))$. Consider γ_* as a parametrization say $\gamma_* : [0, 1] \to \mathbb{R}^2$. Choose $N_* \in \mathbb{N}$ such that $|\gamma_*(t) - \gamma_*(t')| < \delta$ whenever $|t - t'| \leq 1/N_*$.

Now consider any $\varepsilon' > 0$. From Proposition 1.3.12 we can take a partition of $[a, b]$, say $Q = \{a = t_0, \ldots, t_n = b\}$, such that the length of the polygonal curve ψ_Q, whose vertices are the points $\gamma_*(t_i)$ with $0 \leq i \leq n$, satisfies

$$\operatorname{length}(\gamma_*) < \operatorname{length}(\psi_Q) + \tfrac{\varepsilon'}{|a|}. \tag{2.9}$$

We may assume also that the partition Q satisfies $|t_i - t_{i-1}| < 1/N_*$ for all $i \in \{1, \ldots, n\}$, so that

$$|\gamma_*(t_i) - \gamma_*(t_{i-1})| < \delta \text{ for all } i \in \{1, \ldots n\}. \tag{2.10}$$

Now we show that $|P(z) - P(w)| > |a||z - w|$ whenever $z, w \in \gamma_*$ and $0 < |z - w| < \delta$. Let $z, w \in \mathbb{C}$ such that $|z - w| < \delta$ and assume that $z \neq w$. Let ξ be the middle point between z and w, i.e. $\xi = \frac{z+w}{2}$. Notice that

$$P'(\xi)(z-w) = (2\xi+a)(z-w) = (z+w+a)(z-w) = z^2 - az - (w^2+az) = P(z) - P(w).$$

Therefore,

$$(2\xi + a)(z-w) = P(z) - P(w). \tag{2.11}$$

We can see that $|\xi| > |a|$. Indeed, since ξ is the middle point between z and w and $|z-w| < \delta$, we get $|z-\xi| < \delta/2$ Hence, as $|z| > |a| + \delta$, we conclude that $|\xi| > |a|$. This, together with (2.11), implies:

$$|P(z) - P(w)| = |2\xi + a||z-w| \geq (2|\xi| - |a|)|z-w| > |a||z-w|$$

whenever $|z-w| < \delta$. Hence, in view of (2.10), we have:

$$|P(\gamma_*(t_i)) - P(\gamma_*(t_{i-1}))| > |a||\gamma_*(t_i) - \gamma_*(t_{i-1})|,$$

for all $i \in \{1, \ldots, n\}$.

Therefore, using Lemma 1.3.10 and recalling the inequality in (2.9) we have:

$$\begin{aligned}\operatorname{length}(f(\gamma)) &\geq \sum_{i=1}^{n} |P(\gamma_*(t_i)) - P(\gamma_*(t_{i-1}))| > \sum_{i=1}^{n} |a||\gamma_*(t_i) - \gamma_*(t_{i-1})| \\ &= |a| \operatorname{length}(\psi_Q) > |a| \left(\operatorname{length}(\gamma_*) - \tfrac{\varepsilon'}{|a|}\right) = |a| \operatorname{length}(\gamma_*) - \varepsilon'.\end{aligned}$$

Since this is true for all $\varepsilon' > 0$, in view of (2.8), we gather that:

$$\operatorname{length}(f(\gamma)) \geq |a| \operatorname{length}(\gamma_*) \geq \frac{\pi}{2}|a|^2 =: N'.$$

Thus, in both cases we have $\operatorname{length}(f(\gamma)) \geq \frac{\pi}{2}|a|^2 = N'$. By the Lipschitz property of

f, following Remark 1.3.9, we conclude that:

$$|p_1 - p_2| = \text{length}(\gamma) \geq \tfrac{1}{L}\,\text{length}(f(\gamma)) \geq \frac{N'}{L} = \varepsilon.$$

Thus $|p_1 - p_2| \geq \varepsilon$, which is what we wanted to show. $\square$

This new version of Proposition 1.2.9 leads to the following new version of Corollary 1.2.12 that we will be using later on. Notice that the key difference between Proposition 1.2.9 and Proposition 2.4, as well as the difference between Corollary 1.2.12 and Corollary 2.5 is that in the second versions ε is independent of x.

Corollary 2.5. *Let $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ be a 2-fold Lipschitz quotient mapping. There exist $N > 0$ and $\varepsilon > 0$ such that for every $x \in \mathbb{R}^2$ with $\|x\| > N$ we have $\partial(f(B_r(x))) = f(\partial B_r(x))$ for all $r \leq \varepsilon$.*

Moreover, if c denotes the co-Lipschitz constant of f, then

$$\|f(x_1) - f(x_2)\| \geq c\|x_1 - x_2\|,$$

whenever $\|x_1\|, \|x_2\| > N$ and $\|x_1 - x_2\| < \varepsilon$.

Proof. The proof can be worked out in the same way as in Corollary 1.2.12 but using Proposition 2.4 instead of Proposition 1.2.9. $\square$

In connection with Corollary 1.2.12 and Remark 1.2.13, we would like to point out that Corollary 2.5 shows that there exist $N > 0$ and $\varepsilon > 0$ such that for all $x \in \mathbb{R}^2 \setminus B_N(0)$ the mapping $f|_{B_\varepsilon(x)}$ is bi-Lipschitz, considered as a map onto $f(B_\varepsilon(x))$. This sort of bi-Lipschitz behaviour can also be found under other conditions different from the co-Lipschitz condition. For instance in [11] it is proved that, in the Euclidean setting, the so-called Lipschitz regular mappings can be "nicely" decomposed into bi-Lipschitz mappings. Moreover, in [3] the same conclusion is obtained under weaker assumptions.

A similar result for bounded length distortion mappings is proved in [24] and [25]. For further reference in this topic see [8, Chapter 7].

After this interlude about injectivity, we go back to pave the way for the proof of Theorem 2.7. The next theorem, Theorem 2.6, which generalises [20, Lemma 2], is probably the theorem that we use the most for our geometric intuition in the rest of this work. This theorem is somehow the geometric version of the statement "every Lipschitz quotient on the plane can be written as a homeomorphism followed by a polynomial of one complex variable". Basically it states that far away from the origin, the behaviour of Lipschitz quotients is close to that of a polynomial z^n.

Theorem 2.6. *If $f : (\mathbb{R}^2, \|\cdot\|_1 \to (\mathbb{R}^2, \|\cdot\|_2)$ is an n-fold Lipschitz quotient mapping with co-Lipschitz constant equal to c, then:*

(1) *There exist $M > 0$ and $R' > 0$ such that for all $\rho > R'$ we have:*

$$\operatorname{Ind}_0 f(\partial B^1_\rho(0)) = n \ \ \text{and} \ \ f(\partial B^1_\rho(0)) \subseteq (\mathbb{R}^2 \setminus B^2_{c(\rho-M)}(0)).$$

(2) *For any $\varepsilon \in (0,1)$ there exists an $R'_\varepsilon > 0$ such that for all $\rho \geq R'_\varepsilon$ we have:*

$$\operatorname{Ind}_0 f(\partial B^1_\rho(0)) = n \ \ \text{and} \ \ f(\partial B^1_\rho(0)) \subseteq (\mathbb{R}^2 \setminus \overline{B^2}_{c_\varepsilon \rho}(0)), \ \ \text{where } c_\varepsilon = c(1-\varepsilon).$$

Here $B^1_r(x)$ denotes the ball centred at x of radius r under the norm $\|\cdot\|_1$.

Proof. We may assume that $f(0) = 0$, this would change M by at most $\|f(0)\|_2$ which is a constant. From Lemma 2.3, we know that $f = P \circ h$ where $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z$ and h is a homeomorphism with $h(0) = 0$. Take $R_1 > 0$ such that

$$\sum_{k=1}^{n-1} |a_k|/r^{n-k} < \tfrac{1}{2} \ \text{ for all } r \geq R_1. \tag{2.12}$$

As $\|\cdot\|_1$ is equivalent to $|\cdot|$, there is a $\beta > 0$ such that $\beta\|p\|_1 \leq |p|$ for all $p \in \mathbb{R}^2$. Set $R := R_1/\beta$, from Lemma 2.3 we know that there is an $R' > R$, which may be calculated using (2.4), such that $\|h(p)\|_1 > R$, whenever $\|p\|_1 \geq R'$.

For $\rho > R$ consider the boundary of the ball $B^1_\rho(0)$ with respect to the norm $\|\cdot\|_1$. Let γ denote the curve $\gamma := h(\partial B^1_\rho(0))$. Then, for any $w \in \gamma$ we have $|w| \geq \beta\|w\|_1 > \beta R = R_1$, hence:

$$|P(w) - w^n| = |\sum_{k=1}^{n-1} a_k w^k| \leq |w|^n \sum_{k=1}^{n-1} \frac{|a_k|}{|w|^{n-k}} \leq |w|^n \sum_{k=1}^{n-1} \frac{|a_k|}{{R_1}^{n-k}} < \tfrac{1}{2}|w|^n.$$

This implies that $\mathrm{Ind}_0\, P(\gamma) = \mathrm{Ind}_0\{w^n : w \in \gamma\} = n$.

Thus for any $\rho > R'$ we have $\mathrm{Ind}_0\, f(\partial B^1_\rho(0)) = n$ and, by Lemma 2.2, we know that there exists $M > 0$ (given by (2.2)) such that $\|f(p)\|_2 \geq c(\|p\|_1 - M)$ for all $p \in \partial B^1_\rho(0)$, therefore $f(\partial B^1_\rho(0)) \subseteq (\mathbb{R}^2 \setminus B^2_{c(\rho-M)}(0))$.

Now the statement (2) of this Lemma follows easily. Let $\varepsilon \in (0,1)$ and take the relevant R_ε as in Lemma 2.2. If we take R and R' as before and we define $R'_\varepsilon := \max\{R', R_\varepsilon\}$ then from Lemma 2.2 we have:

$$f(\partial B^1_\rho(0)) \subseteq (\mathbb{R}^2 \setminus \overline{B^2}_{c_\varepsilon \rho}(0)) \ \text{ for all } \rho > R'_\varepsilon.$$

Also, since $R'_\varepsilon > R'$, from statement (1) we know that $\mathrm{Ind}_0\, f(\partial B^1_\rho(0)) = n$ whenever $\rho > R'_\varepsilon$. $\square$

Now the proof of the main result of this chapter follows easily from the above lemmas and some results from Chapter 1.

Theorem 2.7. *Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on $\mathbb{R}^2$ and let $\mathscr{L}_1$ and $\mathscr{L}_2$ denote the length of the boundary of the unit ball $B^i_1(0)$, under the norm $\|\cdot\|_i$, $i = 1, 2$, defined as in Theorem 2.6.*

If $f : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}^2, \|\cdot\|_2)$ is an L-Lipschitz and c-co-Lipschitz n-fold mapping then,

$$\frac{c}{L} \leq \frac{\mathscr{L}_1}{n\mathscr{L}_2}.$$

In particular, if $\|\cdot\|_1 = \|\cdot\|_2$, then $c/L \leq 1/n$.

Proof. First notice that if we multiply the function f by a constant number, the ratio c/L stays, so we can assume that $L = 1$. Suppose now that, on the contrary, $\frac{c}{L} = c > \frac{\mathscr{L}_1}{n\mathscr{L}_2}$ and take $\varepsilon > 0$ such that $\varepsilon < (1 - \frac{\mathscr{L}_1}{nc\mathscr{L}_2})$.

From Theorem 2.6 we know that there is some positive number $R'_\varepsilon > 0$ such that, for all $\rho > R'_\varepsilon$ we have: $\operatorname{Ind}_0 f(\partial B^1_\rho(0)) = n$ and $f(\partial B^1_\rho(0)) \subseteq (\mathbb{R}^2 \setminus \overline{B^2}_{c(1-\varepsilon)\rho}(0))$.

Recall further that from Proposition 1.2.9 we can consider a constant $R_I > 0$ such that f is locally injective at x for all $x \in \mathbb{R}^2 \setminus B^1_{R_I}(0)$. Hence, the length of a curve $f(\gamma)$, where γ is contained outside $B^1_{R_I}(0)$, is defined. See also Remark 1.3.9.

Take $\rho > \{R'_\varepsilon, R_I\}$, from Lemma 1.3.3, we can consider an injective parametrization, $\gamma : [a, b] \to \partial B^1_\rho(0)$, of the boundary of $B^1_\rho(0)$. Since $\rho > R'_\varepsilon$, we know that $\operatorname{Ind}_0 f(\gamma) = n$ and $f(\gamma)$ is contained outside the ball $\overline{B^2}_{c(1-\varepsilon)\rho}(0)$. Therefore, from Corollary 1.3.16, we gather that:

$$\begin{aligned}\operatorname{length}_{\|\cdot\|_2}(f \circ \gamma) \geq n \left(\operatorname{length}_{\|\cdot\|_2}(\partial B^2_{c_\varepsilon \rho}(0))\right) = n\,(c_\varepsilon \rho \mathscr{L}_2) = nc(1-\varepsilon)\rho\mathscr{L}_2 \\ > nc\frac{\mathscr{L}_1}{nc\mathscr{L}_2}\rho\mathscr{L}_2 = \rho\mathscr{L}_1.\end{aligned}$$

On the other hand, since f is a Lipschitz mapping with Lipschitz constant equal to one, then f cannot increase the length of γ —see for instance (1.10) in Remark 1.3.9— so we have $\operatorname{length}_{\|\cdot\|_2}(f \circ \gamma) \leq \operatorname{length}_{\|\cdot\|_1}(\gamma) = \rho\mathscr{L}_1$. This is a contradiction, thus we must have $c/L \leq \mathscr{L}_1/n\mathscr{L}_2$. ◻

Along the rest of this work we will use Theorem 2.7 only in the case when $\|\cdot\|_1 = \|\cdot\|_2$.

We would like to point out that this theorem might not give further information about the ratio c/L for the case $\|\cdot\|_1 \neq \|\cdot\|_2$, namely when $\mathscr{L}_1 > n\mathscr{L}_2$. However, we can find norms $\|\cdot\|_1, \|\cdot\|_2$ on the plane such that $\|\cdot\|_1 \neq \|\cdot\|_2$ and $\mathscr{L}_1 = \mathscr{L}_2$, so Theorem 2.7 does give an effective bound for the ratio of constants c/L in those cases. We include more detailed comments in this regard in Chapter 6.

Sometimes it will be convenient to think about the result in Theorem 2.7, for the case $\|\cdot\|_1 = \|\cdot\|_2$, in the following way.

Corollary 2.8. *Let f be a Lipschitz quotient mapping from the plane to itself. If L and c are the Lipschitz and co-Lipschitz constants of f under any given norm $\|\cdot\|$ on $\mathbb{R}^2$, then:*

$$\tfrac{c}{L} > \tfrac{1}{n+1} \ \textit{ implies } \ \#f^{-1}(x) \leq n \ \textit{ for all } \ x \in \mathbb{R}^2.$$

Proof. This is the contrapositive version of Theorem 2.7 for the case $\|\cdot\|_1 = \|\cdot\|_2$. ☐

Corollary 2.8 shows that the Euclidean scale $\rho_n = 1/(n+1)$ from [19, Theorem 2] is in fact a universal scale, in the sense that it works for any Lipschitz quotient mapping $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$, where $\|\cdot\|$ is an arbitrary norm. Now we would like to know if the scale ρ_n is sharp in all cases (recall that we already know that for the Euclidean case this scale is sharp). The next chapter deals with this problem for the particular case of the supremum norm.

Chapter 3
Lipschitz quotient mappings on $\mathbb{R}^2$ with the supremum norm

In this section we will focus on the relation between the cardinality of the inverse image of a point under a Lipschitz quotient and the ratio of Lipschitz and co-Lipschitz constants of this mapping in the particular case when the Lipschitz quotient mapping is defined from the plane endowed with the supremum norm to itself. As we shall see, these relations do not work in the same way for the supremum norm as they do for the Euclidean norm.

The main result of this section, Theorem 3.2.5, shows that, unlike the Euclidean case, there does not exist a two-fold Lipschitz quotient mapping f on $\mathbb{R}^2$ endowed with the supremum norm and ratio of constants equal to $1/2$.

3.1 An example of a two-fold Lipschitz quotient mapping

In the Euclidean case we have examples of n-fold Lipschitz quotient mappings from the plane to itself, such that the ratio between the co-Lipschitz and Lipschitz constants is equal to $1/n$; the standard examples are given by $f_n(re^{i\theta}) = re^{in\theta}$. In particular $f_2(re^{i\theta}) = re^{2i\theta}$ is a 2-fold Lipschitz quotient mapping with ratio of constants $1/2$. However, if we define a function on $\mathbb{R}^2$ endowed with the supremum norm, that behaves in an "analogous" fashion (i.e a map such that each curve describing the boundary of a ball centred at the origin is mapped onto a curve that goes two times around the same ball) we get a different ratio of constants. We shall see this in Example 3.1.1.

Example 3.1.1. Define a Lipschitz quotient mapping f to fix any square centred at the origin and to "double" the length of any piece of the curve $\partial B_r(0)$, $r > 0$, starting at the bottom right corner of the square. Formally, the function is defined as follows: Divide the plane into the eight regions R_i, $i = 1, ..., 8$ given by:

$$R_1 = \{(x, y) \in \mathbb{R}^2 : x > |y|, y \leq 0\} \cup \{(0, 0)\},$$
$$R_2 = \{(x, y) \in \mathbb{R}^2 : x \geq |y|, y > 0\},$$
$$R_3 = \{(x, y) \in \mathbb{R}^2 : y > |x|, x \geq 0\},$$
$$R_4 = \{(x, y) \in \mathbb{R}^2 : y \geq |x|, x < 0\},$$

and so on. See Figure 3.1 for an illustration. Then the function is defined as:

$$f(x,y) = \begin{cases} (x, x+2y) & \text{if } (x,y) \in R_1 \\ (x-2y, x) & \text{if } (x,y) \in R_2 \\ (-y, 2x-y) & \text{if } (x,y) \in R_3 \\ (-2x-y, -y) & \text{if } (x,y) \in R_4 \\ (-x, -x-2y) & \text{if } (x,y) \in R_5 \\ (-x+2y, -x) & \text{if } (x,y) \in R_6 \\ (y, -2x+y) & \text{if } (x,y) \in R_7 \\ (2x+y, y) & \text{if } (x,y) \in R_8 \end{cases} \tag{3.1}$$

We prove that at every point $x \neq 0$ the local Lipschitz and local co-Lipschitz constants of this mapping are equal to 3 and 1, respectively. The following lemma is very useful.

Lemma 3.1.2. *Let $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ be defined by $f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$, with $a, b, c, d \in \mathbb{R}$ and assume that $\{(a, b), (a, c), (c, d), (b, d)\} \cap \{(0, 0)\} = \emptyset$ (so that f is bijective). Then f is a Lipschitz quotient mapping under the supremum norm with*

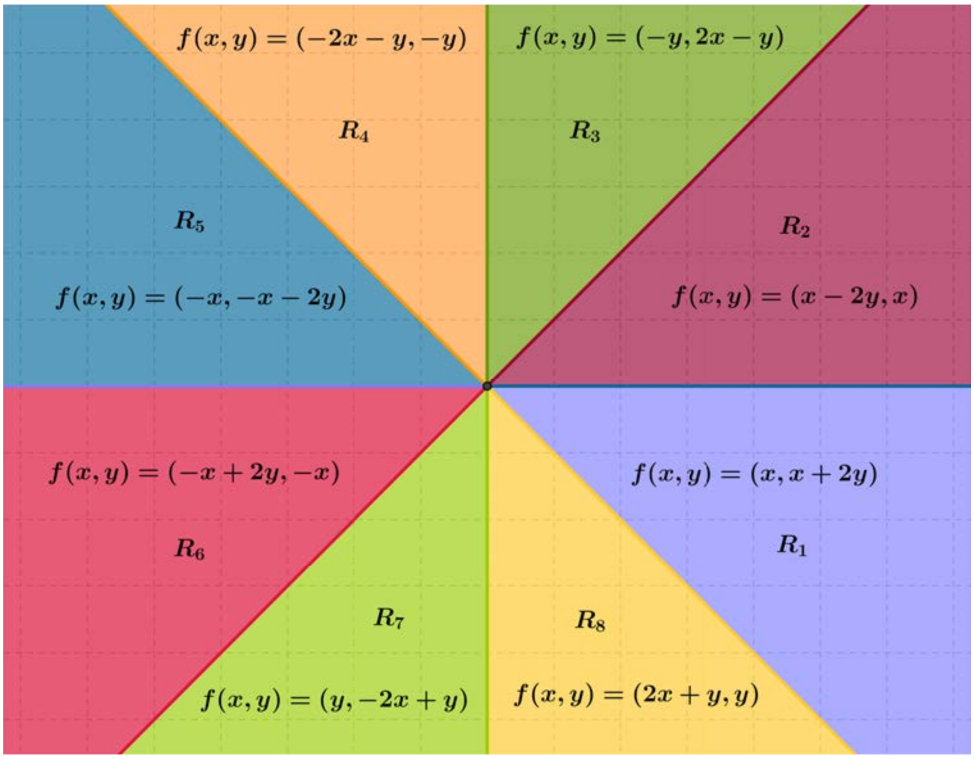


Figure 3.1: Lipschitz quotient mapping which "fixes" the squares

Lipschitz constant equal to $\max\{|a|+|b|,|c|+|d|\}$ *and co-Lipschitz constant equal to* $\min\left\{\frac{|bc-ad|}{|a|+|c|},\frac{|bc-ad|}{|d|+|b|}\right\}$.

Proof. Let f be as in the hypothesis. First notice that since f is linear, for any $r,L>0$ we have:

$$f(B_r(x))\subseteq B_{Lr}(f(x)) \text{ if and only if } f(B_1(0))\subseteq B_L(0). \tag{3.2}$$

For, assume that $f(B_1(0))\subseteq B_L(0)$ and take $y\in B_r(x)$, then $\frac{1}{r}(y-x)\in B_1(0)$, and so $\frac{1}{r}(f(y)-f(x))=f(\frac{1}{r}(y-x))\in B_L(0)$. We conclude that $f(y)\in B_{Lr}(f(x))$, so $f(B_r(x))\subseteq B_{Lr}(f(x))$.

Let L denote the Lipschitz constant of f, from (3.2) it follows that

$$\begin{aligned} L &= \inf\{L' > 0 : f(B_r(x)) \subseteq B_{L'r}(f(x)) \forall x \in \mathbb{R}^2, r > 0\} \\ &= \inf\{L' > 0 : f(B_1(0)) \subseteq B_{L'}(0)\} \\ &= \inf\{L' > 0 : f(B_1(0)) \subseteq L'B_1(0)\} = \|f\|_{op}, \end{aligned}$$

where $\|\cdot\|_{op}$ denotes the operator norm on linear maps from $(\mathbb{R}^2, \|\cdot\|_\infty)$ to itself.

Now, f is bijective and the inverse of f, being linear, is L_*-Lipschitz for some $L_* > 0$. Therefore, from the argument above and Corollary 1.2.14 it follows that f is co-Lipschitz and that the co-Lipschitz constant of f, c, is given by $c = 1/L_* = 1/\|f^{-1}\|_{op}$.

Finally notice that in this case the operator norm of f is the ∞-norm of the matrix determined by f, hence $\|f\|_{op} = \max\{|a| + |b|, |c| + |d|\}$. Similarly, we can see that $\|f^{-1}\|_{op} = \max\left\{\frac{|a|+|c|}{|bc-ad|}, \frac{|d|+|b|}{|bc-ad|}\right\} = \min\left\{\frac{|bc-ad|}{|a|+|c|}\frac{|d|+|b|}{|d|+|b|}\right\}$. Which is what we wanted to prove. $\square$

Proposition 3.1.3. *The function $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ defined in Example 3.1.1 is a Lipschitz quotient mapping with Lipschitz constant equal to* 3 *and co-Lipschitz constant equal to* 1*.*

Proof. Denote by $\mathcal{L}_i$ the different rays that define the boundary of each region R_i by $\mathcal{L}_i = \overline{R}_i \cap \overline{R}_{i+1}$ for $i \in \{1, \cdots, 7\}$ and $\mathcal{L}_8 = R_8 \cap R_1$ (see Figure 3.1 for the picture of R_i). Let us agree that the function f is defined by $f(x_1, x_2) = (a_i x_1 + b_i x_2, c_i x_1 + d_i x_2)$ if $x = (x_1, x_2) \in R_i$, so a_i, b_i, c_i, d_i denote the relevant coefficients in (3.1) that define the function f on the region R_i.

Notice that from Lemma 3.1.2 it follows that f is locally Lipschitz and locally co-Lipschitz at any point in the interior of the region R_i, for any $i = 1, \ldots, 8$. Furthermore, for $x \in \mathrm{Int}(R_i)$ the local Lipschitz constant, L_x, and the local co-Lipschitz constant, c_x, at the point x, satisfy:

$$L_x \leq \sup\{|a_i| + |b_i|, |c_i| + |d_i|\} = 3 \tag{3.3}$$

$$c_x \geq \min\left\{\frac{|b_i c_i - a_i d_i|}{|a_i| + |c_i|}, \frac{|b_i c_i - a_i d_i|}{|d_i| + |b_i|}\right\} = 1 \tag{3.4}$$

We can easily see that inequality (3.3) remains true for all $x \in \mathbb{R}^2$. Indeed, consider a point $x = (x_1, x_2) \in \mathcal{L}_i$ and assume first that $x \neq 0$. From the definition of f and $\mathcal{L}_i$, we can see that:

$$f(x) = (a_i x_1 + b_i x_2, c_i x_1 + d_i x_2) = (a_{i+1} x_1 + b_{i+1} x_2, c_{i+1} x_1 + d_{i+1} x_2),$$

so that, if $y \in \mathbb{R}^2$ is such that $\|y - x\|_\infty < \frac{1}{2}\|x\|_\infty$, it does not matter if the point y belongs to R_i or to R_{i+1}, we will always have $\|f(x) - f(y)\|_\infty \leq 3\|x - y\|_\infty$. We conclude that for all $x \in \mathbb{R}^2 \setminus \{0\}$ the local Lipschitz constant satisfies $L_x \leq 3$. However, it is clear that for $x = 0$ this inequality is satisfied as well; in fact since $\|f(y)\|_\infty = \|y\|_\infty$ for all $y \in \mathbb{R}^2$ we see that the local Lipschitz constant of f at zero is equal to 1. Hence, from Lemma 3.1.2, it follows that f is a Lipschitz mapping with Lipschitz constant less than or equal to 3.

We are now left to show that f is 1-co-Lipschitz at the points belonging to $\mathcal{L}_i$ for $i \in \{1, \cdots, 8\}$. For $x = 0$ this is clear since for all $r > 0$, f maps balls of radius r centred at zero, to balls of radius r centred at zero. Therefore f is locally co-Lipschitz at $x = 0$ and the co-Lipschitz constant at zero is equal to 1. Now, to prove this for $x \neq 0$ we divide the proof into eight cases depending on the ray $\mathcal{L}_i$ which the point x belongs to. We will only deal here with one case, $i = 6$, all other cases can be carried out in a similar fashion.

Case 1: $x \in \mathcal{L}_6 = \{(t, t) : t \leq 0\}$. We need to show that there exists a constant r_x such that for all $r < r_x$ we have $B_r(x) \subseteq f(B_r(x))$. For this, divide the plane (the co-domain of f) in 4 regions $\mathfrak{R}_j$ defined as $\mathfrak{R}_j := (R_{2j-1} \cup R_{2j})$ for $j \in \{1, 2, 3, 4\}$. Notice that $\mathbb{R}^2 = \cup_{j=1}^4 \mathfrak{R}_j$, so if we prove that there is a positive constant r_x such that for all

$r < r_x$ we have

$$B_r(f(x)) \cap \Re_j \subseteq f(B_r(x)), \quad \text{for all} \quad j \in \{1,2,3,4\}. \tag{3.5}$$

then f is locally co-Lipschitz at x with co-Lipschitz constant less than or equal to 1. As we will show now, this is true if we set $r_x := \frac{1}{2}\|x\|_\infty$.

First we prove (3.5) for $j = 2$. Let $r \leq r_x = \frac{1}{2}\|x\|_\infty$. Take $w \in B_r(f(x)) \cap \Re_j$ and consider the point $y = (y_1, y_2)$ defined by $y_1 = -w_2$, $y_2 = \frac{1}{2}(w_1 - w_2)$. We will see that $w \in f(B_r(x))$ by showing that $\|y - x\|_\infty < r$ and that $w = f(y)$.

Since $x = (x_1, x_2) \in \mathcal{L}_6$, we have $x_1 = x_2$ and $f(x) = (x_1, -x_1)$, so that:

$$\|y - x\|_\infty = \sup\{|w_2 + x_1|, |\tfrac{1}{2}(w_1 - w_2) - x_1|\},$$

and we have:

$$\begin{aligned}
|w_2 + x_1| &\leq \sup\{|w_1 - x_1|, |w_2 + x_1|\} = \|w - f(x)\|_\infty < r. \\
|\tfrac{1}{2}(w_1 - w_2) - x_1| &\leq \tfrac{1}{2}(|w_1 - x_1| + |-w_2 - x_1|) \\
&\leq \sup\{|w_1 - x_1|, |w_2 + x_1|\} = \|w - f(x)\|_\infty < r.
\end{aligned}$$

Thus $y \in B_r(x)$.

Now, to show that $f(y) = w$, notice that the hypothesis $w \in \Re_2$ leads to $-w_2 \leq w_1 < w_2$, hence:

$$y_2 = \tfrac{1}{2}(w_1 - w_2) < 0$$

$$-y_1 = w_2 \geq |\tfrac{1}{2}(w_1 - w_2)| = |y_2|$$

Thus, $y \in R_6$, therefore, recalling (3.1): $f(y) = (-y_1 + 2y_2, -y_1) = (w_1, w_2)$. We conclude that $y \in f^{-1}(w) \cap B_r(x)$. Therefore (3.5) is true for $j = 2$.

Now, for $j = 3$, take $w \in \mathfrak{R}_3 = R_5 \cup R_6$ and consider the point $y = (y_1, y_2)$ given by $y_2 = w_1$, $y_1 = \frac{1}{2}(w_1 - w_2)$. Then:

$$\|y - x\|_\infty = \sup\{|\tfrac{1}{2}(w_1 - w_2) - x_1, |w_1 - x_1||\}.$$

Since $f(x_1, x_2) = (x_1, -x_1)$ we have:

$$\begin{aligned} |w_1 - x_1| \leq \sup\{|w_1 - x_1|, |w_2 + x_1|\} = \|w - f(x)\|_\infty < r. \\ |\tfrac{1}{2}(w_1 - w_2) - x_1| \leq \tfrac{1}{2}(|w_1 - x_1| + |w_2 + x_1|) \\ \leq \sup\{|w_1 - x_1|, |w_2 + x_1|\} = \|w - f(x)\|_\infty < r. \end{aligned}$$

Thus $\|y - x\|_\infty < r$, i.e. $y \in B_r(x)$.

To show that $f(y) = w$ notice that the hypothesis $w \in \mathfrak{R}_3$ implies $w_1 \leq w_2 < -w_1$, then:

$$y_2 - y_1 = \tfrac{1}{2}(w_1 + w_2) < 0 \text{ and}$$
$$y_1 = \tfrac{1}{2}(w_1 - w_2) \text{ and } y_2 = w_1 < 0$$

Thus, $y \in R_7$, therefore $f(y) = (y_2, -2y_1 + y_2) = (w_1, w_2)$, and so (3.5) is satisfied for $j = 3$.

Finally, for $j \in \{1, 4\}$, notice that since $x \in \mathcal{L}_6$, we have $x_1 = x_2 = -\|x\|_\infty$ and $f(x) = (x_1, -x_1)$. This, together with the fact that $\|w - f(x)\|_\infty < r \leq \frac{1}{2}\|x\|_\infty$ for any $w \in B_r(f(x))$ leads to:

$$|w_1 - x_1| \leq \|w - f(x)\|_\infty < -\tfrac{1}{2}x_1 \Rightarrow w_1 < \tfrac{1}{2}x_1 < 0;$$
$$|w_2 + x_1| \leq \|w - f(x)\|_\infty < -\tfrac{1}{2}x_1 \Rightarrow w_2 > -\tfrac{1}{2}x_1 > 0.$$

On the other hand we have:

$$\mathfrak{R}_1 = R_1 \cup R_2 \subseteq \{(z_1, z_2) \in \mathbb{R}^2 : z_1 > 0\};$$
$$\mathfrak{R}_4 = R_7 \cup R_8 \subseteq \{(z_1, z_2) \in \mathbb{R}^2 : z_2 < 0\}.$$

This means that, for $r < R_x$, $B_r(f(x)) \cap \mathfrak{R}_{\mathrm{j}} = \emptyset$, for $j \in \{1, 4\}$, therefore (3.5) is true for $j = 1, 4$.

We conclude that for all $r < \frac{1}{2}\|x\|_\infty$, $B_r(f(x)) \cap \mathfrak{R}_{\mathrm{j}} \subseteq f(B_r(x))$, for all $j \in \{1, 2, 3, 4\}$; thus f is locally co-Lipschitz at every point $x \in \mathcal{L}_6$ and the co-Lipschitz constant, c_x, at the point x, satisfies $c_x \geq 1$.

The proof of the remaining cases, $x \in R_i$, $i \neq 6$ can be carried out in the same way. So we conclude that f is locally co-Lipschitz at every point $x \in \mathbb{R}^2$ and that the local co-Lipschitz constant, c_x, of f at x satisfies $c_x \geq 1$.

Summing up, we have shown that the mapping f is a locally Lipschitz quotient mapping on the plane, and thus, by Propositions 1.2.6 and 1.2.7, a Lipschitz quotient mapping. We have also shown that for every x in the plane the local Lipschitz and co-Lipschitz constants at the point x satisfy $L_x \leq 3$ and $c_x \geq 1$, so, if L and c denote the Lipschitz and co-Lipschitz constants of the map f, we must have $L \leq 3$ and $c \geq 1$.

Since f fixes the norm of each point on the plane, it is easy to see that the co-Lipschitz constant of f is also less than or equal to 1. Indeed, since $f(B_r(0)) = B_r(0)$ for all $r > 0$, then $f(B_r(0))$ does not contain any ball centred at $f(0) = 0$ with radius bigger that r. Hence $c \leq 1$ and we conclude that $c = 1$.

On the other hand, it is also easy to see that $L = 3$, because for each $x \in \mathbb{R}^2 \setminus \{0\}$ there exists $y \in \mathbb{R}^2$ such that $\|f(x) - f(y)\|_\infty = 3\|x - y\|_\infty$; for example, in R_1 if $x = (x_1, x_2)$,

$y = (y_1, y_2)$, with $x \neq y$ and we take $x_1 - y_1 = x_2 - y_2$ then:

$$\begin{aligned} \|f(x) - f(y)\|_\infty &= \|x_1 - y_1, x_1 - y_1 + 2(x_2 - y_2)\|_\infty \\ &= \sup\{|x_1 - y_1|, 3|x_1 - y_1|\} \\ &= 3\|x - y\|_\infty. \end{aligned}$$

Thus f is a Lipschitz quotient mapping with ratio of constants equal to $\frac{1}{3}$.

Notice that in each region R_i there is always a direction $(\pm 1, \pm 1)$ in which the Lipschitz constant is equal to 3. More precisely, given $x \in \mathbb{R}^2 \setminus \{0\}$ we can always find a direction $v \in (\pm 1, \pm 1)$ such that for all y in the same region as x with $y = (\lambda v x)$ for some $\lambda \in \mathbb{R}$, we have $\|f(x) - f(y)\|_\infty = 3\|x - y\|_\infty$. Furthermore, in each region, one of the coordinate functions that define f is given by $(x_1, x_2) \mapsto \pm x_i$ with $i \in \{1, 2\}$, and it is clear that the co-Lipschitz constant of f must be smaller than or equal to the co-Lipschitz constant of each of its coordinate functions, thus $c = 1$. So, indeed, we have shown that the local Lipschitz and co-Lipschitz constants of f at any non-zero point $x \in \mathbb{R}^2$ are equal to 3 and 1 respectively. □

3.2 Two-fold Lipschitz quotient mappings cannot have ratio of constants equal to $^1/_2$

In the second part of this chapter we will show that it is not possible for a Lipschitz quotient mapping from the plane endowed with the supremum norm, to itself, to have $\max \# f^{-1}(p) = 2$ and ratio of constants equal to $^1/_2$.

First, we are going to prove that a Lipschitz quotient mapping satisfying these two properties cannot send the corners of the squares centred at the origin far from the lines $y = x$ and $y = -x$. Later on, we will generalise this property for the case of norms whose unit ball is a regular polygon consisting of $4m$ sides, and prove a similar statement in

Proposition 4.2.9 for $m > 1$.

Throughout this section for any pair of subsets $A, B \subseteq \mathbb{R}^2$ the notation $\operatorname{dist}(A, B)$ will mean the distance between the sets A and B under the supremum norm, this is:

$$\operatorname{dist}(A, B) = \operatorname{dist}_\infty(A, B) = \inf\{\|a - b\|_\infty : a \in A, b \in B\}.$$

Proposition 3.2.1. *Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a 2-fold L-Lipschitz and c-co-Lipschitz mapping with respect to the supremum norm $\|\cdot\|_\infty$.*

If $c/L = \frac{1}{2}$ then there exist constants κ and R' such that for all $\rho \geq R'$ if $p \in \mathcal{D}_\rho$ then $\operatorname{dist}(g(p), \mathcal{D}_{c\rho}) < \kappa$, where $\mathcal{D}_\rho$ is defined as

$$\mathcal{D}_\rho = \{(x, y) \in \mathbb{R}^2 : |x| = |y| > \rho\}. \tag{3.6}$$

Proof. Assume first that g is a Lipschitz quotient mapping with Lipschitz constant $L = 1$ and co-Lipschitz constant $c = 1/2$ that maps zero to zero.

Take M and R' as in statement (1) of Theorem 2.6. Let $p_0 = (x_0, y_0) \in \mathcal{D}_\rho$, where $\rho > R'$, we set $r := \|p_0\|_\infty > \rho$ and $a := \operatorname{dist}_\infty(g(p_0), \mathcal{D}_{c\rho})$; assume $a > 0$. Take the points $p_1 = (x_0 - a_x, y_0)$ and $p_2 = (x_0, y_0 - a_y)$, where $|a_x| = |a_y| = a$ and a_x, a_y have the same sign as x_0 and y_0, respectively. Notice that $p_1, p_2 \in \partial B_r(0)$, indeed, by the Lipschitz property, we have $\|g(p_0)\|_\infty = \|g(p_0) - g(0)\|_\infty \leq \|p_0\|_\infty = r$, thus the distance between $g(p_0)$ and some corner of the square $\partial B_{cr}(0)$ is less than or equal to $r - cr$, therefore $a \leq r - cr = r/2$. Consequently $\|p_1\|_\infty = \|p_0\|_\infty = \|p_2\|_\infty = r$.

Consider the set $\mathcal{D}_0 = \{(x, y) \in \mathbb{R}^2 : |x| = |y| > 0\}$ and let $R_1, \ldots, R_4$ be the closure of each of the connected components of $\mathbb{R}^2 \setminus (\mathcal{D}_0 \cup \{(0, 0)\}$. Let us agree that R_1 is the region that contains the point $(0, 1)$ and that the remaining indices are placed counterclockwise.

Note that:

$$\|g(p_0) - g(p_i)\|_\infty \le \|p_0 - p_i\|_\infty = a = \operatorname{dist}(g(p_0), \mathcal{D}_{r/2}), \quad \text{for} \;\; i \in \{1, 2\},$$

so all the three points $g(p_i)$, $i \in \{0, 1, 2\}$, are in one of the four regions, say in R_ξ. It follows from Corollary 1.3.11 that $\mathcal{H}_1(\partial B_r(0)) = 8r$, so we can let $\gamma : [0, 8r] \to \partial B_r(0)$ be the 1-Lipschitz parametrization of $\partial B_r(0)$ given by Corollary 1.3.18, starting at p_1 so that $\gamma(t_i) = p_i$ where $t_1 = 0$, $t_0 = a$ and $t_2 = 2a$. Then, by Theorem 2.6, the curve $g \circ \gamma$ is a curve contained outside $B_{c(r-M)}(0)$ with $\operatorname{Ind}_0 g \circ \gamma = 2$.

Now, let $q_1 := g(\gamma(t_1)) = g(p_1)$ and $q_2 := g(\gamma(t_2)) = g(p_2)$. From Lemma 1.3.14 we infer that:

$$\|q_1 - q_2\|_\infty + \operatorname{length}_\infty(g \circ \gamma|_{[t_2, 8r]}) \ge (2)(8)(\tfrac{1}{2}(r - M)) = 8r - 8M.$$

In addition, since g is 1-Lipschitz we have:

$$\|q_1 - q_2\|_\infty = \|g(\gamma(t_1)) - g(\gamma(t_2))\|_\infty \le \|\gamma(t_1) - \gamma(t_2)\|_\infty = a.$$

Hence, $\operatorname{length}_\infty(g \circ \gamma|_{[t_2, 8r]}) \ge 8r - 8M - a$.

On the other hand, since g and γ are 1-Lipschitz, the argument from Remark 1.3.9 leads to:

$$\operatorname{length}_\infty(g \circ \gamma|_{[t_2, 8r]}) \le 8r - 2a.$$

So we conclude that $8r - 2a \ge 8r - 8M - a$, i.e. $a \le 8M$.

Thus, the conclusion of the present lemma is satisfied if we set $\kappa > 8M$.

Finally, consider any L-Lipschitz and c-co-Lipschitz mapping g such that $c/L = 1/2$ and assume $\max \# g^{-1}(p) = 2$. We know that for the mapping $g_1 := \frac{1}{L}(g - g(0))$ there are constants R_1' and κ_1 such that $\operatorname{dist}(g_1(p), \mathcal{D}_{(c/L)r}) < \kappa_1$, for all $p \in \mathcal{D}_r$ with $r > R'$.

Then it is clear that the constants $R' := R'_1$ and $\kappa := L\kappa_1 + \|g(0)\|_\infty$, would work for the mapping g. $\square$

Now that we know that a 2-fold Lipschitz quotient f, with ratio of constants $c/L = 1/2$ must map corners of squares "close to corners", we can actually say something more about the behaviour of a function f satisfying the conditions of Proposition 3.2.1. As the next corollary shows, such a mapping f should also map the corners of squares close to corners of squares in a "certain order". In this sense, the behaviour of such a function is very similar to the one of the 2-fold mapping defined in Example 3.1.1. Before stating this result, we introduce new notation.

Definition 3.2.2. For $\rho > 0$ and $i \in \{0, 1, 2, 3\}$, we define the following sets.

We will denote by $\mathcal{L}_i^\rho$, the different components of the set $\mathcal{D}_\rho$ defined by (3.6), in the following way:

$$\mathcal{L}_0^\rho = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = -x_2 \geq \rho\} \qquad \mathcal{L}_1^\rho = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 \geq \rho\}$$
$$\mathcal{L}_2^\rho = \{(x_1, x_2) \in \mathbb{R}^2 : -x_1 = x_2 \geq \rho\} \qquad \mathcal{L}_3^\rho = \{(x_1, x_2) \in \mathbb{R}^2 : -x_1 = -x_2 \geq \rho\}.$$

We will also denote by P_i^ρ, the corners of the square of radius ρ centred at the origin (in the supremum norm sense), starting with the bottom right corner and placing the indices counterclockwise and starting with the bottom right corner.

Finally, for any given $\varepsilon > 0$ we define the region $\mathcal{R}_i^\rho(\varepsilon) \subseteq \mathbb{R}^2$ as:

$$\mathcal{R}_i^\rho(\varepsilon) = \{x \in \mathbb{R}^2 \ : \ \operatorname{dist}(x, \mathcal{L}_i^\rho) < \varepsilon\}, \ \ i = 0, 1, 2, 3. \tag{3.7}$$

The following statement, proved for the supremum norm, is later proved for the case of polygonal n-norm with $n = 4m$, in Lemma 4.2.10.

Lemma 3.2.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a 2-fold L-Lipschitz and c-co-Lipschitz mapping with respect to the supremum norm.*

If $c/L = \frac{1}{2}$, then there exist $j_0 \in \{0, \ldots, 3\}$ and constants κ and R' such that for all $\rho \geq R'$ we have that $\operatorname{dist}(f(P_i^\rho), \mathcal{L}_{j(i)}^{c\rho}) < \kappa$, where $j(i) = (2i + j_0) \mod 4$; in other words:

$$f(P_i^\rho) \in \mathcal{R}_{j(i)}^{c\rho}(\kappa) \ \ \textit{where } j(i) = 2i + j_0 \mod 4.$$

Proof. Let κ and R' be as is Proposition 3.2.1, and set the constants:

$$\varepsilon \in (0, \tfrac{1}{7}) \ \text{ and } \ R > \max\left\{R', R'_\varepsilon, \tfrac{\kappa}{c\varepsilon}\right\},$$

where R'_ε is as in Theorem 2.6.

Let $\rho > R$, $\rho' = (1 - \varepsilon)\rho$ and consider the regions $\mathcal{R}_i^{c\rho}(\kappa)$ defined by (3.7). Notice that, since $\rho > R > \kappa/c\varepsilon$, we have $c\rho' = c(1 - \varepsilon)\rho < c\rho - \kappa$, therefore, for all i, the region $\mathcal{R}_i^{c\rho}(\kappa)$ is contained outside the square $B_{c\rho'}(0)$, see Figure 3.2. Hence:

$$\operatorname{dist}_\infty(\mathcal{R}_i^{c\rho}(\kappa), \mathcal{R}_{i+1}^{c\rho}(\kappa)) > \operatorname{dist}_\infty(\mathcal{R}_i^{c\rho}(\kappa), \mathcal{L}_{i+1}^{\rho'}) > \operatorname{dist}_\infty(\mathcal{L}_i^{c\rho'}(\kappa), \mathcal{L}_{i+1}^{\rho'}) = 2c\rho'. \tag{3.8}$$

We have shown in Proposition 3.2.1 that for $i \in \{0, 1, 2, 3\}$ we have $f(P_i^\rho) \in \mathcal{R}_j^{c\rho}(\kappa)$ for some $j \in \{0, \ldots, 3\}$. Let j_0 be the index of the region that contains $f(P_0^\rho)$, i.e $f(P_0^\rho) \in \mathcal{R}_{j_0}^{c\rho}(\kappa)$. For simplicity of notation let us assume that $j_0 = 0$, we will show that in this case we actually have $f(P_i^\rho) \in \mathcal{R}_{(2i) \mod 4}^{c\rho}(\kappa)$ for all $i \in \{0, 1, 2, 3\}$. Assume on the contrary that $f(P_1^\rho) \in R_1^{c\rho}$, for example.

From Corollary 1.3.18, we can consider the 1-Lipschitz parametrization of $\partial B_\rho(0)$ with starting point at P_0, say $\gamma : [0, 8\rho] \to \partial B_\rho(0)$, given by (1.16). It is easy to see that $\gamma(2\rho) = P_1^\rho$, therefore $f(\gamma(2\rho)) = f(P_1^\rho) \in \mathcal{R}_1^{c\rho}(\kappa)$. We will consider the two pieces γ_1

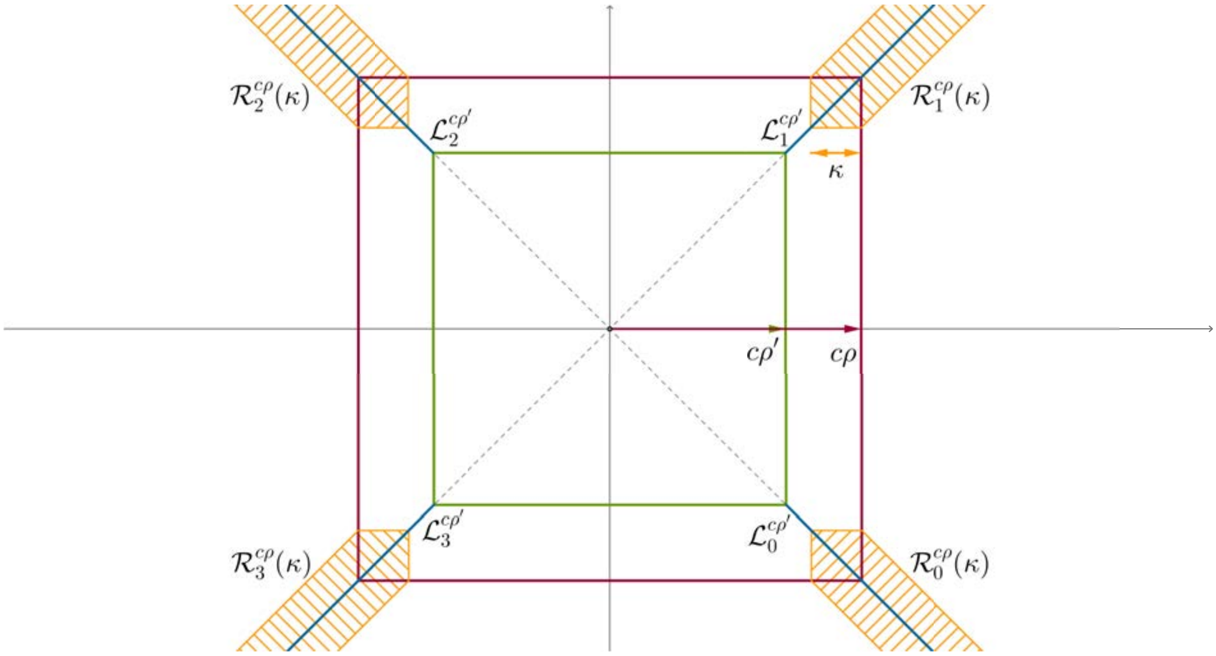


Figure 3.2

and γ_2 of γ given by

$$\gamma_1 = \gamma|_{[0,2\rho]} \text{ and } \gamma_2 = \gamma|_{[2\rho,8\rho]}.$$

From Theorem 2.6, we know that $f(\gamma) \subseteq \mathbb{R}^2 \setminus B_{c\rho'}(0)$ and $\operatorname{Ind}_0 \gamma = 2$. Therefore either γ_1 or γ_2 has index at least 1 around the origin.

Assume first that the curve $f \circ \gamma$ is oriented counterclockwise.

If γ_2 has index at least 1 around the origin, then since we are assuming that $f(\gamma(2\rho)) \in \mathcal{R}_1^{c\rho}(\kappa)$, and $f(\gamma(0)) \in \mathcal{R}_0^{c\rho}(\kappa)$, the curve $f \circ \gamma_2$ must go from $\mathcal{R}_1^{c\rho}(\kappa)$ to $\mathcal{R}_0^{c\rho}(\kappa)$ in the counterclockwise direction outside the square $\partial B_{\rho'}(0)$, plus one complete turn around $\partial B_{\rho'}(0)$ hence, recalling (3.8):

$$\begin{aligned} \operatorname{length}_\infty(f \circ \gamma_2) \geq\, & \operatorname{dist}_\infty(\mathcal{R}_1^{c\rho}(\kappa), \mathcal{L}_2^{c\rho'}) + \operatorname{dist}_\infty(\mathcal{L}_2^{c\rho'}, \mathcal{L}_3^{c\rho'}) + \operatorname{dist}_\infty(\mathcal{L}_3^{c\rho'}, \mathcal{R}_0^{c\rho}(\kappa)) \\ & + \operatorname{length}_\infty(\partial B_{c\rho'}(0) > 6c\rho' + 8c\rho' = 14c\rho'. \end{aligned} \tag{3.9}$$

On the other hand, since f is L-Lipschitz and γ_2 is 1-Lipschitz, by Remark 1.3.9, we have:

$$\text{length}_\infty f \circ \gamma_2 \leq L(8\rho - 2\rho) = L6\rho. \tag{3.10}$$

From (3.9) and (3.10) we have $14c(1-\varepsilon)\rho < L6\rho$, therefore:

$$\frac{c}{L} \leq \frac{6}{14(1-\varepsilon)} < \frac{6}{14(1-\frac{1}{7})} = \frac{1}{2}.$$

But we are assuming $c/L = 1/2$, thus in this case $f(P_1^\rho) \notin \mathcal{R}_1^{c\rho}(\kappa)$.

Now, if γ_1 has index at least 1 around the origin, following the same idea we get:

$$\begin{aligned} \text{length}_\infty(f \circ \gamma_1) \geq & \text{length}_\infty(\partial B_{c\rho'}(0)) + \text{dist}_\infty(\mathcal{R}_0^{c\rho}(\kappa), \mathcal{R}_1^{c\rho}(\kappa) \\ > & 8c\rho' + 2c\rho' = 10c\rho'. \end{aligned} \tag{3.11}$$

From the Lipschitz condition we have:

$$\text{length}_\infty f \circ \gamma_1 \leq L(2\rho - 0) = L2\rho. \tag{3.12}$$

So now from equations (3.11) and (3.12) we get

$$\frac{c}{L} \leq \frac{1}{5(1-\varepsilon)} < \frac{1}{5(1-\frac{1}{7})} < \frac{1}{2}.$$

This is again a contradiction, so we conclude that in any case $f(P_1^\rho) \notin \mathcal{R}_1^{c\rho}$.

Now, following similar ideas, we prove that $f(P_1^\rho) \notin \mathcal{R}_i^{c\rho}(\kappa)$, for $i \in \{0, 3\}$. Indeed, notice that if $f \circ \gamma_1$ goes from $f(\gamma_1(0))$ to $f(\gamma_1(2\rho))$ in the counterclockwise direction around $\partial B_{c\rho'}(0)$, then the same idea of the previous case would lead to:

$$6c\rho' < \text{length}_\infty(f \circ \gamma_1) \leq 2L\rho$$

and we get again:

$$\frac{c}{L} < \frac{2}{6(1-\varepsilon)} < \frac{1}{3(1-\frac{1}{7})} < \frac{1}{2},$$

which is a contradiction. The other option is that $f \circ \gamma_1$ goes from $f(\gamma_1(0))$ to $f(\gamma_1(2\rho))$ in the clockwise direction first and then —since we are assuming that $f \circ \gamma$ is oriented in the counterclockwise direction— from $\mathcal{L}_0^{c\rho'}$ complete two entire turns around $\partial B_{c\rho'}(0)$. In this case we would have:

$$16c\rho' < \text{length}_\infty(f \circ \gamma_2) \leq L6\rho.$$

$$\frac{c}{L} < \frac{6}{16(1-\varepsilon)} < \frac{3}{8(1-\frac{1}{7})} < \frac{1}{2}.$$

Since this is impossible we conclude that $f(P_1^\rho) \in \mathcal{R}_2^{c\rho}$, whenever $f \circ \gamma$ is oriented in the counterclockwise direction. However, if $f \circ \gamma$ is oriented clockwise, and we assume that $f(P_1^\rho) \in \mathcal{R}_i^{c\rho}(\kappa)$ with $i \neq 2$ then we will get the same contradictory inequalities that we found for the case $f \circ \gamma$ oriented counterclockwise and $f(P_i^\rho) \in \mathcal{R}_{4-i}^{c\rho}(\kappa)$. Thus in any case we have $f(P_1^\rho) \in \mathcal{R}_2^{c\rho}(\kappa)$.

We can follow the same argument for $i = 2$ and $i = 3$, to get $f(P_2^\rho) \in \mathcal{R}_0^{c\rho}(\kappa)$ and $f(P_3^\rho) \in \mathcal{R}_2^{c\rho}(\kappa)$. So we conclude that whenever $j_0 = 0$ we have:

$$f(P_i^\rho) \in \mathcal{R}_{(2i) \mod 4}^{c\rho}(\kappa) \text{ for all } i \in \{0,1,2,3,4\}.$$

Furthermore, by continuity, we can assure that this remains true for all $\rho > R'$. So this finishes the proof for $j_0 = 0$.

It is clear that for $j_0 > 0$ we only need to perform a rotation of $-j_0\pi/4$ and then back, so this last rotation will add j_0 to the index of the region $\mathcal{R}_j^{c\rho}(\kappa)$, therefore $f(P_i^\rho) \in \mathcal{R}_{(2i)+j_0 \mod 4}^{c\rho}(\kappa)$ for all $\rho > R'$. □

Proposition 3.2.4. *Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a 2-fold L-Lipschitz and c-co-Lipschitz mapping* *with respect to the supremum norm. Assume further that for all* $\varepsilon > 0$ *and for all* $M > 0$ *there is a* $\rho > M$ *such that:*

i) $|\arg(g(P_i^\rho)) + \pi/4| < \varepsilon, \quad i \in \{0, 2\}.$

ii) $|\arg(g(P_i^\rho)) - 3\pi/4| < \varepsilon, \quad i \in \{1, 3\}.$

Then $c/L \leq 1/3$*. (Here,* P_i^ρ *is as in Definition 3.2.2.)*

Proof. As before assume that $L = 1$ and $g(0) = 0$. We must show that $c \leq 1/3$. Let $\varepsilon > 0$ and take the relevant R'_ε as in Theorem 2.6. By hypothesis we can pick some $\rho > R'_\varepsilon$ which satisfies the conditions (i) and (ii) of the present proposition. Notice that Theorem 2.6 implies that for $i \in \{0, 1, 2, 3\}$ we have $\|g(P_i^\rho)\|_\infty \geq c(1 - \varepsilon)\rho$.

Let $\rho' = (1 - \varepsilon)\rho$ and consider the square $\partial B_{2\rho'}(0)$. By the co-Lipschitz property of g, we know that $g(\overline{B}_{2\rho'}(0)) \supseteq \overline{B}_{2c\rho'}(0)$. Since $P_1^{2c\rho'} \in \overline{B}_{2c\rho'}(0)$, there is a point $x_0 \in \overline{B}_{2\rho'}(0)$ such that $g(x_0) = P_1^{2c\rho'}$. Notice that, since $x_0 \in \overline{B}_{2\rho'}(0) \subseteq B_{2\rho}(0)$, there is a corner P_k^ρ of the square $\partial B_\rho(0)$ such that $\|x_0 - P_k^\rho\|_\infty \leq \rho$, see Figure 3.3. Hence, by the Lipschitz property, we have:

$$\|P_1^{2c\rho'} - g(P_k^\rho)\|_\infty = \|g(x_0) - g(P_k^\rho)\|_\infty \leq \|x_0 - P_k^\rho\|_\infty \leq \rho. \tag{3.13}$$

Let Q_0 and Q_1 be the intersections between the square $\partial B_{c\rho'}(0)$ and the rays $y = \tan(-\pi/4 + \varepsilon)x$, $y' = \tan(3\pi/4 - \varepsilon)x$ with $x \geq 0$ (see the right hand side of Figure 3.3). Since $g(P_k^\rho)$ satisfies either $i)$ or $ii)$ of the hypothesis, we know that $\|P_1^{2c\rho'} - g(P_k^\rho)\|_\infty \geq \|P_1^{2c\rho'} - Q_0\|_\infty = \|P_1^{2c\rho'} - Q_1\|_\infty$. Then we have:

$$\begin{aligned}
\|P_1^{2c\rho'} - g(P_k^\rho)\|_\infty &\geq \|P_1^{2c\rho'} - Q_0\|_\infty \\
&\geq \|P_1^{2c\rho'} - P_0^{c\rho'}\|_\infty - \|P_0^{c\rho'} - Q_0\|_\infty = 3c\rho' - \|P_0^{c\rho'} - Q_0\|_\infty \\
&= 3c\rho' - c\rho'(1 - \tan(\pi/4 - \varepsilon)) = c\rho'(2 + \tan(\pi/4 - \varepsilon)).
\end{aligned}$$

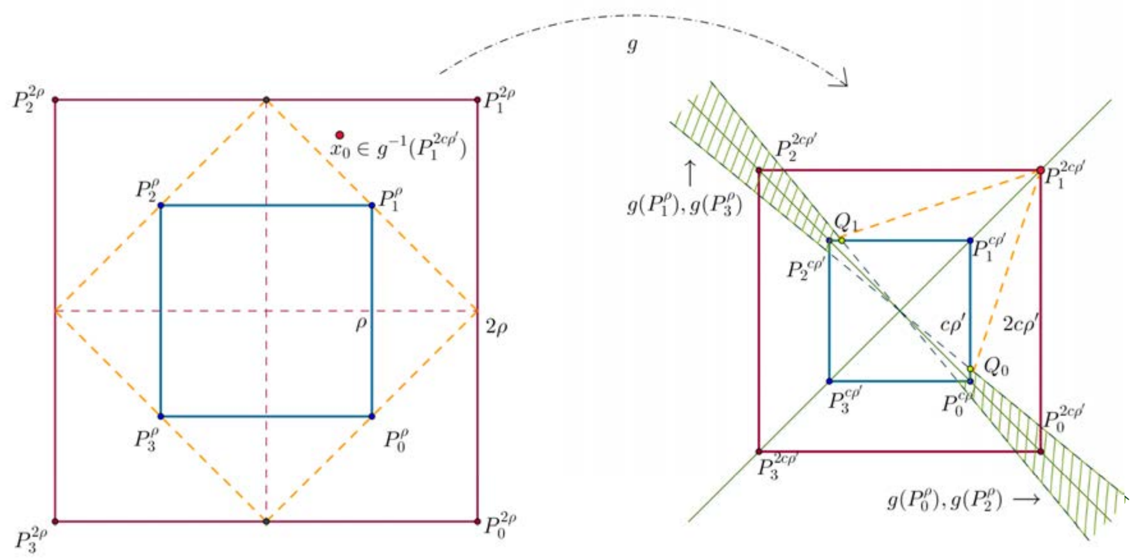


Figure 3.3

Hence, from (3.13), and recalling that $\rho' = (1-\varepsilon)\rho$, we gather that:

$$\rho \geq c(1-\varepsilon)\rho(2 + \tan(\pi/4 - \varepsilon))$$

Thus the co-Lipschitz constant of g satisfies:

$$c \leq \frac{1}{1-\varepsilon} \times \frac{1}{2+\tan(\pi/4-\varepsilon)}. \tag{3.14}$$

Therefore, since this inequality holds for every ε, and

$$\lim_{\varepsilon \to 0} (1-\varepsilon)(2+\tan(\pi/4-\varepsilon)) = 3,$$

we conclude that $c \leq 1/3$. □

We finish this chapter by proving that every 2-fold Lipschitz quotient mapping with Lipschitz and co-Lipschitz constants equal to L and c under the supremum norm, has ratio of constants ${}^{c}/_{L} < {}^{1}/_{2}$ (strictly less than ${}^{1}/_{2}$).

This shows that, even when the bounds —from Corollary 2.8— for the ratio of constants of an n-fold Lipschitz quotient work for any norm on the plane, there are norms for which some of the bounds are not achieved by any n-fold Lipschitz quotient mapping.

Theorem 3.2.5. *If $g : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$, is a 2-fold L-Lipschitz and c-co-Lipschitz mapping then $c/L < 1/2$.*

Proof. Let g be a 2-fold Lipschitz quotient mapping under the supremum norm with Lipschitz and co-Lipschitz constants equal to L and c, respectively. From Theorem 2.7, we know that $c/L \leq 1/2$. Let us assume, for a contradiction, that $c/L = 1/2$. In this case from Lemma 3.2.3 there exist $R' > 0$, $\kappa > 0$ and $j_0 \in \{0, 1, 2, 3\}$ such that

$$g(P_i^\rho) \in \mathcal{R}_{j(i)}^\rho(\kappa) \text{ whenever } \rho > R', \text{ and } j(i) = 2i + j_0 \mod 4. \tag{3.15}$$

Recall that $\mathcal{R}_i^\rho(\kappa)$ is defined by (3.7). It is not hard to see that condition (3.15) implies that g satisfies the conditions of Proposition 3.2.4. Indeed, given $\varepsilon \in (0, \pi/2)$ and $M > 0$, take

$$\rho > \max\{R', \tfrac{\sqrt{2}c\kappa}{\delta}, M\},$$

where $\delta = \tan(\varepsilon)$. Let us assume for simplicity that $j_0 = 0$, so that $g(P_i^\rho) \in \mathcal{R}_{2i}^{c\rho}(\kappa)$. Then, for $i \in \{0, 2\}$ we have $g(P_i^\rho) \in \mathcal{R}_0^{c\rho}(\kappa)$, so we know that there is a point p with $\arg(p) = -\pi/4$ such that $g(P_i^\rho) \in B_\kappa(p)$, therefore:

$$\left|\tan\left(\arg(g(P_i^\rho)) + \tfrac{\pi}{4}\right)\right| = \left|\tan\left(\arg(g(P_i^\rho)) - \arg(p)\right)\right| < \tfrac{\sqrt{2}\kappa}{c\rho} < \delta.$$

The same argument shows that for $i \in \{1, 3\}$ we have:

$$\left|\tan\left(\arg(g(P_i^\rho)) - \tfrac{3\pi}{4}\right)\right| < \delta.$$

Thus, we conclude that for $\rho > M$ we have:

$$|\arg(g(P_i^\rho)) + \pi/4| < \varepsilon,\ i \in \{0, 2\} \quad \text{and} \quad |\arg(g(P_i^\rho)) - \tfrac{3\pi}{4}| < \varepsilon,\ i \in \{1, 3\}.$$

By Proposition 3.2.4 this implies $c/L \leq 1/3$, which is impossible. □

We have shown then, that for the supremum norm there is no 2-fold Lipschitz quotient that achieves the bound $1/2$ of Corollary 2.8. This fact opens new questions.

First, can we find a sharp value ρ_1^∞ such that for any L- Lipschitz and c-co-Lipschitz mapping $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$, the assumption $c/L > \rho_1^\infty$ implies $\#f^{-1}(x) \leq 2$?

Another natural question is whether the Euclidean norm is the only norm on the plane that achieves the bound of Corollary 2.8; or similarly, whether the supremum norm is the only norm that does not always achieve the bounds.

Later on, in Chapter 5, we will work on the first question where we state and prove some results that indicate that ρ_1^∞ should be equal to $1/3$. But first, we devote the next chapter to the research that we have done on the second question.

Chapter 4
Polygonal norms

In the previous chapter, in Theorem 3.2.5, we proved that with respect to the supremum norm $\|\cdot\|_\infty$ on the plane, unlike in the Euclidean case, every 2-fold Lipschitz quotient mapping defined on the plane will have ratio of constants strictly less than $1/2$. A natural question is: Is the supremum norm the only norm on the plane with this property? Or, in the opposite direction, is the Euclidean norm the only norm on the plane such that there is a 2-fold Lipschitz quotient mapping with ratio of constants equal to $1/2$? As we shall prove in this chapter, the answer to both questions is negative. We will find examples of such norms by considering "polygonal norms", i.e. norms whose unit ball is a polygon.

Definition 4.1. For $n \in \mathbb{N}$, n even, let the *n-norm*, denoted by $\|\cdot\|_n$, be the norm in $\mathbb{R}^2$ whose unit ball centred at the origin, $\partial B_1^n(0)$, is the regular n-gon with a vertex at $(0,1)$. In this way the ℓ_1 norm —also known as the rectilinear norm, or the taxicab norm— will be denoted by $\|\cdot\|_4$, for example.

Given a curve γ on $\mathbb{R}^2$ we will use the notation $\text{length}_n(\gamma)$ instead of $\text{length}_{\|\cdot\|_n}(\gamma)$ as defined in Definition 1.3.7.

In a similar way as we did for the supremum norm in Example 3.1.1, we define, for any even $n \in \mathbb{N}$, *the doubling mapping for the n-norm*, which is a two-fold mapping that behaves in an analogous way to the exponential mapping $f(re^{i\theta}) \to re^{i2\theta}$, but relative to the n-norm. We define this mapping in the following way.

Definition 4.2. Let $\mathscr{L}_n := \text{length}_n(\partial B_1^n(0))$, and for each constant $r > 0$ consider the curve $\gamma_r : [0, 2r\mathscr{L}_n] \to \partial B_r^n(0)$ such that:

1. $\operatorname{Ind} \gamma_r(0) = 2$;
2. γ_r is a 1-Lipschitz mapping;
3. $\gamma_r(0) = \gamma_r(r\mathscr{L}_n) = \gamma_r(2r\mathscr{L}_n) = (r, 0)$.

Now consider the doubling mapping $f_n : (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ defined in the following way: given $x \in \mathbb{R}^2$ with $\|x\|_n = r$, take $t_x \in [0, r\mathscr{L}_n)$ such that $\gamma_r(t_x) = x$. Notice that t_x is uniquely defined since γ_r is injective along $[0, r\mathscr{L}_n)$ and $\gamma_r([0, r\mathscr{L}_n)) = \partial B_r^n(0)$. We set $f_n(x) := \gamma_r(2t_x)$.

In what follows, when working under an n-norm and whenever we consider a polygon $\partial B_r^n(0)$ we will denote its vertices by

$$V_0^r, V_1^r, \ldots, V_{n-1}^r, \tag{4.1}$$

starting with the vertex that lies on the positive side of the x-axis and going counterclockwise; sometimes, when the radius r is fixed we will simply denote them by $V_0, V_1, \ldots, V_{n-1}$. In the same way, when we consider polygons $\partial B_r^n(x)$ that are not centred at the origin we will enumerate their vertices starting with the vertex that, when the centre of the polygon is translated to the origin, lies in the positive side of the x-axis and placing the remaining indices counterclockwise. Also, let us agree that whenever we are considering vertices of balls under the n-norm the subindices are understood to be modulous n.

Sometimes we refer to $B_r^n(x)$ and sometimes to $\partial B_r^n(x)$ as polygons. When the norm that we are working with is fixed we may write $B_r(0)$ instead of $B_r^n(0)$ to simplify the notation. Finally, before we start working with the polygonal norms, it will be useful to recall some very basic properties of regular polygons:

* The Euclidean length of a side of a regular polygon with n sides and radius r is given by $2r\sin(\pi/n)$.

* An apothem of a regular polygon is a segment joining the centre with the middle point of a side and it has Euclidean length equal to $r\cos(\pi/n)$.

4.1 Polygonal norms with $4m+2$ sides

We will show now that there are non-Euclidean norms on the plane for which, as in the Euclidean case, there exists a 2-fold Lipschitz quotient mapping f satisfying $c/L = 1/2$. Indeed, for all the n-norms with $n = 4m+2$ the doubling mapping f_n defined by Definition 4.2 satisfies $c/L = 1/2$. This section is devoted to the proof of this result. We first prove a technical lemma.

Lemma 4.1.1. *For any given $x \in \mathbb{R}^2$ and $r > 0$ let W_i^r be the vertices of the polygon $\partial B_r(x)$ centred at x, placing the indices counterclockwise. If $\mathcal{L}_i^r$ denotes the line through the vertex W_i^r with slope $-\frac{\cos(\pi/n)}{\sin(\pi/n)}$ (i.e. parallel to the side $[W_0^r, W_1^r]$ of $\partial B_r(x)$). Then, for all $k \in \{1, \ldots, m-1\}$ the line $\mathcal{L}_k^r$ is to the left of the line $\mathcal{L}_{k+1}^{2r}$.*

Proof. For simplicity of notation we denote the coordinates of points relative to x as if x were the origin, so for $k \in \{0, \ldots m\}$, we have $W_k^r = r\left(\cos(\frac{2k\pi}{n}), \sin(\frac{2k\pi}{n})\right)$.

Notice that the intersections of the x-axis (relative to the point x) with the lines $\mathcal{L}_k^r$ and $\mathcal{L}_{k+1}^{2r}$, say x_k and z_{k+1} respectively, are given by:

$$\begin{aligned} x_k &= r\left(\cos(\tfrac{2k\pi}{n}) + \sin(\tfrac{2k\pi}{n})\tan(\tfrac{\pi}{n})\right), \\ z_{k+1} &= 2r\left(\cos(\tfrac{2(k+1)\pi}{n}) + \sin(\tfrac{2(k+1)\pi}{n})\tan(\tfrac{\pi}{n})\right), \end{aligned} \tag{4.2}$$

so we need to show that $x_k < z_{k+1} = 2x_{k+1}$ for all $k \in \{1, \ldots, m-1\}$. See Figure 4.1.

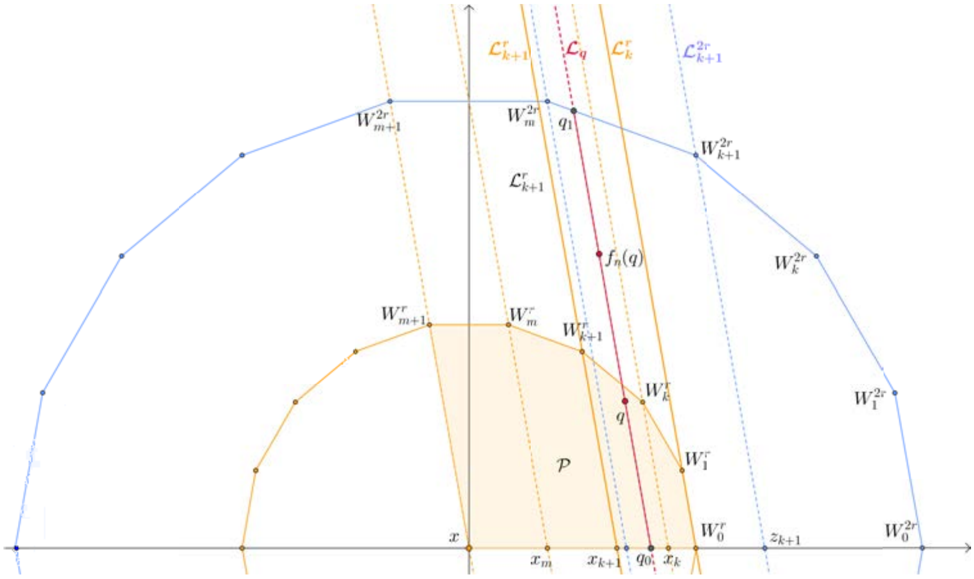


Figure 4.1

For $k = m-1$ we have:

$$x_k = x_{m-1} = r\left(\cos\left(\tfrac{2(m-1)\pi}{4m+2}\right) + \sin\left(\tfrac{2(m-1)\pi}{4m+2}\right)\tan\left(\tfrac{\pi}{4m+2}\right)\right) = r\left(\sin(\tfrac{3\pi}{n}) + \cos(\tfrac{3\pi}{n})\tan(\tfrac{\pi}{n})\right)$$

$$z_{k+1} = 2x_m = 2r\left(\cos\left(\tfrac{2m\pi}{4m+2}\right) + \sin\left(\tfrac{2m\pi}{4m+2}\right)\tan\left(\tfrac{\pi}{4m+2}\right)\right) = 4r\sin(\tfrac{\pi}{n}),$$

so in this case, the inequality $x_k < z_{k+1}$ is equivalent to

$$\sin(\tfrac{3\pi}{n}) + \cos(\tfrac{3\pi}{n})\tan(\tfrac{\pi}{n}) < 4\sin(\tfrac{\pi}{n}),$$

which is satisfied since $\sin(\frac{\pi}{n}) > 0$ and:

$$\begin{aligned}\sin(\tfrac{3\pi}{n}) + \cos(\tfrac{3\pi}{n})\tan(\tfrac{\pi}{n}) &= 3\sin(\tfrac{\pi}{n}) - 4\sin^3(\tfrac{\pi}{n}) + \left(4\cos^3(\tfrac{\pi}{n}) - 3\cos(\tfrac{\pi}{n})\right)\tan(\tfrac{\pi}{n}) \\ &= 4\sin(\tfrac{\pi}{n})\left(\cos^2(\tfrac{\pi}{n}) - \sin^2(\tfrac{\pi}{n})\right) = 4\sin(\tfrac{\pi}{n})\cos(\tfrac{2\pi}{n}) < 4\sin(\tfrac{\pi}{n}).\end{aligned}$$

Now, to show that $x_k < z_{k+1}$ for $k \in \{1, \ldots, m-2\}$, we show that $\cos(\frac{2k\pi}{n}) < 2\cos(\frac{2(k+1)\pi}{n})$.

Indeed, if $k \leq m-2$ then, using $\frac{2(m-2)}{n} = \frac{2m-4}{4m+2} = \frac{1}{2} - \frac{5}{n}$:

$$\begin{aligned}
2\cos(\tfrac{2(k+1)\pi}{n}) - \cos(\tfrac{2k\pi}{n}) &= 2\left(\cos(\tfrac{2k\pi}{n})\cos(\tfrac{2\pi}{n}) - \sin(\tfrac{2k\pi}{n})\sin(\tfrac{2\pi}{n})\right) - \cos(\tfrac{2k\pi}{n}) \\
&= \cos(\tfrac{2k\pi}{n})\left(2\cos(\tfrac{2\pi}{n}) - 1\right) - 2\sin(\tfrac{2k\pi}{n})\sin(\tfrac{2\pi}{n}) \\
&\geq \cos(\tfrac{2(m-2)\pi}{n})\left(2\cos(\tfrac{2\pi}{n}) - 1\right) - 2\sin(\tfrac{2(m-2)}{\pi})\sin(\tfrac{2\pi}{n}) \\
&= \sin(\tfrac{5\pi}{n})\left(2\cos(\tfrac{2\pi}{n}) - 1\right) - 2\cos(\tfrac{5\pi}{n})\sin(\tfrac{2\pi}{n}) \\
&\geq \sin(\tfrac{4\pi}{n})\left(2\cos(\tfrac{2\pi}{n}) - 1\right) - 2\cos(\tfrac{4\pi}{n})\sin(\tfrac{2\pi}{n}) \\
&= 2\left(\sin(\tfrac{2\pi}{n})(2\cos^2(\tfrac{2\pi}{n}) - \cos(\tfrac{2\pi}{n})) - \cos^2(\tfrac{2\pi}{n})\sin(\tfrac{2\pi}{n}) + \sin^3(\tfrac{2\pi}{n})\right) \\
&= 2\sin(\tfrac{2\pi}{n})\left(\cos^2(\tfrac{2\pi}{n}) - \cos(\tfrac{2\pi}{n}) + \sin^2(\tfrac{2\pi}{n})\right) \\
&= 2\sin(\tfrac{2\pi}{n})\left(1 - \cos(\tfrac{2\pi}{n})\right) \\
&> 0.
\end{aligned}$$

In the penultimate inequality we used that for $n \geq 10$ we have $\sin(\frac{5\pi}{n}) \geq \sin(\frac{4\pi}{n})$ and $2\cos(\frac{2\pi}{n}) > 1$. For $n = 6$, we used $2\cos(\frac{2\pi}{n}) = 1$.

Hence, $\cos(\frac{2k\pi}{n}) \leq 2\cos(\frac{2(k+1)\pi}{n})$ and clearly $\sin(\frac{2k\pi}{n}) \leq 2\sin(\frac{2(k+1)\pi}{n})$ holds for all $k \in \{1, \ldots, m-2\}$. Therefore, from (4.2), we see that $x_k < z_{k+1}$ for all $k \in \{1, \ldots, m-2\}$ and so the line $\mathcal{L}_k^r$ is to the left of the line $\mathcal{L}_{k+1}^{2r}$. □

Now we introduce some notation that we will be using in the proof of the next theorem.

Definition 4.1.2. For each $k \in \{0, \ldots, n-1\}$ and $r > 0$ let v_k^r (or simply v_k when the radius r is fixed) denote the midpoint of the side $[V_k^r, V_{k+1}^r]$ of $\partial B_r(0)$ and let $\mathcal{T}_k$ denote the line through the origin and the point v_k. Finally denote by $\mathcal{D}_k$ be the ray through the origin and the vertex V_k of $\partial B_r(0)$.

We will also consider, for each $k \in \{0, \ldots, n-1\}$ the open region $\mathcal{R}_k$ enclosed by the lines $\mathcal{D}_k$ and $\mathcal{T}_k$ and the open region $\mathcal{R}'_k$ region enclosed by the lines $\mathcal{T}_k$ and $\mathcal{D}_{k+1}$.

See Figure 4.2 for an illustration of all this new notation.

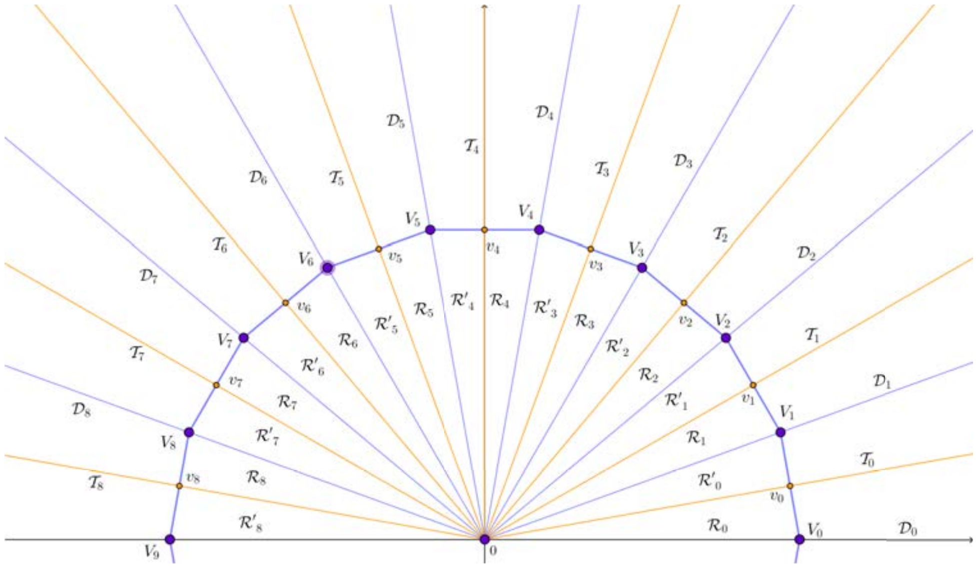


Figure 4.2

Now we can prove that for any $n = 4m + 2$ the doubling mapping f_n under the n-norm is a 2-Lipschitz and 1-co-Lipschitz mapping. The proof of this result is long because we need to consider many cases. We divided the proof in two main parts, one for the Lipschitz constant and the other for the co-Lipschitz constant. In each of these parts we will deal with the corresponding local constants of the mapping f_n at a point p. We will divide each main part into cases depending on the region of the plane —in terms of the regions described in Definition 4.1.2— the point p belongs to.

Theorem 4.1.3. *For $n = 4m + 2, m \in \mathbb{N} \setminus \{0\}$, the Lipschitz constant of the doubling mapping f_n under the n-norm is equal to 2 and the co-Lipschitz constant is equal to 1.*

Proof. Consider a point $p \in \mathbb{R}^2$ such that $\|p\|_n = \rho > 0$. Let $V_0, \ldots, V_{n-1}$ be the vertices of $\partial B_\rho(0)$ numbered as usual and for $k \in \{0, \ldots n-1\}$, let v_k denote the midpoint of the side $[V_k, V_{k+1}]$.

I. Lipschitz constant of f_n.

We will show that the local Lipschitz constant of f_n at the point p is 2. We divide the proof into the following cases: $p \in \mathcal{R}_0$, $p \in \mathcal{R}'_0$, $p \in \mathcal{T}_0$, $p \in \mathcal{D}_0$ and finally $p \in$

$(\mathcal{D}_k, \cup\mathcal{R}_k \cup \mathcal{T}_k \cup \mathcal{R}'_k,)$, for $k \in \{1, \dots, n-1\}$.

Case I.1. *The local Lipschitz constant of f_n at p is equal to 2 for $p \in \mathcal{R}_0$.*

In this case we have $p \in (V_0, v_0)$. Let $R_p > 0$ be such that for all $0 < r \leq R_p$, the polygon $B_r(p)$ is contained in the region $\mathcal{R}_0$. Notice that, by choosing R_p in this way, we can assure that any $q \in B_{R_p}(p)$ belongs to the first half of the first side of the polygon $\partial B_{\|q\|_n}(0)$.

To prove that f_n is 2-Lipschitz at p, we show that for $0 < r < \frac{1}{2}R_p$ the image of $B_r(p)$ is a subset of $B_{2r}(f_n(p))$. We will denote the vertices of the polygon $\partial B_r(p)$ of radius r centred at p by U_i^r, $i = 0, \dots, n-1$ and the vertices of the polygon $\partial B_r(f_n(p))$ centred at $f_n(p)$ by W_i^r, $i = 0, \dots, n-1$. See Figure 4.3. Notice that the lines $\mathcal{L}_i^r = (U_i^r, W_i^r)$ are parallel to the first side $[V_0, V_1]$ of $\partial B_\rho(0)$, as $[p, f_n(p)]$ is a subset of $[V_0, V_1]$.

As a first step, we show that the image of the polygon

$$\mathcal{P} = p, U_0^r, U_1^r, \dots, U_{m+1}^r \tag{4.3}$$

(shown in green in Figure 4.3), which is roughly the first quarter of $\partial B_r(p)$, is a subset of $B_{2r}(f_n(p))$.

Let $q \in \mathcal{P}$ and denote by $\mathcal{L}_q$ the line parallel to $[V_0, V_1]$ —and so to all the lines $\mathcal{L}_i^r$— that goes through q, and let q_1, q_2 be the intersection points between $\mathcal{L}_q$ and $\partial B_{2r}(f_n(p))$:

$$q_1, q_2 \in \mathcal{L}_q \cap \partial B_{2r}(f_n(p)) \text{ such that } y(q_1) > y(f_n(p)) > y(q_2).$$

We will also denote by $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$ the horizontal lines through p, $f_n(p)$ and the origin respectively. See Figure 4.3. Finally, once r is fixed, for each line $\mathcal{L}_i^r$, define the intersection points:

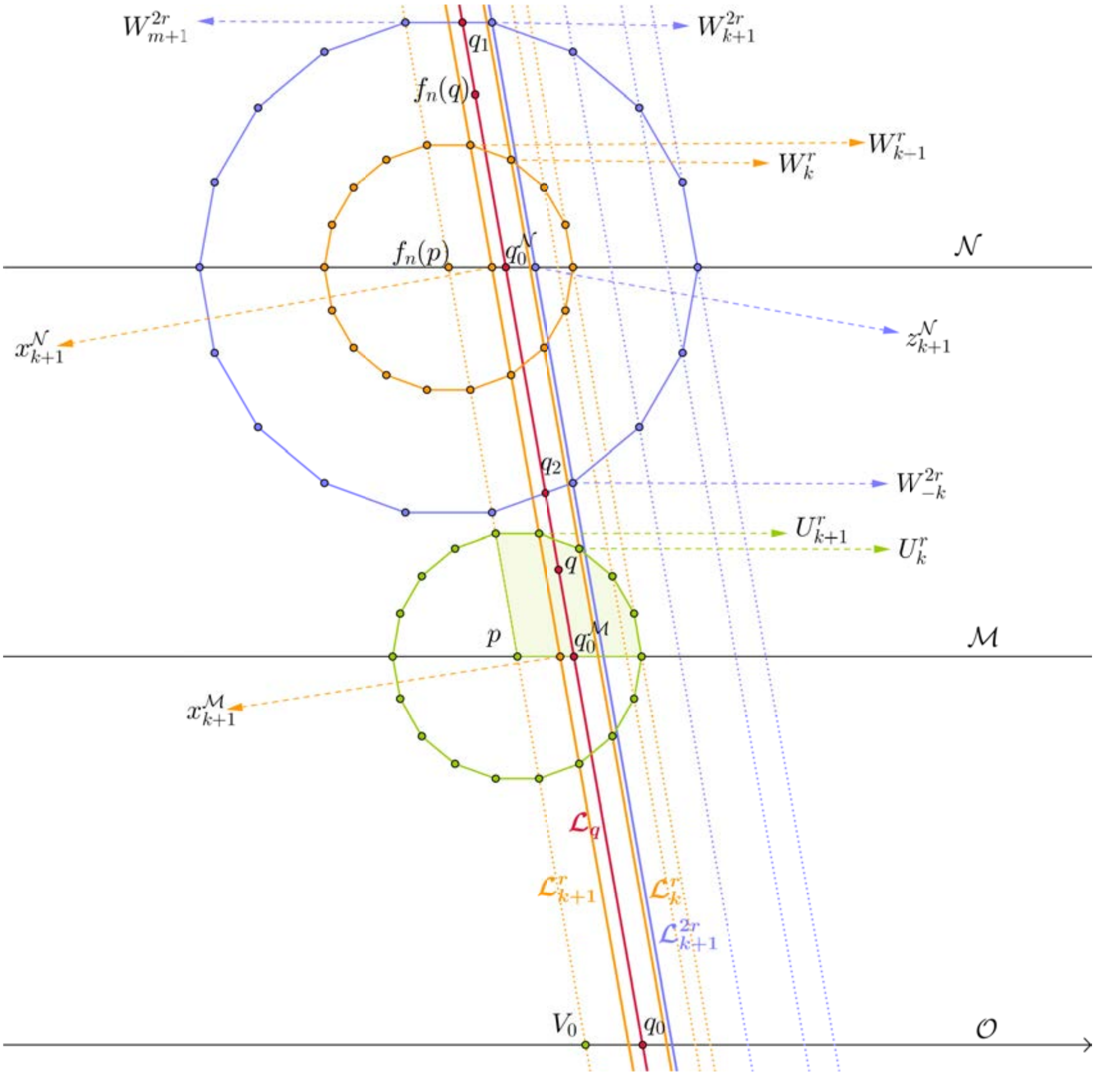


Figure 4.3

$$
\begin{aligned}
x_i &:= \mathcal{L}^r_i \cap \mathcal{O} \qquad & x^{\mathcal{M}}_i &:= \mathcal{L}^r_i \cap \mathcal{M} \qquad & x^{\mathcal{N}}_i &:= \mathcal{L}^r_i \cap \mathcal{N} \\
z_i &:= \mathcal{L}^{2r}_i \cap \mathcal{O} \qquad & z^{\mathcal{M}}_i &:= \mathcal{L}^{2r}_i \cap \mathcal{M} \qquad & z^{\mathcal{N}}_i &:= \mathcal{L}^{2r}_i \cap \mathcal{N} \\
q_0 &:= \mathcal{L}_q \cap \mathcal{O} \qquad & q^{\mathcal{M}}_0 &:= \mathcal{L}_q \cap \mathcal{M} \qquad & q^{\mathcal{N}}_0 &:= \mathcal{L}_q \cap \mathcal{N}
\end{aligned}
\tag{4.4}
$$

Hence, in c this notation, we have

$$x_{m+1} = V_0; \quad x^{\mathcal{M}}_{m+1} = p; \quad x^{\mathcal{N}}_{m+1} = f_n(p)$$

Since $p \in [V_0, v_0]$ and $r < R_p$, we know that the point $q \in B_r(p)$ is on the first half of the first side of the polygon $\partial B_{\|q\|_n}(0)$, see Figure 4.3 for an illustration. Therefore, $f_n(q)$ is still on the first side of $\partial B_{\|q\|_n}(0)$.

Notice that in order to prove that $f_n(q) \in B_{2r}(f_n(p))$ it is enough to show that

$$\|q_0 - q_1\|_n \geq \|q_0 - f_n(q)\|_n \geq \|q_0 - q_2\|_n. \tag{4.5}$$

Recall that by definition of the doubling mapping f_n, since $r < R_p$, the point $f_n(q)$ is on the first side of $\partial B_{\|q\|}(0)$ on the line $\mathcal{L}_q$ and satisfies $\|q_0 - f_n(q)\|_n = 2\|q_0 - q\|_n$.

We now prove the first inequality of (4.5). Recall also that $q \in \mathcal{P}$, where $\mathcal{P}$ is the polygon defined in (4.3). Take $k \in \{1, \ldots, m\}$ such that the line $\mathcal{L}_q$ is between the lines $\mathcal{L}^r_{k+1}$ and $\mathcal{L}^r_k$. Assume first that $k \in \{1, \ldots, m-1\}$. In this case $\mathcal{L}_q$ is to the left of the line $\mathcal{L}^r_k$ and we know by Lemma 4.1.1 that $\mathcal{L}^r_k$ is to the left of $\mathcal{L}^{2r}_{k+1}$. So the points of intersection of $\partial B_{2r}(f_n(p))$ with $\mathcal{L}_q$, which are the points q_1 and q_2, are to the left of the intersection between $\partial B_{2r}(f_n(p))$ and $\mathcal{L}^{2r}_{k+1}$ which are the points W^{2r}_{k+1} and W^{2r}_{4m+2-k}. We can see that

$$\|q^{\mathcal{N}}_0 - q_1\|_n \geq \|z^{\mathcal{N}}_{k+1} - W^{2r}_{k+1}\|_n, \tag{4.6}$$

indeed, observe first that the y-coordinate of the points W^{2r}_k increases for $k \in \{0, \ldots, m\}$, so that the y-coordinate of q_1 is greater than or equal to the y-coordinate of W^{2r}_{k+1} . Hence, if we consider the intersection point, say Q, between $\mathcal{L}^{2r}_{k+1}$ and the horizontal line through q_1, we have $W^{2r}_{k+1} \in [z^{\mathcal{N}}_{k+1}, Q]$. On the other hand, by the translation invariance of the norm, $\|q^{\mathcal{N}}_0 - q_1\|_n = \|z^{\mathcal{N}}_{k+1} - Q\|_n$, so we gather that:

$$\|q^{\mathcal{N}}_0 - q_1 - q_1\|_n = \|z^{\mathcal{N}}_{k+1} - W^{2r}_{k+1}\|_n + \|W^{2r}_{k+1} - Q\|_n \geq \|z^{\mathcal{N}}_{k+1} - W^{2r}_{k+1}\|_n.$$

Therefore we have (4.6). Now, in the same way, we can see that $\|x^{\mathcal{M}}_{k+1} - U^r_{k+1}\|_n \geq$

$\|q_0^{\mathcal{M}} - q\|_n$, because $\mathcal{L}_{k+1}^r$ is to the left of $\mathcal{L}_q$. Hence,

$$\begin{aligned}\|q_0^{\mathcal{N}} - q_1\|_n \geq \|z_{k+1}^{\mathcal{N}} - W_{k+1}^{2r}\|_n &= 2\|x_{k+1}^{\mathcal{N}} - W_{k+1}^r\|_n \\ &= 2\|x_{k+1}^{\mathcal{M}} - U_{k+1}^r\|_n \geq 2\|q_0^{\mathcal{M}} - q\|_n.\end{aligned}$$

Therefore,

$$\begin{aligned}\|q_0 - q_1\|_n &= \|q_0 - q_0^{\mathcal{N}}\|_n + \|q_0^{\mathcal{N}} - q_1\|_n \\ &= 2\|q_0 - q_0^{\mathcal{M}}\|_n + \|q_0^{\mathcal{N}} - q_1\|_n \\ &\geq 2\|q_0 - q_0^{\mathcal{M}}\|_n + 2\|q_0^{\mathcal{M}} - q\|_n \\ &= 2\|q_0 - q\|_n = \|q_0 - f_n(q)\|_n.\end{aligned}$$

We conclude that the first inequality in (4.5) is satisfied for all $q \in \mathcal{P}$ such that q is in between the lines $\mathcal{L}_{k+1}^r$ and $\mathcal{L}_k^r$ with $k \in \{1, \ldots, m-1\}$. The remaining case, q between $\mathcal{L}_m^r$ and $\mathcal{L}_{m+1}^r$, is easy. Simply notice that in this case, q belongs to the parallelogram $p, x_m^{\mathcal{M}}, U_m^r, U_{m+1}^r$ and that the intersection between $\mathcal{L}_m^r$ and $\partial B_{2r}(f(p))$, q_1, certainly occurs on the side W_m^{2r}, W_{m+1}^{2r} of $\partial B_{2r}(f_n(p))$. Therefore,

$$\begin{aligned}\|q_0 - q_1\|_n &= \|q_0 - q_0^{\mathcal{N}}\|_n + \|q_0^{\mathcal{N}} - q_1\|_n \\ &= 2\|q_0 - q_0^{\mathcal{M}}\|_n + \|f_n(p) - W_{m+1}^{2r}\|_n \\ &= 2\left(\|x_{m+1} - p\|_n + \|p - U_{m+1}^r\|_n\right) \\ &= 2\left(\|x_{m+1} - U_{m+1}^r\|_n\right) \geq 2\|q_0 - q\|_n \\ &= \|q_0 - f_n(q)\|_n.\end{aligned}$$

We are now left to show the second inequality in (4.5), this follows from:

$$\begin{aligned}\|q_0 - f_n(q)\|_n = 2\|q_0 - q\|_n &= 2\left(\|q_0 - q_0^{\mathcal{M}}\|_n + \|q_0^{\mathcal{M}} - q\|_n\right)\\ &\geq 2\|q_0 - q_0^{\mathcal{M}}\|_n = 2\|x_{m+1} - p\|_n\\ &= \|V_0 - f_n(p)\|_n \geq \|q_0 - q_2\|_n.\end{aligned}$$

We conclude that for all $q \in \mathcal{P}$ (defined in (4.3)) both inequalities in (4.5) are satisfied, thus $f_n(\mathcal{P}) \subseteq B_{2r}(f_n(p))$.

Now, we are going to show that the image the fourth quarter of $B_r(p)$ is a subset of $B_{2r}(f_n(p))$. For this it will be convenient to denote the vertices of the polygons $\partial B_r(p)$, $\partial B_r(f_n(p))$ and $\partial B_{2r}(f_n(p))$ with negative indices, so that, for $k \in \{0, \ldots, 2m+1\}$ the vertices with index $4m+2-k$ will be denoted with the index $-k$. Let $\mathcal{P}' \subseteq B_r(p)$ be the polygon whose vertices are $U_0^r, U_{-1}^r, \ldots, U_{-m}^r, p$. We will show that for all $q \in \mathcal{P}'$, we have $f_n(q) \in B_{2r}(f_n(p))$. As before, let $\mathcal{L}_q$ be the line parallel to the first side of the polygon $\partial B_r(p)$, and recall the notation in (4.4). Again, it is enough to show the inequalities in (4.5). Take $k \in \{0, \ldots, m\}$ such that $\mathcal{L}_q$ is between the lines $\mathcal{L}_{k+1}^r$ and $\mathcal{L}_k^r$. Notice that $\mathcal{L}_{k+1}^r$ goes through the vertex U_{k+1}^r and also through the vertex U_{-k}^r of $\partial B_r(p)$ and the same for the lines $\mathcal{L}_k^{2r}$ and the vertices of $B_{2r}(f_n(p))$. Now, since $\mathcal{L}_q$ is to the left of $\mathcal{L}_k^r$, and the latter is to the left of $\mathcal{L}_{k+1}^{2r}$, we get:

$$\|q_2 - q_0^{\mathcal{N}}\|_n \geq \|W_{-k}^{2r} - z_{k+1}^{\mathcal{N}}\|_n;$$

this is because the y-coordinate of the vertices W_{-k}^{2r} decreases for $k \in \{0, \ldots, m\}$. For the same reason $\|U^r_{-k} - x^{\mathcal{M}}_{k+1}\|_n \geq \|q - q^{\mathcal{M}}_0\|_n$, so we gather:

$$\begin{aligned}\|q_2 - q^{\mathcal{N}}_0\|_n \geq \|W^{2r}_{-k} - z^{\mathcal{N}}_{k+1}\|_n &= 2\|W^r_{-k} - x^{\mathcal{N}}_{k+1}\|_n \\ &= 2\|U^r_{-k} - x^{\mathcal{M}}_{k+1}\|_n \geq 2\|q - q^{\mathcal{M}}_0\|_n.\end{aligned}$$

So altogether, we have:

$$\begin{aligned}\|q_0 - q_2\|_n = \|q_0 - q^{\mathcal{N}}_0\|_n - \|q_2 - q^{\mathcal{N}}_0\|_n &\leq \|q_0 - q^{\mathcal{N}}_0\|_n - 2\|q - q^{\mathcal{M}}_0\|_n \\ = 2\left(\|q_0 - q^{\mathcal{M}}_0\|_n - \|q - q^{\mathcal{M}}_0\|_n\right) &= 2\|q_0 - q\|_n = \|q_0 - f_n(q)\|.\end{aligned} \tag{4.7}$$

Hence, the second inequality in (4.5) is satisfied. To show the first inequality, notice that:

$$\begin{aligned}\|q_0 - f_n(q)\|_n = 2\|q_0 - q\|_n &\leq 2\|q_0 - q^{\mathcal{M}}_0\|_n \\ &= \|q_0 - q^{\mathcal{N}}_0\|_n \leq \|q_0 - q_1\|_n.\end{aligned} \tag{4.8}$$

Thus, for all $q \in \mathcal{P}'$, we have $\|q_0 - q_1\|_n \geq \|q_0 - f_n(q)\| \geq \|q_0 - q_2\|_n$. We conclude that $f_n(\mathcal{P} \cup \mathcal{P}') \subseteq B_{2r}(f_n(p))$.

Finally, we are going to show that the local Lipschitz constant L_p of f_n at p is equal to 2. First we show that for all $0 < r_0 < \frac{1}{2}R_p$, we have $f_n(B_{r_0}(p)) \subseteq B_{2r_0}(f(p))$.

Fix $r_0 < \frac{1}{2}R_p$ and pick $q \in B_{r_0}(p)$. Notice that as $B_{r_0}(q) \subset B_{R_p}(p)$, we certainly have $r_0 < R_q$. If $\|q\|_n \geq \|p\|_n$, then q belongs either to the fourth or to the first "quarter" of the polygon $B_{r_0}(p)$ (i.e. q belongs to one of the polygons $\mathcal{P}$ or $\mathcal{P}'$ that we have considered before). Hence, as we have just shown, $f_n(q) \in B_{2r_0}(f_n(p))$ and we are done.

On the other hand, if $\|q\|_n \leq \|p\|_n$, pick $r \in (\|p - q\|_n, r_0)$ and consider the ball $B_r(q)$. Now p belongs to the fourth or to the first "quarter" of the polygon $B_r(q)$ and, since $r < r_0 < \frac{1}{2}R_p$, we have $r < R_q$. Therefore, swapping the role of p and q, we get $f_n(p) \in B_{2r}(f_n(q))$, so that $\|f_n(p) - f_n(q)\|_n < 2r < 2r_0$, hence $f_n(q) \in B_{2r_0}(f_n(p))$.

Hence, for all $r_0 < \frac{1}{2}R_p$, we have $f_n(q) \in B_{2r_0}(f_n(p))$, whenever $q \in B_{r_0}(p)$. Thus we proved that $f_n(B_{r_0}(p)) \subseteq B_{2r_0}(f_n(p))$ for all $r_0 < \frac{1}{2}R_p$, and this shows that $L_p \leq 2$. To show that L_p is in fact 2, notice that the vertex $U^{r_0}_{m+1}$ of $\partial B_{r_0}(p)$ is mapped to the point $f_n(U^{r_0}_{m+1})$ on $\mathcal{L}_{m+1}$ such that $\|V_0 - f_n(U^{r_0}_{m+1}\|_n = 2\|V_0 - U^{r_0}_{m+1}\|_n$, hence:

$$\|V_0 - f_n(U^{r_0}_{m+1})\|_n = 2\|V_0 - U^{r_0}_{m+1}\|_n = 2\left(\|V_0 - p\|_n + r_0\right) = \|V_0 - f_n(p)\|_n + 2r_0.$$

Therefore $f_n(U^{r_0}_{m+1}) = W^{2r_0}_{m+1}$, and we have:

$$\|f_n(p) - f_n(U^{r_0}_{m+1})\|_n = 2\|p - U^{r_0}_{m+1}\|_n. \tag{4.9}$$

We conclude that $L_p = 2$ for all $p \in \mathcal{R}_0$. This finishes the proof for this case.

Case I.2. *The local Lipschitz constant of f_n at p is equal to 2 for $p \in \mathcal{R}'_0$.*

For this we will consider the functions:

$$\begin{aligned} &\mathrm{Sym}_{\mathcal{T}}, \text{ which will denote the symmetric reflection about the line } \mathcal{T} \\ &\mathrm{Rot}_k, \text{ which will denote the rotation by } \tfrac{2k\pi}{n} \text{ radians around the origin.} \end{aligned} \tag{4.10}$$

Notice that Rot_k and $\mathrm{Sym}_{\mathcal{T}}$ are linear isometric isomorphisms, in particular for all $x \in \mathbb{R}^2$ we have $\|\mathrm{Rot}_k(x)\|_n = \|x\|_n = \|\mathrm{Sym}_{\mathcal{L}}(x)\|_n$ for all integer k, and for all line $\mathcal{L} = \mathcal{T}_k$ or $\mathcal{L} = \mathcal{D}_k$.

Define the function:

$$f^*_n(x) = \mathrm{Sym}_{\mathcal{T}_1}\left(\mathrm{Rot}_1\left(f_n\left(\mathrm{Sym}_{\mathcal{T}_0}(x)\right)\right)\right).$$

As we will see now,

$$\text{for all } p \in \mathcal{R}'_0 \text{ we have } f^*_n(p) = f_n(p). \tag{4.11}$$

Recall that $\rho = \|p\|_n$ and that for $p \in \mathcal{R}_0'$, the point $f_n(p)$ is the point on the side $[V_1, V_2]$ of $\partial B_\rho(0)$ such that

$$\|V_0 - V_1\|_n + \|V_1 - f_n(p)\|_n = 2\|V_0 - p\|_n.$$

Notice that

$$\mathrm{Sym}_{\mathcal{T}_1}(\mathrm{Rot}_1(f_n([V_0, v_0]))) = \mathrm{Sym}_{\mathcal{T}_1}(\mathrm{Rot}_1([V_0, V_1])) = \mathrm{Sym}_{\mathcal{T}_1}([V_1, V_2]) = [V_1, V_2].$$

Since $\mathrm{Sym}_{\mathcal{T}_0}(p) \in [V_0, v_0]$, we gather that:

$$f_n^*(p) = \mathrm{Sym}_{\mathcal{T}_1}\left(\mathrm{Rot}_1\left(f_n\left(\mathrm{Sym}_{\mathcal{T}_0}(p)\right)\right)\right) \in [V_1, V_2].$$

Also, since v_0 is the midpoint between V_0 and V_1,

$$\|V_0 - V_1\|_n + \|V_1 - f_n^*(p)\|_n = 2\|V_0 - v_0\|_n + \|V_1 - f_n^*(p)\|_n. \tag{4.12}$$

It is clear that

$$f_n^*(v_0) = \mathrm{Sym}_{\mathcal{T}_1}(\mathrm{Rot}_1(f_n(v_0))) = \mathrm{Sym}_{\mathcal{T}_1}(\mathrm{Rot}_1(V_1)) = V_1.$$

On the other hand,

$$\begin{aligned} \|V_0 - f_n(\mathrm{Sym}_{\mathcal{T}_0}(p))\|_n &= 2\|V_0 - \mathrm{Sym}_{\mathcal{T}_0}(p)\|_n \\ &= 2\left(\|V_0 - v_0\|_n - \|v_0 - \mathrm{Sym}_{\mathcal{T}_0}(p)\|_n\right) \\ &= 2\Big(\|V_0 - v_0\|_n - \|v_0 - p\|_n\Big). \end{aligned} \tag{4.13}$$

Now, since Sym and Rot_1 are linear isometries, from (4.13), it follows that:

$$\begin{aligned}
\|V_1 - f_n^*(p)\|_n &= \|f_n^*(v_0) - f_n^*(p)\|_n = \|f_n(\mathrm{Sym}_{\mathcal{T}_0}(v_0)) - f_n(\mathrm{Sym}_{\mathcal{T}_0}(p))\|_n \\
&= \|f_n(v_0) - f_n(\mathrm{Sym}_{\mathcal{T}_0}(p))\|_n = \|V_1 - f_n(\mathrm{Sym}_{\mathcal{T}_0}(p))\|_n \\
&= \|V_0 - V_1\|_n - \|V_0 - f_n(\mathrm{Sym}_{\mathcal{T}_0}(p))\|_n \qquad (4.14)\\
&= \|V_0 - V_1\|_n - 2\left(\|V_0 - v_0\|_n - \|v_0 - p\|_n\right) \\
&= 2\|v_0 - p\|_n.
\end{aligned}$$

Substituting this in (4.12) we conclude that:

$$\begin{aligned}
\|V_0 - V_1\|_n + \|V_1 - f_n^*(p)\|_n &= 2\left(\|V_0 - v_0\|_n + \|v_0 - p\|_n\right) \\
&= 2\|V_0 - p\|_n.
\end{aligned}$$

Hence, for $p \in \mathcal{R}_0'$, we have $f_n^*(p) \in [V_1, V_2]$ and $\|V_0 - V_1\|_n + \|V_1 - f_n^*(p)\|_n = 2\|V_0 - p\|_n$. Thus, $f_n^*(p) = f_n(p)$ for all $p \in \mathcal{R}_0'$.

This implies that for all $p \in \mathcal{R}_0'$ the local Lipschitz constant of f_n at p is less than or equal to 2. For, given $p \in \mathcal{R}_0'$ let $R_p > 0$ be such that $B_{R_p}(p) \subseteq \mathcal{R}_0'$. Then, if $\|p - q\|_n < \frac{1}{2}R_p$, we have:

$$\begin{aligned}
\|f_n(p) - f_n(q)\|_n &= \|f_n^*(p) - f_n^*(q)\|_n \qquad (4.15)\\
&= \|\,\mathrm{Sym}_{\mathcal{T}_1}\left(\mathrm{Rot}_1\left(f_n\left(\mathrm{Sym}_{\mathcal{T}_0}(p)\right)\right)\right) - \mathrm{Sym}_{\mathcal{T}_1}\left(\mathrm{Rot}_1\left(f_n\left(\mathrm{Sym}_{\mathcal{T}_0}(q)\right)\right)\right)\|_n \\
&= \|f_n\left(\mathrm{Sym}_{\mathcal{T}_0}(p)\right) - f_n\left(\mathrm{Sym}_{\mathcal{T}_0}(q)\right)\|_n \leq 2\|\,\mathrm{Sym}_{\mathcal{T}_0}(p) - \mathrm{Sym}_{\mathcal{T}_0}(q)\|_n \\
&= 2\|p - q\|_n,
\end{aligned}$$

the last inequality here uses that $\mathrm{Sym}_{\mathcal{T}_0}(p) \in \mathcal{R}_0$, which follows from Case I.1.

Case I.3. *The local Lipschitz constant of f_n at p is equal to 2 for $p \in \mathcal{T}_0$.*

Now we deal with the points p on the line $\mathcal{T}_0$. Let $R_p > 0$ be such that $B_{R_p}(p) \subseteq (\mathcal{R}_0 \cup \mathcal{R}_0' \cup \mathcal{T}_0)$, and consider $q \in B_r(p)$ with $0 < r < \frac{1}{4}R_p$. Notice that the image under f_n of $B_r(p) \cap \mathcal{R}_0$ is a subset of $\mathcal{R}_0 \cup \mathcal{R}_0' \cup \mathcal{T}_0$; in other words if $q \in \mathcal{R}_0$, then $f_n(q)$ belongs to the first side of the polygon $\partial B_{\|q\|_n}(0)$. Hence, to show that $f_n(q) \in B_{2r}(f(p))$ it is enough to show that (4.5) is satisfied —were q_0, q_1 and q_2 are defined by (4.4). Therefore, we can repeat the same geometric argument that we used in Case I.1 to show the Lipschitz condition for points $p \in \mathcal{P}'$, to conclude that $\|f_n(p) - f_n(q)\|_n \leq 2\|p - q\|_n$. See (4.7) and (4.8). Now for $q \in \mathcal{R}_0'$ we again use the function f_n^* to argue that $\|f_n(p) - f_n(q)\|_n \leq 2\|p - q\|_n$. Finally, if $q \in \mathcal{T}_0$ we simply observe that p and q are collinear with the origin, so that $\|p - q\|_n = |\|p\|_n - \|q\|_n|$. Also notice that, by definition of the mapping f_n, we know that $f_n(p), f_n(q) \in \mathcal{D}_1$ so we also have $\|f_n(p) - f_n(q)\|_n = |\|f_n(p)\| - \|f_n(q)\|_n|$. Now, since the mapping f_n fixes the norm of each point, we gather that:

$$\|f_n(p) - f_n(q)\|_n = \Big| \|f_n(p)\|_n - \|f_n(q)\|_n \Big| = \Big| \|p\|_n - \|q\|_n \Big| = \|p - q\|_n < r.$$

This shows that for all $p \in \mathcal{T}_0$, $f(B_r(p)) \subseteq B_{2r}(f(p))$, whenever $r < \frac{1}{4}R_p$.

Case I.4. *The local Lipschitz constant of f_n at p is equal to 2 for $p \in \mathcal{D}_0 = \mathcal{O}$.*

Consider a point $p \in \mathcal{O} = \mathcal{D}_0$, which is the positive side of the x-axis (so that, in this case we have $p = V_0 = f_n(p)$. Let $R_p > 0$ be such that $B_{R_p}(p) \subseteq (\mathcal{R}_{n-1}' \cup \mathcal{O} \cup \mathcal{R}_0)$. Take $q \in B_r(p)$, with $0 < r < \frac{1}{4}R_p$. We first consider the points $q \in B_r(p)$ such that $\|q\|_n \geq \|p\|_n$. If we assume further that $q \in \mathcal{R}_0$ then $q \in \mathcal{R}_0 \cap \mathcal{P}$ (were $\mathcal{P}$ is, as before, defined by (4.3)). Hence we can repeat the whole argument used for the case $p \in \mathcal{R}_0$ and $q \in \mathcal{P}$ to show that $\|f_n(p) - f_n(q)\|_n \leq 2\|p - q\|_n$. Now, if $q \in \mathcal{R}_{n-1}'$, recall the functions defined in(4.10) and observe that $f_n(q) = \mathrm{Sym}_{\mathcal{O}}(f_n(\mathrm{Sym}_{\mathcal{O}}(q)))$, and clearly

$\mathrm{Sym}_{\mathcal{O}}(q) \in \mathcal{R}_0$, therefore:

$$\begin{aligned}\|f_n(p) - f_n(q)\|_n &= \|p - \mathrm{Sym}_{\mathcal{O}}(f_n(\mathrm{Sym}_{\mathcal{O}}(q))\|_n = \|p - f_n(\mathrm{Sym}_{\mathcal{O}}(q))\|_n \\ &= 2\|p - \mathrm{Sym}_{\mathcal{O}}(q)\|_n = 2\|p - q\|_n.\end{aligned}$$

Finally, if $q \in \mathcal{O}$, since the ray $\mathcal{O}$ is fixed under f_n we have $\|f_n(p) - f_n(q)\|_n = \|p - q\|_n$ for all $q \in \mathcal{O}$. Therefore, we have $f_n(q) \in B_{2r}(f_n(p))$ for all $q \in B_r(p)$, with $\|q\|_n \geq \|p\|_n$, (i.e. for all $q \in \mathcal{P} \cup \mathcal{P}'$, using previous notation).

It is clear that we can extend this to the points $q \in B_r(p)$ with $\|q\|_n < \|p\|_n$, in the same way as we did for the case $p \in \mathcal{R}_0$. This shows that the local Lipschitz constant of f_n at any point $p \in \mathcal{O}$ is less than or equal to 2.

So far, we have shown that the local Lipschitz constant L_p of f_n at any point $p \in (\mathcal{O} \cup \mathcal{R}_0 \cup \mathcal{T}_0 \cup \mathcal{R}'_0)$ satisfies $L_p \leq 2$. As we will see, this is enough to cover all cases of $p \in \mathbb{R}^2$.

Case I.5. *The local Lipschitz constant of f_n at p is equal to 2 for $p \in (\mathcal{D}_k \cup \mathcal{R}_k \cup \mathcal{T}_k \cup \mathcal{R}'_k)$, $k \in \{1, \ldots, n-1\}$.*

Let $k \in \{1, \ldots, n-1\}$ and consider the function

$$g_n(p) = \mathrm{Rot}_{2k}(f_n(\mathrm{Rot}_{-k}(p))),$$

where Rot_{-k} is defined by (4.10). It is not hard to see that for $p \in (\mathcal{D}_k \cup \mathcal{R}_k \cup \mathcal{T}_k \cup \mathcal{R}'_k)$ we have $g_n(p) = f_n(p)$. Indeed, if $p \in \mathcal{D}_k \cup \mathcal{R}_k$ then $\mathrm{Rot}_{-k}(p) \in [V_0, v_0] \subseteq (\mathcal{D}_0 \cup \mathcal{R}_0)$, hence $f_n(\mathrm{Rot}_{-k}(p)) \in [V_0, V_1]$ and

$$g_n(p) \in \mathrm{Rot}_{2k}([V_0, V_1]) = [V_{2k}, V_{2k+1}].$$

Here all the indices are taken modulo n. It remains to show that the length of the curve that goes along $\partial B_\rho(0)$ with starting point V_0 and end point $g_n(p)$ is equal to twice the length of the curve along $\partial B_\rho(0)$ with starting point V_0 and end point p; this is:

$$\sum_{i=0}^{2k-1} \|V_i - V_{i+1}\|_n + \|V_{2k} - g_n(p)\|_n = 2\left(\sum_{i=0}^{k-1} \|V_i - V_{i+1}\|_n + \|V_k - p\|_n\right),$$

which follows from:

$$\begin{aligned}\|V_{2k} - g_n(p)\|_n &= \|V_{2k} - \mathrm{Rot}_{2k}\left(f_n(\mathrm{Rot}_{-k}(p))\right)\|_n = \|V_0 - f_n(\mathrm{Rot}_{-k}(p))\|_n \\ &= 2\|V_0 - \mathrm{Rot}_{-k}(p)\|_n = 2\|V_k - p\|_n,\end{aligned}$$

and the fact that

$$\sum_{i=0}^{2k-1} \|V_i - V_{i+1}\|_n = 2k\|V_0 - V_1\|_n = \sum_{i=0}^{k-1} \|V_i - V_{i+1}\|_n. \tag{4.16}$$

Similarly, if $p \in \mathcal{T}_k \cup \mathcal{R}'_k$ we get $f_n(\mathrm{Rot}_{-k}(p)) \in f_n([v_0, V_1]) \subseteq [V_1, V_2]$ so

$$g_n(p) \in \mathrm{Rot}_{2k}([V_1, V_2]) = [V_{2k+1}, V_{2k+2}],$$

and we also have

$$\|V_0 - V_1\|_n + \|V_1 - f_n(\mathrm{Rot}_{-k}(p))\|_n = 2\|V_0 - \mathrm{Rot}_{-k}(p)\|_n. \tag{4.17}$$

Notice that in this case in order to prove that $g_n(p) = f_n(p)$ we need to show that

$$\sum_{i=0}^{2k} \|V_i - V_{i+1}\|_n + \|V_{2k+1} - g_n(p)\|_n = 2\left(\sum_{i=0}^{k-1} \|V_i - V_{i+1}\|_n + \|V_k - p\|_n\right),$$

which follows now from (4.17), since:

$$\|V_{2k+1} - g_n(p)\|_n = \|V_{2k+1} - \mathrm{Rot}_{2k}\left(f_n(\mathrm{Rot}_{-k}(p))\right)\|_n = \|V_1 - f_n(\mathrm{Rot}_{-k}(p))\|_n$$
$$= 2\|V_0 - \mathrm{Rot}_{-k}(p)\|_n - \|V_0 - V_1\|_n$$

so we gather that

$$\begin{aligned}\sum_{i=0}^{2k} \|V_i - V_{i+1}\|_n + \|V_{2k+1} - g_n(p)\|_n| &= 2k\|V_0 - V_1\|_n + 2\|V_0 - \mathrm{Rot}_{-k}(p)\|_n - \|V_0 - V_1\|_n \\ &= (2k-1)\|V_0 - V_1\|_n + 2\|V_0 - \mathrm{Rot}_{-k}(p)\|_n \\ &= 2\left(\sum_{i=0}^{k-1} \|V_i - V_{i+1}\|_n + \|V_k - p\|_n\right),\end{aligned}$$

as we wanted.

Thus $g_n(p) = f_n(p)$ for all $p \in (\mathcal{D}_k \cup \mathcal{R}_k \cup \mathcal{T}_k \cup \mathcal{R}'_k)$.

Now take $p \in (\mathcal{D}_k \cup \mathcal{R}_k \cup \mathcal{T}_k \cup \mathcal{R}'_k)$ and $R'_p > 0$ such that $B_{R'_p}(p) \subseteq (\mathcal{D}_k \cup \mathcal{R}_k \cup \mathcal{T}_k \cup \mathcal{R}'_k)$. Let $p_* := \mathrm{Rot}_{-k}(p)$, hence $p_* \in (\mathcal{O} \cup \mathcal{R}_0 \cup \mathcal{T}_0 \cup \mathcal{R}'_0)$. From the previous cases it follows that there exists $R_{p_*} \in (0, R'_p)$ such that

$$f_n(B_r(p_*)) \subseteq B_{2r}(f_n(p_*)), \quad \text{whenever} \quad r < R_{p_*}. \tag{4.18}$$

We now notice that

$$\mathrm{Rot}_{2k}(f_n(B_r(p_*))) = \mathrm{Rot}_{2k}(f_n(B_r(\mathrm{Rot}_{-k}(p)))) = \mathrm{Rot}_{2k}(f_n(\mathrm{Rot}_{-k}(B_r(p))))$$
$$= g_n(B_r(p)) = f_n(B_r(p))$$

and that

$$\begin{aligned}\mathrm{Rot}_{2k}(B_{2r}(f_n(p_*))) &= \mathrm{Rot}_{2k}(B_{2r}(f_n(\mathrm{Rot}_{-k}(p))))B_{2r}(\mathrm{Rot}_{2k}(f_n(\mathrm{Rot}_{-k}(p))))\\ &= B_{2r}(g_n(p)) = B_{2r}(f_n(p)).\end{aligned}$$

From this two equations above and (4.18) we conclude that $f_n(B_r(p) \subseteq B_{2r}(f_n(p))$ and this finishes the proof for this case.

Finally, it is clear that the local Lipschitz constant of f_n at the origin is equal to 1, so it follows that for any $p \in \mathbb{R}^2$ the local Lipschitz constant of the mapping f_n at p satisfies $L_p \leq 2$. From Proposition 1.2.6 it follows that the global Lipschitz constant L of f_n satisfies the same inequality. Since we have shown in (4.9) that there are points such that $\|f_n(p) - f_n(q)\|_n = 2\|p - q\|_n$ we conclude that $L = 2$.

II. Co-Lipschitz constant of f_n.

Now we show that for all $p \in \mathbb{R}^2$ the local co-Lipschitz constant, c_p, of the mapping f_n at the point p satisfies $c_p \geq 1$. This is obvious for $p = 0$, so we show it for $p \in \mathbb{R}^2 \setminus \{0\}$. We consider the same 5 cases as we did for the Lipschitz constant.

Case II.1. *The local co-Lipschitz constant of f_n at p is less than or equal to 1 for all $p \in \mathcal{R}_0$.*

Let $p \in \mathcal{R}_0$ and, as before, let $0 < R_p$ be such that $B_{R_p}(p) \subseteq \mathcal{R}_0$. We show that for $r < R_p$, if $q' \in B_r(f_n(p))$ then there exists a $q \in B_r(p)$ such that $f_n(q) = q'$.

Let $s = \|q'\|_n$ and consider the polygon $\partial B_s(0)$, whose vertices are $q_0 = V_0^s, \ldots, V_{n-1}^s$. Denote by $\mathcal{L}_q$ the line through the vertices V_0^s and V_1^s and recall the notation in (4.4) and Figure 4.3. Of course we somewhat abuse the notation here but we are going to find $q \in \mathcal{L}_q$ so this will justify our choice for naming this line. We also define the points q_1^*, q_2^* as the intersection points between $\mathcal{L}_q$ and $\partial B_r(p)$, where q_2^* is above the line $\mathcal{M}$ and q_1^* below it. In the same way let q_3^*, q_4^* be the points that belong to $\mathcal{L}_q \cap \partial B_r(f_n(p))$, where

q_4^* is above the line $\mathcal{N}$ and q_3^* below it.

Take $q \in \mathcal{L}_q$ such that $\|q_0 - q\|_n = \frac{1}{2}\|q_0 - q'\|_n$. By definition of the doubling mapping f_n we know that $f_n(q) = q'$. We need to show that $q \in B_r(p)$; we do this by showing that

$$\|q_0 - q_1^*\|_n \leq \|q_0 - q\|_n \leq \|q_0 - q_2^*\|_n. \tag{4.19}$$

Notice that from our choice of q, we have $\|q_0 - q\|_n = \frac{1}{2}\|q_0 - q'\|_n$, also by symmetry we have $\|q_0^{\mathcal{N}} - q_4^*\|_n = \|q_0^{\mathcal{M}} - q_2^*\|_n$, therefore:

$$\begin{aligned}\|q_0 - q\|_n &= \tfrac{1}{2}\|q_0 - q'\|_n \leq \tfrac{1}{2}\left(\|q_0 - q_0^{\mathcal{N}}\|_n + \|q_0^{\mathcal{N}} - q_4^*\|\right)\\ &= \|q_0 - q_0^{\mathcal{M}}\|_n + \tfrac{1}{2}\|q_0^{\mathcal{M}} - q_2^*\|_n \leq \|q_0 - q_0^{\mathcal{M}}\|_n + \|q_0^{\mathcal{M}} - q_2^*\|_n = \|q_0 - q_2^*\|_n.\end{aligned}$$

On the other hand, using now that $\|q_0^{\mathcal{N}} - q_3^*\|_n = \|q_0^{\mathcal{M}} - q_1^*\|_n$, we get:

$$\begin{aligned}\|q_0 - q\|_n &= \tfrac{1}{2}\|q_0 - q'\|_n \geq \tfrac{1}{2}\left(\|q_0 - q_0^{\mathcal{N}}\|_n - \|q_0^{\mathcal{N}} - q_3^*\|_n\right)\\ &= \|q_0 - q_0^{\mathcal{M}}\|_n - \tfrac{1}{2}\|q_0^{\mathcal{M}} - q_1^*\|_n \geq \|q_0 - q_0^{\mathcal{M}}\|_m - \|q_0^{\mathcal{M}} - q_1^*\|_n = \|q_0 - q_1^*\|_n.\end{aligned}$$

We conclude that (4.19) is satisfied, thus $B_r(f_n(p)) \subseteq f_n(B_r(p))$. This shows that for all $p \in \mathcal{R}_0$, we have $c_p \geq 1$.

Case II.2. *The local co-Lipschitz constant of f_n at p is less than or equal to 1 for all $p \in \mathcal{R}_0'$.*

Now for $p \in \mathcal{R}_0'$, take $R_p > 0$ such that $B_{R_p}(p) \subseteq \mathcal{R}_0'$. Recall that for all $p \in \mathcal{R}_0'$ we have $f_n(p) = f_n^*(p)$ (see (4.11)), therefore, since $\mathrm{Sym}_{\mathcal{T}_0}(p) \in \mathcal{R}_0$, we know:

$$B_r\left(f_n(\mathrm{Sym}_{\mathcal{T}_0}(p))\right) \subseteq f_n\left(B_r(\mathrm{Sym}_{\mathcal{T}_0}(p)\right),$$

hence,

$$\mathrm{Sym}_{\mathcal{T}_1}\left(\mathrm{Rot}_1\left(B_r\left(f_n(\mathrm{Sym}_{\mathcal{T}_0}(p)\right)\right)\right) \subseteq \mathrm{Sym}_{\mathcal{T}_1}\left(\mathrm{Rot}_1\left(f_n\left(B_r(\mathrm{Sym}_{\mathcal{T}_0}(p))\right)\right)\right).$$

It follows, using again the isometric properties, that

$$B_r\left(\mathrm{Sym}_{\mathcal{T}_1}\left(\mathrm{Rot}_1\left(f_n(\mathrm{Sym}_{\mathcal{T}_0}(p))\right)\right)\right) \subseteq \mathrm{Sym}_{\mathcal{T}_1}\left(\mathrm{Rot}_1\left(f_n\left(\mathrm{Sym}_{\mathcal{T}_0}(B_r(p))\right)\right)\right),$$

and we conclude that

$$B_r(f_n(p)) = B_r(f_n^*(p)) \subseteq f_n^*(B_r(p)) = f_n(B_r(p)).$$

which is what we wanted. Hence, $c_p \geq 1$ for all $p \in \mathcal{R}_0'$.

Case II.3. *The local co-Lipschitz constant of f_n at p is less than or equal to 1 for all $p \in \mathcal{T}_0$.*

This case can be worked out in a similar way as the case $p \in \mathcal{R}_0$. We now let $R_p > 0$ be such that $B_{R_p}(p) \subseteq \mathcal{R}_0 \cup \mathcal{T}_0 \cup \mathcal{R}_0'$ and take $0 < r < \frac{1}{4}R_p$. We will show that $f_n(B_r(p)) \supseteq B_r(f_n(p))$. Take $q' \in B_r(f_n(p))$ and notice that $f_n(p) = V_1$, therefore the line $\mathcal{D}_1$ divides the ball $B_r(f_n(p))$ into two polygons, say $\mathcal{P}_1$ and $\mathcal{P}_2$, where $\mathcal{P}_1$ is the half of $B_r(f_n(p))$ below $\mathcal{D}_1$ and $\mathcal{P}_2$ is the half above, so that $B_r(f_n(p)) = \mathcal{P}_1 \cup \mathcal{P}_2$. We divide the proof of this case into two subcases (see Figure 4.4 for an illustration of the second case):

If q' belongs to $\mathcal{P}_1$, we just define q as we did in the case $p \in \mathcal{R}_0$ and repeat the proof to show (4.19), which implies $q \in B_r(p)$.

The case $q' \in \mathcal{P}_2$ is somehow similar, but now we define q to be the point in $\mathcal{L}_q$ such that:

$$\|q_0 - q\|_n = \frac{1}{2}\left[\|q_0 - V_1^s\|_n + \|V_1^s - q'\|_n\right].$$

We also keep the definition of the points q_1^*, q_2^* and q_3^* exactly as before, but now we define

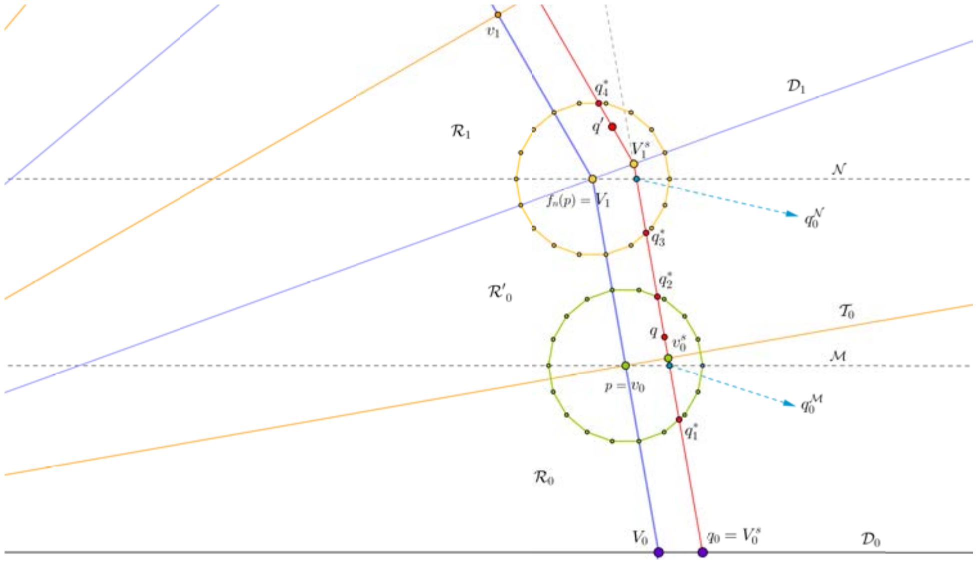


Figure 4.4

the point q_4^* as the intersection between $\partial B_r(f_n(p))$ and the segment line $[V_1^s, V_2^s]$. In this way $f_n(q) = q'$, and to show that $q \in B_r(p)$ we must show again that (4.19) is satisfied. Indeed, notice that in this case, since the ball $\partial B_r(f_n(p))$ is symmetric with respect to $\mathcal{D}_1$ and $\partial B_r(p)$ is symmetric with respect to $\mathcal{T}_0$. Also notice that $\partial B_r(p) = \partial B_r(v_0)$ is a translation of $\partial B_r(f_n(p)) = \partial B_r(V_1)$, so we have:

$$\begin{aligned}
\tfrac{1}{2}\|q_0^{\mathcal{N}} - q'\|_n &= \tfrac{1}{2}\Big(\|q_0^{\mathcal{N}} - V_1^s\|_n + \|V_1^s - q'\|_n\Big) \qquad (4.20)\\
&= \tfrac{1}{2}\Big(\|q_0^{\mathcal{N}} - V_1^s\|_n + \|V_1^s - q'\|_n\Big) \leq \tfrac{1}{2}\Big(\|q_0^{\mathcal{N}} - V_1^s\|_n + \|V_1^s - q_4^*\|_n\Big)\\
&= \tfrac{1}{2}\Big(\|q_0^{\mathcal{N}} - V_1^s\|_n + \|V_1^s - q_3^*\|_n\Big)\\
&= \tfrac{1}{2}\Big(\|q_0^{\mathcal{N}} - V_1^s\|_n + \|V_1^s - q_0^{\mathcal{N}}\|_n + \|q_0^{\mathcal{N}} - q_3^*\|_n\Big)\\
&= 2\|q_0^{\mathcal{M}} - v_0^s\|_n + \tfrac{1}{2}\|q_0^{\mathcal{N}} - q_3^*\|_n \leq 2\|q_0^{\mathcal{M}} - v_0^s\|_n + \|q_0^{\mathcal{N}} - q_3^*\|_n\\
&= \|q_0^{\mathcal{M}} - v_0^s\|_n) + (\|q_0^{\mathcal{M}} - v_0^s\|_n + \|q_0^{\mathcal{M}} - q_1^*\|_n) = \|q_0^{\mathcal{M}} - v_0^s\|_n + (\|v_0^s - q_1^*\|_n)\\
&= \|q_0^{\mathcal{M}} - v_0^s\|_n + (\|v_0^s - q_2^*\|_n)
\end{aligned}$$

Therefore:

$$\begin{aligned}\|q_0 - q\|_n &= \tfrac{1}{2}\Big(\|q_0 - q_0^{\mathcal{N}}\|_n + \|q_0^{\mathcal{N}} - V_1^s\|_n + \|V_1^s - q'\|_n\Big) \\ &\leq \|q_0 - q_0^{\mathcal{M}}\|_n + \|q_0^{\mathcal{M}} - v_0^s\|_n + (\|v_0^s - q_2^*\|_n) = \|q_0 - q_2^*\|_n\end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned}\|q_0 - q\|_n &= \tfrac{1}{2}\Big(\|q_0 - q_0^{\mathcal{N}}\|_n + \|q_0^{\mathcal{N}} - V_1^s\|_n + \|V_1^s - q'\|_n\Big) \\ &\geq \|q_0 - q_0^{\mathcal{M}}\|_n + \|q_0^{\mathcal{M}} - v_0^s\|_n \geq \|q_0 - v_0^s\|_n \geq \|q_0 - q_1^*\|_n\end{aligned}$$

Therefore $f_n(B_r(p)) \supseteq B_r(f_n(p))$ for all $p \in \mathcal{T}_0$, as we wanted.

Case II.4. *The local co-Lipschitz constant of f_n at p is less than or equal to 1 for all $p \in \mathcal{D}_0$.*

Let $p \in \mathcal{D}_0 = \mathcal{O}$; in this case we take $R_p > 0$ such that $B_{R_p}(p) \subseteq (\mathcal{R}'_{n-1} \cup \mathcal{D}_0 \cup \mathcal{R}_0)$. Let $r < \frac{1}{4}R_p$ and $q \in B_r(p)$. Now we have $f_n(p) = p$, so we must show that $B_r(p) \subseteq f_n(B_r(p))$. Take $q' \in B_r(p)$, since $r < \frac{1}{4}R_p$, then q' belongs to the first or to the last side of the polygon $\partial B_{\|q'\|_n}(0)$. Define q to be the point on the same side of the polygon $\partial B_{\|q'\|_n}(0)$ as q' such that

$$\|q - V_0^{\|q'\|_n}\|_n = \tfrac{1}{2}\|q' - V_0^{\|q'\|_n}\|_n.$$

In this way it is clear that $q \in B_r(p)$ and $f_n(q) = q'$, hence $B_r(p) \subseteq f_n(B_r(p))$.

Case II.5. *The local co-Lipschitz constant of f_n at p is less than or equal to 1 for all $p \in (\mathcal{D}_k \cup \mathcal{R}_k \cup \mathcal{T}_k \cup \mathcal{R}'_k),\ k \in \{1, \ldots, n-1\}$.*

Notice that in this case we can follow the same argument as in the case $p \in \mathcal{R}'_0$ using the function

$$f_n(p) = g_n(p) := \mathrm{Rot}_{2k}\left(f_n\left(\mathrm{Rot}_{-k}(p)\right)\right),$$

which we have already shown satisfies $f_n = g_n$ for all $p \in (\mathcal{D}_k \cup \mathcal{R}_k \cup \mathcal{T}_k \cup \mathcal{R}'_k)$, $k > 1$.

This finishes the proof for all cases.

We conclude that the local co-Lipschitz constant c_p of f_n at any point $p \in \mathbb{R}^2$ satisfies $c_p \geq 1$. In particular, the local co-Lipschitz constant of f_n at the origin is equal to 1, so from Proposition 1.2.6, it follows that the global co-Lipschitz constant of the doubling mapping f_n is equal to 1, for all $n = 4m + 2$ with $m \in \mathbb{N} \setminus \{0\}$. □

4.2 Polygonal norms with $4m$ sides

We have shown that for any polygonal norm with $4m + 2$ sides there exists a 2-fold Lipschitz quotient mapping with ratio of constants equal to $1/2$. Now, we are going to show in Theorem 4.2.12, that for all remaining n-norms on the plane and, moreover, for all regular polygonal norms in the plane with $4m$ sides (i.e. for all norms whose unit ball is a regular n-gon with n divisible by 4) every two-fold Lipschitz quotient has ratio of constants strictly less than $^1/_2$. We first show in Theorem 4.2.11 that for $n = 4m$ there is no 2-fold Lipschitz quotient mapping, under the n-norm, that achieves the $^1/_2$ ratio of constants bound of Theorem 2.7.

It will be clear that in order to prove this result, we had in mind Theorem 3.2.5, and moreover, the whole structure of the results in Chapter 3. However, since we do not derive a formula for the $4m$-norm of a given point in terms of its coordinates —unlike the case of the supremum norm where such a formula is very easy to write down— we will calculate only those distances under the $4m$-norm which we will subsequently need in the proof of statements leading to Theorem 4.2.11. In the first part of this section we will learn how to measure some distances under a $4m$-norm.

Lemma 4.2.1. *For $n = 4m$, $m \in \mathbb{N}$, consider the plane under the n-norm $(\mathbb{R}^2, \|\cdot\|_n)$ and for $r > 0$ let $B_r(0)$ denote the ball of radius $r > 0$ under the n-norm. Denote the vertices* *of* $\partial B_r(0)$ *by* $V_0, V_1 \ldots, V_{n-1}$, *as in* (4.1).

Given $a \in (0, \|V_0 - V_1\|_n)$, *let* P_1 *and* P_2 *be the points on the sides* $[V_{n-1}, V_0]$ *and* $[V_0, V_1]$ *of* $\partial B_r(0)$, *respectively, such that* $\|V_0 - P_i\|_n = a$. *Then* $\|P_1 - P_2\|_n = 2a\cos^2(\pi/n)$.

Proof. Since the segment $[P_1, P_2]$ is parallel to the diameter $\mathcal{D}$ of the polygon $\partial B_r(0)$ formed by the vertices $V_{n/4}$, $V_{3n/4}$ we have:

$$\frac{\|P_1 - P_2\|_n}{\|\mathcal{D}\|_n} = \frac{|P_1 - P_2|}{|\mathcal{D}|},$$

where $|\cdot|$ denotes the Euclidean norm. Therefore, $\|P_1 - P_2\|_n = |P_1 - P_2|$, as $\|\mathcal{D}\|_n = |\mathcal{D}|$.

On the other hand, since the segment $[V_0, P_2]$ is parallel to the apothem $\mathcal{A}$ of the polygon $\partial B_r(0)$ through the middle point of the side $[V_{n/4}, V_{n/4+1}]$, we have:

$$\frac{\|V_0 - P_2\|_n}{\|\mathcal{A}\|_n} = \frac{|V_0 - P_2|}{|\mathcal{A}|};$$

recalling that $|\mathcal{A}| = r\cos(\pi/n)$, this is:

$$|V_0 - P_2| = \frac{\|V_0 - P_2\|_n |\mathcal{A}|}{\|\mathcal{A}\|_n} \tag{4.21}$$

$$= \frac{a(r\cos(\pi/n))}{r} = a\cos(\pi/n). \tag{4.22}$$

This also shows that $|V_0 - P_1| = a\cos(\pi/n)$.

Now let U be the intersection point between the x-axis and the perpendicular through P_2. Looking at the triangle P_2, U, V_0, we see that $|P_2 - U| = |P_2 - V_0|\sin(\frac{(n-2)\pi}{2n}) = \sin(\frac{\pi}{2} - \frac{\pi}{n})$. This, together with (4.21), gives

$$|P_2 - U| = |P_2 - V_0| = |V_0 - P_2|\cos(\tfrac{\pi}{n}) = a\cos^2(\pi/n).$$

Since the triangles P_2, U, V_0 and P_1, U, V_0 are congruent we gather that: $\|P_1 - P_2\|_n =$

$|P_1 - P_2| = 2a\cos^2(\pi/n)$. $\square$

In the rest of this section we are going to use a similar notation to the one in Definition 4.1.2:

Definition 4.2.2. Fix $n \in \mathbb{N}$. For $k \in \{0, \dots, n-1\}$ let $\mathcal{D}_k$ denote, as before, the line through the origin that forms an angle of $2k\pi/n$ with the x-axis, i.e. $\mathcal{D}_k$ is the ray through the origin and the vertex V_k^r of the polygon $\partial B_r^n(0)$. In the same way, denote by $\mathcal{T}_k$ the ray through the origin and the middle point v_k^r of the k-th side of the polygon $\partial B_r^n(0)$.

For $\rho > 0$ we define the sets $\mathcal{D}_k^\rho$ and $\mathbb{D}^\rho$ as:

$$\mathcal{D}_k^\rho = \{x \in \mathbb{R}^2 : x \in \mathcal{D}_k \text{ and } \|x\|_n \geq \rho\}; \; \mathbb{D}^\rho = \bigcup_{k=0}^{n-1} \mathcal{D}_k^\rho. \tag{4.23}$$

Finally, for $k \in \{0, \dots, n\}$, let $\mathcal{R}_k$ be the unbounded open region enclosed by the lines $\mathcal{D}_k$ and $\mathcal{T}_k$. Similarly, let $\mathcal{R}'_k$ denote the unbounded open region enclosed by the lines $\mathcal{T}_k$ and $\mathcal{D}_{k+1}$ (with $\mathcal{D}_n := \mathcal{D}_0$). See Figure 4.2 for an illustration of this notation in the case $n = 4m + 2$.

Sometimes it will be convenient to extend all the previous notation to any index $k \in \mathbb{N}$ considering the $k \mod 4$ indexed item, so for example, for all $k \in \mathbb{N}$ we define $\mathcal{D}_k^\rho := \mathcal{D}_j^\rho$, where $j \in \{0, \dots, n-1\}$ and $j \equiv k \mod n$.

For the following lemma and the next proposition we will be using the following construction.

Construction 4.2.3. Given $r > 0$ consider a regular n-gon centred at the origin with radius r and a vertex in the x-axis. As usual, denote its vertices by $V_0^r, V_1^r, \dots, V_{n-1}^r$, we shall also denote the midpoint of the side $[V_i^r, V_{i+1}^r]$ by v_i^r. Using the notation in Definition 4.2.2, consider the intersection point between the line $\mathcal{T}_k$ and the vertical line through V_0^r. Since this intersection point belongs to $\mathcal{T}_k$, it is the midpoint v_0^s of the first side of a polygon $\partial B_s(0)$ for some $s > r$. See Figure 4.5.

Lemma 4.2.4. *Given the notation in Construction 4.2.3, if* $n = 4m$ *for some* $m \in \mathbb{N}$ *then* $s = r(1+\tan^2(\pi/n))$, $\|V_0^r - v_0^s\|_n = r\tan(\frac{\pi}{n})$ *and* $\|V_0^r - V_1^s\|_n = r\tan(\frac{\pi}{n})(2+\tan^2(\frac{\pi}{n}))$.

Proof. Throughout the proof we will be working with the n-norm denoted by $\|\cdot\|_n$, and with the Euclidean norm denoted by $|\cdot|$, as usual. Given the Construction 4.2.3, since v_0^s belongs to the vertical line through V_0^r, it is easy to see that

$$\|V_0^r - v_0^s\|_n = |V_0^r - v_0^s| = r\tan(\pi/n).$$

Now let $d := \|V_0^r - V_1^s\|_n$, consider the polygon $B_d(V_0^r)$ and and denote by Q its $\frac{n}{4}$-th vertex, see Figure 4.5. Let H be the intersection point between the horizontal line through V_1^s and the vertical line through V_0^r. Finally, let V^* denote the intersection between the x-axis and the vertical line through V_1^s.

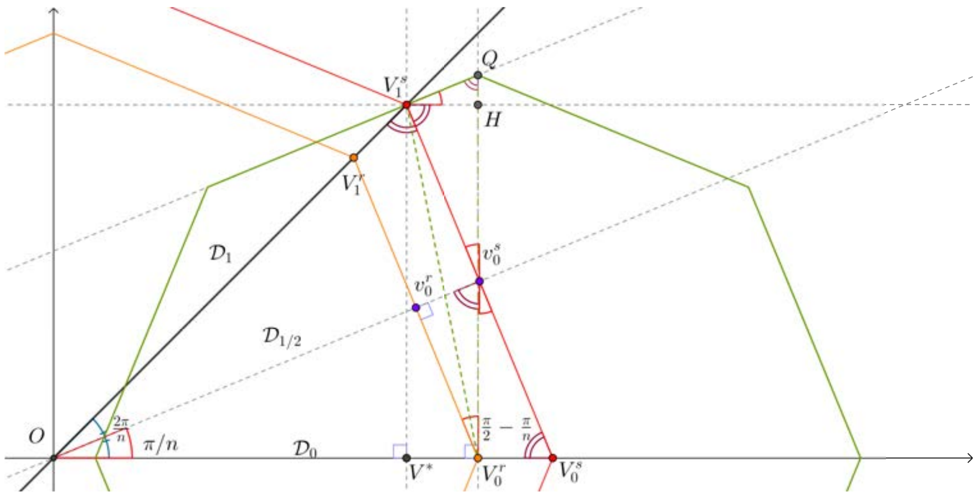


Figure 4.5

Now to find the value of s, consider the triangle whose vertices are V_0^r, v_0^s and V_0^s. From the construction we have,

$$\angle V_0^r V_0^s v_0^s = \tfrac{\pi}{2} - \tfrac{\pi}{n};\ \angle v_0^s V_0^r V_0^s = \tfrac{\pi}{2};\ \angle V_0^r v_0^s V_0^s = \tfrac{\pi}{n}.$$

Now looking at the triangle v_0^r, V_0^r, v_0^s, we see that we also have:

$$\angle v_0^r v_0^s V_0^r = \frac{\pi}{2} - \frac{\pi}{n};\ \angle V_0^r v_0^r v_0^s = \frac{\pi}{2};\ \angle v_0^r V_0^r v_0^s = \frac{\pi}{n}.$$

Therefore, these triangles are similar, and we have:

$$|V_0^s - V_0^r| = \frac{|v_0^s - V_0^r||v_0^s - v_0^r|}{|V_0^r - v_0^r|}. \tag{4.24}$$

Since v_0^r is the middle point of the side $[V_0^r, V_1^r]$, we know that the Euclidean distance $|V_0^r - v_0^r|$ is equal to $r\sin(\pi/n)$. Hence, looking at the triangle v_0^r, V_0^r, v_0^s we gather that:

$$|v_0^s - v_0^r| = \frac{\sin(\pi/n)(r\sin(\pi/n))}{\cos(\pi/n)} = \frac{r\sin^2(\pi/n)}{\cos(\pi/n)}.$$

Substituting these values in (4.24), we get:

$$s - r = |V_0^s - V_0^r| = \frac{r\tan(\pi/n)\left(\frac{r\sin^2(\pi/n)}{\cos(\pi/n)}\right)}{r\sin(\pi/n)} = r\tan^2(\pi/n).$$

Therefore $s = r + |V_0^s - V_0^r| = r(1+\tan^2(\pi/n)) = r\sec^2(\pi/n)$.

Finally, to find the n-distance between V_0^r and V_1^s, notice first that the triangles V_0^r, V_0^s, v_0^s and H, V_1^s, v_0^s are congruent, so $|H - V_1^s| = s - r = r\tan^2(\pi/n)$. Now, looking at the triangle H, V_1^s, Q we get:

$$|H - Q| = (s-r)\tan(\pi/n) = r\tan^3(\pi/n)$$

In the same way, from the triangle O, V_1^s, V^*, we get

$$|V_1^s - V^*| = s\sin(2\pi/n) = r\sec^2(\pi/n)\sin(2\pi/n) = 2r\tan(\pi/n).$$

So the n-distance between V_0^r and V_1^s is given by:

$$\begin{aligned}\|V_0^r - V_1^s\|_n = \|Q - V_0^r\|_n = |Q - V_0^r| &= |Q - H| + |H - V_r^0| \\ &= |Q - H| + |V_1^s - V^*| \\ &= r\left(\tan^3(\pi/n) + 2\tan(\pi/n)\right) = r\tan(\pi/n)(2 + \tan^2(\pi/n)).\end{aligned}$$

This is what we wanted. $\square$

Lemma 4.2.5. *Let $n = 4m$ for some $m \in \mathbb{N}$ and $u > 0$. Define*

$$w_0 := u\cos(2\pi/n) \quad \textit{and} \quad w_1 := u\sec(2\pi/n) \tag{4.25}$$

and let $w \in [w_0, w_1]$. If $q \in \mathcal{D}_1$, then:

1. *$\|V_0^u - q\|_n \geq \|V_0^u - V_1^w\|_n$, whenever $\|q\|_n \geq w$.*

2. *for $p \in \mathcal{D}_0$ with $\|p\|_n \geq u$ we have $\|p - q\|_n \geq \|V_0^u - V_1^w\|_n$, whenever $w_1 \geq \|q\|_n \geq w$.*

Proof. We denote the origin by the letter O. Again notice that we will be using both the Euclidean norm $|\cdot|$ and the n-norm $\|\cdot\|_n$ throughout the proof. Before starting the proof it is worth making the following observation.

Remark 4.2.6. The constants w_0, w_1 are defined by (4.25) in such a way that if we consider the points $V_1^{w_0}, V_1^{w_1}$ on $\mathcal{D}_1$, then the angle $O, V_0^u, V_1^{w_1}$ is a right angle and the angle $V_1^{w_0}, V_0^u, V_1^{w_1}$ is equal to $\frac{2\pi}{n}$ radians, see Figure 4.6. Therefore, whenever we consider a polygon centred at V_0^u, say $B_r(V_0^u)$ with $r \geq \|V_0^u - V_1^{w_0}\|_n$, the intersection between the boundary of this polygon and the line segment $[V_1^{w_0}, V_1^{w_1}]$ (green segment in Figure 4.6), will occur on the $(\frac{n}{4}+1)$-th side of the polygon $B_r(V_0^u)$. This is because, if we define $d := \|V_0^u - V_1^{w_0}\|_n$, then the polygon $B_d(V_0^u)$ has $V_1^{w_0}$ as its $(\frac{n}{4}+1)$-th vertex.

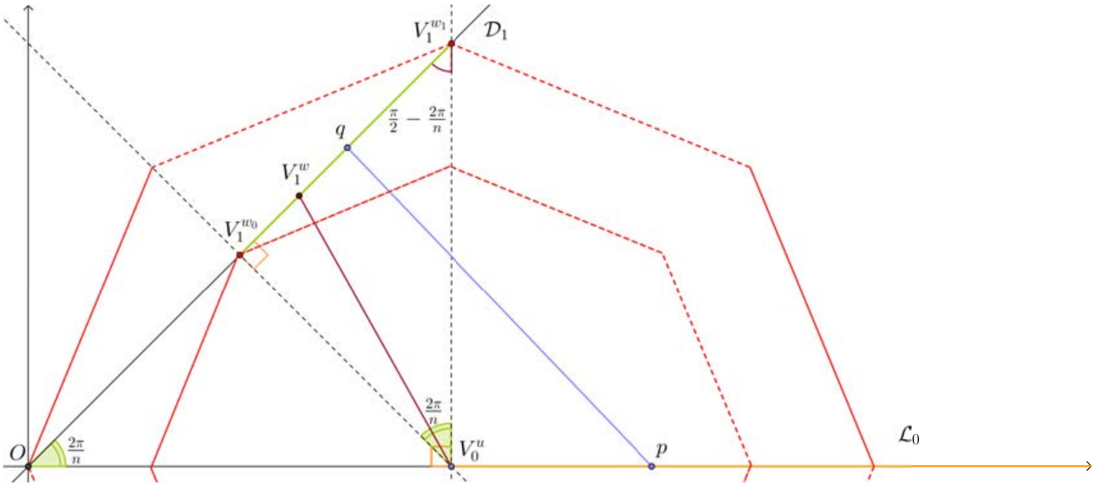


Figure 4.6

First we are going to show the first inequality of the statement, i.e. that for any point q that belongs to $\mathcal{D}_1$ such that $\|q\|_n \geq w$ we have:

$$d^* = \|q - V_0^u\|_n \geq \|V_0^u - V_1^w\|_n =: d. \tag{4.26}$$

We may suppose $\|q\|_n > w$, otherwise $q = V_1^w$ and we are done. See Figure 4.7.

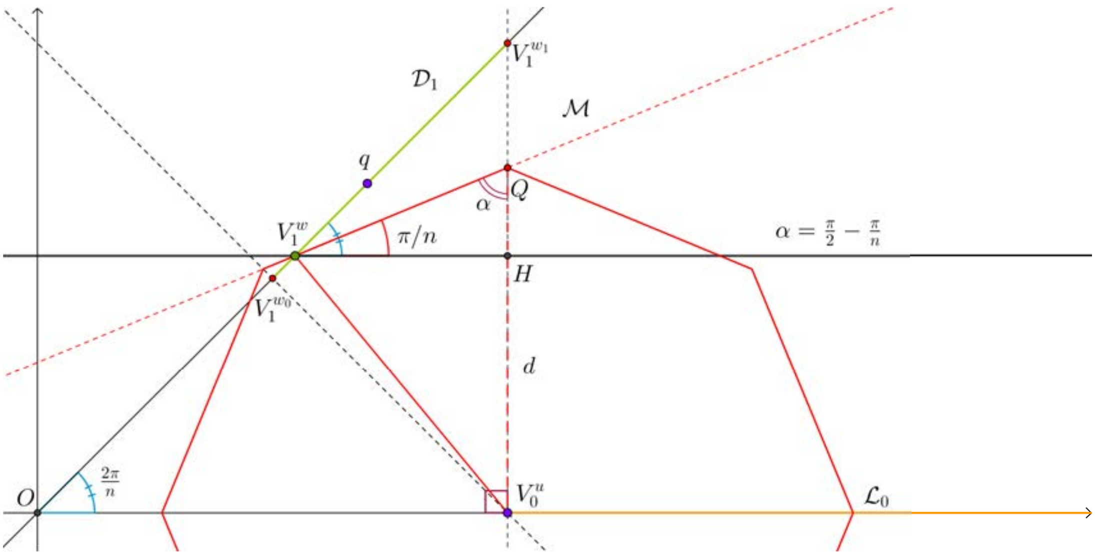


Figure 4.7

Consider the polygon $\partial B_d(V_0^u)$, denote its $(\frac{n}{4})$-th vertex by Q and let H denote the intersection point between the horizontal line through V_1^w and the vertical line through V_0^u. Also denote by $\mathcal{M}$ the line containing the segment $[V_1^w, Q]$.

Since V_1^w belongs to the line segment $[V_1^{w_0}, V_1^{w_1}]$, by the observation made at the beginning of this proof, we know that V_1^w belongs to the $(\frac{n}{4}+1)$-th side of the polygon $\partial B_d(V_0^u)$, therefore the angle $\angle HV_1^wQ = \pi/n$.

On the other hand, since $q \in \mathcal{D}_1$ and $\|q\|_n \geq w$, we have

$$\angle HV_1^wq = \angle V_0^uOq = \tfrac{2\pi}{n} > \tfrac{\pi}{n}.$$

Thus, $q \notin B_d(V_0^u)$, since q belongs to the closed half plane above the line $\mathcal{M}$, and the polygon $B_d(V_0^u)$ belongs to the half plane below $\mathcal{M}$, which means that $d^* \geq d$, as stated in (4.26).

Our next step is to show that for any point $p \in \mathcal{D}_0$ with $\|p\|_n \geq u$ we have:

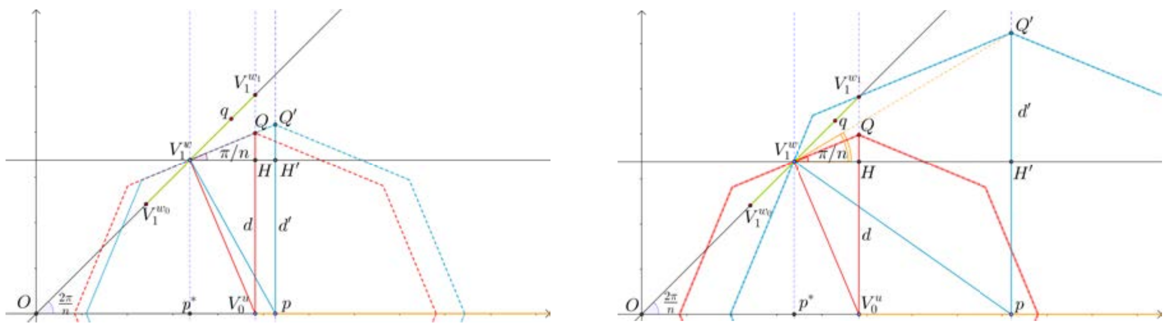
$$d' := \|p - V_1^w\|_n \geq \|V_0^u - V_1^w\|_n =: d. \tag{4.27}$$

Let H and Q be as before and, in the same way, let H' denote the intersection point between the horizontal line through V_1^w and the vertical line through p. See Figure 4.8(a). Also denote by Q' the $n/4$-th vertex of the polygon $\partial B_{d'}(p)$ and finally, let p^* be the intersection point between the x-axis and the vertical line through V_1^w.

Notice that if $0 \leq \angle Q'pV_1^w \leq 2\pi/n$, then V_1^w belongs to the $(\frac{n}{4}+1)$-th side of the polygon $\partial B_{d'}(p)$. In this case we have (see Figure 4.8(a)):

$$\angle pOV_1^w = \tfrac{2\pi}{n}\ ;\ \angle V_1^wQH = \angle V_1^wQ'H' = \tfrac{\pi}{2} - \tfrac{\pi}{n} \ \text{ and } \ \angle H'V_1^wQ' = \angle HV_1^wQ = \tfrac{\pi}{n} \tag{4.28}$$

It is also clear that if $\angle Q'pV_1^w > 2\pi/n$, then V_1^w is no longer on the $(\frac{n}{4}+1)$-th side, so



(a) (b)

Figure 4.8

instead of the last two equations in (4.28), we have:

$$\angle H'V_1^wQ' \geq \angle HV_1^wQ = \tfrac{\pi}{n},$$

see Figure 4.8(b).

In both cases we have:

$$d' = \|p - V_1^w\|_n = |p - Q'| \geq |V_0^u - Q| = d,$$

as p and V_0^u belong to the x-axis and Q' is higher than Q. Thus, (4.27) is satisfied in any case.

Summarising, (4.26) proves the first statement of the present Lemma. To prove the second statement, fix $u > 0$ and let $w \in [w_0, w_1]$ and $q \in \mathcal{D}_0$ be given and assume that $w \leq \|q\|_n \leq w_1$. Now rewrite (4.27) replacing w by $\|q\|_n$:

$$\|p - V_1^{\|q\|_n}\|_n \geq \|V_0^u - V_1^{\|q\|_n}\|_n,$$

and notice that $V_1^{\|q\|_n} = q$, so that we in fact have the following inequality: $\|p - q\|_n \geq \|V_0^u - q\|_n$, and by the first part of the present lemma we have $\|p - q\|_n \geq \|V_0^u - q\|_n \leq \|V_0^u - V_1^w\|_n$. $\square$

Before the next result it is worth recalling the notation given in Definition 4.2, where we defined for each even n, the constant $\mathscr{L}_n$ so that $\text{length}_n\, \partial B^n_r(0) = r\mathscr{L}_n$. In Corollary 4.2.8 we will show that the shortest path joining the rays $\mathcal{D}^r_k$ and $\mathcal{D}^r_{k+1}$ is precisely the side of the polygon $\partial B^n_r(0)$. In other words, that the n-distance between the rays $\mathcal{D}^r_k$ and $\mathcal{D}^r_{k+1}$ is equal to the n-length of a side of the polygon $\partial B^n_r(0)$. We first calculate, in the next lemma, the n-length of a side of a polygon $\partial B^n_r(x)$.

Lemma 4.2.7. *If $n \in \mathbb{N}$ is divisible by 4, then for any $r > 0$, the n-length of a side of the polygon $\partial B^n_r(0)$, is given by:*

$$\tfrac{1}{n} r\mathscr{L}_n = 2r\tan(\pi/n). \tag{4.29}$$

Proof. Let $n = 4m$ for some $m \in \mathbb{N}$ and let $r > 0$. Notice that in order to measure the n-length of a side of the polygon $\partial B^n_r(0)$ under the n-norm, we can repeat the argument used in Lemma 4.2.1. So, denoting the Euclidean norm by $|\cdot|$, we have:

$$\frac{\|V^r_0 - V^r_1\|_n}{\|\mathcal{A}_r\|_n} = \frac{|V^r_0 - V^r_1|}{|\mathcal{A}_r|},$$

where $\mathcal{A}_r$ is an apothem of $B_r(0)$ parallel to the side $[V^r_0, V^r_1]$ of $\partial B^n_r(0)$. Hence, $\|\mathcal{A}_r\|_n = r$ and we have:

$$\|V^r_0 - V^r_1\|_n = \frac{r|V^r_0 - V^r_1|}{|\mathcal{A}_r|} = \frac{r(2r\sin(\pi/n))}{r\cos(\pi/n)} = 2r\tan(\pi/n). \tag{4.30}$$

Thus, the n-length of a side of a regular n-gon is given by $\frac{1}{n} r\mathscr{L}_n = 2r\tan(\pi/n)$. □

Corollary 4.2.8. *Let $n = 4m$ and let $r > 0$. If for some k, $p \in \mathcal{D}_k$ and $q \in \mathcal{D}_{k+1}$, are such that $\|p\|_n \geq r$ and $\|q\|_n \geq r$, then $\|p - q\|_n \geq \frac{1}{n} r\mathscr{L}_n = 2r\tan(\pi/n)$.*

Hence,

$$\text{dist}_n(\mathcal{D}^r_k, \mathcal{D}^r_{k+1}) = 2r\tan(\tfrac{\pi}{n}) = \tfrac{1}{n} r\mathscr{L}_n.$$

Proof. For $n = 4$, $m = 1$ we know that $\|\cdot\|_4$ is the ℓ_1-norm, so we can easily calculate distances, indeed, for any k we have:

$$\begin{aligned}\operatorname{dist}_n(\mathcal{D}_k^r, \mathcal{D}_{k+1}^r) = \operatorname{dist}_4(\mathcal{D}_0^r, \mathcal{D}_1^r) &= \inf\{|x_1 - y_1| + |x_2 - y_2| : x_2 = 0 = y_1, x_1, y_2 \geq r\}\\ &= 2r = 2r\tan(\pi/n).\end{aligned}$$

Assume now that $n > 4$, $m > 1$ and let $p \in \mathcal{D}_k$ and $q \in \mathcal{D}_{k+1}$. Take $r' = \min\{\|q\|_n, \|p\|_n\}$; we know that $r' \geq r$. It is clear that by symmetry we only need to show the statement for $k = 0$. We divide the proof into cases.

Case 1. $r' = \|p\|_n$. In this case we can apply the first statement of Lemma 4.2.5, using $u = w = r'$ so that $p = V_0^{r'} = V_0^u$, $w \geq w_0 = u\cos(\frac{2\pi}{n})$, $w \leq w_1 = \frac{u}{\cos(\frac{2\pi}{n})}$ and $\|q\|_n \geq r' = w$. We get:

$$\|p - q\|_n = \|V_0^{r'} - q\|_n \geq \|V_0^{r'} - V_1^{r'}\|_n = 2r'\tan(\pi/n) \geq 2r\tan(\pi/n),$$

where the second equality comes from (4.29).

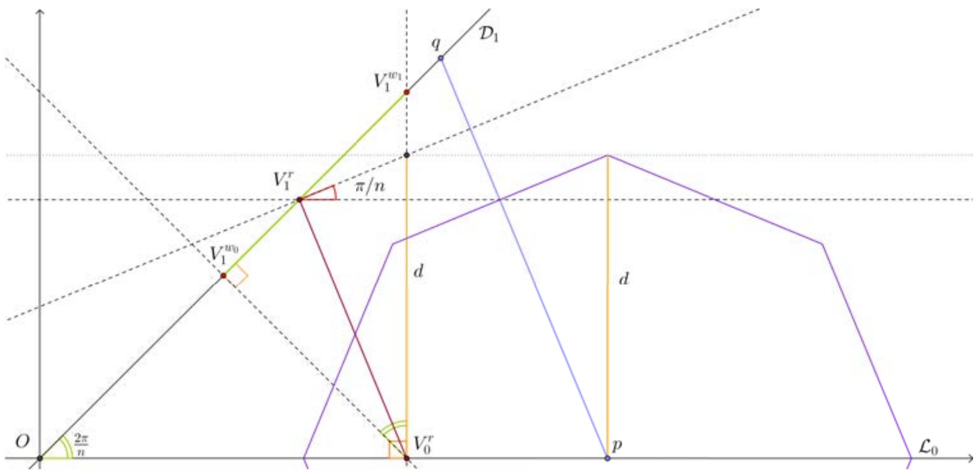


Figure 4.9

Case 2. $r' = \|q\|_n$. Let $d := \|V_0^r - V_1^r\|_n = 2r\tan(\pi/n)$ (see (4.29)) and consider again the notation as in the hypothesis of Lemma 4.2.5, but now set $w = u = r$. We know that $\|p\|_n \geq r = u$ and $\|q\|_n \geq r = w \in [w_0, w_1] = [u\cos(\frac{2\pi}{n}), u\sec(\frac{2\pi}{n})]$. If in addition, $\|q\|_n \leq w_1$ then $\|q\|_n \in [w, w_1] = [r, r\sec(2\pi/n)]$, and we can apply the second statement in Lemma 4.2.5 to get:

$$\|p - q\|_n \geq \|V_0^r - V_1^r\|_n = 2r\tan(\pi/n).$$

Finally, assume that $\|q\|_n > w_1$. See Figure 4.9 for an illustration of this situation. Recall that according to Lemma 4.2.5, $V_1^{w_1}$ is, in this case, the intersection between the perpendicular line through V_0^r and the ray $\mathcal{D}_1$, which exists since we are assuming $n > 4$ (and hence $2\pi/n < \pi/2$). See Figure 4.9. Hence, if $\|q\|_n > w_1$ then the y-coordinate of q, $y(q)$, is greater than the y-coordinate of $V_1^{w_1}$, $y(V_1^{w_1})$, in fact:

$$y(q) > y(V_1^{w_1}) = r\tan(\tfrac{2\pi}{n}) = \frac{2r\tan(\frac{\pi}{n})}{1 - \tan^2(\frac{\pi}{n})} > 2r\tan(\tfrac{\pi}{n}) = d.$$

The last inequality holds since we are assuming that $n > 4$, therefore $\tan(\frac{\pi}{n}) \in (0, 1)$. Hence, $q \notin \bar{B}_d(p)$, since the highest point of a polygon $\partial B_d(p)$ will have y-coordinate equal to d. Therefore, $\|p - q\|_n > d = 2r\tan(\pi/n)$.

Thus, in any case we have $\|p - q\|_n \geq \|V_0^r - V_1^r\|_n = 2r\tan(\frac{\pi}{n})$. Since $V_i^r \in \mathcal{D}_i^r$, $i \in \{1, 2\}$ we conclude that $\mathrm{dist}_n(\mathcal{D}_0^r, \mathcal{D}_1^r) = 2r\tan(\frac{\pi}{n})$ and this finishes the proof. $\square$

After this brief survey about how to measure distances with respect to $4m$-norms we can finally go back to the study of the constants of Lipschitz quotient mappings under $4m$-norms. The next proposition is an analogous version of Proposition 3.2.1 for general $4m$-norms instead of the supremum norm.

Proposition 4.2.9. *Let $n = 4m$ for some $m \in \mathbb{N}$ and let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be an L-Lipschitz and c-co-Lipschitz two-fold mapping with respect to the norm $\|\cdot\|_n$. If $c/L = \frac{1}{2}$ then there exist positive constants κ (defined by (4.32)) and R' such that for all $\rho \geq R'$ if $p \in \mathbb{D}^\rho$ we* *have:*

$$\mathrm{dist}(g(p), \mathbb{D}^{\rho'}) < \kappa,$$

where $\rho' = c(\rho - M)$ *and* $M = \max\{\|p\|_n : g(p) = g(0)\}$.

Proof. First assume that g is a Lipschitz quotient mapping with Lipschitz constant $L = 1$, co-Lipschitz constant $c = 1/2$ and $g(0) = 0$.

Recall that the constant M given by Theorem 2.6 is given by $M = \max\{\|p\|_n : g(p) = g(0)\}$. Hence, we can assume that M and R' are as in the conclusion (1) of Theorem 2.6.

Let $\rho > R'$ and take a point $p \in \mathbb{D}^\rho$, therefore p is a vertex of $\partial B_r(0)$ for some $r \geq \rho$. Since we may perform a rotation of any integer multiple of $2\pi/n$ radians without affecting the Lipschitz and co-Lipschitz constants of g, we may assume without loss of generality that p is the vertex V_0^r of $\partial B_r(0)$.

Set $a := \mathrm{dist}_n(g(V_0^r), \mathbb{D}^{\rho'})$, where $\rho' = c(\rho - M)$. If $a = 0$, there is nothing to prove. Assume $a > 0$. We will define κ in (4.31), but first we show that $a < \|V_0^r - V_1^r\|_n$.

By Lemma 4.2.7 we know that $\|V_0^r - V_1^r\|_n = 2r\tan(\pi/n)$. On the other hand, since $g(V_0^r)$ is not in $\mathbb{D}^{\rho'}$, then $g(V_0^r)$ lies between two of the lines $\mathcal{D}_k$, say $g(V_0^r)$ lies in the region enclosed by $\mathcal{D}_k$ and $\mathcal{D}_{k+1}$ (where k and $k+1$ are taken modulo n). Let $r' := \|g(V_0^r)\|_n$ and consider the polygon $\partial B_{r'}(0)$, whose vertices are $V_i^{r'}$, $i = 0, \dots, n-1$. Notice that, from Theorem 2.6, we know that $r' \geq c(r - M) = \rho'$, therefore both vertices of $\partial B_{r'}(0)$, $V_k^{r'}$ and $V_{k+1}^{r'}$, belong to the set $\mathbb{D}^{\rho'}$, hence:

$$0 < a \leq \|g(V_0^r) - V_k^{r'}\|_n < \|V_{k+1}^{r'} - V_k^{r'}\|_n = 2r'\tan(\pi/n).$$

Since g is a 1-Lipschitz mapping, we have $r' = \|g(V_0^r)\|_n \leq \|V_0^r\|_n = r$, therefore:

$$a < 2r'\tan(\pi/n) \leq 2r\tan(\pi/n) = \|V_0^r - V_1^r\|_n.$$

Thus a is strictly less than the length of a side of the polygon $\partial B_r(0)$ under the n-norm. Once this has been proven, we can take the points P_1 and P_2 (as in Lemma 4.2.1) on the sides $[V_{n-1}^r, V_0^r]$ and $[V_0^r, V_1^r]$ of $\partial B_r(0)$, respectively, such that $\|P_i - V_0^r\|_n = a$.

Let $\gamma : [0, \rho\mathscr{L}_n] \to \partial B_\rho(0)$ be the 1-Lipschitz parametrization of the boundary of the polygon $B_\rho(0)$ with starting point at P_1 given by Corollary 1.3.18 so that $\gamma(0) = P_1$, $\gamma(a) = V_0^r$ and $\gamma(2a) = P_2$. Then, by Theorem 2.6, the curve $g \circ \gamma$ is contained outside of $B_{\rho'}(0)$ with $\operatorname{Ind}_0 g \circ \gamma = 2$.

Now, let $q_1 := g(\gamma(0)) = g(P_1)$ and $q_2 := g(\gamma(2a)) = g(P_2)$, hence $\|q_1\|_n, \|q_2\|_n > \rho'$. Denote by $\mathcal{U}$ the closure of the unbounded region enclosed between the rays $\mathcal{D}_k$, $\mathcal{D}_{k+1}$ and the side $[V_k^{\rho'}, V_{k+1}^{\rho'}]$ of the polygon $\partial B_{\rho'}(0)$. We know that $g(V_0^r) \in \mathcal{U}$. Since g is 1-Lipschitz, for $i = 1, 2$ we have

$$\|q_i - g(V_0^r)\|_n \leq \|P_i - V_0^r\|_n = a = \operatorname{dist}_n(g(V_0^r), \mathbb{D}^{\rho'})$$

so we conclude that $q_1, q_2 \in \mathcal{U}$. Even more, since the region $\mathcal{U}$ is convex we know that $[q_1, q_2] \subseteq \mathcal{U}$. This means that both $g \circ \gamma$ and the segment $[q_1, q_2]$ are contained outside $B_{\rho'}(0)$. Hence if we replace the part of the curve $g \circ \gamma(t)$ that is the image of the points $t \in [0, 2a]$ with the line segment $[q_1, q_2]$ we get a curve of index 2 around the origin which is contained outside $B_{\rho'}(0)$, so from Lemma 1.3.14 we infer that:

$$\|q_1 - q_2\|_n + \operatorname{length}_n(g \circ \gamma|_{[2a, \rho\mathscr{L}_n]}) \geq 2(\rho'\mathscr{L}_n).$$

Using Lemma 4.2.1 and the Lipschitz condition we have:

$$\|q_1 - q_2\|_n = \|g(P_1) - g(P_2)\|_n \leq \|P_1 - P_2\|_n = 2a\cos^2(\pi/n).$$

Hence,

$$\text{length}_n(g \circ \gamma|_{[2a, \mathscr{L}_n \rho]}) \geq 2\mathscr{L}_n(c(\rho - M)) - 2a \cos^2(\pi/n).$$

Also notice that, since g and γ are 1-Lipschitz, we have:

$$\text{length}_n(g \circ \gamma|_{[2a, \rho\mathscr{L}_n]}) \leq \rho\mathscr{L}_n - 2a.$$

So we conclude that

$$\rho\mathscr{L}_n - 2a \geq 2\mathscr{L}_n(c(\rho - M)) - 2a \cos^2(\pi/n).$$

Since we are assuming that $c = 1/2$, the last inequality can be written as: $a \leq \frac{\mathscr{L}_n M}{2(1-\cos^2(\pi/n))}$.

Thus, for the mapping g the conclusion of the present lemma is satisfied if we take $\kappa_1 \in (0, \frac{1}{2}\mathscr{L}_n M(\sin(\pi/n))^{-2})$. Notice that from (4.29) we know that $\mathscr{L}_n = 2n\tan(\pi/n)$, therefore in case $g(0) = 0$ and $L = 1$:

$$\kappa_1 = \frac{2nM}{\sin(\pi/n)\cos(\pi/n)}. \tag{4.31}$$

Now, consider any L-Lipschitz and c-co-Lipschitz two-fold mapping g such that $c/L = 1/2$. Define the Lipschitz quotient mapping $g_1 := \frac{1}{L}(g - g(0))$, which is a 1-Lipschitz, 1/2-co-Lipschitz mapping that maps zero to zero. We have shown that for this mapping, if $M = \max\{\|p\|_n : g_1(p) = 0\}$ then there exist constants $R' > 0$ and $\kappa_1 > 0$ such that $\text{dist}(g_1(p), \mathbb{D}^{\frac{1}{2}(\rho - M)}) < \kappa_1$, for all $p \in \mathbb{D}^\rho$ with $\rho > R'$. Now define the constant

$$\kappa := L\frac{2nM}{\sin(\pi/n)\cos(\pi/n)} + \|g(0)\|_n, \tag{4.32}$$

and take $p \in \mathbb{D}^\rho$ with $\rho > R'$. As we have just shown for the mapping g_1, we can find $p' \in \mathbb{D}^{\frac{1}{2}(\rho - M)}$ such that $\|g_1(p) - p'\|_n < \kappa_1$. Now $Lp' \in \mathbb{D}^S$, where $S = L\frac{1}{2}(\rho - M)$. Since we assume $c/L = 1/2$, we have $L\frac{1}{2}(\rho - M) = c(\rho - M)$, so $Lp' \in \mathbb{D}^{\rho'}$ and

$$\|g(p) - Lp'\|_n = \|Lg_1(p) + g(0) - Lp'\|_n \le L\|g_1(p) - p'\|_n + \|g(0)\|_n < L\kappa_1 + \|g(0)\|_n = \kappa.$$

Hence $\mathrm{dist}(g(p), \mathbb{D}^{\rho'}) < \kappa$. $\square$

As we mentioned before, the previous proposition is an analogous version of Proposition 3.2.1 for $4m$-norms and, as in Chapter 3, this will allow us to prove now an analogous result to Lemma 3.2.3 for $4m$-norms. Notice that in Lemma 3.2.3, unlike in Lemma 4.2.10, we have only one possibility for the location of the point $f(V_i^\rho)$ because in the case $n = 4$ we have that $4 - 2i$ is congruent to $2i \mod 4$.

Lemma 4.2.10. *Let $n = 4m$ for some $m \in \mathbb{N} \setminus \{1\}$, let $f : (\mathbb{R}^2, \|\cdot\|_n \to (\mathbb{R}^2, \|\cdot\|_n)$ be a 2-Lipschitz and 1-co-Lipschitz two-fold mapping with $f(0) = 0$ and let $M = \max\{\|p\|_n : f(p) = 0\}$. There exist positive constants R_0, κ and κ_2 such that for all $\rho \ge R_0$, if $\mathrm{dist}_n(f(V_0^\rho), \mathcal{D}_0^{\rho - M})) < \kappa$, then one of the following is satisfied:*

1. $\mathrm{dist}_n(f(V_i^\rho), \mathcal{D}_{2i}^{\rho-M})) < \kappa$ *and* $\mathrm{dist}_n(f(v_i^\rho), \mathcal{D}_{(2i+1)}^{\rho-M}) < \kappa_2$ $\forall i \in \mathbb{N}$
2. $\mathrm{dist}_n(f(V_i^\rho), \mathcal{D}_{n-2i}^{\rho-M})) < \kappa$ *and* $\mathrm{dist}_n(f(v_i^\rho), \mathcal{D}_{n-(2i+1)}^{\rho-M}) < \kappa_2$ $\forall i \in \mathbb{N}$.

Here V_i^ρ and v_i^ρ are the i-th vertex and middle point of the i-th side of the polygon $\partial B_\rho(0)$, respectively.

Proof. Take κ and R' from Proposition 4.2.9 and let

$$R_0 > \max\left\{R', 10M, \frac{8n\kappa}{\mathcal{L}_n}, 2Mn + \frac{2n\kappa}{\mathcal{L}_n}\right\}.$$

Pick $\rho > R_0$ and consider the vertex V_0^ρ of $\partial B_\rho(0)$. Assume, as in the hypothesis of the present Lemma, that $\mathrm{dist}_n(f(V_0^\rho), \mathcal{D}_0^{\rho-M})) < \kappa$. We first show that either for all $i \in \mathbb{N}$, we have $\mathrm{dist}_n(f(V_i^\rho), \mathcal{D}_{2i}^{\rho-M})) < \kappa$, or that for all $i \in \mathbb{N}$ we have $\mathrm{dist}_n(f(V_i^\rho), \mathcal{D}_{n-2i}^{\rho-M})) < \kappa$.

We will prove that this is true for $i = 1$ and, as we shall see, this will be enough since we will be able to repeat the argument inductively "in one direction".

From Proposition 4.2.9 we know that $\mathrm{dist}_n(f(V_1^\rho), \mathbb{D}^{\rho-M})) < \kappa$. As $\mathbb{D}^{\rho-M}$ is the union of the rays $\mathcal{D}_k^{\rho-M}$ we know that there exists $i \in \mathbb{N}$ such that $\mathrm{dist}_n(f(V_1^\rho), \mathcal{D}_i^{\rho-M})) < \kappa$. Hence, we need to show that $i \in \{2, n-2\}$.

Let $\gamma : [0, \rho\mathscr{L}_n] \to \partial B_\rho(0)$ be a 1-Lipschitz curve that goes once around $\partial B_\rho(0)$, with starting point $\gamma(0) = V_0^\rho$, hence $\gamma(\rho\mathscr{L}_n/n) = V_1^\rho$. Let us define the curves $\gamma_1 = \gamma|_{[0,\rho\mathscr{L}_n/n]}$ and $\gamma_2 = \gamma|_{[\rho\mathscr{L}_n/n, \rho\mathscr{L}_n]}$.

From Theorem 2.6 we know that $f \circ \gamma$ is a closed curve contained outside $B_{\rho-M}(0)$ with $\mathrm{Ind}_0 f \circ \gamma = 2$. Notice that $\rho > R_0 > \frac{2n\kappa}{\mathscr{L}_n}$, hence by Corollary 4.2.8, we get:

$$\kappa < \tfrac{\mathscr{L}_n \rho}{2n} = \tfrac{1}{2} \mathrm{dist}_n(\mathcal{D}_k^\rho, \mathcal{D}_{k+1}^\rho); \quad k = 0, 1, \ldots, n-1. \tag{4.33}$$

Now, the curve $f \circ \gamma$ has index 2 around the origin and the points $f(\gamma(0)) = f(V_0^\rho)$ and $f(\gamma(\rho\mathscr{L}_n/n)) = f(V_1^\rho)$ are at most κ far from the rays $\mathcal{D}_0^{\rho-M}$ and $\mathcal{D}_i^{\rho-M}$, respectively. Therefore —depending on the direction the curve $f \circ \gamma$ is oriented in— the curve $f \circ \gamma_1$ must intersect either all the rays $\mathcal{D}_k^{\rho-M}$ with $0 < k < i$, or all the rays $\mathcal{D}_k^{\rho-M}$ with $n > k > i$.

Assume first that $f \circ \gamma_1$ intersects all the rays $\mathcal{D}_k^{\rho-M}$ with $0 < k < i$. Using again Corollary 4.2.8 and assuming $i \neq 0$, we gather that:

$$\begin{aligned} \mathrm{length}_n(f \circ \gamma_1) &\geq \sum_{k=0}^{i-1} \mathrm{dist}_n(\mathcal{D}_k^{\rho-M}, \mathcal{D}_{k+1}^{\rho-M}) - 2\kappa \\ &= i\tfrac{(\rho-M)\mathscr{L}_n}{n} - 2\kappa. \end{aligned} \tag{4.34}$$

On the other hand, since f is 2-Lipschitz and γ_1 is 1-Lipschitz, we have:

$$\mathrm{length}_n(f \circ \gamma_1) \leq 2\,\mathrm{length}_n(\gamma_1) \leq 2\frac{\rho\mathscr{L}_n}{n}. \tag{4.35}$$

From (4.34) and (4.35) we infer that:

$$2 \geq \frac{i(\rho - M)\frac{\mathscr{L}_n}{n} - 2\kappa}{\rho\frac{\mathscr{L}_n}{n}} = i\frac{\rho - M}{\rho} - \frac{2\kappa}{\rho\frac{\mathscr{L}_n}{n}} \tag{4.36}$$

$$\geq i(1 - \tfrac{M}{10M}) - \frac{2\kappa}{(\frac{8n\kappa}{\mathscr{L}_n})\frac{\mathscr{L}_n}{n}} = i(1 - \tfrac{1}{10}) - \tfrac{1}{4}. \tag{4.37}$$

Hence $i \leq \frac{2+1/4}{1-1/10} = 2 + \frac{1}{2}$. Therefore, in this case, $i \in \{0, 1, 2\}$.

Now the curve $f \circ \gamma_2$ is a curve with starting point $f(V_1^\rho)$, and this point is at most κ-far from $\mathcal{D}_i^\rho$. Also, the end point of $f \circ \gamma_1$ is $f(V_0^\rho)$, which is at most κ-far from $\mathcal{D}_0^\rho$. Since we are assuming that $f \circ \gamma$ is oriented counterclockwise, the curve γ_2 goes from somewhere close to $\mathcal{D}_i^\rho$, intersects all the rays $\mathcal{D}_k$ with $i < k < n$, then pass again through $\mathcal{D}_0^\rho$ and complete another turn around $B_{\rho-M}(0)$. In this case, using the same argument as before, we get:

$$\begin{aligned} \text{length}_n(f \circ \gamma_2) &\geq \sum_{k=i}^{2n-1} \text{dist}_n(\mathcal{D}_k^{\rho-M}, \mathcal{D}_{k+1}^{\rho-M}) - 2\kappa \\ &= (2n - i)\tfrac{(\rho-M)\mathscr{L}_n}{n} - 2\kappa. \end{aligned} \tag{4.38}$$

On the other hand, using the Lipschitz condition we get

$$\text{length}_n(f \circ \gamma_2) \leq 2\,\text{length}_n(\gamma_2) \leq 2\frac{(n-1)\rho\mathscr{L}_n}{n}. \tag{4.39}$$

So now, if we assume that $i \neq 2$, we find from the above equations, (4.38) and (4.39), that:

$$2(n-1)\rho\tfrac{\mathscr{L}_n}{n} \geq (2n - i)(\rho - M)\tfrac{\mathscr{L}_n}{n} - 2\kappa.$$

And this, since we are assuming $i \in \{0, 1\}$, implies:

$$\tfrac{2\kappa n}{\mathscr{L}_n} + M(2n - i) \geq \rho(2n - i - 2(n-1)) = \rho(2 - i) \geq \rho.$$

Thus $\rho \leq \frac{2\kappa n}{\mathcal{L}_n} + 2nM < R_0$. This is a contradiction since we chose $\rho > R_0$, hence in this case $i = 2$.

Assume now that the curve $f \circ \gamma$ is oriented in the opposite direction. Notice that if we let $j := n - i$, following the same idea, we see that (4.34) becomes:

$$\begin{aligned} \text{length}_n(f \circ \gamma_1) &\geq \sum_{k=0}^{j-1} \text{dist}_n(\mathcal{D}_{n-k}^{\rho-M}, \mathcal{D}_{n-k-1}^{\rho-M}) - 2\kappa \\ &= j\tfrac{(\rho-M)\mathcal{L}_n}{n} - 2\kappa. \end{aligned} \tag{4.40}$$

In the same way (4.38) becomes:

$$\begin{aligned} \text{length}_n(f \circ \gamma_2) &\geq \sum_{k=j}^{2n-1} \text{dist}_n(\mathcal{D}_{n-k}^{\rho-M}, \mathcal{D}_{n-k-1}^{\rho-M}) - 2\kappa \\ &= (2n-j)\tfrac{(\rho-M)\mathcal{L}_n}{n} - 2\kappa. \end{aligned} \tag{4.41}$$

Since the inequalities (4.35) and (4.39) stay unchanged, from the above argument we know that $j = 2$, hence $i = n - 2$.

We conclude that $f(V_1^\rho)$ is either κ-close to the ray $\mathcal{D}_2^{\rho-M}$ or κ-close to the ray $\mathcal{D}_{n-2}^{\rho-M}$ depending on the orientation of $f \circ \gamma$ (which is fixed). Hence, we can follow inductively this argument to show that either for all $i \in \mathbb{N}$ we have $\text{dist}_n(f(V_i^\rho), \mathcal{D}_{(2i)}^{\rho-M}) < \kappa$ or for all $i \in \mathbb{N}$ we have $\text{dist}_n(f(v_i^\rho), \mathcal{D}_{n-2i}^{\rho-M}) < \kappa$.

Now, using what we have just proved, we will show that the middle point v_i^ρ of each side of the polygon $\partial B_r(0)$ satisfies the conclusion of the present lemma. Actually we will show that, setting κ_2 according to (4.42), we will have that if $f(V_i^\rho)$ is κ-close to the ray $\mathcal{D}_{2i}^{\rho-M}$ then the point $f(v_i^\rho)$ is κ_2-close to $\mathcal{D}_{2i+1}^{\rho-M}$ for all $i \in \mathbb{N}$, and by symmetry it will be clear that in the other case, when $f(V_i^\rho)$ is κ-close to the ray $\mathcal{D}_{n-2i}^{\rho-M}$, we have $f(v_i^\rho)$, κ_2-close to the ray $\mathcal{D}_{n-(2i+1)}^{\rho-M}$.

Define the constant

$$\kappa_2 := M\tfrac{\mathcal{L}_n}{n} + \kappa + 1 = 2M\tan(\tfrac{\pi}{n}) + \kappa + 1. \tag{4.42}$$

Again, it will be enough to show that $f(v_i^\rho)$ is κ_2-close to $\mathcal{D}_{2i+1}^{(\rho-M)}$ for $i = 0$ only. Assume for a contradiction that for some $\rho > R_0$ we have

$$\operatorname{dist}_n(f(v_0^\rho), \mathcal{D}_1^{\rho-M}) \geq \kappa_2. \tag{4.43}$$

From Theorem 2.6, we know that $f([V_0^\rho, V_1^\rho])$ is contained outside $B_{\rho-M}(0)$ and from the first part of the present lemma, we know that the points $f(V_0^\rho)$ and $f(V_1^\rho)$ are at most κ-far from the rays $\mathcal{D}_0^{\rho-M}$ and $\mathcal{D}_2^{\rho-M}$, respectively.

Let W_0 and W_2 be the points on the rays $\mathcal{D}_0$ and $\mathcal{D}_2$, respectively, such that

$$\|W_0 - f(V_0^\rho)\|_n < \kappa;\ \ \|W_2 - f(V_1^\rho)\|_n < \kappa \text{ and } \ \|W_i\|_n \geq \rho - M, \text{ for } \ i = 0, 2. \tag{4.44}$$

Recall that we chose $\rho > R_0$ so that the points $f(V_0^\rho)$ and $f(V_1^\rho)$ are on different sides of $\mathcal{D}_1$ (see (4.33)). Hence the curve $f([V_0^\rho, V_1^\rho])$ must intersect the ray $\mathcal{D}_1^{\rho-M}$, so there must be a point $v^* \in [V_0^\rho, V_1^\rho]$ such that $f(v^*) \in \mathcal{D}_1^{\rho-M}$.

Assume first that v^* belongs to the first half of the segment $[V_0^\rho, V_1^\rho]$. Note that $\|f(v^*)\|_n, \|W_0\|_n \geq \rho - M$ and that, by (4.43), we have $\|f(v_0^\rho) - f(v^*)\|_n \geq \kappa_2$. Using now Lemma 1.3.10 and Corollary 4.2.8, and (4.43) we get:

$$\begin{aligned}
\operatorname{length}_n(f([V_0^\rho, v_0^\rho])) &= \operatorname{length}_n(f([V_0^\rho, v^*])) + \operatorname{length}_n(f([v^*, v_0^\rho])) \\
&\geq \|W_0 - f(v^*)\|_n - \|W_0 - f(V_0^\rho)\|_n + \|f(v_0^\rho) - f(v^*)\|_n \\
&\geq (\rho - M)\tfrac{\mathcal{L}_n}{n} - \kappa + \kappa_2 > (\rho - M)\tfrac{\mathcal{L}_n}{n} + M\tfrac{\mathcal{L}_n}{n} = \rho\tfrac{\mathcal{L}_n}{n}.
\end{aligned}$$

Therefore,

$$\text{length}_n(f([V_0^\rho, v_0^\rho])) > \rho \frac{\mathscr{L}_n}{n} = \|V_0^\rho - V_1^\rho\|_n = 2\|V_0^\rho - v_0^\rho\|_n.$$

This is not possible, since f is a 2-Lipschitz mapping. It is clear that if we now assume that v^* belongs to the second half of the segment $[V_0^\rho, V_1^\rho]$ we can follow the same argument, considering the point W_2 instead of W_0, to reach a contradiction. Thus $f(v_0^\rho)$ is κ close to $\mathcal{D}_1^{\rho-M}$ Thus, if for all $i \in \mathbb{N}$ $f(V_i^\rho)$ is κ-close to the ray $\mathcal{D}_{2i}^{\rho-M}$, then

$$\text{dist}_n(f(v_i^\rho), \mathcal{D}_{2i+1}^{\rho-M}) < \kappa_2,$$

for all $\rho > R_0$ and $i \in \mathbb{N}$.

This finishes the proof because if we now assume that $\text{dist}_n(f(V_i^\rho), \mathcal{D}_{n-2i}^{\rho-M}) < \kappa$, i.e. if $f \circ \gamma$ is oriented in the opposite direction, then the same argument follows for V_{n-1} and v_{n-1}^ρ instead of V_1^ρ and v_0^ρ, so we get:

$$\text{dist}_n(f(v_i^\rho), \mathcal{D}_{n-(2i+1)}^{\rho-M}) < \kappa_2, \quad \text{for all } \rho > R_0 \text{ and } i \in \mathbb{N}.$$

□

With the last two results in hand we are now able to show that the ratio of constants of any Lipschitz quotient mapping under a $4m$-norm is strictly less than $1/2$. For the proof of this result we will use a construction that is valid only for $4m$-norms with $m > 1$. However, we can use Proposition 1.2.4 to derive the same result for the 4-norm (or ℓ_1-norm) from Theorem 3.2.5. See, further, Theorem 4.2.12.

Theorem 4.2.11. *Let $n = 4m$ for some $m \in \mathbb{N} \setminus \{1\}$, and let $\|\cdot\|_n$ denote the n-norm on $\mathbb{R}^2$.*

If $f : (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ is an L-Lipschitz and c-co-Lipschitz 2-fold mapping then $c/L < 1/2$.

Proof. By Theorem 2.7 we know that $c/L \leq 1/2$. Assume on the contrary that $c/L = 1/2$.

Without loss of generality we can assume further that $f(0) = 0$, $c = 1$ and $L = 2$.

From Theorem 2.6, we know that there exists $R > 0$ such that $f(\partial B_\lambda(0)) \subseteq \mathbb{R}^2 \setminus B_{\lambda-M}(0)$ for all $\lambda > R$, with $M := \max\{\|p\|_n : p \in f^{-1}(0)\}$.

By Proposition 4.2.9 we know that there exists $R' > 0$ and a constant κ such that for some $k \in \{0, \dots, n-1\}$ we have $\mathrm{dist}_n(f(V_0^\lambda), \mathcal{D}_k^{\lambda-M}) < \kappa$. Since we may perform a rotation of $\frac{2k\pi}{n}$ radians without changing the Lipschitz and co-Lipschitz constants of f, we can assume without loss of generality that $k = 0$, so that $\mathrm{dist}_n(f(V_0^\lambda), \mathcal{D}_0^{\lambda-M}) < \kappa$.

Now, from Lemma 4.2.10, there exists $R_0 > R'$ such that for $\lambda > R_0$ and $k \in \mathbb{N}$ we have $\mathrm{dist}_n(f(V_k^\lambda), \mathcal{D}_{2k}^{\lambda-M}) < \kappa$ and $\mathrm{dist}_n(f(v_k^\lambda), \mathcal{D}_{2k+1}^{\lambda-M}) < \kappa_2$, or $\mathrm{dist}_n(f(V_k^\lambda), \mathcal{D}\lambda - M_{n-2k}) < \kappa$ and $\mathrm{dist}_n(f(v_k^\lambda), \mathcal{D}_{n-(2k+1)}^{\lambda-M}) < \kappa_2$; for the definition of these constants see (4.31) and (4.42).

Now we set the new constants

$$\kappa_1 := \frac{\kappa}{\cos(\pi/n)} \ ; \ \kappa' = \max\{\kappa, \kappa_1, \kappa_2\} \ ; \ \delta := \frac{2\kappa'}{\tan(\pi/n)} + M + 1 \ ; \tag{4.45}$$
$$R^* := \max\left\{\delta, \tfrac{4\delta}{\tan^3(\pi/n)}, \tfrac{M+\delta\cos(2\pi/n)}{1-\cos(2\pi/n)}, \tfrac{M}{\tan^2(2\pi/n)}\right\}.$$

Of course, $\kappa_1 > \kappa$ and $\frac{4\delta}{\tan^3(\pi/n)} > \delta$ for $n \geq 4$ but we add these constants in the definitions of κ' and R^* respectively, in order to simplify the proofs of subsequent inequalities.

Pick $r > \max\{R, R_0, R^*\}$ and consider the polygon $\partial B_r(0)$. Since $r > R'$ we know that either *1* or *2* of the statement of Lemma 4.2.10 is satisfied for all the vertices V_i^r, v_i^r of $\partial B_r(0)$. For simplicity, we will work out this proof under the assumption that *1* is satisfied but it will be clear that by symmetry the same proof will work under the assumption *2* of Lemma 4.2.10. So assume *1* of Lemma 4.2.10, hence, in particular, the first vertex V_0^r and the middle point v_0^r of the first side of the polygon $\partial B_r(0)$ satisfy:

$$\mathrm{dist}_n(f(V_0^r), \mathcal{D}_1^{r-M}) < \kappa \quad \text{and} \quad \mathrm{dist}_n(f(v_0^r), \mathcal{D}_1^{r-M}) < \kappa_2. \tag{4.46}$$

See Figure 4.10 for an illustration.

Now, as we shall see, it is not possible that both points, $f(V_0^r)$ and $f(v_0^r)$, are outside $\overline{B}_{r+\delta}(0)$. For, if $f(V_0^r), f(v_0^r) \in \mathbb{R}^2 \setminus \overline{B}_{r+\delta}(0)$, then, let $W_0 \in \mathcal{D}_0^{\rho-M}$ be as in (4.44)), so that $\|W_0 - f(V_0^r)\|_n < \kappa$, similarly let W_1 be the point on $\mathcal{D}_1^{\rho-M}$ such that $\|f(v_0^r) - W_1\|_n < \kappa_2$. Then using the second part of the statement of Lemma 4.2.5 for $u = w = r+\delta$, $p = W_0$ and $q = W_1$, we get:

$$\begin{aligned}\|f(V_0^r) - f(v_0^r)\|_n &> \|W_0 - W_1\|_n - (\kappa + \kappa_2) \geq \|V_0^{r+\delta} - V_1^{r+\delta}\|_n - 2\kappa' \\ &= 2(r+\delta)\tan(\tfrac{\pi}{n}) - 2\kappa' \geq 2r\tan(\tfrac{\pi}{n}) + 2\left(\frac{2\kappa'}{\tan(\frac{\pi}{n})}\right)\tan(\tfrac{\pi}{n}) - 2\kappa' \\ &> 2r\tan(\tfrac{\pi}{n}) = 2\|V_0^r - v_0^r\|_n.\end{aligned}$$

This is impossible since f is a 2-Lipschitz mapping. Therefore $f(V_0^r) \in \overline{B}_{r+\delta}(0)$ or $f(v_0^r) \in \overline{B}_{r+\delta}(0)$.

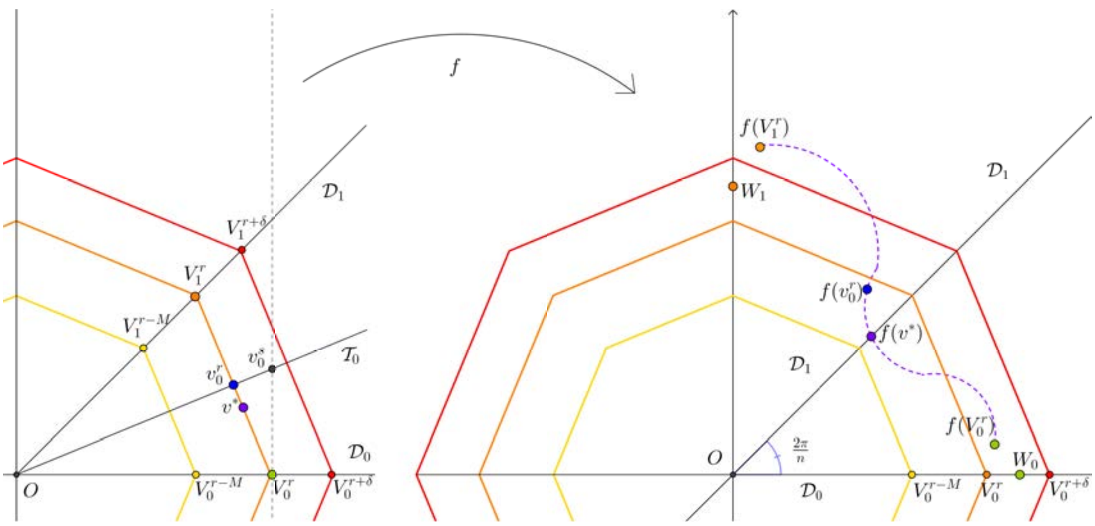


Figure 4.10

Case 1. Assume that $f(V_0^r) \in \overline{B}_{r+\delta}(0)$.

In this case $f(V_0^r)$ is at most κ-far from the ray $\mathcal{D}_0^{r-M}$, and we also have $r - M \leq \|f(V_0^r)\|_n \leq r + \delta$, therefore $|\|f(V_0)\|_n - \|V_0^r\|_n| = |\|f(V_0)\|_n - r \leq \max\{\delta, M\}$. This implies that the point $f(V_0)$ lays in a region which is a subset of $B_\kappa(V_0^{r-M}) \cup \mathcal{E}$, where

$\mathcal{E}$ is the rectangle determined by the horizontal lines that are at distance κ from the x-axis and the vertical line through $V_0^{r+\delta}$ and V_0^{r-M}. It is easy to see that in both cases, $f(V_0) \in \mathcal{E}$ or $f(V_0) \in B_\kappa(V_0^{r-M})$, we have:

$$\|f(V_0^r) - V_0^r\|_n \leq \max\{M + \kappa, \delta + \kappa\} = \delta + \kappa \tag{4.47}$$

Now (going back to the domain of f), let v_0^s be the intersection point between $\mathcal{T}_0$ and the vertical line through V_0^r. By Lemma 4.2.4, we know that $s = r(1 + \tan^2(\pi/n))$, and $\|V_0^r - v_0^s\|_n = r\tan(\pi/n)$. On the other hand, from Lemma 4.2.10 we also know that $f(v_0^s)$ is at most κ_2-far from the ray $\mathcal{D}_1^{s-M}$ and that this point, $f(v_0^s)$, belongs to the complement of $B_{s-M}(0)$, by Theorem 2.6. Hence, by (4.46) there exists $s' > 0$ such that the vertex $V_1^{s'}$ satisfies:

$$\|f(v_0^s) - V_1^{s'}\|_n < \kappa_2 \text{ and } \|V_1^{s'}\|_n = s' \geq s - M. \tag{4.48}$$

From (4.48) and (4.47) we have:

$$\begin{aligned} \|f(V_0^r) - f(v_0^s)\|_n &\geq \|f(V_0^r) - V_1^{s'}\|_n - \|f(v_0^s) - V_1^{s'}\|_n \geq \|f(V_0^r) - V_1^{s'}\|_n - \kappa_2 \\ &\geq \|V_0^r - V_1^{s'}\|_n - \|f(V_0^r) - V_0^r\|_n - \kappa_2 \\ &\geq \|V_0^r - V_1^{s'}\|_n - (\delta + \kappa) - \kappa_2. \end{aligned} \tag{4.49}$$

Now we will use the first statement of Lemma 4.2.5 with $u = r$, $q = V_1^{s'}$ and $w = s - M$. In order to use this Lemma, we first need to check that the inequalities

$$s' \geq s - M \text{ and } \cos(\tfrac{2\pi}{n}) \leq 1 + \tan^2(\tfrac{\pi}{n}) - M/r \leq \sec(\tfrac{2\pi}{n}) \tag{4.50}$$

are satisfied. The first of these follows from (4.48). To prove the remaining inequalities, the notice that, since we chose

$$r > R^* \geq \frac{M + \delta\cos(2\pi/n)}{1 - \cos(2\pi/n)} > \frac{M}{1 - \cos(2\pi/n)},$$

we have $\frac{M}{r} < 1 - \cos(2\pi/n)$, hence $\cos(2\pi/n) < 1 - \frac{M}{r} < 1 + \tan^2(\frac{\pi}{n}) - M/r$. The last inequality in (4.50) follows from the fact that $\cos(2\pi/n) = \cos^2(\pi/n) - \sin^2(\pi/n) \leq \cos^2(\pi/n)$. Therefore,

$$\sec(2\pi/n) = \frac{1}{\cos(2\pi/n)} \geq \frac{1}{\cos^2(\pi/n)} = \sec^2(\pi/n) = 1 + \tan^2(\pi/n).$$

which proves the second inequality. This allows us to use Lemma 4.2.5 to conclude that $\|V_0^r - V_1^{s'}\|_n \geq \|V_0^r - V_1^{s-M}\|_n$. This last inequality together with (4.49), gives us:

$$\begin{aligned}
\|f(V_0^r) - f(v_0^s)\|_n &\geq \|V_0^r - V_1^{s'}\|_n - (\delta + \kappa) - \kappa_2 \geq \|V_0^r - V_1^{s-M}\|_n - (\delta + \kappa) - \kappa_2 \\
&\geq \|V_0^r - V_1^s\|_n - \|V_1^s - V_1^{s-M}\|_n - (\delta + \kappa + \kappa_2) \\
&= \|V_0^r - V_1^s\|_n - M - (\delta + \kappa + \kappa_2).
\end{aligned}$$

Now, recalling the definition of the constants in (4.45) we can see that $\kappa, \kappa_2 < \kappa' < \delta$ and $M < \delta$. Also, from Lemma 4.2.4 we know that $\|V_0^r - V_1^s\|_n = r\tan(\pi/n)(2 + \tan^2(\pi/n))$. Finally, since we chose $r > R^*$, we have, $r\tan^3(\pi/n) > 4\delta$. From these observations, following the last inequality, we get:

$$\begin{aligned}
\|f(V_0^r) - f(v_0^s)\|_n &\geq \|V_0^r - V_1^s\|_n - 4\delta \geq r\tan(\pi/n)(2 + \tan^2(\pi/n)) - 4\delta \\
&= 2r\tan(\pi/n) + r\tan^3(\pi/n) - 4\delta > 2r\tan(\pi/n).
\end{aligned}$$

Again from Lemma 4.2.4, we know that $\|V_0^r - v_0^s\|_n = r\tan(\pi/n)$, hence, from the last inequality we conclude that $\|f(V_0^r) - f(v_0^s)\|_n > 2\|V_0^r - v_0^s\|_n$, which is not possible since we are assuming that f is a 2-Lipschitz mapping.

Case 2. Assume that $f(V_0^r) \notin \overline{B}_{r+\delta}(0)$ and, therefore, $f(v_0^r) \in \overline{B}_{r+\delta}(0)$.

First we are going to show that

$$\|V_1^{r-M} - V_0^{r+\delta}\|_n > \|V_1^r - V_0^r\|_n + 2\kappa'. \tag{4.51}$$

As we did in Lemma 4.2.5, let $d = \|V_1^{r-M} - V_0^{r+\delta}\|_n$ and consider the polygon $\partial B_d(V_0^{r+\delta})$. Consider the vertical line through $V_0^{r+\delta}$ and let Q denote the vertex of $\partial B_d(V_0^{r+\delta})$ that belongs to this vertical line (see Figure 4.11).

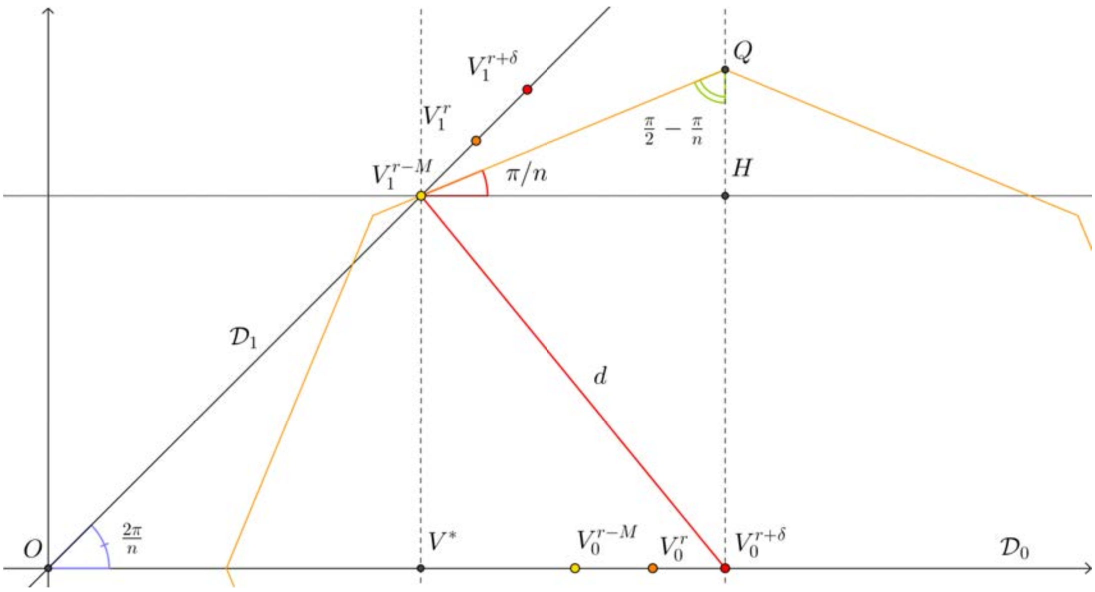


Figure 4.11

Now, consider the horizontal line through V_1^{r-M} and let H be the intersection between this line and the segment $[V_0^{r+\delta}, Q]$. Finally let V^* be the intersection between the x-axis and the vertical line through V_1^{r-M}.

Recall that we chose $r > R^*$, so that $r > \frac{M+\delta\cos(2\pi/n)}{1-\cos(2\pi/n)}$, therefore:

$$(r+\delta)\cos(2\pi/n) \leq r - M < r \leq (r+\delta)\sec(2\pi/n). \tag{4.52}$$

If we let $u = r + \delta$ and $w = r - M$, these last inequalities would mean, using the notation of Lemma 4.2.5, that $w \in [w_0, w_1]$. Therefore, following Remark 4.2.6 we conclude that the point V_1^{r-M} is on the $(\frac{n}{4} + 1)$-th side of the polygon $B_d(V_0^{r-\delta})$

Notice that given this construction, we have:

$$\angle V_1^{r-M} QH = \frac{\pi}{2} - \frac{\pi}{n}; \ \ \angle HV_1^{r-M} Q = \frac{\pi}{n}; \ \ \angle V_1^{r-M} OV^* = \frac{2\pi}{n}; \ \ \angle OV_1^{r-M} V^* = \frac{\pi}{2} - \frac{2\pi}{n}.$$

Therefore:

$$\|Q - H\|_n = |Q - H| = |V_0^{r+\delta} - V^*| \tan(\pi/n) \tag{4.53}$$

$$\|H - V_0^{r+\delta}\|_n = |V^* - V_1^{r-M}| = (r - M) \sin(2\pi/n). \tag{4.54}$$

Now, to find the value of $|V^* - V_0^{r+\delta}|$, we look at the triangle V^*, O, V_1^{r-M} and we find out that $|V^* - O| = (r - M) \cos(\frac{2\pi}{n})$. Therefore,

$$|V_0^{r+\delta} - V^*| = r + \delta - (r - M) \cos(\frac{2\pi}{n}).$$

Substituting this value in (4.53), we get:

$$\begin{aligned} |Q - H| &= \left[r \left(1 - \cos(\tfrac{2\pi}{n})\right) + M \cos(\tfrac{2\pi}{n}) + \delta \right] \tan(\tfrac{\pi}{n}) \\ &= \left[r \left(2 \sin^2(\tfrac{\pi}{n})\right) + M \left(\cos^2(\tfrac{\pi}{n}) - \sin^2(\tfrac{\pi}{n})\right) + \delta \right] \tan(\tfrac{\pi}{n}). \end{aligned}$$

Hence, using this last equality and (4.54):

$$\begin{aligned}
d &= \|V_1^{r-M} - V_0^{r+\delta}\|_n = |Q - H| + |H - V_0^{r+\delta}| \\
&= \left[r\left(2\sin^2(\tfrac{\pi}{n})\right) + M\left(\cos^2(\tfrac{\pi}{n}) - \sin^2(\tfrac{\pi}{n})\right) + \delta\right]\tan(\tfrac{\pi}{n}) + (r-M)2\sin(\tfrac{\pi}{n})\cos(\tfrac{\pi}{n}) \\
&= 2r\left[\frac{\sin^3(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\cos^2(\frac{\pi}{n})}{\cos(\frac{\pi}{n})}\right] - M\left[\frac{\cos^2(\frac{\pi}{n})\sin(\frac{\pi}{n}) + \sin^3(\frac{\pi}{n})}{\cos(\frac{\pi}{n})}\right] + \delta\tan(\tfrac{\pi}{n}) \\
&= (2r - M + \delta)\tan(\tfrac{\pi}{n}).
\end{aligned}$$

Recalling the definition of δ and that the n-length of a side of a polygon of radius r is equal to $2r\tan(\pi/n)$, we gather that:

$$\begin{aligned}
\|V_1^{r-M} - V_0^{r+\delta}\|_n - \|V_1^r - V_0^r\|_n &= (\delta - M)\tan(\pi/n) \\
&> \left[\left(\tfrac{2\kappa'}{\tan(\pi/n)} + M\right) - M\right]\tan(\pi/n) = 2\kappa'.
\end{aligned}$$

This gives (4.51), as we wanted.

Now, from Proposition 4.2.9, we know that $f(V_0^r)$ is at most κ-far from the ray $\mathcal{D}_0$ and, in this case, we have $\|f(V_0^r)\|_n > r + \delta$, so that we can find a point $V_0 \in \mathcal{D}_0$, with $\|V_0\|_n \geq r + \delta$ such that $\|f(V_0^r) - V_0\|_n < \kappa$. In the same way, given that in this case $f(v_0^r) \in B_{r+\delta}(0)$, we can find a point $V_1 \in \mathcal{D}_1$ such that $\|f(v_0^r) - V_1\|_n < \kappa_2$ and $r - M \leq \|V_1\|_n < r + \delta$. Recall that in (4.52) we already checked the conditions to use Lemma 4.2.5 for $u = r + \delta$ and $w = r - M$, and we can take $p = V_0$ and $q = V_1$. This gives:

$$\|V_0 - V_1\|_n \geq \|V_0^{r+\delta} - V_1^{r-M}\|_n.$$

Combining this inequality with (4.51) we obtain:

$$\|f(V_0^r) - f(v_0^r)\|_n \geq \|V_0 - V_1\|_n - (\kappa + \kappa_2) \geq \|V_0^{r+\delta} - V_1^{r-M}\|_n - (\kappa + \kappa_2)$$
$$> \|V_0^r - V_1^r\|_n + 2\kappa' - (\kappa + \kappa_2) \geq \|V_0^r - V_1^r\|_n = 2\|V_0^r - v_0^r\|_n.$$

This is not possible since f is a 2-Lipschitz mapping.

In any case we arrive at a contradiction, so we conclude that $c/L < 1/2$. □

We can easily derive now the more general result.

Theorem 4.2.12. *Let $n = 4m$ for some $m \in \mathbb{N}$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ whose unit ball is a (possibly rotated) regular polygon with n sides. Every 2-fold Lipschitz quotient mapping $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ has ratio of constants strictly less than $1/2$.*

In particular, this includes the cases of the ℓ_1 and ℓ_∞ norms.

Proof. This follows from Theorem 3.2.5, Theorem 4.2.11 and Proposition 1.2.4. □

We have shown then that for every norm whose unit ball is a regular polygon with $4m$ sides, every Lipschitz quotient mapping with $\max_{x\in\mathbb{R}^2} \#g^{-1}(x) = 2$ will satisfy $c/L < 1/2$. In the previous chapter we were able to calculate the exact value of the Lipschitz and co-Lipschitz constants of the doubling mapping for the supremum norm, getting $c = 1, L = 3$, see Proposition 3.1.3. In the next proposition we find sharper bounds for the constants of the doubling mapping f_n under the n-norm, for any $n = 4m$.

Proposition 4.2.13. *If $n = 4m$ for some $m \in \mathbb{N}$, then the Lipschitz and co-Lipschitz constants, L_n and c_n, of the doubling mapping f_n satisfy $L_n \geq 2 + \tan^2(\frac{\pi}{n})$ and $c_n \leq 1$. Hence,*

$$\frac{c_n}{L_n} \leq \frac{1}{2 + \tan^2(\frac{\pi}{n})}.$$

Proof. First notice that the inequality $c_n \leq 1$ is obvious since for all $r > 0$, $f_n(\partial B_r(0)) = \partial B_r(0)$, so $B_{cr}(0)$ is not contained in $f_n(B_r(0))$ for any $c > 1$.

To prove that $L_n \geq 2 + \tan^2(\frac{\pi}{n})$, take a point $p_0 = (\rho, 0) = V_0^\rho$ with $\rho > 0$. By the definition of the doubling mapping we know that $f_n(p_0) = p_0$. Recall the notation given in Definition 4.2.2 and let $R > 0$ be sufficiently small so that $B_R(p_0)$ does not intersect the rays $\mathcal{T}_0$ and $\mathcal{T}_{n-1}$. Take $r < R$ and consider the polygon $\partial B_r(p_0)$. Let us denote the vertices of the polygon $\partial B_r(p_0)$ by $U_0, \ldots, U_{n-1}$ placing the indices as usual. Since $r < R$ we know that, if p denotes the vertex $U_{n/4}$ of $\partial B_r(p_0)$, then p belongs to the region $\mathcal{R}_0$. Hence if $\rho' := \|p\|_n = \|U_{n/4}\|_n$, then $p \in [V_0^{\rho'}, v_0^{\rho'}]$. Therefore $f_n(p)$ is the point on the segment $[V_0^{\rho'}, V_1^{\rho'}]$ such that $\|V_0^{\rho'} - f_n(p)\|_n = 2\|V_0^{\rho'} - p\|_n$. So we have the three points $V_0^{\rho'}$, p and $f_n(p)$ on the same line segment $[V_0^{\rho'}, V_1^{\rho'}]$ and $\|p - V_0^{\rho'}\|_n = \|p - f_n(p)\|_n$. See Figure 4.12 for an illustration.

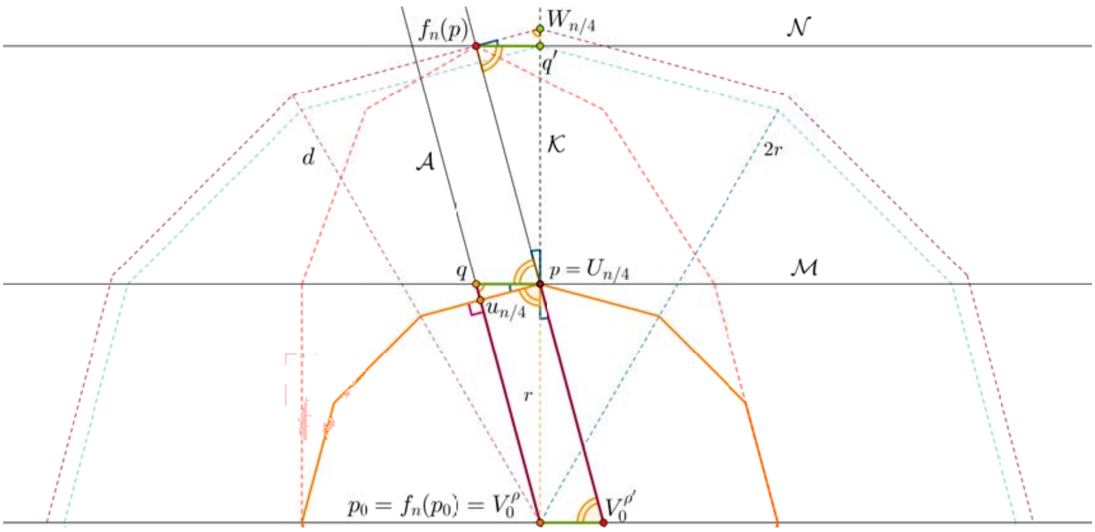


Figure 4.12

Now denote by $\mathcal{M}$ and $\mathcal{N}$ the lines parallel to the x-axis that pass through the points p and $f_n(p)$, respectively. Also let $\mathcal{K}$ be the line perpendicular to $\mathcal{M}$ and $\mathcal{N}$ through the point p_0 and denote by q' the intersection between $\mathcal{K}$ and $\mathcal{N}$. Finally, consider the line $\mathcal{A}$ that contains the points p_0 and $u_{n/4}$, where $u_{n/4}$ denotes the middle point of the side $[U_{n/4}, U_{n/4+1}]$ of the polygon $B_r(p_0)$.

Notice that $\mathcal{A}$ is parallel to the line segment $[V_0^{\rho'}, V_1^{\rho'}]$ that contains $f_n(p)$ and that $\mathcal{K}$

intersects the line segment $[V_0^{\rho'}, V_1^{\rho'}]$ at the point p, to the right of $u_{n/4}$ and below $f_n(p)$. Therefore $f_n(p)$ lies between the lines $\mathcal{A}$ and $\mathcal{K}$, hence: the left angle between $\mathcal{M}$ and the segment $[p, f_n(p)]$ is equal to the angle $\angle p\, V_0^{\rho'}\, p_0 = \frac{\pi}{2} - \frac{\pi}{n}$, so looking at the triangle $f_n(p), p, p_0$, we get:

$$\angle f_n(p)\, p_0\, q' < \angle q\, V_0^{\rho}\, p = \tfrac{\pi}{2} - (\tfrac{\pi}{2} - \tfrac{\pi}{n}) = \tfrac{\pi}{n};$$

so we have $\angle f_n(p)\, p_0\, q' < \pi/n < 2\pi/n$. Hence if we define $d := \|p_0 - f_n(p)\|_n$, the last inequality means that the point $f_n(p)$ belongs to the $(\frac{n}{4}+1)$-th side of the polygon $\partial B_d(p_0)$. In other words, if we denote by $W_0, \ldots, W_{n-1}$ the vertices of the polygon $\partial B_d(p_0)$, then $f_n(p) \in [W_{n/4}, W_{(n/4)+1}]$. Recalling that $f_n(p_0) = p_0$, we now have:

$$d := \|f_n(p_0) - f_n(p)\|_n = |p_0 - W_{n/4}| = |p_0 - p| + |p - q'| + |q' - W_{n/4}|,$$

where $|\cdot|$ denotes the Euclidean norm. So now we need to compute each of the values on the right hand side. We already know that $|p_0 - p| = r$, and since the triangles $f_n(p), q', p$ and $p, V_0^{\rho'}, p_0$ are congruent, we get $|p - q'| = |p_0 - p| = r$, so the previous equation turns into:

$$\|f_n(p_0) - f_n(p)\|_n = 2r + |q' - W_{n/4}| \tag{4.55}$$

To find $|q' - W_{n/4}|_n$, consider again the line $\mathcal{A}$ and let $q \in \mathcal{A} \cap \mathcal{M}$. Notice that

$$\angle p\, p_0\, q = \angle p\, V_0^{\rho}\, u_{n/4} = \tfrac{1}{2}\angle p\, p_0\, U_{(n/4)+1} = \pi/n,$$
$$\angle u_{n/4}\, p\, p_0 = \tfrac{\pi}{2} - \tfrac{\pi}{n} \;\text{ and }\; \angle p\, u_{n/4}\, p_0 = \tfrac{\pi}{2} = \angle p\, u_{n/4}\, q.$$

In particular $\mathcal{A}$ is parallel to the line segment $[V_0^{\rho'}, p]$, so that the quadrilateral $p_0, q, p, V_0^{\rho'}$ is a parallelogram and $|p - q| = |p_0 - V_0^{\rho'}|$. Also notice that since the triangles $f_n(p), q', p$ and $V_0^{\rho'}, p_0, p$ are congruent we have $|p - q| = |f_n(p) - q'|$. On the other hand looking at the triangles $q, u_{n/4}, p$ and $W_{n/4}, q', f_n(p)$ we see:

$$\begin{aligned}
\angle q\, u_{n/4}\, p &= \tfrac{\pi}{2} = \angle W_{n/4}\, q'\, f_n(p) \\
\angle q\, p\, u_{n/4} &= \tfrac{\pi}{2} - (\tfrac{\pi}{2} - \tfrac{\pi}{n}) = \tfrac{\pi}{n} = \angle W_{n/4}\, f_n(p)\, q' \\
\angle u_{n/4}\, q\, p &= \tfrac{\pi}{2} - \tfrac{\pi}{n} = \angle f_n(p)\, W_{n/4}\, q'
\end{aligned}$$

We calculate first the side $|u_{n/4} - q|$ of the triangle $q, u_{n/4}, p$: Since $|u_{n/4} - p| = r\sin(\frac{\pi}{n})$ (recall that the side of the polygon $\partial B_r(0)$ has Euclidean length $2r\sin(\frac{\pi}{n})$), we get:

$$|p - q| = \frac{|u_{n/4} - p|}{\cos(\frac{\pi}{n})} = \frac{r\sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})}\tan(\tfrac{\pi}{n}).$$

Now since $|p - q| = |f_n(p) - q'|$, we can calculate the side $|q' - W_{n/4}|$ of the triangle $W_{n/4}, q', f_n(p)$ and we get

$$|q' - W_{n/4}| = \tan(\tfrac{\pi}{n})|f_n(p) - q'| = r\tan^2(\tfrac{\pi}{n}).$$

Finally, substituting this value in equation (4.55) we conclude that:

$$\|f_n(p_0) - f(p)\|_n = 2r + r\tan^2(\tfrac{\pi}{n}) = \left(2 + \tan^2(\tfrac{\pi}{n})\right)\|p_0 - p\|_n.$$

Thus, the Lipschitz constant, L_n, of the doubling mapping f_n must satisfy $L_n \geq 2 + \tan^(\frac{\pi}{n})$ and therefore $c_n/L_n \leq 1/(2 + \tan^2(\frac{\pi}{n}))$. □

The following conjecture is a generalisation of Theorem 4.2.12 and Proposition 4.2.13:

Conjecture 4.2.14. *Let $n = 4m$ for some $m \in \mathbb{N}$, and let $\|\cdot\|_n$ denote the n-norm on $\mathbb{R}^2$.*

If $f : (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ is an L-Lipschitz and c-co-Lipschitz 2-fold mapping *then*

$$\frac{c}{L} \leq \frac{1}{2 + \tan^2(\frac{\pi}{n})}.$$

The equality is achieved for $f = f_n$, where f_n is the doubling mapping defined by Definition 4.2.

Remark 4.2.15. In Proposition 4.2.13 we only showed that for $n = 4m$, we have $c_n \leq 1$ and $L_n \geq 2 + \tan^2(\frac{\pi}{n})$, where L_n and c_n are the Lipschitz and co-Lipschitz constants of the doubling mapping. A detailed analysis of various points $p \in \mathbb{R}^2$ shows that the co-Lipschitz constant c and the Lipschitz constant L of f_n do satisfy:

$$c = 1 \quad \text{and} \quad L = 2 + \tan^2(\tfrac{\pi}{n}). \tag{4.56}$$

However, we decided not to include the proof of (4.56) since it would only be relevant if we had a way to prove that for any L-Lipschitz and c-co-Lipschitz 2-fold mapping we have $c/L \leq c_n/L_n$, where c_n and L_n are the constants of the doubling mapping f_n. The next chapter shows some positive results in this direction for the case $n = 4$.

Chapter 5
Is 1/3 an upper bound for the ratios of constants of 2-fold Lipschitz quotient mappings on $(\mathbb{R}^2, \|\cdot\|_\infty)$?

In view of Proposition 3.2.4, the Example 3.1.1 defined in Section 3.1 and other examples we have considered, our conjecture is that for any 2-fold Lipschitz quotient mapping f from the plane to itself, endowed with the supremum norm, the ratio between the co-Lipschitz and Lipschitz constants of f is less than or equal to 1/3. In this chapter we present some partial results in this direction.

We show that, if we assume that the mapping f maps squares centred at the origin into squares centred at the origin (but not necessarily fixing them), then $c/L \leq 1/3$. On the other hand we get the same inequality for the ratio between co-Lipschitz and Lipschitz constants if we make some differentiability assumptions. We divided this chapter into two sections, in the first one we will work under assumptions slightly weaker than in Proposition 3.2.4. In the second we will work with differentiability assumptions.

5.1 Results mapping squares to squares

As we have just mentioned, in this section we will see, in Proposition 5.1.3, that we can relax the hypothesis of Proposition 3.2.4 and derive the same result. For this we will need a couple of lemmas regarding the behaviour of Lipschitz quotients along boundaries of balls. As we shall see in the next Lemma, a Lipschitz quotient mapping that maps balls centred at the origin into balls centred at the origin must do it in an "increasing" fashion, in the following sense.

Lemma 5.1.1. *Let $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ be a Lipschitz quotient mapping with $f(0) = 0$. Assume there is a function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $r > 0$ we have $f(\partial B_r(0)) = \partial B_{\omega(r)}(0)$. Then ω is strictly increasing.*

Proof. First notice that for all $r > 0$, $f(B_r(0)) = B_{r_*}(0)$ for some $r_* > 0$. Indeed,

$$f(B_r(0)) = f\left(\bigcup_{0<s<r} \partial B_s(0)\right) \cup \{0\} = \left(\bigcup_{0<s<r} f(\partial B_s(0))\right) \cup \{0\} = \left(\bigcup_{0<s<r} \partial B_{\omega(s)}(0)\right) \cup \{0\}.$$

Since f is a continuous open mapping, $f(B_r(0))$ must be a connected open set containing 0. We conclude that $f(B_r(0))$ is an open ball around the origin, say $B_{r_*}(0)$.

Hence, for $r > 0$ we have:

$$f(\overline{B}_r(0)) = f(B_r(0)) \cup f(\partial B_r(0)) = B_{r_*}(0) \cup \partial B_{\omega(r)}(0).$$

Since $f(\overline{B}_r(0))$ is a connected set, we infer that $\omega(r) \le r_*$. On the other hand, since $f(\overline{B}_r(0))$ is closed, the same equation above implies that $\omega(r) \ge r_*$ (otherwise, we get $f(\overline{B}_r(0)) = B_{r_*}(0)$, and the latter is not a closed set). Thus $\omega(r) = r_*$ and we conclude that $f(B_r(0)) = B_{\omega(r)}(0)$ for all $r > 0$.

Now, if we assume that $0 < r_1 < r_2$, then

$$\partial B_{\omega(r_1)}(0) = f(\partial B_{r_1}(0)) \subseteq f(B_{r_2}(0)) = B_{\omega(r_2)}(0) \subseteq B_{\omega(r_1)}(0).$$

so that $\omega(r_1) < \omega(r_2)$. Thus ω is an increasing function. $\square$

Furthermore, using Proposition 1.2.11 we can prove the following generalization of Lemma 5.1.1.

Lemma 5.1.2. *Let $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ be a Lipschitz quotient mapping with $f(0) = 0$, such that for all $r > 0$ we have $f(\partial B_r(0)) = \varphi_r$ where φ_r is a simple closed curve with index one around the origin. If $0 < r < s$, then $\varphi_r \subseteq I(\varphi_s)$. Here $I(\varphi)$ denotes the interior of the bounded component of $\mathbb{R}^2 \setminus \varphi$. In other words, $I(\varphi)$ is the bounded component of $\mathbb{R}^2 \setminus \varphi$ as a subset of the space $\mathbb{R}^2 \setminus \varphi$.*

Proof. First we are going to show that, given the conditions of the statement, if $0 < r < s$, then $\varphi_r \cap \varphi_s = \emptyset$. Assume on the contrary that there exists a point $y \in \varphi_r \cap \varphi_s$.

On the one hand, since $y \in \varphi_s$, from Proposition 1.2.11 we get:

$$y \in \varphi_s = f(\partial B_s(0)) = \partial(f(B_s(0))).$$

On the other hand, since $y \in \varphi_r$, we have:

$$y \in \varphi_r = f(\partial B_r(0)) \subseteq f(B_s(0)).$$

Thus, $y \in (\partial f(B_s(0))) \cap f(B_s(0))$, which is impossible since f is an open mapping.

Now we show that $f(B_r(0)) \subseteq I(\varphi_r)$ for all $r > 0$. Let $r > 0$ and take $x \in B_r(0)$. Assume for a contradiction that $f(x) \notin I(\varphi_r)$. Since $B_r(0)$ is convex, the line segment $[0, x]$ is contained in $B_r(0)$. On the other hand, since $0 \in I(\varphi_r)$ and $f(x) \notin I(\varphi_r)$, there exists a point $x' \in [0, x]$ such that $f(x') \in \varphi_r$. Therefore, if $m = \|x'\|$,

$$x' \in \partial B_m(0) \text{ and } f(x') \in f(\partial B_m(0)) = \varphi_m.$$

Since $\varphi_r \cap \varphi_s = \emptyset$ for $r \neq s$, we have $m = r$. Hence, $x' \in \partial B_r(0)$ and $x' \in [0, x] \subseteq B_r(0)$. This is impossible, thus $f(B_r(0)) \subseteq I(\varphi_r)$. Finally, if we now assume that $0 < r < s$, we have:

$$\varphi_r = f(\partial B_r(0)) \subseteq f(B_s(0)) \subseteq I(\varphi_s),$$

as we wanted. $\square$

We are now able to show that the ratio of constants of a two fold Lipschitz quotient mapping under the supremum norm, that maps squares centred at the origin to squares centred at the origin cannot be bigger than $1/3$.

Proposition 5.1.3. *Let $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ be a 2-fold Lipschitz quotient mapping. Assume there is a function $\omega : \mathbb{R} \to \mathbb{R}$ such that for all $r > 0$ we have $f(\partial B_r(0)) = \partial B_{\omega(r)}(0)$. Then the Lipschitz constant L and the co-Lipschitz constant c of f satisfy $c/L \leq 1/3$.*

Proof. As before we may assume that $f(0) = 0$ and $c = 1$, we need to show that $L \geq 3$. Assume indirectly that $L < 3$, then by Proposition 3.2.4 we know that there exists $\rho > 0$ such that some corner of the square $\partial B_\rho(0)$, say $P = P_j^\rho$, is not mapped to a corner, i.e.

$$\operatorname{dist}(f(P), A_0) > 0 \quad \text{where} \quad A_0 = \{(x, y) \in \mathbb{R}^2 : |x| = |y|\}.$$

Since we may perform a rotation through integer multiples of $\pi/2$ without changing the Lipschitz and co-Lipschitz constants of f, we will assume that $P = \rho V$, where $V := (1, -1)$ and that $f(P) \in \{(x, y) \in \mathbb{R}^2 : y < 0,\ |x| < |y|\}$, i.e. $f(P)$ belongs to a horizontal side of $\partial B_{\omega(\rho)}(0)$. Let $0 < \kappa < \operatorname{dist}(f(P), A_0)$, consider the square $B_\kappa(P)$ and denote its corners by $P_0, \ldots, P_3$, starting at the bottom right corner and placing the indices counterclockwise. By the co-Lipschitz property we know that $\overline{B}_\kappa(f(P)) \subseteq f(\overline{B}_\kappa(P))$. Denote the corners of the square $\partial B_\kappa(f(P))$ by $Q_0, \ldots, Q_3$ placing the indices as before. Note that $[Q_1, Q_2]$ is a horizontal line segment. Finally, for $i \in \{0, \ldots, 3\}$ pick $q_i \in \overline{B}_\kappa(P)$ such that $f(q_i) = Q_i$, see Figure 5.1.

From the Lipschitz property we get:

$$3\|q_1 - q_2\|_\infty > L\|q_1 - q_2\|_\infty \geq \|f(q_1) - f(q_2)\|_\infty = \|Q_1 - Q_2\|_\infty = 2\kappa,$$

therefore,

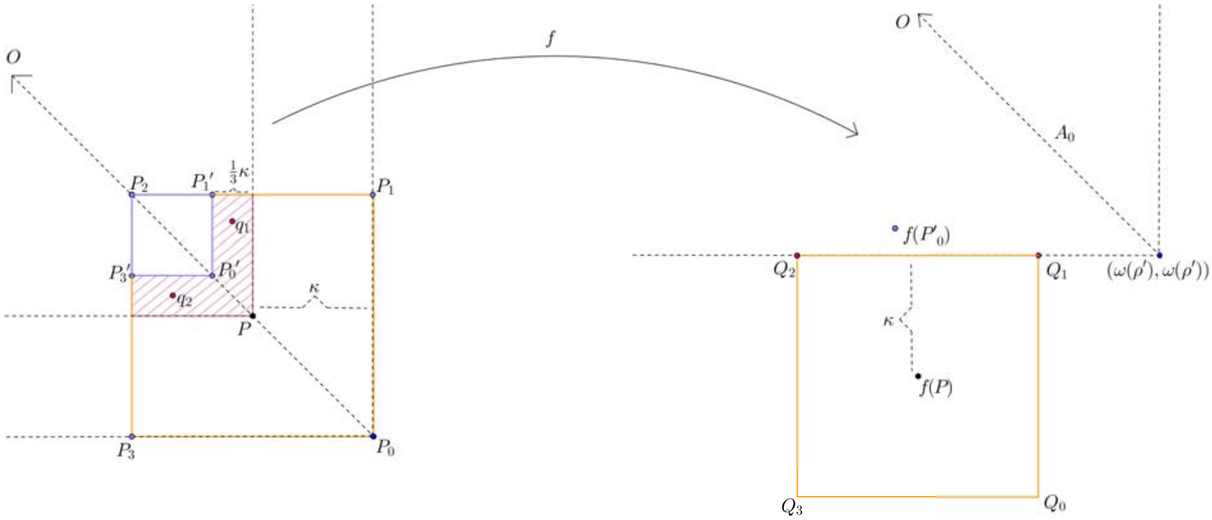


Figure 5.1

$$\|q_1 - q_2\|_\infty > \frac{2}{3}\kappa. \tag{5.1}$$

Observe that we have $\|Q_1\|_\infty = \|Q_2\|_\infty$, and so $\|q_1\|_\infty = \|q_2\|_\infty$ (because from Lemma 5.1.1 we know that ω is injective). We show now that this observation together with (5.1), implies that $\rho' := \|q_i\|_\infty \geq \rho - \frac{1}{3}\kappa$, $i = 1, 2$. Indeed, if we assume that $\rho' < \rho - \frac{1}{3}\kappa$, then both points q_1, q_2 belong to $(B_{\rho - \frac{1}{3}\kappa}(0) \cap \overline{B}_\kappa(P))$ —this is, in Figure 5.1 the green square on the left. Therefore, for $i = 1, 2$, we would get:

$$q_i \in B_{\frac{1}{3}\kappa}(\tilde{P}_r) \quad \text{where} \quad r := \rho - \tfrac{2}{3}\kappa, \quad \text{and} \quad \tilde{P}_r := \tfrac{r}{\rho}P = rV. \tag{5.2}$$

The latter implies $\|q_1 - q_2\|_\infty \leq \frac{2}{3}\kappa$, which we have just shown in (5.1) is impossible. This establishes

$$\rho' = \|q_1\|_\infty = \|q_2\|_\infty \geq \rho - \tfrac{1}{3}\kappa.$$

Now consider the square $\partial B_{\frac{1}{3}\kappa}(\tilde{P}_r)$, where r is as in (5.2), and denote its corners by

$P_0', \dots, P_3'$ in the same way as before. Since

$$\|P_0'\|_\infty = r + \tfrac{1}{3}\kappa = \rho - \tfrac{1}{3}\kappa \le \rho',$$

from Lemma 5.1.1, we get:

$$\omega(\|P_0'\|_\infty) \le \omega(\rho') = \|Q_1\|_\infty = \|f(P)\|_\infty - \kappa,$$

(see Figure 5.1.1 for an illustration). This means that $\|f(P_0')\|_\infty \le \|f(P)\|_\infty - \kappa$. Hence:

$$\|f(P) - f(P_0')\|_\infty \ge \|f(P)\|_\infty - \|f(P_0')\|_\infty \ge \kappa.$$

On the other hand, by the Lipschitz property of f we have:

$$\|f(P) - f(P_0')\|_\infty \le L\|P - P_0'\|_\infty < 3\|P - P_0'\|_\infty = 3(\tfrac{1}{3}\kappa) = \kappa.$$

Thus $\kappa \le \|f(P) - f(P_0')\|_\infty < \kappa$, a contradiction. Therefore we must have $L \ge 3$. □

5.2 Results using differentiability assumptions

In this section we will be assuming some differentiability properties of the Lipschitz quotient mapping f. More precisely, we will assume the existence of some differentiability points when f is restricted to some curve γ. Hence, whenever γ is a fixed parametrized curve, and p is a point belonging to the image of γ, say $p = \gamma(t_0)$, it will be convenient to say that $f \circ \gamma$ is differentiable at p, meaning that $f \circ \gamma$ is differentiable at t_0.

These sort of differentiability assumptions come from the observation that the corners of big enough squares centred at the origin might play a particular role regarding the behaviour of such mappings. For instance, see Propositions 3.2.1, 3.2.4 and 4.2.9. In the general case we do not have much information about the behaviour of these mappings around corners of centred squares. However, we might be able to say something about the ratio of the Lipschitz and co-Lipschitz constants of the Lipschitz quotient f, if we could find a linear approximation of the restriction of f to $\partial B_r(0)$ at some corner of the square $\partial B_r(0)$. We will see, in Propositions 5.2.1 and 5.2.2, that this is the case if we assume that the relevant derivative has some specific directions. Notice that even if we can not guarantee the existence of such differentiability points, since the mapping $f \circ \gamma : [0,1] \to \mathbb{R}^2$ is Lipschitz, in the sense that for every $t_1, t_2 \in [0,1]$ we have $\|f \circ \gamma(t_1) - f \circ \gamma(t_2)\| \leq \alpha L |t_1 - t_2|$, where α is the Lipschitz constant of γ, we have that, $f \circ \gamma$ is differentiable at almost all $t \in [0,1]$.

The following proposition is a generalization of Proposition 5.1.3 in the case when the images of corners are not on the main diagonals (see Proposition 3.2.4).

Proposition 5.2.1. *For every $\rho > 0$ let γ_ρ be the curve describing the square $\partial B_\rho(0)$. Let $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ be a 2-fold Lipschitz quotient mapping and let N be as in Corollary 2.5. Assume that for some $\rho > N$ the curve $f \circ \gamma_\rho$ is differentiable at some corner P of the square $\partial B_\rho(0)$ and that the tangent at this point is parallel to one of the sides of $\partial B_{\|f(P)\|_\infty}(0)$. Then, $c/L \leq 1/3$.*

Proof. As before we will assume that $f(0) = 0$ and $c = 1$, and we will then see that the assumption $L < 3$ leads to a contradiction. Let $L = 3 - \alpha < 3$. Let N and $\varepsilon > 0$ be as in Corollary 2.5, therefore f is injective in every neighbourhood $B_\varepsilon(x)$ with $\|x\|_\infty > N$. By hypothesis we can find a point P with $\|P\|_\infty := \rho > N$ such that $f \circ \gamma_\rho$ is differentiable at P. Consider a 1-Lipschitz parametrization $\varphi : [-4\rho, 4\rho] \to \partial B_\rho(0)$ of $\partial B_\rho(0)$ such that $\varphi(0) = P$ oriented counterclockwise, so that the tangent to the curve $f \circ \varphi$ at this point $f(P)$ is parallel to a side of the square $\partial B_\rho(0)$.

Assume that the derivative of the curve $f \circ \varphi$ at the point P is parallel to the x-axis, so the tangent vector to the curve $f \circ \varphi$ at the point P has the form $(k, 0)$ for some $k \in \mathbb{R}$. Therefore, if we denote by $x(f)$ and $y(f)$ the coordinate functions of f, $f(z) = (x(f(z)), y(f(z)))$, then we can find $\delta \in (0, \varepsilon)$ such that:

$$x(f(\varphi(t))) = x(f(\varphi(0))) + kt + o_1(t) = x(f(P)) + kt + o_1(t), \tag{5.3}$$

$$y(f(\varphi(t))) = y(f(\varphi(0))) + o_2(t) = y(f(P)) + o_2(t), \tag{5.4}$$

with $|o_1(t)| < \frac{|kt|}{2}$ and $|o_2(t)| < \frac{\alpha}{2}|t|$, whenever $0 < |t| < \delta$.

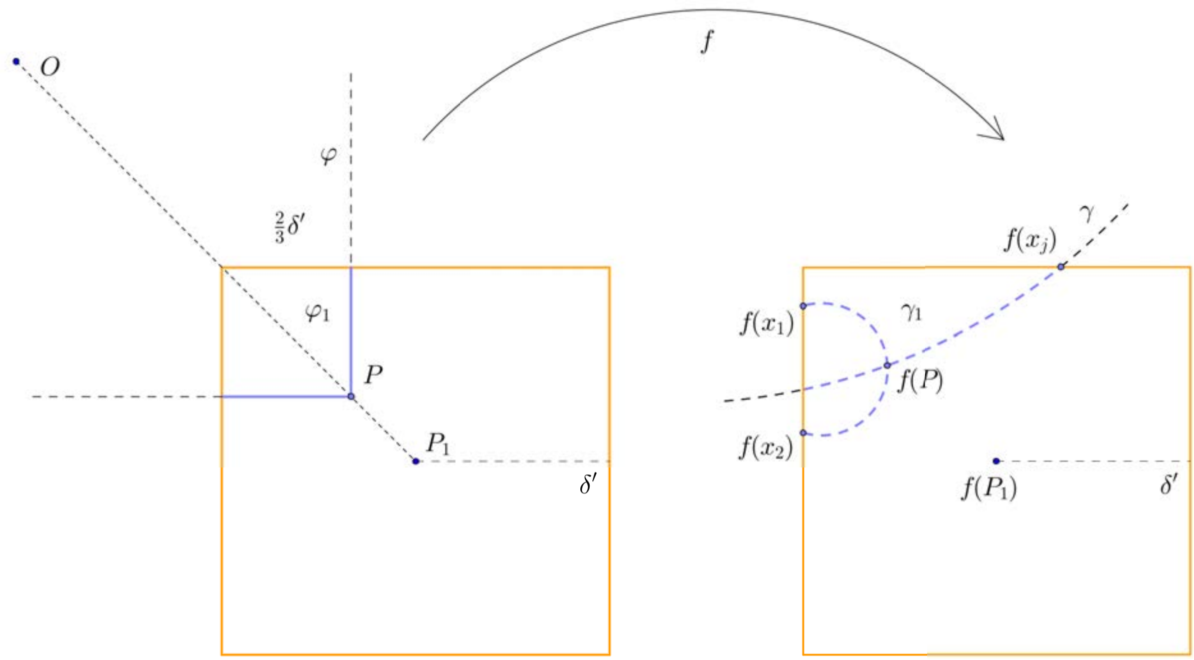


Figure 5.2

Let $0 < \delta' < \min\{\frac{2}{3}\delta, \frac{\delta}{8}|k|\}$ and take the corner P_1 of the square $\partial B_{\rho+\delta'/3}(0)$ that is in the same direction as P, i.e. $P_1 = (1 + \delta'/3\rho)P$, see Figure 5.2. Now, consider the image of P_1 under f and the square $\partial B_{\delta'}(f(P_1))$ centred at $f(P_1)$ with radius δ'. Let $\gamma := f(\varphi)$ and define :

$$\varphi_1 := (\partial B_\rho(0) \cap \overline{B}_{\delta'}(P_1))$$

$$\gamma_1 := f(\varphi_1).$$

First notice that the curve γ intersects the boundary of the square $B_{\delta'}(f(P_1))$ in at least two points. Indeed, it is clear that $f(P)$ belongs to the square $B_{\delta'}(f(P_1))$, because:

$$\|f(P) - f(P_1)\|_\infty \leq L\|P - P_1\|_\infty = (3-\alpha)\tfrac{1}{3}\delta' = \delta' - \tfrac{\alpha\delta'}{3} < \delta' \tag{5.5}$$

and from (5.3), we know that for every $t \in (-\delta, \delta)$ we have:

$$|x(f(\varphi(t))) - x(f(\varphi(0)))| = |kt + o_1(t)| \geq |kt| - |o_1(t)| > \tfrac{1}{2}|kt|.$$

In particular, for $t_1^* = -\delta/2$ and $t_2^* = \delta/2$ we have:

$$|x(f(\varphi(t_i^*))) - x(f(\varphi(0)))| > \tfrac{1}{2}|kt_i^*| = \tfrac{\delta}{4}|k| = 2\tfrac{\delta}{8}|k| > 2\delta',$$

so that both points $f(\varphi(t_1^*))$ and $f(\varphi(t_2^*))$ lie outside $\overline{B}_{\delta'}(f(P_1))$.

This means that both pieces, $f(\varphi((-\delta,0)))$ and $f(\varphi((0,\delta)))$, of the curve γ go from the outside to the inside of the square $B_{\delta'}(f(P_1))$, and the other way around respectively, so both curves must intersect the boundary of the square $B_{\delta'}(f(P_1))$ by the Jordan Curve Theorem.

We have shown then that there exist $t_1 \in (-\delta, 0)$ and $t_2 \in (0, \delta)$ such that $f(\varphi(t_i)) \in \partial B_{\delta'}(f(P_1))$. As we will see, using the local injectivity of f we can actually assure that these points satisfy $t_1 \in [-\frac{2\delta'}{3}, 0)$ and $t_2 \in (0, \frac{2\delta'}{3}]$. Notice that the images of $\varphi(t_i)$, $i \in \{1,2\}$ under f belong to the square $\overline{B}_{\delta'}(f(P_1))$, and from the co-Lipschitz property we have $\overline{B}_{\delta'}(f(P_1)) \subseteq f(\overline{B}_{\delta'}(P_1))$ so the inverse images under f of the points $f(\varphi(t_i))$, $i = 1,2$ must intersect $\overline{B}_{\delta'}(P_1)$. This is enough to conclude that $t_1 \in [-\frac{2\delta'}{3}, 0)$ and $t_2 \in (0, \frac{2\delta'}{3}]$. Indeed, notice that since φ is 1-Lipschitz and $t_1 \in (-\delta, 0), t_2 \in (0, \delta)$, then for $i = 1,2$ we have $\|\varphi(t_i) - \varphi(P)\|_\infty \leq |t_i| < \delta$, so $\varphi(t_i) \in B_\delta(P)$. On the other hand, since $\delta < \varepsilon$, then f is injective in $B_\delta(P)$. Hence, for $i = 1,2$ the only point in $B_\delta(P)$ that is mapped to $f(\varphi(t_i))$ is in fact $\varphi(t_i)$ and it should belong to $\overline{B}_{\delta'}(P_1)$. Since

$$\varphi \cap \overline{B}_{\delta'}(P_1) = \{\varphi(t) : t \in [-\tfrac{2\delta'}{3}, \tfrac{2\delta'}{3}]\},$$

we conclude that $t_1 \in (-\delta, 0) \cap [-\frac{2\delta'}{3}, \frac{2\delta'}{3}]$ and $t_2 \in (0, \delta) \cap [-\frac{2\delta'}{3}, \frac{2\delta'}{3}]$. See Figure 5.2.

Now, let $x_i := \varphi(t_i)$ and notice that the images of x_1, x_2 under f cannot belong both to the left vertical side of the square $\partial B_{\delta'}(f(P_1))$. See Figure 5.2. For, if $f(x_i)$, $i = 1, 2$ belong to the left vertical side of the square $\partial B_{\delta'}(f(P_1))$ and $P \in B_{\delta'}(f(P_1))$, using (5.3) we get:

$$0 < x(f(P)) - x(f(x_1)) = x(f(P)) - x(f(\varphi(t_1))) = -kt_1 - o_1(t_1') < -kt_1 + \tfrac{1}{2}|k|(-t_1)$$
$$0 < x(f(P)) - x(f(x_2)) = x(f((P)) - x(f(\varphi(t_2))) = -kt_2 - o_1(t_2') < -kt_2 + \tfrac{1}{2}|k|(t_2).$$

Since $t_1 < 0 < t_2$, these inequalities yield:

$$0 < -t_1(\tfrac{1}{2}|k| + k) \;\Rightarrow\; -k < \tfrac{1}{2}|k|$$
$$0 < t_2(\tfrac{1}{2}|k| - k) \;\Rightarrow\; k < \tfrac{1}{2}|k|$$

and we get $|k| < \frac{1}{2}|k|$, a contradiction. A similar argument shows that both points cannot belong to the right hand side of the square $\partial B_{\delta'}(f(P_1))$.

Furthermore, it is easy to see that it is also not possible that the points $f(x_1)$, $f(x_2)$ belong to two opposite sides of the square $\partial B_{\delta'}(f(P_1))$. Indeed, since for any two points, $x_1, x_2 \in \varphi_1$ we have $\|x_1 - x_2\|_\infty \leq \frac{2}{3}\delta'$, by the Lipschitz property we have:

$$\|f(x_1) - f(x_2)\|_\infty \leq L\|x_1 - x_2\|_\infty < 3(\tfrac{2}{3}\delta') = 2\delta'.$$

Thus the distance between the images of any two points of the curve φ_1 is strictly less than

$2\delta'$, so the points $f(x_1), f(x_2)$ cannot lie on two opposite sides of the square $\partial B_{\delta'}(f(P_1))$.

Therefore the only option remaining is that at least one of these two points, $f(x_1)$ or $f(x_2)$ belongs to the top side or the bottom side of the square $\partial B_{\delta'}(f(P_1))$.

Since $|t_i| \leq \frac{2\delta'}{3} < \delta$ we can apply (5.4) for $t = t_i$, $j \in \{1, 2\}$, and get:

$$|y(f(x_i)) - y(f(P))| = |o_2(t_i)| < \tfrac{\alpha}{2}(\tfrac{2}{3}\delta') = \tfrac{\alpha\delta'}{3}. \tag{5.6}$$

On the other hand the inequality (5.5) implies:

$$|y(f(P)) - y(f(P_1))| \leq \|f(P) - f(P_1)\|_\infty \leq \delta' - \tfrac{\alpha\delta'}{3}. \tag{5.7}$$

Therefore, from the inequalities (5.6) and (5.7) we gather that $|y(f(x_i)) - y(f(P_1))| < \delta'$, which means that $f(x_i)$ cannot be a point of the top or bottom side of the square $\partial B_{\delta'}(f(P_1))$, as we have $|b - y(f(P_1))| = \delta'$ for all points (a, b) belonging to the top or bottom side of the square $\partial B_{\delta'}(f(P_1))$.

We conclude that if the derivative of the curve $f \circ \gamma$ at P is parallel to the x-axis, then L could not be smaller than 3. Since we can preform a rotation of $\frac{\pi}{2}$ without affecting the LIpschitz and co-Lipschitz constants of f. We conclude that $L > 3$ also in the case where the derivative of $f \circ \gamma$ at P is parallel to the y-axis. □

The next proposition shows that if we now assume that the tangent at some corner P is ${}^{\pi}/_{4}$ —instead of π or ${}^{\pi}/_{2}$ as in Proposition 5.2.1— we can derive the same colclusion that ${}^{c}/_{L} \leq {}^{1}/_{3}$.

Proposition 5.2.2. *For every $\rho > 0$ let γ_ρ be the curve describing the square $\partial B_\rho(0)$. Let $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ be a 2-fold Lipschitz quotient mapping. Let N be as in Proposition 2.4 and assume that for some $\rho > N$ the curve $f \circ \gamma_\rho$ is differentiable at some corner P of the square $\partial B_\rho(0)$ and that the tangent at this point has slope equal to $\pi/4$.*

Then, $c/L \leq 1/3$.

Proof. We assume again that $f(0) = 0$ and $c = 1$, and we carry out the proof by contradiction repeating a similar construction to the one on Proposition 5.2.1. Let $L = 3 - \alpha < 3$ and N as in Corollary 2.5, so we know that there exists $\varepsilon > 0$ such that f is injective in every neighbourhood $B_\varepsilon(x)$, whenever $\|x\|_\infty > N$. By hypothesis we can find a corner point P with $\|P\|_\infty := \rho > N$ such that $f \circ \gamma_\rho$ is differentiable at P and that the tangent at this point has slope equal to $\pi/4$. Let $\gamma : [-4\rho, 4\rho] \to \partial B_\rho(0)$ be a 1-Lipschitz parametrization of the curve $\partial B_\rho(0)$ with $\gamma(0) = P$ oriented counterclockwise, so that the tangent vector to the curve $f \circ \gamma$ at this point P is equal to (k, k) for some $k \in \mathbb{R}$.

First we will assume that $k > 0$, therefore, if we denote by $x(f)$ and $y(f)$ the coordinate functions of f, $f(z) = (x(f(z)), x(f(z)))$, then we can find $\delta \in (0, \varepsilon)$ such that:

$$x(f(\gamma(t))) = x(f(\gamma(0))) + kt + o_1(t) = x(f(P)) + kt + o_1(t) \tag{5.8}$$

$$y(f(\gamma(t))) = y(f(\gamma(0))) + kt + o_2(t) = y(f(P)) + kt + o_2(t) \tag{5.9}$$

with $|o_1(t)|, |o_2(t)| < \min\{k|t|, \frac{\alpha}{8}|t|\}$, whenever $|t| < \delta$.

As in Proposition 5.2.1, let $\delta' < \frac{2}{3}\min\{\delta, \varepsilon\}$ and consider the square $\partial B_{\delta'}(f(P))$ centred at $f(P)$ with radius δ'. See Figure 5.3. Denote the corners of this square by A, B, C, D starting from the bottom right corner and going counterclockwise. Now, (back to the domain of f) consider the square $\partial B_{\rho-\delta'/3}(0)$ and take the corner P_1 of $\partial B_{\rho-\delta'/3}(0)$ that is in the same direction as P, i.e. $P_1 = (1 - \delta'/3\rho)P$.

It is clear that the point $f(P_1)$ belongs to the square $B_{\delta'}(f(P)) = ABCD$. Indeed, since $L < 3$, we have:

$$\|f(P) - f(P_1)\|_\infty < 3\|P - P_1\|_\infty = \delta'.$$

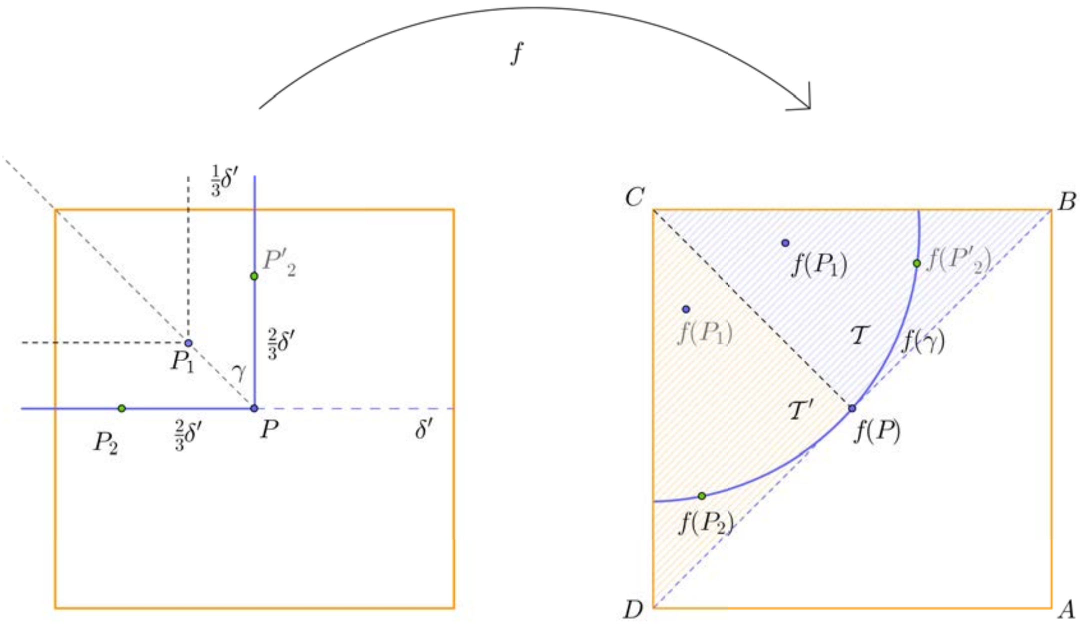


Figure 5.3

We may assume that $f(P_1)$ belongs to the upper triangle B, C, D; otherwise we can perform a π radians rotation without changing the Lipschitz and co-Lipschitz constants of f. (Notice that here, differently to Proposition 5.2.1, we do not have any further assumptions about the position of the corner P).

Now, assume that $f(P_1)$ belongs to the triangle $\mathcal{T}$ whose vertices are $f(P), B$ and C and consider the point $P_2 := (x(P) - \frac{2}{3}\delta', y(P))$, where $x(P), y(P)$ are the x-coordinate and the y-coordinate of the point P respectively. Hence $\|P - P_2\|_\infty = \frac{2}{3}\delta'$, by definition of the curve γ this means that $\gamma(t_2) = P_2$, where $t_2 := -\frac{2}{3}\delta'$.

Now, since P_2 lies inside the neighbourhood $B_\delta(P)$, from (5.8) and (5.9) we have:

$$\|f(P) - f(P_2)\|_\infty = \max\{|kt_2 + o_i(t_2)| : i = 1, 2\}.$$

Therefore:

$$\begin{aligned}
\|f(P)-f(P_2)\|_\infty - |y(f(P))-y(f(P_2))| \leq & \Big| |x(f(P))-x(f(P_2))| - |y(f(P))-y(f(P_2))| \Big| \\
= & \Big| |kt_2+o_1(t_2)| - |kt_2+o_2(t_2)| \Big| \\
\leq & |kt_2+o_1(t_2)-(kt_2+o_2(t_2))| \\
\leq & |o_1(t_2)|+|o_2(t_2)| < \tfrac{\alpha}{4}|t_2|.
\end{aligned}$$

Hence,

$$|y(f(P))-y(f(P_2))| > \|f(P)-f(P_2)\|_\infty - \tfrac{\alpha}{4}|t_2|. \tag{5.10}$$

On the other hand, since we are assuming that $f(P_1)$ lies inside the triangle $\mathcal{T}$ then

$$0 < \|f(P_1)-f(P)\|_\infty = |y(f(P_1))-y(f(P))| = y(f(P_1))-y(f(P)). \tag{5.11}$$

Also, from (5.9), and the fact that $-\delta < -\frac{2}{3}\delta' = t_2 < 0$ it follows that

$$\begin{aligned}
y(f(P_2)) = y(f(P)) + kt_2 + o_2(t_2) &\leq y(f(P)) + kt_2 + |o_2(t_2)| \\
&< y(f(P)) + kt_2 + k|t_2| = y(f(P)),
\end{aligned}$$

using $k>0$. Hence, since we are assuming that $f(P_1)\in\mathcal{T}$ we have:

$$y(f(P_2)) < y(f(P)) < y(f(P_1)). \tag{5.12}$$

Finally, notice that from Corollary 2.5, since $\delta' < \varepsilon$, we know that for every pair of points, say x_1, x_2, belonging to $\overline{B}_{\delta'}(P)$ we have $\|f(x_1)-f(x_2)\|_\infty \geq \|x_1-x_2\|_\infty$, in particular:

$$\|f(P_2) - f(P)\|_\infty \geq \|P_2 - P\|_\infty = \tfrac{2}{3}\delta', \tag{5.13}$$
$$\|f(P_1) - f(P)\|_\infty \geq \|P_1 - P\|_\infty = \tfrac{1}{3}\delta'.$$

From (5.12), (5.11), (5.10) and (5.13) it follows that:

$$\begin{aligned}
\|f(P_2) - f(P_1)\|_\infty &\geq |y(f(P_2)) - y(f(P_1))| = y(f(P_1)) - y(f(P_2)) \\
&= (y(f(P_1)) - y(f(P))) + (y(f(P)) - y(f(P_2))) \\
&= \|f(P_1) - f(P)\|_\infty + (y(f(P)) - y(f(P_2))) \\
&> \|f(P_1) - f(P)\|_\infty + \|f(P) - f(P_2)\|_\infty - \tfrac{\alpha}{4}|t_2| \\
&\geq \tfrac{1}{3}\delta' + \tfrac{2}{3}\delta' - \tfrac{\alpha}{4}(\tfrac{2}{3}\delta') > \delta'(1 - \tfrac{1}{3}\alpha) = (3-\alpha)\tfrac{1}{3}\delta'.
\end{aligned}$$

This is a contradiction since by the Lipschitz condition we have:

$$\|f(P_2) - f(P_1)\|_\infty \leq (3-\alpha)\|P_2 - P_1\|_\infty = (3-\alpha)\tfrac{1}{3}\delta'.$$

In a similar way, we reach a contradiction assuming that $f(P_1)$ lies inside the triangle $\mathcal{T}'$ whose vertices are C, $f(P)$ and D. Indeed, instead of the point P_2 consider now the point P_2' defined as $P_2' := (x(P), y(P) + \frac{2}{3}\delta')$. Then $P_2' = \gamma(t_2')$ where $t_2' := \frac{2}{3}\delta'$. Then (5.10) can be changed to:

$$\begin{aligned}
&\|f(P) - f(P_2')\|_\infty - |x(f(P)) - x(f(P_2'))| \\
\leq&\Big||y(f(P)) - y(f(P_2'))| - |x(f(P)) - x(f(P_2'))|\Big| \\
=&\Big||kt_2' + o_2(t_2')| - |kt_2' + o_1(t_2')|\Big| \\
\leq&|o_1(t_2')| + |o_2(t_2')| < \tfrac{\alpha}{4}|t_2'|.
\end{aligned} \tag{5.14}$$

Also in this case, since $f(P_1) \in \mathcal{T}'$, the estimate (5.11) becomes:

$$\|f(P_1) - f(P)\|_\infty = |x(f(P_1)) - x(f(P))| = x(f(P)) - x(f(P_1)), \tag{5.15}$$

and (5.12) would be changed into:

$$x(f(P_1)) < x(f(P)) < x(f(P_2')), \tag{5.16}$$

this is because, given that $k > 0$, we have:

$$\begin{aligned} x(f(P_2')) &= x(f(P)) + kt_2' + o_1(t_2') \geq x(f(P)) + kt_2' - |o_1(t_2')| \\ &> x(f(P)) + kt_2' - k|t_2'| = x(f(P)), \end{aligned}$$

again using $k > 0$. Thus, using (5.15), (5.16), (5.14) and then (5.13), we have:

$$\begin{aligned} \|f(P_2') - f(P_1)\|_\infty &\geq |x(f(P_2')) - x(f(P_1))| = x(f(P_2')) - x(f(P_1)) \\ &= (x(f(P_2')) - x(f(P))) + (x(f(P)) - x(f(P_1))) \\ &> \|f(P_2') - f(P)\|_\infty - \tfrac{\alpha}{4}|t_2'| + \|f(P) - f(P_1)\|_\infty \\ &\geq \tfrac{2}{3}\delta' - \tfrac{\alpha}{4}|t_2'| + \tfrac{1}{3}\delta' > \delta'(1 - \tfrac{1}{3}\alpha) = (3 - \alpha)\tfrac{1}{3}\delta'. \end{aligned}$$

Again, this is a contradiction since

$$\|f(P_2') - f(P_1)\|_\infty \leq (3 - \alpha)\tfrac{1}{3}\delta'.$$

Thus we have proved that the statement is true for $k > 0$. It is clear that if we assume $k < 0$ then we only need to repeat the argument above but choosing the point P_2, when P_1 belongs to the triangle $\mathcal{T}'$ and the point P_2' when P_1 belongs to the triangle $\mathcal{T}$. □

After the last two propositions, the idea would be to generalise these results to the case when the tangent at the image of some corner P of $\partial B_r(0)$ has an arbitrary slope. The next result points in that direction, still, we will use stronger assumptions than in the previous results. We assume further that there is a neighbourhood of a corner P such that any other corner inside that neighbourhood is mapped to a point with x-coordinate equal to the x-coordinate of $f(P)$.

Proposition 5.2.3. *For every $\rho > 0$ let γ_ρ be the curve describing the square $\partial B_\rho(0)$. Let $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ be a 2-fold Lipschitz quotient mapping. Assume that for a sufficiently large $\rho \in \mathbb{R}^+$, there is a corner P of the square γ_ρ and a neighbourhood $B_\delta(P)$ of P such that:*

1. *For all $\lambda \in (\rho - \delta, \rho + \delta)$ the image under f of the path $\gamma_\lambda \cap B_\delta(P)$ is a straight line with gradient $m \in (0, 1)$.*

2. *For each $\lambda \in (\rho - \delta, \rho + \delta)$ there is a real number $\xi(\lambda)$ such that $f(P + (\lambda - \rho, \lambda - \rho)) = f(P) + (0, \xi(\lambda))$.*

Then the Lipschitz constant, L, and the co-Lipschitz constant, c, of f satisfy $c/L \leq 1/3$.

Proof. Assume that $f(0) = 0$ and $c = 1$. We carry out the proof by contradiction, so assume $L < 3$. Let N and ε be given by Corollary 2.5. By hypothesis we can find a corner point P with $\|P\|_\infty := \rho > N + 1$ and a neighbourhood $B_\delta(P)$ such that the conditions *1* and *2* of the proposition are satisfied. Without loss of generality we may assume that $\delta < \varepsilon < 1$ and we also can assume that $P = (\rho, \rho)$, since the mapping defined as the composition of a rotation by any integer multiple of $\frac{\pi}{2}$ followed by f has the same Lipschitz and co-Lipschitz constants as f.

Fix any $r \in (0, \frac{3}{2}\delta)$ and consider the square whose upper right corner is the point P and has radius $\frac{1}{3}r$, i.e. the square $B_{\frac{1}{3}r}(Q)$ where $Q := P - (\frac{1}{3}r, \frac{1}{3}r)$. Denote by C the bottom right corner of the square $B_{\frac{1}{3}r}Q$ and by B and D upper and bottom left corners, respectively. See Figure 5.4.

For a point p, let $x(p)$ and $y(p)$ be the x-coordinate and the y-coordinate of p, respectively. Let $k_1 = \frac{1}{3}(m+1)$ and $k_2 = \frac{1}{3}(1-m)$. For $i \in \{1,2\}$. We will consider the square $\mathcal{S}_i$ whose bottom left corner is D and has side $(\frac{2}{3}+k_i)r$. Finally, denote by A_i the upper right corner of the square $\mathcal{S}_i$. Notice that $f(P)$ and $f(A_i)$ have the same x-coordinate. Indeed, since

$$A_i = D + (\tfrac{2}{3}+k_i)r(1,1) = P + k_i r(1,1),$$

from the hypothesis *2* of the present lemma, we only need to check that $|k_i r| \leq \delta$. However, since $m \in (0,1)$, we have

$$|k_i| = |\tfrac{1}{3}(1 \pm m)| < \tfrac{2}{3} \ \text{ and } \ r < \tfrac{2}{3}\delta.$$

Therefore, $x(P) = x(A_i)$. On the other hand, we may assume without loss of generality that $y(f(A_1)) > y(f(P))$, since now the mapping defined as the composition of f followed by a rotation of π radians has the same Lipschitz and co-Lipschitz constants as f. See Figure 5.4 for an illustration.

Notice that once we have assumed that the y-coordinate of $f(A_1)$ is greater then the y-coordinate of $f(P)$ then the same must happen with the y-coordinate of $f(A_2)$. Indeed, suppose that $y(f(A_2)) \leq y(f(P))$, and let φ be a parametrization of the line segment that joins P with A_1 (passing through A_2). Then, by the hypothesis *2*, $f(\varphi)$ would be mapped into a curve describing a straight vertical line segment with starting point $f(P)$, then going down through $f(A_2)$ and then up to $f(A_1)$. Therefore, there is another point in between A_1 and A_2 which is mapped to $f(P)$. This is impossible since f is injective in $\mathcal{S}_1$. We conclude that $y(f(A_2)) > y(f(P))$. A very similar argument shows that the points $f(B)$ and $f(C)$ are on different sides of $f(P)$. Indeed, assume that $f(C)$ and $(f(B)$

are on the same side of $f(P)$ and let γ'_ρ be the path contained in γ_ρ that goes from B to C. Then, by hypothesis *1*, $f(\gamma'_\rho$ is contained in a straight line with gradient m and since we are assuming that $f(P)$ is not in between $f(B)$ and $f(C)$, the curve $f(\gamma'_\rho$ must pass through $f(B)$, then $f(P)$, and then back to $f(C)$, so all the points on the line segment $[f(P), f(C)]$ have two preimages, which is not possible since f is injective on $\mathcal{S}_\rangle$.

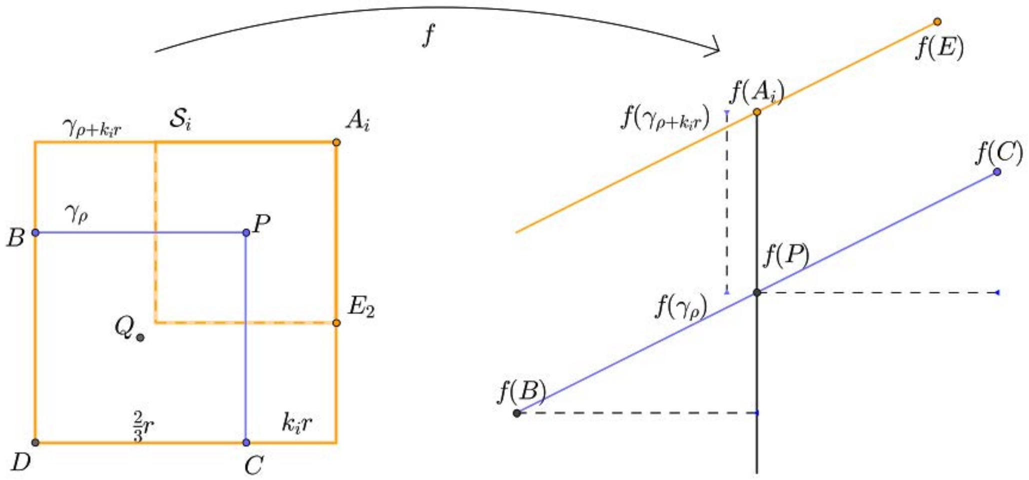


Figure 5.4

Once this is clear, we first consider the square $\mathcal{S}_1$; we are going show that

$$|x(f(P)) - x(f(C))| = |x(f(A_1)) - x(f(C))| \geq (1 + \tfrac{1}{3}m)r. \tag{5.17}$$

The first equality is obvious since we are assuming that $f(P)$ and $f(A_1)$ have the same x-coordinate. Notice that since f is injective in $B_\delta(P)$, by Corollary 2.5 we know that

$$\|f(A_1) - f(C)\|_\infty \geq c\|A_1 - C\|_\infty = (\tfrac{2}{3} + k_1)r = (1 + \tfrac{1}{3}m)r. \tag{5.18}$$

To prove the inequality in (5.17) we consider two cases.

Case A1. $y(f(C)) > y(f(A_1))$.

In this case, using hypothesis *1* of the present Lemma, we have

$$y(f(C))-y(f(A_1)) < y(f(C)-y(f(P)) = m\,(x(f(C)) - x(f(P))) < |x(f(A_1))-x(f(C))|.$$

Therefore $\|f(A_1) - f(C)\|_\infty = |x(f(A_1)) - x(f(C))|$ and from (5.18) we conclude that (5.17) is satisfied.

Case A2. $y(f(C)) \leq y(f(A_1))$.

Notice that in this case we have:

$$y(f(A_1)) - y(f(C)) = y(f(A_1)) - y(f(P)) - m|x(f(P)) - x(f(C))|, \tag{5.19}$$

see Figure5.4 for an illustration. We know that $(y(f(A_1)) - y(f(P))) \leq L\|A_1 - P\|_\infty < 3k_1 r$. On the other hand, using again Corollary 2.5, we have

$$|x(f(P)) - x(f(C))| = \|f(P) - f(C)\|_\infty \geq c\|P - C\|_\infty = \tfrac{2}{3}r.$$

The first equality is satisfied since $f(P)$ and $f(C)$ lie, by hypothesis, on a line with gradient $0 < m < 1$. Therefore

$$|y(f(A_1)) - y(f(P))| - m|x(f(P)) - x(f(C))| < (3k_1 - m\tfrac{2}{3})r = (1 + \tfrac{1}{3}m)r.$$

Using (5.19), this proves $|y(f(A_1))-y(f(C))| < (1+\frac{1}{3}m)r$. On the other hand, from (5.18) we have $\|f(A_1) - f(C)\|_\infty \geq (1 + \frac{1}{3}m)r$. Therefore, we must have

$$(1 + \tfrac{1}{3}m)r \leq \|f(A_1) - f(C)\|_\infty = |x(f(A_1)) - x(f(C))|,$$

and this finishes the proof of (5.17).

Our next goal is to consider the square $\mathcal{S}_2$, and to show that

$$|x(f(P)) - x(f(B))| = |x(f(A_2)) - x(f(B))| \geq (1 - \tfrac{1}{3}m)r. \tag{5.20}$$

Again the first equality is clear. Notice that:

$$|y(f(A_2)) - y(f(B))| = |y(f(A_2)) - y(f(P))| + |y(f(P)) - y(f(B))|,$$

see Figure 5.4. Now, since $f(P)$ and $f(B)$ have the same x-coordinate, we have $\|f((P)) - f(B))\|_\infty = |y(f(P)) - y(f(B))|$ and, since P and B belong to $\gamma_\rho \cap B_\delta(P)$, from hypothesis *1* we get $|y(f(P)) - y(f(B))| = m|x(f(P)) - x(f(B))| = m|x(f(A_2)) - x(f(B))|$, the last equality here uses $x(f(P)) = x(f(A_2))$. Therefore,

$$|y(f(A_2)) - y(f(B))| = m|x(f(A_2)) - x(f(B))| + \|f(P) - f(A_2)\|_\infty. \tag{5.21}$$

Using Corollary 2.5 we have:

$$\begin{aligned}\|f(A_2) - f(B)\|_\infty &= \max\{|x(f(A_2)) - x(f(B))|, |y(f(A_2)) - y(f(B))|\} \\ &\geq c\|A_2 - B\|_\infty = (\tfrac{2}{3} + k_2)r = (1 - \tfrac{1}{3}m)r. \end{aligned} \tag{5.22}$$

Again we have two cases in which we establish (5.20).

Case B1. $|x(f(A_2)) - x(f(B))| > |y(f(A_2)) - y(f(B))|$.

In this case we have $\|f(A_2) - f(B)\|_\infty = |x(f(A_2)) - x(f(B))|$. Hence, from (5.22) we conclude $|x(f(A_2)) - x(f(B))| \geq (1 - \frac{1}{3}m)r$, which proves (5.20).

Case B2. $|x(f(A_2)) - x(f(B))| \leq |y(f(A_2)) - y(f(B))|$.

Now, using (5.21) and (5.22) we have:

$$m|x(f(A_2)) - x(f(B))| + \|f(P) - f(A_2)\|_\infty = |y(f(A_2)) - y(f(B))| \geq (1 - \tfrac{1}{3}m)r.$$

This implies

$$|x(f(A_2)) - x(f(B))| \geq \tfrac{1}{m}\left((1 - \tfrac{1}{3}m)r - \|f(P) - f(A_2)\|_\infty\right). \tag{5.23}$$

Furthermore, we can show that $\|f(P) - f(A_2)\|_\infty < (3-2m)\|P - A_2\|_\infty = (3-2m)k_2 r$. Indeed, if we now consider the square $B_{k_2 r}(P)$ and we denote by E_2 the bottom right corner of the square (see Figure 5.4) then

$$\|f(P) - f(E_2)\|_\infty \leq L\|P - E_2\|_\infty < 3k_2 r. \tag{5.24}$$

However, if we assume that $\|f(P) - f(A_2)\|_\infty \geq (3 - 2m)\|P - A_2\|_\infty$ then, since $\gamma_{\rho+k_2 r}$ is also mapped to a straight line with gradient m and the x-coordinates of both points, $f(P)$ and $f(A_2)$, are the same, we have the following:

$$\begin{aligned}
\|f(E_2) - f(P)\|_\infty \geq& |y(f(E_2)) - y(f(P))| = |y(f(A_2)) - y(f(E_2))| + \|f(A_2) - f(P)\|_\infty \\
=& m\|f(A_2) - f(E_2)\|_\infty + \|f(A_2) - f(P)\|_\infty \\
\geq& m(c\|A_2 - E_2\|_\infty) + (3 - 2m)\|P - A_2\|_\infty \\
=& m(2k_2 r) + (3 - 2m)k_2 r = 3k_2 r.
\end{aligned} \tag{5.25}$$

This is not possible, as (5.25) contradicts (5.24). Hence,

$$\begin{aligned}
\|f(P) - f(A_2)\|_\infty <& (3 - 2m)\|P - A_2\|_\infty = (3 - 2m)k_2 r = (3 - 2m)(\tfrac{1}{3}(1 - m))r \\
=& (\tfrac{2}{3}m^2 - \tfrac{5}{3}m + 1)r.
\end{aligned}$$

Using (5.23) and the latter inequality we get:

$$\begin{aligned}|x(f(A_2)) - x(f(B))| &\geq \tfrac{r}{m}(1 - \tfrac{1}{3}m - (\tfrac{2}{3}m^2 - \tfrac{5}{3}m + 1))\\ &= \tfrac{r}{m}(\tfrac{4}{3}m - \tfrac{2}{3}m^2) = \tfrac{2}{3}(2-m)r\\ &= ((1-\tfrac{1}{3}m) + \tfrac{1}{3}(1-m))r > (1-\tfrac{1}{3}m)r,\end{aligned}$$

which proves (5.20).

Finally, from the inequalities (5.17) and (5.20) we gather that:

$$\begin{aligned}\|f(C) - f(B)\|_\infty &\geq |x(f(C)) - x(f(B))| = |x(f(C)) - x(f(P))| + |x(f(P)) - x(f(B))|\\ &\geq (1+\tfrac{1}{3}m)r + (1-\tfrac{1}{3}m)r = 2r.\end{aligned}$$

On the other hand, since $L < 3$, we have

$$\|f(C) - f(B)\|_\infty \leq L\|C - B\|_\infty < 3(\tfrac{2}{3}r) = 2r.$$

This is a contradiction, therefore we negated the initial assumption $L < 3$. This finishes the proof of Proposition 5.2.3. ◻

We believe that Proposition 5.2.3 can be generalised so that, using linear approximations of $f \circ \gamma$, we can prove that:

Conjecture 5.2.4. *For every $\rho > 0$ let γ_ρ be the curve describing the square $\partial B_\rho(0)$. Let $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ be a 2-fold Lipschitz quotient mapping. Let N be as in Proposition 2.4 and assume that for some $\rho > N$ the curve $f \circ \gamma_\rho$ is differentiable at some corner P of the square $\partial B_\rho(0)$ and that the tangent at this point has gradient equal to m with $m \in (0,1)$. Then, $c/L \leq 1/3$.*

With this conjecture proved we would cover, using rotations, all possible directions of the tangent's slope at $f(P)$. This is work in progress.

We would like to point out that even though we have not been able to prove that every 2-fold Lipschitz quotient mapping $f : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ with constants L and c, satisfies ${}^{c}/_{L} \leq {}^{1}/_{3}$, the results in this chapter show that it would be difficult to find a 2-fold Lipschitz quotient mapping with ${}^{c}/_{L} > {}^{1}/_{3}$.

Even more, if we look at the general picture, putting together the results in this chapter with the result in Proposition 3.2.4, we see that if f maps corners "close to the main diagonals", then ${}^{c}/_{L} \leq {}^{1}/_{3}$. On the other hand, if Conjecture 5.2.4 is true and f does not map a corner to a corner then —assuming that $f \circ \gamma$ is differentiable at that point— we again have ${}^{c}/_{L} \leq {}^{1}/_{3}$. Recall that we can always find a differentiability point of $f \circ \gamma$ as close to a corner as we want, so the differentiability assumption is always satisfied at points arbitrarily close to a corner.

Chapter 6
Final comments and further work

From the main results of this work, Theorems 2.7, 4.1.3 and 4.2.12, we can state the following general conclusion:

There is a universal scale of real numbers $0 < \ldots < \rho_k < \ldots < \rho_1 < 1$ such that, given any norm $\|\cdot\|$ on the plane, if $f : (\mathbb{R}^2, \|\cdot\|) \to: (\mathbb{R}^2, \|\cdot\|)$ is an L-Lipschitz and c-co-Lipschitz mapping with $c/L > \rho_k$ then $\#f^{-1}(x) \leq k$ for all $x \in \mathbb{R}^2$. The values of this scale are given by $\rho_k = {}^1\!/_{(k+1)}$. Even more, this scale is sharp not only for the Euclidean norm but also, in the case $k = 2$, the scale is sharp for any norm whose unit ball is a regular $(4m + 2)$-gon. However, there exist norms on the plane for which the ratio of constants c/L of any 2-fold Lipschitz quotient mapping does not achieve the bound $\rho_1 = {}^1\!/_2$. For instance, any polygonal norm whose unit ball is a regular $4m$-gon satisfies this property.

This conclusion gives place to some questions that the present work could also help to answer. One question is: Can we find sharp scales ρ_k^{4m} for the $4m$-polygonal norms?

As we have seen in Chapter 5, there are positive results that indicate that the second value of such a scale for the supremum norm should be $\rho_1^4 = 1/3$. This fact would also support Conjecture 4.2.14, stated in Chapter 4. That conjecture comes from the observation, in Theorem 2.6, that far from the origin, a k-fold Lipschitz quotient mapping behaves as a complex polynomial of degree k. So the image of the boundary of a big enough ball $\partial B_r(0)$ must wind k times around the origin and must do it going "almost outside" the ball $B_{cr}(0)$, where c is the co-Lipschitz constant of the mapping. It sounds plausible to think that the best way (i.e. "without stretching more than needed") to wind around $\partial B_{cr}(0)$ would be to go along $\partial B_{cr}(0)$ with constant speed k.

In the same way as we did in Definition 4.2, we can define for any $k \in \mathbb{N}$ and any norm $\|\cdot\|$ a "k-fold winding mapping", that we shall denote by $f_{\|\cdot\|,k}$, in the following way: Recall that given a norm $\|\cdot\|$ on $\mathbb{R}^2$, we defined $\mathscr{L}_{\|\cdot\|} := \mathcal{H}_1^{\|\cdot\|}(\partial B_1^{\|\cdot\|}(0))$. For a fixed $k \in \mathbb{N}$ and for each constant $r > 0$ consider the curve $\gamma_r : [0, kr\mathscr{L}_{\|\cdot\|}] \to \partial B_r^{\|\cdot\|}(0)$ such that:

1. $\operatorname{Ind} \gamma_r(0) = k$;
2. γ_r is a 1-Lipschitz mapping;
3. $\gamma_r(ir\mathscr{L}_{\|\cdot\|}) = (r, 0)$ for all $i \in \{0, \ldots, k\}$.

We define the k-fold winding mapping $f_{\|\cdot\|,k} : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ in the following way: given $x \in \mathbb{R}^2$ with $\|x\|_n = r$, take $t_x \in [0, r\mathscr{L}_{\|\cdot\|})$ such that $\gamma_r(t_x) = x$. Notice that t_x is uniquely defined since γ_r is injective along $[0, r\mathscr{L}_{\|\cdot\|})$ and $\gamma_r([0, r\mathscr{L}_{\|\cdot\|})) = \partial B_r^{\|\cdot\|}(0)$. We set $f_{\|\cdot\|,k}(x) := \gamma_r(kt_x)$.

Notice that in this definition (condition 3.) we are fixing the "starting point" $(1, 0)$ from where the k-fold winding mapping starts increasing the length by a factor of k. It might well happen that changing the starting point in this definition affects the Lipschitz and co-Lipschitz constants of the mapping. For the case of the polygonal norms we believe that the Lipschitz constant, L'_k, and the co-Lipschitz constant, c'_k, of a k-fold winding mapping with a different starting point satisfy $L'_k \geq L_k$ and $c'_k \leq c_k$, where c_k and L_k are the co-Lipschitz and Lipschitz constants of the k-fold winding mapping $f_{\|\cdot\|,k}$, although we have not formally verified this. However, for the case of the supremum norm, it follows from Proposition 5.1.3, that every 2-fold winding mapping (previously called doubling mapping) has ratio of constants less than or equal to $1/3$.

In this context, Conjecture 4.2.14 could be stated in a more general way:

Conjecture 6.1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and let $f_{\|\cdot\|,k}$ be the k-fold winding mapping in the norm $\|\cdot\|$.*

If $f : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ is an L-Lipschitz and c-co-Lipschitz k-fold mapping then

$$\frac{c}{L} \leq \frac{c_k}{L_k},$$

where L_k and c_k denote the Lipschitz and co-Lipschitz constants of the k-fold winding mapping $f_{\|\cdot\|,k}$.

Even more, by doing the relevant changes in the proof of Proposition 4.2.13, it would not be hard to derive the more general statement:

Proposition 6.2. *If $n = 4m$ for some $m \in \mathbb{N}$, then the Lipschitz and co-Lipschitz constants, $L_{n,k}$ and $c_{n,k}$, of the k-fold winding mapping $f_{n,k}$ satisfy $L_n \geq k+(k-1)\tan^2(\frac{\pi}{n})$ and $c_n \leq 1$. Hence,*

$$\frac{c_{n,k}}{L_{n,k}} \leq \frac{1}{k+(k-1)\tan^2(\frac{\pi}{n})}.$$

This proposition, together with Conjecture 6.1, would imply the following:

Conjecture 6.3. *Let $n = 4m$ for some $m \in \mathbb{N}$. If $f : (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ in an L-Lipschitz and c-co-Lipschitz k-fold mapping with respect to the n-norm $\|\cdot\|_n$, then*

$$\frac{c}{L} \leq \frac{1}{k+(k-1)\tan^2(\frac{\pi}{n})}.$$

For example, for the supremum norm the sharp scale of values, ρ_k^4, $k \in \mathbb{N}$, such that $c/L > \rho_k$ implies $\#f^{-1}(x) \leq k$ for all $x \in \mathbb{R}^2$, would be given by:

$$0 < \ldots < \rho_k^4 = \tfrac{1}{2k+1} < \ldots < 1/5 < 1/3 < 1.$$

Figure 6.1 shows how the sharp scale for the supremum norm would be shifted. The coloured dots on the right hand side of each subinterval — $(1/k+1, 1/k]$, for the Euclidean norm, and $(1/2k+1, 1/2k-1]$ for the supremum norm— show the place where the ratio c/L of the k-fold winding mappings are.

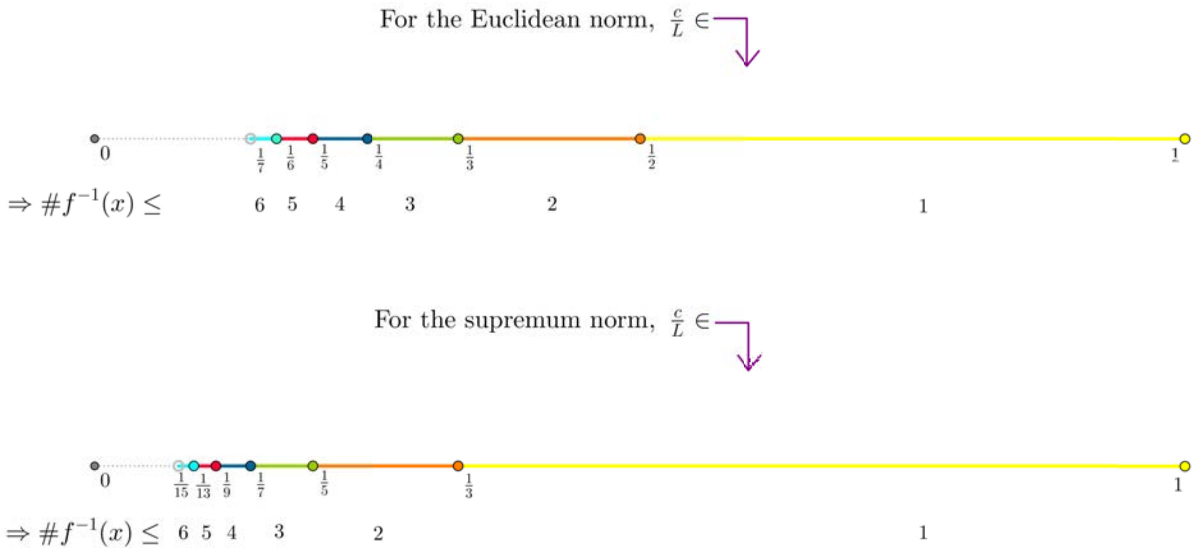


Figure 6.1

However, we need to be careful with Conjecture 6.1 because even though the general idea is very intuitive, we need to bear in mind that with non-Euclidean norms the results are not quite intuitive, see Remark 1.3.9.

Another question that arose from this work already in Chapter 2, is what can we say if we now consider Lipschitz quotient mappings $f : (\mathbb{R}^2, \|\cdot\|_1) \to (\mathbb{R}^2, \|\cdot\|_2)$, where $\|\cdot\|_1 \neq \|\cdot\|_2$?

In this case it is more complicated to calculate the exact Lipschitz and co-Lipschitz constants of a given Lipschitz quotient mapping. However, since $\text{length}_\infty(\partial B_1^\infty(0)) = 8$, recalling the notation used in Theorem 2.7, we see that $\mathscr{L}_E/\mathscr{L}_\infty = \pi/4 < 1$, where $\mathscr{L}_E$ stands for the Euclidean length of $\partial B_1^{Eucl}(0)$. Hence, from Theorem 2.7 it follows that: For every k-fold Lipschitz quotient mapping $f : (\mathbb{R}^2, |\cdot|) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ the ratio between the Lipschitz and co-Lipschitz constants of f is less than or equal to $\pi/4k < 1/k$. Moreover, for any n-norm with $n = 4m+2$ we can easily see that we also have $= \text{length}_n(\partial B_1^n(0)) = \text{length}_E(\partial B_1^n(0)) = 2n\sin(\frac{\pi}{n})$, this follows from the fact that the sides of an $4m+2$ sided polygon are parallel to a radius of the polygon and a remark after Definition 4.2. Hence we have the following Corollary from Theorem 2.7.

Corollary 6.4. *Let $\|\cdot\|_n$ denote the n-polygonal norm with $4m+2$. For every k-fold Lipschitz quotient mapping $f : (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, |\cdot|)$ the ratio between the co-Lipschitz constant c and the Lipschitz constant L of f satisfies:*

$$\frac{c}{L} \leq \frac{2n\sin(\pi/n)}{k(2\pi)} < \frac{1}{k}.$$

As we mentioned in Chapter 2 (see comment before Corollary 2.8, if $\mathscr{L}_1 > k\mathscr{L}_2$, then Theorem 2.7 does not give any useful information about the ratio of constants. It will be interesting to study the general behaviour of these constants,$\mathscr{L}_1$ and $\mathscr{L}_2$ and see if this result leads to further results about the ratio between the Lipschitz and co-Lipschitz constants of a k-fold mapping.

A more conceptual question related to this work is: what are the underlying properties of a norm that determine the relevant sharp bounds $\rho_k^{\|\cdot\|}$ for the ratio of constants of a k-fold Lipschitz quotient mapping? We have not found a clear answer to this question but there are few remarks that we would like to make.

First of all, it is clear that these bounds are not related to the way the norm $\|\cdot\|$ measures the distance between points, but more to the "particular geometric shape" the unit ball, $B_1^{\|\cdot\|}(0)$, has. For instance the supremum norm, ℓ_∞, and the ℓ_1-norm are very different in the way they measure distances but, since these norms have the same unit ball shape, they achieve the same bounds for the ratio of Lipschitz and co-Lipschitz constants (see Proposition 1.2.4). So we need to distinguish between norms paying attention to the shape and the geometry more than to the distance between points. In this direction, the Banach-Mazur distance between Banach spaces could be useful. In the particular case of $\mathbb{R}^2$, the Banach Mazur distance, d, between the spaces $X := (\mathbb{R}^2, \|\cdot\|)$ and $Y := (\mathbb{R}^2, \|\cdot\|_*)$ can be defined as:

$$d(X,Y) := \inf\{ab : \tfrac{1}{b}B_1^{\|\cdot\|}(0) \subseteq TB_1^{\|\cdot\|_*}(0) \subseteq aB_1^{\|\cdot\|}(0); T \in GL(X,Y); a,b > 0\},$$

where $GL(X,Y)$ denotes the set of linear operators between X and Y.

In this way d measures how different is (in shape) the unit ball $B_1^{\|\cdot\|}(0)$ to the unit ball $B_1^{\|\cdot\|_*}(0)$. Even more, $d(X,Y) \geq 1$ for any pair of two-dimensional Banach spaces and $d(\ell_1^2, \ell_\infty^2) = 1$, see [17]. In addition, in [27, Corollary] it is shown that, in a sense, the farthest norms on the plane are the hexagonal and quadrangular norms. More precisely, it is shown that:

For any pair of symmetric convex bodies in the plane C and D there are linear images of them, say C' and D' such that $d(C', D') \leq 3/2$ with equality only if C' is a linear image of a regular hexagon and D' a linear image of a square.

This result could also link up well with our estimates for the sharp bounds of the hexagon and the square, which are in a sense the farthest. On the other hand, the fact that $\rho^{4m+2} = \rho^{4n+2}$ for all $n, m \in \mathbb{N}$ does not seem to be clearly justified from this point of view.

Furthermore, if the shape of the unit ball $B_1^{\|\cdot\|}(0)$ determines the sharp bounds $\rho_k^{\|\cdot\|}$, one would expect that, for big enough n, the sharp bounds ρ_1^n of the polygonal norm with n sides are closer to the Euclidean bound $\rho_1 = 1/2$, than the bounds of a polygonal norm with few sides, say ρ_1^6. In other words, at the beginning of this research, we expected the bounds ρ_1^n of the polygonal norms with n sides to increase as n increases, so that $\rho_1^{n+1} > \rho_1^n$ for all $n \in \mathbb{N}$ and $\rho_1^n \to \rho_1 = 1/2$ as $n \to \infty$. Therefore, we found it somehow surprising that polygonal norms with $4m$ sides behave differently than the polygons with $4m+2$ sides, and that, for instance, $\rho_1^6 = \rho_1 = 1/2$. On the other hand, if we assume that Conjecture 6.3 we can approximate ρ_k with the values ρ_k^n with $n = 4m$, this is:

$$\lim_{m\to\infty} \rho_k^{4m} = \lim_{m\to\infty} \frac{1}{k+1+k\tan^2(\frac{\pi}{4m})} = \frac{1}{k+1} = \rho_k.$$

In conclusion, the estimates of the values of the sharp scale $\rho_n^{\|\cdot\|}$ seem to be related to the particular geometric properties of the unit ball $B_1^{\|\cdot\|}(0)$, for example —in the case of polygonal norms— having all sides parallel to a radius, or all sides parallel to an apothem. This possible conclusion makes it more difficult to have reliable conjectures for general convex bodies in the plane. However it seems that in order to achieve the $1/2$ ratio of Lipschitz and co-Lipschitz constants, a norm should satisfy very strict regularity properties, so we expect that in most cases the $1/2$ ratio —and in general, the $1/n$ ratio— will not be achieved. In this direction, it will be interesting to study "perturbed" polygons. In the case of the $4m+2$ sided polygons, taking into account the previous comment, we would hope for a small perturbation of the regular polygon to prevent any 2 fold Lipschitz quotient to achieve the $1/2$ ratio.

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