THE RADIATIVE HEATING OF PLANE-PARALLEL AND SPHERICAL ATMOSPHERE

David H. Morgan

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DECLARATION

I hereby declare that the following Thesis has been composed by myself, that the work of which it is a record has been done by myself, and that it has not been accepted in any previous application for a higher degree. I was admitted as a research student under Ordinance General No. 12 on 1st October 1969 to undertake research work on radiative transfer under the supervision of Professor D.W.N. Stibbs, and was admitted as a candidate for the Degree of Doctor of Philosophy on the satisfactory completion of my first two terms as a research student under that ordinance. The research work was performed at the University Observatory, St. Andrews.

CERTIFICATE

I certify that David H. Morgan has fulfilled the conditions of Ordinance General No. 12 and the Resolution of the University Court No. 1, and the Senate Regulations governing that ordinance, and that he is qualified to submit this Thesis in candidature for the Degree of Doctor of Philosophy in the University of St. Andrews.

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CHAPTER I

RADIATIVE HEATING

The subject of radiative heating is concerned with the heating and cooling of atomic, molecular and particulate matter by the processes of absorption and emission of radiation, which will be seen later to be very important heating mechanisms of such material. A quantitative description of these processes requires a detailed knowledge of the optical properties of the material involved in the interaction with the radiation. This must be supplied by mathematical models appropriate to the astrophysical context. In most problems the matter is of sufficient a density to consider it to constitute an atmosphere, in which case the radiation field incident upon each particular element of matter is controlled by the optical properties of the remainder of the atmosphere. In this way the theory of the radiative heating of an atmosphere is closely associated with the theory of radiative transfer.

The theory of radiative transfer has developed from the study of stellar atmospheres into a large and complex branch of astrophysics and is now deeply involved in the study of planetary nebulae, circumstellar shells, interstellar dust clouds and planetary atmospheres. The theory of planetary atmospheres is of major significance at the present time because the nearby planets are the only astronomical bodies for which direct measurements are available. During the past decade various probes have been sent to the nearby planets with the purpose of making measurements of the physical conditions that exist within their atmospheres. It is of fundamental importance to modern astronomy that the theories of radiative transfer and radiative heating stand up to these direct tests. It is also of great value to find the temperatures of other astronomical objects that are heated by radiative processes. The dust particles of interstellar space are heated in this way and the temperatures of these particles are important in a number of astronomical contexts. For example, the formation of hydrogen molecules is, at present, considered to take place in association with dust grains, and the relevant physical processes of absorption, adsorption and evaporation depend critically on the temperatures of the grains. The interpretation of the results of radiative heating calculations must be performed with reference to other heating processes. If it is found that other heating mechanisms provide a significant source of heat, and hence radiation by thermal emission, then the whole transfer problem must be reformulated and re-solved.

In general, the intensity of a radiation field in an atmosphere is a function of position in the atmosphere, direction and frequency. The optical characteristics of the atmospheric constituents are also, in general, functions of position and frequency. We shall see that this situation presents too complex a problem to be handled by transfer theory as known at present, and that it will be necessary to introduce several physical and mathematical approximations. These approximations will be introduced at an appropriate stage in the development of the theory but it will be useful to introduce a geometrical approximation at this juncture because it is mathematically necessary for the theory to be applied within the framework of a particular co-ordinate system. The theories of stellar and planetary atmospheres have been established in plane-parallel geometry and rectangular cartesian co-ordinates. Stars and planets are approximately spherical in shape but the geometrical thicknesses of their atmospheres are so small with respect to their radii that it is a good approximation to consider their atmospheres as infinite slabs with parallel planar surfaces. This would not be true for circumstellar shells and interstellar dust clouds. Consequently the theories of radiative transfer in these objects have been established in spherical geometry and spherical polar co-ordinates. This geometry provides a good approximation for circumstellar shells but not necessarily for interstellar dust clouds. Nevertheless, the representation of these clouds by spherical atmospheres is the simplest procedure that can be adopted. Moreover, the complexity of the physics and mathematics of the radiative heating problem is such that the geometrical approximation is usually the last to be questioned.

After formulating the equation of transfer in these two geometries we shall consider the large variety of methods available for the solution of the equation of transfer in each geometry. Discussions of the physical nature of the scattering processes in planetary atmospheres and dust clouds; the frequency dependence of the radiative heating problem; and the topic of radiative heating itself, both in general terms and in relation to other mechanisms of heating will follow and form the main body of this introductory chapter. We shall then be in a position to discuss previous work in this field and to present the rationale behind the selection of the methods to be used in Chapters II and III for the solution of the appropriate equations of transfer.

1. The Equation of Transfer

The equation of transfer is the fundamental equation of the theory of energy transport in any medium. It appears in all the branches of physical science where energy is carried by particles of any description. The two branches in which it occurs most frequently are radiative transfer and neutron diffusion, the particles transporting the energy being photons and neutrons respectively. The equation of transfer is formulated from an "Eulerian point of view" for it considers only the energy that is involved in an interaction between a particle and the medium through which the particle is carrying the energy, and the way in which that energy is transformed by the interaction. It does not depend on the nature of the particle carrying the energy nor on the nature of the interaction involved in the energy transfer. The relation between these ideas of energy transfer and those ideas that depend on the precise nature of the interactions between the individual particles has been discussed at length by Kourganoff (1952) and by Samuelson (1967b). This matter will not be discussed here, but it will be necessary to quote the definitions of the quantities involved in the formation of the equation of transfer because they are fundamental to the whole subject. These definitions can be found in all the standard reference texts on radiative transfer, examples of which are works by Chandrasekhar (1960), Kourganoff (1952), Unsöld (1955), Woolley and Stibbs (1953), Sobolev (1963), Pecker and Schatzman (1959) and Mihalas (1970).

The fundamental quantity required is the amount of energy, dE_ν, in a specified frequency interval, ($\nu, \nu + d\nu$), which is transported across an element of area, $d\sigma$, and in directions confined to an element of solid angle, $d\omega$, during a time, dt.

This energy, dE_ν, is expressed in terms of the specific intensity, I_ν, by

$$dE_\nu = I_\nu \cos\theta \, d\nu \, d\sigma \, d\omega \, dt, \tag{I-1}$$

where θ is the angle which the direction considered makes with the outward normal to $d\sigma$. The specific intensity is usually referred to as the intensity and we shall adopt this unambiguous abbreviation. The construction involved in defining the intensity also defines a pencil of radiation.

A pencil of radiation traversing a medium will be weakened by its interaction with matter. If the intensity, I_ν, becomes $I_\nu + dI_\nu$ after traversing a thickness, ds, in the direction of its propagation, then

$$dI_\nu = -k_\nu \rho I_\nu \, ds, \tag{I-2}$$

where ρ is the density of the material and k_ν is the mass absorption coefficient of the material for radiation of frequency ν. Equation (I-2) defines the mass absorption coefficient. The radiation lost to the radiation field is either absorbed or scattered. When it is absorbed it is converted by the material into some other form of energy, which may take the form of radiation of a different frequency. When it is scattered it is changed in direction but often remains of the same frequency. Of course, it would not be prudent to consider the scattering to involve all but the original direction, so the radiation in the original direction is the sum of the radiation which is not scattered and that which is scattered without a change of direction. The distinction between absorption and scattering must be made with great care, particularly with regard to frequency dependence. This will be carried out in Section I.3. but for the purposes of this expository section on the equation of transfer all that need be stated is that k_ν as defined above includes absorption losses due to time absorption and scattering. It will hereafter be known as the extinction coefficient and "true" will be omitted from "true absorption coefficient".

The emission coefficient, j_ν , is defined in such a way that an element of mass, dm, emits in directions confined to an element of solid angle, $d\tilde{\omega}$, in the frequency interval (ν, $\nu + d\nu$) and in time, dt , an amount of radiant energy given by

$$j_\nu \, dm \, d\omega \, d\nu \, dt. \tag{I-3}$$

The source of this energy is unimportant for the purposes of this section. The ratio of the emission coefficient to the extinction coefficient is an important quantity in the theory of radiative transfer and is known as the source function,

$$B_\nu = j_\nu / k_\nu . \tag{I-4}$$

These definitions permit the derivation of the equation of transfer, which involves the equating of the differential change in energy content of an infinitesimal volume of matter with the difference between the energy it absorbs and the energy it emits. Consider a small cylindrical element of cross-section, $d\sigma$, and length, ds , in the medium. From the definition of the intensity, the difference in the radiant energy in the frequency interval, (ν, $\nu + d\nu$), crossing the two faces normally in a time,

dt, and confined to a solid angle, $d\omega$, can be expressed as $(dI_\nu/ds)\, d\nu\, d\sigma\, d\omega\, dt\, ds$. This increase in energy must be equal to the energy emitted by the element in the same frequency interval, into the same solid angle and in the same time, minus the energy absorbed by the element in the same frequency interval, solid angle and time. The absorbed energy is given by the expression, $k_\nu \rho\, ds \times I_\nu\, d\nu\, d\sigma\, d\omega\, dt$, while the emitted energy is given by $j_\nu \rho\, d\sigma\, ds\, d\nu\, d\omega\, dt$. Thus the equation for energy balance in the cylinder is

$$\frac{dI_\nu}{ds}\, ds\, d\nu\, d\sigma\, d\omega\, dt = -k_\nu \rho\, ds\, I_\nu\, d\sigma\, d\omega\, dt + j_\nu \rho\, d\sigma\, ds\, d\nu\, d\omega\, dt,$$

which is commonly written in the form,

$$\frac{-\,dI_\nu}{k_\nu \rho\, ds} = I_\nu - j_\nu/k_\nu = I_\nu - B_\nu, \qquad \text{(I-5)}$$

and is known as the equation of transfer.

The solution of this equation can be written down straightway, for the equation of transfer in this form is merely a first order differential equation whose integrating factor is $\exp(-\tau_\nu(s,s'))$ where

$$\tau_\nu(s,s') = \int_{s'}^{s} k_\nu \rho\, ds. \qquad \text{(I-6)}$$

This quantity is known as the optical distance between s and s′ and it is a measure of the distance between s and s′ in terms of the extinction of radiation of frequency, ν. If $I_\nu(0)$ is the intensity in the positive s-direction at $s = 0$, then the solution of equation (I-5) is

$$I_\nu(s) = I_\nu(0)\, e^{-\tau_\nu(s,0)} + \int_0^s B_\nu(s')\, e^{-\tau_\nu(s,s')} k_\nu \rho \, ds' . \qquad \text{(I-7)}$$

The physical meaning of this equation is clear. Equation (I-7) merely states that the intensity at a distance, s, along a line in a medium, in the direction of positive s along the line, is equal to the intensity at the end of the line, $s = 0$, in the positive s-direction, attenuated by the optical distance between $s = 0$ and s, plus the integral sum of the emission coefficient at all points between $s = 0$ and s, attenuated by the optical distance between that point and s.

Were the source function known then equation (I-7) could be solved either analytically or numerically, and the intensity, $I_\nu(s)$, could be found exactly. However, in most problems the source function is known only as a function of the intensity itself and equation (I-7) becomes an integral equation for the intensity. It is not uncommon for the source function to be a function of the integral of the intensity as well as the intensity, in which case equation (I-5) is an integro-differential equation. Both equations (I-5) and (I-7) are far from easy to solve even in the most simplified physical contexts. Before giving details of the various methods available for the solution of these equations we shall adapt equation (I-5) to the two geometries that will be considered in the later chapters of the thesis.

Firstly, we consider the equation of transfer in a plane-parallel atmosphere, for which rectangular cartesian co-ordinates are the most suitable. The intensity will, in general, be a function of three position co-ordinates, (x, y, z), and two direction co-ordinates. However, in most problems and in particular, the problems tackled in this thesis, the intensity will not be a function of the position co-ordinates, (x, y), because the incident radiation on the (x.y) plane will always be uniform and the atmospheric parameters will be assumed to remain constant throughout each layer parallel to the surface of the atmosphere. The effect of lateral inhomogeneity in plane-parallel atmospheres has been considered by Wilson (1963), and the scattering problem for a searchlight beam, which is an example of non-uniform incident radiation, has been considered by Rybicki (1971). The remaining position variable, z, is defined as the distance along a line normal to the surface of the atmosphere, measured positively outwards from the surface. The radiative transfer is considered to take place along a line set at an angle, θ , to this axis and at an azimuthal angle, ϕ , to some arbitary azimuthal direction. These geometrical parameters are shown in Fig. 1. In keeping with standard procedure we define

$$\mu = \cos\theta. \tag{I-8}$$

Thus the equation of transfer for a plane-parallel atmosphere becomes

$$\frac{-\mu}{k_\nu \rho}\frac{dI_\nu(z,\mu,\phi)}{dz} = I_\nu(z,\mu,\phi) - B_\nu(z,\mu,\phi). \tag{I-9}$$

Secondly, we consider the equation of transfer in a spherical atmosphere, for which spherical polar co-ordinates are the most suitable. In the majority of the problems considered in the later part of the thesis, all the radiation fields are axially symmetric. Consequently, we can assume the atmosphere to be homogenous throughout each spherical shell centred on the origin, so that we have complete spherical symmetry. In this case the intensity is a function of the radial co-ordinate, r , and the direction co-ordinate, θ , only. Fig. 2 shows the co-ordinate system of the axially symmetric spherical atmosphere and Fig. 3 shows the construction of the geometry around point P, suitably enlarged. The s-direction is the line along which we consider the radiative transfer to take place and the distance, R , is the radius of the outer surface of the atmosphere. Again we use equation (I-8). The distance along s is a function of r and μ so that the total differential of s is

$$\frac{d}{ds} = \frac{\partial r}{\partial s}.\frac{\partial}{\partial r} + \frac{\partial \mu}{\partial s}.\frac{\partial}{\partial \mu} .$$

It is clear that $dr = ds.\cos\theta = \mu\, ds$ and that the angle $P\hat{O}P$ is equal to $-d\theta$. Therefore we write $-r\, d\theta = ds.\sin\theta$ and hence

$$\frac{d}{ds} = \mu\frac{\partial}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial}{\partial \mu} ,$$

so that the equation of transfer in a spherically symmetric atmosphere is

$$\mu\frac{\partial I_\nu(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_\nu(r,\mu)}{\partial \mu} = -k_\nu \rho\left[I_\nu(r,\mu) - B_\nu(r,\mu)\right]. \qquad \text{(I-10)}$$

2. Solutions for the Source Function

The source function is, in general, a function of the intensity of the radiation field so that a solution of the equation of transfer in either its differential equation, (I-5), or integral equation, (I-7), form is a very difficult problem. Furthermore, the source function is often a function of the mean intensity of the radiation field which is denoted by J (z) and is defined by equation (I-11). Equation (I-11) also defines two other quantities that are frequently encountered in radiative transfer studies, namely $H_\nu(z)$ and $K_\nu(z)$.

$$J_\nu(z) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} I_\nu(z,\mu,\phi)\,d\mu\,d\phi \;; \quad H_\nu(z) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} \mu\, I_\nu(z,\mu,\phi)\,d\mu\,d\phi \;;$$

$$K_\nu(z) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} \mu^2\, I_\nu(z,\mu,\phi)\,d\mu\,d\phi . \qquad \text{(I-11)}$$

These three quantities are the first three direction moments of the intensity. The equation of transfer has been used most extensively in plane-parallel atmospheres so we shall outline a number of the methods available for its solution in plane-parallel geometry and then discuss the extension of these methods to spherical geometry.

It will be convenient to introduce the variable, τ_ν , which is the optical depth. It is the optical distance measured from the surface of an atmosphere along the negative z-axis, and by analogy with equation (I-6) is defined by

$$d\tau_\nu = -k_\nu \rho\, dz \;; \quad \tau_\nu = \int_0^z k_\nu \rho\, dz . \qquad \text{(I-12)}$$

It will also be convenient to omit the subscript ν from all symbols used in this and the following section so that the symbols refer to the monochromatic quantities or the integrated quantities. The matter of their frequency dependence will be the subject of Section I.4. Hence equation (I-5) becomes

$$\mu \frac{dI}{d\tau}(\tau, \mu, \phi) = I(\tau, \mu, \phi) - B(\tau, \mu, \phi) \qquad \text{(I-13)}$$

When the source function includes the moments of the intensity as defined by equation (I-11), the equation of transfer becomes an integro-differential equation involving the variables τ, μ and ϕ. The earliest methods of solution of this equation involve the replacement of the intensity of the radiation field, which is an unknown function of direction by an approximate one of simple angular dependence, and thus reduce the integro-differential equation, (I-13), to one or more ordinary differential equations for directional independent quantities. The first of these methods is the Schuster-Schwarzchild two-stream approximation. The total radiation field is replaced by a radiation field in the direction, $\mu = +1$, whose intensity is equal to the mean intensity of the radiation in the hemisphere, $\mu > 0$, and another appropriately defined radiation field in the other hemisphere. The equation of transfer is formulated for these two radiation fields independently and the two resulting differential equations, which are usually elementary, are solved by standard mathematical techniques. Details of this and the other methods mentioned here are available in the standard reference texts.

A similar approximate method is that due to Eddington. The intensity is assumed to be isotropic and equal to I_1 in the hemisphere, $\mu > 0$, and isotropic and equal to I_2 in the hemisphere, $\mu < 0$. This representation of the intensity, together with the equations (I-11), gives the relation

$$K(\tau) = (1/3)\, J(\tau), \tag{I-14}$$

which is known as Eddington's approximation. It is clear from its construction that it is an exact relation for an isotropic radiation field. The integral operators, L_0 and L_1, defined as

$$L_0\left\{f(\mu,\phi)\right\} = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} f(\mu,\phi)\, d\mu\, d\phi\ ;$$

$$L_1\left\{f(\mu,\phi)\right\} = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} \mu\, f(\mu,\phi)\, d\mu\, d\phi\ . \tag{I-15}$$

are applied to equation (I-13) and produce two coupled ordinary differential equations that can be solved by using the relation (I-14). The approximate intensity representation also yields an approximate boundary condition. If there is no radiation incident upon the surface of the atmosphere then I_2 is zero and we obtain the relation

$$J(0) = 2\, H(0), \tag{I-16}$$

which is known as the Eddington approximate boundary condition. The Eddington approximation is more general than the two-stream approximation. As shown by Woolley and Stibbs (1953) equation

(I-14) is exactly true when the intensity can be represented by a series expansion of Legendre polynomials of argument, μ, provided that the term involving $P_2(\mu)$ is excluded. Thus the Eddington approximation is valid for quite anisotropic radiation fields. However, the approximate boundary condition is true only when there is no incident radiation on the surface and when the emergent radiation is isotropic.

A third method involving an approximation for the angular dependence of the intensity is due to Chandrasekhar (1960). The intensity is considered to be 2n streams in the directions given by the abscissae of the Gaussian mechanical quadrature. These abscissae are, in fact, the zeros of the Legendre polynomial $P_{2n}(\mu)$. The mean intensity can therefore be represented by

$$J(\tau) = \frac{1}{2} \sum_{\substack{j=-n \\ \neq 0}}^{+n} a_j I(\tau, \mu_j), \tag{I-17}$$

where a_j and μ_j are the weights and abscissae of the Gaussian mechanical quadrature of order n, and the equation of transfer can be replaced by the system for 2n coupled linear differential equations

$$\mu_i \frac{dI(\tau, \mu_i)}{d\tau} = I(\tau, \mu_i) - B(\tau, \mu_i); \quad i = \pm 1, \pm 2, \ldots \pm n. \tag{I-18}$$

When n is unity this approximation reduces to a two-stream approximation that is very similar to the Schuster-Schwarzchild approximation. The higher the value of n the better the approximation but the more difficult the solution because there are always 2n differential equations. Details of the solution of such sets of equations for a variety of problems have been described by Chandrasekhar.

The discrete ordinate method described above is essentially a generalization of the two-stream approximation to a 2 n -stream approximation. The Eddington approximation has been generalized in a similar way by Huang (1968). In the Eddington approximation the radiation field is assumed to be constant over two hemispheres so that the generalized method assumes it to be constant over smaller ranges of direction, the number of these ranges giving half the order of the approximation. The advantage in this method over the discrete ordinate method lies in the low order approximations when its advantage is of the same nature and magnitude as that of the Eddington approximation over the two-stream approximation.

Equations (I-7) and (I-11) combine to form

$$J(\tau) = \frac{1}{2}\int_0^{\tau_0} B(t)\, E_1(|\tau - t|)\, dt, \qquad \text{(I-19)}$$

provided that I(0) = 0, as is invariably the case. The functions $E_n(t)$ are the exponential integral functions, details of which are given in the Appendix; and τ_0 is the total optical thickness of the atmosphere. This equation is frequently referred to as Milne's first integral equation and details of its construction are given in Section II.4. It is often written in the form

$$J(\tau) = \Lambda_\tau \{ B(t) \},$$

where

$$\Lambda_\tau \{ f(t) \} = \frac{1}{2}\int_0^{\tau_0} f(t)\, E_1(|\tau - t|)\, dt. \qquad \text{(I-20)}$$

The operator, $\Lambda_\tau \{ f(t) \}$ is known as the lambda operator. In a great many problems the source function can be expressed by

$$B(\tau) = J(\tau) + J_1(\tau)$$

in which case the following integral equation can be derived from equation (I-20).

$$B(\tau) = \Lambda_\tau \{ B(t) \} + J_1(\tau). \tag{I-21}$$

Such an equation exists when the source function is isotropic. However, when the source function is anisotropic equation (I-21) takes the form

$$B(\tau,\mu) = \frac{1}{2} \int_{-1}^{+1} \int_{0}^{\tau_0} d\mu' \, p(\mu,\mu') \, dt \, B(t,\mu') f(\tau - t, \mu') \tag{I-22}$$

where $\quad f(\tau,\mu) = e^{-\tau/\mu} / |\mu|, \; (\tau/\mu > 0) \; ; \; = 0, \; (\tau/\mu < 0).$

The solution of this equation, or its simpler form, (I-21), is shown by Busbridge (1960) to be the Neumann series solution. That is

$$B(\tau,\mu) = \sum_{n=1}^{\infty} \Lambda_\tau^{n-1} \{ J_1(t) \} = \sum_{n=1}^{\infty} J_n(\tau), \tag{I-23}$$

where $\quad J_n(\tau) = \Lambda_\tau \{ J_{n-1}(t) \},$

and the lambda operator is defined by equation (I-20) or, more generally, by the integral part of equation (I-22), which must be evaluated numerically. Van de Hulst (1948) has shown that this solution represents the sum of the contributions to the radiation field from light scattered n times, and that the source function for the light scattered n times is given by one lambda operation upon the source function for the light scattered $n-1$ times. The quantity $J_1(\tau)$ is taken to be the source function for the light that is not scattered at all. This method of solution has been used by Irvine (1968a) who performed the lambda operation by use of a double numerical integration scheme for each value of n. The exactness of the method is determined by the accuracy of the numerical computations and the number of terms used for the series of lambda operations. The number of terms required for convergence to an accurate solution was found to be small for optically thin atmospheres but too large to render the method practicable for optically thick atmospheres because light is scattered a large number of times in optically thick atmospheres and each term in the series represents the contribution from each order of scattering. However, Hansen (1969a) has developed a method due to Van de Hulst whereby the exact solution for multiple scattering problems in optically thick atmospheres can be obtained. This method involves the numerical evaluation of reflection and transmission functions for an atmosphere of optical thickness $2\tau_0$ from the same functions for an atmosphere of optical thickness τ_0. This is known as the doubling method, and results for optically thick atmospheres can be obtained fairly rapidly from the results for optically thin atmospheres which have been found by the Neumann series method. The relative speed and accuracy of this doubling method has been established by Irvine (1968a) and by Hansen (1969b). The lambda operator can be used in a slightly different way that gives rise to a series of methods that are frequently known as iterative methods. These are particularly useful when the source function is independent of direction and the analytical form of the lambda operator can be used. We have seen that successive applications of the lambda operator on J_1 yields the Neumann series solution which converges to the exact solution, but only does so at any practicable rate when the atmosphere is optically thin. If an approximate solution is inserted into equation (I-21) another iterative series is obtained. The solutions obtained by the two-stream and Eddington approximations are ideal for this purpose and are frequently used. The initial approximation is now quite good so the series converges rapidly even for optically thick atmospheres. However, it does not necessarily converge to the exact solution. The reason behind the poor convergence of the Neumann series is that the initial approximation, $J_1(\tau)$ is not close to the final solution in cases where multiple scattering is important. However, in the iteration methods based on approximate solutions a complete analytical solution is never possible. Even when the initial approximation is of a very simple analytical form, the lambda operator, by virtue of the exponential integral function will never yield analytical solutions simple enough to be iterated analytically a second time. Thus, numerical integration techniques must be employed for second and subsequent lambda iterations. For stellar atmosphere problems dealing with conservative isotropic "scattering" and constant net flux Kourganoff (1952) indicates that lambda iterations on the source function found by using Eddington's approximation, converge to the exact source function as the number of iterations tends to infinity. The lambda iteration method is also discussed by Mihalas (1970) in relation to stellar atmosphere problems. He points out that a constant error in the approximate solution is halved at the surface but unchanged deep inside the atmosphere, by one application of the lambda operator. A lambda operator improves the solution at the boundary but leaves it unchanged at large distances away from the surface so that a very great number of iterations is needed to improve the solution deep inside the atmosphere. Thus it is expedient to select an approximation that is best deep in the atmosphere, though perhaps weak near the surface. Such an approximation is the Eddington approximation, and the improvement in its solution for stellar limb darkening has been demonstrated by Mihilas.

Several variations on the lambda iteration technique are discussed by Kourganoff (1952), who also discussed another class of methods known as variational methods. These are also outlined by Sobolev (1963). The principle of these methods is as follows. A functional is created which takes an extreme value when the exact solution of the equation to be solved is inserted into it. If, for example, the equation to be solved is

$$B(\tau) = \frac{1}{2}\int_0^\infty B(t)\,E_1(|t-\tau|)\,dt,$$

then a suitable functional is

$$\sigma = \int_0^\infty \left[\, \Phi(\tau)/F - 1 \,\right]^2 d\tau.$$

The equation to be solved is Milne's first integral equation. It is known that when that is satisfied then so is Milne's second integral equation, which is

$$F = 2\int_{\tau}^{\infty} B(t)\,E_2(t-\tau)\,dt \;-\; 2\int_{0}^{\tau} B(t)\,E_2(\tau-t)\,dt .$$

Writing $$\Phi(\tau) = 2\int_{\tau}^{\infty} B(t)E_2(t-\tau)\,dt - 2\int_{0}^{\tau} B(t)\,E_2(\tau-t)\,dt ,$$

the functional, σ, will become zero when $\Phi(\tau) = F$. This will occur when the source function, $B(t)$, in Milne's second integral equation is the exact solution because the functional, σ, will be greater than zero when the source function is not the correct one. The source function is expressed as a series of n suitably chosen functions with arbitrary coefficients. The functional is thus a function of these coefficients and is made to approach zero by adjusting them in an appropriate manner. The series for $B(\tau)$ which gives a minimum value for σ is the best that can be found for that particular set of functions. The variational method has its greatest advantage when the equation to be solved is relatively straightforward and when a functional can be chosen that can be evaluated analytically. This is made possible through the choice of the arbitrary functions. Numerical work is involved only in minimising the functional, though this too could be analytical for very simple arbitrary functions. The method has proved very useful in work on stellar atmospheres but an equation such as (I-22) would require numerical integration to find the functional as well as numerical processes to minimise it. It would seem that the Neumann series solution with doubling is a better method than the variational method as it requires no more numerical calculations and yet provides the exact solution which the variational method does not. The accuracy of the variational method has been discussed by Kourganoff (1952) who concludes that the solutions converge rapidly as n increases, even for small values of n . As in the Eddington approximation the deviation from the exact solution is greatest near the surface, so the solution is ideally suited for a lambda iteration, which is usually possible analytically as the chosen functions are usually simple. Kourganoff has shown that the iterated variational solution for n = 6 is very close indeed to the exact solution for the grey stellar atmosphere problem.

The advances in computational techniques made during recent years have stimulated the development of a large number of methods of solution of the equation of transfer that are essentially numerical in character. They are based on the numerical solution of the equation of transfer written in a slightly more amenable form, and have arisen in connexion with stellar atmosphere problems. Several are detailed by Mihilas (1970). Their main advantage is that they are capable of dealing with problems too complex to be dealt with by the other methods. For example, they can cope with the sets of equations that result when the frequency dependence of radiative transfer is taken into account. These sets of equations are usually interdependent and strongly coupled. This class of methods fall into two categories: those favouring a solution from the integral equation form of the equation of transfer and those favouring a solution from the differential equation form.
Of the former type, one method is due to Kurucz (1969) which involves the replacement of the integral in the lambda-operator by the sum of integrals over a set of discrete depth intervals and the source function within each depth interval by a quadratic interpolation formula. These integrals are then solved

analytically as they are of a simple form and yield a set of linear algebraic equations for the source function values in each of the depth intervals, which are solved numerically. The approximation involves the use of the interpolation coefficients and the set of discrete source function values, but this can be made minimal by extending the number of numerical divisions in the problem.

When considered as an integro-differential equation the equation of transfer involves an integral of the intensity over all directions. Chandrasekhar's discrete ordinate method involved the replacement of this integral by a Gaussian quadrature sum and the replacement of the integro-differential equation by a set of 2n coupled ordinary differential equations. The analytical solution of these equations is difficult for all but small values of n and even then possible for certain problems only. With the aid of numerical techniques the 2n coupled differential equations could be solved exactly. However, the boundary conditions that are associated with the vast majority of problems in radiative transfer are such that one boundary condition needs to be applied at the upper surface of the atmosphere and the other at the lower surface of the atmosphere which is infinity if the atmosphere is semi-infinite. This two-point boundary condition makes the numerical solution of the set of differential equations very much more difficult, and consequently, various techniques that attempt to surmount this difficulty have been proposed. Two such techniques are outlined by Mihalas (1970). The first involves a Riccati transformation and was introduced by Rybicki (1965). The 2n differential equations are written as two matrix differential equations, one for the outward intensities, $\mu > 0$, and the other for the inward intensities, $\mu < 0$. When the vector representing the outgoing intensities is represented by a matrix operating on the vector representing the incoming intensities plus an auxilliary vector, the first matrix differential equation can be transformed into two simultaneous initial value matrix differential equations. As these have initial value boundary conditions they can be integrated numerically without any serious difficulty and consequently the whole intensity distribution can be determined.

The second method is due to Feautrier (1964) and involves the replacement of the differential operators in the equation of transfer by difference operators. This is accomplished by substituting a set of depth points for the continuous depth variable. The set of differential equations are thus reduced to a set of matrix equations for which the two-point boundary condition presents no problem. They are solved numerically. The advantage of these modern numerical methods lies in their flexibility. Initially they proceed along the same lines as the discrete ordinate method to form the set of 2 n differential equations, but whereas the older methods were unable to cope with the more complex problems involving the interlocking of various radiation fields, these numerical methods can deal with such problems with no drastic change in the theory. This great flexibility arises from the introduction of sets of discrete angle, depth and frequency points. However, it also means that large amounts of computer time are required, particularly in the case of the non-grey, non-coherent, non-LTE problems of stellar atmospheres. In order to reduce this large computation time, a method has been developed by Auer and Mihalas (1970) for use in complex model non-LTE stellar atmosphere calculations, and which involves the use of an Eddington-like approximation for the angular dependence of the intensity instead of a set of discrete direction points. The Eddington approximation, equation (I-14), can be written as $K(\tau) = f(\tau)\, J(\tau)$, where $f(\tau) = (1/3)$. After solving for the source function using this approximation and then for the intensity using Feautrier's method, Auer and Mihilas calculated the quantities $J(\tau)$, $K(\tau)$ and the new function, $f(\tau)$. They then repeated the process as an iteration scheme and found that the solutions converged after two or three iterations. The Eddington boundary condition written in the form, $J(0) = g\, H(0)$, was also used in the iteration process. It was found that this iteration scheme was much faster than the equivalent method of using a high order angle quadrature for the initial approximation which results in the solution of large number of differential equations. Moreover, the Eddington approximation itself was found to give good results for most of the spectrum of the emergent stellar radiation.

The advantage of these numerical methods lies in their ability to cope with complex physical problems. Their use has been developed in stellar atmosphere studies but problems on planetary atmospheres are less complex in that the radiative transfer treatment seldom concerns frequency interdependence. The great advantage of the numerical techniques is therefore almost nullified so that planetary atmosphere work has retained the simpler methods outlined earlier. Irvine (1968b) and Kawata and Irvine (1970) have compared several of the methods available for the study of multiple scattering in planetary atmospheres. They used the doubling method based on the Neumann series solution for optically thin atmospheres and compared its results for the total albedo of the atmosphere with those obtained by Romanova's method, (1962), the Eddington approximation, the two-stream approximation and a modified version of the two-stream approximation as developed by Sagan and Pollock (1967). The total albedo of the atmosphere is defined as the ratio of the reflected flux to the incident flux; and all calculations were performed for scattering that was strongly peaked in the forward direction. Romanova's method is one in which highly anisotropic scattering is accounted for in an approximate way so that computing time may be saved; and Irvine did find this to be true and satisfactorily accurate. The accuracy of the other approximate methods was seen to vary from situation to situation. In general, the Eddington approximation was the most accurate for isotropic scattering and fairly good for most angles of incidence. However, for forward scattering with considerable absorption in thin atmospheres it was found to be rather poor; the two-stream giving the best results. Consequently, the Eddington approximation was found to be the best of approximate methods overall, and it was found to be most accurate for conservative isotropic scattering and normal incidence in thick atmospheres.

The final class of methods to be mentioned here do not find the source function by solving the equation of transfer directly but find it from the solution of an equation for the intensity of the radiation which is based on another physical concept. Chandrasekhar (1960) has developed exact solutions for the emergent radiation from a plane-parallel atmosphere by means of principles of invariance. These were first introduced by Ambartzumian (1943). In a series of papers, Bellmann et al have extended these ideas to develop theories based on the Principle of Invariant Imbedding. This principle is stated by Bellmann and Kalaba (1956) as follows. "Given a system, S, whose state at a time, t, is specified by a state-vector, x, we consider a process which consists of a family of transformations applied to the state-vector. Suitably enlarging the dimension of the original vector by means of additional components, the state-vectors are made elements of a space which is mapped into itself by the family of transformations. In this way we obtain an invariant process by imbedding the original process within the new family of processes. The functional equations governing the new process are the analytical expression of this invariance". The method involves adding an amount, Δx, to a linear dimension, x, and writing down an expression, for example, the reflection at the point, $x + \Delta x$, in terms of the reflection at the point, x, plus first order processes within Δx. As Δx is allowed to approach zero, an equation for the reflection function is found. This can be solved numerically and is more general than the results from the principle of invariance itself. Moreover, it can be applied to source functions within the medium whereas methods derived from the principle of invariance cannot. Essentially, the method is based on differentiation with respect to the total thickness of the atmosphere and physically means the building up of the atmosphere by the successive addition of very thin layers. However, as noted by Van de Hulst and Grossman (1968), it progresses very slowly, as does the Neumann series solution. Hence it is better to use the doubling method where possible.

These methods based on the principle of invariant imbedding are very flexible but involve the numerical solution of complex integral equations. Similar function equations have been derived using probability functions, as proposed by Sobolev (1963).

The methods outlined here range from the type where a simple physical approximation permits an analytical solution, to the type where the whole problem is solved numerically, using a computer. The general advantage of the former is simplicity and of the latter, flexibility together with the ability to cope with complex problems. The best method for a particular problem clearly depends on the nature of that particular problem and on the purpose of the required solution. The selection of the method to be used in this work will be discussed later, in Section I.7.

Methods of solution for the source function in spherical geometry:- The introduction of spherical geometry into radiative transfer problems arose from the failure of the plane-parallel approximation in certain cases where the curvature of the atmosphere could not be neglected. For problems in normal stellar atmospheres the plane-parallel approximation is satisfactory. The earliest problems to use spherical geometry were those in stellar interiors and extended stellar envelopes, and it was not long before planetary nebula and interstellar cloud problems were formulated in spherical geometry. Most problems in spherical geometry have enjoyed complete spherical symmetry and most have involved a central source, either as a point source, as in planetary nebulae or as a diffuse field of specified outward flux incident on the inner surface of a spherical shell.

The methods that have been applied to these problems are varied, as were those used for plane-parallel atmospheres. Huang (1969b) has used the Eddington approximation in circumstellar envelope problems, and Underhill (1948) has used the second approximation of the discrete ordinate method in studies of line absorption in extended stellar atmospheres. Both these methods are of the class that use simple approximations. For a simpler problem in planetary nebulae, the more complex spherical harmonic method has been used by Sen (1949) and the half-range spherical harmonic method has also been applied to planetary nebulae and extended stellar atmospheres by Wilson and Sen (1965,a,b). An important paper by Chapman (1966) is concerned with the use of an iterative procedure, based on the Eddington approximation, and equivalent to the lambda-iteration method of plane-parallel atmospheres. The particular atmosphere that he studied was a spherical shell surrounding a black-body core. This method provided results for the intensity, as well as the moments of the intensity, and it was found that the radiation was strongly peaked along an outward radial direction; so strongly, in fact, that the ratio, K/J was closer to unity than to one third. Chapman then accounted for this in a somewhat empirical manner. However, this peaking of the radiation field stimulated a numerical method, proposed by Hummer and Rybicki (1971) for conservative isotropic scattering and extended by Cassinelli and Hummer (1971) for non-conservative scattering. The method involves an iterative procedure whereby the ratio, K/J, known as the Eddington factor and designated, $f(\tau)$, is ascribed an assumed form as a function of optical depth. The equation of transfer is then integrated numerically to find the mean intensity and the source function, from which the intensity is recalculated by Feautrier's technique (1964). The function $f(\tau)$ is then recalculated and the procedure repeated. Iterative procedures such as those of Hummer and Rybicki and Chapman do not strictly adhere to the flux integral which can be solved exactly in conservative problems. This was noted by Wilson, Tung and Sen (1972) who also show that the half-range spherical harmonic method automatically takes into account any peaking of the radiation field. Nevertheless, the departures of the flux from the correct fluxes in the iterative methods are very small.

The only methods that have been applied to complex, non-conservative, anistropic scattering problems are those that involve the principle of invariant imbedding and those of a probabilistic nature. Such methods are very general and can be applied to problems in any geometry without any new difficulty. Leong and Sen (1969) have applied the probabilistic model to a spherical cloud with a central source. The methods of this type also have a straightforward application to problems with external sources. For example, Uesugi and Tsujita (1969) have considered a spherical atmosphere illuminated by a searchlight beam and Bellman, Kagiwada, Kalaba and Ueno (1969) have dealt with illumination by a crown of rays. However many of the integral equations derived by these methods are still awaiting numerical techniques suitable for their solution. Monte Carlo techniques are readily applicable to very complex geometrical situations, and Mattilla (1970) has studied scattering by an externally illuminated spherical cloud using such techniques. Nevertheless, it is in this range of problems in which the illumination is external, for which the least amount of information is available and all the problems considered in this thesis involve externally illuminated atmospheres.

It is natural to enquire into the possibility of replacing a spherical atmosphere by an equivalent plane-parallel atmosphere. Several authors have shown that this is possible in problems for which the integral equations of radiative transfer take the same form in each geometry. Gruschinske and Ueno (1970) have used such a technique to give exact numerical results for various problems by using the invariant imbedding approach. However, Minin (1964) has shown that this similarity between the two geometries is only present for isotropic scattering problems so that methods involving geometrical transformations are limited in their applicability. The selection of the method that we shall adopt for use in this thesis will be delayed until the general discussion in Section I.7.

3. Isotropic and Anisotropic Scattering

The absorption coefficient was defined in Section I.1 by equation (I-2) and it was stated there that this equation defined the total absorption or extinction coefficient per unit mass of the medium. The equation merely states that the loss of intensity from a radiation field passing through a medium is proportional to the intensity of the radiation and the distance traversed in the medium, provided the distance is infinitesimal so that the intensity and physical nature of the medium remain constant over that distance. It was also stated that this loss was due to absorption or scattering, or a combination of both. When the loss is due to absorption only, the extinction coefficient is written, κ_ν , and referred to as the absorption coefficient. It replaces

k_ν in equation (I-2). Similarly, when the loss is due to scattering only, the extinction coefficient is written, σ_ν , and referred to as the scattering coefficient. The linearity of these phenomena is evident from the definitions and requires that $k_\nu = \kappa_\nu + \sigma_\nu$ when both absorption and scattering are present. We define $\tilde{\omega}_\nu$ to be the albedo for single scattering of radiation of frequency ν , such that

$$\tilde{\omega}_\nu = \sigma_\nu / (\kappa_\nu + \sigma_\nu) . \tag{I-24}$$

The albedo is thus the fraction of the intensity lost to the pencil of radiation that is lost by a single scattering process. Alternatively, it can be said that the albedo for single scattering is the probability that a photon lost to the radiation field will re-emerge as a scattered photon.

The scattered radiation will not necessarily be scattered in the same direction as the incident radiation, nor will it necessarily be scattered equally in all directions, but will be scattered as a function of the angle between the directions of the incident and scattered radiation. If this angle is Θ , the function governing the directional dependence of the scattering is written, $p_\nu(\Theta)$, and known as the phase function for single scattering. It must be normalized so that the radiation scattered in all directions is given by the albedo. Thus, we have

$$\frac{1}{4\pi} \int p_\nu(\Theta)\, d\omega = \tilde{\omega}_\nu , \tag{I-25}$$

where $d\omega$ is the differential element of solid angle and the integral is over all solid angles. The phase function for isotropic scattering is, $p_\nu(\Theta) = \tilde{\omega}_\nu$; and another commonly used phase function, Rayleigh's phase function, is

$$p_\nu(\Theta) = \frac{3}{4}\,(1 + \cos^2\Theta).$$

This is an example of a conservative scattering phase function and it is accordingly normalized to unity.

A general phase function is frequently described by a series expansion of Legendre polynomials

$$p(\Theta) = \sum_{l=0}^{\infty} \tilde{\omega}_l \, P_l(\cos\Theta). \qquad \text{(I-26)}$$

We have omitted any frequency subscripts for clarity. The frequency dependence of these and other quantities will be discussed in the following section. In our co-ordinate systems directions are specified by the two co-ordinates, μ and ϕ. With primed quantities referring to the incident direction of the radiation and unprimed quantities to the emergent direction of the scattered radiation, equation (I-26) becomes

$$p(\mu,\phi;\mu',\phi') = \sum_{l=0}^{\infty} \tilde{\omega}_l \, P_l \left[\mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi-\phi') \right]$$

On expanding this by means of the addition theorem for spherical harmonics, we find

$$p(\mu,\phi;\mu',\phi') = \sum_{l=0}^{\infty} \tilde{\omega}_l \left\{ P_l(\mu)P_l(\mu') + \right.$$

$$\left. + 2\sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\mu)\,P_l^m(\mu')\cos m(\phi-\phi') \right\} \qquad \text{(I-27)}$$

This general phase function is only tractable for certain of the methods of solution of the transfer equation outlined in the preceding section. In fact, it cannot be used in the method that will be employed in Chapter II for the study of the radiative heating of plane-parallel atmospheres. It has already been stated that the problems in spherical geometry will involve axially symmetric radiation fields, which means that the radiation fields are independent of azimuth. Consequently, the phase function may be integrated over azimuth before introducing it into the theory of the problem at hand. This integration over azimuth results in a great simplification of equation (I-27). Hence

$$p(\mu, \mu') = \frac{1}{2\pi}\int_0^{2\pi} p(\mu, \phi; \mu', \phi')\, d\phi = \sum_{l=0}^{\infty} \tilde{\omega}_l P_l(\mu) P_l(\mu'), \qquad \text{(I-28)}$$

scattering

for axially symmetric radiation fields. This general phase function, (I-26) is normalized by use of equations (I-27) and (I-25) to yield $\tilde{\omega}_0$ as its albedo.

For many real atmospheres, phase functions are highly anisotropic. The atmospheres are frequently assumed to consist of spherical dielectric particles whose radius is of the order of 4 μ. The Mie theory of scattering is applicable to such particles and this theory always produces phase functions that have strong forward peaks and, to a lesser extent, backward peaks. Potter (1970) uses such a phase function for scattering in the atmosphere of Venus. The forward peak is sufficiently strong and narrow that Potter approximates the real function by truncating it and adding a delta-function spike in the forward direction. He considers the radiation scattered into the spike not to be

WRONG: α is fraction that is isotropically ~~isotropically~~ scattered; $1-\alpha$ is the remainder that goes into δ-fn spikes fwd & backward, & β is the fraction of $1-\alpha$ that goes into forward direction.

Hence $\alpha = 1$ isotropic, $\alpha = 0$ $\beta = 1$ fwd spike only

$\alpha = 0$ $\beta = 0$ backward spike only scattered at all and alters the albedo and total thickness of the atmosphere accordingly. Most treatments of complex phase functions involve their expansion as series such as that of equation (I-26). When strong forward scattering is present a large number of terms are needed to make the series expansion accurate. Such a large number of terms results in excessive computation time. However, this computation time is drastically reduced by the use of the delta-function approximation, whereby a 350 term expansion can be reduced to a 50 term expansion. Potter's results using the approximate phase functions agree to within one percent of those obtained by using the real phase functions except in cases where the radiation incident on the atmosphere is incident at a grazing angle and where the radiation emerging from the atmosphere is normal to or grazing the surface of the atmosphere. The general conclusion of his work is that a delta-function approximation for a sharp peak in a phase function is a good and valuable approximation.

The phase functions used in the study of planetary atmospheres do not differ greatly from the isotropic phase function once the two peaks are accounted for by delta-functions. As the delta-function approximation as described above has been found to be good, it is reasonable to postulate a schematic phase function consisting of an isotropically scattering part and two delta-function spikes, one forward and the other backward. Defining, $\tilde{\omega}$ as the albedo, α, the fraction of the scattered radiation that
isotropically, so that the fraction not going
is scattered into the spikes, and β, the fraction of the radiation scattered into the spikes that is scattered into the forward spike, the normalized schematic phase function is

$$p(\mu, \phi; \mu', \phi') = \tilde{\omega}\left\{ \alpha + 4\pi(1-\alpha)\left[\beta\,\delta(\mu-\mu')\,\delta(\phi-\phi') + \right.\right.$$

$$\left.\left. + (1-\beta)\,\delta(\mu+\mu')\,\delta(\phi-\phi'-\pi)\right]\right\}. \qquad \text{(I-29)}$$

The azimuthally independent form of this phase function suitable for axially symmetric radiation fields is

$$p(\mu, \mu') = \tilde{\omega}\left\{ \alpha + 2(1-\alpha)\left[\beta\,\delta(\mu-\mu') + (1-\beta)\,\delta(\mu+\mu')\right]\right\}. \qquad \text{(I-30)}$$

Three important special cases of equations (I-29) and (I-30) are; (1) $\alpha = 1$, which corresponds to isotropic scattering; (2) $\alpha = 0$, $\beta = 1$ which corresponds to scattering into the forward spike only; and (3) $\alpha = 0$, $\beta = 0$ which corresponds to the physically unrealistic case of scattering into the backward spike only. The second case can be considered to be one of no scattering provided that optical depth scale is suitably adjusted.

Phase functions are commonly described by two parameters, the albedo, $\tilde{\omega}$, and the asymmetry parameter, g . The asymmetry parameter is a measure of the forward throwing nature of the phase function and is defined as

$$g = \frac{1}{2}\int_{-1}^{+1} \cos\Theta\; p(\cos\Theta, 1)\; d\cos\Theta . \qquad \text{(I-31)}$$

Thus, for the schematic phase function given by equation (I-30) we have

$$g = \tilde{\omega}(1-\alpha)(2\beta-1), \qquad \text{(I-32)}$$

and for the general phase function given by equation (I-26) we have

$$g = \frac{1}{2}\int_{-1}^{+1} P_1(\cos\Theta) \sum_{l=0}^{\infty} \tilde{\omega}_l P_l(\cos\Theta)\, d\cos\Theta$$

$$= \frac{1}{3}\tilde{\omega}_1 . \qquad \text{(I-33)}$$

Is it possible to construct a schematic phase function that exhibits the same properties as a general phase function? If not, what restrictions must be imposed on the phase function in order to make possible the representation of a general phase function by a schematic one? In most problems involving scattering, part or all the emission coefficient is given by the scattering of a radiation field. For axially symmetric radiation fields this part of the emission coefficient is

$$j^{(s)} = k\rho \cdot \frac{1}{2}\int_{-1}^{+1} p(\mu,\mu')\, I(\mu')\, d\mu' . \qquad \text{(I-34)}$$

Consider first a phase function given by the general expansion, (I-26), the axially symmetric form of which is given by equation (I-28). An arbitrary radiation field can be represented by a sum of Legendre polynomials also;

$$I(\mu) = \sum_{n=0}^{N} a_n P_n(\mu) , \qquad \text{(I-35)}$$

so that inserting equations (I-28) and (I-35) into equation (I-34), we have

$$j^{(s)}(\mu) = \frac{1}{2} k\rho \int_{-1}^{+1} \sum_{l=0}^{\infty} \tilde{\omega}_l P_l(\mu) P_l(\mu') . \sum_{n=0}^{N} a_n P_n(\mu') . d\mu'$$

$$= k\rho \sum_{l=0}^{N} \tilde{\omega}_l a_l P_l(\mu) / (2l+1). \qquad \text{(I-36)}$$

Consider now the schematic phase function whose axially symmetric form is given by equation (I-30). The emission coefficient for scattering in this case is

$$j^{(s)}(\mu) = \frac{1}{2} k\rho \int_{-1}^{+1} \tilde{\omega} \left\{ \alpha + 2(1-\alpha) \left[\beta \delta(\mu-\mu') + \right.\right.$$

$$\left.\left. + (1-\beta) \delta(\mu+\mu') \right] . \sum_{n=0}^{N} a_n P_n(\mu') \, d\mu' \right.$$

$$= k\rho \left\{ \tilde{\omega} \alpha a_0 + 2\tilde{\omega}(1-\alpha) \sum_{n=0}^{N} \left[\beta a_n P_n(\mu) + \right.\right.$$

$$\left.\left. + (1-\beta) a_n P_n(-\mu) \right] \right\}.$$

Now, $P_n(-\mu) = (-1)^n P_n(\mu)$. Therefore

$$j^{(s)}(\mu) = k\rho \left\{ \tilde{\omega} \alpha a_0 + 2\tilde{\omega}(1-\alpha) \sum_{n=0}^{N} \left[\beta + (-1)^n (1-\beta) \right] a_n P_n(\mu) \right\}$$

$$= k\rho \sum_{n=0}^{N} b_n P_n(\mu). \qquad \text{(I-37)}$$

Equations (I-36) and (I-37) are of the same form and are identical if each pair of the N+1 coefficients are the same in the two series. Thus we can write N + 1 equations relating the three unknowns of the second series ($\tilde{\omega}$, α, β) to the N+1 unknowns of the first series, $(\tilde{\omega}_l)$. These can only be solved when N=2, whence we have

$$\tilde{\omega}_0 a_0 \equiv \tilde{\omega} a_0 \alpha + \tilde{\omega} a_0 (1-\alpha) = \tilde{\omega} a_0 \quad , \quad \text{so that} \quad \tilde{\omega} \equiv \tilde{\omega}_0 ,$$

$$\tilde{\omega}_1 a_1 / 3 \equiv \tilde{\omega} (1-\alpha)(2\beta-1) a_1 \quad , \quad \text{so that} \quad (2\beta-1) \equiv \tilde{\omega}_1 / 3 \tilde{\omega}_0 (1-\alpha),$$

and $$\tilde{\omega}_2 a_2 / 5 \equiv \tilde{\omega} (1-\alpha) a_2 , \quad \text{so that} \quad \alpha = (5\tilde{\omega}_0 - \tilde{\omega}_2) / 5\tilde{\omega}_0 .$$

Thus we can say that if an axially symmetric peaked radiation field can be represented by the sum of three Legendre polynomials, then the real phase function can be replaced by a schematic one and there will be no change in the radiation scattered from that field. A great many of the peaked radiation fields encountered in radiative transfer studies can be approximated fairly well by three Legendre polynomials, so the schematic phase function is a reasonable approximation. It is only when the radiation fields themselves become strongly peaked that there is a greater discrepancy between the results from the two phase functions. A corollary of this result is that the schematic phase function produces the same emission coefficient as a general phase function in an arbitrary axially symmetric radiation field if the phase function can be expressed in terms of three Legendre polynomials. This result follows by terminating the series of equation (I-28) at term M and noting that the two series, equations (I-36) and (I-37) can be made identical if M equals two for any value of N as well as vice versa. There are considerable advantages in using the schematic phase function for it is relatively simple to include it in analytical work and long computational times will not be required when numerical calculations are involved.

A further point that can be appropriately demonstrated at this point will appear very important in our later work in the thesis. It applies to problems that have axially symmetric radiation fields and is thus important in the theory of spherical atmospheres as developed in Chapter III. The integrals over angle, of the emission coefficient unweighted and weighted with μ, are often needed. For the general phase function, the scattering emission coefficient is given by equation (I-34) and the phase function by equation (I-28). Thus

$$\frac{1}{2}\int_{-1}^{+1} j^{(s)}(\mu)\,d\mu = \frac{1}{2}k\rho\int_{-1}^{+1}\int_{-1}^{+1} I(\mu')\sum_{l=0}^{\infty}\tilde{\omega}_l P_l(\mu)P_l(\mu')\,d\mu'd\mu$$

$$= k\rho\,\tilde{\omega}_0\,J, \qquad \text{(I-38)}$$

and

$$\frac{1}{2}\int_{-1}^{+1} \mu\, j^{(s)}(\mu)\,d\mu = \frac{1}{2}k\rho\int_{-1}^{+1}\int_{-1}^{+1} \mu' I(\mu')\sum_{l=0}^{\infty}\tilde{\omega}_l P_l(\mu)P_l(\mu')\,d\mu'd\mu$$

$$= \frac{1}{3}k\rho\,\tilde{\omega}_1 H = k\rho g H. \qquad \text{(I-39)}$$

The schematic phase function is given by equation (I-30) and the same two quantities for this phase function are

$$\frac{1}{2}\int_{-1}^{+1} j^{(s)}(\mu)\,d\mu = \frac{1}{2}k\rho\int_{-1}^{+1}\int_{-1}^{+1} I(\mu')\,\tilde{\omega}\left\{\alpha + 2(1-\alpha)\left[\beta\delta(\mu-\mu') + (1-\beta)\,\delta(\mu+\mu')\right]\right\}d\mu'\,d\mu,$$

$$= k\rho\tilde{\omega}\,J, \qquad \text{(I-40)}$$

and
$$\frac{1}{2}\int_{-1}^{+1} \mu\, j^{(s)}(\mu)\,d\mu = \frac{1}{2}k\rho\int_{-1}^{+1}\int_{-1}^{+1} \mu I(\mu')\,\tilde{\omega}\left\{\alpha + 2(1-\alpha)\times \left[\beta\,\delta(\mu-\mu') + (1-\beta)\,\delta(\mu+\mu')\right]\right\}d\mu'\,d\mu$$

$$= k\rho\,\tilde{\omega}(1-\alpha)(2\beta-1)\,H = k\rho g H. \qquad \text{(I-41)}$$

Equations (I-38) to (I-41) show that the two phase functions give the same expressions for these quantities if theyhave the same albedo and the same asymmetry parameter. The other integrals in the series can also be found, The schematic phase function has only three arbitrary parameters so that only three pairs of integrals can be equated for the two phase functions. However, it will transpire that only the two integrals evaluated above will be needed in order to solve the equation of transfer. This means that, under any restrictions imposed by the method, the solution of the equation of transfer will be independent of the phase function apart from the values of the albedo and the asymmetry parameter. The significance of this will be examined later when the equation of transfer is solved for spherically symmetric atmospheres.

The relationship between anisotropic and isotropic scattering has been studied by a number of authors. Van de Hulst and Grossman

(1968) have reported the existence of similarity relations between plane-parallel atmospheres containing anisotropic scatterers and similar atmospheres containing isotropic scatterers. The intensity of the reflected radiation from a finite plane-parallel atmosphere containing anisotropic scatterers is very similar to that of the radiation reflected from a plane-parallel atmosphere of different total optical thickness containing isotropic scatterers of a different albedo. The similarity relations are simple relationships between the two optical thicknesses and the two albedos. It is interesting to note that they involve only the albedo and the asymmetry parameter of the phase function and that the exact shape of the phase function does not affect the reflected radiation. This conclusion is virtually the same as those of the two preceding paragraphs and emphasises the fact that the exact nature of the phase function is not of vital importance to the results of radiative transfer calculations. Hansen (1969b) has calculated absorption line profiles for a cloudy atmosphere using an expansion for the phase function and also using the similarity relations. His conclusions are basically that, although anisotropy does affect the shape of the absorption line, that shape is barely dependent on which of the two methods is used. These results show that the similarity relations are good, certainly in that context, and also that the asymmetry parameter is the most important phase function parameter after the albedo. The existence of the similarity relations and their dependence on $\tilde{\omega}$ and g only, gives support to the result that the solution of the equation of transfer for axially symmetric fields depends on $\tilde{\omega}$ and g only; a result which might have appeared at first sight to be detrimental to the approximation that permitted the result.

Thus we conclude that these similarity relations together with Potter's delta-function results provide a good rationale for the use of the schematic phase function of equation (I-29). This phase function can provide a full range of values of g from zero to unity and is lacking complete generality only in the rest of its shape. Consequently, we shall not concern ourselves with phase functions more complex than the schematic one.

4. Grey and Non-grey Atmospheres

The general equation of transfer for plane-parallel atmospheres is equation (I-9) and this equation involves frequency dependent quantities. The frequency dependence of these quantities arises from their definitions which are constructed to be as general as possible. This Section is concerned with the nature of the frequency dependence of these quantities and the manner in which it affects the equation of transfer. The discussion will be restricted to plane-parallel atmospheres but applies equally to atmospheres in any geometry.

The source function at a particular frequency is, in general, dependent on the intensity of the radiation field at every other frequency and this is basically the reason for the extreme complexity of the frequency dependent problem. We shall consider two limiting cases for the source function. The first is that for an atmosphere in local thermodynamic equilibrium or LTE. In this case, an element of matter emits radiation with a spectrum that depends solely on its temperature, which in turn depends on the energy balance within the element of matter. This spectral distribution is described by the Planck function, $B(\nu, T)$; and the equation of transfer, (I-9) becomes

$$\mu \frac{dI_\nu(z)}{dz} = -\kappa_\nu \rho \left[I_\nu(z) - B(\nu, T) \right], \tag{I-42}$$

where κ_ν is the frequency dependent absorption coefficient. There is no scattering of the radiation in this case.

The other extreme case is that of an atmosphere whose emission is due to scattering only. Before formulating an expression for the source function it is necessary to define a function, $P(\nu, \nu')$, which describes the frequency dependence of an act of scattering. Adopting the definition of such a function as given by Stibbs (1953), we say that $P(\nu, \nu')$ denotes the probability that the average atom in an assembly of absorbing atoms will emit radiation of frequency, ν' , after an absorption has taken place in frequency ν . This function is normalized to unity so that

$$\int_0^\infty P(\nu, \nu')\, d\nu' = 1 . \tag{I-43}$$

It follows from the definition of the phase function, the absorption coefficient and the emission coefficient that the emission coefficient for a scattering atmosphere is

$$j_\nu^{(s)}(z, \mu, \phi) = (\kappa_\nu + \sigma_\nu) \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} \int_0^\infty P(\nu, \nu')\, p(\mu, \phi; \mu', \phi') \times$$

$$\times\, I_\nu(z, \mu', \phi')\, d\mu' d\phi' d\nu', \tag{I-44}$$

where any radiation absorbed through the absorption coefficient,

κ_ν , is completely removed from the radiation field. The source function is given by $j_\nu^{(s)}(z,\mu,\phi)\,/\,(\kappa_\nu+\sigma_\nu)$; and we thus see that the source function is a function of the intensity at every frequency.

We must consider first the special case of coherent scattering. Scattering is said to be coherent when the emitted radiation is of the same frequency as the incident radiation. This is expressed mathematically by the equation

$$p(\nu,\nu') = \delta(\nu-\nu') , \tag{I-45}$$

where $\delta(\nu-\nu')$ is Dirac's delta-function which is normalized to unity by definition. When inserting equation (I-45) into equation (I-44) for the emission coefficient, we obtain

$$j_\nu^{(s)}(z,\mu,\phi) = (\kappa_\nu+\sigma_\nu)\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} p(\mu,\phi;\mu',\phi')\, I_\nu(z,\mu',\phi')\,d\mu' d\phi' . \tag{I-46}$$

Thus the equation of transfer for pure, non-conservative, coherent scattering in a plane-parallel atmosphere is

$$-\mu\frac{dI_\nu(z,\mu,\phi)}{dz} = (\kappa_\nu+\sigma_\nu)\rho\left\{ I_\nu(z,\mu,\phi) - \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} p(\mu,\phi;\mu',\phi')\, I_\nu(z;\mu',\phi')\,d\mu' d\phi' .\right\} \tag{I-47}$$

This equation of transfer for radiation of frequency, ν , is independent of the intensity of radiation at other frequencies. Consequently, problems involving coherent scattering only are relatively simple to solve in terms of the frequency dependence of the scattered radiation.

In general, the source function is a combination of both extreme cases of emission and the problem becomes very much more complex. In line absorption in stellar atmospheres radiation is transferred by means of atomic and ionic electron transitions. This could be considered to be coherent scattering except for the many atomic collisions that exist because the density and temperature in stellar atmospheres are high. These have the effect of broadening the atomic energy levels and making the scattering non-coherent. This subject has received considerable attention, one review article, for example, is that by Spitzer (1944). However, our work is concerned with radiative transfer in planetary atmospheres and interstellar dust clouds. These media are much cooler than stellar atmospheres and are made up of much larger particles, which scatter radiation in a classical manner as described by the Mie theory. The subject of scattering by such particles has been covered in detail by Van de Hulst (1957). Mie scattering is always coherent so we shall consider all scattering in this thesis to be coherent as is standard procedure in studies of planetary atmospheres and interstellar dust clouds.

A different technique is required in order to simplify the equation of transfer of an atmosphere in LTE as given by equation (I-42). A commonly used approximation in this respect is that of the atmosphere being grey. An atmosphere is said to be grey when the absorption coefficients are independent of frequency. The grey equation of transfer is found by integrating equation (I-42) over all frequencies and is

$$-\mu \frac{dI(z)}{dz} = \kappa\rho \left[I(z) - B(T) \right], \tag{I-48}$$

where $I(z) = \int_0^\infty I_\nu(z)\, d\nu$ and $B(T) = \int_0^\infty B(\nu,T)\, d\nu$.

The quantity I(z) is known as the integrated intensity and B(T) as the integrated Planck function which is a simple function of the temperature,

$$B(T) = \sigma T^4/\pi, \tag{I-49}$$

where σ is Stephan's constant. The absorption coefficient, κ, in equation (I-48) is the grey absorption coefficient. In practice, the absorption coefficient is not grey but a strongly frequency dependent quantity. Nevertheless, equation (I-48) is frequently used as the equation of transfer for the integrated intensity and κ is taken to be a mean absorption coefficient suitably defined. This sounds simple but is quite complicated, even in the classical Milne problem. For equation (I-48) to be true with κ representing a mean absorption coefficient it is necessary to have

$$\kappa = \int_0^\infty \kappa_\nu I_\nu\, d\nu \Big/ \int_0^\infty I_\nu\, d\nu = \int_0^\infty \kappa_\nu B(\nu,T)\, d\nu \Big/ B(T).$$

This relation cannot, of course, hold in general.

However, when the problem is concerned with an optically thin atmosphere dominated by emission, the integrated Planck function is greater than the integrated intensity. Consequently, the mean absorption coefficient may be taken to be equal to the Planck mean absorption coefficient which is defined as

$$\bar{\kappa}_P = \int_0^\infty \kappa_\nu B(\nu,T)\,d\nu \Big/ B(T). \tag{I-50}$$

For the other extreme of an optically thick atmosphere in which the temperature varies slowly with depth, the Rosseland mean absorption coefficient may be used, namely

$$\bar{\kappa}_R = \int_0^\infty \kappa_\nu \frac{\partial B(\nu,T)}{\partial T}\,d\nu \Big/ \int_0^\infty \frac{\partial B(\nu,T)}{\partial T}\,d\nu. \tag{I-51}$$

There have been many attempts to formulate suitable mean absorption coefficients for problems intermediate between the two cases mentioned above. These frequently involve the intensity of the radiation field which is the desired solution of the equation of transfer. This in turn depends on the mean absorption coefficient and the direct solution of the equation of transfer. Furthermore, as has been pointed out by Traugott (1968), mean absorption coefficients that involve the intensity itself will be functions of direction as well; and the equation of transfer will be even more complicated than that given by equation (I-48). Recently, Pomraning (1971) has suggested a new method of finding a suitable grey equation of transfer. The method involves a mean absorption coefficient defined by the asymptotic solution of the equation of transfer. The angular dependence of the mean absorption coefficient so defined is accounted for by a process that involves the insertion of an extra term into the source function. This takes a form identical to a scattering source function. The major difficulty encountered in this method of forming a grey equation of transfer is the construction of the asymptotic solution of the equation of transfer, which is fundamental to the definition of the mean absorption coefficient. Pomraning also suggests that such detailed consideration of the mean absorption coefficient is only necessary when the frequency variation of the absorption coefficient is rapid. Otherwise, any suitable mean absorption coefficient such as the Planck or Rosseland coefficient is quite adequate. Additional alternative treatments of non-grey stellar atmosphere problems have been discussed by Kourganoff (1952). These involve iterative procedures starting from the solutions for the grey atmospheres. However, for our studies of planetary atmospheres and interstellar dust clouds it will be sufficient to consider them to be grey and to use a simple mean absorption coefficient.

In conclusion, it should be noted that the grey equation of transfer and the monochromatic equation of transfer for coherent scattering are of the same form but refer to different quantities, namely the integrated and monochromatic intensities respectively. The latter is given by equation (I-47), while the integrated emission coefficient for the coherent scattering atmosphere is the integral of equation (I-44) over all frequencies, which is

$$j^{(s)}(z,\mu,\phi) = (\kappa+\sigma)\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} p(\mu,\phi;\mu',\phi')\,I(z,\mu',\phi')\,d\mu'\,d\phi$$

Accordingly, we have

$$\mu \frac{dI(z,\mu,\phi)}{dz} = -(\kappa+\sigma)\rho \Big\{ I(z,\mu,\phi) - \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} p(\mu,\phi;\mu',\phi')\, I(z,\mu',\phi')\, d\mu'\, d\phi' \Big\} \qquad \text{(I-52)}$$

The solution of this equation is the same function of z as the solution of equation (I-47) for the monochromatic intensity. As a consequence of this we can solve for the monochromatic intensity and obtain the integrated intensity by a suitable frequency integration. This solution for the integrated intensity will differ from the solution of equation (I-52). The reason for this difference is that the variation of the absorption coefficient with frequency will cause the radiation of different frequencies to be attenuated by different amounts after traversing the same geometrical distance. Thus, at any point in the atmosphere the integral of the intensity over frequency will not have the same value as the integrated intensity given by the solution of the grey equation of transfer. However, this discrepancy will be very small if the absorption coefficient is a slowly varying function of frequency. It is such a function in the planetary atmosphere and interstellar dust cloud problems so that we shall use the grey equation. We shall see later that this allows a complete analytical solution of the radiative heating problem. The monochromatic treatment would not allow such a solution, but would be preferable were a complete analytical solution impossible due to other factors.

5. The Problem of Radiative Heating

It is well known that radiation in an enclosure surrounded by walls which are maintained at a constant temperature will be in equilibrium with the walls, and that the equilibrium distribution of the radiation, both as regards quantity and quality (wavelength), is fixed entirely by the temperature, T, of the walls. Such radiation is known as black-body radiation and is described by the Planck function, $B(\nu,T)$. Many of the properties of this black-body radiation have been discussed by Eddington (1926). Although black-body radiation can be in equilibrium with matter, and both the radiation and the matter can be described by the temperature, T, a mixture of radiation of various wavelengths in arbitrary proportions is not, in general, in equilibrium with matter at any temperature and has no unique temperature. However, if it has the same total energy density as radiation of temperature, T, which must be a black-body radiation, T is called its effective temperature. Thus, an isotropic radiation field with arbitrary frequency variation of intensity, I_ν , is ascribed an effective temperature, T, given by

$$\frac{\sigma T^4}{\pi} = \int_0^\infty I_\nu \, d\nu , \tag{I-53}$$

where σ is Stefan's constant. The term, $\sigma T^4/\pi$ is merely the integral of the Planck function $B(T)$, as stated by Stefan's law. The temperature that a particle would adopt in such a radiation field has been discussed by Van de Hulst (1946), Eddington (1926), and by Fabry (1917). A black-body is said to absorb all radiation incident upon it and emit the same total energy, which by Kirchhoff's law is in the form of radiation described by Planck's law.

Therefore, it will adopt a temperature equal to the effective temperature of the radiation field.

Real particles are not black-bodies but are particles that have frequency dependent absorption efficiencies that are less than unity. The absorption efficiency of a particle is defined by Van de Hulst (1957) to be the ratio of its absorption cross-section to its geometrical cross-section, and is denoted by Q_{abs}. For the particle to be in thermal equilibrium it must emit the same total energy as it absorbs. This condition is expressed mathematically by the energy balance equation,

$$\int_0^\infty Q_{abs}\, I_\nu \, d\nu = \int_0^\infty Q_{abs}\, B(\nu,T)\, d\nu . \tag{I-54}$$

The quantity, Q_{abs} is, of course, a function of frequency, and T is defined as the temperature of the particle. The form of this equation follows that proposed by Van de Hulst (1946) but is expressed in terms used more recently by most authors. It can be seen that it reduces to equation (I-53) if the absorption efficiency of the particle is independent of frequency in which case the particle is said to be grey. Thus, a grey body adopts the same temperature that a black-body would in the same radiation field, that temperature being equal to the effective temperature of the radiation. In equation (I-54) the absorption efficiency can be replaced by the absorption coefficient defined in Section I.1 because the latter is merely the former multiplied by an appropriate constant which cancels from each side of the equation. Accordingly, equation (I-54) can be written in the form

$$\int_0^\infty \kappa_\nu I_\nu \, d\nu = \int_0^\infty \kappa_\nu B(\nu,T)\, d\nu . \tag{I-55}$$

In all problems involving the radiative heating of a particle by external radiation sources, the radiation field in which the particle is found is dilute, and in most cases, very dilute. Typically, a grain in interstellar space will be in a radiation field which is approximately black-body radiation of temperature 10^4 ^oK, diluted by a factor $W = 10^{-15}$. Were such a grain a black-body it would take up the effective temperature of the radiation field, which in this case is approximately 3^oK. Thus we see that the absorption by the grain, given by the left-hand side of equation (I-55) involves radiation in the visible and neighbouring parts of the spectrum, whilst the emission from the grain involves radiation in the far infra-red. In most problems, $I_\nu = 0$ when ν is less than some particular frequency, ν_1, and $B(\nu,T) = 0$ when ν is greater than some particular frequency, ν_2. Thus, equation (I-55) can usually be replaced by

$$\int_{\nu_1}^{\infty} \kappa_\nu I_\nu \, d\nu = \int_0^{\nu_2} \kappa_\nu B(\nu,T) \, d\nu . \qquad \text{(I-56)}$$

It will be seen later that transfer problems involving these radiation fields are simplified considerably when ν_1 is greater than ν_2. Fortunately this is true for most physical problems, because the dilution of the radiation is so high. For example, a 10^4 ^oK radiation must have $W < 10^{-6}$ in order to ensure that $\nu_1 > \nu_2$.

A microscopic reason for this degrading of dilute high temperature radiation is given by Woolley and Stibbs (1953) in terms of Rosseland's theorem. This theorem states that, in a very dilute radiation field a quantum system will almost always degrade the radiation and thus, the chance of a radiation field being changed into a dilute hotter field is negligible. The theorem also shows that the probability of thermal photons being upgraded to visible photons is extremely small. This condition will be a fundamental feature of our treatment of radiative heating as will be seen when the equations of transfer pertinent to the problem are formulated in Chapters II and III.

The manner in which the temperature of a real particle depends on its absorption coefficient has been discussed by Fabry (1917). He simplified the discussion by considering the absorption coefficient to be grey and equal to κ_s at frequencies greater than ν_1, and grey and equal to κ_p at frequencies lower than ν_2. We again assume that ν_1 is greater than ν_2 and we define the parameter, n to be their ratio; $n = \kappa_s/\kappa_p$. When n equals unity the material is completely grey and the temperature of the particle is equal to the effective temperature of the radiation field incident upon it. When n increases, κ_p decreases so that $B(\nu,T)$ must increase to maintain energy balance in the particle. Similarly, κ_p increases and $B(\nu,T)$ decreases when n decreases. The outcome of this balancing procedure is that the temperature of the particle is greater than or less than the temperature that a black-body would assume in the same situation, according to whether n is greater or less than unity. This argument has been applied to a situation which at first sight appears to be rather restricted; but a more general frequency dependent absorption coefficient could have been considered and the argument would have reached the same conclusion with the parameter, n, defined as some measure of the average slope of the absorption coefficient plotted as a function of frequency.

In defining the temperature of a particle by means of the energy balance condition, it is necessary to estimate whether or not fluctuations of the energy of the system are comparable to the energy content of the particle. Greenberg (1968) stated that if they were then the temperature could not be so defined. However, he concluded that the definition of temperature from the energy balance condition is acceptable from this point of view by showing that the rate of absorption of radiant energy by a particle typical of those in interstellar space is significantly less than the total heat content of the particle.

As examples of typical calculations for interstellar grains we quote Greenberg (1971) who calculates the temperatures of spherical ice particles which possess a frequency dependent absorption coefficient. The absorption efficiency or spectral emissivity, ϵ, of a 0.1μ radius ice particle at $10^3\,^\circ K$ and $10^\circ K$ is 2.5×10^{-2} and 1.9×10^{-4} respectively. Were these values constant over their respective parts of the spectrum then the parameter n, defined above would assume a value of the order of 10^2. The fact that they are not constant makes the temperature calculations longer and sufficiently complex to necessitate the use of a computer. For such an ice-particle in a dilute black-body radiation field of temperature, $10^3\,^\circ K$, and dilution factor, 10^{-10}, Greenberg's calculations give $10.8\,^\circ K$ as the temperature of the grain. This compares with the value of $3.18^\circ K$ that a black-body would adopt in the same radiation field, and bears out the conclusions of the qualitative argument given above. The precise nature of the frequency dependence of the grain absorption coefficients is fundamental to the determination of their temperatures and a great deal of work is being done at present on various grain models and their absorption coefficients. Over the past decade a large number of models have been proposed. These include carbon grains, carbon grains with ice mantles, silicate grains and mixtures of all three. These and other grain models have been reviewed by Greenberg (1968) and more recently by Wickramasinghe and Nandy (1972). Also, Hoyle and Wickramasinghe (1967) have proposed grains with impurity oscillators that enhance their far infra-red emission and hence lower their temperature. These grain models have to fit observational data obtained for several different phenomena and in particular have to fit interstellar reddening curves. So far, no grain model has been isolated as the best, the problem being remarkably complex in view of the infinite possibility of grain mixtures.

The preceding discussion has been concerned with an isolated particle. This thesis is concerned with the radiative heating of accumulations of such particles in atmospheres. The dependence of the energy balance on the nature of the absorption coefficient is the same for any particle whether it is isolated or in an atmosphere. However, a particle in an atmosphere absorbs a radiation field that varies throughout the atmosphere. This radiation field is controlled by the absorption coefficient of the particles so that the particle temperature depends on the absorption coefficient in two distinct ways. As an example, we shall consider a semi-infinite atmosphere illuminated from above by a dilute stellar radiation field. The term "stellar radiation field" will always refer to radiation in the visible and neighbouring parts of the spectrum, that radiation being of some stellar origin. Again, for simplification of the discussion we shall assume a two-part grey absorption coefficient of parameters κ_s , κ_p and n as defined earlier. The stellar radiation is attenuated on passage through the atmosphere due to absorption by the particles according to the stellar absorption coefficient, κ_s . The reduction of the intensity of the stellar field causes the temperature to fall and thus it would be expected that the temperature would decrease with depth in the atmosphere for any value of n . This would be true were equation (I-56) valid inside an atmosphere; but it is not. The thermal radiation from the other particles in the atmosphere will also be incident upon the particle under consideration and will also heat up the particle. The absorption, and hence, the temperature, will essentially depend on the intensity of the dominant radiation field. Equation (I-55) shows clearly that the dominant field is the stellar field when n is large, and the thermal field, which is the radiation field generated by the thermal emission from the particles, when n is small. The intensity of the thermal radiation field is itself controlled by the parameter, n , the stellar radiation field being fixed by the particular choice of κ_s . The parameter, n , as well as being the ratio of the absorption coefficient in the visible to that in the far infra-red, is also the inverse ratio of the mean free paths of the photons in the two radiation fields. If Λ is the mean free path of a photon, then $n = \Lambda_p / \Lambda_s$. Thus, for small values of n , the mean free path of the thermal photons is shorter than that of the visible photons and a large radiation field can be built up inside the atmosphere in those regions where the stellar photons can reach easily but the thermal photons cannot leave. This is known as the classical greenhouse effect, a phenomenon that greatly increases the temperature of those parts of the atmosphere that are sufficiently far from the surface. It increases the temperature to such an extent that the decrease in temperature due to both the smaller stellar field and the single particle energy balance effect are made insignificant.

When n assumes a value greater than unity the absorption by a particle is dominated by the stellar radiation field. Moreover, the mean free path of the thermal photons is large in comparison with that of the stellar photons so that thermal radiation escapes very easily and cannot be built up into a large field. An inverse greenhouse effect is said to exist and the temperature of the particles follows closely the intensity of the stellar radiation field except where that field is zero, in which case the thermal radiation, though small, is non-zero everywhere due to the fact that ~~ease with which~~ the thermal photons pass through the atmosphere, and this non-zero thermal radiation field prevents the temperature of the particles falling to zero.

It must be stressed that a particle at any point in an atmosphere will assume a temperature above or below the effective temperature of the radiation field at that point, depending on n in the same manner as did an isolated particle. The effective temperature of the radiation field is, of course, the temperature that a black-body would assume at the same point in the atmosphere, though the radiation field would be vastly different were all the particles black-bodies. Nevertheless, a particle deep in an atmosphere whose value of n is less than unity will assume a temperature greater than the effective temperature of the reduced incident radiation due to the build up of the large thermal field, but less than the effective temperature of the total radiation field due to the energy balance condition.

This discussion has not included consideration of scattering of the incident radiation. This is because it would not have altered the qualitative conclusions but would have unnecessarily complicated and confused the argument involved in reaching these conclusions. The effect of scattering will be discussed later.

The energy balance equation has been derived under the assumption that the heating of the particle is due to the absorption of radiation only, and that the cooling of the particle is due to the emission of radiation only. We shall now consider the relative importance of other possible mechanisms for the heating and cooling of interstellar grains. The literature, to date, would imply that these other mechanisms are negligible and that the energy balance within a particle is indeed represented by equation (I-55). Van de Hulst (1949) has considered the heating of interstellar grains in normal interstellar space, by collisions with atoms and ions and by chemical reactions involving captured atoms. His calculations showed that both these heating mechanisms are negligible. Similarly, his calculations for the cooling of interstellar grains by the evaporation of atoms from their surfaces indicate that the dominant cooling mechanism is the emission of thermal radiation. The effect of grain heating by collisions has been calculated more recently by Greenberg (1971) and he too deems it negligible under normal conditions. The physical processes of molecule formation on the surfaces of interstellar grains have been examined by Solomon and Wickramasinghe (1969). Atoms of hydrogen adhere to the surface of a grain by physical rather than chemical absorption processes and they form molecules on the grain surfaces.

Considerable heat of formation is generated by this process, but whether it is imparted to the grain as a whole or to neighbouring atoms, the grain temperature will remain fairly constant due to the resulting cooling by evaporation. Furthermore, a monolayer of hydrogen molecules is prevented from forming on the grain surface until all the hydrogen in the atmosphere is in molecular form. Only then is it possible for shells of hydrogen to be built up around the grain. The important result of such work from our point of view is that the temperature of the grain is barely affected by these processes, so that we can ignore grain heating by molecule formation. A further source of heating is that of cosmic rays passing through the grains. Salpeter and Wickramasinghe (1969) discuss this process and show it to be negligible under most conditions. Thus we may consider the grains to acquire a temperature controlled solely by radiative absorption and emission processes.

Different physical conditions exist in planetary atmospheres where the densities of the particles and the gas are large. Essentially there are two subsystems; the cloud subsystem and the gas subsystem. The processes of radiative heating and cooling apply to the two subsystems separately. Their heating rates are different because the heat capacities of the two subsystems are different and this sets up a non-LTE situation. To deal with energy transfer within either subsystem it would be necessary to account for the flow of energy between them and hence set up a coupled pair of transfer problems. The processes of energy exchange between the two subsystems are: (1) radiative exchange; (2) heat conduction across the particle boundary from the kinetic energy of the gas molecules; and (3) phase changes involving mass exchanges between the two subsystems. If these processes are efficient then the two subsystems will be in LTE and may be treated as one system. If not, then the complex problem suggested earlier will exist. Samuelson (1970), who formulated this argument, showed that process (2) is very efficient in normal planetary atmospheres and that such atmospheres can be considered to be in LTE. The consequence of this is that both subsystems are at the same temperature, which is generated by the radiative heating and cooling of the two subsystems together. The atmosphere can thus be treated as having optical properties equal to the sums of the respective optical properties of the two individual subsystems. The equation of transfer can be formulated and solved for the medium as a whole. Other forms of heating, such as cosmic ray heating, are clearly negligible in the case of planetary atmospheres. Samuelson also states that the LTE preserving mechanisms are far less efficient for interstellar and circumstellar clouds where the gas densities are very much lower. Consequently LTE cannot be assumed for these media. Fortunately, as already explained, the gas density in these media is usually so small that the interaction between the gas and dust subsystems is negligible and the radiative heating of the grains can be treated without reference to the gas. This is certainly true in most interstellar clouds but care must be taken in circumstellar shell problems where gas densities may be higher.

Before concluding this section mention should be made of the phenomenon of scattering and its relation to radiative heating. The grains and cloud particles are known to scatter a substantial fraction of the visible radiation incident upon them. This affects the transfer aspects of the problem in that it governs the amount of radiation that is absorbed and also the penetration of the stellar radiation into the atmosphere. The scattered radiation is emergent from the particle in one sense, but it is not included in the energy balance equation because it would be imprudent to consider the temperature to be dependent on a dilute radiation field, the temperature of the particle being a measure of its thermal emission. A fraction of the thermal radiation absorbed by a particle may also be considered to be scattered and consequently not contribute to the temperature of the particle. Such a situation has been considered by Samuelson (1967a), though it is not standard amongst other works on radiative heating and infra-red radiative transfer. It must be stressed that all scattering processes affect the temperature of a particle only in as much as they control the radiation field incident on the particle. The temperatures of particles in a particular radiation field depend only on the absorption coefficients.

6. Critique of Previous Work

The quantity of literature concerned with radiative transfer problems is so vast that we must restrict the discussion of this section to those papers that deal directly with radiative heating problems, even though scattering theory is used in radiative heating theory. As mentioned in the previous section the topic of radiative heating has been centred on the heating of planetary atmospheres and dust clouds, interstellar and circumstellar. Firstly we shall consider several of the papers devoted to the greenhouse effect in planetary atmospheres.

The exact solution for the mean intensity of the thermal radiation and the temperature within a two part grey semi-infinite plane-parallel atmosphere has been obtained by Wildt (1966) by using a form of Hopf's (1934) analytical solution of Milne's integral equation for a grey atmosphere in strict radiative equilibrium. Wildt's solution is restricted to the special case of a non-scattering atmosphere but he quotes extensive results for this problem. More recently, Shultis and Kaper (1969) have obtained the exact solution for the mean intensity of the thermal radiation and temperature within an anisotropically scattering, two part grey, finite plane-parallel atmosphere. They used a complex method derived from techniques developed in the field of neutron transport which produced the solution in the form of an integral equation to be solved by numerical methods. They did not quote any numerical results because no suitable computer programme had been developed. In the event of the production of such a programme, their method will prove extremely valuable.

The lack of comprehensive exact results has meant that it has been necessary to use approximate methods for solving the transfer problems involved in the greenhouse effect. Samuelson (1967a) has used the discrete ordinate method for a two part grey, anisotropically scattering, semi-infinite plane-parallel atmosphere, which he supposed to consist of particulate matter whose optical properties could be calculated from the Mie theory. Another source of infra-red opacity in the Venusian atmosphere is absorption by carbon dioxide and water vapour. In relation to this, a series of papers by Sagan and Pollock were published drawing conclusions from measurements made by the Mariner 5 and Venera 4 Venus probes. In a study of the greenhouse effect on Venus under radiative and convective equilibrium for a variety of frequency dependent absorption coefficients, Sagan (1969) concluded that an approximation of greyness is not good when gaseous absorption is the dominant source of the infra-red opacity but acceptable when particulate matter is the dominant infra-red opacity source, provided that the particles are larger than or of comparable size to the wavelength of the radiation. Furthermore, even a window-grey approximation is often inappropriate in the former case. These conclusions were obtained from solutions of the transfer problems using Eddington's approximation. Pollack (1969a,b) gives a method of calculating the greenhouse effect for non-grey atmospheres; but the models that he used were numerous and diverse so that attempts to compare his results with the measurements of the physical state of the atmosphere are inconclusive with regard to the choice of the most suitable model for the atmosphere. The general conclusion drawn from these papers is that the gases in the atmosphere can provide the infra-red opacity necessary to give rise to the measured greenhouse effect under certain restrictions imposed on the constitution of the atmosphere. Further conclusions are obscure. The non-grey problem has also been treated by Ohring (1969) who assumed the measured temperature profile for the Venusian atmosphere and developed an iterative technique for the radiation fields using the energy balance condition at the surface. He too concluded that the greenhouse effect could be maintained by either gas or dust. A further critique of other work on the greenhouse effect has been given by Pollack (1969b), who also showed that the Venusian atmosphere is not black to the incident sunlight. Thus, the semi-infinite atmosphere used by Samuelson is far from satisfactory even for Venus. We shall see later that the semi-infinite atmosphere is a good approximation to a finite atmosphere with a conservative ground layer, only when the ground behaves as the semi-infinite atmosphere itself. The measurements of the temperature profile of the Venusian atmosphere as quoted by Sagan and Pollack (1969) indicate that the atmosphere is not isothermal near the ground, whereas the predictions of Samuelson's model are that the temperature rises as the depth increases until a maximum temperature is reached at a certain depth and that below this depth the atmosphere is isothermal. As we shall see later, this isothermal region occurs when the fluxes of the infrared and stellar radiation fields are zero. Thus, the Venusian atmosphere can never be replaced by a semi-infinite atmosphere, unless it transpires that the reflection properties of the ground are exactly those of the semi-infinite atmosphere.

The emission and transmission of thermal radiation in clouds and haze in planetary atmospheres has been treated in a completely different manner by Kattawar and Plass (1970). They utilised a Monte Carlo technique to follow the path of a thermal photon after emission, accounting for multiple scattering until it emerged from the atmosphere or was absorbed. They assumed a temperature profile for the atmosphere, which was assumed to be in LTE and calculated the thermal emission from each particle using the black-body function for the appropriate temperature. This treatment, though acceptable for scattering problems with an external source, contradicts the theory of radiative transfer in which the absorbed photons contribute to the source function, the temperature and the thermal emission of each particle. The Monte Carlo method of Kattawar and Plass is only valid in the limiting case where the temperature is dominated by the absorption of visual photons and where there is a large albedo for thermal photon scattering. This is the situation of an inverse greenhouse effect which is not the observed situation for planetary atmospheres.

The equivalent study in spherical geometry is less well documented. Whilst there has been a strong interest in developing radiative transfer solutions for scattering in spherical atmospheres there have been very few calculations of the infra-red radiation generated in such atmospheres. The astronomical objects to have received some interest in this context are circumstellar shells and interstellar grains. Huang has studied the former in a series of three papers (1969a,b, 1971). The first of these deals with the two limiting cases of optically thin and optically thick shells, and the second with the more interesting intermediate case. The properties that he ascribed to the grains of the shell are not those of typical interstellar grains. He claimed that circumstellar grains are very much larger than typical interstellar grains and hence, that they scatter isotropically, emit infra-red radiation isotropically and possess a completely grey extinction coefficient. It would not be a difficult task to extend Huang's calculations to encompass grains that scatter anisotropically and have a two part grey extinction coefficient. Huang used the Eddington approximation to solve the transfer equations and assumed the radiation incident on the inner surface of the shell to be diffuse and of a known flux. This assumption is only valid when the shell is very close to the surface of the central star. The resulting solution for the mean intensity of the visible radiation in the shell involves a homogeneous Bessel equation which can be solved analytically; and the solution for the mean intensity of the thermal radiation in the shell is found from the solutions for the visible radiation and the total radiation. A serious criticism of Huang's theory arises from the work of Chapman (1966) on extended stellar atmospheres. For the problem of constant net energy flow, which is the spherical equivalent of the problem of constant net flux, Chapman has shown that the outward flowing radiation field becomes progressively more forward peaked as its distance from the central star increases. Thus the ratio K/J which Huang has assumed to be one third, is closer to unity at large values of the radius. As mentioned in Section I.2 several other methods are known to be able to overcome this phenomenon.

Huang's third paper deals with the case of distant envelopes for which the solid angle subtended by the central star at any point in the envelope or shell is sufficiently small that the star may be taken to behave as a point source. Consequently, the radiation incident on the inner boundary of the shell is no longer diffuse but radially directed. The reduced incident radiation, which is the unscattered part of the radiation from the star, is known exactly and the equation of transfer for the diffuse, scattered radiation field can be solved. It reduces to an inhomogeneous Bessel equation whose solution can be evaluated by numerical integration. Although the numerical evaluation of the solution is more complex in this case the peaking of the radiation field is removed or certainly reduced by separating out the highly directional reduced incident radiation. The method for close shells could be improved in this way by using an assumed angular distribution for the incident radiation, treating the reduced incident radiation exactly and consequently reducing the peaking of the outward flowing radiation for which the Eddington approximation is made.

In treating the diffuse radiation field as two distinct entities, one of which is scattered optical radiation, and the other, thermally emitted infra-red radiation, care must be taken to ensure that they do occupy separate parts of the spectrum. This can be done by considering the frequency range of the calculated black-body functions. If the dilution of the optical radiation is too small, then an overlap in frequency between the two fields will occur, and the equations of transfer will be very much more complex. Huang did not mention this point nor did he quote absolute temperatures. Consequently, his theory may be erroneous in certain cases.

Finally, we turn to the subject of interstellar grains. It was noted in the previous section that no particular grain model has been selected as the obvious candidate and that extensive review articles on interstellar grains are available. The behaviour of interstellar grains with regard to their infra-red emission has been studied by Krishna Swamy (1970) and, more recently, by Greenberg (1971) who investigated the influence of the incident radiation and optical properties of the grains upon their temperature. The effects of grain shape have also been studied by Greenberg and Shah (1971). All these calculations utilise non-grey absorption coefficients but are restricted to grains in free space. Greenberg (1971) does make an attempt to estimate the grain temperatures within clouds but he does so in a very rough manner. Werner and Salpeter (1969) claim to have solved the radiative transfer problem in detail for a spherical dust cloud illuminated externally by a uniform isotropic dilute radiation field typical of that pervading interstellar space. Their model was one of a cloud of constant density, and frequency dependent absorption coefficient, albedo and asymmetry parameter. They treated the anisotropic phase function in a schematic way, which as we noted in Section I.3 is a good approximation; and they based their solution for the scattered radiation on an interpolation between the Neumann series solution for multiple scattering in a semi-infinite plane-parallel atmosphere and the solution for singly scattered radiation in a spherical atmosphere. The former solution, they assumed to be valid for optically thick spherical atmospheres and the latter for optically thin spherical atmospheres. They calculated the reduced incident radiation by numerical integration, but it transpires that an analytical integration was possible in their case of constant density atmospheres. The nature of the integrand for the reduced incident radiation mean intensity makes the numerical integration, though straightforward in theory, complex for optically thick and intermediate atmospheres at most values of the radius so that this numerical treatment could have proved an unnecessary source of numerical error. Their treatment of the transfer of the infra-red radiation consisted of a two part iteration scheme. For a particle whose infra-red absorption coefficient is very much smaller than its visible absorption coefficient, as is generally true of interstellar grains, the source function for the infra-red radiation is dominated by the mean intensities of the reduced incident and scattered radiation fields. Consequently, Werner and Salpeter calculated the temperatures that the grains would have adopted had there been no thermal radiation field, and then they calculated the thermal radiation field at every point in the atmosphere by integrating the thermal source function derived from their estimates of the grain temperatures, along all lines of sight assuming that there was no interaction between the radiation and the intervening grains. They then calculated new grain temperatures based on the new radiation field at every point. Thus we see that they have treated the scattered radiation in neither an exact manner nor even in an approximate radiative transfer manner; and that they have completely ignored the transfer aspects of the thermal radiation field. In fact, the former is the more serious weakness for grains whose infra-red opacities are as low as those that they used. This leaves considerable scope for producing a model interstellar dust cloud involving standard radiative transfer theory, either exact or approximate.

7. Outline of the Present Work

This Chapter has been concerned with some of the general aspects of radiative heating and the associated radiative transfer problems. In Section I.1 the fundamental quantities of the subject were defined and the equation of transfer for a general problem was derived in both plane-parallel and spherical geometries. The solution of this equation has been the goal of a large number of astrophysicists and indeed scientists in a diversity of disciplines, and several of the more frequently used methods of solution were outlined in Section I.2. It was seen that these methods were of great diversity, ranging from those utilising a simple approximation in order to permit an analytical solution, to those of extreme generality whose evaluation was only possible by extensive numerical computation. Section I.3 was devoted to the topic of scattering, especially that of anisotropic scattering; its implications and means of treating the additional complexity that it introduces to a transfer problem. Aspects of the frequency dependence of transfer problems were considered in Section I.4, in particular, the topics of coherent scattering and grey absorption coefficients. Section I.5 dealt with the problem of radiative heating in a qualitative manner, discussing the behaviour of real and black matter in a particular radiation field. It was seen that the scattered radiation field and the thermal radiation field occupy distinct parts of the spectrum for many problemsof astronomical significance. This division of the spectrum into two parts enables the radiative heating problem to be treated as two transfer problems only partially coupled. The scattering problem can be treated independently of the remainder of the problem but the transfer of the thermal radiation involves the solution for the scattered radiation field. Lastly, in the previous section, a number of recent publications dealing with the radiative heating of planetary atmospheres, and, circumstellar and interstellar clouds, were discussed.

Before selecting the most suitable method available for solving the twin transfer problems of radiative heating it is necessary to have regard to the aims of the solution. These aims are naturally dictated to, by the contents of the previous section which dealt with prior work in the field. The aim of a problem in theoretical astrophysics can, broadly speaking, fall into two categories. Firstly, there is the ideal of producing a mathematically accurate and physically realistic model of a particular astronomical object. Such a model must, of necessity, be very complex and use the most accurate available values of the physical quantities required as input data. In the absence of suitable observational measurements, the most complete current theoretical model of each facet of the problem must be used. For example, in the theory of scattering in cloudy atmospheres both Potter (1969) and Samuelson (1967a) have used a scattering phase function calculated using the Mie theory. It is a complex function and is usually represented by a suitable series expansion of Legendre polynomials, which, as mentioned in Section I.3, frequently requires handling by complex computational techniques. For this physically realistic type of model it is essential to use the most accurate method of solution of the equations of the problem, which in the case of transfer equations is usually one of the numerical methods delineated in Section I.2. It was stressed there that such methods were designed to handle the more general problems that arise for physically realistic models.

The other ideal is to investigate the physical processes involved in the astronomical object and to determine the role that each physical parameter assumes in controlling the physical state of the object, which in our case is the temperature of the atmosphere. When a numerical method is used, the relative importance of the atmospheric parameters is often obscured, whereas an analytical method frequently clarifies the situation. The equation of transfer is an integro-differential equation and as such can only be solved analytically in the presence of one or more approximations. The question of non-greyness illustrates this point. When the absorption and scattering coefficients are frequency dependent there is no unique albedo and consequently no way to determine precisely the effect of scattering on the solutions of the equations. When the atmosphere is grey there is one albedo and its role in the problem can be investigated simply and clearly.

It has been the aim of most work on the subject of radiative heating, to provide a simple, yet accurate model of the atmosphere in question. Samuelson has, to a certain extent, showed how the atmospheric parameters control the temperature but his theory has been applied to a semi-infinite atmosphere only, which, as we have seen, is inadequate even for an atmosphere as thick as that of Venus. In the field of interstellar dust clouds there is an even greater need for a simple, explanatory model because the limited work in this field has been directed to produce real physical values by using complex grain models and has been accomplished at the expense of radiative transfer theory. Although an approximate model may not realise physically accurate solutions there is a great value in knowing the dependence of the solution upon the physical parameters involved in the problem, besides the ensuing understanding of the physical principles themselves. An approximate solution will yield a considerable amount of information showing which parameters are important under given conditions, and consequently which parameters need to be determined accurately before an accurate physically realistic model is constructed. It will be the aim of this study to ascertain the role of each atmospheric parameter in determining the temperature profiles of simple models of plane-parallel and spherical atmospheres by means of a simple analytical solution of the appropriate equations of transfer. In doing so, the approximate temperature profiles will themselves be valuable as first results in problems where results are hitherto unavailable. Furthermore, it is hoped that the resulting information concerning the atmospheric parameters will be of value in illuminating those physical studies that are most important in the construction of more accurate model atmospheres.

The model that has been chosen to be investigated has the following properties. The absorption coefficients are grey over each of two separate parts of the spectrum but not necessarily grey over the whole spectrum. These two parts of the spectrum correspond to the visible and nearby frequencies and the far infra-red frequencies. The scattering of the visible radiation is anisotropic with a phase function represented by the schematic one of equation (I-29). It was seen in Section I.3 that this phase function was capable of producing results identical to those obtained with an arbitrary phase function when the radiation field was axially symmetric and could be represented by a three term series of Legendre polynomials. Since such radiation fields are fairly common in radiative transfer problems the schematic representation of anisotropy is excellent. The infra-red radiation is generated by thermal emission from the matter in the atmosphere and is not scattered by that matter, though is frequently re-absorbed. We shall use Eddington's approximation to solve the appropriate equations of transfer for the scattered and the thermal radiation fields because it yields simple analytical solutions for the moments of each of the radiation fields. We saw in Section I.2 that it was generally the best method of its type. Furthermore, it readily lends itself to an analytical solution for the intensity as a function of position and direction, thus enabling the emergent radiation to be obtained as a function of direction. All further details of the model will be introduced at a more convenient moment. These few properties described here are, in fact, the major approximations that will be needed.

Chapter II will be concerned with the radiative heating of plane-parallel atmospheres illuminated by dilute parallel stellar radiation. The first section will be a discussion of the illuminating radiation and the second, a discussion of the nature of the absorption coefficients involved in the radiative heating problem. After defining the precise physical problem of the heating of a plane-parallel atmosphere, the equations of transfer for the scattered and thermal radiation fields will be solved using Eddington's approximation, firstly for a semi-infinite atmosphere and secondly for a finite atmosphere. This will occupy section II.3. The finite atmosphere will be treated with and without a partially reflecting ground layer at its lower surface. The mean intensities of the two radiation fields will be illustrated and their dependence on the atmospheric parameters will be discussed in detail. The temperature distribution within these atmospheres will be calculated and suitably illustrated in Section II.4. It was mentioned in Section I.2 that the source function as derived by Eddington's approximation is an ideal operand for the Lambda operator and that this operation improves the solution. Consequently, we shall apply this procedure to the thermal radiation field and discuss the ensuing results. This will occur in Section II.4 also. For several decades the exact solutions for the emergent scattered radiation from both semi-infinite and finite plane-parallel atmospheres have been known. These solutions are based on the principles of invariance as established by Chandrasekhar (1960). In Section II.5 we shall develop similar methods by which the emergent thermal radiation can be found. These will be applicable in certain circumstances only, and will be described in Section II.5.1. In Section II.5.2 we shall calculate the emergent radiation intensity as found from the approximate source functions of Section II.4; and in Section II.5.3 we shall discuss and compare the results for the exact and approximate solutions for the intensity of the emergent thermal radiation from these plane-parallel atmospheres.

Chapter III will be concerned with the radiative heating of spherical atmospheres illuminated externally by a radiation field similar to that of the interstellar radiation field. The type example of this problem is the interstellar dust cloud and the treatment of the spherical atmosphere problem will be orientated towards the appropriate solutions for interstellar clouds, though will still be of sufficiently general a nature to be applicable to other objects of astronomical interest. Section III.1 will include a discussion of the interstellar radiation field and its relation to typical dust clouds; and Section III.2 will cover points on the absorption coefficients of grains typical of those in interstellar clouds. Sections III.3, III.4 and III.5 are counterparts of those sections in Chapter II, and will give accounts of the solutions of the equations of transfer for the two radiation fields, the temperature distributions and the emergent radiation fields respectively. However, there are no counterparts to the exact solutions of Sections II.5.1 for spherical atmospheres. The results will be discussed in relation to the importance of the various parameters involved in the theory; and comparisons will be made between the results of Chapter II for plane-parallelatmospheres and the results

of Chapter III for spherical atmospheres. However, these comparisons will not be strict comparisons between two similar problems because the incident radiation is different for the two geometries. An important approximation will be made before the solutions of Section III.3 are performed. That is, the absorption coefficients and density will be assumed to be independent of position in the atmosphere. The need for this assumption will be made apparent at the appropriate time. In Section III.6 we shall deal with attempts to relax this restriction; and in Section III.7 we shall consider the possible effects of varying the geometrical distribution of the incident radiation.
There will be a short summary of each Chapter at its conclusion and the thesis will be completed with several concluding remarks.

CHAPTER II

PLANE-PARALLEL ATMOSPHERES

The form of the equation of transfer pertinent to considerations of the radiative heating of plane-parallel atmospheres is that given by (I-9) in terms of the geometrical depth, z , measured normally to the surface. In order to simplify this, we shall assume the atmosphere to be grey in two discrete parts of the spectrum. The first of these is centred on the visible part of the spectrum and we shall use the adjective "stellar" to describe both scattered and unscattered radiation of these frequencies. The appropriate absorption and scattering coefficients will be assigned the subscript, s. However the subscript, s , appended to the intensity and its associated moments will not refer to the stellar radiation field but to the scattered radiation field or even part of that field. This will be defined clearly in Section II.3. In previous work on planetary atmospheres the stellar radiation field has been referred to as the solar radiation field, and the radiation field generated in the infra-red,which is the second region of the spectrum in which the atmosphere is assumed to be grey,, by thermal emission of the atmosphere has been referred to as the planetary radiation field and has usually been ascribed the subscript p. Such terminology will not be applicable to interstellar dust clouds which are the type example of Chapter III. Consequently we shall use the names stellar and thermal or infrared to refer to the two radiation fields; these names being equally suitable in each context. However, in order to adhere to a standard notation we shall maintain the use of the subscript p for the thermal radiation field in planetary atmospheres and also use it to depict the thermal radiation field in interstellar dust clouds.

1. The Incident Radiation

The radiation incident on the surface of an atmosphere will, in general, vary from one atmosphere to another in both its geometrical distribution and its photon content. The latter is essentially its dilution and its spectral distribution. This Chapter is concerned with plane-parallel atmospheres of which planetary atmospheres are the archetype so that this discussion will be limited to typical radiation fields incident on the planets. Of the planets in the solar system, Venus has received most attention with regard to its atmosphere. Consequently, most of the data quoted here will be those used in studies of the Venusian atmosphere. However, the radiation incident on the other planetary atmospheres will be different chiefly in its dilution allowing the discussion to be quite general in its application.

To a first approximation, the radiation flux incident on the surface of a planetary atmosphere is that from the Sun, diluted according to the inverse square of the distance between the Sun and the planet. For the planet Venus, the geometrical dilution factor is equal to the square of the radius of the Sun, divided by the square of the distance between the Sun and Venus, and equals 4×10^{-5}. For such dilution the Sun's radiation at the position of Venus can be regarded as parallel; the angle of divergence of the beam being 23 arcsecs. Consequently, the geometry of the typical planetary problem is one of a plane-parallel atmosphere, illuminated at all points on its upper surface by parallel radiation incident at an angle, arc cos μ_0 , to the normal to the surface and at an angle, ϕ_0 , in azimuth to an arbitrary zero of azimuth. The incident radiation field is said to be of integrated flux, πF across a surface normal to its direction of propagation so that the incident radiation field is written

$$I^{inc.}_{red.}(0, -\mu, \phi) = \pi F \, \delta(\mu - \mu_0) \, \delta(\phi - \phi_0) ,$$

and $$I^{inc.}_{red.}(0, +\mu, \phi) = 0 , \qquad \text{(II-1)}$$

where $\mu > 0$.

The transfer problems of this Chapter will all involve this incident radiation field.

We noted in Chapter I that scattering is a linear phenomenon and that we can therefore treat two radiation fields of the same frequency in the same atmosphere as separate entities. Suppose the incident radiation is given by the superposition of several beams and is given by

$$I^{inc.}_{red.}(0, -\mu, \phi) = \pi \sum_{i=1}^{N} F_i \, \delta(\mu - \mu_i) \, \delta(\phi - \phi_i) ,$$

then the intensity of the scattered radiation field at a point in the atmosphere characterised by the position co-ordinate, τ , is

$$I(\tau, \mu, \phi) = \sum_{i=1}^{N} I_i(\tau, \mu, \phi) ,$$

where $I_i(\tau, \mu, \phi)$ is the intensity of the scattered radiation field that would result from the incident beam, $\pi F_i\ \delta(\mu-\mu_i)\ \delta(\phi-\phi_i)$. Thus we have a simple way of adapting the theory to cater for other forms of incident radiation.

We have a wide selection of suitable spectral distributions for the incident radiation. In the case of Venus, or any planet of the Solar System we have the observed solar spectrum at our disposal. For accurate model atmosphere calculations this is clearly the one to use. A variety of theoretical model stellar atmospheres are available and it would make an interesting study to investigate the effect of the spectral type of the central star on the thermal properties of a planet. The simplest of these is the black-body spectrum corresponding to the colour temperature of the central star. However, for the study of grey planetary atmospheres the spectral distribution of the incident radiation is immaterial; and, as noted in Section I.4, frequency dependent absorption and scattering coefficients have very little effect on the resultant radiation fields for coherent scattering, which is usually assumed to be valid for planetary atmospheres. The spectral distribution of the incident radiation field is thus far less important than the value of the integrated net flux. It is this latter quantity that controls the average temperature of the planet. However, this conclusion requires one qualification. The spectral distribution might lie within the spectral divisions created by the two part grey absorption coefficients. When there is incident radiation of frequencies close to that of the border between the "stellar" and "thermal" parts of the spectrum then the problems mentioned in the previous Chapter will arise. Fortunately, both the temperature of the Sun and the dilution are sufficiently high to prevent this happening in typical planetary problems. An incident radiation flux field located entirely in the thermal part of the spectrum, however, is readily acceptable by the model.

The atmosphere of a planet will always have a ground at its lower surface, though the atmosphere of Venus is frequently assumed to be semi-infinite. The semi-infinite approximation is sometimes adopted by considering the ground layer to exhibit optical properties similar to those of the semi-infinite atmosphere below a certain depth. However, it is a simple matter to treat the atmosphere as finite and to consider the ground as a radiation source in its own right, though, of course, its own source of energy is that radiation passing through the atmosphere. A variety of ground models are available, but we shall restrict our work to the simplest and most common. The ground is assumed to reflect a fraction, λ, of the stellar radiation flux incident upon it and to absorb the remainder. It is also assumed to absorb all the infra-red radiation incident upon it and to emit isotropically and thermally all the energy that it has absorbed. The reflection of the stellar radiation is considered to obey Lambert's law, as detailed by Chandrasekhar (1960), and according to which the reflected radiation is isotropic and independent of the direction of the incident radiation. The opposite physical extreme is that of specular reflection which is often assumed to occur at the interface between the earth's atmosphere and an ocean. However, Lambert's law is more appropriate for the standard planetary problem. An extension of the model to include complex reflection functions is possible but would create more additional mathematics than would be profitable in view of the approach of the thesis to the transfer problems.

2. The Greenhouse Parameter

The radiative heating of an atmosphere that is grey in two parts of the spectrum involves the two grey absorption coefficients, κ_s and κ_f, and the grey scattering coefficient, σ_s. We require two parameters to link these three coefficients. The choice of the first is simple. It is $\tilde{\omega}$, the albedo for single scattering as defined in Section I.3.by

$$\tilde{\omega} = \sigma_s / (\kappa_s + \sigma_s). \tag{II-2}$$

The choice of the second is less straightforward because it is affected by the choice of the depth scale in the atmosphere. In solving transfer problems, rather than using the geometrical depth, -z, it is convenient to use the optical depth as the position variable. This is defined by

$$d\tau = - \kappa_i \rho \, dz , \tag{II-3}$$

where κ_i can be either the absorption, the scattering or the extinction coefficient. It has already been mentioned that it is generally convenient to divide the stellar radiation field into the reduced incident radiation and the scattered radiation field. With the choice of the extinction coefficient for κ_i in equation (II-3) the reduced radiation field is independent of the albedo and depends solely on the optical depth. This situation is a useful asset, so we define

$$d\tau = - (\kappa_s + \sigma_s) \rho \, dz . \tag{II-4}$$

The choice of the second parameter would therefore lie between $(\kappa_s + \sigma_s)/\kappa_p$ and κ_s/κ_p. The former is the ratio of the extinction coefficients in the two parts of the spectrum and the latter, the ratio of the absorption coefficients. These two ratios are the same when there is no scattering and have been called the greenhouse parameter by Wildt (1966) and by Stibbs (1971), who point out that this greenhouse parameter is equal to the ratio, Λ_p/Λ_s where Λ_p and Λ_s are the mean free paths of the thermal and stellar photons respectively. When the greenhouse parameter is less than unity, the thermal photons are "trapped" within the atmosphere with respect to the stellar photons; and when the greenhouse parameter is greater than unity the thermal photons can disperse easily, again with respect to the stellar photons. The former effect is known as the greenhouse effect from which the parameter derives its name. When stellar photon scattering is present the ratio of the mean free path is

$$n = (\kappa_s + \sigma_s)/\kappa_p. \tag{II-5}$$

This is the more important physical parameter of the two and accordingly, we define the greenhouse parameter, n , by equation (II-5).

In general, the greenhouse parameter is a function of position and direction. The positional dependence arises from the fact that all three coefficients are mutually independent functions of depth, and the directional dependence from the theoretical impossibility of deriving true grey absorption coeffcients. However, as discussed in Section I.4 we shall assume that genuine direction independent absorption coefficients can be formed and that real frequency averaged coefficients will not differ significantly from these. It will be found that the solutions of the equations of transfer involve integrals of the type, $\int n(\tau)\, B(\tau)\, d\tau$. These can be evaluated numerically for any function, $n(\tau)$, but can only be evaluated analytically for certain simple functions. The aim of this thesis was stated to be the attempt to gain insight into the roles of the various atmospheric parameters involved in the problem. Consequently, we shall assume the greenhouse parameter to be independent of depth. This is not a serious physical liability in view of the fact that the optical depth accounts for the major depth variation of the density and absorption coefficients by definition. We shall also assume the albedo to be depth independent. These two assumptions are frequently made, for example, by Wildt (1966), Stibbs (1971) and Samuelson (1967a).

At this juncture it is necessary to ascertain the range of values that the two parameters, $\tilde{\omega}$ and n should cover. The Venusian atmosphere, as a prominent candidate for a greenhouse model has received a large amount of study. Sobolev (1963) quotes a value of 0.989 for the albedo for single scattering in the Venusian atmosphere, and subsequent estimates of this quantity have all been of the same order of magnitude. In studies of line absorption profiles typical values of the albedo for scattering in the continuum, $\tilde{\omega}_c$, are 0.99, 0.999 and 1.0. Hansen (1969b) has showed that values of $\tilde{\omega}_c$ down to 0.976 barely affect the absorption line profiles and provide correct values for the total planetary albedo, which is the fraction of the incident flux on the surface of the atmosphere, that is reflected by the planet as a single entity. He used values of $\tilde{\omega}_o$, the albedo at the centre of an absorption line, of the order of 0.9. Such values of $\tilde{\omega}_o$ and $\tilde{\omega}_c$ would give values of a grey albedo ranging from 0.95 to 0.999. When the albedo is close to unity the absorption coefficient κ_p ? is a very small fraction of the extinction coefficient κ_s. Now, the value of the absorption coefficient is very important in determining the temperature of the atmosphere in that it actually measures the amount of energy absorbed by an element of matter within the atmosphere. Consequently, even though the albedo may not critically affect the shapes of the absorption lines, it is imperative that evaluations of the thermal characteristics of a physically realistic model atmosphere use very accurate values of the albedo, if the albedo is close to unity as indeed it is for the Venusian atmosphere. $n = \kappa_s/\kappa_p$

Samuelson (1967a) has shown that the ratio of the extinction coefficients in the visible κ_s to that in the infra-red κ_p will not ? fall below unity for any size of particle. He did so using the Mie theory, and with his value of $\tilde{\omega}$ of 0.99 a typical value of the ratio of the absorption coefficients would be 0.01. Wildt (1966) used such a value for his exact theory without scattering but exact correlation between different conditions is difficult due to the scattering, particularly when it is highly forward peaked. The inclusion of gaseous infra-red opacity may render the greenhouse parameter smaller still. This results from Pollack's (1969b) studies of the Venusian atmosphere, which also show the extreme complexity of a real planetary atmosphere.

For grains in interstellar space, typical values of $\tilde{\omega}$ and n quoted in the literature are quite unlike those for planetary atmospheres. Werner and Salpeter (1969) use albedo values of the order of 0.5 and show in Fig. 1 of their paper, values of the greenhouse parameter ranging from 10^2 to 10^8. This vast range arises from the dramatic variation of the infra-red absorption coefficient with frequency and the strong effect of impurities in the grains. It shows that the choice of the mean absorption coefficient is important in the interpretation of observations by comparison with theory. The large number of grain models available at present has also helped in maintaining a large range of suitable values for the greenhouse parameter and albedo. For scattering in the Coalsack and Libra cloud, Mattilla (1970) concludes that the albedo is of the order of 0.65; Martin (1971) suggests a value of 2.5 for the greenhouse parameter for a grain with κ_P measured at 10μ; and Greenberg (1971) uses a value of approximately one hundred for the greenhouse parameter. In the field of circumstellar shells, Huang (1969b), argues that the grains are relatively large and consequently have values of n of the order of unity.

The third parameter required is the asymmetry parameter of the phase function. By far the majority of authors use phase functions calculated from the Mie theory which always produces phase functions with large forward peaks. Potter (1969) and Samuelson (1967a) use such phase functions. Potter's phase function also has a peak in the backward direction but Samuelson's does so only for a few particle sizes. Typical values of g for these phase functions are 0.9 whereas those quoted by Werner and Salpeter (1969) for interstellar grains are near 0.4.

We have seen that there exists a wide range of values of each of the three optical parameters, certainly covering different objects but sometimes covering different models for the same object. The calculations to be performed later in the Chapter will include the full range of possible values for each of these parameters. This will be done for three reasons. Firstly, it will enable us to gain a full insight into the effects of the parameters even though the values of the parameters applicable to planetary atmospheres are not applicable to interstellar grains and vice versa. By using the full set of parameters in each geometry the effects of the geometry will not be obscured by the more important effects of the optical parameters. Secondly, it will enable us to consider the complete range of physical properties of each parameter. Although the phase functions with negative values of g are not physically significant they are not mathematically meaningless and will bring to light any special effects that may occur when ? is zero, and may be obscured if the range of values of g were terminated at zero. Thirdly, and by no means trivially, it will furnish a useful check on the numerical procedures in the computer programmes that provide the results of the analytical solutions. This is very useful because an error may be significant but undetectable within a limited range of values of one of the parameters but it is extremely unlikely that an error would remain unnoticed when subjected to the full range of parameter values, which in the case of the greenhouse parameter can extend from 10^{-4} to 10^{4}.

3. The Source Function

The radiative heating problem, as formulated in the preceding sections is summarised as follows. A parallel beam of dilute stellar radiation of integrated net flux, πF, per unit area normal to itself, is incident upon the surface of a plane-parallel atmosphere at an angle, $\cos^{-1}\mu_0$, to the inward normal and at an azimuthal angle, ϕ_0 , to an arbitrary zero of azimuth. The atmosphere is grey to this radiation and has grey absorption and scattering coefficients, κ_s , and σ_s respectively. The absorbed radiation is degraded by the absorbing matter and re-emitted thermally and isotropically in the far infra-red. The atmosphere is grey to radiation of these frequencies also, with a grey absorption coefficient, κ_p . The three absorption coefficients are related by the albedo, $\tilde{\omega}$, and the greenhouse parameter, n , both of which are assumed to be independent of depth in the atmosphere. They are defined by equations (II-2) and (II-5) respectively. The optical depth is measured in terms of the stellar extinction coefficient as defined by equation (II-4). The scattering of the stellar radiation is anisotropic and obeys the schematic phase function of equation (I-29).

The combination of the anisotropy of the scattering and the azimuthal dependence of the incident radiation creates the first difficulty encountered in solving the problem by causing the scattered radiation field to be dependent on azimuth. Chandrasekhar (1960), when treating the scattering problem for a general phase function by the discrete ordinate method, divided the scattered radiation into a series of components of specified azimuth dependence and solved the equation of transfer for each component independently. A similar technique can be used here and fortunately a very simple solution occurs due to the simple nature of the phase function. The radiation that is continuously scattered by the delta-function spikes of the schematic phase function will remain in the same line in the atmosphere as the reduced incident radiation. The sum of this part of the scattered radiation field and the reduced incident radiation field will be known as the linear radiation field and denoted by $I_{lin}(\tau, -\mu_0, \phi_0)$ and $I_{lin}(\tau, +\mu_0, \phi_0+\pi)$. The remainder of the scattered radiation is that scattered from the linear radiation field in the isotropic part of the phase function. This is the azimuthally independent part of the radiation field even though second and subsequent scatterings may be anisotropic. We shall call this radiation field the scattered radiation field even though that is not strictly correct and shall denote it by $I_s(\tau, \mu)$. We have thus separated the azimuthally dependent part of the radiation field from the azimuthally independent part. The former can be found exactly but we must treat semi-infinite and finite atmospheres separately.

3.1. Semi-infinite Atmospheres It follows from the definitions of the source function and schematic phase function, and equation (I-46) that the source function for the linear radiation field is

$$B_{lin}(\tau, -\mu_0, \phi_0) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} \tilde{\omega}\, 4\pi(1-\alpha) \Big\{ \beta\, \delta(-\mu_0-\mu')\, \delta(\phi-\phi') + (1-\beta)\, \delta(-\mu_0+\mu')\, \delta(\phi_0-\phi'-\pi)\Big\}\, d\mu'\, d\phi'$$

$$= \tilde{\omega}(1-\alpha)\Big[\beta\, I_{lin}(\tau, -\mu_0, \phi_0) + (1-\beta)\, I_{lin}(\tau, \mu_0, \phi_0+\pi)\Big]$$

and $$B_{lin}(\tau, +\mu_0, \phi_0+\pi) = \tilde{\omega}(1-\alpha)\Big[\beta\, I_{lin}(\tau, \mu_0, \phi_0+\pi) + (1-\beta)\, I_{lin}(\tau, -\mu_0, \phi_0)\Big].$$

The azimuth co-ordinate will be omitted hereafter without any loss of significance because only one azimuthal value is possible for each value of μ_0. The two equations of transfer for upward and downward flowing radiation are

$$\mu_0 \frac{dI_{lin}(\tau,\mu_0)}{d\tau} = I_{lin}(\tau,\mu_0) - \tilde{\omega}(1-\alpha)\beta\, I_{lin}(\tau,\mu_0) -$$

$$- \tilde{\omega}(1-\alpha)(1-\beta)\, I_{lin}(\tau,-\mu_0), \qquad \text{(II-6)}$$

and
$$-\mu_0 \frac{dI_{lin}(\tau,-\mu_0)}{d\tau} = I_{lin}(\tau,-\mu_0) - \tilde{\omega}(1-\alpha)\beta\, I_{lin}(\tau,-\mu_0) -$$

$$- \tilde{\omega}(1-\alpha)(1-\beta)\, I_{lin}(\tau,\mu_0). \qquad \text{(II-7)}$$

Equations (II-6) and (II-7) constitute a pair of simultaneous linear first order differential equations which can be solved easily. They combine to give

$$\left[\frac{d}{d\tau} - \frac{\sigma^2}{\mu_0^2}\right] I_{lin}(\tau,-\mu_0) = 0, \qquad \text{(II-8)}$$

where
$$\sigma^2 = [1-\tilde{\omega}(1-\alpha)\beta]^2 - \tilde{\omega}^2(1-\alpha)^2(1-\beta)^2$$

$$= [1-\tilde{\omega}(1-\alpha)]\,[1-\tilde{\omega}(1-\alpha)(2\beta-1)] \qquad \text{(II-9)}$$

The general solution of this second order differential equation is

$$I_{lin}(\tau,-\mu_0) = Ae^{\sigma\tau/\mu_0} + Be^{-\sigma\tau/\mu_0} \qquad \text{(II-10)}$$

where A and B are two arbitrary constants which must be found by using two boundary conditions. The first boundary condition is based on the requirement that, in the limit as τ tends to infinity, $I_{lin}(\tau,-\mu_0)$ tends to zero. This is true for all values of the scattering parameters with the exception of the special case when $\tilde{\omega} = 1$ and $\alpha = 0$. In that case equation (II-10) is not the solution of equation (II-8). The arbitrary constant, A, must equal zero for this boundary condition to be fulfilled. The second boundary condition is applied at the surface of the atmosphere and is, that the downward flowing radiation must be equal in intensity to the incident radiation which is πF. Thus we have

$$I_{lin}(\tau, -\mu_0) = \pi F e^{-\sigma\tau/\mu_0} \qquad \text{(II-11)}$$

This result together with equation (II-7) gives the intensity in the upward direction as

$$I_{lin}(\tau, \mu_0) = \pi F \frac{\left[1 - \tilde{\omega}(1-\alpha)\beta - \sigma\right]}{\tilde{\omega}(1-\alpha)(1-\beta)} e^{-\sigma\tau/\mu_0} . \qquad \text{(II-12)}$$

From these two equations we obtain the moments of the linear radiation field, the first two of which are

$$J_{lin}(\tau) = \frac{1}{4} F e^{-\sigma\tau/\mu_0} \frac{\left[1 - \tilde{\omega}(1-\alpha)(2\beta-1) - \sigma\right]}{\tilde{\omega}(1-\alpha)(1-\beta)} \qquad \text{(II-13)}$$

and

$$H_{lin}(\tau) = -\frac{1}{4}\mu_0 F e^{-\sigma\tau/\mu_0} \frac{\left[\tilde{\omega}(1-\alpha) - 1 + \sigma\right]}{\tilde{\omega}(1-\alpha)(1-\beta)} . \qquad \text{(II-14)}$$

There are two special cases for the solutions for the linear field. When the scattering is isotropic, α is equal to unity and equations (II-12) to (II-14) assume indeterminate forms. In this physical situation the linear radiation field and the reduced incident radiation field are identical. There is no upward reduced incident radiation and the constant, σ, is equal to unity.

Hence, the first two moments of the reduced incident radiation field are

$$J^{inc}_{red}(\tau) = \frac{1}{4} F e^{-\tau/\mu_0} \quad \text{and} \quad H^{inc.}_{red.}(\tau) = -\frac{1}{4}\mu_0 F e^{-\tau/\mu_0}. \qquad \text{(II-15)}$$

The symbol $I^{inc}_{red}(\tau, \mu, \phi)$ and its associated symbols are chosen to represent the reduced incident radiation; and in the case of isotropic scattering the moments $J_{lin}(\tau)$ and $H_{lin}(\tau)$ are given by equation (II-15) also.

Again equations (II-12) to (II-14) assume indeterminate forms when the backward scattering spike is absent. This corresponds to the parameter β being unity and the linear scattering being forward only so that the upward linear radiation field is again zero. In this case the constant σ is equal to $(1-\tilde{\omega})$ and the first two moments of the linear radiation field are

$$J_{lin}(\tau) = \frac{1}{4} F e^{-\sigma\tau/\mu_0} \quad \text{and} \quad H_{lin}(\tau) = -\frac{1}{4}\mu_0 F e^{-\sigma\tau/\mu_0} \qquad \text{(II-16)}$$

We are now in a position to consider the emission coefficient and source function for the scattered radiation field. The emission coefficient for this radiation field is made up of the radiation scattered isotropically from the linear radiation field and the total radiation scattered from the scattered radiation field itself. The albedos for these two processes are $\tilde{\omega}\alpha$ and $\tilde{\omega}$ respectively. The emission coefficient, which is not a function of azimuth, is therefore

$$j_s(\tau,\mu) = (k_s+\sigma_s)\rho \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} \tilde{\omega}\alpha\, I_{lin}(\tau,\mu',\phi')\,d\mu'\,d\phi' +$$

$$+ (k_s+\sigma_s)\rho \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} p(\mu,\phi;\mu',\phi')\, I_s(\tau,\mu')\,d\mu'\,d\phi'. \qquad \text{(II-17)}$$

Hence, we obtain

$$B_s(\tau,\mu) = \tilde{\omega}\alpha J_{lin}(\tau) + \tilde{\omega}\alpha J_s(\tau) + \tilde{\omega}(1-\alpha)\beta I_s(\tau,\mu) +$$

$$+ \tilde{\omega}(1-\alpha)(1-\beta) I_s(\tau,-\mu), \qquad \text{(II-18)}$$

so that the equation of transfer is

$$\mu \frac{dI_s(\tau,\mu)}{d\tau} = I_s(\tau,\mu) - \tilde{\omega}(1-\alpha)\beta\, I_s(\tau,\mu) - \tilde{\omega}(1-\alpha)(1-\beta) I_s(\tau,-\mu) -$$

$$- \tilde{\omega}\alpha J_s(\tau) - \tilde{\omega}\alpha J_{lin}(\tau). \qquad \text{(II-19)}$$

A similar equation for the thermal radiation can also be formulated and this will be done later in this section. It will be seen to involve the solution of equation (II-19). The two equations of transfer form a pair of simultaneous integro-differential equations whose inter-relationship arises from the degrading of the unknown of the first equation, the scattered radiation, to the unknown of the second, the thermal radiation. Hence the equation of transfer for the thermal radiation uses the solution of equation (II-19). Were there some mechanism whereby the inverse process could take place, the source function of equation (II-18) would include a term involving the thermal radiation field and the two integro-differential equations would be truly coupled. Fortunately, this degree of complexity does not arise in situations involving highly dilute incident radiation, so that the coupling is only partial and equation (II-19) can be solved directly.

We shall solve equation (II-19) using the Eddington approximation. We reduce the integro-differential equation to a pair of total differential equations by applying the two moment integral operators, defined by equation (I-15), to equation (II-19). Thus, we obtain

$$\frac{dH_s(\tau)}{d\tau} = (1-\tilde{\omega})\,J_s(\tau) - \tilde{\omega}\alpha\,J_{lin}(\tau) \qquad \text{(II-20)}$$

and

$$\frac{dK_s(\tau)}{d\tau} = \left[1-\tilde{\omega}(1-\alpha)(2\beta-1)\right] H_s(\tau).$$

These can be solved by use of the Eddington approximation which links $K_s(\tau)$ and $J_s(\tau)$ by the relation, $K_s(\tau) = J_s(\tau)/3$. Accordingly, we obtain

$$\frac{dJ_s(\tau)}{d\tau} = 3\gamma\,H_s(\tau), \qquad \text{(II-21)}$$

where

$$\gamma = 1-\tilde{\omega}(1-\alpha)(2\beta-1).$$

The constant, γ, is related to the asymmetry parameter, g, by the relation, $\gamma = (1-g)$; and it transpires that the solution of equations (II-20) and (II-21) involve γ rather than g, so for this reason we shall use γ rather than $(1-g)$. Equations (II-20) and (II-21) combine to form

$$\left[\frac{d^2}{d\tau^2} - \epsilon^2\right] J_s(\tau) = -3\tilde{\omega}\alpha\gamma\,J_{lin}(\tau), \qquad \text{(II-23)}$$

where $$\epsilon^2 = 3\gamma(1-\tilde{\omega}) \,, \tag{II-24}$$

and the function $J_{lin}(\tau)$ is given by the appropriate equation from (II-13) to (II-16). These equations are all of the same form and can be expressed as

$$J_{lin}(\tau) = \frac{\lambda F}{4} e^{-\sigma\tau/\mu_0} \,.$$

The solution of equation (II-23) is

$$J_s(\tau) = C e^{\epsilon\tau} + D e^{-\epsilon\tau} + \frac{3\tilde{\omega}\alpha\gamma\lambda\mu_0^2 F e^{-\sigma\tau/\mu_0}}{4(\epsilon^2\mu_0^2 - \sigma^2)} \,, \tag{II-25}$$

where C and D are arbitrary constants. To find these we require two boundary conditions. Firstly, we must ensure that, in the limit as τ tends to infinity the quantity, $J_s(\tau)e^{-\epsilon\tau}$ tends to zero. This is satisfied if C is equal to zero. This boundary condition is commonly used in scattering problems and has been discussed by Chandrasekhar (1960). Secondly, we apply the Eddington approximate boundary condition at the surface where it can be applied because there is no downward scattered radiation at the surface. The condition is given by equation (I-16) and gives an expression for D, of

$$D = \frac{-3\tilde{\omega}\alpha\gamma\lambda\mu_0^2 F(3\gamma + 2\sigma/\mu_0)}{4(\epsilon^2\mu_0^2 - \sigma^2)(3\gamma + 2\epsilon)} \tag{II-26}$$

The two special cases that gave rise to the spherical forms of $J_{lin}(\tau)$ merely affect the expression for the constant, λ. However, a mathematical indeterminate form exists for equations (II-25) and (II-26) when $(\epsilon^2\mu_0^2 - \sigma^2) = 0$. The solution for this special case proceeds exactly as before and yields

$$J_s(\tau) = \frac{3\tilde{\omega}\alpha\gamma\lambda F\left[(1-2\epsilon/3\gamma)-(1+2\epsilon/3\gamma)(1+2\epsilon\tau)\right]e^{-\epsilon\tau}}{16\epsilon^2(1+2\epsilon/3\gamma)} \qquad \text{(II-27)}$$

Equation (II-27) arises for real values of μ_0 given by

$$\mu_0^2 = \left[1-\tilde{\omega}(1-\alpha)\right]/3(1-\tilde{\omega}).$$

For example, by $\cos^{-1}\mu_0 = 45^{\circ}$ when $\tilde{\omega} = 0.5$ and $\alpha = 0.5$. In general the solution for the mean intensity of the scattered radiation field is given by

$$J_s(\tau) = D e^{-\epsilon\tau} + E' e^{-\sigma\tau/\mu_0}, \qquad \text{(II-28)}$$

with the constants, D and E' assuming one of the forms given above.

We are now in a position to obtain a solution for the thermal radiation field. The emission coefficient for this field is comprised of two parts. Firstly, the thermal radiation field is absorbed and re-radiated conservatively and isotropically; and secondly, the energy absorbed from the stellar radiation field is converted into thermal radiation which is likewise radiated isotropically. The emission coefficient in this case is a function of τ only, because the emission process is isotropic. It is given by

$$j_p(\tau) = \kappa_s\rho\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1}\left[I_s(\tau,\mu',\phi') + I_{lin}(\tau,\mu',\phi')\right]d\mu'\,d\phi' +$$

$$+\kappa_p\rho\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} I_p(\tau,\mu',\phi')\,d\mu'\,d\phi', \qquad \text{(II-29)}$$

and hence the source function is given by

$$B_p(\tau) = J_p(\tau) + n(1-\tilde{\omega})\left[J_s(\tau) + J_{lin}(\tau)\right] \qquad \text{(II-30)}$$

where n is defined by equation (II-5). The equation of transfer for the thermal radiation field thus becomes

$$\mu \frac{dI_p(\tau,\mu)}{d\tau} = \frac{1}{n} I_p(\tau,\mu) - \frac{1}{n} J_p(\tau) - (1-\tilde{\omega})\left[J_s(\tau) + J_{lin}(\tau)\right]. \qquad \text{(II-31)}$$

This equation is written in terms of τ, as defined earlier, which is the optical depth for extinction of the stellar radiation. It is for this reason that the right-hand side of the equation involves the factor, $(1/n)$.

The solution of equation (II-31) follows the same procedure as the solution of equation (II-19) for the scattered radiation field. The two integral operators, L_0 and L_1, applied to equation (II-31) yield

$$\frac{dH_p(\tau)}{d\tau} = -(1-\tilde{\omega})\left[J_s(\tau) + J_{lin}(\tau)\right] \qquad \text{(II-32)}$$

and

$$\frac{dK_p(\tau)}{d\tau} = \frac{3}{n} H_p(\tau)$$

which, in Eddington's approximation becomes

$$\frac{dJ_p(\tau)}{d\tau} = \frac{3}{n} H_p(\tau) \qquad \text{(II-33)}$$

Owing to the conservative nature of the process of the absorption and re-radiation of the thermal radiation field, equation (II-32) does not involve $J_p(\tau)$ and thus can be solved independently of equation (II-33). In general we have

$$J_s(\tau) + J_{lin}(\tau) = De^{-\epsilon\tau} + Ee^{-\sigma\tau/\mu_0} \tag{II-34}$$

where
$$E = E' + \lambda F/4 \tag{II-35}$$

Using equation (II-34) and integrating equation (II-32) we obtain

$$H_p(\tau) = \frac{(1-\tilde{\omega})De^{-\epsilon\tau}}{\epsilon} + \frac{(1-\tilde{\omega})E\mu_0 e^{-\sigma\tau/\mu_0}}{\sigma} + \frac{nG'}{3}. \tag{II-36}$$

Integrating equation (II-33) directly, using equation (II-36) we find

$$J_p(\tau) = \frac{-3(1-\tilde{\omega})De^{-\epsilon\tau}}{n\epsilon^2} - \frac{3(1-\tilde{\omega})\mu_0^2 Ee^{-\sigma\tau/\mu_0}}{n\sigma^2} + G'\tau + G. \tag{II-37}$$

where G and G$'$ are the two constants of integration. These are found by applying two boundary conditions the first of which is expressed by the equation,

$$H_s(\tau) + H_p(\tau) + H_{lin}(\tau) = 0, \tag{II-38}$$

which states that the net flux of energy at any depth in the atmosphere is zero. This is a direct consequence of the principle of conservation of energy which must be applied because there are no energy sources or sinks in the atmosphere; and it applies at all optical depths because the atmosphere is homogeneous and contains no local heat sources or sinks or heat transfer mechanisms such as convection. The algebra involved in applying the condition of conservation of energy as expressed by equation (II-38) is simple though tedious, and leads to the condition that G' must equal zero. This restraint on the system prevents $J_p(\tau)$ increasing with depth, so that no boundary condition at large optical depths need be required. The second boundary condition to be applied is the approximate Eddington boundary condition, $J_p(0) = 2H_p(0)$, which leads to the expression

$$G = 3(1-\tilde{\omega})\left[\frac{D(1/n + 2\epsilon/3)}{\epsilon^2} + \mu_0^2 \frac{E(1/n + 2\sigma/3\mu_0)}{\sigma^2}\right] \qquad \text{(II-39)}$$

This completes the solution for the mean intensities of the scattered and thermal radiation fields.

Before discussing the properties of these solutions it will be profitable to investigate a possible variation in the method of solution of the equations of transfer. Adding equations (II-20) and (II-32) gives

$$\frac{dH_T(\tau)}{d\tau} = -[1 - \tilde{\omega}(1-\alpha)]\, J_{lin}(\tau)$$

where $$H_T(\tau) = H_S(\tau) + H_p(\tau).$$

This equation can be solved exactly to give

$$H_T(\tau) = \frac{[1-\omega(1-\alpha)]\lambda F e^{-\sigma\tau/\mu_0}}{4} + C_1 .$$

Equations (II-14) to (II-16) give the appropriate expressions for $H_{lin}(\tau)$ from which we see that

$$H_T(\tau) = C_1 - H_{lin}(\tau).$$

Conservation of energy demands that C_1 be zero. In this way we see that the condition of conservation of energy can be applied to the problem before any approximation is made and that the flux, $H_T(\tau)$ is known exactly. However, $H_S(\tau)$ cannot be found exactly because the two moment equations for the scattered radiation field are coupled. The two approximate second moment equations combine to give

$$\frac{dJ_T(\tau)}{d\tau} = 3\left[\gamma H_S(\tau) + \frac{1}{n} H_P(\tau)\right],$$

or

$$\frac{1}{\gamma} J_S(\tau) + n J_P(\tau) = 3\int H_T(\tau)\,d\tau .$$

Hence

$$J_P(\tau) = C_2 - \frac{1}{n\gamma} J_S(\tau) - 3\frac{\left[1-\tilde{\omega}(1-\alpha)\right]}{4n}\lambda F e^{-\sigma\tau/\mu_0} . \qquad \text{(II-40)}$$

A tedious reduction shows that this expression for the mean intensity of the thermal radiation field is exactly the same as that given by equation (II-37). Although these two methods of solution involve the same approximations and arrive at the same answers, they do draw light on different facets of the mathematics of the problem and the application of the boundary conditions. In particular the second method shows that energy is conserved exactly in the solution for the fluxes, a point that is obscured in the first method.

We are now in a position to consider the physical significance of the solutions for the two radiation fields. We have seen how useful it was to construct the linear radiation field which includes the azimuthally dependent part of the scattered radiation field. Neither this, nor the azimuthally independent scattered radiation hitherto referred to as the scattered radiation field, is physically meaningful; the two radiation fields that are physically meaningful are the reduced radiation field and the genuine scattered radiation field. The latter is given by

$$J_{sc}(\tau) = J_s(\tau) + J_{tin}(\tau) - J_{red}^{inc}(\tau) , \qquad \text{(II-41)}$$

and will be referred to as the scattered radiation field for the remainder of this subsection.

The numerical values of the mean intensities of the scattered and thermal radiation fields have been found by assigning each parameter a numerical value and evaluating the appropriate expression for the mean intensities by means of a computer programme. A set of values of each parameter was used, though not every combination of parameters was utilised because some were included for use in special cases. The values of the parameters used in the calculation are given in Table I.

The mean intensity of the scattered radiation field is shown graphically as a function of τ in Figs. 4 to 6. These graphs show the dependence of the radiation field on albedo, phase function and angle of incidence respectively. The surface values and those at small optical depths are important, so the optical depth scale was chosen to be linear between $\tau = 0.0$ and $\tau = 1.0$. In order to include a wide range of values of τ the scale beyond $\tau = 1.0$ was chosen to be logarithmic. It is this change of scale that causes the discontinuities at the point $\tau = 1.0$.

The greater the albedo for single scattering, the larger the number of photons that are scattered into the scattered radiation field, and the larger the number of collisions that a photon

TABLE I

Values of atmospheric parameters used in computer programmes for calculating the mean intensities of the radiation fields in semi-infinite atmospheres.

τ	$\tilde{\omega}$	α	β	μ	n
0.0	0.1	0.0	0.0	1.0000	10^{4}
0.1	0.3	0.2	0.2	0.9484	10^{2}
0.2	0.5	0.4	0.5	0.8660	10^{0}
0.5	0.7	0.6	0.8	0.6248	10^{-2}
1.0	0.9	0.8	1.0	0.3548	
1.5	0.95	1.0			
2.0	0.99				
5.0					
10.0					
20.0					
50.0					

undergoes before it is absorbed. A larger number of collisions before absorption gives rise to a greater penetration into the atmosphere. Both these phenomena are borne out in Fig. 4 in which the radiation field is seen to increase with albedo at all optical depths and especially at large optical depths. The rate of increase of the radiation field with albedo, $\partial J_{sc}(\tau, \tilde{\omega}) / \partial \tilde{\omega}$ reaches its maximum at $\tilde{\omega} = 1.0$. Consequently the value of the albedo is critically important when it is close to unity. This is the case in planetary atmospheres and thus physically realistic model planetary atmospheres require very accurate values of the albedo.

Fig. 5 shows the mean intensity of the scattered radiation field as a function of τ, for several phase functions, with $\mu_o =$ 1.0, and $\tilde{\omega} = 0.9$. The phase functions shown vary from the extreme cases of complete forward and complete backward scattering to pure isotropic scattering. The effect of varying the parameter, β, can be seen closely. When $\beta = 1.0$ the delta-function scattering is all forward so that, for a given value of α the penetration of the radiation into the atmosphere is at its maximum. Thus, at the surface, the scattered radiation field is smaller than it is when $\beta = 0.5$, the case corresponding to a symmetrical phase function; whilst, in the interior of the atmosphere, it is greater. The opposite is true when $\beta = 0.0$. It can therefore be said that the scattered radiation field contains an energy that is almost independent of the value of β. This can be seen from the graphs of Fig. 5 which intersect at intermediate values of τ. The total energy content of the scattered radiation field is not completely independent of β because surface losses depend on β. This is seen most clearly when $\alpha = 0.0$. In this case, when $\beta = 1.0$ all the scattered radiation is directed away from the surface and there is no radiation loss from the surface. It is for this reason that $J_{sc}(0) = 0.0$ in this case, However, when β is less than unity there will be some loss through the surface and this loss will be an inverse function of β. Consequently, the scattered radiation as a whole increases as β increases.

The effect of α on $J_{sc}(\tau)$ is two-fold. Firstly, the variation of $J_{sc}(\tau)$ with β increases dramatically as α tends to zero because the absolute value of the asymmetry parameter, g, increases as α tends to zero. Secondly, a decrease of α makes the radiation more penetrating when $\beta = 0.5$. Now, when $\beta = 0.5$ the phase function is symmetric, g equals zero and a variation of the radiation field with α can only be understood in terms of surface effects. The radiation field can be seen to increase with α at the surface and to decrease with α deep in the atmosphere when $\tau = 10.0$. When some of the downward scattered radiation is scattered by the spike of the phase function, the penetration will be greater than when all the downward scattered radiation is scattered by the isotropic part of the phase function. The opposite can be said of the upward scattered radiation, and for $\beta = 0.5$ the two effects will neutralise themselves leaving $J_{sc}(\tau)$ independent of α. However, near the surface the upward scattered radiation is modified by surface loss. For small values of α there is more surface loss than for high values of α. This is most easily understood when the incident radiation is normal to the surface and Fig. 5 refers to this case. The radiation scattered upwards has least chance of being scattered or absorbed before it emerges from the surface, if it is scattered along a path normal to the surface. The fraction of the scattered radiation scattered along such a path is greatest when $\alpha = 0.0$. Hence, the radiation field near the surface increases as α increases, and the radiation field in the interior of the atmosphere decreases as α increases. In Section I.3 it was seen that axially symmetric fields produced solutions for $J_{sc}(\tau)$ that are independent of α when $\beta = 0.5$. The reason for this is that the preferential surface loss described here no longer occurs for axially symmetric incident radiation fields. Hence, for such fields, g is a unique measure of the effect of anisotropy.

The effect of μ_0 on the scattered radiation field is dependent on both the phase function and the position in the atmosphere. For large and intermediate values of τ the qualitative effect is independent of the phase function. The mean intensity of the scattered radiation simply decreases as the value of μ_0 decreases. The smaller the value of μ_0, the greater the quantity, τ/μ_0 , which is the optical distance that the incident radiation must traverse in order to reach depth τ in the atmosphere. Consequently, the radiation field will suffer greater absorption loss before reaching depth τ when μ_0 is small than when μ_0 is large. The situation is more complex when τ is small. As we have already seen, the scattered radiation field increases with depth as it is built up from the reduced incident radiation field until it begins to decrease with depth as absorption losses become greater. It transpires that, as μ_0 decreases, the optical distance along the line of the incident radiation between the surface and a point at optical depth τ , will increase, and that if this distance is small, the scattered radiation field will be built up to a greater extent, the smaller the value of μ_0 .

Fig. 6 shows the mean intensity of the scattered radiation field plotted as a function of τ for several values of μ_o, for values of $\tilde{\omega} = 0.9$, $\alpha = 0.0$ and $\beta = 1.0$. The quantity, $\partial J_{sc}(\tau, \mu_o)/\partial \mu_o$ is seen to be negative for small values of τ and positive for large values of τ. This situation only occurs when α is set to zero from the set of available values of α listed in Table I. When α is one of the non-zero values there is no crossing of the curves as there is in Fig. 6. Instead, the curves for small values of μ_o are always below those for high values of μ_o. The reason for this is that when α is not zero or not close to zero some radiation is scattered "sideways" out of the line of the incident radiation. As μ_o decreases some of this sideways scattered radiation has a progressively greater chance of escape from the surface and it transpires that the increase in energy loss through the surface as μ_o decreases is greater than the increase in build-up of the scattered radiation field for the same decrease of μ_o. Fig. 6 has been drawn for β equal to unity. Were β smaller the intersections of the curves would have been located at lower values of τ. This follows from the fact that the build-up of the scattered radiation field is greatest for high values of β. It is measured by the slope of the graphs of Fig. 5, which are $dJ_{sc}(\tau)/d\tau$.

These results can be investigated mathematically. Firstly, consider the case of linear scattering only, which is the only case in the range of available phase functions for which $J_{sc}(\tau)$ was evaluated in which the interesting intersections of Fig. 6 occur. Equations (II-13), (II-15) and (II-40) give

$$J_{sc}(\tau,\mu_o) = \frac{1}{4} F e^{-\sigma\tau/\mu_o} \frac{[1-\tilde{\omega}(2\beta-1)-\sigma]}{\tilde{\omega}(1-\beta)} - \frac{1}{4} F e^{-\tau/\mu_o}$$

The maximum occurs when the derivative of this equation is equated to zero, whence

$$\frac{dJ_{sc}(\tau,\mu_o)}{d\mu_o} = 0 = \frac{F}{4\mu_o^2}\left\{\frac{\sigma\tau[1-\tilde{\omega}(2\beta-1)-\sigma]e^{-\sigma\tau/\mu_o}}{\tilde{\omega}(1-\beta)} - \tau e^{-\tau/\mu_o}\right\}$$

or $$e^{\tau(1-\sigma)/\mu_o} = \tilde{\omega}(1-\beta)\,/\,\sigma[1-\tilde{\omega}(2\beta-1)-\sigma] = \psi .$$

Thus, the value of μ_o which gives the maximum value of $J_{sc}(\tau,\mu_o)$ is μ_o^{max} which is

$$\mu_o^{max} = \tau(1-\sigma)\,/\,\log\psi . \qquad \text{(II-42)}$$

To provide a numerical example we choose $\beta = 0.5$ and $\tilde{\omega} = 0.9$. The appropriate values for σ and $\log\psi$ are 0.3162 and 0.7324 respectively. From equation (II-42) we find that the maximum value of $J_{sc}(\tau,\mu_o)$ occurs at values of μ_o^{max} of 0.093, 0.93 and 4.67, when values of τ are 0.1, 1.0 and 5.0 respectively. These numbers agree with the qualitative conclusions of the preceding paragraph. The form of equation (II-42) draws to mind a point that is easily glossed over in qualitative arguments and often obscured in numerical results; namely, the existence of a finite value of μ_o^{max} when τ is very small indeed. This is, of course, due to the fact that, even with very small values of τ and μ_o, when μ_o is small enough, (0.09 in the example) the attenuation factor will create a reduction of $J_{sc}(\tau,\mu_o)$. It is not advantageous to apply the same technique to the general case in which α is non-zero, as the expressions become very complex.

The maximum of $J_{sc}(\tau)$ with respect to τ can be found by equating the derivative of $J_{sc}(\tau)$ with respect to τ, to zero. For general anisotropic scattering this gives

$$\tau_{max} = \frac{\mu_0 \log\left[-\epsilon\mu_0 D F/\sigma E' \right]}{\left[\epsilon\mu_0 - (1-\sigma) \right]} , \qquad \text{(II-43)}$$

where D and E$'$ are always of opposite sign.

We now focus our attention on the thermal radiation field, the behaviour of which can readily be detected by inspection of the formulae for $J_p(\tau)$. When n is large the mean intensity of the thermal radiation field is approximately constant throughout the atmosphere and this constant is of the order of 0.5. When n is small, $J_p(\tau)$ is about 0.5 at the surface, rises rapidly with depth until it is of the order of 50 at an optical depth of about 10, and remains at that high value at all points deeper in the atmosphere. The physical processes behind this behaviour can also be deduced from the equations of the preceding analysis. Consider equation (II-37). Remembering that G$'$ equals zero, it is clear that $J_p(\tau)$ tends to the value of the constant, G , as τ becomes larger. Furthermore it is clear that $J_p(\tau)$ is approximately equal to G when n is large even for small values of τ . Finally, the τ-dependent terms of this equation are of comparable magnitude to G , as given by equation (II-39), when n and τ are both small, so that $J_p(\tau)$ is small in these circumstances. The physics behind these results can be seen most clearly by studying equations (II-33) and (II-38). The first of these states that the energy density gradient necessary to maintain a certain flux in the opposite direction is proportional to that flux. The second is the mathematical expression of the principle of conservation of energy. From this, together with the expressions for $H_s(\tau)$ and $H_{(in}(\tau)$ it is clear that no stellar radiation penetrates below a certain optical depth, τ_c , and that the flux of the thermal radiation field is also zero below that depth. This zero flux means, by equation (II-33), that the mean intensity of the thermal radiation field is constant below τ_c . At points between τ_c and the surface the thermal flux is non-zero and the dependence of the mean intensity or energy density gradient upon n is shown by equation (II-33). In fact this gradient is inversely proportional to n because $H_p(\tau)$ is independent of n as can be seen from equation (II-32). The flux, $H_p(\tau)$ never attains a very high value so that the gradient, $dJ_p(\tau)/d\tau$ is always very small when n is large and large when n is small. Now, at the surface, the boundary condition links $J_p(o)$ with $H_p(o)$ which, as we have seen is independent of n . Consequently $J_p(o)$ is independent of n . Thus, when n is small, $J_p(\tau)$ rises rapidly from a small value until τ is equal to τ_c and $J_p(\tau)$ has reached a large value. The physical principle behind this phenomenon is that the energy density gradient of a radiation field needed to maintain a given flux through an atmosphere, is inversely proportional to the mean free path of the photons of the radiation field in that atmosphere. This is true for isotropic scattering (or isotropic and conservative, absorption plus emission, which amounts to the same phenomenon as far as radiative transfer in grey atmospheres is concerned) in as much as equation (II-33), which is based on Eddington's approximation, remains true. This physical principle is precisely that upon which Van de Hulst (1968) based his similarity relations, for he expressed the energy density gradient as being proportional to the fraction $(1-g)$. The asymmetry parameter is a measure of the mean free path in the direction of transfer, so therefore the two principles are the same.

The effect of scattering on the thermal radiation field arises from its effect on the thermal radiation flux via the principle of conservation of energy. Its importance is demonstrated in Fig. 7 which shows the quantity, $\mathfrak{J}(\tau, \tilde{\omega}) = J_p(\tau, \tilde{\omega}) / J_p(\tau, 0{\cdot}1)$, plotted as a function of $(1-\tilde{\omega})$. The quantity, $\mathfrak{J}$, is a more suitable quantity to show the physical effects of $\tilde{\omega}$ rather than $J_p(\tau, \tilde{\omega})$ because the latter extends over a wide range of values. The logarithmic scale for $(1-\tilde{\omega})$ is used to show the importance of $\tilde{\omega}$ as it approaches unity, these values of $\tilde{\omega}$ being the most important in planetary atmosphere studies. The diagram shows $J_p(\tau, \tilde{\omega})$ as a function of $\tilde{\omega}$ for discrete values of τ; the whole being shown for normal incidence and isotropic scattering. The essential features of Fig. 7 are independent of the phase function and angle of incidence. Firstly, consider the continuous curves for which n equals 10^{-2}, and in particular, that curve for $\tau = 0$. The inward flux of radiation at the surface is a constant and by virtue of conservation of energy an increase in the emergent stellar flux results in a decrease in the emergent thermal flux, and this, by virtue of the boundary condition results in a decrease of the mean intensity of the thermal radiation field at the surface. As the albedo for single scattering increases so does the emergent thermal flux. Hence the gradient $dJ_p(0, \tilde{\omega})/d\tilde{\omega}$ is negative. The situation inside the atmosphere is more complex. At small values of τ, the decrease of the mean intensity of the stellar radiation field is greatest for the smallest value of $\tilde{\omega}$. This arises from the fact that, in a certain small optical distance the absorption loss is proportional to $(1-\tilde{\omega})$. Consequently, the inward stellar flux and the outward thermal flux increase as the albedo decreases. From equation (II-33) we see that the thermal energy density gradient likewise increases. This together with the fact that $J_p(0)$ increases as the albedo decreases gives the result that the gradient $\partial J_p(\tau,\tilde{\omega})/\partial\tilde{\omega}$ is negative when τ is small. However, the situation is different for large values of τ for which the slopes of the stellar intensities with τ are zero for small values of $\tilde{\omega}$ but still non-zero for larger values of $\tilde{\omega}$ due to the associated increased penetration. Hence, the opposite conclusion is reached; that the gradient, $\partial J_p(\tau,\tilde{\omega})/\partial\tilde{\omega}$ is positive for large values of τ . There is however, a slight difference between the two arguments. In the case when τ was small the negative gradient, $\partial J_p(0,\tilde{\omega})/\partial\tilde{\omega}$ enhanced the negative gradient, $\partial J_p(\tau,\tilde{\omega})/\partial\tilde{\omega}$. However, on reversal of the albedo dependence of the gradient, $dJ_p(\tau)/d\tau$, the surface value dependence on the albedo counteracts this gradient dependence. Consequently, it is only for very large values of τ that the gradient $\partial J_p(\tau,\tilde{\omega})/\partial\tilde{\omega}$ is positive for all values of $\tilde{\omega}$. For lesser values of τ the surface value effects exceed the gradient effects for high values of the albedo so that the curves shown in Fig. 7 have maxima around $\tilde{\omega}$ = 0.9. The results from Fig. 7 emphasize our earlier deduction that the albedo is a very critical parameter when it is near unity. Only in the case of τ = 10 is the dependence of J_p upon the albedo small when the albedo is large. Moreover, we see that, for very thick atmospheres such as that of Venus, the mean intensity of the thermal radiation varies with the albedo in opposite senses at the surface and deep in the atmosphere.

When n is large the curves for all values of τ are identical to that of $J_p(0)$ for $n = 10^{-2}$ because, as we have already noted,

$J_p(\tau)$ is a constant for $n = 10^4$ and $J_p(0)$ is independent of n . The special case in which n equals unity will be discussed later. We will first investigate the effects of the phase function on the thermal radiation mean intensity. This is shown in Fig. 8 for $\mu_0 = 1.0$ and $\tilde{\omega} = 0.9$. The families of curves are for $n = 10^4$, 1 and 10^{-2} and are at depth $\tau = 50$ where the mean intensity is at its maximum value. The family of curves for $n = 10^4$ apply for all values of τ . The scale of the ordinate is different for the three families but this does not affect the qualitative conclusions which are clarified by superimposing the three families of curves. Fig. 8 shows J_p plotted as a function of α for the three values of β; 0.0, 0.5 and 1.0. Consider first the case of $n = 10^4$ for which we have already noted that the thermal radiation mean intensity is controlled by the emergent stellar flux. We have seen that the stellar flux is greatest when β is zero so that by considering the principle of conservation of energy and the Eddington boundary condition we see that $J_p(0)$ and also $J_p(\tau)$, because n is large, is greatest when β is unity. Similarly, the emergent stellar flux is smallest for small values of α, and β equal to unity, and greatest for small values of α when β is zero. The same argument as before shows the curves of Fig. 8 to be physically reasonable; and they apply equally well to each value of n . However, when n is 10^{-2} the extent of the dependence of $J_p(\tau)$ upon β is far greater, particularly when β is near unity. The consequence for model planetary atmospheres is clear. They involve strong forward throwing phase functions so that the phase function must be accurately determined in order for accurate values of $J_p(\tau)$ to be produced. However, we saw in Section I.3 that the exact shape of the phase function was not important so we conclude that model planetary atmospheres require very accurate values of the asymmetry parameter as well as the albedo.

The variation of the mean intensity of the thermal radiation field with μ_0 is shown in Fig. 9, in which the quantity, $\zeta = J_p(\tau,\mu_0)/J_p(\tau,1)$ is plotted as a function of τ for $\tilde{\omega}$ = 0.9, a = 1.0, n = 10^{-2} and several values of μ_0. The incident flux contains a factor μ_0 so that the thermal flux includes this factor also. This is shown clearly in Fig. 9. The quantity, ζ , exhibits very little dependence on τ and, of course, would show none were n = 10^4. Consequently, we can write down an approximate equation to account for the effect of μ_0 in $J_p(\tau)$; which is

$$J_p(\tau,\mu_0) = \eta\,\mu_0\, J_p(\tau, 1.0) , \qquad \text{(II-44)}$$

where η is a fraction slightly less than unity.

Finally, we consider the special case when n equals unity. It would appear from Fig. 7 that $\xi \leq 1.0$. However this is not a general conclusion; it merely applies to the special case of isotropic scattering. Equivalent graphs for which β is greater than 0.5 would be similar to those for n = 10^{-2} ; whilst those for which β is less than 0.5 would be similar to that for n = 10^4. The arguments applied earlier to the effects of albedo and phase function on $J_p(\tau)$ are still valid for the case of n = 1.0. Nevertheless, it will be profitable to investigate the characteristics of the equations of the problem for the special case of isotropic scattering and n equal to unity. It is easy to show that the constant, G , is 5/4 for normal incidence and it is for this reason that the quantity, ξ , equals unity for large values of τ , as shown in Fig. 7; the constant being independent of the albedo. For this problem in general, it can be shown that

$$\lim_{\tau \to \infty} J_p(\tau) = \frac{1}{4}\mu_0 F(3\mu_0 + 2), \tag{II-45}$$

where the convergence to the limit is moderately rapid. Consequently, we obtain an expression for η, as defined by equation (II-44), which is

$$\eta = (3\mu_0 + 2)/5. \tag{II-46}$$

The reason for the independence of equation (II-45) with $\tilde{\omega}$ is that the transfer of the thermal radiation is exactly the same as the transfer of the stellar radiation, the equation of transfer for the sum of the intensities of the two fields being that for conservative isotropic scattering. Thus, when the stellar intensity is zero, the thermal intensity has attained its constant value which is independent of the albedo. The two boundary conditions used were $J_s(0) = 2H_s(0)$ and $J_p(0) = 2H_p(0)$. Were the factor, 2, in the boundary conditions a different factor, ϕ say, equation (II-45) would have been

$$\lim_{\tau \to \infty} J_p(\tau) = \frac{1}{4}\mu_0 F(3\mu_0 + \phi).$$

For normal incidence the limiting value of $J_p(\tau)$ given by this equation is equal to the limiting value of $J_p(\tau)$ given by equation (II-45) for the Eddington boundary condition, multiplied by a factor, $(3 + \phi)/5$. This result can be manipulated to show that an error of x% in the value of ϕ used in the boundary condition leads to an error of $2x/5$% in the resultant limiting value of $J_p(\tau)$

for this special case. The transmitted error is higher when μ_0 is less than unity, but in general, we may conclude that the Eddington approximate boundary condition will affect the solutions deep in the atmosphere to a lesser extent than it does the solutions at the surface.

3.2. Finite Atmospheres We have already noted that a semi-infinite atmosphere is not a good model for a planetary atmosphere, even for that of Venus. We shall now consider the same radiative heating problem for finite atmospheres with a ground layer, the properties of which were given in Section II.1. We shall solve the problem in the same way as we did for semi-infinite atmospheres and then we shall consider the special case of a finite atmosphere with no ground. This has no value in planetary atmosphere studies but is included for completeness by which it emphasises many of the salient features of the radiative heating problem.

The linear radiation field is defined in the same way as before and we shall derive exact expressions for its intensity. There are two further radiation fields in this problem, namely, the reduced visible ground radiation field and the reduced thermal ground radiation field. By virtue of the isotropy of the emission from the ground, the radiation scattered from these fields is independent of azimuth, so that there is no need to include any contribution from these in the azimuthally dependent linear radiation field. The scattering of the linear radiation field is one example of the general problem of one-dimensional radiative transfer along a line of finite length and with radiation incident upon both ends of the line. It will be expedient to solve the general situation because other examples of this problem will occur in Chapter III. Let x_0 be the total optical length of the linear medium and let x be the optical distance measured in terms of the extinction coefficient from one end. Let $I^+(x)$ represent the intensity of the radiation flowing in the positive x-direction at a point x, and let $I^-(x)$ represent the intensity of the radiation flowing in the opposite direction at the point x. Let I_0^+ be the intensity of the radiation incident upon the medium in the positive x-direction at the origin, $x = 0$, and let I_0^- be the intensity of the radiation incident upon the medium in the negative x-direction at the point, $x = x_0$. These are defined in each problem and hence create the two boundary conditions.

$$I^+(0) = I_0^+ \quad \text{and} \quad I^-(x_0) = I_0^- . \tag{II-47}$$

Let $\tilde{\omega}_1$ be the albedo for single scattering and let β be the fraction of the scattered radiation that is scattered forwards, the remaining fraction, $(1 - \beta)$, being the scattered backwards. The subscript, unity, on the albedo serves as a reminder that it is not necessarily the same quantity as the albedo for single scattering in the complete problem of scattering in a finite atmosphere. It is, in fact, equal to $\tilde{\omega}(1 - \alpha)$ in terms of the parameters defined earlier for the schematic phase function, and represents the albedo for scattering into the delta-function spikes.

The two equations of transfer for the intensities in the positive and negative x-directions are

$$\frac{dI^+(x)}{dx} = -I^+(x) + B^+(x) \quad \text{and} \quad -\frac{dI^-(x)}{dx} = -I^-(x) + B^-(x)$$

respectively, where $B^{\pm}(x)$ are the two source functions. The source function, $B^+(x)$, is the sum of the radiation scattered forwards from

$I^+(x)$ and the radiation scattered backwards from $I^-(x)$, whilst the source function, $B^-(x)$, is the sum of the radiation scattered forwards from $I^-(x)$ and the radiation scattered backwards from $I^+(x)$. That is,

$$B^{\pm}(x) = \tilde{\omega}_1 \beta \, I^{\pm}(x) + \tilde{\omega}_1 (1-\beta) \, I^{\mp}(x).$$

Hence, we obtain

$$\frac{dI^+(x)}{dx} = -(1-\tilde{\omega}_1\beta) I^+(x) + \tilde{\omega}_1 (1-\beta) I^-(x) \tag{II-48}$$

and $$-\frac{dI^-(x)}{dx} = -(1-\tilde{\omega}_1\beta) I^-(x) + \tilde{\omega}_1 (1-\beta) I^+(x).$$

These two equations combine to give

$$\frac{d^2 I^+(x)}{dx^2} - \sigma^2 I^+(x) = 0,$$

the solution of which is

$$I^+(x) = C_1 e^{\sigma x} + C_2 e^{-\sigma x}, \tag{II-49}$$

where $$\sigma^2 = (1-\tilde{\omega}_1\beta)^2 - \tilde{\omega}_1^2 (1-\beta)^2, \tag{II-50}$$

and C_1 and C_2 are constants of integration. Equations (II-48) and (II-49) combine to give

$$I^-(x) = \frac{1}{\tilde{\omega}_1 (1-\beta)} \left\{ C_1 e^{\sigma x} [1-\tilde{\omega}_1\beta + \sigma] + C_2 e^{-\sigma x} [1-\tilde{\omega}_1\beta - \sigma] \right\} \tag{II-51}$$

Applying the two boundary conditions given by equations (II-47) to equations (II-50) and (II-51) we obtain

$$C_1 = \frac{\tilde{\omega}_1(1-\beta)I_0^- - (1-\tilde{\omega}_1\beta-\sigma)I_0^+ e^{-\sigma x_0}}{(1-\tilde{\omega}_1\beta+\sigma)e^{\sigma x_0} - (1-\tilde{\omega}_1\beta-\sigma)e^{-\sigma x_0}} \qquad \text{(II-52)}$$

and $\qquad C_2 = I_0^+ - C_1.$

Consider the following special cases.

(i) $I_0^+ = 0$; $I_0^- = 0$. This is the case of no incident radiation; and equations (II-49) to (II-52) yield the trivial solution

$$C_1 = C_2 = 0 , \qquad I^+(x) = I^-(x) = 0.$$

(ii) $I_0^- = 0$. This is the case of incident radiation upon one end only for which we obtain

$$C_1 = \frac{-(1-\tilde{\omega}_1\beta-\sigma)I_0^+ e^{-\sigma x_0}}{(1-\tilde{\omega}_1\beta+\sigma)e^{\sigma x_0} - (1-\tilde{\omega}_1\beta-\sigma)e^{-\sigma x_0}} \qquad \text{(II-53)}$$

and $\qquad C_2 = I_0^+ - C_1 .$

(iii) $I_0^+ = 0$. This is the case of radiation incident upon the end, $x = x_0$ only, for which we obtain

$$C_1 = \frac{\tilde{\omega}_1(1-\beta)I_0^-}{(1-\tilde{\omega}_1\beta+\sigma)e^{\sigma x_0} - (1-\tilde{\omega}_1\beta-\sigma)e^{-\sigma x_0}} \qquad \text{(II-54)}$$

and $$C_2 = - C_1.$$

(iv) $I_0^+ = I_0^- = I_0$. We have the same incident radiation upon each end of the medium and therefore, we have

$$C_1 = \frac{I_0 \left\{ \tilde{\omega}_1(1-\beta) - (1-\omega_1\beta-\sigma)\, e^{-\sigma x_0} \right\}}{\left\{ (1-\tilde{\omega}_1\beta+\sigma)e^{\sigma x_0} - (1-\tilde{\omega}_1\beta-\sigma)e^{-\sigma x_0} \right\}} \tag{II-55}$$

and $$C_2 = I_0 - C_1.$$

The linear radiation field in our problem of a finite plane-parallel atmosphere with parallel radiation incident upon its upper surface corresponds to the special case (ii) with $I_0^+ = \pi F$. The albedo in question is the albedo for scattering into the delta-function spikes, which is $\tilde{\omega}_1 = \tilde{\omega}\,(1 - \alpha)$. The geometry of the problem demands that $x = \tau/\mu_0$ and that the radiation field exists for $\mu = \mu_0$ only and for $\phi = \phi_0$ or $\phi = \phi_0 + \pi$ only. Consequently, the first two moments of the linear radiation field are

$$J_{lin}(\tau) = \frac{C_1\left[1-\tilde{\omega}(1-\alpha)(2\beta-1)+\sigma\right]e^{\sigma\tau/\mu_0}}{4\pi\,\tilde{\omega}\,(1-\alpha)(1-\beta)} +$$

$$+ \frac{C_2\left[1-\tilde{\omega}(1-\alpha)(2\beta-1)-\sigma\right]e^{-\sigma\tau/\mu_0}}{4\pi\,\tilde{\omega}\,(1-\alpha)\,(1-\beta)} \tag{II-56}$$

and $$H_{lin}(\tau) = \frac{C_1\mu_0\left[1-\tilde{\omega}(1-\alpha)+\sigma\right]e^{\sigma\tau/\mu_0}}{4\pi\,\tilde{\omega}\,(1-\alpha)(1-\beta)} + \frac{C_2\mu_0\left[1-\tilde{\omega}(1-\alpha)-\sigma\right]e^{-\sigma\tau/\mu_0}}{4\pi(1-\alpha)(1-\beta)} \tag{II-57}$$

where C_1 and C_2 are given by equations (II-53) with $I_0^+ = \pi F$.

The reduced incident radiation field can be written down immediately. It is merely the incident intensity exponentially attenuated from the point of entry into the atmosphere to the depth in question. We thus have

$$I_{red}^{inc}(\tau, \mu, \phi) = \pi F e^{-\tau/\mu_0} \delta(\mu - \mu_0) \delta(\phi - \phi_0),$$

$$J_{red}^{inc}(\tau) = \frac{1}{4} F e^{-\tau/\mu_0} \quad \text{and} \quad H_{red}^{inc}(\tau) = -\frac{1}{4}\mu_0 F e^{-\tau/\mu_0}. \qquad \text{(II-58)}$$

Before proceding with the solution for the scattered radiation we shall consider several special cases. Firstly, in the limit as τ_0 tends to infinity, the constants C_1 and C_2 tend to zero and π F respectively. Thus, equations (II-49) and (II-51) tend to equations (II-11) and (II-12), so that the limiting forms of the expressions for finite atmospheres agree with those obtained for semi-infinite atmospheres in the preceding sub-section. Secondly, in the case of isotropic scattering, σ is unity and the albedo, $\tilde{\omega}_1$, is zero. This causes equation (II-51) to be indeterminate. However, when the scattering is isotropic the linear radiation field is identical to the reduced incident radiation field given by equation (II-58). Another apparent singularity arises in the third special case for which β = 1.0. The solution can be found by appropriately adjusting the equations of transfer, equations (II-48) and then proceeding as before. Hence, we obtain

$$I^+(x) = I_0^+ e^{-(1-\tilde{\omega}_1)x} \quad \text{and} \quad I^-(x) = 0,$$

so that $$J_{lin}(\tau) = \frac{1}{4} F e^{-\sigma\tau/\mu_0} \quad \text{and} \quad H_{lin}(\tau) = -\frac{1}{4}\mu_0 F e^{-\sigma\tau/\mu_0}, \qquad \text{(II-59)}$$

where $\sigma = 1 - \tilde{\omega}_1$, in this case. These results are obvious from physical reasoning. When the spike scattering is forward only the situation is the same as though the scattered radiation were not scattered. Thus, the incident radiation is attenuated by the absorption coefficient and isotropic part of the scattering coefficient rather than the extinction coefficient.

We are now in a suitable position to construct the source function for the scattered radiation and solve the equation of transfer. The scattered radiation is, of course, not the true scattered radiation field because part of this is included in the linear radiation field. The most convenient way to treat the ground radiation is to consider it an external source of isotropic stellar intensity, G_s , and isotropic thermal radiation, G_p . The values of G_s and G_p are found by applying boundary conditions. The emission coefficient for the scattered radiation is made up of three terms; the radiation scattered isotropically from the linear radiation field; the radiation scattered anisotropically from the scattered radiation field; and the radiation scattered anisotropically from the reduced visible ground radiation. Hence

$$j_s(\tau,\mu) = (\kappa_s+\sigma_s)\rho \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} \tilde{\omega}\alpha\, I_{(\ell)}(\tau,\mu',\phi')\, d\mu'\, d\phi' +$$

$$+ (\kappa_s+\sigma_s)\rho \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} p(\mu,\phi;\mu',\phi')\, I_s(\tau,\mu',\phi')\, d\mu'\, d\phi' +$$

$$+ (\kappa_s+\sigma_s)\rho \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} p(\mu,\phi;\mu',\phi')\, G_s\, e^{-(\tau_0-\tau)/\mu'}\, d\mu'\, d\phi'.$$

The limits of the last integral are $\mu' = 0$ and $\mu' = 1$, rather than $\mu' = -1$ and $\mu' = +1$, because the integrand is zero for all negative values of μ'. The source function, $B_s(\tau,\mu)$, is the ratio of this emission coefficient to the extinction coefficient, $(\kappa_s+\sigma_s)\rho$;

and the phase function used above is the schematic one of equation (I-29). The third integral in the emission coefficient deserves mention. By virtue of the three-part nature of the phase function, this integral will divide into three separate integrals. The one that arises from the isotropic part of the phase function involves an integral that is known as an exponential integral function. Details of the exponential integral functions, their definitions and their properties are found in the Appendix. The other two integrals of this third integral of the emission coefficient involve a type of delta function which we shall define as

$$\delta_{\pm} = 1 \; ; \qquad \mu \gtrless 0 \, ,$$

$$\delta_{\pm} = 0 \; ; \qquad \mu \lessgtr 0 \, .$$

It is introduced merely to aid the mathematical expression of the equation of transfer; and it will be cancelled during the solution of the equation. Using these functions, the source function becomes

$$B_s(\tau,\mu) = \tilde{\omega}\alpha J_{lin}(\tau) + \tilde{\omega}\alpha J_s(\tau) + \tilde{\omega}(1-\alpha)\beta I_s(\tau,+\mu) +$$

$$+ \tilde{\omega}(1-\alpha)(1-\beta) I_s(\tau,-\mu) + \tfrac{1}{2}\tilde{\omega}\alpha G_s E_2(\tau_0-\tau) +$$

$$+ \delta_+ \tilde{\omega}(1-\alpha)\beta G_s e^{-(\tau_0-\tau)/\mu} + \delta_- \tilde{\omega}(1-\alpha)(1-\beta) G_s e^{(\tau_0-\tau)/\mu} \, . \qquad \text{(II-60)}$$

Consequently, the equation of transfer for anisotropic scattering in a finite plane-parallel atmosphere with a ground layer is

$$\mu \frac{dI_s(\tau,\mu)}{d\tau} = I_s(\tau,\mu) - \tilde{\omega}\alpha J_s(\tau) - \tilde{\omega}(1-\alpha)\beta I_s(\tau,+\mu) -$$

$$- \tilde{\omega}(1-\alpha)(1-\beta) I_s(\tau,-\mu) - \tilde{\omega}\alpha J_{lin}(\tau) - \tfrac{1}{2}\tilde{\omega}\alpha G_s E_2(\tau_0-\tau) -$$

$$-\tilde{\omega}(1-\alpha)\beta\delta^+ G_s e^{-(\tau_0-\tau)/\mu} - \tilde{\omega}(1-\alpha)(1-\beta)\delta^- G_s e^{(\tau_0-\tau)/\mu}. \qquad \text{(II-61)}$$

We shall solve this equation by Eddington's method in the same way as we did in the preceding subsection for semi-infinite atmospheres. Firstly, we integrate the equation of transfer by applying the L_0-operator, to obtain

$$\frac{dH_s(\tau)}{d\tau} = (1-\tilde{\omega}) J_s(\tau) - \tilde{\omega}\alpha J_{lin}(\tau) - \frac{1}{2}\tilde{\omega} G_s E_2(\tau_0-\tau), \qquad \text{(II-62)}$$

and secondly, we integrate the equation of transfer by applying the L_1-operator, to obtain

$$\frac{dK_s(\tau)}{d\tau} = \left[1-\tilde{\omega}(1-\alpha)(2\beta-1)\right] H_s(\tau) - \frac{1}{2}\tilde{\omega}(1-\alpha)(2\beta-1) G_s E_3(\tau_0-\tau).$$

Using the Eddington approximation, equation (I-14), and defining

$$\gamma = 1-\tilde{\omega}(1-\alpha)(2\beta-1) \qquad \text{(II-63)}$$

we obtain

$$\frac{dJ_s(\tau)}{d\tau} = 3\gamma H_s(\tau) - \frac{3}{2}(1-\gamma) G_s E_3(\tau_0-\tau). \qquad \text{(II-64)}$$

Equations (II-62) and (II-64) combine to give

$$(D^2 - \delta^2)\, J_s(\tau) = -3\tilde{\omega}\alpha\gamma J_{lin}(\tau) - \frac{3}{2}\left[1-\gamma(1-\tilde{\omega})\right] G_s\, E_2(\tau_0 - \tau), \qquad \text{(II-65)}$$

where

$$\delta^2 = 3\gamma(1-\tilde{\omega}). \qquad \text{(II-66)}$$

Equation (II-65) is a second order inhomogeneous total differential equation with constant coefficients and can be solved by normal analytical techniques. The solution involves further transcendental functions, the F_n-functions, which are integrals of products of exponential and exponential integral functions. They are defined in Section 2 of the Appendix and several of their properties are also listed. To avoid long strings of constants, equation (II-56) for the mean intensity of the linear radiation field will be written

$$J_{lin}(\tau) = A_6\, e^{\sigma\tau/\mu_0} + A_7\, e^{-\sigma\tau/\mu_0}, \qquad \text{(II-67)}$$

where A_6 and A_7 can be found from equations (II-56), (II-58) or (II-59) which ever is appropriate. The solution of equation (II-65) is

$$J_s(\tau) = A_1 e^{\delta\tau} + A_2 e^{-\delta\tau} + A_3 e^{\sigma\tau/\mu_0} + A_4 e^{-\sigma\tau/\mu_0} +$$

$$+ A_5 G_s \left\{ e^{\delta(\tau_0-\tau)} F_2\left[-\delta, (\tau_0-\tau)\right] - e^{-\delta(\tau_0-\tau)} F_2\left[\delta, (\tau_0-\tau)\right] \right\}, \qquad \text{(II-68)}$$

where A_1 and A_2 are arbitrary constants, and

$$A_3 = 3\tilde{\omega}\alpha\gamma\mu_0^2 A_6 / (\delta^2\mu_0^2 - \sigma^2),$$

$$A_4 = 3\tilde{\omega}\alpha\gamma\mu_0^2 A_7 / (\delta^2\mu_0^2 - \sigma^2), \qquad \text{(II-69)}$$

and $$A_5 = -3[1-\gamma(1-\tilde{\omega})]/4\delta .$$

The boundary conditions to be applied to this solution all involve the scattered flux, $H_s(\tau)$, which is found by using equations (II-66) and (II-68). The derivatives of the F_n-functions are given in the Appendix, so that we have

$$H_s(\tau) = \frac{\delta A_1 e^{\delta\tau}}{3\gamma} - \frac{\delta A_2 e^{-\delta\tau}}{3\gamma} + \frac{\sigma A_3 e^{\sigma\tau/\mu_0}}{3\gamma\mu_0} -$$

$$- \frac{\sigma A_4 e^{-\sigma\tau/\mu_0}}{3\gamma\mu_0} + \frac{(1-\gamma) G_s}{2\gamma} E_3(\tau_0-\tau) -$$

$$- \frac{\delta A_5 G_s}{3\gamma}\left\{e^{\delta(\tau_0-\tau)} F_2[-\delta,(\tau_0-\tau)] + e^{-\delta(\tau_0-\tau)} F_2[\delta,(\tau_0-\tau)]\right\} \quad \text{(II-70)}$$

The conditions required to determine G_s , A_1 and A_2 are the two Eddington approximate boundary conditions pertinent to the two surfaces of the atmosphere and the equation defining the parameter λ. The two Eddington approximate boundary conditions are

$$J_s(0) = 2H_s(0) \quad \text{and} \quad J_s(\tau_0) = -2H_s(\tau_0). \qquad \text{(II-71)}$$

The parameter, λ, is defined as the ratio of the stellar radiation flux reflected by the ground, to the stellar radiation flux incident on the ground. Now the outward flux from the ground, by definition of G_s , is

$$\int_0^{2\pi}\int_{-1}^{+1} \mu G_s \, d\mu \, d\phi = \pi G_s , \qquad \text{(II-72)}$$

and the flux incident upon the ground is $-4\pi H_s(\tau_0) - 4\pi H_{lin}(\tau_0)$. Equating the emergent flux with λ times the incident flux, we obtain

$$G_s = -4\lambda \left[H_s(\tau_0) + H_{lin}(\tau_0) \right]. \qquad \text{(II-73)}$$

Equations (II-71) and (II-73) provide three equations for three unknowns and the solution for these unknowns involves no difficulty. Thus we have completed the solution for the first two moments of the scattered radiation field.

The emission coefficient of the thermal radiation is comprised of five terms. The radiation from the three stellar radiation fields, the linear, the scattered and the reduced visible ground radiation fields, is converted into thermal radiation and emitted isotropically. The first three terms of the emission coefficient are from these three sources. The other two sources consist of the radiation absorbed from the thermal and reduced thermal ground radiation fields and then emitted conservatively and isotropically. We have defined G_p as the intensity of the thermal radiation emitted isotropically from the ground, so that the emission coefficient for the thermal radiation is

$$j_p(\tau) = \kappa_s \rho \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} I_{lin}(\tau, \mu', \phi')\, d\mu'\, d\phi' + \kappa_s \rho \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} I_s(\tau, \mu')\, d\mu'\, d\phi' +$$

$$+ \kappa_s \rho \frac{1}{4\pi} \int_0^{2\pi} \int_0^{1} G_s\, e^{-(\tau_0 - \tau)/\mu'}\, d\mu'\, d\phi' + \kappa_p \rho \frac{1}{4\pi} \int_0^{2\pi} \int_0^{1} G_p\, e^{-(\tau_0 - \tau)/\mu'n}\, d\mu'\, d\phi' +$$

$$+ \kappa_p \rho \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} I_p(\tau, \mu')\, d\mu'\, d\phi'. \qquad \text{(II-74)}$$

The factor, n, in the attenuation coefficient in the fourth term of this equation is due to the fact that the thermal ground radiation is attenuated according to κ_p whereas the optical depth scale is measured in terms of $(\kappa_s + \sigma_s)$. The source function, which is the emission coefficient divided by the absorption coefficient, κ_p, is thus

$$B_p(\tau) = J_p(\tau) + \frac{1}{2} G_p E_2[(\tau_0-\tau)/n] +$$

$$+ n(1-\tilde{\omega})\left[J_s(\tau) + J_{lin}(\tau) + \frac{1}{2} G_s E_2(\tau_0-\tau)\right], \qquad \text{(II-75)}$$

and hence, the equation of transfer for the thermal radiation is

$$\mu \frac{dI_p(\tau,\mu)}{d\tau} = \frac{1}{n} I_p(\tau,\mu) - \frac{1}{n} J_p(\tau) - \frac{1}{2n} G_p E_2[(\tau_0-\tau)/n] -$$

$$- (1-\tilde{\omega}) J_s(\tau) - (1-\tilde{\omega}) J_{lin}(\tau) - \frac{1}{2}(1-\tilde{\omega}) G_s E_2(\tau_0-\tau). \qquad \text{(II-76)}$$

This equation is solved by applying the two integral operators, L_0 and L_1, and replacing the resultant moment $K_p(\tau)$ in the second equation by $J_p(\tau)/3$ according to the Eddington approximation. Hence, we obtain

$$\frac{dH_p(\tau)}{d\tau} = -\frac{G_p}{2n} E_2[(\tau_0-\tau)/n] - \frac{1}{2} G_s(1-\tilde{\omega}) E_2(\tau_0-\tau) -$$

$$- (1-\tilde{\omega}) J_s(\tau) - (1-\tilde{\omega}) J_{lin}(\tau), \qquad \text{(II-77)}$$

and

$$\frac{dJ_p(\tau)}{d\tau} = \frac{3}{n} H_p(\tau). \qquad \text{(II-78)}$$

Using equations (II-67) and (II-68) for $J_{lin}(\tau)$ and $J_s(\tau)$ respectively we integrate equation (II-77) to obtain

$$H_p(\tau) = B_1 - \frac{1}{2} G_p E_3[(\tau_0-\tau)/n] -$$

$$-(1-\tilde{\omega}) G_s \left[\frac{1}{2} - \frac{2A_s}{\delta}\right] E_3(\tau_0-\tau) - \frac{(1-\tilde{\omega})}{\delta}\left[A_1 e^{\delta\tau} - A_2 e^{-\delta\tau}\right] -$$

$$-\frac{(1-\tilde{\omega})}{\sigma}\mu_0\left[(A_3+A_6)e^{\sigma\tau/\mu_0} - (A_4+A_7)e^{-\sigma\tau/\mu_0}\right] +$$

$$+\frac{(1-\tilde{\omega})A_s G_s}{\delta}\left\{e^{\delta(\tau_0-\tau)} F_2[-\delta,(\tau_0-\tau)] + e^{-\delta(\tau_0-\tau)} F_2[\delta,(\tau_0-\tau)]\right\}, \quad \text{(II-79)}$$

where details of the integrations of the exponential integral functions and the F_n-functions are given in the Appendix.

Equation (II-79) involves two unknown constants, B_1 and G_p; and these must be determined by two boundary conditions. However, at this stage in the solution only one boundary condition can be applied successfully, and that is the condition of conservation of energy. So far, our solutions have been valid for atmospheres with and without a ground. However, the application of the principle of conservation of energy is different in the two cases. We shall postpone consideration of atmospheres with no ground, and first consider atmospheres with a ground layer. For these atmospheres the principle of conservation of energy expresses itself mathematically as the condition of zero net flux, which is

$$H_{lin}(\tau) + H_s(\tau) + H_p(\tau) + H_{ground}(\tau) = 0, \quad \text{(II-80)}$$

where $4\pi H_{ground}$ is the flux of the combined reduced ground radiation fields. It is given by

$$H_{ground}(\tau) = \frac{1}{2} G_s E_3(\tau_0-\tau) + \frac{1}{2} G_p E_3[(\tau_0-\tau)/n]. \qquad \text{(II-81)}$$

The substitution of equations (II-57), (II-70), (II-79) and (II-81) into equation (II-80), followed by a lengthy algebraic reduction yields the condition that the constant, B_1 , must be zero for energy to be conserved in the atmosphere.

It is interesting to note that this result is independent of G_p . Consider the energy balance at the ground surface. The emergent thermal flux is equal to the incident thermal flux plus the absorbed fraction, (1 - λ) of the incident stellar flux. The emergent thermal flux is

$$\int_0^{2\pi}\int_0^1 \mu G_p \, d\mu \, d\phi = \pi G_p , \qquad \text{(II-82)}$$

the incident thermal flux is $-4\pi H_p(\tau_0)$, and the incident stellar flux is $-4\pi [H_s(\tau_0) + H_{lin}(\tau_0)]$. The energy balance is therefore

$$\pi G_p = -4\pi(1-\lambda)\left[H_s(\tau_0) + H_{lin}(\tau_0)\right] - 4\pi H_p(\tau_0). \qquad \text{(II-83)}$$

However, this is not an equation for G_p . The total flux of ground radiation, $4\pi H_{ground}(\tau_0)$ is given by $\pi(G_s + G_p)$ from equations (II-72) and (II-82). Thus an equation for $H_{ground}(\tau_0)$ can be written involving equations (II-73) and (II-83).

$$4\pi H_{ground}(\tau_0) = \pi(G_p + G_s)$$

$$= -4\pi\lambda\left[H_s(\tau_0) + H_{lin}(\tau_0)\right] - 4\pi(1-\lambda)\left[H_s(\tau_0) + H_{lin}(\tau_0)\right] - 4\pi H_p(\tau_0).$$

A glance at this equation shows it to be nothing more than equation (II-80), the equation of constant net flux, which does not involve the constant, G_p . Thus we see that the greater the flux out of the ground the greater the flux reflected back into the ground by the atmosphere. Nevertheless, it is still quite surprising that this state of affairs should exist. It is quite analogous, however, with a similar arbitrariness encountered by Chandraskhar (1960) in work on the exact solutions of similar problems using the principles of invariance. These principles will be discussed and used in Section II.5.1. Chandrasekhar found that, for conservative problems, the principles of invariance yielded arbitrary solutions that could not be resolved by appeal to the flux-integral alone, but by appeal to the K-integral as well. The situation here is similar. The problem is conservative and is exact because we are considering fluxes only. This was shown to be so by the second method outlined in Section II.3.1. and is true here also, even though it is not proved directly. Our solutions are arbitrary and we shall see that a unique solution is only possible after using the second moment integral of the equation of transfer, which is the K-integral, and is expressed in an approximate form by equation (II-78). Furthermore, the arbitrariness vanishes if the thermal radiation transfer is considered to be non-conservative.

The integration of equation (II-78) using equation (II-79) gives

$$J_p(\tau) = B_2 - \frac{3}{2} G_p E_4 [(\tau_0-\tau)/n] -$$

$$- \frac{3(1-\tilde{\omega})G_s}{n}\left[\frac{1}{2} - \frac{2A_5}{\delta}\right] E_4(\tau_0-\tau) - \frac{3(1-\tilde{\omega})}{n\delta^2}\left[A_1 e^{\delta\tau} + A_2 e^{-\delta\tau}\right] -$$

$$- \frac{3(1-\tilde{\omega})\mu_0^2}{n\sigma^2}\left[(A_3+A_6)e^{\sigma\tau/\mu_0} + (A_4+A_7)e^{-\sigma\tau/\mu_0}\right] -$$

$$- \frac{3(1-\tilde{\omega})A_5 G_s}{n\delta^2}\left\{e^{\delta(\tau_0-\tau)} F_2[-\delta,(\tau_0-\tau)] - e^{-\delta(\tau_0-\tau)} F_2[\delta,(\tau_0-\tau)]\right\}. \quad \text{(II-84)}$$

The constants, B_2 and G_p are found by using the two Eddington approximate boundary conditions

$$J_p(0) = 2H_p(0) \quad \text{and} \quad J_p(\tau_0) = -2H_p(\tau_0). \quad \text{(II-85)}$$

This procedure is straightforward and completes the solution of the equations of transfer for the scattered and thermal radiation fields in an anisotropically scattering finite plane-parallel atmosphere with a conservative Lambertian ground layer.

An important special case is that of the finite atmosphere with no ground at its lower surface. The scattering problem is exactly the same as the scattering problem of the finite atmosphere with a ground layer of albedo, λ, equal to zero. Consequently, the source function is given by equation (II-60), the equation of transfer by equation (II-61) and the mean intensity and flux of the scattered radiation field by equations (II-68) and (II-70) respectively. In these equations the parameter, λ, and hence the intensity, G_s, are zero. The linear radiation field is, of course, unchanged. However, the two problems are not the same for the thermal radiation field. In the case of the ground being absent, the intensity, G_p , is zero also, but with this restriction, equations (II-75), (II-76) and (II-79) still represent the source function, the equation of transfer and the thermal flux respectively. The major change in the physics of the problem arises in the expression of the principle of conservation of energy, which is no longer the equation of zero net flux, equation (II-80). Energy must be conserved in the atmosphere, so we state that the net flux into the atmosphere at its upper surface must equal the net flux out of the atmosphere at its lower surface. Accordingly, we have

$$H_{lin}(0) + H_s(0) + H_p(0) = H_{lin}(\tau_0) + H_s(\tau_0) + H_p(\tau_0) . \qquad \text{(II-86)}$$

It transpires that this equation is independent of B_1 and that the net flux at any depth is constant and equal to B_1 . This arises from the conservative nature of the problem. The finite plane-parallel atmosphere with a conservative ground and the semi-infinite plane parallel atmosphere both had a constant net flux but that flux was zero. The mean intensity of the thermal radiation is found by using equations (II-78) and (II-79), and is given by

$$J_p(\tau) = B_2 + \frac{3B_1\tau}{n} - \frac{3(1-\tilde{\omega})}{n\,\delta^2}\left[A_1 e^{\delta\tau} + A_2 e^{-\delta\tau}\right] -$$

$$- \frac{3(1-\tilde{\omega})}{n\sigma^2}\mu_0^2\left[(A_3+A_6)e^{\sigma\tau/\mu_0} + (A_4+A_7)e^{-\sigma\tau/\mu_0}\right] . \qquad \text{(II-87)}$$

The two constants, B_1 and B_2, are found by using the two Eddington approximate boundary conditions, equations (II-85). The difference between the forms of the equation for the mean intensity of the thermal radiation field in finite plane-parallel atmospheres with and without a ground is, apart from the presence of the ground radiation terms in the former case, the linear term in the optical depth, which is non-zero in the latter case. The constant, B_1, which multiplies τ in this term is given by

$$B_1 = \frac{(1-\tilde{\omega})}{(\tau_0 + 4n/3)} \left\{ \frac{A_1}{\delta^2} \left[e^{\delta\tau_0}(1+2n\delta/3) - (1-2n\delta/3) \right] + \right.$$

$$+ \frac{A_2}{\delta^2} \left[e^{-\delta\tau_0}(1-2n\delta/3) - (1+2n\delta/3) \right] +$$

$$+ \frac{\mu_0^2 (A_3 + A_6)}{\sigma^2} \left[e^{\sigma\tau/\mu_0}(1+2n\sigma/3\mu_0) - (1-2n\sigma/3\mu_0) \right] +$$

$$\left. + \frac{\mu_0^2 (A_4 + A_7)}{\sigma^2} \left[e^{-\sigma\tau/\mu_0}(1-2n\sigma/3\mu_0) - (1+2n\sigma/3\mu_0) \right] \right\}$$

The constants, A_1, A_3 and A_6 all involve negative exponentials of τ_0 so that, when τ_0 is large, B_1 is small. This is in agreement with the semi-infinite limit in which B_1 equals zero. Clearly, the difference term between the two problems is greatest when n and τ_0 are small, and negligible when n is large, whether τ_0 be large or small.

There are two other special cases where the general theory requires modification. Firstly, there is the case in which the phase function parameters, (α, β) are $(0, 1)$. The solution for the scattered radiation field can be obtained exactly. The downward radiation field is the linear radiation field and the upward stellar radiation field is the reduced visible ground radiation and radiation scattered from it. The first of these is given by

$$I_{lin}(\tau, -\mu) = \pi F e^{-\sigma\tau/\mu_0},$$

where $\sigma = (1-\tilde{\omega})$; and the last by

$$I_s(\tau, +\mu) = G_s \left[e^{-\sigma(\tau_0-\tau)/\mu} - e^{-(\tau_0-\tau)/\mu} \right]. \quad \text{(II-88)}$$

The intensity of the radiation from the ground is given by the flux balance at the ground surface and is

$$G_s = \lambda\mu_0 F e^{-\sigma\tau_0/\mu_0}. \quad \text{(II-89)}$$

The mean intensity of the scattered radiation field is found by integrating equation (II-88) over all directions and substituting the expression for G_s given by equation (II-89). Hence, we obtain

$$J_s(\tau) = \frac{1}{2} F\lambda\mu_0 e^{-\sigma\tau_0/\mu_0} \left[E_2[\sigma(\tau_0-\tau)] - E_2(\tau_0-\tau) \right]. \quad \text{(II-90)}$$

It is this procedure of integrating the intensity to obtain the mean intensity that prevents similar analytical expressions being obtained for other values of β. In such cases, the intensity of the scattered radiation can be found exactly but its complex dependence on μ prohibits its analytical integration. However, the general method copes with the problem quite adequately in those cases. The solution for the mean intensity of the thermal radiation for the special case, $(\alpha, \beta) = (0, 1)$, follows the general method but involves the following equations which are of a different form from the general ones on account of the different form of equation (II-90) for $J_s(\tau)$. The equation of transfer is

$$\mu \frac{dI_p(\tau,\mu)}{d\tau} = \frac{1}{n} I_p(\tau,\mu) - \frac{1}{n} J_p(\tau) - \frac{G_p}{2n} E_2[(\tau_0-\tau)/n]$$

$$- \frac{1}{2}(1-\tilde{\omega}) G_s E_2[\sigma(\tau_0-\tau)] - \frac{1}{4}(1-\tilde{\omega}) F e^{-\sigma\tau/\mu_0} , \qquad \text{(II-91)}$$

from which we obtain

$$H_p(\tau) = -\frac{1}{2} G_p E_3[(\tau_0-\tau)/n] - \frac{1}{2}(1-\tilde{\omega}) G_s E_3[\sigma(\tau_0-\tau)] +$$

$$+ \frac{\mu_0 (1-\tilde{\omega}) F e^{-\sigma\tau/\mu_0}}{4\sigma} + B_1 . \qquad \text{(II-92)}$$

The principle of conservation of energy as expressed by equation (II-80), proves that the constant B_1 must be zero, so that the mean intensity of the thermal radiation field is given by

$$J_p(\tau) = B_2 - \frac{3}{2} G_p E_4[(\tau_0-\tau)/n] - \frac{3 G_s E_4}{2n\sigma}[\sigma(\tau_0-\tau)] - \frac{3\mu_0^2 F e^{-\sigma\tau/\mu_0}}{4n\sigma} , \qquad \text{(II-93)}$$

in which the constants, B_2 and G_p , are found from the two Eddington approximate boundary conditions, equation (II-85).

The final special case for consideration is that in which $\delta^2\mu_0^2 - \sigma^2 = 0$, a condition that produces a singularity in the solutions. The method is unchanged and produces

$$J_s(\tau) = A_1' e^{\delta\tau} + A_2' e^{-\delta\tau} + A_3' \tau e^{\delta\tau} + A_4' \tau e^{-\delta\tau} +$$

$$+ A_5' G_5 \left\{ e^{\delta(\tau_0-\tau)} F_2[-\delta, (\tau_0-\tau)] - e^{-\delta(\tau_0-\tau)} F_2[\delta, (\tau_0-\tau)] \right\}, \quad \text{(II-94)}$$

where $A_3' = -3\tilde{\omega}\alpha\gamma A_6 / 2\delta$,

$$A_4' = 3\tilde{\omega}\alpha\gamma A_7 / 2\delta,$$

and $A_5' = A_5$.

The constants A_1' and A_2' are found from the Eddington approximate boundary conditions; and the solution for the thermal radiation is as it was for the general problem.

We are now in a position to discuss the results of the foregoing analysis. The equations for the mean intensities of the three radiation fields were evaluated by a computer for sets of discrete values of the atmospheric parameters. The depth points used were integral tenths of the total optical thickness of the atmosphere, τ_0, for which values of 0.1, 1.0, 5.0, 10.0 and 50.0 were used. Otherwise, the atmospheric parameters were allocated those values given in Table I for semi-infinite atmospheres. Firstly, we shall consider the scattered radiation field, which for the purpose of this discussion will revert to the true scattered radiation field as defined by equation (II-41) and denoted by $J_{sc}(\tau)$. It will prove valuable to compare the results for the finite atmosphere with those for the semi-infinite atmosphere. Consequently, we shall denote any quantity in a semi-infinite atmosphere by the superscript, ∞.

Fig. 10 shows the mean intensity of the scattered radiation field in a finite plane-parallel atmosphere with no ground, as a function of optical depth, for values of τ_0 of 0.1, 1.0, 5.0, 10.0 and ∞. All curves are for normal incidence and an albedo of 0.9. The continuous curves refer to isotropic scattering and the broken curves to complete back scattering, $(\alpha, \beta) = (0, 0)$. The case of all forward scattering would produce curves that would superimpose upon the curve for isotropic scattering in a semi-infinite atmosphere. Both sets of curves show that $J_{sc}(\tau)$ approaches $J_{sc}^{\infty}(\tau)$ for all values of τ, as τ_0 tends to infinity and that this approach is always from below. In fact, the graph of $J_{sc}(\tau)$ for $\tau_0 = 50$ would be superimposed upon $J_{sc}^{\infty}(\tau)$ to within the available accuracy of the graph. As the thickness of an atmosphere increases, so the loss of radiation at the lower surface decreases, and once the scattered and reduced incident radiation fields are attenuated to negligible quantities the atmosphere is effectively semi-infinite. Moreover, it is evident from Fig. 10 that the difference in the results for a thick finite atmosphere and a semi-infinite atmosphere is smallest at the upper surface and greatest at the lower surface. This is a natural consequence of the source of the difference between the two atmospheres being the lower surface itself. Atmospheres for which τ_0 is less than or equal to unity differ greatly from semi-infinite atmospheres. The reduced incident radiation that leaves the lower surface is large and has not traversed a distance great enough for a substantial scattered radiation field to form. Consequently, the scattered radiation field in thin atmospheres is small at all optical depths.

The effect of anisotropy on the scattered radiation field is shown in Fig. 11. The quantity plotted as ordinate, ψ , is defined as the fraction, $[J_{sc}^{\infty}(\tau_0) - J_{sc}(\tau_0)] / J_{sc}^{\infty}(\tau_0)$ and is a measure of the reduction of $J_{sc}(\tau)$ from its value in a semi-infinite atmosphere at depth τ_0 to its value at the lower surface of a finite atmosphere of the same total optical thickness, τ_0 ; this reduction being due to the truncation of the atmosphere at $\tau = \tau_0$. The effect is greatest at the lower surface, and therefore ψ is defined at the lower surface. In Fig. 11, ψ is plotted as a function of α for various values of β and τ_0 , and for $\tilde{\omega}$ = 0.9 and μ_0 = 1.0. The total effect of anisotropy is given by the combination of Figs. 5 and 11. Clearly, ψ is greatest when the backscattering content of the phase function is greatest. To understand Fig. 11 it is best to consider the scattered radiation to consist of radiation scattered from upward flowing radiation and also from downward flowing radiation. At the lower surface of the atmosphere the former is absent and consequently the scattered radiation field is lower than the scattered radiation field at the same depth in a semi-infinite atmosphere which is derived from both upward and downward flowing radiation fields. Clearly, the phase function that scatters most from the upward flowing radiation field will give rise to the smallest scattered radiation at the lower surface. Such phase functions are those with the smallest values of β, and of those of a particular value of β, those with the smallest value of α. The converse is also true, and in the limit of $(\alpha, \beta) = (0,1)$, ψ takes the value of zero. This is the limiting case of all forward scattering where the scattered radiation is independent of the atmosphere below. Thus, τ_0 has no effect on $J_{sc}(\tau)$ in this case. The curves of Fig. 11 for which β = 0.5 show that ψ is not a unique function of g ; the asymmetry parameter being zero for all values of α when $\beta = 0.5$. A similar point was discussed in the previous subsection. The final point to note from Fig. 11 is the decrease of ψ with τ_0 for all values of the phase function parameters. This was seen in Fig. 10 also but here we observe that the effect of anisotropy is greater for optically thin atmospheres. This can be attributed to the influence of the anisotropy depending on the upward flowing radiation field lost due to truncation of the atmosphere and this loss being greatest for optically thin atmospheres.

The effects of the albedo and the angle of incidence upon the scattered radiation field are essentially the same for finite atmospheres as they were for semi-infinite atmospheres. We may now consider the effects of the inclusion of a ground layer at the lower surface of a finite atmosphere. These are shown in Fig. 12 for isotropic scattering of albedo, 0.9, and normal incidence, $\mu_0 = 1.0$. The presence of the ground introduces an extra radiation field, the reduced visible ground radiation, which also gives rise to scattered radiation. The greater the value of λ, the greater, G_s and thus, the greater the radiation scattered from the reduced visible ground radiation. This is borne out by Fig. 12 for all values of τ_0 , though the effect of the ground is small for very small and very large values of τ_0 . In the former case the reduced visible ground radiation leaves the atmosphere through its upper surface before it is sufficiently attenuated to give rise to a substantial contribution to the scattered radiation field. Thus, the scattered radiation field remains small. However, it is not merely doubled but quadrupled when λ changes from zero to unity. This is due to the isotropic nature of the reflected radiation from the ground. That radiation which is diffusely reflected at a grazing angle to the ground has a large optical distance to traverse before reaching the surface. Consequently, the fraction of the reduced ground visible radiation, whose flux is of the same order of magnitude as the reduced incident flux, that gives rise to scattered radiation is greater than the equivalent fraction of the reduced incident radiation. For large optical thicknesses all the reduced incident and scattered radiation is absorbed before it penetrates to the ground. Consequently, the value of λ does not affect the scattered radiation field at all. To see the true effect of the ground on atmospheres of intermediate thickness we must include the reduced visible ground radiation field with the scattered radiation field. Fig. 13 shows the sum of the mean intensities of these two fields plotted against optical depth for the same set of atmospheric parameters as Fig. 12. It can be seen that this stellar radiation field considerably exceeds $J_{sc}^{\infty}(\tau)$, in particular for optically thin atmospheres. In the semi-infinite case, the atmosphere below a particular depth, τ_0 , reflects a fraction of the downward radiation flux at that depth. The ground at the lower surface of a finite atmosphere of total optical thickness, τ_0 , also reflects a fraction of the downward flux at that depth; this time the fraction is λ. The reflection by the semi-infinite atmosphere depends on the scattering parameters of the atmosphere and the directional distribution of the downward flowing radiation. Nevertheless, it can be said that if the ground albedo, λ, is sufficiently greater than the total albedo of the semi-infinite atmosphere then the stellar radiation in the finite atmosphere will exceed that at the same optical depth in the semi-infinite atmosphere. Again, if λ is sufficiently smaller than the total albedo of the semi-infinite atmosphere, the converse is true. A glance at Fig. 13 shows that a value of λ of about 0.4 gives rise to radiation fields of the same order of magnitude as those in a semi-infinite atmosphere. This is also true for optically thick atmospheres, but deviations from $J_{sc}^{\infty}(\tau)$ are much smaller. We can conclude from Fig. 13 that, as far as the stellar radiation is concerned, an atmosphere of optical thickness greater than twenty may be replaced by a semi-infinite atmosphere, and one of optical thickness greater than ten, if λ is neither close to zero nor unity. These approximate limits would be smaller for smaller scattering albedos and vice versa.

The mean intensity of the thermal radiation field is described graphically in Figs. 14 to 16. This quantity is $J_p(\tau)$ as given by equation (II-84) and hence excludes the reduced thermal ground radiation. Firstly, we shall consider the greenhouse parameter, n, to be large, in fact 10^4, in which case all the terms involving τ in equation (II-84) are negligible unless those containing positive exponentials are sufficiently large to remain significant when divided by n. This condition is only true when both τ_0 and τ are very large. The results for $J_p(\tau)$ when $n = 10^4$ show that $J_p(\tau)$ is independent of τ. Even for a value of τ_0 of 50 the change in $J_p(\tau)$ from the upper to the lower surface is only of the order of a percentage.

Fig. 14 shows $J_p(\tau)$ at $\tau/\tau_0 = 0.5$ plotted as a function of τ_0 for $n = 10^4$, $\tilde{\omega} = 0.9$, $\mu_0 = 1.0$, $\alpha = 0$ and for $\beta = 0.0$, 0.5 and 1.0; and $\lambda = 0.9$, 0.5 and 0.0. The results for the atmosphere with no ground are the same as those for the atmosphere with $\lambda = 0.0$. Two clear conclusions can be drawn. Firstly, for optically thin atmospheres the scattering phase function is unimportant and for a specified value of the albedo for single scattering, the thermal radiation depends only on the ground albedo, λ. Secondly, the converse is true. For optically thick atmospheres, the thermal radiation, $J_p(\tau)$, depends critically on the phase function but is independent of both the ground albedo and the optical thickness of the atmosphere. The emission of the thermal radiation depends on the absorption of five radiation fields; the reduced incident; the reduced visible ground; the scattered; the reduced thermal ground and the thermal, radiation fields. For large values of n, the first three of these sources of thermal radiation are dominant, but of these only the first two are important in optically thin atmospheres. It is only the third that is dependent upon the phase function to any significant extent; so therefore the phase function is irrelevent as far as optically thin atmospheres with large values of n are concerned. The reduced visible ground radiation depends directly on λ and hence, $J_p(\tau)$ does likewise for small values of τ_0 . Although not shown in Fig. 14, both the absolute value of $J_p(\tau)$ and the dependence of $J_p(\tau)$ upon λ are far greater for smaller values of the scattering albedo. In such cases the absorption is greater and yet not sufficiently large to reduce G_s noticeably, because the atmosphere is thin. Hence, $J_p(\tau)$ increases as $\tilde{\omega}$ decreases and λ increases. Furthermore, the limiting optical thickness below which the phase function can be ignored is 0.5 for $\tilde{\omega}$ = 0.1 and λ = 0.5, whereas it is 0.2 for $\tilde{\omega}$ = 0.9 with the same value of λ. This follows from the relative importance of the scattered radiation field, which is the most strongly phase function dependent of the relevant terms in the emission coefficient. The situation is different for optically thick atmospheres. In such atmospheres no stellar radiation reaches the ground, so that the ground behaves merely as a conservative diffuse reflector of the thermal radiation field. Thus, λ is not important for optically thick atmospheres. For a particular value of $\tilde{\omega}$ it can be seen that $J_p(\tau)$ increases with an increase in the value of g , the asymmetry parameter. This result stems from the flux balance condition. As g is increased, so the stellar radiation scattered out of the atmosphere through its upper surface is decreased and the outward thermal flux is increased. The boundary condition requires that $J_p(0)$ increases also, and, as n is large, $J_p(\tau)$ increases at all values of τ when g is increased. Fig. 14 shows this clearly; that for $\tau_0 = 50$ for example, $J_p(\tau)$ is greatest when $\beta = 1.0$ and smallest when $\beta = 0.0$. Now the value of g is directly proportional to $\tilde{\omega}$ but the flux of stellar radiation scattered out of the atmosphere increases drastically with $\tilde{\omega}$. Thus, $J_p(\tau)$ is greatest for small values of $\tilde{\omega}$. The exception to this is the case of $\beta = 1.0$, when no scattered radiation is lost through the upper surface of the atmosphere and all the scattered radiation is absorbed by the atmosphere. The high value of n and the associated independence of $J_p(\tau)$ upon τ cause $J_p(\tau)$ to be independent of $\tilde{\omega}$ also, in this special case. The value of τ_0 greater than which $J_p(\tau)$ is independent of λ is clearly dependent on g. The greater the value of $\tilde{\omega}$ and the greater the value of β, the more penetrating is the stellar radiation. The thermal radiation, $J_p(\tau)$ is only independent of λ in atmospheres where the stellar radiation does not penetrate to the ground, so that for $\beta = 1.0$ and $\tilde{\omega} = 0.9$, $J_p(\tau)$ depends on λ even in atmospheres as thick as $\tau_0 = 50.0$.

The atmosphere with no ground layer deserves special mention.

It is surprising that it gives the same values of $J_\rho(\tau)$ as do those cases with a ground albedo of $\lambda = 0$, because the two atmospheres are completely different with regard to the thermal radiation. When $\lambda = 0$ the ground reflects all the thermal radiation incident upon it and no stellar radiation, but when there is no ground there is no reflection of either radiation field. For large values of n, the atmospheres are optically thin to the infra-red radiation even though they may be optically thick to the stellar radiation. Consequently, the contribution to the diffuse thermal radiation field from absorption of the reduced thermal ground radiation is negligible. The quantity, $J_\rho(\tau)$ does not include the reduced thermal ground radiation field so that we have the result that $J_\rho(\tau)$ is the same for no ground as it is for a ground of $\lambda = 0$, when n is very large.

By comparison of Figs. 8 and 14 we note that the limiting values of $J_\rho(\tau)$ for thick atmospheres in Fig. 14 do not approach the appropriate values of $J_\rho(\tau)$ for the semi-infinite atmospheres, $J_\rho^\infty(\tau)$, but approach a value exactly one half of $J_\rho^\infty(\tau)$. This applies for all values of λ and for the case of no ground also. We are considering the case of $n = 10^4$, for which the atmosphere is very thin to the thermal radiation and $J_\rho(\tau)$ is essentially a constant equal to one half the emergent flux of thermal radiation. For an optically thick finite atmosphere the thermal radiation is generated in the upper ten or twenty units of optical distance. The radiation that is emitted upwards will almost all escape from the upper surface because n is large, and will produce a certain emergent flux. This contribution to the emergent flux will be the same for a semi-infinite atmosphere. The radiation that is emitted downwards will be replaced by either a ground layer or the semi-infinite atmosphere below the point under consideration. In each case the reflected flux is equal to the downward flux due to the conservative nature of both physical systems, and in each case, n being large, this downward flux is equal to the upward flux due to the isotropic nature of the thermal emission process. However for a finite atmosphere with a ground the reflected field is not considered to be part of the thermal radiation field. Thus, the emergent flux of radiation in the thermal part of the spectrum from an optically thick atmosphere of $\tau_0 \simeq 50.0$ and of $n = 10^4$, is the same as that from a semi-infinite atmosphere. However, the emergent thermal radiation flux, $H_p(\tau)$, is one half that from the semi-infinite atmosphere, the other half being the reduced thermal ground radiation. It is this latter flux that escapes from the lower surface of a finite atmosphere with no ground so that the value of $J_p(\tau)$ is the same for optically thick atmospheres of $\tau_0 \simeq 50$ and $n = 10^4$, with and without a ground of albedo, $\lambda = 0$.

When $n = 10^{-2}$, the function, $J_p(\tau)$ is far more complex and is strongly dependent on optical depth. As for semi-infinite atmospheres, the small value for n enables a large thermal radiation field to be maintained away from the upper surface of the atmosphere. The physical principles controlling the thermal radiation field are the same as those controlling the same field in a semi-infinite atmosphere and were discussed in detail in that context. However, the large thermal radiation field cannot be maintained near the lower surface of the atmosphere. This decrease in intensity near the lower boundary, and indeed the whole intensity profile, is shown in Fig. 16. Before discussing this, we shall consider Fig. 15 which is the counterpart of Fig. 14 for

$n = 10^{-2}$. However, the ordinate is now the maximum mean thermal intensity attained in the atmosphere. The values of λ represented are 0.9, 0.5 and 0.1 and the case of no ground is also included as it bears very little relation to that of $\lambda = 0.0$. The similarity between Figs. 14 and 15 is striking. Most of the underlying physics in Fig. 15 is the same as that discussed in connexion with Fig. 14. When the atmosphere is optically thin the scattered radiation field is negligible, the phase function irrelevant but the ground albedo important. When the atmosphere is optically thick, the ground albedo is irrelevant whilst the phase function is important. However, Fig. 15 does differ from Fig. 14 in three ways. The major λ-dependent sources of the diffuse thermal radiation in optically thin atmospheres are the two reduced ground radiation fields of which the thermal field is the more important when n is small because it is entirely converted into diffuse thermal radiation. Hence $J_p(\tau)$ increases as λ increases in this case. Secondly, the case of no ground does not give the same results as the cases with ground layers for the values of τ_0 available but does give results that seem to approach those for the cases with ground layers, as τ_0 increases. Thirdly, the maximum radiation field in a finite atmosphere of $\tau_0 = 50$ with a ground layer is identical to that in a semi-infinite atmosphere at optical depth, $\tau = 50$; whereas for $n = 10^4$ they were exactly half those values. This latter effect for $n = 10^4$ was attributed to the inclusion of one half the total thermal radiation field in $J_p(\tau)$ and calling the other half the reduced thermal ground radiation. It will be seen in Fig. 16 that this still applies when $n = 10^{-2}$ but only very close to the ground, because the reduced thermal ground radiation is attenuated in a very short optical distance from the ground, and that absorbed thermal radiation is propagated conservatively under the classification of $J_p(\tau)$. Fig. 15 shows the maximum of $J_p^{\infty}(\tau)$ which is therefore equal to $J_p(\tau)$. At first sight it would appear that the function, $J_p(\tau)$ for the case of no ground should remain below the functions $J_p(\tau)$ for the cases with a ground, and should do so for all values of τ_0 . For optically thin atmospheres this difference is large but for optically thick atmospheres it is small because the lower boundary does not affect the properties of the upper part of the atmosphere. This becomes clearer when Fig. 16 is considered. It shows the profiles of $J_p(\tau)$ with $J_p(\tau) / J_p^{max}$ as ordinate and τ/τ_0 as abscissa, for the phase function (α, β) = (0.0, 0.5), normal incidence, albedo, $\tilde{\omega}$ = 0.9, for the case of a finite atmosphere with no ground and one with a ground of arbitrary albedo, λ. The profiles are independent of λ. For optically thick atmospheres we have seen that J_p^{max} is equal to $J_p^{\infty}(\infty)$ the constant value which $J_p(\tau)$ attains deep in a semi-infinite atmosphere. It can be seen that $J_p(\tau)$ rises to its maximum in about one fifth of the optical thickness of the atmosphere. Now, we have already noted that a large energy density gradient is necessary to maintain a certain flux through an atmosphere in which the mean free path of the photons is small. This steep gradient is maintained into the atmosphere until the stellar radiation field has been attenuated to a negligible quantity at which point the fluxes of both radiation fields are zero. Then the thermal radiation field remains at this constant value at all points deeper in the atmosphere, that constant value being $J_p^{\infty}(\infty)$ if the atmosphere is semi-infinite. The same is true for the total thermal

thermal radiation in a finite atmosphere when a ground is present. This can be seen in Fig. 16 where the decrease in $J_p(\tau)/J_p^{max}$ at the lower boundary is due to the division of the thermal radiation into the two thermal radiation fields as explained earlier. These effects are clearly independent of λ. Moreover, $J_p(\tau)$ in the no ground case, rises to the same maximum value because this maximum depends on the scattered radiation field only, which for optically thick atmospheres is independent of λ. However, having attained this maximum, $J_p(\tau)$, in the no ground case, decreases steadily towards the lower surface, almost to zero, the gradient being sufficiently large to maintain the small but finite flux that must pass through the atmosphere. This flux, which is determined by $H_s(\tau)$, $H_{Hd}^{inc}(\tau)$ and the principle of conservation of energy, controls the negative gradient of $J_p(\tau)$, and unless τ_0 is one hundred or greater this gradient is not steep enough to attain $J_p^{\infty}(\infty)$ at the depth great enough for this value to have been attained by the mean intensity gradient from the upper surface. Thus $J_p(\tau)$ never attains this value in Fig. 15 for the values of τ_0 chosen. For optically thin atmospheres, the maximum in Fig. 16 is close to the lower, ground surface. These thin atmospheres are still optically thick to the thermal radiation so the fall in $J_p(\tau)$ at $\tau \simeq \tau_0$ is again due to the change in designation of the thermal radiation leaving the ground, but the maximum is much nearer the lower surface due to the balance of this last effect and the steep gradient needed to maintain the thermal flux which is non-zero at all optical depths in these optically thin atmospheres. This balance also produces the result that $J_p(\tau)$ never reaches $J_p^{\infty}(\infty)$. The functions, $J_p(\tau)$ in the optically thin atmospheres with no ground layer are almost symmetrical about the point, τ/τ_0 = 0.5 so that the outward flowing fluxes of thermal radiation are equal and opposite. However, the magnitude of the thermal radiation field in this case is small despite n being small because the atmosphere is so thin that very little stellar radiation is absorbed. Those atmospheres of τ_0 = 1, 5 and 10 are intermediate between the two atmospheres discussed and this is clearly shown in Fig. 16. Were anisotropy amongst the variables of Fig. 16, then it would be seen that the maxima occur nearer the lower surface when g is large and positive, and nearer the upper surface when g is large and negative. This is due to the change in profile of the stellar radiation field, which, of course, is the source of the thermal radiation field.

4. The Temperature Profile

In Section I.5, which was concerned with the problem of radiative heating, the subject of temperature was discussed. It was seen that it is common practice to define the temperature of a particle by the energy balance equation, (I-55). This equation states that the temperature of a particle is defined as the temperature of a black-body which would emit the same amount of radiation as the particle. This is not to say that it is the temperature which a black-body would have in the same situation. The difference between such temperatures were fully discussed in Chapter I. In adopting equation (I-55) as the temperature defining equation we are assuming LTE to hold for the atmosphere. In the problem that we have been discussing the absorption coefficient has been taken to be constant over all appropriate frequencies, which are located in the infra-red region of the spectrum. Consequently, we say that the emission coefficient is

$$j_p = \kappa_p B_p = \kappa_p B(T), \tag{II-96}$$

where B(T) is the integrated Planck function and B_p , the source function of the thermal radiation which we have obtained by solution of the equation of transfer in the previous section. Equation (II-96) is merely an expression of Kirchhoff's law and the definition of the source function. It leads to

$$\sigma T^4/\pi = B_p(\tau). \tag{II-97}$$

Now, we have shown that the source function, $B_p(\tau)$, is proportional to F, the flux in the incident beam, so we therefore have the result that the temperature is proportional to the fourth root of that flux. It is convenient to replace F by the effective temperature of the incident radiation field which is defined by the relation,

$$F = \sigma T_e^4/\pi. \tag{II-98}$$

Thus, we have

$$\frac{B_p(\tau)}{F} = (T/T_e)^4. \tag{II-99}$$

It will be convenient to omit the constant, T_e , so that we shall use the symbol T to refer to the temperature in units of T_e. Accordingly, we have

$$\frac{\sigma}{\pi} T^4 = B_p(\tau) \qquad \text{(II-100)}$$

Thus the temperature is found directly from the results of the previous section.

4.1. Semi-infinite Atmospheres The source function for the thermal radiation field in a semi-infinite plane-parallel atmosphere is given by equation (II-30) and the resulting temperature is given by

$$\frac{\sigma}{\pi} T^4 = J_p(\tau) + n(1-\tilde{\omega})\left[J_s(\tau) + J_{lin}(\tau)\right]. \qquad \text{(II-101)}$$

For large values of n , it is clear that the stellar field is the more important source of heating unless the stellar field is extremely small as is the case deep in the atmosphere. As we saw in the previous section, $J_p(\tau)$ is approximately constant throughout the atmosphere and is of the order of 0.5. Consequently, it is only important in the heating of the atmosphere when the stellar field is very small, of the order of 5×10^{-5} . For small values of n , the converse is true. The thermal radiation field is the more important, and, as we have seen, is very large throughout most of the atmosphere.

Figs. 17 to 19 show the effect of the albedo on the temperature profiles of the atmosphere for $n = 10^4$; 10^{-2} and 1.0 respectively. All three are drawn for isotropic scattering and normal incidence. In Fig. 17, n is large and it is clear that T varies inversely with $\tilde{\omega}$ near the surface and deep within the atmosphere whilst it varies directly with $\tilde{\omega}$ at intermediate values of the optical depth of the order of 5.0. Deep in the atmosphere the only radiation field present is the thermal radiation field which, as we saw in Fig. 7 varies inversely with the albedo. This was discussed in the previous section. The inverse dependence of the temperature upon the albedo near the surface is due to the factor, $(1-\tilde{\omega})$ in equation (II-101). The energy balance equation (I-55) shows that the temperature depends on the ratio of the absorption coefficients in the stellar and thermal parts of the spectrum. This ratio is equal to $n(1-\tilde{\omega})$ with n defined as the ratio of the two extinction coefficients. As the albedo increases so the material becomes a poorer absorber of the stellar radiation and hence the energy content and temperature of the material decrease. The anomalous result for intermediate values of τ is due to the increased penetration of the stellar radiation that accompanies an increase of the albedo. When $\tilde{\omega}$ is greater than 0.9 there is a significant scattered field at $\tau = 10$ whereas when $\tilde{\omega}$ is less than 0.9 there is virtually no scattered radiation at that depth. Clearly this effect of the albedo is the greatest of the three at these optical depths. It can be seen that a maximum exists in each $T(\tau)$ curve for very large values of the albedo. The dominant radiation field is $[J_s(\tau) + J_{in}(\tau)]$ and as this includes the reduced incident radiation field it would be expected that the gradient, $dT(\tau)/d\tau$ would be negative for all values of the albedo. However, when the albedo is very large the scattered radiation field increases more rapidly than the reduced incident radiation field decreases, so that a maximum exists in the sum of the two radiation fields. This arises from the "sideways" scattered radiation that contributes more to the scattered radiation field than that which is lost by absorption. This can only happen when the absorption loss is very small and therefore the albedo high. The depth at which $T(\tau)$ attains is maximum is given by

$$\tau_{max} = \frac{\log[-\sigma E/\mu_0 \epsilon D]}{[\sigma/\mu_0 - \epsilon]} , \qquad \text{(II-102)}$$

where the constants are given in Section II.3.1. For isotropic scattering and normal incidence this reduces to

$$\tau_{max} = \log[2(3+2\epsilon)/5\tilde{\omega}\epsilon]/(1-\epsilon).$$

Hence, a maximum only occurs when the albedo is greater than two thirds. Consequently, we have a maximum at τ_{max} = 1.157 when $\tilde{\omega}$ = 0.99. This is verified in Fig. 17.

Fig. 18 shows the temperature profiles of a similar atmosphere for which $n = 10^{-2}$. In this case $J_p(\tau)$ is large and is the dominant term in equation (II-101) at all depths. Consequently, the albedo dependence of the temperature is the same as that of $J_p(\tau)$ which was displayed in Fig. 7 and discussed in Section II.3.1.

Fig. 19 is somewhat intermediate between the two previous figures, for neither the stellar nor the thermal radiation field is dominant. At small optical depths the decrease of the temperature resulting from an increase in the albedo, is large. We have seen in Fig. 7 that $J_p(\tau)$ decreases as the albedo increases. The opposite is true for the stellar radiation field but that field is multiplied by the factor $(1-\tilde{\omega})$ so that the increase with albedo is overcome and a net decrease results for the temperature profile. Deep in the atmosphere the temperature is independent of the albedo. At such optical depths the albedo dependent scattered radiation field is zero so that the albedo dependence of the temperature is the same as that of the thermal radiation field. When the scattering is isotropic the stellar and thermal photons are transferred in similar ways so that the albedo has no effect on the thermal radiation field. For other phase functions the transference of the two fields is not the same so that the temperature deep in the atmosphere will depend upon the albedo.

The effect of anisotropy as a whole is shown in Fig. 20 which is drawn for $\tilde{\omega} = 0.9$, a value for which the effects of anisotropy are large. The temperature profiles are shown for the three values of n, 10^4, 1.0 and 10^{-2} and for values of β of 0.0, 0.5 and 1.0. Again the values of 0.0 and 1.0 were chosen for α and μ for convenience. Deep in the atmosphere the effect of anisotropy is greatest when n is less than or equal to unity, whilst near the surface it is greatest when n is larger than unity. This arises from the effect of anisotropy on the particular term that is dominant in each solution. Discussion of these points was made in the previous section with reference to each radiation field, and will not be repeated here. It can be seen that, when n is less than or equal to unity, the forward scattering phase function produces far greater temperatures than the other phase functions. The reason is essentially due to the very small loss of scattered radiation through the surface when the scattering is peaked in the forward direction.

In general, we can conclude that the temperature depends strongly on the albedo, particularly near the surface; and strongly on the phase function or its asymmetry parameter, particularly deep in the atmosphere when n is small. However, the value of the greenhouse parameter is the most important factor in determining the temperature of a planetary atmosphere. Once this is established for a particular atmosphere we can see from the above results at which optical depths the albedo and other parameters are their most critical in determining the temperature of the atmosphere.

An interesting situation arises when μ_0 is small and n is equal to unity. Fig. 21 shows the appropriate temperature profiles for a semi-infinite atmosphere in which there is isotropic scattering with an albedo of 0.9. In general, the temperature decreases as μ_0 decreases because the flux of energy entering the atmosphere is proportional to μ_0 and the temperature depends directly on that flux. This is true for all values of n, but when n is equal to unity and μ_0 is of the order of 0.1, a minimum temperature is seen to exist at optical depths of about 0.25. For such low values of μ_0 the incident radiation, on entry into the atmosphere has but to traverse a small vertical depth before it is attenuated to zero. Hence, this radiation heats up the surface layers only. That radiation derived from it can penetrate into the atmosphere more or less as before. The thermal radiation so derived can build up a temperature gradient so that a minimum is observed in the temperature profile. When n is small, the reduced incident radiation field barely contributes to the temperature; and when n is large, the thermal radiation field is equally unimportant. Consequently the minima are observed when n is equal to unity or close to unity, only. Fig. 22 shows the minima in relation to anisotropy for $\tilde{\omega} = 0.9$, $\mu_0 = 0.15$ and again, $n = 1.0$. The larger the value of α, the greater the minimum in the temperature profile. This follows from the fact that, when α is small, the scattered radiation remains closer to the surface. Furthermore, when α is zero and β is less than 0.5, the temperature gradient is always negative. In this case the thermal radiation is generated so close to the surface that a sufficiently large fraction of its escapes to ensure that the thermal radiation field never becomes large enough to establish a positive temperature gradient.

We have seen that, when n is unity and the scattering is isotropic the transfer of the two radiation fields is the same and that the total radiation field is independent of the albedo. Although the total radiation can be treated as one field as far as radiative transfer is concerned, only part of that field is considered to contribute to the temperature and that part is certainly dependent on the albedo. A similar situation can be envisaged for the thermal radiation field with any value of n . Consider a fraction, $\tilde{\omega}_p$, of the absorbed thermal radiation to be emitted isotropically without contributing to the temperature of the absorbing material, and let the remainder, also emitted isotropically, contribute to the temperature. The former part is a scattered part because it does not affect the material during its interaction with the material, whilst the latter part is absorbed and, together with the absorbed stellar radiation, is thermally emitted at a temperature controlled by the relative absorption and emission properties of the material. The parameter, $\tilde{\omega}_p$, is therefore a thermal scattering albedo. In this situation, the source function for the thermal radiation is not equal to the integrated Planck function as in equation (II-96). The intensity of the thermal radiation field at any point in the atmosphere is independent of the value of $\tilde{\omega}_p$ because, as far as the equation of transfer is concerned the two processes of conservative isotropic scattering and absorption plus conservative isotropic emission are identical. This applies to grey atmospheres only, or, at least to atmospheres that are grey in the infra-red. When considering monochromatic radiative transfer the two processes are not equivalent because their respective emission coefficients are completely different functions of frequency. The temperature, however, does depend on the value of $\tilde{\omega}_p$. In the situation where the thermal scattering is anisotropic the thermal radiation field is transferred in a different manner and consequently the mean intensity of the radiation field is dependent on $\tilde{\omega}_p$ and, of course, the phase function involved. However, it is our aim to consider the effect of $\tilde{\omega}_p$ on the temperature profiles. Therefore it will suffice to consider the scattering to be isotropic and thus introduce no new equation of transfer. The equation of transfer for this situation is

$$\mu \frac{dI_p}{dx}(x,\mu) = -(\kappa_p + \sigma_p)\rho\, I_p(x,\mu) + \sigma_p \rho\, J_p(x) + \kappa_p \rho\, J_p(x) + \kappa_s \rho \left[J_s(x) + J_{lim}(x) \right]. \quad \text{(II-103)}$$

The second term on the right-hand side of this equation is the contribution to the source function of the scattered thermal radiation, the third term is that of the absorbed thermal radiation, which is emitted isotropically, and the fourth term is that of the absorbed stellar radiation field, which is transformed into thermal radiation by the absorbing material. The parameter, n is defined as the ratio of the extinction coefficients; that is

$$\eta = (\kappa_s + \sigma_s) \,/\, (\kappa_p + \sigma_p). \quad \text{(II-104)}$$

With this definition of n the equation of transfer becomes

$$\mu \frac{dI_p(\tau,\mu)}{d\tau} = \frac{1}{n} I_p(\tau,\mu) - \frac{1}{n} J_p(\tau) - (1-\tilde{\omega}_s)\left[J_s(\tau) + J_{lin}(\tau)\right], \qquad \text{(II-105)}$$

where $\tilde{\omega}_s$ is the albedo for single scattering of the stellar photons. This equation is identical to that used before and its solution is therefore given by equation (II-37). The temperature, however, is not given by equation (II-101) but by

$$\frac{\sigma}{\pi} T^4 = J_p(\tau) + \frac{n(1-\tilde{\omega}_s)}{(1-\tilde{\omega}_p)}\left[J_s(\tau) + J_{lin}(\tau)\right], \qquad \text{(II-106)}$$

where $\tilde{\omega}_p$ is the albedo for single scattering of the thermal photons and is defined as

$$\tilde{\omega}_p = \sigma_p / (K_p + \sigma_p). \qquad \text{(II-107)}$$

Since all the radiation fields are independent of $\tilde{\omega}_p$, the temperature is influenced by $\tilde{\omega}_p$ only by its presence in the denominator in equation (II-106).

Clearly, the effect of $\tilde{\omega}_p$ is negligible when n is very small. Its effect when n is 10^4 and 1.0 is shown in Figs. 23 and 24 respectively. These show the temperature plotted against optical depth for normal incidence, isotropic scattering of $\tilde{\omega}_s$ = 0.9, and for several values of $\tilde{\omega}_p$. In Fig. 23, for $n = 10^4$, an increase in temperature of 10% is incurred when $\tilde{\omega}_p$ changes from 0.0 to 0.3 but in Fig. 24, for n = 1.0, the equivalent increase is only 1%. These increases are greatest near the surface, because that is where the stellar radiation is most important and $\tilde{\omega}_p$ influencesthe temperature through the stellar radiation field.

At large optical depths the temperature is independent of the value of $\tilde{\omega}_p$. The diagrams show that as $\tilde{\omega}_p$ increases so the gradient $dT/d\tilde{\omega}_p$ increases, and, as can be seen from equation (II-106), the temperature reaches infinity when $\tilde{\omega}_p$ equals unity. This physically implausible situation is, however, never reached. It arises when $\tilde{\omega}_p$ = 1.0 which means that κ_p is zero; and, because the emission coefficient, given by Kirchhoff's law, is $\kappa_p B(T)$ the temperature must be infinite in order to maintain an emission from the material equal to the absorption of the stellar radiation. However, as the temperature increases, so the maximum frequency of the Planck function increases and the situation will arise when this frequency exceeds the lowest frequency of the incident radiation. The model thus breaks down when the temperature increases too much so that the range of $\tilde{\omega}_p$ is limited by the model. The maximum value of $\tilde{\omega}_p$ is still close to unity because the highest temperature calculated was merely 12.0 and that was for $\tilde{\omega}_s$ = 0.1 and $\tilde{\omega}_p$ = 0.9. Such a non-zero value of $\tilde{\omega}_p$ has been considered by Samuelson (1967a) and he has used a maximum value of $\tilde{\omega}_p$ of 0.5 in connexion with planetary atmospheres. For these atmospheres n is not large. Consequently, the effect of $\tilde{\omega}_p$ on the temperature profiles can be neglected. This also justifies the neglection of anisotropy in the treatment of the scattering of the thermal radiation. The additional mathematical computation would have been considerable and yet would have yielded little numerical change in the temperature profiles.

<u>An Iterated Solution for the Temperature Distribution</u>:- It was mentioned in Chapter I that the solutions obtained by the Eddington approximation are amenable to an iteration procedure and that the first of these iterations can usually be performed analytically.

The integral form of the equation of transfer as given by equation (I-7) is, for the semi-infinite plane-parallel atmosphere,

$$I(\tau, +\mu) = \int_{\tau}^{\infty} e^{-(t-\tau)/\mu} B(t) dt/\mu ,$$

(II-108)

$$I(\tau, -\mu) = I(0, -\mu) e^{-\tau/\mu} + \int_{0}^{\tau} e^{-(\tau-t)/\mu} B(t)\, dt/\mu ,$$

and the mean intensity is given by

$$J(\tau) = \frac{1}{2}\int_{-1}^{+1} I(\tau, \mu)\, d\mu . \qquad \text{(II-109)}$$

These equations apply to axially symmetric radiation fields and those whose source functions are isotropic. This is the case for the thermal radiation field in a semi-infinite plane-parallel atmosphere, and it is an iteration on this radiation field that we require. The variable, τ, in equations (II-108) is measured in terms of the extinction coefficient of the radiation field in question. We have been using the symbol, τ, to represent the optical depth measured in terms of the extinction coefficient of the incident radiation field. In order to maintain this convention the equations (II-108) for the thermal radiation field, become

$$I_p(\tau, +\mu) = \int_{\tau}^{\infty} e^{-(t-\tau)/n\mu} B_p(t)\, dt/n\mu ,$$

(II-110)

$$I_p(\tau, -\mu) = I_p(0, -\mu) e^{-\tau/n\mu} + \int_{0}^{\tau} e^{-(\tau-t)/n\mu} B_p(t)\, dt/n\mu .$$

Combining equations (II-110) and (II-109), together with the boundary condition, $I_p(0,-\mu) = 0$, we obtain

$$J_p(\tau) = \frac{1}{2n}\int_0^\infty B_p(t)\,E_1\left[\frac{|\tau-t|}{n}\right]dt. \qquad \text{(II-111)}$$

The source function for the thermal radiation field is given by equation (II-30);

$$B_p(\tau) = J_p(\tau) + n(1-\tilde{\omega})\left[J_s(\tau) + J_{lin}(\tau)\right].$$

Hence
$$B_p(\tau) = n(1-\tilde{\omega})\left[J_s(\tau) + J_{lin}(\tau)\right] + \Lambda_\tau\left\{B_p(t)\right\}, \qquad \text{(II-112)}$$

where
$$\Lambda_\tau\left\{f(t)\right\} = \frac{1}{2n}\int_0^\infty f(t)\,E_1\left[\frac{|\tau-t|}{n}\right]dt, \qquad \text{(II-113)}$$

and $\Lambda_\tau\{\ldots\ldots\}$ is known as the lambda operator. This is not exactly the same operator as the conventional lambda operator discussed in detail by Kourganoff (1952) but it is entirely equivalent to that operator in principle. The difference between the two exists merely in the factor, n , which arises due to the particular choice of the variable, τ . which was utterly absurd.

Equations (II-111) and (II-112) describe an iteration procedure involving the source function. The source function for the thermal radiation field is found by substituting equations (II-13), (II-25) and (II-37) into equation (II-30), and is

$$B_p(t) = Q_1 e^{-\epsilon t} + Q_2 e^{-\sigma t/\mu_0} + Q_3 , \qquad \text{(II-114)}$$

where $\quad Q_1 = (1-\tilde{\omega}) D [n - 3/n\epsilon^2]$,

$\quad Q_2 = (1-\tilde{\omega}) E [n - 3\mu_0^2/n\sigma^2]$,

and $\quad Q_3 = G$.

The application of equation (II-112) involves the exponential integral function, details of which are given in the Appendix. The required result for the iterated source function is

$$B_p^{(1)}(\tau) = n(1-\tilde{\omega}) [J_s(\tau) + J_{lin}(\tau)] +$$

$$+ \tfrac{1}{2} Q_1 e^{-\epsilon\tau} \{ F_1[\epsilon n, \tau/n] + \log[1+\epsilon n]/\epsilon n \} +$$

$$+ \tfrac{1}{2} Q_2 e^{-\sigma\tau/\mu_0} \{ F_1[\sigma n/\mu_0, \tau/n] + \log[1+\sigma n/\mu_0].\mu_0/\sigma n \} +$$

$$+ \tfrac{1}{2} Q_3 [2 - E_3(\tau/n)]. \qquad \text{(II-115)}$$

The final three terms in this equation constitute the new expression for the mean intensity of the thermal radiation field. Consider first the case when n is large. In Section II.3.1 we saw that $J_p^{(0)}(\tau) \approx Q_3$ for large values of n, where the superscript zero denotes those functions belonging to the first solutions as obtained in that section. The new function, $J_p^{(1)}(\tau)$ depends upon the ratio, τ/n, the optical depth measured in terms of the thermal absorption coefficient. When this ratio is large $J_p^{(1)}(\tau)$ tends to Q_3 and the standard property of the lambda operator, that the source function remains unchanged deep in the atmosphere, is still true. For values of this ratio, τ/n, that are small, the final term in equation (II-115) tends to $\frac{1}{2}Q_3$, and $J_p^{(1)}(\tau)$ is one half $J_p^{(0)}(\tau)$. This situation exists when τ is large, even though τ/n may be small. When τ itself is small, the exponential terms in equation (II-115) are no longer small, and it transpires that they are greater than $\frac{1}{2}Q_3$ so that $J_p^{(1)}(\tau)$ is greater than $J_p^{(0)}(\tau)$ for small values of τ . However, when n is large, the temperature is dominated by the stellar radiation field at optical depths down to 10 units so that the temperature at optical depths of less than 10 units is virtually unaffected by the lambda operator. The lambda operator only has effect at depth where the stellar radiation is zero and where the atmosphere above is optically thin to the thermal radiation. The function $J_p^{(1)}(\tau)$ is shown in Fig. 23 for values of τ as large as 10^6 . The regions in which the lambda operator increases or decreases the temperature, as discussed above, are quite clear.

Consider now the case when n is small. For all but small values of τ the following limits are reached: $E_2(\infty) = 0$; $F_1[0,\infty] = 1$; and $\log[1+\epsilon n]/\epsilon n$ tends to unity. Hence, $J_p^{(1)}(\tau)$ tends to $J_p^{(0)}(\tau)$ for all values of τ provided that the ratio, τ/n , is large. For small values of τ/n, the term, $E_2(\tau/n)$, tends to unity and $J_p^{(1)}(\tau)$ is reduced to a value less than $J_p^{(0)}(\tau)$ A different situation arises for intermediate values of τ ; that is, for those values of τ that yield the ratio, τ/n , approximately unity. The F_n-functions are no longer at their limiting values of unity and the result is that $J_p^{(1)}(\tau)$ exceeds $J_p^{(0)}(\tau)$ for these values of τ . The depression of $J_p(0)$ is of the order of 4 or 5% and is the point at which the lambda operator has the greatest effect. Consequently, the temperature profile is altered only slightly by one lambda operation, and the results are not of sufficient magnitude to warrant their graphical representation.

The lambda operator affects the temperature profile for the case in which n equals unity in the same manner as that for the case in which n is small but the effect extends to greater optical depths. The results of a lambda operation on such an atmosphere are shown in Fig. 24. As in Fig. 23 this is for $\mu_0 = 1.0$, $\tilde{\omega} = 0.9$, and isotropic scattering. In general we conclude that the lambda operator takes effect within an optical depth of one or two units measured in terms of the infra-red absorption coefficient, and that this effect has most effect on the thermal radiation field when n is large but most effect on the temperature profiles when n is unity.

4.2. <u>Finite Atmospheres</u> The temperature distribution is given by the fourth root of the source function for the thermal radiation which, in the case of a finite plane-parallel atmosphere with a ground at the lower surface is,

$$T^4 = \mathcal{J}_p(\tau) + \tfrac{1}{2} G_p E_2[(\tau_0 - \tau)/n] + $$

$$+ n(1-\tilde{\omega})[\mathcal{J}_s(\tau) + \mathcal{J}_{lim}(\tau) + \tfrac{1}{2} G_s E_2(\tau_0 - \tau)]. \qquad \text{(II-116)}$$

Again, it is evident that the stellar radiation fields are dominant when n is large and the thermal radiation fields when n is small.

In Fig. 25 the temperature profiles of an atmosphere of n $= 10^4$ under normal illumination are plotted as a function of fractional optical depth for the case of isotropic scattering of albedo, 0.9. The temperature profiles are drawn for each of the standard total optical thicknesses and for values of λ of 0.1, 0.5 and 0.9, plus the no ground case. Most of the features of these graphs have been discussed in relation to the individual radiation fields of which the source function is comprised. The optically thick atmospheres are unaffected by λ because no stellar radiation can penetrate to the ground; but it is only for these that the atmosphere with no ground is significantly different from that with a ground of $\lambda = 0$. In this case, when there is no ground there is no thermal radiation reflected at the lower surface, and because n is large and the stellar radiation field is zero, the omission of this contribution to the heating of the atmosphere is important. As the total optical thickness of the atmosphere increases the amount of stellar radiation reaching the ground increases so that the effect of λ becomes increasingly more important. For example, when $\tau_0 = 5.0$ and $\lambda = 0.9$ the atmosphere close to the ground is hotter than that a small distance farther from the ground. When the atmosphere is optically thin the reduced visible ground radiation is far larger than the scattered radiation and consequently the temperature of the atmosphere, which is virtually isothermal, is controlled by λ rather than τ_0. For smaller values of the scattering albedo the general effects are the same but the atmospheres are much hotter, especially near the upper surface, due to the factor $(1 - \tilde{\omega})$ in equation (II-116). For anisotropic scattering the changes in the temperature profiles are even simpler. A forward throwing phase function effectively makes the atmosphere optically thinner so that when $(\alpha, \beta) = (0, 1)$ the temperature profiles for atmospheres of of 50.0 and 5.0 are very similar to those of isotropically scattering atmospheres of τ_0 of 10.0 and 1.0 respectively.

Fig. 26 shows the equivalent results as those of Fig. 25 for an atmosphere with $n = 10^{-2}$. However, the values of λ chosen for this diagram are 0.1 and 1.0. In general, the temperature of the atmosphere increases towards the lower surface showing the existence of the greenhouse effect. Again, the ground parameter, λ, does not affect those atmospheres sufficiently thick to prevent the stellar radiation field reaching the ground surface. However, the temperatures of thin atmospheres are not dominated by λ only as they were in Fig. 25 but by λ and τ_0 . This happens because, even though the atmosphere may be optically thin to the stellar radiation it is very thick to the thermal radiation and this thickness controls the thermal flux through its control of the scattered flux and the principle of conservation of energy. The temperature profiles of the atmospheres with no ground layers now differ considerably for those with a ground layer, and do so for all optical thicknesses. When $n = 10^{-2}$ the temperature is essentially the fourth root of the sum of the mean intensities of the two thermal radiation fields. Consequently the main features of Fig. 26 are the same as those of the thermal radiation fields discussed in Section II.3.2.

The intermediate case of $n = 1.0$ is shown in Fig. 27, which is similar to Figs. 25 and 26 but shows curves for $\lambda = 0.1$ only. There is, for each value of τ_0 , a positive temperature gradient indicating a greenhouse effect, but in absolute terms the temperature deviates only a small amount from unity. The greenhouse effect is thus very much smaller than it was when n was 10^{-2} . The most interesting feature of this set of graphs is the behaviour of the temperature near the ground surface. Two effects contribute to this behaviour, the first of which occurs when n is 10^{4} also and therefore, is a feature of the stellar radiation field. The stellar radiation diffusely reflected from the ground is reflected isotropically whereas stellar radiation incident on the ground, in particular, the reduced incident radiation, is incident at angles close to the normal to the ground. Were the reflection similar to that of a mirror, the mean intensity of the stellar radiation would not be expected to rise near the ground, but the isotropic diffuse reflection causes a relative trapping or localization of part of the reflected radiation in those layers of the atmosphere close to the ground. Consequently the temperature rises near the ground when n is such that the stellar radiation contributes to the temperature and when the atmosphere is sufficiently thin to allow the stellar radiation to reach the ground. However, this behaviour is modified by another effect which is the cause of the minimum in the curve for $\tau_0 = 50$. The mean intensity of the thermal radiation field for large values of τ_0 and τ as given by equation (II-84) is approximately

$$B_2 - \frac{3}{2} G_p E_4 [(\tau_0 - \tau)/n] .$$

Consequently, the source function, as given by equation (II-75), is

$$B_p(\tau) = B_2 - \frac{3}{2} G_p E_4 [(\tau_0 - \tau)/n] + \frac{1}{2} G_p E_2[(\tau_0 - \tau)/n]. \quad \text{(II-117)}$$

When $n = 10^4$, even with $\tau_0 = 50$, the optical distance, $(\tau_0 - \tau)/n$ is very small. Consequently, $B_p(\tau) \approx B_2$ and we observe no minimum in the temperature profile. When $n = 10^{-2}$, the optical distance, $(\tau_0 - \tau)/n$ is sufficiently large to render both exponential integrals zero, so that, again $B_p(\tau) \approx B_2$; but now this is for intermediate and small values of τ only. However, when $n = 1.0$, the exponential integrals are not negligible, except for optically thin atmospheres where (II-117) is not valid. Thus we obtain a minimum in the temperature profiles of optically thick atmospheres at an optical distance of unity, or a little less, above the ground. This minimum is a standard feature of the Eddington approximation for certain conservative problems. In the simple case of a conservative scattering semi-infinite plane-parallel atmosphere illuminated from above by isotropic radiation, the solution for the total radiation field, in Eddington's approximation is

$$J_T(\tau) = B - \frac{1}{4} E_2(\tau)\left[3E_4(\tau)/E_2(\tau) - 1\right].$$

The ratio, $E_4(\tau)/E_2(\tau)$, extends from one third, when τ is zero, to unity when τ is infinity. Consequently, $J_T(\tau)$ is equal to B when τ is zero, and infinity, but less than B, when τ is some intermediate value. The same atmosphere when illuminated by parallel radiation of net flux, πF , yields

$$J_T(\tau) = B - \frac{1}{4} e^{-\tau/\mu_0} (3\mu_0^2 - 1),$$

for the mean intensity of the total radiation field in Eddington's approximation. The gradient of this is positive or negative according to whether μ_0 is less than or greater than $1/\sqrt{3}$. The minimum for isotropic incidence is clearly the integral effects of many such parallel beams. As can be seen from Fig. 27 these minima are combined with the other ground effect when τ_0 is less than 50.0. These surface effects are similar for all phase functions. However, when the scattering is all forward an optical thickness is not sufficient to prevent the stellar radiation reaching the ground. Consequently, the minimum is not noticeable in this case.

The presence of a ground layer emitting radiation thermally introduces the concept of a ground temperature. This matter has been discussed by Milne (1930) who has shown that a semi-infinite isothermal body behaves as a black body for each frequency, ν , for which κ_ν , its absorption coefficient, is not zero, provided that scattering is neglected. This result assumes LTE to hold within the body, an assumption that is reasonable for a planetary surface. The restriction of there being no scattering applies to transfer of the radiation within the body. The emission governed by G_s in our problem is not of this category but is a surface reflection of part of the stellar radiation incident upon it. This reflected light plays no part in determining the temperature of the ground. Only the absorbed radiation affects the ground temperature and we have not considered any scattering of the thermal radiation field in the finite atmosphere problem. Hence, we allocate a temperature, T_g , to the ground such that

$$T_g^{\,4} = G_p / F . \tag{II-118}$$

This temperature is shown plotted as a function of τ in Fig. 28 for isotropic scattering, $\tilde{\omega}$ = 0.9, μ_0 = 1.0, λ = 0.1, and $n = 10^4$, 1.0 and 10^{-2} . The ground temperatures are shown as continuous lines and the surface temperatures of the atmospheres in contact with the ground are shown as broken lines. For most values of n and τ there is a temperature discontinuity between the ground and the atmosphere with which it is in contact. In real atmospheres local conduction effects will tend to smooth out these discontinuities However, no discontinuities exist for optically thick atmospheres because no stellar radiation reaches the ground. The source function for optically thick atmospheres is given by equation (II-117) and gives

$$B_p(\tau_0) = T^4(\tau_0) = B_2 .$$

The constant B_2 is found from equations (II-79), (II-84) and (II-85) appropriately adjusted for this special case, and equals G_p / F. By comparison with equation (II-118) we conclude that $T_g = T(\tau_0)$ for optically thick atmospheres. The reason behind this equality lies in the isotropic nature of the emitted radiation and the assumed isotropic nature of the thermal radiation field used in the Eddington boundary condition. This must apply to the thermal radiation field for all values of n. Consequently, there is a close agreement between T_g and $T(\tau_0)$ for all values of τ_0 when n is 10^{-2}, a value for which the stellar radiation field does not affect the temperature to any extent. Also, we see that $T(\tau_0)$ is far greater than T_g when n is 10^4 and the atmosphere is optically thin. When n is large the infra-red absorption coefficient is very small, so that a small amount of absorbed visible radiation produces a high temperature. The ratio of the absorption coefficients in this case is $n(1-\tilde{\omega})$ However, the ground absorbs a fraction, $(1 - \lambda)$, of the incident visible radiation and all the incident thermal radiation, so that the ratio between its absorption coefficients is $(1 - \lambda)$. The fact that it is a good absorber and emitter of thermal radiation means that T_g cannot rise in the same way as $T(\tau_0)$ does. We conclude that a ground layer behaves in a similar manner to a semi-infinite atmosphere whose greenhouse parameter is unity and whose albedo for single scattering is λ. The similarity is not an exact relation because the semi-infinite atmosphere only reflects isotropically under certain conditions which are never prevalent at the lower surface of a finite atmosphere. Results similar to those given in Fig. 28 are available for all values of $\tilde{\omega}$ and λ. Since it is the values of $n(1-\tilde{\omega})$ and $(1 - \lambda)$ that determine the temperatures of the atmosphere and the ground respectively, the values of T_g and $T(\tau)$ are very close when n is small and when $\tilde{\omega} = \lambda$. It is for this reason that Fig. 28 was drawn for $\lambda = 0.1$ and $\tilde{\omega} = 0.9$, which was the case that would show up the differences between $T(\tau)$ and T_g to the greatest extent.

An Iterated Solution for the Temperature Distribution:- The lambda operator may also be used to iterate the solutions for the source function for the thermal radiation field in a finite plane-parallel atmosphere in a manner analogous to that used in Section II.4.1 for semi-infinite atmospheres. The lambda operator has a different form but is constructed in the same way as that for semi-infinite atmospheres. The formal solution of the equation of transfer for thermal radiation in a finite plane-parallel atmosphere is given by equation (I-7) suitably adapted, and is

$$I_p(\tau,\mu) = \int_{\tau}^{\tau_0} B_p(t)\, e^{-(t-\tau)/n\mu}\, dt/n\mu ,$$

$$I_p(\tau,-\mu) = \int_{0}^{\tau} B_p(t)\, e^{-(\tau-t)/n\mu}\, dt/n\mu . \qquad \text{(II-119)}$$

Substituting these expressions for the intensity in the defining equation for the mean intensity, we obtain

$$J_p(\tau) = \frac{1}{2n}\int_0^{\tau_0} B_p(t)\, E_1[\,|\tau-t|/n\,]\, dt. \qquad \text{(II-120)}$$

$$\Lambda_\tau\{f(t)\} = \frac{1}{\tau}\int_0^\tau f(t)\,\Sigma_1\left[|\tau - t|\right] dt$$

The source function for the thermal radiation in a finite plane-parallel atmosphere with a ground layer is given by equation (II-75), and is

$$B_p(t) = J_p(t) + \tfrac{1}{2} G_p E_2 [(\tau_0 - t)/n] +$$

$$+ n(1-\tilde{\omega}) \left[J_s(t) + J_{lin}(t) + \tfrac{1}{2} G_s E_2 (\tau_0 - t) \right] , \qquad \text{(II-121)}$$

so that we obtain the iteration scheme,

$$B_p(\tau) = \Lambda_\tau \{ B_p(t) \} + \tfrac{1}{2} G_p E_2 [(\tau_0 - t)/n] +$$

$$+ n(1-\tilde{\omega}) \left[J_s(\tau) + J_{lin}(\tau) + \tfrac{1}{2} G_s E_2 (\tau_0 - \tau) \right] , \qquad \text{(II-122)}$$

where $\Lambda_\tau \{ f(t) \} = \frac{1}{2n} \int_0^{\tau_0} f(t) E_1 \left[|\tau - t|/n \right] dt .$

This is the lambda operator for finite plane-parallel atmospheres and is a truncated form of the lambda operator for semi-infinite atmospheres given by equation (II-113).

The source function is a function of optical depth of the form

$$B_p(t) = Q_1 e^{\delta t} + Q_2 e^{-\delta t} + Q_3 e^{\sigma t/\mu_0} + Q_4 e^{-\sigma t/\mu_0} +$$

$$+ Q_5 E_2(\tau_0 - t) + Q_6 E_4(\tau_0 - t) + Q_7 E_2[(\tau_0 - t)/n] +$$

$$+ Q_9 \left\{ e^{\delta(\tau_0 - t)} F_2[-\delta, (\tau_0 - t)] - e^{-\delta(\tau_0 - t)} F_2[\delta, (\tau_0 - t)] \right\} +$$

$$+ Q_8 E_4[(\tau_0 - t)/n] + Q_{10} t + Q_{11} ,$$

where the constants, Q_i can be found in Section II.3.2. for the cases of finite plane-parallel atmospheres with and without ground layers. The substitution of this source function into equation (II-122) leads to the integrals of products of exponential integral functions. Some of these have been tabulated and are known functions of ordinary exponential integrals but others have not and must be evaluated numerically. However, in the special case of a finite plane-parallel atmosphere without a ground surface, the exponential integral functions vanish and the source function,

$$B_p(t) = Q_1 e^{\delta t} + Q_2 e^{-\delta t} + Q_3 e^{\sigma t/\mu_0} + Q_4 e^{-\sigma t/\mu_0} + Q_5 t + Q_6 , \quad \text{(II-123)}$$

yields an analytical solution for equation (II-122). The iterated mean intensity of the thermal radiation field is thus

$$J_p(\tau) = \tfrac{1}{2} Q_1 e^{\delta \tau} \left\{ F_1[-n\delta, \tau/n] + F_1[n\delta, (\tau_0 - \tau)/n] \right\} +$$

$$+ \tfrac{1}{2} Q_2 e^{-\delta \tau} \left\{ F_1[n\delta, \tau/n] + F_1[-n\delta, (\tau_0 - \tau)/n] \right\} +$$

$$+ \frac{1}{2} Q_3 e^{\sigma\tau/\mu_0} \left\{ F_1[-n\sigma/\mu_0, \tau/n] + F_1[n\sigma/\mu_0, (\tau_0-\tau)/n] \right\} +$$

$$+ \frac{1}{2} Q_4 e^{-\sigma\tau/\mu_0} \left\{ F_1[n\sigma/\mu_0, \tau/n] + F_1[-n\sigma/\mu_0, (\tau_0-\tau)/n] \right\} +$$

$$+ \frac{1}{2} Q_5 \left\{ 2\tau - \tau_0 E_2[(\tau_0-\tau)/n] + n E_3[\tau/n] - n E_3[(\tau_0-\tau)/n] \right\} +$$

$$+ \frac{1}{2} Q_6 \left\{ 2 - E_2(\tau/n) - E_2[(\tau_0-\tau)/n] \right\}. \qquad \text{(II-124)}$$

The effect of a lambda operation on the temperature profile of a finite atmosphere is very similar to that of a lambda operation on those of a semi-infinite atmosphere. When n is small the effects are small and are localized in the regions of the atmosphere very close to its upper surface. When n is large the effects on the thermal radiation field are to increase it substantially unless τ/n is large. However, when n is large the thermal radiation field has little effect on the temperature profiles unless τ and τ_0 are large. In such cases the temperature is increased considerably. For example, when τ_0 = 50 the temperature descends from 4.6 to 0.8 instead of 0.6. Thus, we conclude that the lambda operator changes the temperature very little.

However, it is well known, from the theory of the lambda operator as given in detail by Kourganoff (1952), that the lambda operator reduces the function on which it is operating, at the origin but leaves it unchanged at higher values of τ . The preceding results would appear to contradict this. In fact they do not because we are not comparing the resultant function of the lambda operation with the initial function whether we consider the mean intensity of the thermal radiation or the source function. The original source function is

$$B_p^{(0)}(t) = J_p(t) + n(1-\tilde{\omega})\left[J_s(t) + J_{lin}(t)\right],$$

and the iterated source function is

$$B_p^{(1)}(t) = \Lambda_t\left\{J_p(t)\right\} + n(1-\tilde{\omega})\,\Lambda_t\left\{J_s(t) + J_{lin}(t)\right\} +$$

$$+ \; n(1-\tilde{\omega})\left[J_s(t) + J_{lin}(t)\right].$$

The final term in the iterated source function is added to the result of the lambda operation so that, when n is large the iterated source function is greater than the original source function. Similarly, the second term is added to the result of the iteration for the thermal radiation mean intensity so that this too is increased even though $\Lambda_t\{J_p(t)\}$ is less than $J_p(t)$ There is a slight reduction in the source function after a lambda operation when n is very small because the decrease in $J_p(t)$ to $\Lambda_t\{J_p(t)\}$ is greater than the two final terms of the last equation. Thus the effect of the lambda operator is not always that of reducing a function near the origin.

5. The Emergent Radiation

The theory of Sections II.3 and II.4 has produced approximate results for the temperatures and mean intensities of the radiation fields within plane-parallel atmospheres. Whilst there is no simple exact theory for these solutions, there is an exact theory available for calculating the emergent radiation from plane-parallel atmospheres. We shall discuss this theory and then compare its results with similar results that will be obtained from our approximate source functions. This will prove to be a very useful check on the accuracy of the approximate method.

5.1. The Exact Solution It will be seen that the intensity of the emergent radiation from a plane-parallel atmosphere can be expressed exactly in terms of certain functions that are solutions of several non-linear integral equations. Whilst it is true to say that the appropriate functions for semi-infinite atmospheres are the limiting cases of those for finite atmospheres, it will prove instructive to consider the semi-infinite atmospheres separately.

5.1.1. Semi-infinite Atmospheres The intensity of the radiation diffusely reflected from a semi-infinite plane-parallel scattering atmosphere illuminated by a parallel beam of radiation has been derived exactly by Chandrasekhar (1960) using a method based on the principles of invariance. For the problem of diffuse reflection, the principle of invariance is stated as follows:- "The law of diffuse reflection by a semi-infinite plane-parallel atmosphere must be invariant to the addition (or subtraction) of layers of arbitrary thickness to (or from) the atmosphere". This principle was first formulated by Ambartzumian (1943) and can be expressed mathematically in terms of a pre-defined reflection function. For a semi-infinite plane-parallel atmosphere illuminated by a parallel beam of radiation of net flux, πF per unit area normal to itself, which is incident on the surface of the atmosphere in a direction, $(-\mu_0, \phi_0)$, the intensity of the radiation reflected from the atmosphere in a directior (μ, ϕ) is given by

$$I(0, \mu, \phi) = \frac{F}{4\mu} . S(\mu, \phi; \mu_0, \phi_0). \qquad \text{(II-125)}$$

This is the definition of the reflection function, $S(\mu, \phi; \mu_0, \phi_0)$ where the direction co-ordinates are the same as those used earlier in the Chapter. For isotropic scattering, the reflection function, S, can be shown to be given by

$$\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) S(\mu, \phi, \mu_0, \phi_0) = \tilde{\omega} H(\mu) H(\mu_0), \tag{II-126}$$

where $\tilde{\omega}$ is the albedo and $H(\mu)$ is a known function defined by the integral equation

$$H(\mu) = 1 + \frac{1}{2}\tilde{\omega}\mu H(\mu) \int_0^1 \frac{H(\mu')\,d\mu'}{\mu + \mu'} \;. \tag{II-127}$$

The azimuthal co-ordinates have been omitted from equation (II-126) because the reflection function is independent of azimuth when the scattering is isotropic. The theory has also been done for anisotropic scattering by Chandrasekhar (1960) who has used several simple phase functions and a general phase function in the form of a series of Legendre polynomials. The resulting integral equations are far more complex and it would be easier to treat anisotropy by applying the similarity relations of Van de Hulst and Grossmann (1968) to the solutions for isotropic scattering. However these would not render exact solutions though they would be very reliable as proved by Hansen (1969b).

The exact solution for the emergent intensity of the thermal radiation produced by absorption of the stellar radiation is a more complex problem. For the special case in which scattering is absent, the problem has been solved by Stibbs (1971). This was based on the model for the classical greenhouse effect involving two grey absorption coefficients as we have been using. The emergent thermal radiation is given in terms of a reflection function in a manner analogous to that used for the scattering atmosphere. This reflection function, which is independent of azimuth because the thermal emission is isotropic, is defined by

$$I_p(0, \mu) = \frac{F_s}{4\mu} R(\mu, \mu_0), \tag{II-128}$$

where πF_s is the net flux of the illuminating beam of dilute stellar radiation and is shown to be

$$R(\mu, \mu_0) = \frac{\mu \mu_0}{\mu + \mu_0/n} H(\mu) H(\mu_0/n), \tag{II-129}$$

where n is the greenhouse parameter.

Attempts to find the emergent thermal radiation from such an atmosphere when there is scattering of the stellar radiation as well as thermal radiation generation have been made but have yielded an integral equation that was not soluble in terms of known functions. However, solutions of the problem are available in certain special cases. Firstly, when n is equal to unity and the scattering is isotropic, the thermal radiation distributes itself throughout the atmosphere in exactly the same way as does the scattered radiation. Thus, the total radiation emerging from the surface of the atmosphere is identical to that reflected from a conservative isotropic scattering atmosphere. The intensity of the emergent radiation is the difference between the intensities of the emergent total radiation and the emergent scattered radiation. Writing the H-functions as functions of direction and albedo, we have

$$I_p(0, \mu) = \frac{F_s \mu_0}{\mu + \mu_0} \left[H(\mu, 1) H(\mu_0, 1) - H(\mu, \tilde{\omega}) H(\mu_0, \tilde{\omega}) \right], \tag{II-130}$$

for the intensity of the emergent thermal radiation from an atmosphere with isotropic scattering of albedo, $\tilde{\omega}$ and greenhouse parameter, unity.

There is a second special case for which the problem with scattering can be solved and that is the case of linear scattering, the case for which the parameter α of the schematic phase function, equation (I-29), is zero. For the remainder of this section we shall express all optical depths as τ_p , the optical depth measured in terms of κ_p . The linear radiation field in a semi-infinite atmosphere is given by equations (II-11) to (II-13), which expressed in terms of τ_p are

$$I_{lin}(\tau_p, -\mu_0) = \pi F_s e^{-n\sigma\tau_p/\mu_0},$$

$$I_{lin}(\tau_p, +\mu_0) = \pi F_s \eta e^{-n\sigma\tau_p/\mu_0}, \qquad \text{(II-131)}$$

and $$J_{lin}(\tau) = \frac{1}{4} F_s \psi e^{-n\sigma\tau_p/\mu_0},$$

where $$\eta = [1-\tilde{\omega}\beta-\sigma] / \tilde{\omega}(1-\beta),$$

$$\psi = [1-\tilde{\omega}(2\beta-1)-\sigma] / \tilde{\omega}(1-\beta), \qquad \text{(II-132)}$$

and $$\sigma^2 = (1-\tilde{\omega})[1-\tilde{\omega}(2\beta-1)].$$

The equation of transfer for the thermal radiation field is given by equation (II-31), which, expressed in terms of the variable, τ_p is

$$\mu \frac{dI_p(\tau_p,\mu)}{d\tau_p} = I_p(\tau_p,\mu) - J_p(\tau_p) - n(1-\tilde{\omega})J_{lin}(\tau_p). \qquad \text{(II-133)}$$

There are two components of the linear radiation field in this problem, one upward component and one downward component, whereas there was no upward component when scattering was absent. Nevertheless, the principle of invariance is expressed mathematically in terms of the downward component and is unaffected by the upward component. The mathematical expression of the principle of invariance and the ensuing algebra proceeds by a method analogous to that used by Stibbs (1971). We therefore have

$$I_p(\tau_{p_1} + \mu) = \frac{F_s}{4\mu} e^{-n\sigma\tau_p/\mu_0} R(\mu, \mu_0) + $$

$$+ \frac{1}{2\mu}\int_0^1 I_p(\tau_p, -\mu') S(\mu, \mu') d\mu' , \qquad \text{(II-134)}$$

for the principle of invariance, and

$$\left(\frac{1}{\mu} + \frac{n\sigma}{\mu_0}\right) R(\mu, \mu_0) = \left\{ n\gamma(1-\tilde{\omega}) + \int_0^1 R(\mu', \mu_0) d\mu'/\mu' \right\} \times$$

$$\times \left\{ 1 + \int_0^1 S(\mu, \mu') d\mu'/\mu' \right\} , \qquad \text{(II-135)}$$

as the resulting expression for the reflection function, $R(\mu, \mu_0)$. Let

$$T(\mu_0) = n\gamma(1-\tilde{\omega}) + \int_0^1 R(\mu', \mu_0) d\mu'/\mu' ,$$

so that
$$R(\mu, \mu_0) = \frac{\mu(\mu_0/n\sigma)}{\mu + (\mu_0/n\sigma)} H(\mu) T(\mu_0) , \qquad \text{(II-136)}$$

where $H(\mu)$ is the H-function for conservative isotropic scattering.

Hence, $$T(\mu_0) = n\psi(1-\tilde{\omega}) + \frac{1}{2}\left(\frac{\mu_0}{n\sigma}\right) T(\mu_0) \int_0^1 \frac{H(\mu')\,d\mu'}{\mu' + (\mu_0/n\sigma)} ,$$

or, $$\frac{n\psi(1-\tilde{\omega})}{T(\mu_0)} = 1 - \frac{1}{2}\left(\frac{\mu_0}{n\sigma}\right) \int_0^1 \frac{H(\mu')\,d\mu'}{\mu' + (\mu_0/n\sigma)} . \qquad \text{(II-137)}$$

Now $$\frac{1}{H(\mu)} = 1 - \frac{1}{2}\mu \int_0^1 \frac{H(\mu')\,d\mu'}{\mu' + \mu} .$$

Consequently, $$\frac{1}{H(\mu_0/n\sigma)} = 1 - \frac{1}{2}\left(\frac{\mu_0}{n\sigma}\right) \int_0^1 \frac{H(\mu')\,d\mu'}{\mu' + (\mu_0/n\sigma)} . \qquad \text{(II-138)}$$

Equating the right-hand sides of equations (II-137) and (II-138), we obtain

$$T(\mu_0) = n\psi(1-\tilde{\omega})\,H(\mu_0/n\sigma) .$$

Hence $$R(\mu, \mu_0) = \frac{n\psi(1-\tilde{\omega})\,\mu\mu_0}{n\sigma\mu + \mu_0} H(\mu) H(\mu_0/n\sigma) , \qquad \text{(II-139)}$$

and $$I_p(0, \mu) = \frac{1}{4} F_s\, n\psi(1-\tilde{\omega}) \frac{\mu_0}{n\sigma\mu + \mu_0} H(\mu) H(\mu_0/n\sigma) . \qquad \text{(II-140)}$$

Thus, we have the exact solution for the emergent thermal radiation from an atmosphere with non-conservative linear scattering. This is not the most general solution but it does give solutions for the full range of values of the asymmetry parameter, g, through the relations

$$\sigma^2 = (1-\tilde{\omega})(1-g) \quad ; \quad g = \tilde{\omega}\,(2\beta - 1) .$$

5.1.2. Finite Atmospheres The method of obtaining the exact solution for the emergent radiation from a finite plane-parallel atmosphere is similar to that used for semi-infinite atmospheres but it is far more complex. It is, of course, necessary to evaluate the radiation diffusely transmitted through a finite atmosphere as well as that reflected by it. As with the theory of semi-infinite atmospheres, this theory relies heavily on the work of Chandrasekhar (1960). We have only sketched the theory for semi-infinite atmospheres and shall do the same for the theory of the emergent scattered radiation from finite atmospheres. However, we shall give the complete theory of the emergent thermal radiation from finite atmospheres because all these theories are essentially the same, and that of the emergent thermal radiation from finite atmospheres is the most complex and is not available in the literature. Having solved the problem for a finite plane-parallel atmosphere we shall find it relatively easy to add a ground layer at the lower surface of the atmosphere. Firstly, however, we shall give an outline of the functions involved in the theory of the emergent scattered radiation from a finite plane-parallel atmosphere.

Reflection and transmission functions analogous to that of equation (II-125) are defined and four equations that embody the principle of invariance are constructed. These together with the equation of transfer give four integral equations for the reflection and transmission functions. For isotropic scattering these render the following solution:

$$\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) S(\tau_0; \mu, \mu_0) = \tilde{\omega}\left[X(\mu)X(\mu_0) - Y(\mu)Y(\mu_0)\right] \quad \text{(II-141)}$$

and $$\left(\frac{1}{\mu} - \frac{1}{\mu_0}\right) T(\tau_0; \mu, \mu_0) = \tilde{\omega}\left[Y(\mu)X(\mu_0) - X(\mu)Y(\mu_0)\right],$$

where $X(\mu)$ and $Y(\mu)$ are defined by a pair of integral equations:

$$X(\mu) = 1 + \tfrac{1}{2}\tilde{\omega}\mu \int_0^1 \left[X(\mu)X(\mu') - Y(\mu)Y(\mu')\right] \frac{d\mu'}{\mu + \mu'}, \qquad \text{(II-142)}$$

and $$Y(\mu) = e^{-\tau_1/\mu} + \tfrac{1}{2}\tilde{\omega}\mu \int_0^1 \left[Y(\mu)X(\mu') - X(\mu)Y(\mu')\right] \frac{d\mu'}{\mu - \mu'}. \qquad \text{(II-143)}$$

These functions are known as Chandrasekhar's X- and Y-functions and the equations defining them vary from phase function to phase function. The quantity, τ_0 is the total optical thickness of the atmosphere measured in terms of the extinction coefficient of the scattered radiation. Naturally, the reflection and transmission functions are functions of τ_0. The angles, $\cos^{-1}\mu$ and $\cos^{-1}\mu_0$ are the angles of emergence and incidence respectively, and all the functions are azimuthally independent for isotropic scattering. Chandrasekhar has also solved the scattering problem for more complex phase functions, but these can be accounted for more easily by the similarity relations.

We now give the detailed theory of the exact solution for the emergent thermal radiation from a finite plane-parallel atmosphere with no scattering. The method follows that of Chandrasekhar (1960), outlined above, and that of Stibbs (1971) for the equivalent problem in semi-infinite atmospheres. The model atmosphere is the same as that used in previous sections. That is, the atmosphere is plane-parallel and of finite optical thickness, τ_0. A parallel beam of dilute stellar radiation of net flux, πF_s, across unit area normal to itself, is incident upon the upper surface of the atmosphere at an angle, $\cos^{-1}\mu_0$, to the normal to the surface. This radiation is attenuated by absorption and the absorbed radiation is emitted conservatively and isotropically as infra-red radiation. The atmosphere is assumed to be grey in the visible or stellar part of the spectrum, of absorption coefficient, κ_s ; and grey in the infra-red or thermal part of the spectrum, of absorption coefficient, κ_p . The ratio of these absorption coeffients is n , the greenhouse parameter which is assumed to be constant throughout the atmosphere. All optical depths will be measured in terms of the thermal absorption coefficient and designated τ , so that the incident radiation field at depth τ is of net flux, $\pi F_s e^{-n\tau/\mu_0}$ per unit area normal to itself. The thermal radiation field is independent of azimuth because it is emitted isotropically. Consequently, we shall omit any possible azimuthal dependence in the reflection and transmission functions.

Define a reflection function, $S(\tau_0; \mu, \mu_0)$, and a transmission function, $T(\tau_0; \mu, \mu_0)$, for such an atmosphere, such that

$$I_p(0, +\mu) = \frac{F_p}{4\mu} S(\tau_0; \mu, \mu_0), \qquad \text{(II-144)}$$

and

$$I_p(\tau_0, -\mu) = \frac{F_p}{4\mu} T(\tau_0; \mu, \mu_0), \qquad \text{(II-145)}$$

where $\cos^{-1}\mu$ is the angle of emergence and μ is greater than zero.

Those intensities described by $+\mu$ are all upward flowing, and those described by $-\mu$ are all downward flowing. The flux, πF_p , is the flux of thermal radiation incident on the upper surface of the atmosphere at an angle of $\cos^{-1}\mu_0$ to the normal. In the problem, F_p is actually zero but it is necessary to use such a definition for S and T. Define a reflection and transmission function for the conversion of stellar to thermal radiation, such that

$$I_p(0, +\mu) = \frac{F_s}{4\mu} R(\tau_0; \mu, \mu_0), \tag{II-146}$$

and
$$I_p(\tau_0, -\mu) = \frac{F_s}{4\mu} Q(\tau_0; \mu, \mu_0). \tag{II-147}$$

The principle of invariance is now invoked in four ways:-

(i) The intensity of the upward flowing thermal radiation at depth, τ , is equal to that of the radiation reflected and converted from the reduced incident radiation and reflected from the downward diffuse thermal radiation field at depth τ by the atmosphere below depth τ . Expressed mathematically this becomes

$$I_p(\tau, +\mu) = \frac{F_s}{4\mu} e^{-n\tau/\mu_0} R(\tau_0-\tau; \mu, \mu_0) + \\ + \frac{1}{2\mu}\int_0^1 S(\tau_0-\tau; \mu, \mu') I_p(\tau, -\mu')\, d\mu'. \tag{II-148}$$

(ii) The intensity of the downward flowing thermal radiation at depth τ is equal to that of the radiation transmitted and converted from the incident radiation plus that of the radiation diffusely reflected from the upward diffuse thermal radiation at depth τ , by the atmosphere above depth τ . That is

$$I_p(\tau,-\mu) = \frac{F_s}{4\mu} Q(\tau_0;\mu,\mu_0) + $$

$$+ \frac{1}{2\mu}\int_0^1 S(\tau;\mu,\mu') I_p(\tau,+\mu')\,d\mu' . \qquad \text{(II-149)}$$

(iii) The intensity of the thermal radiation reflected from the whole atmosphere is equal to that of the thermal radiation reflected by the atmosphere above depth τ, plus that of the thermal radiation which passes through that same part of the atmosphere from below. The latter quantity is made up of the upward thermal radiation field at depth τ transmitted directly and also diffusely. Hence

$$\frac{F_s}{4\mu} R(\tau_0;\mu,\mu_0) = \frac{F_s}{4\mu} R(\tau;\mu,\mu_0) + e^{-\tau_0/\mu} I_p(\tau,+\mu) + $$

$$+ \frac{1}{2\mu}\int_0^1 T(\tau;\mu,\mu') I_p(\tau,+\mu')\,d\mu' . \qquad \text{(II-150)}$$

(iv) The intensity of the thermal radiation diffusely transmitted through the whole atmosphere is equal to that of the thermal radiation diffusely transmitted from the reduced incident radiation at depth τ , through the atmosphere below τ , plus that of the downward thermal radiation field at depth τ transmitted directly through the atmosphere below depth τ plus that of the radiation diffusely transmitted from the downward thermal radiation field at depth τ through the atmosphere below depth τ . That is

$$\frac{F_s}{4\mu} Q(\tau_0;\mu,\mu_0) = \frac{F_s}{4\mu} e^{-n\tau/\mu_0} Q(\tau_0-\tau;\mu,\mu_0) +$$

$$+ e^{-(\tau_0-\tau)/\mu} I_p(\tau,-\mu) + \frac{1}{2\mu}\int_0^1 T(\tau_0-\tau;\mu,\mu') I_p(\tau,-\mu')\, d\mu' . \qquad \text{(II-151)}$$

Differentiating equations (II-148) to (II-151) and passing the first and last to the limit of $\tau = 0$, and the other two to the limit of $\tau = \tau_0$, we obtain

$$\left[\frac{dI_p}{d\tau}(\tau,+\mu)\right]_0 = \frac{F_s}{4\mu}\left[-\frac{n}{\mu_0} R(\tau_0;\mu,\mu_0) - \frac{\partial R}{\partial \tau_0}(\tau_0;\mu,\mu_0)\right] +$$

$$+ \frac{1}{2\mu}\int_0^1 S(\tau_0;\mu,\mu')\left[\frac{dI_p}{d\tau}(\tau,-\mu')\right]_0 d\mu' , \qquad \text{(II-152)}$$

$$\left[\frac{dI_p}{d\tau}(\tau,-\mu)\right]_{\tau_0} = \frac{F_s}{4\mu}\frac{\partial Q}{\partial \tau_0}(\tau_0;\mu,\mu_0) +$$

$$+ \frac{1}{2\mu}\int_0^1 S(\tau_0;\mu,\mu')\left[\frac{dI_p}{d\tau}(\tau,+\mu')\right]_{\tau_0} d\mu' , \qquad \text{(II-153)}$$

$$0 = \frac{F_s}{4\mu}\frac{\partial R}{\partial \tau_0}(\tau_0;\mu,\mu_0) + e^{-\tau_0/\mu}\left[\frac{dI_p}{d\tau}(\tau,+\mu)\right]_{\tau_0} -$$

$$-\frac{1}{\mu} e^{-\tau_0/\mu} I_p(\tau,+\mu) + \frac{1}{2\mu}\int_0^1 T(\tau_0;\mu,\mu')\left[\frac{dI_p}{d\tau}(\tau,+\mu')\right]_{\tau_0} d\mu' \qquad \text{(II-154)}$$

and $$0 = \frac{F_s}{4\mu}\left[\frac{-n}{\mu_0} Q(\tau_0;\mu,\mu_0) - \frac{\partial Q}{\partial \tau_0}(\tau_0;\mu,\mu_0)\right] + e^{-\tau_0/\mu}\left[\frac{dI_p(\tau,-\mu)}{d\tau}\right]_0 +$$

$$+ \frac{1}{2\mu} e^{-\tau_0/\mu} I_p(0,-\mu) + \frac{1}{2\mu}\int_0^1 T(\tau_0;\mu,\mu')\left[\frac{dI_p(\tau,-\mu')}{d\tau}\right]_0 d\mu' . \qquad \text{(II-155)}$$

We now invoke two boundary conditions and make use of the equation of transfer. The two boundary conditions are: that there is no incident thermal radiation on the upper and lower surfaces. That is

$$I_p(0,-\mu) = 0 \qquad \text{and} \qquad I_p(\tau_0,+\mu) = 0 . \qquad \text{(II-156)}$$

The equation of transfer in a plane-parallel atmosphere is

$$\mu \frac{dI_p(\tau,\mu)}{d\tau} = I_p(\tau,\mu) - B_p(\tau) , \qquad \text{(II-157)}$$

and the source function for the thermal radiation is

$$B_p(\tau) = \frac{1}{2}\int_{-1}^{+1} I_p(\tau,\mu')d\mu' + \frac{1}{4} nF_s e^{-n\tau/\mu_0} . \qquad \text{(II-158)}$$

Hence $$\left[\frac{dI_p(\tau,+\mu)}{d\tau}\right]_0 = \frac{F_s}{4\mu^2} R(\tau_0;\mu,\mu_0) - \frac{1}{\mu} B_p(0) ,$$

$$\left[\frac{dI_p(\tau,+\mu)}{d\tau}\right]_{\tau_0} = -\frac{1}{\mu} B_p(\tau_0) ,$$

$$\left[\frac{dI_p(\tau,-\mu)}{d\tau}\right]_0 = \frac{1}{\mu} B_p(0) ,$$

and $$\left[\frac{dI_p(\tau,-\mu)}{d\tau}\right]_{\tau_0} = \frac{-F_s}{4\mu^2} Q(\tau_0;\mu,\mu_0) + \frac{1}{\mu} B_p(\tau_0) .$$

Using these equations for the derivatives of the intensity, equations (II-152) to (II-155) become

$$\frac{1}{4} F_s \left[\left(\frac{1}{\mu} + \frac{n}{\mu_0}\right) R(\tau_0;\mu,\mu_0) + \frac{\partial R}{\partial \tau_0}(\tau_0;\mu,\mu_0)\right]$$

$$= B_p(0) + \frac{1}{2}\int_0^1 S(\tau_0;\mu,\mu') B_p(0) \frac{d\mu'}{\mu'} , \tag{II-159}$$

$$\frac{1}{4} F_s \left[\frac{1}{\mu} Q(\tau_0;\mu,\mu_0) + \frac{\partial Q}{\partial \tau_0}(\tau_0;\mu,\mu_0)\right]$$

$$= B_p(\tau_0) + \frac{1}{2}\int_0^1 S(\tau_0;\mu,\mu') B_p(\tau_0) \frac{d\mu'}{\mu'} , \tag{II-160}$$

$$\frac{1}{4} F_s \frac{\partial R}{\partial \tau_0}(\tau_0;\mu,\mu_0) = e^{-\tau_0/\mu} B_p(\tau_0) +$$

$$+ \frac{1}{2}\int_0^1 T(\tau_0;\mu,\mu') B_p(\tau_0) \frac{d\mu'}{\mu'} , \tag{II-161}$$

and $$\frac{1}{4} F_s \left[\frac{n}{\mu_0} Q(\tau_0;\mu,\mu_0) + \frac{\partial Q}{\partial \tau_0}(\tau_0;\mu,\mu_0)\right]$$

$$= e^{-\tau_0/\mu} B_p(0) + \frac{1}{2}\int_0^1 T(\tau_0;\mu,\mu') B_p(0) \frac{d\mu'}{\mu'} . \tag{II-162}$$

The source function at the appropriate values of τ is given by equations (II-158), (II-144), (II-145) and (II-156), so that

$$B_p(0) = \frac{1}{4}F_s\left[n + \frac{1}{2}\int_0^1 R(\tau_0;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right], \tag{II-163}$$

and

$$B_p(\tau_0) = \frac{1}{4}F_s\left[ne^{-n\tau_0/\mu_0} + \frac{1}{2}\int_0^1 Q(\tau_0;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right]. \tag{II-164}$$

These two equations together with equations (II-159) to (II-162) give four integral equations for the reflection and transmission functions;

$$\left(\frac{1}{\mu}+\frac{n}{\mu_0}\right)R(\tau_0;\mu,\mu_0) + \frac{\partial R(\tau_0;\mu,\mu_0)}{\partial\tau_0}$$

$$= \left\{n + \frac{1}{2}\int_0^1 R(\tau_0;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right\}\left\{1 + \frac{1}{2}\int_0^1 S(\tau_0;\mu,\mu')\frac{d\mu'}{\mu'}\right\}, \tag{II-165}$$

$$\frac{\partial R(\tau_0;\mu,\mu_0)}{\partial\tau_0} = \left\{ne^{-n\tau_0/\mu_0} + \frac{1}{2}\int_0^1 Q(\tau_0;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right\}\times$$

$$\times\left\{e^{-\tau_0/\mu} + \frac{1}{2}\int_0^1 T(\tau_0;\mu,\mu')\frac{d\mu'}{\mu'}\right\}, \tag{II-166}$$

$$\frac{n}{\mu_0}Q(\tau_0;\mu,\mu_0) + \frac{\partial Q(\tau_0;\mu;\mu_0)}{\partial\tau_0}$$

$$= \left\{n + \frac{1}{2}\int_0^1 R(\tau_0;\mu'',\mu_0)\frac{d\mu''}{\mu''}\right\}\left\{e^{-\tau_0/\mu} + \frac{1}{2}\int_0^1 T(\tau_0;\mu,\mu')\frac{d\mu'}{\mu'}\right\}, \tag{II-167}$$

and $$\frac{1}{\mu} Q(\tau_0;\mu,\mu_0) + \frac{\partial Q}{\partial \tau_0}(\tau_0;\mu,\mu_0)$$

$$= \left\{ n e^{-n\tau_0/\mu_0} + \frac{1}{2}\int_0^1 Q(\tau_0;\mu'',\mu_0)\frac{d\mu''}{\mu''} \right\} \left\{ 1 + \frac{1}{2}\int_0^1 S(\tau_0;\mu,\mu')\frac{d\mu'}{\mu'} \right\}. \quad \text{(II-168)}$$

Now, the reflection function, S , and the transmission function, T, are defined in exactly the same way as they are for the scattering problem. The transfer of the thermal radiation is exactly the same as the transfer of scattered radiation when the scattering is conservative and isotropic. Thus

$$1 + \frac{1}{2}\int_0^1 S(\tau_0;\mu,\mu')\frac{d\mu'}{\mu'} = X(\mu), \quad \text{(II-169)}$$

and $$e^{-\tau_0/\mu} + \frac{1}{2}\int_0^1 T(\tau_0;\mu,\mu')\frac{d\mu'}{\mu'} = Y(\mu). \quad \text{(II-170)}$$

Let $$n + \frac{1}{2}\int_0^1 R(\tau_0;\mu',\mu)\frac{d\mu'}{\mu'} = W(\mu), \quad \text{(II-171)}$$

and let $$n e^{-n\tau_0/\mu_0} + \frac{1}{2}\int_0^1 Q(\tau_0;\mu,\mu')\frac{d\mu'}{\mu'} = Z(\mu). \quad \text{(II-172)}$$

Eliminating the function derivatives from equations (II-165) to (II-168) and using the definitions, equations (II-169) to (II-172), we obtain

$$\left(\frac{1}{\mu} + \frac{n}{\mu_0}\right) R(\tau_0;\mu,\mu_0) = W(\mu_0)X(\mu) - Z(\mu_0)Y(\mu), \quad \text{(II-173)}$$

and $$\left(\frac{n}{\mu_0} - \frac{1}{\mu}\right) Q(\tau_0;\mu,\mu_0) = W(\mu_0)Y(\mu) - Z(\mu_0)X(\mu). \quad \text{(II-174)}$$

Substituting equations (II-173) and (II-174) into equations (II-171) and (II-172) respectively, we obtain

$$W(\mu_0) = n + \frac{1}{2}\int_0^1 \frac{\mu_0}{\mu_0 + n\mu'}\left[W(\mu_0)X(\mu') - Z(\mu_0)Y(\mu')\right]d\mu', \qquad \text{(II-175)}$$

and $$Z(\mu_0) = ne^{-n\tau_0/\mu_0} + \frac{1}{2}\int_0^1 \frac{\mu_0}{n\mu' - \mu_0}\left[W(\mu_0)Y(\mu') - Z(\mu_0)X(\mu')\right]d\mu'. \qquad \text{(II-176)}$$

Equations (II-142) and (II-143), which define the X- and Y-functions, for the conservative case and argument, μ_0/n, are

$$nX(\mu_0/n) = n +$$

$$+ \frac{1}{2}\int_0^1 \frac{\mu_0}{\mu_0 + n\mu'}\left[nX(\mu_0/n)X(\mu') - nY(\mu_0/n)Y(\mu')\right]d\mu'. \text{(II-177)}$$

and $$nY(\mu_0/n) = ne^{-n\tau_0/\mu_0}$$

$$-\frac{1}{2}\int_0^1 \frac{\mu_0}{n\mu' - \mu_0}\left[nY(\mu_0/n)X(\mu') - nX(\mu_0/n)Y(\mu')\right]d\mu'. \qquad \text{(II-178)}$$

A comparison of the pair of equations, (II-175) and (II-176), with the pair, (II-177) and (II-178), indicates that $W(\mu_0)$ and $Z(\mu_0)$ satisfy the same two equations as $nX(\mu_0/n)$ and $nY(\mu_0/n)$. Furthermore, as these equations are ordinary equations in the unknown variables, we can show that

$$W(\mu_0) = nX(\mu_0/n) \quad \text{and} \quad Z(\mu_0) = nY(\mu_0/n). \qquad \text{(II-179)}$$

However, this is not the final solution because the X- and Y-functions for conservative scattering are not unique. Chandrasekhar has proved that, if $X(\mu)$ and $Y(\mu)$ are solutions of the integral equations (II-142) and (II-143), then so are the functions $X(\mu) + q\mu\left[X(\mu) + Y(\mu)\right]$ and $Y(\mu) - q\mu\left[X(\mu) + Y(\mu)\right]$, where q is an arbitrary constant. The functions involved in equations (II-175) and (II-176) are not unique, and so the functions, $W(\mu_o)$ and $Z(\mu_o)$ are not unique, but are

$$W(\mu_o) = n X_s(\mu_o/n) + \frac{q\mu_o}{n}\left[n X_s(\mu_o/n) + n Y_s(\mu_o/n)\right], \qquad \text{(II-180)}$$

and $$Z(\mu_o) = n Y_s(\mu_o/n) - \frac{q\mu_o}{n}\left[n X_s(\mu_o/n) + n Y_s(\mu_o/n)\right], \qquad \text{(II-181)}$$

where the subscript s denotes a standard solution as defined by Chandrasekhar. The emergent thermal radiation can thus be expressed, by using equations (II-180), (II-181), (II-173), (II-174), (II-146) and (II-147) and the non-unique forms of the X- and Y-functions, as

$$I_p(0, +\mu) = \frac{F_s\mu_o}{4}\left\{\frac{\left[X_s(\mu_o/n)X_s(\mu) - Y_s(\mu_o/n)Y_s(\mu)\right]}{\mu + \mu_o/n} + q\left[X_s(\mu) + Y_s(\mu)\right]\left[X_s(\mu_o/n) + Y_s(\mu_o/n)\right]\right\} \qquad \text{(II-182)}$$

and $$I_p(\tau_o, -\mu) = \frac{F_s\mu_o}{4}\left\{\frac{\left[Y_s(\mu_o/n)X_s(\mu) - X_s(\mu_o/n)Y_s(\mu)\right]}{\mu_o/n - \mu} - q\left[X_s(\mu) + Y_s(\mu)\right]\left[X_s(\mu_o/n) + Y_s(\mu_o/n)\right]\right\}. \qquad \text{(II-183)}$$

The principle of invariance cannot provide a unique solution for the emergent radiation in the conservative problem. We can find an expression for q by utilising the flux and K-integrals which the equation of transfer admits in conservative cases. By applying the L_0-operator to the equation of transfer, which is given by equations (II-157) and (II-158) we obtain the flux integral;

$$\frac{dH_p(\tau)}{d\tau} = -\frac{1}{4} n F_s e^{-n\tau/\mu_0}$$

or
$$H_p(\tau) = \frac{1}{4}\mu_0 F_s \left[e^{-n\tau/\mu_0} - \gamma_1 \right], \tag{II-184}$$

and by applying the L_1-operator to the equation of transfer, we obtain the K-integral;

$$\frac{dK_p(\tau)}{d\tau} = H_p(\tau),$$

or
$$K_p(\tau) = \frac{\mu_0 F_s}{4}\left[\frac{-\mu_0}{n} e^{-n\tau/\mu_0} - \gamma_1\tau + \gamma_2 \right], \tag{II-185}$$

where γ_1 and γ_2 are constants of integration. At the two boundaries of the atmosphere, equations (II-184) and (II-185) become

$$H_p(0) = \frac{1}{4} F_s \mu_0 (1-\gamma_1), \tag{II-186}$$

$$H_p(\tau_0) = \frac{1}{4} F_s \mu_0 \left[e^{-n\tau_0/\mu_0} - \gamma_1 \right] \tag{II-187}$$

$$K_p(0) = \frac{1}{4} F_s \mu_0 \left[-\frac{\mu_0}{n} + \gamma_2 \right] , \qquad \text{(II-188)}$$

and
$$K_p(\tau_0) = \frac{1}{4} \mu_0 F_s \left[-\frac{\mu_0}{n} e^{-n\tau_0/\mu_0} - \gamma_1 \tau_0 + \gamma_2 \right] . \qquad \text{(II-189)}$$

These same quantities can also be found from their definitions, (I-11), and equations (II-182) and (II-183). In this way we are able to obtain four equations involving the constant q which we can equate with equations (II-186) to (II-189) and hence obtain expressions for q, γ_1 and γ_2. To do so we utilise various theorems for the X- and Y-functions which have been proved by Chandrasekhar (1960). Hence, we obtain

$$H_p(0) = \frac{1}{4} F_s \mu_0 \left\{ 1 + \frac{1}{2} q(\alpha_1 + \beta_1) \left[X_s(\mu_0/n) + Y_s(\mu_0/n) \right] \right\} , \qquad \text{(II-190)}$$

$$H_p(\tau_0) = \frac{1}{4} F_s \mu_0 \left\{ e^{-n\tau_0/\mu_0} + \frac{1}{2} q(\alpha_1 + \beta_1) \left[X_s(\mu_0/n) + Y_s(\mu_0/n) \right] \right\} , \qquad \text{(II-191)}$$

$$K_p(0) = \frac{1}{4} F_s \mu_0 \left\{ \frac{1}{2} \alpha_1 X_s(\mu_0/n) - \frac{1}{2} \beta_1 Y_s(\mu_0/n) - \mu_0/n + \right.$$
$$\left. + \frac{1}{2} q(\alpha_2 + \beta_2) \left[X_s(\mu_0/n) + Y_s(\mu_0/n) \right] \right\} , \qquad \text{(II-192)}$$

and
$$K_p(\tau_0) = \frac{1}{4} F_s \mu_0 \left\{ \frac{1}{2} \beta_1 X_s(\mu_0/n) - \frac{1}{2} \alpha_1 Y_s(\mu_0/n) - \right.$$

$$\left. - \frac{\mu_0}{n} e^{-n\tau_0/\mu_0} - \frac{1}{2} q(\alpha_2 + \beta_2) \left[X_s(\mu_0/n) + Y_s(\mu_0/n) \right] \right\} , \qquad \text{(II-193)}$$

where α_n and β_n are the moments of order n of $X_s(\mu)$ and $Y_s(\mu)$ respectively. These moments are defined by

$$\alpha_n = \int_0^1 X_s(\mu)\,\mu^n\,d\mu \quad \text{and} \quad \beta_n = \int_0^1 Y_s(\mu)\,\mu^n\,d\mu . \tag{II-194}$$

Equations (II-186) and (II-190) give, as do equations (II-187) and (II-191),

$$\gamma_1 = -\tfrac{1}{2}(\alpha_1+\beta_1)\,q\left[X_s(\mu_0/n) + Y_s(\mu_0/n)\right], \tag{II-195}$$

and equations (II-188) and (II-192) give

$$\begin{aligned}\gamma_2 = {} & \tfrac{1}{2}\alpha_1 X_s(\mu_0/n) - \tfrac{1}{2}\beta_1 Y_s(\mu_0/n) + \\ & + \tfrac{1}{2}q(\alpha_2+\beta_2)\left[X_s(\mu_0/n) + Y_s(\mu_0/n)\right],\end{aligned} \tag{II-196}$$

whilst equations (II-189) and (II-193) give

$$\begin{aligned}-\gamma_1\tau_0 + \gamma_2 = {} & \tfrac{1}{2}\beta_1 X_s(\mu_0/n) - \tfrac{1}{2}\alpha_1 Y_s(\mu_0/n) - \\ & - \tfrac{1}{2}q(\alpha_2+\beta_2)\left[X_s(\mu_0/n) + Y_s(\mu_0/n)\right].\end{aligned} \tag{II-197}$$

Substituting equations (II-195) and (II-196) into equation (II-197), we obtain an expression for the constant q in terms of the moments of the standard X- and Y-functions. Thus

$$q = \frac{-(\alpha_1-\beta_1)}{(\alpha_1+\beta_1)\tau_0 + 2(\alpha_2+\beta_2)} . \tag{II-198}$$

This expression is the same as that obtained by Chandrasekhar for the problem of conservative isotropic scattering.

Details of the X- and Y-functions will be given in the third part of this section but it can be stated now that their evaluation is sufficiently complex that tables are used for all practical purposes. The standard solutions were defined from mathematical considerations and it transpires that they are seldom experienced in practice. The tables all quote the conservative X- and Y-functions as those functions that satisfy the standard problem of conservative isotropic scattering. These we shall denote by $X_c(\mu)$ and $Y_c(\mu)$. Now we have shown that the constant q is the same for these functions as it is for the functions $W(\mu_0)$ and $Z(\mu_0)$ so that we conclude that

$$W(\mu_0) = n\, X_c(\mu_0/n) \quad \text{and} \quad Z(\mu_0) = n\, Y_c(\mu_0/n). \qquad \text{(II-199)}$$

Hence, the emergent thermal radiation from a finite plane-parallel atmosphere with no scattering is

$$I_p(0,+\mu) = \frac{1}{4} F_s \mu_0 \left[X_c(\mu_0/n)\, X_c(\mu) - Y_c(\mu_0/n)\, Y_c(\mu) \right] / (\mu + \mu_0/n), \qquad \text{(II-200)}$$

and $$I_p(\tau_0,-\mu) = \frac{1}{4} F_s \mu_0 \left[Y_c(\mu_0/n)\, X_c(\mu) - X_c(\mu_0/n)\, Y_c(\mu) \right] / (\mu_0/n - \mu).$$

We may now consider the addition of a ground layer at the lower surface of the atmosphere. This extra feature will be dealt with in a manner closely resembling that with which Chandrasekhar (1960) treated the ground layer at the lower surface of a scattering atmosphere. The properties of the ground layer are the same as those of the ground layer introduced in Section II.3.2. That is, the ground behaves as a Lambertian surface reflecting with intensity, G_s, a fraction, λ, of the stellar radiation incident upon it and absorbing the remaining fraction. It also absorbs all the thermal radiation incident upon it and emits all the absorbed radiation thermally and isotropically with intensity, G_T.

It will be convenient to introduce several shorthand definitions before establishing the physics of the amended problem. We define

$$s(\mu) = \frac{1}{2}\int_0^1 S(\tau_0;\mu,\mu')\,d\mu' \;;\quad t(\mu) = \frac{1}{2}\int_0^1 T(\tau_0;\mu,\mu')\,d\mu' \;;$$

$$r(\mu) = \frac{1}{2}\int_0^1 R(\tau_0;\mu,\mu')\,d\mu' \;;\quad q(\mu) = \frac{1}{2}\int_0^1 Q(\tau_0;\mu,\mu')\,d\mu' \;;$$

$$r_1(\mu) = \frac{1}{2}\int_0^1 R(\tau_0;\mu',\mu)\,d\mu' \;;\quad q_1(\mu) = \frac{1}{2}\int_0^1 Q(\tau_0;\mu',\mu)\,d\mu' \;;$$

$$\bar{r} = 2\int_0^1 r(\mu)\,d\mu \;; \quad \text{and} \quad \bar{s} = 2\int_0^1 s(\mu)\,d\mu . \qquad \text{(II-201)}$$

It is necessary to introduce both $r(\mu)$ and $r_1(\mu)$ because the function $R(\tau_0;\mu,\mu_0)$ does not satisfy the principle of reciprocity. That is to say

$$R(\tau_0;\mu,\mu_0) \neq R(\tau_0;\mu_0,\mu) .$$

This can be seen from equation (II-173). The same is also true for the function $Q(\tau_0; \mu, \mu_0)$; but the functions, $S(\tau_0; \mu, \mu_0)$ and $T(\tau_0; \mu, \mu_0)$ do satisfy the principle of reciprocity. Consequently, $s_1(\mu) \equiv s(\mu)$ and $t_1(\mu) \equiv t(\mu)$.

The intensity of the emergent thermal radiation from an atmosphere with no ground will be assigned the superscript, zero. The ground can be included in a statement similar to those expressing the principle of invariance. That is, the emergent thermal radiation from the upper surface of an atmosphere with a ground at its lower surface is equal to the emergent thermal radiation from the atmosphere in the absence of the ground plus the thermal ground radiation diffusely transmitted through the atmosphere plus the visible ground radiation diffusely transmitted and degraded by the atmosphere plus the thermal ground radiation directly transmitted and reduced. This statement can be expressed mathematically by

$$I_p(0,+\mu) = I_p^0(0,+\mu) + \frac{1}{2\mu}\int_0^1 T(\tau_0; \mu, \mu') G_p \, d\mu' + \frac{1}{2\mu}\int_0^1 Q(\tau_0; \mu, \mu') G_s \, d\mu' + G_p e^{-\tau_0/\mu} ,$$

or

$$I_p(0,+\mu) = I_p^0(0,+\mu) + G_p \xi(\mu) + G_s q(\mu)/\mu , \qquad \text{(II-202)}$$

where

$$\xi(\mu) = t(\mu)/\mu + e^{-\tau_0/\mu} . \qquad \text{(II-203)}$$

A similar physical statement and mathematical expression can be constructed for the intensity of the diffuse thermal radiation field emergent from the lower surface of the atmosphere and hence incident upon the ground. This quantity must be equal to the emergent thermal radiation from the lower surface of the atmosphere in the absence of the ground plus the thermal ground radiation diffusely reflected by the atmosphere plus the visible ground radiation diffusely reflected and degraded by the atmosphere. This statement can be expressed mathematically by

$$I_p(\tau_0, -\mu) = I_p^0(\tau_0, -\mu) + \frac{1}{2\mu}\int_0^1 G_p S(\tau_0; \mu, \mu')\,d\mu' +$$

$$+ \frac{1}{2\mu}\int_0^1 G_s R(\tau_0; \mu, \mu')\,d\mu' ,$$

or $$I_p(\tau_0, -\mu) = I_p^0(\tau_0, -\mu) + G_p s(\mu)/\mu + G_s r(\mu)/\mu . \qquad \text{(II-204)}$$

In order to complete the solution we must derive expressions for G_s and G_p . This is done by considering the energy balance into and out of the ground. The flux of the reduced incident radiation into the ground is $\pi \mu_0 F_s \exp(-n\tau_0/\mu_0)$, and it is a fraction, λ, of this which must be equal to the upward flux of stellar radiation reflected by the ground, which is πG_s . Hence

$$G_s = \lambda \mu_0 F_s e^{-n\tau_0/\mu_0} . \qquad \text{(II-205)}$$

The upward flux of thermal radiation from the ground is πG_p , and this must be equal to the downward flux of thermal radiation incident upon the ground, which is given by equation (II-204), plus the fraction, $(1 - \lambda)$, of the downward flux of the reduced incident radiation at the ground. This arises from the conservative nature of the ground. Hence

$$\pi G_p = (1-\lambda)\mu_0 \pi F_s e^{-n\tau_0/\mu_0} + 2\pi\int_0^1 \mu I_p^0(\tau_0,-\mu)\,d\mu +$$

$$+ 2\pi\int_0^1 G_p\, s(\mu)\,d\mu + 2\pi\int_0^1 G_s\, r(\mu)\,d\mu .$$

or $$G_p = \frac{\left[(1-\lambda)\mu_0 F_s e^{-n\tau_0/\mu_0} + F_s q_1(\mu_0) + G_s \bar{r}\right]}{\left[1-\bar{s}\right]} . \qquad \text{(II-206)}$$

Although equations (II-202), (II-205) and (II-206) express the complete solution for the emergent thermal radiation from a finite plane-parallel atmosphere with a ground, it will be expedient to show how the functions defined by equations (II-201) are obtained in practice. Those that are derived from the standard scattering reflection and transmission functions can be expressed in terms of the X- and Y-functions and their moments. We have,

$$t(\mu) = \frac{1}{2}\int_0^1 T(\tau_0;\mu,\mu')\,d\mu'$$

and $$T(\tau_0;\mu,\mu_0) = \frac{\mu\mu_0}{\mu-\mu_0}\left[X_c(\mu_0)Y_c(\mu) - X_c(\mu)Y_c(\mu_0)\right]$$

for conservative isotropic scattering, so that

$$t(\mu) = \frac{1}{2}\int_0^1 \frac{\mu\mu'\,d\mu'}{\mu-\mu'}\left[X_c(\mu')Y_c(\mu) - X_c(\mu)Y_c(\mu')\right]$$

$$= \mu\left\{-e^{-\tau_0/\mu} + \left[\tfrac{1}{2}\beta_0 X_c(\mu) + (1-\tfrac{1}{2}\alpha_0)Y_c(\mu)\right]\right\} . \qquad \text{(II-207)}$$

The last result is given by Chandrasekhar (1960), as are many other properties of the X- and Y-functions. Hence

$$\zeta(\mu) = \frac{1}{2}\left[\beta_0 X_c(\mu) + (2-\alpha_0) Y_c(\mu)\right]. \qquad \text{(II-208)}$$

We must remember that the moments are now moments of the X_c - and Y_c -functions. Similarly

$$s(\mu) = \frac{1}{2}\int_0^1 \frac{\mu\mu' d\mu'}{\mu+\mu'}\left[X_c(\mu)X_c(\mu') - Y_c(\mu)Y_c(\mu')\right]$$

$$= \mu\left\{1 - \left[(1-\tfrac{1}{2}\alpha_0)X_c(\mu) + \tfrac{1}{2}\beta_0 Y_c(\mu)\right]\right\}, \qquad \text{(II-209)}$$

and

$$\bar{s} = 2\int_0^1 s(\mu)\, d\mu$$

$$= 1 - 2\alpha_1(2-\alpha_0) - 2\beta_0\beta_1 . \qquad \text{(II-210)}$$

The functions $q(\mu)$, $q_1(\mu)$ and $\bar{r}$ are more complex by virtue of the presence of the parameter n . Equations (II-147), (II-183) and (II-199) provide expressions for $q(\mu)$ and $q_1(\mu_0)$.

$$q(\mu) = \frac{1}{2}\int_0^1 \frac{\mu\mu' d\mu'}{\mu'/n - \mu}\left[Y_c(\mu'/n)X_c(\mu) - X_c(\mu'/n)Y_c(\mu)\right], \qquad \text{(II-211)}$$

and

$$q_1(\mu_0) = \frac{1}{2}\int_0^1 \frac{\mu'\mu_0 d\mu'}{\mu_0/n - \mu'}\left[Y_c(\mu_0/n)X_c(\mu') - X_c(\mu_0/n)Y_c(\mu')\right].$$

The second of these is integrable in the same way as were $s(\mu)$ and $t(\mu)$, so that

$$q_1(\mu_0) = \mu_0 \left\{ -e^{-n\tau_0/\mu_0} + \tfrac{1}{2}\beta_0 X_c(\mu_0/n) + \right.$$

$$\left. + (1 - \tfrac{1}{2}\alpha_0) Y_c(\mu_0/n) \right\}. \qquad \text{(II-212)}$$

However, the first integral must be evaluated by numerical methods unless n is unity. An analytical method of solution would involve changing the upper limits of the integral to $1/n$ and no further progress could be made. Now

$$\bar{r} = 2\int_0^1 r(\mu)\,d\mu = 2\int_0^1\int_0^1 R(\tau_0, \mu, \mu')\,d\mu\,d\mu'$$

$$= 2\int_0^1 r_1(\mu')\,d\mu' .$$

As with $q(\mu)$ and $q_1(\mu)$, $r_1(\mu)$ can be integrated analytically but $r(\mu)$ cannot. Hence

$$r_1(\mu') = \frac{1}{2}\int_0^1 \frac{\mu\mu'\,d\mu'}{\mu'/n + \mu}\left[X_c(\mu'/n)\,Y_c(\mu) - Y_c(\mu'/n)\,X_c(\mu) \right]$$

$$= \mu' - \tfrac{1}{2}\mu' \left[(2-\alpha_0)\,X_c(\mu'/n) + \beta_0\,Y_c(\mu'/n) \right],$$

and

$$\bar{r} = 1 - \int_0^1 \mu' \left[(2-\alpha_0)\,X_c(\mu'/n) + \beta_0\,Y_c(\mu'/n) \right] d\mu'. \qquad \text{(II-213)}$$

Equation (II-213) must be integrated numerically. The need to perform this and other integrals numerically is not a serious drawback to the exactness of the solution, especially when it is remembered that the so-called "analytical" solutions are not genuinely analytical. They merely express the solution in terms of standard moments of the X- and Y-functions. These moments must be evaluated by interpolation on suitable tables which have been constructed by numerical integration of the X- and Y-functions. Thus, even though the foregoing analysis is exact, the intensity of the emergent thermal radiation is only available to an accuracy determined by the tables of the X- and Y-functions and their moments, together with the accuracy of the method of interpolation utilised in the numerical programmes.

The theory for semi-infinite atmospheres was extended to include linear scattering. This is possible for finite atmospheres only when β is unity and the scattering all forward. This is because the source function has an extra term which vanishes when β is unity. This extra term hinders the progress of the theory. Nevertheless, for β equal to unity the solution proceeds as before to give

$$W(\mu_0) = \sigma n\, X_c(\mu_0/n\sigma) \quad \text{and} \quad Z(\mu_0) = \sigma n\, Y_c(\mu_0/n\sigma) ,$$

where $\sigma = (1-\tilde{\omega})$. Briefly, this can be seen from the fact that the stellar radiation is attenuated by $\sigma n \tau$, so that n is replaced by $n\sigma$ in the appropriate equations. Only in the equation for the source function is n not so changed, but this equation involves the factor, $n(1 - \tilde{\omega})$, which equals $n\sigma$ in this case. Consequently, n is replaced by $n\sigma$ throughout the theory and the intensity of the emergent thermal radiation from the upper surface of a finite plane-parallel atmosphere with no ground, but with forward scattering, is

$$I_p(0,+\mu) = \frac{1}{4}\mu_0 F_s \frac{\left[X_c(\mu_0/n\sigma)X_c(\mu) - Y_c(\mu_0/n\sigma)Y_c(\mu)\right]}{\left[\mu + \mu_0/n\sigma\right]}. \quad \text{(II-214)}$$

It is reasonable that n should be replaced by $n\sigma$ because the only real physical change is in the relationship between the absorption coefficients. The scattered radiation can be accounted for by assuming that it is not scattered but that the absorption coefficient has become $(1-\tilde{\omega})(\kappa_s+\sigma_s)$, which is equal to the extinction coefficient employed in measuring optical depths. Hence, n is changed to the ratio of the new extinction coefficients which is $n\sigma$. This did not happen in semi-infinite atmospheres when β was less than unity.

There is one further case that we can usefully consider. That is the case of isotropic scattering when n is equal to unity. The transfer of the stellar and thermal radiation fields is the same so that the total emergent radiation is given by the solution for the emergent scattered radiation in the conservative isotropic scattering problem. Thus the emergent thermal radiation is given by the difference between the two solutions for the emergent radiation in the isotropic scattering problem for conservative and non-conservative scattering. Writing the X- and Y-functions as functions of μ and $\tilde{\omega}$ we have

$$I_p(0,+\mu) = \frac{\mu_0 F_s}{4(\mu+\mu_0)}\Big\{X_c(\mu_0,1)X_c(\mu,1) - Y_c(\mu_0,1)Y_c(\mu,1) -$$

$$- X_c(\mu_0,\tilde{\omega})X_c(\mu,\tilde{\omega}) + Y_c(\mu_0,\tilde{\omega})Y_c(\mu,\tilde{\omega})\Big\} \quad \text{(II-215)}$$

The extension of the solution to the atmosphere with a ground layer follows readily in the same manner as before.

5.2. The Approximate Solution As with the exact solution we shall find it profitable to consider semi-infinite and finite plane-parallel atmospheres separately.

5.2.1. Semi-infinite Atmospheres The emergent intensity from the surface of a semi-infinite plane-parallel atmosphere is given by equation (II-108), which represents the formal solution of the equation of transfer, with the variable, τ set to zero. This equation applies to both radiation fields in the greenhouse problem with scattering, where the variables, t and τ are measured in terms of the appropriate extinction coefficient. We shall revert to our original convention of measuring τ in terms of the extinction coefficient for the stellar radiation because our solutions for the source function involve this convention. Thus, the intensity of the emergent radiation from a semi-infinite plane-parallel atmosphere is

$$I_s(0, +\mu) = \int_0^\infty B_s(t)\, e^{-t/\mu}\, dt/\mu\,, \qquad \text{(II-216)}$$

and

$$I_p(0, +\mu) = \int_0^\infty B_p(t)\, e^{-t/n\mu}\, dt/n\mu\,, \qquad \text{(II-217)}$$

for the scattered and thermal radiation fields respectively.

In Section II.3.1 the approximate solutions for the source functions were derived, and by inserting these in the above equations we arrive at approximate solutions for the emergent intensity.

The mean intensity of the radiation at the surface can be found by direct integration of the emergent intensity over all directions.

This process is, of course, the same as the lambda operation described in Section II.4.1.

For the scattered radiation field, equation (II-216) can only be used when the scattering is isotropic, because all other forms of the phase function yield source functions that include the intensity as well as the mean intensity and hence render equation (II-216) an integral equation for the intensity. However, we have solved for the scattering source function when the scattering is isotropic, and we have

$$B_s(t) = \tilde{\omega} J_s(t) + \tilde{\omega} J_{red}^{inc}(t) ,$$

where $J_s(t)$ and $J_{red}^{inc}(t)$ are given by equations (II-25) and (II-15) respectively. This can be written in the form

$$B_s(t) = Q_1 e^{-\epsilon t} + Q_2 e^{-\sigma t/\mu_0} , \qquad \text{(II-218)}$$

with Q_1 and Q_2 appropriately defined. Substituting this equation for the source function into equation (II-216) we obtain

$$I_s(0, +\mu) = \frac{Q_1}{(\epsilon\mu + 1)} + \frac{Q_2}{(\sigma\mu/\mu_0 + 1)} , \qquad \text{(II-219)}$$

for the emergent scattered radiation from a semi-infinite plane-parallel atmosphere with non-conservative isotropic scattering.

The source function for the thermal radiation field, however, is isotropic whatever the phase function of the scattering of the stellar radiation. Consequently, the emergent thermal radiation can be found for all scattering phase functions. The source function for the thermal radiation is given by equation (II-30), which is,

$$B_p(t) = J_p(t) + n(1-\tilde{\omega})\left[J_s(t) + J_{in}(t) \right].$$

It is convenient to express this in the form

$$B_p(t) = Q_1 e^{-\epsilon t} + Q_2 e^{-\sigma t/\mu_0} + Q_3 , \tag{II-220}$$

where $\quad Q_1 = \left[n(1-\tilde{\omega}) - 3(1-\tilde{\omega})/n\epsilon^2 \right] D ,$

$\qquad\qquad Q_2 = \left[n(1-\tilde{\omega}) - 3(1-\tilde{\omega})\mu_0^2/n\sigma^2 \right] E ,$

and $\qquad Q_3 = G ,$

the constants D, E and G being given in Section II.3.1.

Substituting equation (II-220) into equation (II-217), we obtain

$$I_p(0,+\mu) = \frac{Q_1}{(\epsilon n\mu + 1)} + \frac{Q_2}{(\sigma n\mu/\mu_0 + 1)} + Q_3 . \tag{II-221}$$

Thus we have the approximate solution for the emergent thermal radiation from a semi-infinite plane-parallel atmosphere, with non-conservative anisotropic scattering.

5.2.2. <u>Finite Atmospheres</u> Firstly, we consider the case in which there is no ground layer at the lower surface of the atmosphere. The intensities of the emergent scattered and thermal radiation fields from the upper surface of a finite plane-parallel atmosphere are given by the appropriate forms of equation (II-119), which are

$$I_s(0,+\mu) = \int_0^{\tau_0} B_s(t)\, e^{-t/\mu}\, dt/\mu , \tag{II-222}$$

and $$I_p(0, +\mu) = \int_0^{\tau_0} B_p(t)\, e^{-t/n\mu}\, dt/n\mu\,, \tag{II-223}$$

and the intensities of the emergent scattered and thermal radiation fields from the lower surface of the atmosphere, again given by equation (II-119), are

$$I_s(\tau_0, -\mu) = \int_0^{\tau_0} B_s(t)\, e^{-(\tau_0-t)/\mu}\, dt/\mu\,, \tag{II-224}$$

and $$I_p(\tau_0, -\mu) = \int_0^{\tau_0} B_p(t)\, e^{-(\tau_0-t)/n\mu}\, dt/n\mu\,. \tag{II-225}$$

Equations (II-222) and (II-224) can only be integrated when the scattering is isotropic, in which case the source function can be expressed in the form

$$B_s(t) = Q_1 e^{\delta t} + Q_2 e^{-\delta t} + Q_3 e^{-t/\mu_0}\,, \tag{II-226}$$

where the constants Q are found from equations (II-57), (II-60), (II-68) and (II-69). The intensity of the emergent scattered radiation is thus

$$I_s(0, +\mu) = Q_1 \frac{\left[e^{\tau_0(\delta - 1/\mu)} - 1\right]}{(\delta\mu - 1)} - Q_2 \frac{\left[e^{-\tau_0(\delta + 1/\mu)} - 1\right]}{(\delta\mu + 1)} - Q_3 \frac{\left[e^{-\tau_0(1/\mu + 1/\mu_0)} - 1\right]}{(\mu/\mu_0 + 1)} \tag{II-227}$$

from the upper surface of the atmosphere; and

$$I_s(\tau_0,-\mu) = Q_1\frac{\left[e^{\delta\tau_0}-e^{-\tau_0/\mu}\right]}{(\delta\mu+1)} - Q_2\frac{\left[e^{-\delta\tau_0}-e^{-\tau_0/\mu}\right]}{(\delta\mu-1)} -$$

$$- Q_3\frac{\left[e^{-\tau_0/\mu_0}-e^{-\tau_0/\mu}\right]}{(\mu/\mu_0-1)} \, . \qquad \text{(II-228)}$$

from the lower surface of the atmosphere. There are special forms of these equations which arise when $\mu = \mu_0$ and $\mu = 1/\delta$ due to indeterminacies, but these special cases present no difficulty.

The source function for the thermal radiation is always isotropic and assumes the form

$$B_p(t) = Q_1 e^{\delta t} + Q_2 e^{-\delta t} + Q_3 e^{\sigma t/\mu_0} + Q_4 e^{-\sigma t/\mu_0} + Q_5 t + Q_6 \, . \qquad \text{(II-229)}$$

Substituting this into equations (II-223) and (II-225) gives

$$I_p(0,+\mu) = Q_1\frac{\left[e^{(\delta-1/n\mu)\tau_0}-1\right]}{(\delta n\mu-1)} - Q_2\frac{\left[e^{-(\delta+1/n\mu)\tau_0}-1\right]}{(\delta n\mu+1)} +$$

$$+ Q_3\frac{\left[e^{\tau_0(\sigma/\mu_0-1/n\mu)}-1\right]}{(\sigma n\mu/\mu_0-1)} - Q_4\frac{\left[e^{-(\sigma/\mu_0+1/n\mu)\tau_0}-1\right]}{(\sigma n\mu/\mu_0+1)} -$$

$$- Q_5\left[(\tau_0+n\mu)e^{-\tau_0/n\mu}-n\mu\right] - Q_6\left[e^{-\tau_0/n\mu}-1\right] \, , \qquad \text{(II-230)}$$

and
$$I_p(\tau_0,-\mu) = Q_1\frac{\left[e^{\delta\tau_0}-e^{-\tau_0/n\mu}\right]}{(\delta n\mu+1)} - Q_2\frac{\left[e^{-\delta\tau_0}-e^{-\tau_0/n\mu}\right]}{(\delta n\mu-1)} +$$

$$+ Q_3\frac{\left[e^{\sigma\tau_0/\mu_0}-e^{-\tau_0/n\mu}\right]}{(\sigma n\mu/\mu_0+1)} - Q_4\frac{\left[e^{-\sigma\tau_0/\mu_0}-e^{-\tau_0/n\mu}\right]}{(\sigma n\mu/\mu_0-1)} +$$

$$+ Q_5\left[\tau_0-n\mu(1-e^{-\tau_0/n\mu})\right] + Q_6\left[1-e^{-\tau_0/n\mu}\right] \, , \qquad \text{(II-231)}$$

for the intensity of the emergent thermal radiation from the upper and lower surfaces of a finite plane-parallel non-conservative anisotropic scattering atmosphere.

The intensity of the emergent radiation from the upper surface of a finite plane-parallel atmosphere with a ground layer at its lower surface, is found in exactly the same manner as that for the emergent radiation from the atmosphere with no ground. For isotropic scattering the source function for the scattered radiation, given by equation (II-60), can be expressed in the form

$$B_s(t) = Q_1 e^{\delta t} + Q_2 e^{-\delta t} + Q_3 e^{-t/\mu_0} + Q_4 E_2(\tau_0 - t) +$$

$$+ Q_5 \left\{ e^{\delta(\tau_0 - t)} F_2[-\delta, (\tau_0 - t)] - e^{-\delta(\tau_0 - t)} F_2[\delta, (\tau_0 - t)] \right\}. \quad \text{(II-232)}$$

Substituting this into equation (II-222) we obtain

$$I_s(0, +\mu) = Q_1 \frac{\left[e^{\tau_0(\delta - 1/\mu)} - 1 \right]}{(\delta\mu - 1)} - Q_2 \frac{\left[e^{-\tau_0(\delta + 1/\mu)} - 1 \right]}{(\delta\mu + 1)} -$$

$$- Q_3 \frac{\left[e^{-\tau_0(1/\mu_0 + 1/\mu)} - 1 \right]}{(\mu/\mu_0 + 1)} + Q_4 \frac{e^{-\tau_0/\mu}}{\mu} F_2[1/\mu, \tau_0] -$$

$$- \frac{Q_5}{(\delta\mu + 1)} \left\{ e^{-\tau_0/\mu} F_2[1/\mu, \tau_0] - e^{\delta\tau_0} F_2[-\delta, \tau_0] \right\} +$$

$$+ \frac{Q_5}{(\delta\mu - 1)} \left\{ e^{-\tau_0/\mu} F_2[1/\mu, \tau_0] - e^{-\delta\tau_0} F_2[\delta, \tau_0] \right\}. \quad \text{(II-233)}$$

for the intensity of the emergent scattered radiation. Similarly, the source function for the thermal radiation for any scattering phase function, can be expressed in the form

$$B_p(t) = Q_1 e^{\delta t} + Q_2 e^{-\delta t} + Q_3 e^{\sigma t/\mu_0} + Q_4 e^{-\sigma t/\mu_0} +$$

$$+ Q_5 E_2(\tau_0 - t) + Q_6 E_2[(\tau_0 - t)/n] + Q_7 E_4(\tau_0 - t) + Q_8 E_4[(\tau_0 - t)/n] +$$

$$+ Q_9 \left\{ e^{\delta(\tau_0 - t)} F_2[-\delta, (\tau_0 - t)] - e^{-\delta(\tau_0 - t)} F_2[\delta, (\tau_0 - t)] \right\} + Q_{10} , \qquad \text{(II-234)}$$

which, substituted into equation (II-223), gives

$$I_p(0, +\mu) = Q_1 \frac{\left[e^{\tau_0(\delta - 1/n\mu)} - 1 \right]}{(\delta n\mu - 1)} - Q_2 \frac{\left[e^{-\tau_0(\delta + 1/n\mu)} - 1 \right]}{(\delta n\mu + 1)} +$$

$$+ Q_3 \frac{\left[e^{\tau_0(\sigma/\mu_0 - 1/n\mu)} - 1 \right]}{(\sigma n\mu/\mu_0 - 1)} - Q_4 \frac{\left[e^{-\tau_0(\sigma/\mu_0 + 1/n\mu)} - 1 \right]}{(\sigma n\mu/\mu_0 + 1)} +$$

$$+ Q_5 \frac{e^{-\tau_0/n\mu}}{n\mu} F_2[1/n\mu, \tau_0] + Q_6 \frac{e^{-\tau_0/n\mu}}{\mu} F_2[1/\mu, \tau_0/n] +$$

$$+ Q_7 \frac{e^{-\tau_0/n\mu}}{n\mu} F_4[1/n\mu, \tau_0] + Q_8 \frac{e^{-\tau_0/n\mu}}{\mu} F_4[1/\mu, \tau_0/n] +$$

$$+ \frac{Q_9}{(\delta^2 n^2 \mu^2 - 1)} \left\{ (\delta n\mu - 1) e^{\delta\tau_0} F_2[-\delta, \tau_0] + (\delta n\mu + 1) e^{-\delta\tau_0} F_2[\delta, \tau_0] - \right.$$

$$\left. - 2\delta n\mu\, e^{-\tau_0/n\mu} F_2[1/n\mu, \tau_0] \right\} + Q_{10} \left[1 - e^{-\tau_0/n\mu} \right] , \qquad \text{(II-235)}$$

for the intensity of the emergent thermal radiation.

All these emergent intensities refer to the diffuse radiation fields designated with the subscripts s or p in Section II.3.2. They do not represent the total radiation emerging from the atmosphere in the part of the spectrum to which they refer. The intensity of the stellar radiation emerging from the lower surface of a finite plane-parallel atmosphere with no ground layer is given by equation (II-228) plus the term $(1/4)\pi F \exp(-\tau_0/n\mu_0)\,\delta(\mu-\mu_0)\,\delta(\phi-\phi_0)$ which is the intensity of the reduced incident radiation field at the lower surface. The intensity of the stellar radiation emerging from the upper surface of a finite plane-parallel atmosphere with a ground layer at its lower surface is given by equation (II-233) plus the term, $G_s \exp(-\tau_0/\mu)$, which is the intensity of the reduced visible ground radiation field at the upper surface. Similarly the intensity of the thermal radiation emerging from the upper surface of a finite plane-parallel atmosphere with a ground layer at its lower surface is given by equation (II-235) plus the term $G_p \exp(-\tau_0/n\mu)$, which is the intensity of the reduced thermal ground radiation field at the upper surface.

5.3. Comparison of the Solutions The derivation and description of both the approximate and the exact solutions for the angular distribution of the intensity of radiation emerging from plane-parallel scattering atmospheres has been the subject of a large number of authors. In this thesis the topic of scattering has been considered in relation to the problem of radiative heating. Consequently, we shall neither quote nor discuss the results for the emergent scattered radiation from plane-parallel atmospheres, but restrict the discussion of this section to a consideration of the emergent thermal radiation. Furthermore, the exact solutions for the emergent thermal radiation from a semi-infinite plane-parallel atmosphere with no scattering have been discussed by Stibbs (1971). The inclusion of linear scattering does not affect these results to any great extent, so we shall restrict ourselves to giving a comparison of the approximate and the exact solutions for the emergent thermal radiation in certain cases only, rather than giving a comprehensive account of either or both solutions.

5.3.1. Semi-infinite Atmospheres The numerical evaluation of the approximate solution for the intensity of the emergent thermal radiation from a semi-infinite plane-parallel atmosphere, as given by equation (II-221), is elementary. The numerical evaluation of the exact solution, as given by equation (II-140) is less straightforward. It involves the H-functions for arguments greater than unity. The H-functions were first evaluated for arguments between zero and unity, but investigations into the planetary problem led to their evaluation for arguments greater than unity. They have been tabulated by Stibbs (1963) for arguments ranging from zero to one hundred. However, in equation (II-140) the argument is $\mu_0/n\sigma$, which will frequently exceed one hundred when $n = 10^{-2}$. Thus, our results are extensive only for values of n greater than 10^{-2}; but for the case of $n = 10^{-2}$ we do have some results for small angles of incidence. For large values of n, the quantity, $H(\mu_0/n\sigma)$ is approximately $H(0)$ which equals unity.

Fig. 29 shows the emergent thermal radiation, $I_p(0,\mu)$, as a function of μ, the cosine of the angle of emergence. The graphs are plotted for normal incidence, for $n = 10^4$, for values of β of 0.1, 0.5 and 1.0, and for values of $\tilde{\omega}$ of 0.5 and 0.9. A prominent feature of each curve is the rapid increase in the intensity as grazing angles of emergence are approached. A value of 0.25 for μ corresponds to an angle of emergence of 75°. For such small values of μ the representation of a real atmosphere by a plane-parallel one ceases to be good so that these large emergent intensities are not a cause for concern. Fig. 29 applies to the case of $n = 10^4$ which is not typical of planetary atmospheres. These are the atmospheres most frequently represented by plane-parallel ones in the context of radiative heating. It has been shown, for example, ~~by~~ see Mihalas (1970), that the main contribution to the emergent scattered radiation from an atmosphere, emanates from an optical depth equal to μ. For the thermal radiation defined in terms of our optical depth unit, this principle shows that the main contribution issues from an optical depth of μ/n. When n is very large, the layers near the surface are much hotter than those deep in the atmosphere. Consequently, the main contribution to the emergent thermal radiation increases as μ decreases and is large when μ is very small. We have seen earlier, in Fig. 20, that the temperature of the atmosphere at most optical depths is greatest when the asymmetry parameter is unity and smallest when it is zero. Consequently, the emergent thermal radiation is greatest when the phase function parameters, (α, β) are equal to (0.0, 1.0). The converse is true when μ is small because the depth μ/n then lies in the surface region where the temperature decreases as the asymmetry parameter increases. Again, the smaller the value of the albedo, the greater is the contribution of the absorbed stellar radiation to the thermal radiation source function and hence the greater the emergent thermal radiation. Thus, $I_p(0, \mu)$ is greater when $\tilde{\omega} = 0.5$ than when $\tilde{\omega} = 0.9$ as seen in Fig. 29. However, this result does not apply when $\beta = 1.0$, which, for a semi-infinite atmosphere , is similar to the case of no scattering. Consider equation (II-140) which is the exact solution for the emergent thermal radiation. The denominator, $(n\sigma\mu + \mu_o)$ is approximately $n\sigma\mu$; the parameter, ψ and σ are unity and $(1 - \tilde{\omega})$ respectively, so that the equation reduces to

$$I_p(o, +\mu) = \frac{F_s \mu_o}{4\mu} H(\mu) ,$$

for all forward scattering. This result is independent of the albedo, and also of the precise value of n itself. This result is borne out in Fig. 29 also. The agreement between the approximate and the exact solutions is good for both values of $\tilde{\omega}$ shown and for all three values of β. The maximum difference between the two solutions for the thermal radiation emerging normally from a semi-infinite atmosphere is no greater than five percent.

Fig. 30 is the same as Fig. 29 with the value of n equal to unity. In this case the temperature gradient of the atmosphere, $dT/d\tau$, is positive and hence the function, $dI_p(o,\mu)/d\mu$ is also positive. As in the previous case, small values of and high values of β yield the greatest values of $I_p(o, \mu)$, with the special case of $\beta = 1.0$ showing very little dependence of $I_p(o, \mu)$ on the albedo. The agreement between the results of the exact and approximate methods is again good for $\tilde{\omega} = 0.9$ but a little less for $\tilde{\omega} = 0.5$.

Fig. 31 is the same again with $n = 10^{-2}$. However, results are shown for $\tilde{\omega} = 0.5$ and $\mu_o = 0.5$ only, because the arguments of the H-functions exceed one hundred in the cases used in Figs. 29 and 30. Nevertheless, there are sufficient results to show that there is good agreement between the exact and approximate methods, especially when μ is greater than 0.3.

5.3.2. Finite Atmospheres Firstly, we shall consider the finite atmosphere with no ground, for which we shall consider the emergent thermal radiation from the upper surface only. The approximate solution for this is given by equation (II-230) and this can be evaluated without any numerical difficulty. The exact solution for the same quantity is given by equation (II-200), or equation (II-214) if we include forward scattering. These equations involve the X- and Y-functions which are solutions of the pair of coupled integral equations, (II-142) and (II-143). The solution of these equations is a complex mathematical and computational procedure. It has been the goal of many authors to introduce practical methods of computing numerical values for these X- and Y-functions. Such are those methods introduced by Chandrasekhar and Elbert (1952), Mayers (1962), Sobouti (1963), and Bellman et al (1966). They are sufficiently complex that tabulated values of the results are given and it is very much simpler to interpolate on these tables than to generate the functions directly. More recent methods such as those of Cohen (1969) and Caldwell (1971) are simpler to perform but still involve lengthy numerical integration processes. These authors do not provide extensive tabular results. The X- and Y-functions for isotropic scattering are functions of τ_0 and μ. It has been general practice to give results for values of τ_0 less than 2 or 3 and values of μ less than unity. Only Bellman et al give results for large values of τ_0, reaching a maximum of 20.0; and only Sobouti gives results for values of μ greater than unity, reaching a maximum of 20.0.

Consider first the case of $n = 1.0$. Provided there is no scattering, the maximum value of the argument μ is unity. Thus, the results of all authors are applicable and we can produce results for many values of τ_0 less than 20.0. When scattering is present it is necessary to reduce μ_0 to 0.5 in order to obtain the equivalent set of results. The behaviour of the X- and Y-functions as τ_0 tends to infinity, has been studied by Van de Hulst (1964) and Carlstedt and Mullikin (1966). Both give asymptotic expressions for the X- and Y-functions in terms of the H-function. Those of the latter are very general and reduce to those of the former in his case of conservative isotropic scattering. Thus, a complete set of results for all values of τ_0 are available. The case of $n = 10^4$ involves arguments that are very close to zero. Furthermore, in order to obtain results for the set of values of the total optical thickness of the atmosphere as used for the approximate solutions, and measured in terms of the extinction coefficient of the stellar radiation, the X- and Y-functions are required for values of τ_0 ranging from 10^{-6} to 5×10^{-3}. All the tabular values of the X- and Y-functions have 0.1 as their lowest value of τ_0 so that the required tables are not available. Thus, for $n = 10^4$ we are limited to values of τ_0 of 10^3 and greater. For the case of $n = 10^{-2}$, the atmospheres are very thick to the thermal radiation for the standard values of τ_0 and the asymptotic equations can be used. However, they have only been derived for values of μ less than unity. Consequently we are restricted to those tables of Sobouti which are limited to τ_0 less than 3.0×10^{-2}; $\mu / n\sigma$ less than 20.0 or μ/σ less than 0.2. A full set of results awaits an extension of the tables of the X- and Y-functions to larger values of τ_0 and larger values of μ.

The X- and Y-functions that we require are those for conservative isotropic scattering, which we have already seen to be not unique. However, by using the K-integral we proved that they were the same as those for the standard scattering problem, and are the X- and Y-functions that occur in the tables. Consequently, we shall use those functions tabulated by Sobouti for values of $\tau_{0,p}$, as measured in terms of the thermal absorption coefficient, of 0.1 and 1.0.

We also obtained the exact solution for the emergent thermal radiation for isotropic scattering in the special case of n = 1.0. This is given by equation (II-215) which involves the X- and Y-functions for conservative and non-conservative isotropic scattering. Despite the restriction on n , this solution shows the effect of scattering on the emergent thermal radiation to a far greater extent than the solution for the case, $(\alpha, \beta) = (0, 1)$, which is not a physically realistic scattering phase function.

Fig. 32 shows the angular distribution of the emergent thermal radiation from a finite plane-parallel atmosphere under normal incidence with $n = 10^4$. The exact solutions are available for $\beta = 1.0$ only, and as we observed for semi-infinite atmospheres, are almost independent of the albedo. Therefore, Fig. 32 includes results for $\tilde{\omega} = 0.5$ only. The approximate solutions are given for $\beta = 0.0$ and 0.5 as well as 1.0 for completeness. As before, the continuous curves represent the exact solutions and the broken curves the approximate solutions. These optical thicknesses correspond to thermal optical thicknesses of 0.1 and 1.0 respectively In general, the function, $I_p(0, +\mu)$ has a larger negative gradient than the equivalent function for a semi-infinite atmosphere. This is because the source of thermal radiation from absorption of thermal radiation is truncated for all but very small values of μ. The atmospheres are effectively semi-infinite to the stellar radiation which is the dominant source of thermal radiation. The approximate solutions show that the parameter, β, affects the emergent thermal radiation from a finite atmosphere in the same way that it does that from a semi-infinite atmosphere. Again, a comparison between the exact and approximate solutions shows them to be in good agreement, though not sufficiently good to render them indistinguishable, the maximum deviation of the approximate solution from the exact solution being of the order of ten percent. The results are very similar for the two values of τ_0 because the dominant source of the thermal radiation when $n = 10^4$ is the stellar radiation which is independent of τ_0 when τ_0 is greater than 10^2, and we have 10^3 and 10^4 for values of τ_0 .

Fig. 33 shows the same results for $n = 1.0$. The approximate solutions are almost idependent of β so that only the case $\beta = 1.0$ is shown. When $\tau_0 = 0.1$ the temeperature gradient, $dT/d\tau$, is negative, as it was when n was 10^4, and hence, $dI_p(0,\mu)/d\mu$ is also negative. The same is true for the case, $\tau_0 = 1.0$, except when μ represents a grazing angle of emergence, when the emergent thermal radiation reaches a maximum. Fig. 33 also includes the results for isotropic scattering obtained from equation (II-215); and these are generally greater than the corresponding forward scattering results. In general, the agreement between the results from the two methods as shown in Fig. 33, is poorer than it was in Fig. 32, although there is still a great similarity in shape between the intensity distributions of the emergent thermal radiation as determined by the two different methods. The results from the approximate method are all too low by about twenty percent.

Again, the relationship between the isotropic and forward scattering results is the same for each method. We conclude that the approximate solutions for the emergent thermal radiation from an optically thin finite plane-parallel atmosphere display the correct physical characteristics of the problem but are not close enough to the exact solutions to provide accurate quantitative results.

Finally, we consider the solutions of the emergent thermal radiation from a finite plane-parallel atmosphere with a ground layer. The exact and approximate solutions are given by equations (II-204) and (II-235) respectively. The numerical evaluation of the latter is straightforward but, of the former, less so due to the numerical integrations involved in evaluating the direction integrals of the reflection and transmission functions. For simplicity we shall consider the case of no scattering and normal incidence only. Furthermore we shall restrict ourselves to the case of $n = 1.0$ because the limiting range of the tabulated X- and Y-functions does not cover the case, $n = 10^{-2}$, and only permits solution for extremely thick atmospheres of $\tau_0 = 10^3$ and 10^4 when $n = 10^4$. Also, when n is unity the numerical integration of the reflection and transmission functions can be omitted and replaced by analytical integrations whose solutions involve the standard moments of the X- and Y-functions as tabulated. Fig. 34 shows the emergent thermal radiation from such an atmosphere as a function of μ for values of τ_0 of 0.1 and 1.0 and for values of λ of 0.1, 0.5 and 1.0. It also includes the case of the finite atmosphere with no ground layer for comparison purposes. It shows clearly the ability of the ground to reflect and emit thermal radiation.

The difference between the approximate and exact solutions is essentially the same as it was for the no ground case. For $\tau_0 =$ 0.1 the agreement is good at all angles of emergence but for $\tau_0 = 1.0$ the approximate solution is too low by about ten percent. However, it is consistently too low so that the effects of the parameters on the various solutions can be seen just as convincingly from the approximate solutions. On comparing the values of G_p for the two methods it was found that there was effectively no difference when τ_0 was 0.1 for all values of $\tilde{\omega}$ and λ, and a small decrease in G_p from the exact to the approximate method when τ_0 was 1.0. This drop was independent of λ and was of the order of one or two percent.

In general, we may conclude that the qualitative results obtained by the approximate method are acceptable, and in fact, good, but the quantitative results are good in a large number of cases but a little low in certain circumstances. However, it is unfortunate that the exact solutions are not available for a wider range of values of the atmospheric parameters. When the X- and Y-functions are extended then a full set of results will be available. Nevertheless, the comparisons are consistent throughout the range of values available.

6. Summary

We have seen that it is possible to obtain analytical solutions of the equations of transfer for the mean intensities and fluxes of the scattered and thermal radiation fields in plane-parallel atmospheres under parallel illumination by using Eddington's approximation for the two part grey atmosphere with anisotropic scattering according to the schematic phase function. Using these solutions we have obtained the temperatures of the atmospheres as functions of optical depth, with the temperature of an element of mass suitably defined in Section I.5 as the temperature that a black-body would require in order to emit the same total energy in the infra-red. This was accomplished in Sections II.3.1 and II.4.1 for semi-infinite atmospheres and Sections II.3.2 and II.4.2 for finite atmospheres. Results were also obtained for finite atmospheres with ground layers that behaved as Lambertian surfaces, situated at their lower surfaces. These ground layers added no complexity to the problem other than the lengthening of the algebra and the introduction of the exponential integral and its associated functions into the analysis.

The first major point to arise in the development of the theory was the need to divide the stellar radiation field into its azimuth dependent and azimuth independent parts rather than into the reduced incident radiation and the scattered radiation. This procedure was necessary to make the anisotropic scattering problem amenable to solution by the Eddington approximation. It was possible to obtain the intensity of the azimuth dependent part exactly, so that only the remainder of the stellar radiation field was assumed to adhere to the Eddington approximation. This was a direct consequence of the schematic nature of the phase function. Consequently, the greater the absolute value of g , the closer the solution approached the exact solution.

The results for the mean intensities of the scattered and thermal radiation fields were discussed with more emphasis on the latter because the problem of scattering in plane-parallel atmospheres is a classical radiative transfer problem and its solutions are well known. The general conclusion that can be drawn from the results of the scattering problem is that the mean intensity of the scattered radiation field increases along the line of the direction of the incident radiation until a certain optical distance has been traversed, beyond which it decreases with distance due to absorption. The results for optically thick atmospheres with no ground were seen to be similar to those for semi-infinite atmospheres, the scattered radiation field merely being truncated at the appropriate optical depth. However, the mean intensity of the scattered radiation at a depth τ in an optically thin atmosphere was drastically lower than the same quantity at depth τ in a semi-infinite atmosphere.

The greenhouse parameter, n, was seen to be the most important of the atmospheric parameters in controlling the mean intensity of the thermal radiation field through equation (II-34) for semi-infinite atmospheres and equation (II-78) for finite atmospheres. The principle of conservation of energy fixed the thermal radiation flux, for a known stellar radiation flux so that the net flux at all points in the atmosphere was zero. Equations (II-34) and (II-35) express the approximate relation that the gradient of the mean intensity of a radiation field is proportional to its flux, the gradient being measured in terms of the extinction coefficient appropriate to the radiation. Thus a very small mean intensity gradient is necessary to maintain a certain flux through an atmosphere of poor absorbers, and a very large one to maintain the same flux through an atmosphere of good absorbers. Consequently the mean intensity of the thermal radiation field is constant when n is large, and has a rapid increase with optical depth, when n is small, until it reaches the depth where all the fluxes are zero, below which it remains constant.

The temperature profiles of the atmospheres are given by equation (II-101), from which it is clear that the stellar radiation field controls the temperature when n is large and the thermal radiation field when n is small. Hence, when n is large, the temperature decreases with depth until the stellar radiation field is non-existent. Below this point it is prevented from falling to zero by the presence of the thermal radiation field. This situation involves the production of thermal photons that can escape from the atmosphere with very little re-absorption, and is the inverse of the classical greenhouse effect in which the thermal photons produced cannot travel far before they are absorbed and hence maintain a large thermal radiation field. This last situation occurs when n is small, and consequently, a high temperature is maintained away from the surface of the atmosphere. These broad conclusions apply to semi-infinite atmospheres.

To a certain extent they apply to finite atmospheres also. When there is no ground at the lower surface of a finite plane-parallel atmosphere there is a non-zero flux of thermal radiation at all depths. In order to maintain this, there exists a negative thermal radiation mean intensity gradient, again inversely proportional to n , in the lower regions of the atmosphere. We have seen that, when n is small, the maximum temperature attained and the depth at which this maximum occurs depends on the balance of the these two temperature gradients. Of course, when n is large, the thermal radiation field is constant.

When there is a ground at the lower surface of a finite atmosphere a very interesting comparison can be made between the finite atmosphere plus ground system and a semi-infinite atmosphere.

This lies in the conservative natures of the ground and the semi-infinite atmosphere. The ground albedo, λ, has no effect on either radiation field when the atmosphere is optically thick, whilst for atmospheres of intermediate optical thickness, it was seen that a value of λ of 0.4 would give rise to scattered radiation fields similar to those of a semi-infinite atmosphere. No such similarity existed for optically thin atmospheres. The same conclusions apply to the temperature profiles when n is large. When n is small the thermal radiation field dominates the temperature and it was seen that the ground albedo is irrelevant when the atmosphere is optically thick and the phase function irrelevant when the atmosphere is optically thin.

The ground also behaves in a similar way to a semi-infinite atmosphere; the temperatures in an optically thick atmosphere of any value of λ are the same as those in a semi-infinite atmosphere. For optically thin atmospheres the same is true when λ assumes a value of the order of 0.3. The assignment of a temperature to the ground showed that the ground behaved like a semi-infinite atmosphere of greenhouse parameter unity and albedo, λ.
It is only in the extreme cases of $n = 10^4$ and 10^{-2} that the discontinuity in the effective values of n is apparent. When n equals unity there is, of course, no such discontinuity. Thus, when n is 10^4 the ground is very much hotter than the atmosphere in contact with it, provided the atmosphere is not thick. The converse is not true when n is 10^{-2} because in those circumstances the thermal radiation field dominates the temperature and the re-emission of this is conservative in both atmosphere and ground.

Three further general conclusions were reached in Sections II.3 and II.4. Firstly, the Eddington approximation was seen to be unable to differentiate between conservative scattering problems for isotropic incidence and parallel incidence of the same net flux normal to the surface. This was the cause of the minima of Fig. 27. Secondly, the introduction of isotropic scattering of the thermal radiation in addition to the absorption plus isotropic emission processes that contribute to the temperature, was seen to have little effect on the temperature profiles for reasonable values of the thermal scattering albedo when n was large, and no effect at all when n was small. Thirdly, the application of a lambda operator to the source function for the thermal radiation field in the cases of a semi-infinite atmosphere and a finite atmosphere with no ground, was seen to have little effect on the temperature profiles, though it did affect the mean intensity of the thermal radiation field when n was 10^4.

Exact solutions for the intensity of the emergent scattered radiation from plane-parallel atmospheres have been available for many years and also those for the intensity of the emergent thermal radiation from semi-infinite atmospheres with no scattering have been made available more recently. These solutions were based on the principles of invariance. In Section II.5 these methods were extended to give the intensity of the thermal radiation from finite atmospheres, with and without a ground layer, but with no scattering. It was also shown that linear scattering could be incorporated into the theory for semi-infinite atmospheres and forward scattering into the theory for finite atmospheres. These restrictions arose from the complex integral equation that arose when the more general problem of isotropic scattering was considered. This integral equation was neither derived nor quoted in the previous section. However, it was seen that, when n equals unity, the emergent thermal radiation can be obtained for the isotropic scattering case. This was indeed done. The inclusion of a ground layer at the lower surface of a finite plane-parallel atmosphere proved to require only an elementary extension of the theory.

The approximate solutions for the emergent thermal radiation were also obtained, and compared with the exact solutions. It was seen that there was good agreement between the two solutions for semi-infinite atmospheres and for finite atmospheres with $n = 10^4$. However, there was some discrepancy between the two solutions when τ_0 was 1.0 and n was 1.0, but this was such that the interpretation of the role of the atmospheric parameters in controlling the radiation fields was not affected at all. In general, the gradient, $dI_t(0,\mu)/d\mu$ was of the same sign as the gradient, $dT/d\tau$; so that it is the temperature profile that controls the angular distribution of the emergent thermal radiation. Unfortunately, the results of this section were by no means extensive. The X- and Y-functions were needed with arguments ranging from zero to 120 in the case when n was 10^{-2}, and for atmospheres of optical thickness ranging from 10^{-3} to 10^4. They were only available for optical thicknesses ranging from 0.1 to 5.0 and argument values from zero to 20.0. Consequently, the complete numerical expression of the theory for the exact solution for the emergent thermal radiation from finite plane-parallel atmospheres awaits the production of X- and Y-functions for wide ranges of optical thickness and argument.

CHAPTER III

SPHERICAL ATMOSPHERES

The model atmosphere to be studied in this Chapter is that of a spherical atmosphere externally illuminated; and the interstellar dust cloud with no central star, but illuminated by the interstellar radiation field is the astronomical object to which the calculations will refer. Apart from the geometrical factors, the model will be virtually identical to that used in Chapter II; and the definitions of all the particular parameters used in the model will be given, either by repetition or by reference, at the appropriate point in the development of the theory. This will be done for completeness. Where it is possible the differences in the results due to geometry will be noted but such differences are not as common as might be expected because the different restrictions involved in the two geometries ensure that equivalent problems are never realised.

The equation of transfer in spherical geometry is given by equation (I-10); and the presence of a second partial derivative of the intensity immediately makes the radiative heating problem more complicated than it was in plane-parallel geometry. It must be remembered that the equation of transfer, (I-10) is not general for spherical geometry but a special case for spherically symmetric atmospheres. In Section I.2 we discussed the applicability of methods of solution of the plane-parallel equation of transfer to the spherical equation and saw that not many methods were readily adaptable to the spherical atmosphere. However, the Eddington approximate method is easily applied to the spherical atmosphere and has been so applied by Huang (1969b) to circumstellar shells. We shall use this method. Nevertheless, the spherical geometry will necessitate the introduction of several restrictions on the model, even with this simple method of solution. For example, we shall see that the density function, $\rho(r)$, will be severely restricted.

We have obtained the exact solution for the intensity of the emergent radiation from semi-infinite and finite plane-parallel atmospheres by methods based on the principle of invariance. These solutions were formulated in terms of the H-functions and the X- and Y-functions, respectively. To date, no analogous exact solutions are available for the emergent radiation from spherical atmospheres, so that no comparison with the approximate solutions for these quantities is possible. However, we shall see that it is possible to obtain exact and approximate solutions for the mean intensity of the scattered radiation field for certain scattering phase functions, and a comparison between these results will prove a valuable test on the accuracy of the approximate solutions.

1. The Incident Radiation

In order to develop a mathematical model for a typical interstellar dust cloud illuminated by external sources, we must first investigate the nature of this illuminating radiation field. For typical interstellar dust clouds it consists mainly of the sum of the dilute radiation fields from all the stars in the galaxy and thus, is a complex function of wavelength, direction and position in the galaxy. The interstellar radiation field has been studied in a series of three papers by Lambrecht and Zimmerman. In the first of these, Lambrecht and Zimmerman (1954a), the energy density of the interstellar radiation field in the Earth's vicinity was calculated as a function of wavelength. The method employed, involved the counting of the number of stars in various latitude zones, spectral type groups and magnitude groups, as found in the star catalogues. For each group of stars, the number of stars of apparent magnitude, $0^m.0$, that would produce the same energy density in the Earth's vicinity as the group in question, was calculated; and, by assigning these stars a temperature equal to the mean temperature of their spectral group, the energy density was calculated as a function of wavelength for each latitude zone, the stars having been assumed to radiate as black-bodies at their assigned temperatures. The results of this work were later improved, Lambrecht and Zimmerman (1954b), by accounting for interstellar reddening and using recent models for the emission spectra of the appropriate stars rather than black-body spectra. This affected the energy density arising from early-type stars to the greatest extent because such stars emit radiation of a spectrum vastly different to a black-body spectrum. As well as considering the radiation fields from the stars in the three latitude zones, the individual stars, Sirius and Canopus were treated separately because they were of sufficient apparent magnitude to warrant this.

The star counts show that the proportion of early-type stars is very much higher in the lower latitudes than in the higher latitudes. Consequently, the radiation arising from low latitude stars is very much richer in low wavelength radiation, despite the increased reddening and extinction for stars near the galactic plane. Table II gives the ratios of the energy density

TABLE II

Relative contributions to the interstellar radiation field in the vicinity of the Earth at various wavelengths from different regions of the sky.

Wavelength; Å	Source		
	Low Latitudes	Medium Latitudes	Sirius
1000	5.5	3.4	1.4
3000	2.3	1.9	0.95
5000	1.4	1.0	0.57
7000	0.95	1.05	0.21

The figures quoted are the ratios of the energy density from the appropriate source to that from high latitudes. The latitude zones are: low (0°-30°); medium (30°-60°); and high (60°-90°).

of the interstellar radiation field in the vicinity of the Earth at various wavelengths from low and medium latitude stars to that from high latitude stars. It also gives the ratio of the energy density contribution from Sirius to that from the high latitude zones. We see that the low wavelength radiation comes mostly from the low latitude stars and 20% from one star, namely Sirius. However, for wavelengths greater than 4000Å the radiation field is more or less isotropic and, moreover, the contribution from Sirius is very much smaller.

The spectrum of the interstellar radiation field from all stars shows a peak around 1000Å, very little radiation at lower wavelengths, and a plateau of about 25% of the peak energy density, for wavelengths greater than 1200Å. The form of this peak in the energy density spectrum is unimportant to the radiative heating problem, if the optical properties of the constituent matter of the atmosphere are grey or slowly varying functions of wavelength in the 1000Å region of the spectrum. If they do vary considerably in this wavelength region, then the precise form of the peak is very important in the development of accurate model dust clouds. In this thesis the atmosphere is considered to be grey to the dilute stellar radiation. Consequently, it is the integrated intensity rather than the monochromatic intensity, of the interstellar radiation field that must be evaluated.

Table II also shows the directional distribution of the interstellar radiation field. Radiation of wavelength greater than 1200Å is virtually isotropic whereas radiation of wavelength less than 1200Å is highly anisotropic, the greater part coming from low latitude stars, and very little from high latitude stars. Even though the low wavelength radiation is the most energetic part of the interstellar radiation field, and hence an important contributor to the heating of the atmosphere, the range of wavelengths over which the radiation is anisotropic is not very large. Therefore, we may consider the integrated radiation field to be isotropic. The relative sizes of the dust clouds and the galaxy are such that the integrated radiation field can be assumed to uniform at all points on the surface of the cloud. Hence, we shall assume the integrated intensity of the interstellar radiation field incident on the surface of an interstellar dust cloud in the galactic plane, to be isotropic and uniform over the surface of the cloud.

The contribution to the interstellar radiation field in the vicinity of the Sun, from Sirius is of sufficient magnitude to render an approximation of isotropy for the radiation field, rather poor. In general, however, there is no reason for a particular star to contribute such a high proportion of the energy of the interstellar radiation field, so that the radiation field can still be considered to be isotropic. In Section III.7, we shall deal with the situation where an additional external radiation source such as a nearby star influences the temperature profiles of the cloud.

The discussion so far has been restricted to the interstellar radiation field and dust clouds at points in the galactic plane, in fact in the vicinity of the Sun. The interstellar radiation field at points away from the galactic plane has been calculated by Zimmerman (1964). His calculations were based on theoretical formulae for the stellar spatial density as a function of distance from the galactic plane, and were evaluated by numerical integration of all sources in the galaxy. For such points away from the galactic plane, the stellar densities in northern and southern latitudes are now different. Table III shows the ratios of the energy density of the interstellar radiation field from several latitude zones to that from the high northern latitudes, as a function of wavelength, for a point 250 parsecs to the north of the galactic plane. The ratios shown in this table are considerably higher than those shown in Table II. The galactic plane is to the south of the point in question and consequently, the southern latitudes contain many more stars than the northern latitudes and in particular contain virtually all the early-type stars. The interstellar radiation field is thus far from isotropic and any clouds in such positions away from the galactic plane must be treated in a different manner altogether from those situated in the galactic plane. We shall restrict this work to clouds near the galactic plane for which uniform isotropic incident radiation can be assumed. Finally, the interstellar radiation field at all wavelengths was assumed to be independent of longitude in the three papers mentioned hitherto. This is a good approximation in relation to the latitude approximation and simplifies the problem considerably.

Werner and Salpeter (1969) in their study of interstellar dust clouds, also assumed the incident radiation field to be uniform and isotropic. Their calculations were non-grey and they assumed the interstellar radiation field to consist of the sum of three dilute black-body functions. Even though it is the effective temperature of the incident radiation field that controls the range of temperatures attained in the cloud, their treatment of the coherent scattering would have been no more complex using the real spectrum of the interstellar radiation field rather than an approximate spectrum of the same effective temperature. In our case of a grey

TABLE III

Relative contributions to the interstellar radiation field at a point 250pc north of the galactic plane, at various wavelengths from different regions of the sky.

Wavelength Å.	Source				
	Medium Northern Latitudes	Low Northern Latitudes	Low Southern Latitudes	Medium Southern Latitudes	High Southern Latitudes
1000	1.6	3.1	13.3	32.0	29.3
3000	1.5	2.1	9.5	23.5	23.5
5000	1.3	1.3	3.0	6.9	6.4
7000	1.3	1.3	2.5	5.1	5.1

The figures quoted are the ratios of the energy density from the appropriate source to that from high northern latitudes.

atmosphere the spectrum of the incident radiation is irrelevant but the effective temperature crucial.

The interstellar radiation field, as discussed by Lambrecht and Zimmerman is that originating from the stars of the galaxy and is confined to the visible and neighbouring regions of the spectrum. In both interstellar and intergalactic space there exists the universal microwave background radiation discovered by Penzias and Wilson (1965). It has been postulated by Dicke et al (1965) to be the thermal radiation remaining from the fireball phase of the universe predicted by evolutionary cosmologies. Various source models for this radiation have been examined, for example, by Wolfe and Burbidge (1969). It is thought that this universal radiation is of a black-body nature of temperature 2.7°K, and highly isotropic. Measurements have not, as yet, proved the former, but have proved the latter, as shown by Boughn et al (1971). Nevertheless this radiation field has the same energy density as an undilute black-body radiation field of 2.7°K. The stellar radiation cannot penetrate to the central regions of optically thick clouds, so that, if the thermal radiation can penetrate to such depths, the microwave background radiation may increase the temperatures of the particles at those depths. The inclusion of an additional incident radiation field in the microwave and far infra-red regions of the spectrum will be considered in Section III.7. Its inclusion is simple because it is definitely isotropic and uniform.

The interstellar radiation field in the visible part of the spectrum can be roughly approximated to the function, $W.\,B(\nu, 10^4)$ where W is the dilution factor and is equal to 10^{-14}. This yields a value of $\sigma W(10^4)^4/\pi$ for I_0, the intensity of the integrated incident radiation. That is, $I = 10^2\,\sigma/\pi$. The undilute 2.7°K background radiation field has an integrated intensity of $I_1 = \sigma\,(2.7)^4/\pi = 53\sigma/\pi$. Thus, we have $I_1 \approx 0.53 I_0$. This ratio gives the order of magnitude of the ratio of the two input parameters, I_0 and I_1. The incident infra-red radiation may also include thermal radiation from other clouds and dust so that the ratio, I_1/I_0, could reasonably assume a value somewhere between 0.5 and 1.0.

The intensity of the reduced incident radiation and its associated moments can be obtained exactly. The geometry of the spherical atmosphere and the reduced incident radiation field is shown in Fig. 35. The intensity of the reduced incident radiation at a radial distance, r, from the centre of a spherical atmosphere of total radius, R; and in direction, $\cos^{-1}\mu$, to the radial direction is denoted by the symbol, $I^{inc}_{red}(r,\mu)$. The whole system is axially symmetric so that there is no azimuthal dependence in any of the quantities involved in this problem. We shall see in Section III.3 that it is necessary to restrict ourselves to atmospheres in which the product, $k\rho$, is independent of the position co-ordinate, r, where k is the extinction coefficient, which is grey for the incident radiation, and ρ is the density of the atmosphere. We define the optical radius, τ, by

$$d\tau = k\rho\,dr \qquad \text{and} \qquad \tau = \int_0^r k\rho\,dr = k\rho r. \tag{III-1}$$

The total optical radius of the atmosphere is $\tau_0 = k\rho R$ and the optical distance corresponding to a geometrical distance, x, is $\tau_x = k\rho x$. It is this quantity that introduces the difficulties when the density is a function of radius. We shall consider this in Section III.6. The variable, τ, is defined as an optical radius rather than an optical depth because the former case produces equations of a far greater symmetry.

The intensity of the reduced incident radiation field is therefore

$$I_{red}^{inc}(\tau, \mu) = I_0 e^{-\tau_x} \qquad \text{(III-2)}$$

where I_0 is the intensity of the incident radiation on the surface of the cloud. The nth moment of this intensity is defined as

$$I_n(\tau) = \frac{1}{2}\int_{-1}^{+1} \mu^n I_{red}^{inc}(\tau, \mu)\, d\mu$$

$$= \frac{1}{2} I_0 \int_{-1}^{+1} \mu^n e^{-\tau_x}\, d\mu . \qquad \text{(III-3)}$$

These moments can be found analytically by changing the variable of equation (III-3) from μ to τ_x, whence

$$\mu = \frac{-(\tau_0^2 - \tau^2)}{2\tau\tau_x} + \frac{\tau_x}{2\tau} ,$$

and

$$d\mu = \left[\frac{(\tau_0^2 - \tau^2)}{2\tau\tau_x^2} + \frac{1}{2\tau} \right] d\tau_x .$$

These two expressions result from the application of the cosine rule on triangle OPN of Fig. 35. On inserting them into equation (III-3), and making the appropriate change of limits, we obtain

$$I_n(\tau) = \frac{1}{2} I_o \int_{-1}^{+1} \left[-\frac{(\tau_o^2-\tau^2)}{2\tau\tau_x} + \frac{\tau_x}{2\tau} \right]^n \left[\frac{(\tau_o^2-\tau^2)}{2\tau\tau_x^2} + \frac{1}{2\tau} \right] e^{-\tau_x} d\tau_x .$$

By expanding the integrand using the series, $(a + b)^n = \sum_{r=0}^{n} {}_nC_r\, a^r b^{n-r}$ where ${}_nC_r$ are the binomial coefficients given by $n!/r!(n-r)!$, we obtain

$$I_n(\tau) = \frac{I_o}{2(2\tau)^{n+1}} \left\{ \sum_{r=0}^{n} {}_nC_r (-1)^{n-r} (\tau_o^2-\tau^2)^{n-r} \int_{\tau_o-\tau}^{\tau_o+\tau} \tau_x^{2r-n} e^{-\tau_x} d\tau_x \right.$$

$$\left. + \sum_{r=0}^{n} {}_nC_r (-1)^{n-r} (\tau_o^2-\tau^2)^{n-r+1} \int_{\tau_o-\tau}^{\tau_o+\tau} \tau_x^{2r-n-2} e^{-\tau_x} d\tau_x \right\} .$$

The integrals in the above equation can all be solved analytically and yield series of powers and exponentials, or exponential integral functions, according to whether the powers of τ_x in the integrand are, greater than or equal to zero, or less than zero, respectively. Performing these integrations gives an expression for the nth moment of the reduced incident radiation;

$$I_n(\tau) = \frac{I_o}{2(2\tau)^{n+1}} \left\{ \sum_{r=0}^{\frac{1}{2}(n-1-\delta)} (-1)^{n-r} {}_nC_r \left[(\tau_o+\tau)^{n-r} (\tau_o-\tau)^{r+1} \times \right. \right.$$

$$\left. \times E_{n-2r}(\tau_o-\tau) - (\tau_o-\tau)^{n-r} (\tau_o+\tau)^{r+1} E_{n-2r}(\tau_o+\tau) \right] +$$

$$+ \sum_{r=\frac{1}{2}(n+1-\delta)}^{n} (-1)^{n-r} {}_nC_r (\tau_o^2-\tau^2)^{n-r} (2r-n)! \left[e^{-(\tau_o-\tau)} \times \right.$$

$$\left. \times \sum_{m=0}^{2r-n} (\tau_o-\tau)^m / m! - e^{-(\tau_o+\tau)} \sum_{m=0}^{2r-n} (\tau_o+\tau)^m / m! \right] +$$

$$+ \sum_{r=0}^{\frac{1}{2}(n+1-\delta)} (-1)^{n-r} {}_nC_r \Big[(\tau_0+\tau)^{n-r+1} (\tau_0-\tau)^r E_{n-2r+2}(\tau_0-\tau) -$$

$$- (\tau_0-\tau)^{n-r+1} (\tau_0+\tau)^r E_{n-2r+2}(\tau_0+\tau) \Big] + \sum_{r=\frac{1}{2}(n+3-\delta)}^{n} (-1)^{n-r} {}_nC_r \times$$

$$\times (\tau_0^2-\tau^2)^{n-r+1} (2r-n-2)! \Big[e^{-(\tau_0-\tau)} \sum_{m=0}^{2r-n-2} (\tau_0-\tau)^{2m}/m! -$$

$$- e^{-(\tau_0+\tau)} \sum_{m=0}^{2r-n-2} (\tau_0+\tau)^m/m! \Big] \Big\} , \qquad \text{(III-4)}$$

where $\delta = 1$ if n is even, and $\delta = 0$ if n is odd.

Solution of the equation of transfer using the Eddington approximation involves the first three of the intensity moments. Now, $J^{inc}_{red}(\tau)$, $H^{inc}_{red}(\tau)$, and $K^{inc}_{red}(\tau)$ are given by equation (III-4) with n set to 0, 1 and 2 respectively. Thus, we have

$$J^{inc}_{red}(\tau) = \frac{I_0}{4\tau} \Big\{ e^{-(\tau_0-\tau)} - e^{-(\tau_0+\tau)} +$$

$$+ (\tau_0+\tau) E_2(\tau_0-\tau) - (\tau_0-\tau) E_2(\tau_0+\tau) \Big\} , \qquad \text{(III-5)}$$

$$H^{inc}_{red}(\tau) = \frac{I_0}{8\tau^2} \Big\{ (\tau_0-\tau+1) e^{-(\tau_0-\tau)} - (\tau_0+\tau+1) e^{-(\tau_0+\tau)} -$$

$$- (\tau_0+\tau)^2 E_3(\tau_0-\tau) + (\tau_0-\tau)^2 E_3(\tau_0+\tau) \Big\} , \qquad \text{(III-6)}$$

and $$K^{inc}_{red}(\tau) = \frac{I_o}{16\tau^2}\left\{ 2\left[1-(\tau_o-\tau)(\tau-1)\right]e^{-(\tau_o-\tau)} + \right.$$

$$- 2\left[1+(\tau_o+\tau)(\tau+1)\right]e^{-(\tau_o+\tau)} - (\tau_o+\tau)^2(\tau_o-\tau)E_2(\tau_o-\tau) +$$

$$\left. + (\tau_o-\tau)^2(\tau_o+\tau)E_2(\tau_o+\tau) + (\tau_o+\tau)^3 E_4(\tau_o-\tau) - (\tau_o-\tau)^3 E_4(\tau_o+\tau)\right\}. \qquad \text{(III-7)}$$

A form of equation (III-5) which will prove more amenable to the algebraic procedures that will occur in Section III.3, can be found by using the recurrence relation for the exponential integral functions. This is given in the Appendix. Hence, we obtain

$$J^{inc}_{red}(\tau) = \frac{I_o}{2\tau}\left\{ E_3(\tau_o-\tau) - E_3(\tau_o+\tau) + \right.$$

$$\left. + \tau_o\left[E_2(\tau_o-\tau) - E_2(\tau_o+\tau)\right]\right\}. \qquad \text{(III-8)}$$

These three quantities decrease in an exponential-like way from the surface to the centre of the atmosphere. When τ_o exceeds 5.0 the mean intensity of the reduced incident radiation at the centre of the atmosphere is less than 1% of its value at the surface, which approaches 0.5 from above, as τ_o tends to infinity. For optically thin atmospheres the mean intensity of the reduced incident radiation field remains within 10% of I_o, and the flux is very small at all positions in the atmosphere. The symmetry of the physical problem ensures that the flux at the centre of every atmosphere is zero.

The ratio $J^{inc}_{red}(\tau) / K^{inc}_{red}(\tau)$ is plotted in Fig. 36 for several values of τ_0. It is clear that any attempt to approximate this ratio to the constant value of 3.0 would be good for optically thin atmospheres but poor for optically thick atmospheres. In fact, for very thick atmospheres it is approximately 1.0 at most points in the atmosphere. We shall be using the Eddington approximation for solving the equations of transfer for the scattered and thermal radiation fields. The results of Fig. 36 show that the reduced incident radiation field does not adhere to Eddington's approximation, so that the solution for the scattered radiation field would be more accurate than that for the total stellar radiation field. Therefore we shall consider the reduced incident radiation field as a separate radiation field whose moments are given by equations (III-6) to (III-8).

2. The Role of the Greenhouse Parameter

We shall retain the definition of the greenhouse parameter that was used in the previous Chapter. That is, n is defined as the ratio of the extinction coefficients in the "stellar" and "thermal" parts of the spectrum, the extinction coefficients being grey in those two regions of the spectrum. To define such a parameter for a real astronomical problem it is necessary to know the spectral distribution and dilution of the incident radiation field. This, for typical interstellar dust clouds, we have discussed in the previous section and have seen that it is restricted to the visible and nearby wavelengths and that its dilution is sufficient to prevent any thermally emitted radiation occupying the same region of the spectrum. In fact, the two regions of the spectrum applicable to the interstellar dust cloud enjoy a greater separation than those applicable to the planetary atmosphere. It is, of course, necessary to construct grey absorption coefficients for the two regions of the spectrum before n can be defined. This was the subject of Section I.4 and it was seen there that a simple mean absorption coefficient such as the Planck mean absorption coefficient, is very acceptable for the vast majority of problems even though the exact derivation of a genuine mean absorption coefficient is a very complex procedure. Having calculated suitable mean absorption coefficients, we have

$$n = (\kappa_s + \sigma_s) / \kappa_p \ . \qquad \text{(III-9)}$$

We allow no scattering of the thermal radiation. The relaxation of this restriction was discussed in Section II.3.1 in connexion with semi-infinite plane-parallel atmospheres and it was seen that scattering of the thermal radiation was unimportant for normal problems.

In Section II.2 we quoted values of n from various references with regard to several radiative heating problems, and it was seen that n could adopt very small values in some problems and very large values in others. Although very small values of n are not appropriate for interstellar dust clouds, they will still be included in the following numerical work for completeness. The greenhouse parameter is generally taken to be far greater than unity for dust clouds, but it is clear from Fig. 1 of the paper by Werner and Salpeter that its precise value is extremely variable, depending on which of many grain models ~~are~~ is used.

The greenhouse parameter was assumed to be independent of optical depth for the plane-parallel atmospheres of the previous Chapter. We shall assume it to be independent of position in the spherical atmospheres also. Now, we have seen that the reduced incident radiation could only be expressed analytically when the absorption coefficients and density were assumed to remain constant throughout the atmosphere. In the light of this assumption the constancy of the greenhouse parameter follows naturally. In Section III.6 we shall relax this restriction and allow the density to vary throughout the cloud, though still retaining the spherical symmetry. We shall see that a greenhouse parameter, $n(r)$ can be accommodated by the theory. However, it will serve no purpose to allow $n(r)$ to vary with the aims of the thesis those given in Section I.7.

The greenhouse parameter is the most important of the atmospheric parameters and it will be expedient to summarise its effects on the temperature profiles of a semi-infinite atmosphere. The temperature is defined to be the fourth root of the source function for the thermal radiation, given by equation (II-30), which is the energy balance equation, (I-55), applied to the model atmosphere of Chapter II. However, it is not the greenhouse parameter itself that is important in this context but the product, $n(1 - \tilde{\omega})$, which is the ratio of the absorption coefficients, κ_s / κ_p. It is this parameter that controls the relative weights of the two radiation fields in contributing to the temperature. When $n(1-\tilde{\omega})$ is large the absorption of the thermal radiation is poor so that the emission of the thermal radiation, and hence the temperature, is controlled by the absorbed portion of the stellar radiation field. Conversely, when $n(1 - \tilde{\omega})$ is small the emission is dominated by the absorption and re-emission of the thermal radiation field. When n is large this is the only way in which it controls the temperature; that is, through $n(1-\tilde{\omega})$. However, when n is small the value of $n(1-\tilde{\omega})$ is unimportant but that of n is important by its control of the mean intensity of the thermal radiation field as seen in equation (II-33). We conclude that when n is small it is the value of $(\kappa_s + \sigma_s)/\kappa_p$ that is important, and when n is large it is the value of κ_s/κ_p. This conclusion will be endorsed by the results of this chapter.

3. The Source Function

We have already obtained expressions for the intensity of the reduced incident radiation field and its moments, so that we can now proceed with the solution for the scattered radiation field. All radiation fields are axially symmetric so there is no need to construct an azimuth dependent linear radiation field as in Chapter II. All the radiation fields are independent of azimuth in this situation. The atmosphere is grey to the stellar radiation and the scattering is anisotropic according to the phase function (I-29) whose azimuth independent form is

$$p(\mu,\mu') = \tilde{\omega}\left\{\alpha + 2(1-\alpha)\left[\beta\,\delta(\mu-\mu') + (1-\beta)\,\delta(\mu+\mu')\right]\right\}. \qquad \text{(III-10)}$$

The emission coefficient for the scattered radiation field consists of that radiation scattered from the reduced incident radiation field plus that radiation scattered from the scattered radiation field. Following the equivalent arguments of Section II.3.1, we obtain

$$j_s(r,\mu) = \frac{1}{2}\int_{-1}^{+1} p(\mu,\mu')\left[I_s(r,\mu') + I_{red}^{inc}(r,\mu')\right] d\mu' .$$

Using equation (III-10) and integrating, we obtain the source function,

$$B_s(r,\mu) = \tilde{\omega}\alpha J_s(r) + \tilde{\omega}(1-\alpha)\beta I_s(r,\mu) + \tilde{\omega}(1-\alpha)(1-\beta) I_s(r,-\mu) +$$

$$+ \tilde{\omega}\alpha J_{red}^{inc}(r) + \tilde{\omega}(1-\alpha)\beta I_{red}^{inc}(r,\mu) + \tilde{\omega}(1-\alpha)(1-\beta) I_{red}^{inc}(r,-\mu), \quad \text{(III-11)}$$

and the equation of transfer, (I-10), becomes

$$\mu \frac{\partial I_s(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_s(r,\mu)}{\partial \mu} = (\kappa_s+\sigma_s)\rho(r) \times$$

$$\times\left[-I_s(r,\mu) + \tilde{\omega}\alpha J_s(r) + \tilde{\omega}(1-\alpha)\beta I_s(r,\mu) + \tilde{\omega}(1-\alpha)(1-\beta) I_s(r,-\mu) + \right.$$

$$\left. + \tilde{\omega}\alpha J_{red}^{inc}(r) + \tilde{\omega}(1-\alpha)\beta I_{red}^{inc}(r,\mu) + \tilde{\omega}(1-\alpha)(1-\beta) I_{red}^{inc}(r,-\mu)\right]. \quad \text{(III-12)}$$

In Section III.1 it was seen that $J_{red}^{inc}(r)$ could be expressed analytically only when $(\kappa_s+\sigma_s)\rho(r)$ was independent of r. Even though $J_{red}^{inc}(r)$ can be evaluated by numerical integration for any density dunction, $\rho(r)$, a variable density function still introduces further complications. Dividing the equation of transfer, (III-12), by $(\kappa_s+\sigma_s)\rho(r)$ gives both $(\kappa_s+\sigma_s)\rho(r)\,dr$ and $(\kappa_s+\sigma_s)\rho(r)r$ in the denominator of the left-hand side of the equation. The former usually defines $d\tau$ or $-d\tau$, but the latter, if it is to be expressed as a function of τ will depend on the precise form of the function, $(\kappa_s + \sigma_s)\rho(r)$. In Section III.6 we shall consider the problem in which $\rho(r)$ is a function of r , but in this section we shall assume that κ_s , σ_s and ρ are all constants. Thus, we define

$$d\tau = (\kappa_s + \sigma_s)\rho\, dr , \tag{III-13}$$

and hence $$\tau = (\kappa_s + \sigma_s)\rho r ,$$

as in equation (III-1). With this definition of the optical radius, τ , the equation of transfer, (III-12) can be rewritten as

$$\mu \frac{\partial I_s(\tau,\mu)}{\partial \tau} + \frac{(1-\mu^2)}{\tau}\frac{\partial I_s(\tau,\mu)}{\partial \mu} = - I_s(\tau,\mu) +$$

$$+ \tilde{\omega}\alpha J_s(\tau) + \tilde{\omega}(1-\alpha)\beta I_s(\tau,\mu) + \tilde{\omega}(1-\alpha)(1-\beta) I_s(\tau,-\mu) +$$

$$+ \tilde{\omega}\alpha J_{red}^{inc}(\tau) + \tilde{\omega}(1-\alpha)\beta I_{red}^{inc}(\tau,\mu) + \tilde{\omega}(1-\alpha)(1-\beta) I_{red}^{inc}(\tau,-\mu). \tag{III-14}$$

We shall solve this equation by the same method that was employed in Chapter II. Applying the integral operators, L_0 and L_1 defined by equation (I-15), we obtain

$$\frac{dH_s(\tau)}{d\tau} + \frac{2H_s(\tau)}{\tau} = -(1-\tilde{\omega}) J_s(\tau) + \tilde{\omega} J_{red}^{inc}(\tau) , \tag{III-15}$$

and $$\frac{dK_s(\tau)}{d\tau} + \frac{1}{\tau}\left[3K_s(\tau) - J_s(\tau)\right] = -\gamma H_s(\tau) + (1-\gamma)H^{inc}_{red}(\tau),$$

respectively, where $$\gamma = 1 - \tilde{\omega}(1-\alpha)(2\beta-1) = 1 - g, \qquad \text{(III-16)}$$

g being the asymmetry parameter. Applying the Eddington approximation, $K_s(\tau) = (1/3)J_s(\tau)$, to the second moment equation, we obtain

$$\frac{dJ_s(\tau)}{d\tau} = -3\gamma H_s(\tau) + 3(1-\gamma)H^{inc}_{red}(\tau). \qquad \text{(III-17)}$$

Differentiating this equation and using equation (III-15), we obtain the following second order total differential equation for the mean intensity of the scattered radiation field.

$$\frac{d^2 J_s(\tau)}{d\tau^2} + \frac{2}{\tau}\frac{dJ_s(\tau)}{d\tau} - 3\gamma(1-\tilde{\omega})J_s(\tau) = -3\left[1-\tilde{\omega}(1-\gamma)\right]J^{inc}_{red}(\tau).$$

In constructing this differential equation we have used the relation

$$\frac{dH^{inc}_{red}(\tau)}{d\tau} + \frac{2H^{inc}_{red}(\tau)}{\tau} = -J^{inc}_{red}(\tau), \qquad \text{(III-18)}$$

which is derived from equations (III-5) and (III-6). It is convenient to change the dependent variable $J_s(\tau)$ to $f(\tau) = \tau J_s(\tau)$ by using the relation

$$\left[D^2 + \frac{2}{\tau}D\right]f(\tau) = \frac{1}{\tau}D^2\left[\tau f(\tau)\right],$$

so that we obtain

$$\left[D^2 - \epsilon^2\right]\left[\tau J_s(\tau)\right] = -\lambda\tau J^{inc}_{red}(\tau), \qquad \text{(III-19)}$$

where $\quad \epsilon^2 = 3\gamma(1-\tilde{\omega}) \quad$ and $\quad \lambda = 3(1-\gamma)+3\tilde{\omega}\gamma. \quad$ (III-20)

We have now converted the integro-differential equation for the intensity of the scattered radiation field into a total differential equation for the mean intensity. Equation (III-19) is the spherical analogue of equations (II-23) and (II-65) for semi-infinite and finite plane-parallel atmospheres. However, we have $\tau J_s(\tau)$ as the variable for the spherical atmosphere rather than $J_s(\tau)$ which was the variable for the plane-parallel atmospheres. Hummer and Rybicki (1971) have claimed that $\tau^2 J_s(\tau)$ is the most appropriate variable for problems in spherical geometry. Their work was concerned with conservative problems with a central source for which $\tau^2 H(\tau)$ was constant, and in such cases their conclusion is true. As we shall see later, our differential equations for the thermal radiation field, which constitutes a conservative problem also, can be expressed with either $\tau J_p(\tau)$ or $\tau^2 J_p(\tau)$ as the variable. It is the presence of the factor, $1/\tau$, in the source functions $B_s(\tau)$ and $B_p(\tau)$ that makes $\tau J_s(\tau)$ and $\tau J_p(\tau)$ infinitely more preferable as variables than $\tau^2 J_s(\tau)$ and $\tau^2 J_p(\tau)$. Thus we see that it is the external radiation that controls the choice of the most suitable variable.

The most convenient form of $J^{inc}_{red}(\tau)$ to use in equation (III-19) is that given by equation (III-8), so that

$$[D^2-\epsilon^2][\tau J_s(\tau)] = -\tfrac{1}{2}\lambda I_0 \Big\{ E_3(\tau_0-\tau) - E_3(\tau_0+\tau) + \tau_0\big[E_2(\tau_0-\tau) - E_2(\tau_0+\tau)\big]\Big\}, \qquad \text{(III-21)}$$

the general solution of which is

$$\tau J_s(\tau) = C_1 e^{\epsilon\tau} + C_2 e^{-\epsilon\tau} - \frac{\lambda I_0}{4\epsilon}\Big\{ e^{\epsilon(\tau_0-\tau)} F_3[-\epsilon,(\tau_0-\tau)] -$$

$$- e^{-\epsilon(\tau_0-\tau)} F_3[\epsilon,(\tau_0-\tau)] - e^{\epsilon(\tau_0+\tau)} F_3[-\epsilon,(\tau_0+\tau)] + e^{-\epsilon(\tau_0+\tau)} \times$$

$$\times F_3[\epsilon,(\tau_0+\tau)] + \tau_0 \Big[e^{\epsilon(\tau_0-\tau)} F_2[-\epsilon,(\tau_0-\tau)] - e^{-\epsilon(\tau_0-\tau)} F_2[\epsilon,(\tau_0-\tau)] -$$

$$- e^{\epsilon(\tau_0+\tau)} F_2[-\epsilon,(\tau_0+\tau)] + e^{-\epsilon(\tau_0+\tau)} F_2[\epsilon,(\tau_0+\tau)] \Big] \Big\}. \quad \text{(III-22)}$$

The two constants of integration, C_1 and C_2, are determined by the use of two boundary conditions. The first of these is, that the mean intensity of the scattered radiation field must not be infinite at the origin. It is convenient to express equation (III-22) in the form

$$\tau J_s(\tau) = C_1 e^{\epsilon\tau} + C_2 e^{-\epsilon\tau} - \frac{\lambda I_0}{4\epsilon} f(\tau). \quad \text{(III-23)}$$

The indeterminacy that arises in this equation when τ is zero is surmounted by application of L'Hopital's rule, because $f(0)$ and $df(\tau)/d\tau|_0$ are zero also. Hence, the first boundary condition can be satisfied only if

$$C_2 = -C_1. \quad \text{(III-24)}$$

The second boundary that we apply is the Eddington approximate boundary condition

$$J_s(\tau_0) = 2H_s(\tau_0). \qquad \text{(III-25)}$$

Using equations (III-17) and (III-23) to (III-25), we obtain

$$C_1 = \frac{\left[\frac{\lambda I_0}{4\epsilon} \left\{ \left[1 - \frac{3}{2}\tau_0\right] f(\tau_0) - \tau_0 \left.\frac{df(\tau)}{d\tau}\right|_{\tau_0} \right\} - 3(1-\gamma)\tau_0^2 H_{red}^{inc}(\tau_0) \right]}{\left[(1 - \frac{3}{2}\gamma\tau_0 - \epsilon\tau_0) e^{\epsilon\tau_0} - (1 - \frac{3}{2}\gamma\tau_0 + \epsilon\tau_0) e^{-\epsilon\tau_0} \right]}. \qquad \text{(III-26)}$$

A glance at the function $f(\tau_0)$ shows that it includes terms involving the exponential of $2\epsilon\tau_0$. The equation for $J_s(\tau)$, equation (III-23), involves the subtraction of two terms of this dimension, so that, on inserting numerical values for the relevant atmospheric parameters into this equation, a considerable loss of significance occurs when τ_0 is large in a computer programme designed to evaluate the function $J_s(\tau)$. Consequently, we rearrange equation (III-22) into a form more suitable for yielding numerical values. Such a form is:

$$J_s(\tau) = \frac{\lambda I_0}{4\epsilon\tau} \Big\{ e^{2\epsilon\tau_0} e^{\epsilon\tau} \Big[(\zeta - \epsilon\tau_0)\tau_0 \Big(F_2[-\epsilon, (\tau_0+\tau)] - F_2[-\epsilon, 2\tau_0] \Big) + (\zeta - \epsilon\tau_0) \Big[F_3[-\epsilon, (\tau_0+\tau)] - F_3[\epsilon, 2\tau_0] \Big] -$$

$$- e^{2\epsilon\tau_0} e^{-\epsilon\tau} \Big[(\zeta - \epsilon\tau_0)\tau_0 \Big(F_2[-\epsilon, (\tau_0-\tau)] - F_2[-\epsilon, 2\tau_0] \Big) +$$

$$+ (\zeta + 2\epsilon\tau_0) \Big(F_3[-\epsilon, (\tau_0-\tau)] - F_3[-\epsilon, 2\tau_0] \Big) \Big] - e^{-2\epsilon\tau_0} e^{\epsilon\tau}$$

$$\times \Big[(\zeta+\epsilon\tau_0)\tau_0 \big(F_2[\epsilon,(\tau_0-\tau)] - F_2[\epsilon,2\tau_0]\big) + (\zeta+\epsilon\tau_0)\times$$

$$\times \big(F_3[\epsilon,(\tau_0-\tau)] - F_3[\epsilon,2\tau_0]\big)\Big] + e^{-2\epsilon\tau_0} e^{\epsilon\tau} \Big[(\zeta+\epsilon\tau_0)\tau_0 \times$$

$$\times \big(F_2[\epsilon,(\tau_0+\tau)] - F_2[\epsilon,2\tau_0]\big) + (\zeta+\epsilon\tau_0)\big(F_3[\epsilon,(\tau_0+\tau)] -$$

$$- F_3[\epsilon,2\tau_0]\big)\Big] + e^{\epsilon\tau}\Big[(\zeta-\epsilon\tau_0)\big(\tau_0 F_2[\epsilon,(\tau_0-\tau)] +$$

$$+ F_3[\epsilon,(\tau_0-\tau)]\big) - (\zeta+\epsilon\tau_0)\big(\tau_0 F_2[-\epsilon,(\tau_0+\tau)] +$$

$$+ F_3[-\epsilon,(\tau_0+\tau)] + e^{-\epsilon\tau}\Big[(\zeta+\epsilon\tau_0)\big(\tau_0 F_2[-\epsilon,(\tau_0-\tau)] +$$

$$+ F_3[-\epsilon,(\tau_0-\tau)]\big) - (\zeta-\epsilon\tau_0)\big(\tau_0 F_2[\epsilon,(\tau_0+\tau)] + F_3[\epsilon,(\tau_0+\tau)]\big)\Big] -$$

$$- \frac{3\epsilon}{2\lambda}(1-\delta)\left[e^{\epsilon\tau} - e^{-\epsilon\tau}\right]\left[1-2\tau_0^2 - (2\tau_0+1)e^{-2\tau_0}\right] \Big\} \Big/$$

$$\Big\{ (\zeta-\epsilon\tau_0)e^{\epsilon\tau_0} - (\zeta+\epsilon\tau_0)e^{-\epsilon\tau_0} \Big\}, \qquad \text{(III-27)}$$

where
$$\zeta = 1 - 3\gamma\tau_0/2\,, \qquad \text{(III-28)}$$

and we have used the relation

$$H^{inc}_{red}(\tau_0) = 1 - 2\tau_0^2 - (2\tau_0+1)e^{-2\tau_0},$$

which is an elementary reduction of equation (III-6).

It is necessary to evaluate $J_s(\tau)$ for several special cases. It will suffice to record the appropriate solutions. Firstly, at the origin

$$J_s(0) = \tfrac{1}{2}\lambda I_0 \Big\{ e^{2\epsilon\tau_0}(\zeta-\epsilon\tau_0)\big[\tau_0\big(F_2[-\epsilon,\tau_0] - F_2[-\epsilon,2\tau_0]\big) + F_3[-\epsilon,\tau_0] - F_3[-\epsilon,2\tau_0]\big] - e^{-2\epsilon\tau_0}(\zeta+\epsilon\tau_0)\big[\tau_0\big(F_2[\epsilon,\tau_0] - F_2[\epsilon,2\tau_0]\big) + F_3[\epsilon,\tau_0] - F_3[\epsilon,2\tau_0]\big] + (\zeta-\epsilon\tau_0)\times \big[\tau_0 F_2[\epsilon,\tau_0] + F_3[\epsilon,\tau_0]\big] - (\zeta+\epsilon\tau_0)\big[\tau_0 F_2[-\epsilon,\tau_0] + F_3[-\epsilon,\tau_0]\big] - \frac{3\epsilon(1-\delta)}{2\lambda}\big[1-2\tau_0^2 - (2\tau_0+1)e^{-2\tau_0}\big]\Big\} \Big/ \Big\{(\zeta-\epsilon\tau_0)e^{\epsilon\tau_0} - (\zeta+\epsilon\tau_0)e^{-\epsilon\tau_0}\Big\}, \tag{III-29}$$

and secondly, at the surface

$$J_s(\tau_0) = \tfrac{1}{2}\lambda I_0 \Big\{ -e^{-\epsilon\tau_0}\big[\tau_0 F_2[-\epsilon,2\tau_0] + F_3[-\epsilon,2\tau_0]\big] + e^{-\epsilon\tau_0}\big[\tau_0 F_2[\epsilon,2\tau_0] + F_3[\epsilon,2\tau_0]\big] - \frac{3(1-\delta)}{4\lambda\tau_0}\big[e^{\epsilon\tau_0} - e^{-\epsilon\tau_0}\big]\big[1-2\tau_0^2 - (2\tau_0+1)e^{-2\tau_0}\big]\Big\} \Big/ \Big\{(\zeta-\epsilon\tau_0)e^{\epsilon\tau_0} - (\zeta+\epsilon\tau_0)e^{-\epsilon\tau_0}\Big\}. \tag{III-30}$$

Thirdly, we find that an indeterminacy exists for conservative scattering, for which $\epsilon = 0$. Reference will be made to this solution later, but it is not important in the study of radiative heating because, in such problems the scattering is necessarily non-conservative. An indeterminacy would also exist were $(\zeta - \epsilon\tau_0)e^{\epsilon\tau_0} = (\zeta + \epsilon\tau_0)e^{-\epsilon\tau_0}$; but this condition is never fulfilled. For large values of τ_0 , this is clear because the parameter, ζ , is not an exponential in τ_0 . For small values of τ_0, such that $e^{\pm\epsilon\tau_0} = 1 \pm \epsilon\tau_0$, it is necessary that ζ is unity for the indeterminate condition to be fulfilled. However, for ζ to equal unity it is necessary for either γ or τ_0 to be zero, neither of which is possible for non-conservative scattering. Thus, the indeterminacy never arises in practice.

The source function for the thermal radiation is the same as it was for the plane-parallel atmosphere with no ground, which is

$$B_p(\tau) = J_p(\tau) + n(1-\tilde{\omega})\left[J_s(\tau) + J^{inc}_{red}(\tau) \right]. \qquad \text{(III-31)}$$

The equation of transfer for the thermal radiation in a spherically symmetric atmosphere is thus

$$\mu \frac{\partial I_p(\tau,\mu)}{\partial \tau} + \frac{(1-\mu^2)}{\tau} \frac{\partial I_p(\tau,\mu)}{\partial \mu} = -\frac{1}{n} I_p(\tau,\mu) +$$

$$+ \frac{1}{n} J_p(\tau) + (1-\tilde{\omega})\left[J_s(\tau) + J^{inc}_{red}(\tau) \right], \qquad \text{(III-32)}$$

where $d\tau = (\kappa_s + \sigma_s)\rho\, dr$ as before.

It will prove valuable to solve first the special case in which there is no scattering. In this case $\tilde{\omega} = 0$ and $J_s(\tau) = 0$, and equation (III-32) reduces to

$$\mu \frac{\partial I_p(\tau,\mu)}{\partial \tau} + \frac{(1-\mu^2)}{\tau}\frac{\partial I_p(\tau,\mu)}{\partial \mu} = -\frac{1}{n} I_p(\tau,\mu) + \frac{1}{n} J_p(\tau) + J^{inc}_{red}(\tau). \qquad \text{(III-33)}$$

The first two moment equations derived from the equation of transfer in the normal way, are

$$\frac{dH_p(\tau)}{d\tau} + \frac{2H_p(\tau)}{\tau} = J^{inc}_{red}(\tau), \qquad \text{(III-34)}$$

and

$$\frac{dJ_p(\tau)}{d\tau} = -\frac{3}{n} H_p(\tau), \qquad \text{(III-35)}$$

where we have used the Eddington approximation, $K_p(\tau) = (1/3)\,J_p(\tau)$ in forming the second of these. Equation (III-34) can be solved directly when expressed in the form

$$\frac{d^2}{d\tau^2}\left(\tau^2 H_p(\tau)\right) = \tau^2 J^{inc}_{red}(\tau),$$

but it is algebraically simpler to solve the second order differential equation formed by combining equations (III-34) and (III-35), which is

$$D^2\left[\tau J_p(\tau)\right] = -\frac{3\tau}{n} J^{inc}_{red}(\tau). \qquad \text{(III-36)}$$

This equation is the simpler to solve on account of the $1/\tau$ dependence of $J^{inc}_{red}(\tau)$. The solution of equation (III-36) with $J^{inc}_{red}(\tau)$ given by equation (III-8), is

$$J_p^{\circ}(\tau) = \frac{-3I_0}{2n\tau}\left\{ E_5(\tau_0-\tau) - E_5(\tau_0+\tau) + \tau_0\left[E_4(\tau_0-\tau) - E_4(\tau_0+\tau)\right]\right\} + C_1 + C_2/\tau , \qquad \text{(III-37)}$$

where the superscript zero is added to denote the condition, $\tilde{\omega} = 0$. The boundary conditions that we must impose are the same for all radiation fields in the atmosphere, namely that the mean intensity cannot be infinite at the origin and that $J_p(\tau_0) = 2H_p(\tau_0)$, which is Eddington's approximate boundary condition. The former ensures that C_2 is zero and the latter gives an expression for C_1.

We now consider the case in which $\tilde{\omega}$ is not equal to zero. The two moment equations derived from the appropriate equation of transfer, equation (III-32), are

$$\frac{dH_p(\tau)}{d\tau} + \frac{2H_p(\tau)}{\tau} = (1-\tilde{\omega})\left[J_s(\tau) + J_{red}^{inc}(\tau)\right], \qquad \text{(III-38)}$$

and

$$\frac{dJ_p(\tau)}{d\tau} = -\frac{3}{n} H_p(\tau) , \qquad \text{(III-39)}$$

where we have again used the Eddington approximation. Again, these equations may be solved independently, but the $1/\tau$ factor in $J_s(\tau)$ and $J_{red}^{inc}(\tau)$ make the algebra simpler if the two equations are combined to give

$$D^2\left[\tau J_p(\tau)\right] = -\frac{3\tau(1-\tilde{\omega})}{n}\left[J_s(\tau) + J_{red}^{inc}(\tau)\right], \qquad \text{(III-40)}$$

$D^2 f + \frac{3(1-\tilde{\omega})}{n} f = -\frac{3(1-\tilde{\omega})}{n}$ [illegible]

what about the [illegible] ?

[illegible]

or
$$J_p(\tau) = C_1 + C_2/\tau - \frac{3(1-\tilde{\omega})}{n\tau} \iint \tau J^{inc}_{red}(\tau)\, d^2\tau$$

$$- \frac{3(1-\tilde{\omega})}{n\tau} \iint \tau J_s(\tau) d^2\tau . \qquad \text{(III-41)}$$

The double integral of $\tau J^{inc}_{red}(\tau)$ has already been evaluated, so that the first term in equation (III-41) is $(1 - \tilde{\omega}) J_p^\circ(\tau)$ where the constants in equation (III-37) are both set to zero. They can be absorbed into the constants of equation (III-41). The algebra involved in the double integral of $\tau J_s(\tau)$ is involved but leads to the simple relation

$$\iint \tau J_s(\tau)\, d^2\tau = \frac{1}{\epsilon^2} \tau J_s(\tau) - \frac{n\lambda\tau}{3\epsilon^2} J_p^\circ(\tau).$$

Thus, we can express $J_p(\tau)$ in terms of $J_p^\circ(\tau)$ and $J_s(\tau)$. That is

$$J_p(\tau) = C_1 + C_2/\tau - \frac{3(1-\tilde{\omega})}{n\epsilon^2} \left[J_s(\tau) - \frac{n}{3}(\lambda+\epsilon^2) J_p^\circ(\tau) \right]. \qquad \text{(III-42)}$$

Again, the boundary condition at the origin that $J_p(0)$ must remain finite demands that C_2 equals zero. Before applying the second boundary condition, which is Eddington's boundary condition, $J_p(\tau_0) = 2 H_p(\tau_0)$, we must ensure that energy is conserved in the system. This is the same as insisting that the net radiation flux at any point in the atmosphere is zero. The flux of the thermal radiation is found from equations (III-38) and (III-42), and is

$$H_p(\tau) = \frac{(1-\tilde{\omega})}{\epsilon^2} \left[\frac{dJ_p(\tau)}{d\tau} - \frac{n}{3}(\lambda+\epsilon^2) \frac{dJ_p^\circ(\tau)}{d\tau} \right].$$

Now, we have $$\frac{d\mathcal{J}_s(\tau)}{d\tau} = -3\gamma H_s(\tau) + 3(1-\gamma) H_{red}^{inc}(\tau),$$

and $$(1-\tilde{\omega})/\epsilon^2 = 1/3\gamma,$$

from equations (III-17) and (III-20) respectively. Therefore

$$H_p(\tau) = -H_s(\tau) + \frac{(1-\gamma)}{\gamma} H_{red}^{inc}(\tau) - \frac{n(\lambda+\epsilon^2)}{3.3\gamma}\frac{d\mathcal{J}_p(\tau)}{d\tau}.$$

Comparing equations (III-34) and (III-38) we see that

$$H_p^{\circ}(\tau) = -H_{red}^{inc}(\tau), \tag{III-43}$$

and hence, we have

$$\frac{d\mathcal{J}_p^{\circ}(\tau)}{d\tau} = \frac{3}{n} H_{red}^{inc}(\tau). \tag{III-44}$$

Therefore $$H_p(\tau) = -H_s(\tau) + \frac{(1-\gamma)}{\gamma} H_{red}^{inc}(\tau) - \frac{(\lambda+\epsilon^2)}{3\gamma} H_{red}^{inc}(\tau).$$

Now, $\lambda + \epsilon^2 = 3$, so that we have

$$H_p(\tau) + H_s(\tau) + H_{red}^{inc}(\tau) = 0, \tag{III-45}$$

and we have ensured that energy is conserved at all depths τ. The second integration constant, which is found from Eddington's approximate boundary condition is thus given by

$$C_1 = -2H_s(\tau_0) - 2H_{red}^{inc}(\tau_0) + \frac{1}{n\gamma}\left[\mathcal{J}_s(\tau_0) - n\mathcal{J}_p^{\circ}(\tau_0)\right].$$

Thus, we obtain our final form for $J_p(\tau)$, which is

$$J_p(\tau) = C_1 - \frac{1}{n\gamma}\left[J_s(\tau) - n\, J_p^{\circ}(\tau)\right]. \qquad \text{(III-46)}$$

There are no special cases for which this equation is not valid. When $\tilde{\omega} = 0$ it does reduce to equation (III-37); and the parameter, γ, is always non-zero.

When considering plane-parallel atmospheres in Chapter II it was found necessary to separate the azimuthally dependent part of the scattered radiation field from the azimuthally independent part because the solution of the equation of transfer by Eddington's method is only possible for azimuthally independent radiation fields. The azimuthally dependent part could be found exactly for the phase function chosen. As we have seen, the spherical symmetry of the problem annulled the need for such a procedure in this Chapter. Nevertheless, this process can still be implemented, and it will prove a useful comparison with the standard method described above. Moreover, for the case, $a = 0$, the second method will give the exact solution for the scattered radiation field and a comparison of this with the standard method will prove rewarding.

We must first derive expressions for the intensity of the radiation field that consists of the reduced incident radiation plus the radiation multiply scattered in the spikes of the schematic phase function. As in Chapter II, this radiation field will be referred to as the linear radiation field and its intensity moments will be designated the symbols, $J_{lin}(\tau)$, $H_{lin}(\tau)$ and $K_{lin}(\tau)$. The general solution for the intensity of radiation scattered in a one-dimensional medium was developed in Section II.3.2. In this case there is radiation of intensity, I_0, incident on each end of the "line", so that the constants of integration, C_1 and C_2, of equations(II-49) and (II-51) are given by equations (II-55). That is

$$C_1 = \frac{\left\{ \tilde{\omega}(1-\alpha)(1-\beta) - [1-\tilde{\omega}\beta(1-\alpha)-\sigma]e^{-\sigma x_0} \right\}}{\left\{ [1-\tilde{\omega}\beta(1-\alpha)+\sigma]e^{\sigma x_0} - [1-\tilde{\omega}\beta(1-\alpha)-\sigma]e^{-\sigma x_0} \right\}} , \qquad \text{(III-47)}$$

and $\quad C_2 = I_0 - C_1$.

Where x_0 is the optical length of the "line of transfer" in question, which in this case is a chord across the atmosphere and is given by

$$x_0 = 2\sqrt{\tau_0^2 - \tau^2(1-\mu^2)} \quad . \qquad \text{(III-48)}$$

Thus, if x is the distance along that chord at which the intensity is described, and is related to τ and x_0 by

$$x = \frac{1}{2}x_0 + \tau\mu \quad , \qquad \text{(III-49)}$$

the mean intensity of the linear radiation field is

$$J_{lin}(\tau) = \frac{1}{2}\int_0^1 \frac{\left[C_1 e^{\sigma x}[1-\tilde{\omega}\beta(1-\alpha)+\sigma] + C_2 e^{-\sigma x}[1-\tilde{\omega}\beta(1-\alpha)-\sigma]\right]}{\tilde{\omega}(1-\alpha)(1-\beta)} d\mu$$

$$+ \frac{1}{2}\int_0^1 \left[C_1 e^{\sigma x} + C_2 e^{-\sigma x} \right] d\mu . \qquad \text{(III-50)}$$

The nature of x and x_0 as functions of μ, together with the form of the constants C_1 and C_2, make the analytical integration of equation

(III-50) impossible, unless $\beta = 1.0$ when the equations adopt special forms. Otherwise equation (III-50) can be integrated easily using a mechanical quadrature or similar form of numerical integration. Now when $\beta = 1.0$ we have

$$I_{lin}(\tau, \mu) = I_0 \, e^{-[1-\tilde{\omega}(1-\alpha)]\tau},$$

and hence $$\mathcal{J}_{lin}(\tau) = \mathcal{J}^{inc}_{red}(\nu\tau), \qquad \text{(III-51)}$$

where $$\nu = 1 - \tilde{\omega}(1-\alpha). \qquad \text{(III-52)}$$

The emission coefficient for the remainder of the scattered radiation field is comprised of the radiation scattered isotropically from the linear radiation field and the radiation scattered anistropically from the scattered radiation field. By analogy to equations (II-19) and (III-14) the equation of transfer for this part of the scattered radiation field is

$$\mu \frac{\partial I_s(\tau,\mu)}{\partial \tau} + \frac{(1-\mu^2)}{\tau} \frac{\partial I_s(\tau,\mu)}{\partial \mu} = - I_s(\tau,\mu) +$$

$$+ \tilde{\omega}\alpha \, \mathcal{J}_s(\tau) + \tilde{\omega}\alpha \, \mathcal{J}_{lin}(\tau) + \tilde{\omega}(1-\alpha) I_s(\tau,\mu) \qquad \text{(III-53)}$$

where $I_s(\tau,\mu)$ now refers to the intensity of this partially scattered radiation field, and we have set β to unity in order to use the analytical expression for $\mathcal{J}_{lin}(\tau)$. The first two moments of equation (III-53) are

$$\frac{dH_s(\tau)}{d\tau} + \frac{2H_s(\tau)}{\tau} = -(1-\tilde{\omega})J_s(\tau)+\tilde{\omega}\alpha J_{lin}(\tau), \qquad \text{(III-54)}$$

and
$$\frac{dJ_s(\tau)}{d\tau} = -3\gamma H_s(\tau), \qquad \text{(III-55)}$$

where
$$\gamma = 1-\tilde{\omega}(1-\alpha) = \nu .$$

In forming equation (III-55) we have used Eddington's approximation, $K_s(\tau) = (1/3)J_s(\tau)$. We must note that this is not the same approximation as that used in the first method. In the first method we applied it to the whole scattered radiation field, but here we apply it to only part of the scattered radiation field. Combining equations (III-54) and (III-55) we obtain

$$\left[\frac{d^2}{d\tau^2} - \epsilon^2\nu^2\right]\left[\tau J_s(\tau)\right] = -3\tilde{\omega}\alpha\gamma\tau J^{inc}_{red}(\nu\tau),$$

where
$$\epsilon^2\nu^2 = 3\gamma(1-\tilde{\omega}).$$

On changing variables from τ to $t=\nu\tau$, we have

$$\left[\frac{d^2}{dt^2} - \epsilon^2\right]\left[tJ_s(t)\right] = -\frac{3\tilde{\omega}\alpha\gamma t}{\nu^2} J^{inc}_{red}(t). \qquad \text{(III-56)}$$

This equation must be solved with two boundary conditions which are, that $J_s(o)$ is finite, and that $J_s(\tau_0) = 2H_s(\tau_0)$. Now by the first method we derived equation (III-19), which is

$$\left[\frac{d^2}{d\tau^2} - \epsilon^2\right]\left[\tau J_s(\tau)\right] = -\lambda\tau J^{inc}_{red}(\tau) , \qquad \text{(III-57)}$$

and was subject to the same two boundary conditions as equation (III-56). The solution of equation (III-57) with its two boundary conditions was $J_s^1(\tau)$ as given by equation (III-27), where the superscript, unity, refers to functions derived by the first method. The similarity between equations (III-56) and (III-57) and their boundary conditions shows that the solution of equation (III-56) is

$$J_s(t) = \frac{3\tilde{\omega}\alpha\gamma}{\lambda\nu^2} J_s'(t)$$

or

$$J_s(\tau) = \frac{3\tilde{\omega}\alpha\gamma}{\lambda\nu^2} J_s'(\nu\tau) , \qquad \text{(III-58)}$$

where ϵ^2, which appears as a constant in $J_s'(\gamma)$ is now given by $3\gamma(1-\tilde{\omega})/\nu^2$.

The equation of transfer for the thermal radiation is unchanged from equation (III-32) and again reduces to equation (III-40), which in this case is

$$\frac{d^2}{d\tau^2}\left[\tau J_p(\tau)\right] = -\frac{3(1-\tilde{\omega})\tau}{n}\left[\frac{3\tilde{\omega}\alpha\gamma}{\lambda\nu^2} J_s'(\nu\tau) + J^{inc}_{red}(\nu\tau)\right].$$

Substituting $t = \nu\tau$, we have

$$t J_p(t) = C_2 + C_1 t - \frac{3(1-\tilde{\omega})\, 3\tilde{\omega}\alpha\gamma}{\lambda n\nu^4}\iint t J_s'(t)\, d^2t - \frac{3(1-\tilde{\omega})}{n\nu^2}\iint t J^{inc}_{red}(t)\, d^2t .$$

Now
$$-\frac{3}{n}\iint \tau\, \mathcal{J}^{inc}_{Rd}(\tau)\, d^2\tau = \tau\, \mathcal{J}^{\circ}_{p}(\tau),$$

where $J^{\circ}_{p}(\tau)$ is given by equation (III-37) with the constants therein set to zero. The solution for the case of no scattering is, of course, the same for each method. We have also seen that

$$-\frac{3}{n}\iint \tau\, \mathcal{J}'_{s}(\tau)\, d^2\tau = -\frac{3\tau}{n\epsilon^2}\mathcal{J}'_{s}(\tau) + \frac{\lambda\tau}{\epsilon^2}\mathcal{J}^{\circ}_{p}(\tau).$$

Therefore
$$t\,\mathcal{J}_p(t) = C_2 + C_1 t - \frac{3(1-\tilde{\omega})3\tilde{\omega}\alpha\gamma}{n\lambda\epsilon^2\nu^2}\, t\,\mathcal{J}'_s(t) +$$

$$+ \frac{\lambda(1-\tilde{\omega})3\tilde{\omega}\alpha\gamma}{\lambda\nu^4\epsilon^2}\, t\,\mathcal{J}^{\circ}_{p}(t) + \frac{(1-\tilde{\omega})}{\nu^2}\, t\,\mathcal{J}^{\circ}_{p}(t). \qquad \text{(III-59)}$$

By cancelling the appropriate constants in equation (III-59), we obtain

$$\mathcal{J}_p(\tau) = C_1 + C_2/\tau - \frac{1}{n\gamma}\Big[\mathcal{J}_s(\tau) - n\,\mathcal{J}^{\circ}_{p}(\nu\tau)\Big]. \qquad \text{(III-60)}$$

The two constants of integration are found in exactly the same way as in the first method, and are; $C_2 = 0$ and

$$C_1 = -2H_s(\tau_0) - 2H^{inc}_{Rd}(\nu\tau_0) + \frac{1}{n\gamma}\Big[\mathcal{J}_s(\tau_0) - n\,\mathcal{J}^{\circ}_{p}(\nu\tau_0)\Big]. \qquad \text{(III-61)}$$

This completes a solution almost identical in form to the earlier one, but one in which a different approximation has been made, and whose limit in the forward scattering case is exact.

This last method was possible for $\beta = 1.0$ only, because equation (III-50) could be integrated analytically in this special case alone. This restriction arose from the complexity of the integration of $I_{lin}(\tau, \mu)$ to give $J_{lin}(\tau)$. However, when α is zero, the linear radiation field is the only field in the stellar part of the spectrum, and a solution is possible because the relatively complex equation (III-53) is not required in this case. The exact solution for the mean intensity of the linear radiation field is found by integrating equation (III-50). A Gaussian quadrature is the most appropriate method of integration, and it gives satisfactory results provided that special care is taken to account for any directional peaking of $I_{lin}(\tau, \mu)$. Thus, we have the exact solution for $J_{lin}(\tau)$, and for the scattered radiation field after subtraction of $J^{inc}_{red}(\tau)$, for linear scattering which covers the complete range of values of g. The thermal radiation mean intensity is found from equation (III-41) with $J_s(\tau)$ equal to zero and $J^{inc}_{red}(\tau)$ replaced by $J_{lin}(\tau)$. The construction of this equation involved the Eddington approximation and its form as a simple integral arose from the conservative nature of the transfer of the thermal radiation. Thus we obtain $J_p(\tau)$ by a double numerical integration of $J_{lin}(\tau)$.

The special case of conservative scattering deserves mention, even though it is not directly concerned with radiative heating. In this case we have $\tilde{\omega} = 1.0$, $\epsilon = 0.0$ and $\lambda = 3.0$, so that equation (III-19) reduces to

$$D^2 [\tau J_s(\tau)] = -3\tau J^{inc}_{red}(\tau). \qquad \text{(III-62)}$$

This equation is independent of the parameters in the scattering phase function other than the albedo which we have equal to unity. The same could not be said of the equivalent equations for the plane-parallel atmospheres of Chapter II nor of the solution by the second method of this section, because in both cases $J_{red}^{inc}(\tau)$ was replaced by $J_{lin}(\tau)$ and the Eddington approximation was applied to only part of the scattered radiation field. Now, equation (III-62) is identical to equation (III-36) when n is unity. The boundary conditions for the two equations are the same, so that we have the result

$$J_s(\tau)_{\tilde{\omega}=1} = J_p^{\circ}(\tau)_{n=1} , \qquad \text{(III-63)}$$

where $J_p^{\circ}(\tau)$ is given by equation (III-37). This result would be expected for isotropic conservative scattering which is identical to the transfer of the thermal radiation when n is unity. It is not an obvious result for other phase functions, but is a consequence of the combination of the axial symmetry of the problem and the Eddington approximation. It is only an approximate relation because we have the exact solutions for the cases with a equal to zero and these solutions are different. It is, however, an exact relation when a is unity and the scattering isotropic.

Before discussing the results of this section it will prove valuable to repeat the foregoing theory for a general phase function. The value of this was intimated in Section I.3. Consider the general phase function given by equation (I-26), of which the form suitable for axially symmetric fields is

$$p(\mu,\mu') = \sum_{l=0}^{\infty} \tilde{\omega}_l P_l(\mu) P_l(\mu') . \qquad \text{(III-64)}$$

We have seen that the albedo and asymmetry parameter of this phase function are given by $\tilde{\omega}_o$ and $\tilde{\omega}/3$ respectively. The emission coefficient for the scattered radiation given by equation (III-11), with the phase function of equation (III-64), is

$$j_s(\tau,\mu) = \frac{1}{2}(\kappa_s+\sigma_s)\int_{-1}^{+1}\sum_{l=0}^{\infty}\tilde{\omega}_l P_l(\mu)P_l(\mu')\left[I_s(\tau,\mu')+I^{inc}_{red}(\tau,\mu')\right]d\mu',$$

and the equation of transfer, (III-14) is

$$\mu\frac{\partial I_s(\tau,\mu)}{\partial\tau} + \frac{(1-\mu^2)}{\tau}\frac{\partial I_s(\tau,\mu)}{\partial\mu} = -I_s(\tau,\mu) +$$

$$+\frac{1}{2}\sum_{l=0}^{\infty}\tilde{\omega}_l P_l(\mu)\int_{-1}^{+1}P_l(\mu')\left[I_s(\tau,\mu')+I^{inc}_{red}(\tau,\mu')\right]d\mu'. \qquad \text{(III-65)}$$

Applying the moment operators, L_0 and L_1, to this equation and using the orthogonality property of the Legendre polynomials, we obtain

$$\frac{dH_s(\tau)}{d\tau} + \frac{2H_s(\tau)}{\tau} = -(1-\tilde{\omega}_0)J_s(\tau) + \tilde{\omega}_0 J^{inc}_{red}(\tau), \qquad \text{(III-66)}$$

and

$$\frac{dK_s(\tau)}{d\tau} + \frac{1}{\tau}\left[3K_s(\tau)-J_s(\tau)\right] = -\gamma H_s(\tau) + (1-\gamma)H^{inc}_{red}(\tau),$$

where

$$\gamma = 1-\tilde{\omega}_1/3 = 1-g. \qquad \text{(III-67)}$$

Equations (III-66) and (III-67) are the same as equations (III-15) and (III-16). The thermal radiation field depends on the scattering phase function through $J_S(\tau)$ only. Therefore, the solutions for $J_s(\tau)$ and $J_p(\tau)$ are the same for the general phase function as they are for the schematic phase function with the same values of albedo and asymmetry parameter. This result stems from the axial symmetry of the atmosphere and radiation fields, and also from the Eddington approximation which is used to solve equations (III-66). Were a higher approximation available that utilised three moment equations then the two phase functions would still give identical results but three parameters $\tilde{\omega}_l$ would need to be fixed by the three parameters of the schematic phase function. The similarity between the solutions for two different phase functions actually shows the limitation of the Eddington approximation to handle highly anisotropic phase functions. However, we have already seen from the similarity relations that the phase function is not important beyond the albedo and asymmetry parameters, so that this apparent limitation on the Eddington approximation is not serious. In considering the general phase function there is no way in which part of the scattered radiation may be treated exactly as was possible for the schematic phase function.

We shall now discuss the results of the foregoing theory and ascertain the nature of the influence that the individual atmospheric parameters exert on the radiation fields. Naturally, this influence will resemble that seen in the study of plane-parallel atmospheres in Chapter II. Where the results in the two cases are very similar they will not be repeated here. The parameters will be given the same values as their equivalents in Chapter II, which were tabulated in Table I. In the ensuing discussion it must be remembered that the optical scale, τ , refers to optical radii whereas it referred to optical depths in Chapter II. Hence, $\tau = 0$ refers to the centre of the atmosphere.

The mean intensity of the scattered radiation, as given by equations (III-27) to (III-30), is a function of the optical radius of the atmosphere, the albedo, the phase function for single scattering and position in the atmosphere. Figs. 37(a) to (c) show $J_S(\tau)$ for three values of τ_0; 0.1, 5.0 and 50.0. Each is for isotropic scattering and each gives a family of curves of parameter $\tilde{\omega}$. They show essentially the same features as the equivalent curves for plane-parallel atmospheres. As the optical depth increases, $J_s(\tau)$ increases due to the conversion of the reduced incident radiation to scattered radiation. Then, if the atmosphere is sufficiently thick, $J_s(\tau)$ decreases as the scattered radiation is attenuated by absorption. In Fig. 37(a), $\tau_0 = 0.1$ and a maximum is never attained, while in Fig. 37(a), $\tau_0 = 50.0$ and the maximum occurs very close to the surface, in fact within the outermost hundredth of the radius. As $\tilde{\omega}$ increases, so $J_S(\tau)$ increases as a natural consequence of the definitions of $\tilde{\omega}$ and the optical depth scale. Furthermore as $\tilde{\omega}$ increases, the maximum of $J_s(\tau)$ occurs deeper in the atmosphere. This is seen clearly in Fig. 37(b), and is due to the greater penetration of the radiation when $\tilde{\omega}$ is large. The attenuation in these atmospheres of intermediate optical radius is much weaker than that of their plane-parallel counterparts due to radiation crossing the atmosphere. The limiting case of $\tilde{\omega} = 1.0$ is also shown. It is greatly different from the case, $\tilde{\omega} = 0.9$, when the atmospheres are optically thick, because in such atmospheres, attenuation is very important, and, of course, when $\tilde{\omega} = 1.0$ there is no attenuation.

Fig. 38 shows the effect of varying the phase function in an atmosphere of total optical radius, 5.0 and albedo, 0.9. It is much simpler than the equivalent graph for plane-parallel atmospheres, Fig. 5, because it forms a one-parameter family of curves, the parameter being g, the asymmetry parameter. Now $g = \tilde{\omega}(1-\alpha)(2\beta-1)$, so that it takes values ranging from -1.0 to +1.0 when β runs from 0.0 to 1.0 with α = 0.0. The one parameter nature of the curves is due to the axial symmetry of the incident radiation as has been suggested earlier. It can be seen that the curves intersect at $\tau \sim 0.95\,\tau_0$. This occurs for all the values of τ_0 that were used. The gradient, $dJ_s(\tau, g)/dg$ is greater than and less than zero when τ is less than and greater than 0.95 τ_0 respectively. When g is positive the scattered radiation penetrates deeper into the atmosphere and consequently the scattered radiation field is built up more in the interior of the atmosphere but depleted near the surface. This depletion near the surface occurs because the fraction of the scattered radiation that is scattered into the outer shell decreases as g decreases. This effect is less well marked in the spherical atmospheres than the plane-parallel atmospheres because the incident radiation is isotropic in the former case and hence enters the outer shell directly. The effect of anisotropy increases as τ_0 increases. When $\tau_0 = 0.1$ there is very little change in $J_s(\tau)$ but when $\tau_0 = 50.0$, $J_s(0)$ is significantly non-zero when (α, β) = (0, 1) and hence g = +1.0, whereas it is effectively zero for all the other phase function parameters considered. In this case of complete forward scattering the value of $J_s(\tau)$ at the surface of a plane-parallel atmosphere was zero. This is not true in a spherical atmosphere because the scattered radiation can reach the surface by penetration right through the atmosphere, though for optically thick spherical atmospheres $J_s(\tau_0)$ is very close to zero because the penetration across the atmosphere, even at grazing angles to the surface, is very small indeed. In this respect such atmospheres can be regarded as plane-parallel.

The results given so far have been those found by the first and more general method. A second method involving the exact solution for part of the scattered radiation field has also been outlined. The two methods will be referred to as methods I and II respectively, and in the latter the field referred to as the scattered radiation field will not be $J_s(\tau)$ but the true scattered radiation field whose mean intensity is given by

$$J_{sc}(\tau) = J_s(\tau) + J_{lin}(\tau) - J_{med}^{inc}(\tau). \qquad \text{(III-68)}$$

For method I, $J_{sc}(\tau) = J_s(\tau)$. We now compare the results from the two methods remembering that method I applies to either the general or the schematic phase functions whereas method II applies to the schematic phase function only, and then only for the cases when α is zero or β is unity.

Fig. 39 shows the function $J_{sc}(\tau)$ for $\tilde{\omega} = 0.9$, $\beta = 1.0$ and $\tau_0 = 5.0$ for a variety of values of α. The value of 5.0 is chosen for τ_0 again because the results are best portrayed in this case. We conclude that, as α approaches unity the results of the two methods approach each other and in the limit when $\alpha = 1.0$ the two methods are identical and give the same result. In general there is some discrepancy between the exact and approximate solutions for $\alpha = 0.0$ but not sufficient to create any ambiguity in qualitative conclusions drawn from the results. This discrepancy is not systemmatically dependent on τ_0. When α is zero, method II gives the exact solution and $J_{sc}(0)$ by method I is too low for $\tau_0 = 0.1$, 1.0 and 50.0, and too high for $\tau_0 = 5.0$ and 10.0. The percentage error in method I is greatest for $\tau_0 = 50.0$ whereas the absolute error is smallest in this case. In absolute terms method I involves the greatest error for medium values of τ_0, but in percentage terms it involves the greatest error for extreme values of τ_0.

We have seen that the exact solution is available by numerical means for all values of β when α is zero. Fig. 40 shows the exact and approximate solutions for the same atmosphere as that of Fig. 39 but for $\alpha = 0$ and several values of β. It shows that the approximate method is most accurate when β is zero and least accurate when β is unity.

The exact solutions of method II also provide exact values of the ratios, $J_{sc}(\tau)/K_{sc}(\tau)$ and $J_{sc}(\tau_0)/H_{sc}(\tau_0)$, which were assumed to be 3 and 2 respectively in method I. We shall denote these ratios by the functions $r(\tau)$ and $r_0(\tau_0)$ respectively. The ratio, $r(\tau)$ is plotted in Figs. 41(a) to (c) as a function of τ, for $\tau_0 = 0.1$, 5.0 and 50.0 respectively. Each figure shows a family of curves with values of β of 0.0, 0.5 and 1.0, each for $\tilde{\omega} = 0.9$ and α=0.0. For optically thin atmospheres we can see that $r(\tau)$ is equal to 3.0 at the centre of the atmosphere and decreases slowly to a value close to 2.0 at the surface. This behaviour is independent of β. When $\tau_0 = 5.0$, $r(\tau)$ still does not depend on β but remains much closer to 3.0 for most of the atmosphere. When $\tau_0 = 50.0$, $r(\tau)$ is between 1.0 and 2.0 for most values of τ but is 3.0 at the centre and at the surface. In general, we conclude that $r(\tau) = 3.0$ is a reasonable approximation for all but very thick atmospheres when $r(\tau)$ is closer to 1.0 than to 3.0 for most points in the atmosphere.

These results are complemented by Figs. 42(a) and (b), which show R_1 and R_2 plotted as functions of τ for the optical radii,

τ_0 = 0.1 and 5.0. The quantity, R_1 , is defined as the ratio of the outward intensity, $I_{sc}(\tau, +1)$, to the transverse intensity, $I_{sc}(\tau, 0)$; and R_2 as the ratio of the inward intensity, $I_{sc}(\tau, -1)$, to the transverse intensity. They are defined for all values of τ and are measures of the asymmetry of the scattered radiation field. When $R_1 = R_2$ = 1.0 the radiation field is isotropic and this is the situation at the centre of every atmosphere. Both figures are drawn for $\tilde{\omega}$ = 0.9 and β = 0.0, 0.5 and 1.0. The transverse and inward intensities of the scattered radiation field at the surface of the atmosphere are both zero, so that, by definition, R_1 is infinite at the surface, and R_2 indeterminate. However, by application of L'Hopital's rule, the latter becomes zero. We consider first the case for which β = 1.0 in Fig. 42(a). As τ increases, the radiation field becomes progressively more peaked in the outward direction. This corresponds exactly to the deviation of $r(\tau)$ from 3.0 in Fig. 41(a). The radiation is peaked in the outward direction because the scattered radiation increases as it passes deeper into the atmosphere until a limiting optical path is reached. This limit is never reached when τ_0 = 0.1, so that the outward flowing radiation at any point has traversed a greater optical distance than the inward flowing radiation at that point, and consequently is the larger of the two. In Fig. 42(b) it can be seen that this limit is approximately $\tau = 0.9\tau_0$. For values of τ less than this limiting value in optically thick atmospheres, the radiation is peaked in the inward direction because the inward flowing radiation has been attenuated less than the outward flowing radiation at that point and consequently is the greater of the two. Near the surface the intensity of the inward flowing radiation has not built up to the intensity of the attenuated outward flowing radiation, and consequently the scattered radiation field is peaked in the outward direction. For atmospheres of optical thickness greater than 5.0, the region in which the scattered radiation is peaked in the outward direction becomes progressively smaller as τ_0 increases or $\tilde{\omega}$ decreases, both of which reduce the flow of radiation across the atmosphere. When $\tau_0 = 50$, R_1 is effectively zero and R_2 very large indeed, and hence the field very strongly peaked. This peaking is so strong that $r(\tau)$ of Fig. 41(c) is approximately unity. When $\tau_0 = 5.0$, R_1 and R_2 remain close to unity and $r(\tau)$ in Fig. 41(b) stays correspondingly close to 3.0. When $\tilde{\omega} = 0.1$ the values of R deviate from unity to a greater extent. For example, R_2 reaches a maximum value of 3.3 for $\tau_0 = 5.0$. Nevertheless, the ratio $r(\tau)$ remains close to 3.0 in this case also. This is because the relation, $r(\tau) = 3.0$ does not depend on the radiation field being isotropic but expandable in terms of certain Legendre polynomials. Thus, we have evidence to show that R_1 and R_2 must deviate considerably from unity in order to produce a significant deviation of $r(\tau)$ from 3.0.

We now consider the cases in which β is less than unity. The effect of β in Fig. 42(a) is negligible, and in Fig. 42(b) still small. However, β plays an important role when $\tau_0 = 50.0$. When β is unity, R_1 is approximately zero, but as β decreases, R_1 increases to very large values. However, R_2 always remains greater than R_1, the ratio of R_2/R_1 being about 5/2. When both R_2 and R_1 are large the radiation field is peaked in both inward and outward directions with respect to the transverse direction. For most points in an optically thick atmosphere the optical distances to most points on the surface are much greater than that to the nearest point, so that radiation at a particular position in the atmosphere has come mostly from the nearest point on the surface. Since the scattering is linear the radiation scattered from this radiation remains in this direction and we find a radiation field highly peaked in both inward and outward directions. Such a radiation field produces a value near 2.0 for $r(\tau)$. Hence, the lower the value of β, the closer $r(\tau)$ is to 3.0 as seen in Fig. 41(b).

The ratio, $r_o(\tau_o)$ is plotted in Fig. 43 as a function of τ_o for $\tilde{\omega} = 0.9$, and for values of β of 0.0, 0.5, 0.9 and 1.0. For optically thin atmospheres $r_o(\tau_o)$ is independent of β and the assumption that this ratio equals 2.0 is in error by as much as 30%. For optically thick atmospheres the approximation of this ratio to 2.0 is good unless β is greater than 0.9, when it is rather an underestimate.

We are now in a position to make a comparison between the results of the two methods, examples of which are shown in Fig. 40. The error in method I is independent of β for $\tau_o = 0.1$ (not shown), and dependent on β for larger values of τ_o where the cases in which β equals unity provide the greatest errors. These errors are complemented exactly by the variation of $r(\tau)$ and $r_o(\tau_o)$ with β, as discussed above. In general, we conclude that method I is a little inaccurate for optically thin and optically thick atmospheres, but adequate for intermediate atmospheres unless β is unity. Although method II is only available for certain phase functions, the use of method I for other phase functions will include errors that will not deviate greatly from those discussed here. Hence, we can give a qualitative estimate, both in magnitude and direction, of the probable error of any quantity evaluated by method I. However, the results of the two methods are always very similar so that qualitative conclusions based on the results of method I will suffer no distortion from the errors in the absolute values of the quantities concerned. Before considering the thermal radiation fields we must stress that the scattered radiation fields for which method I is in greatest error are those for $\tau_0 = 0.1$ and $\tau_0 = 50.0$. In both cases the scattered radiation field is very small and in the former case it is almost negligible in comparison with the reduced incident radiation field.

Both equations (III-46) and (III-60) indicate the form of the mean intensity of the thermal radiation field as a function of optical depth. We shall consider first the case in which n is large. Both these equations then reduce to; $J_p(\tau) \approx C_1$. The function, $J_p^{\circ}(\tau)$, is given by equation (III-37), in which the constants are zero, and this function is clearly very small when n is large. The constancy of $J_p(\tau)$ when n is large was also a feature of the radiative heating problem in plane-parallel atmospheres. For the case in which n is small such a reduction occurs only at large values of $(\tau_0 - \tau)$. In this case, the functions $J_p^{\circ}(\tau)$ and $J_s(\tau)$ are both very small. This was another feature of the problem in plane-parallel atmospheres, so it is reasonable to enquire whether the equation for the mean intensity of the thermal radiation in a plane-parallel atmosphere could have been expressed in a simple form such as equation (III-60), this being the resulting equation from method II which was the method used in the previous Chapter, though then out of necessity.

The function $J_p^{\circ}(\tau)$ is the solution for the mean intensity of the thermal radiation field in an atmosphere in which there is no scattering. For a semi-infinite atmosphere this can be shown to be

$$J_p^{\circ}(\tau) = \frac{-3\mu_o^2}{4n} F e^{-\tau/\mu_o} .$$

The parameter, ν , in equation (III-60) is $[1 - \tilde{\omega}(1 - a)]$ which equals σ as used in Chapter II for the case $\beta = 1.0$. Therefore

$$J_p^{\circ}(\nu\tau) = \frac{-3}{4n} \mu_o^2 F e^{-\sigma\tau/\mu_o} .$$

Now the final term of equation (II-40) is

$$\frac{-3\mu_o^2}{4n\sigma} e^{-\sigma\tau/\mu_o} = -\frac{3\mu_o^2}{4n\gamma} e^{-\sigma\tau/\mu_o} ,$$

when $\beta = 1.0$. Thus equation (II-40) can be re-written as

$$J_p(\tau) = C_1 - \frac{1}{n\gamma}\left[J_S(\tau) - n J_p^{\circ}(\nu\tau) \right] , \qquad \text{(III-69)}$$

which is the same as equation (III-60). Furthermore, for isotropic scattering, equations (III-46), (III-60) and (II-40) are identical. Now equation (III-60) is valid for $\beta = 1.0$ only, because it is only in this case that $J_{lin}(\tau)$ can be expressed analytically, and the equation above is only valid for the case, $\beta = 1.0$, because, for other values of β the right-hand side of equation (II-40) is not equal to $-3\mu_o^2 \exp(-\tau\nu/\mu_o)/4n\gamma$. We conclude by stating that the simple form for $J_p(\tau)$ in terms of $J_p^{\circ}(\tau)$ and $J_s(\tau)$ is valid only when the whole of the scattered radiation field is subject to the Eddington approximation. In method I this condition applies to all phase functions so that equation (III-46) is true in general, but in method II, whether in spherical or plane-parallel atmospheres, the condition applies only to isotropic scattering or scattering with $\beta = 1.0$. The latter can be accounted for within this condition because the radiation scattered within the spike is effectively not scattered at all.

The functions, $J_p(\tau)$ do not vary greatly from those functions derived for plane-parallel atmospheres, with regard to their dependence on the atmospheric parameters. We shall postpone any discussion of $J_p(\tau)$ when n is small until the temperature profiles are discussed in the following section because $J_p(\tau)$ dominates the temperature in that case. Now when $n = 10^4$

$$J_p(\tau) \approx C_1 = -2H_s(\tau_0) - 2H^{inc}_{red}(\tau_0).$$

The reduced incident radiation field is independent of both albedo and phase function for single scattering, and is negative in sign. The scattered flux is positive in sign and of smaller absolute value than the flux of the reduced incident radiation. Therefore, the dependence of $J_p(\tau)$ upon $\tilde{\omega}$ and g is opposite to the dependence of $H_s(\tau_0)$ upon those parameters. Hence, $J_p(\tau)$ decreases as $\tilde{\omega}$ increases, and increases as g increases. The difference between the results from equations (III-46) and (III-60) for $(\alpha, \beta) = (0, 1)$, for which the difference should be greatest, is small. For $\tilde{\omega} = 0.1$ and 0.5, method II gives the larger values of $J_p(\tau)$ but these only differ from those of method I in the third significant figure. For $\tilde{\omega} = 0.9$, method II gives the lower values of $J_p(\tau)$ and the difference occurs in the second significant figure but only of the order of one unit. The differences do not, of course, depend on τ, in this case. A general comparison of the two methods will be given later.

4. Temperature Profiles

The temperature of an element of matter in an atmosphere was defined in Section II.4 to be the temperature of black-body emitting the same total energy in the infra-red part of the spectrum. Equation (II-97) arose from this definition and is

$$\sigma T^4/\pi = B_p(\tau). \qquad \text{(III-70)}$$

For the plane-parallel atmosphere problems of Chapter II, the source function was always proportional to the quantity, F, which measured the flux in the incident beam of radiation, and we defined an effective temperature for the incident radiation in terms of this quantity. In the case of spherical atmospheres in a uniform isotropic radiation field of intensity, I_o, we adopt a similar convention and define the effective temperature of the incident radiation field, T_e, by

$$I_o = \sigma T_e^4/\pi . \qquad \text{(III-71)}$$

Again, we shall measure the temperature of the atmosphere in units of T_e so that we have

$$T^4 = B_p(\tau). \qquad \text{(III-72)}$$

We shall now discuss the form of the temperature profiles for the spherical atmospheres whose radiation mean intensities were given in Section III.3. as derived using the first of the two methods of solution of the equation of transfer. This was the method in which we assumed the Eddington approximation, $K_s(\tau) = (1/3)\,J_s(\tau)$, to apply to the whole of the scattered radiation field; and was the one that encompassed all values of the schematic phase function parameters, α and β. It was also applicable to a general phase function, as given by equation (I-26) for specified values of $\tilde{\omega}$ and g. We have, from equations (III-31) and (III-72),

$$T^4 = J_p(\tau) + n(1-\tilde{\omega})\left[J_s(\tau) + J^{inc}_{red}(\tau) \right], \qquad \text{(III-73)}$$

where $J_p(\tau)$ is given by equation (III-46), $J_s(\tau)$ by equation (III-27) and $J^{inc}_{red}(\tau)$ by equation (III-8).

Before discussing the temperature profiles of the atmospheres we shall discuss the temperatures attained at the centres of the atmospheres. Figs. 44 and 45 show these central temperatures, T(0), as functions of τ_0 and $\tilde{\omega}$ respectively, both for the case of isotropic scattering and $n = 10^4$. In Fig. 44 the family of curves has $\tilde{\omega}$ as its parameter and in Fig. 45, τ_0. For optically thin atmospheres the central temperature gradient, $dT(o)/d\tilde{\omega}$, is negative because it is $(1 - \tilde{\omega})$ that is the fraction of the stellar radiation that is absorbed. The limiting case of conservative scattering gives the result that the temperature as defined here, is zero. The same albedo dependence of the central temperature arises in very thick atmospheres. These atmospheres are sufficiently thick to prevent any significant amount of stellar radiation penetrating to the centre. Hence, the central temperature depends solely on the mean intensity of the thermal radiation field. This latter field is virtually constant throughout the atmosphere and the value of this constant is governed by the boundary conditions. A larger emergent thermal flux means a larger mean intensity for the thermal radiation field. A small albedo results in a small emergent scattered flux, and, by conservation of energy, a large emergent thermal flux. Hence, the quantity, $dT(0)/d\tilde{\omega}$ is negative for large, as well as small, values of τ_0. However, this gradient is not negative for all values of $\tilde{\omega}$ in atmospheres of intermediate optical radius. In these cases an increase in the albedo permits greater penetration of the scattered radiation, which is the dominant term in the expression for the temperature, as given by equation (III-73). Therefore, $dT(0)/d\tilde{\omega}$ is frequently positive for such values of τ_0. We have implied that the gradient, $dT(0)/d\tilde{\omega}$ is negative for all values of $\tilde{\omega}$ for optically thin and optically thick atmospheres. This is true for optically thin atmospheres, τ_0 = 0.1 and 1.0; but not so for optically thick atmospheres. In this case the gradient, though negative for most values of $\tilde{\omega}$ does become positive when $\tilde{\omega}$ is close to unity because, even for very large values of τ_0 there is a value of $\tilde{\omega}$, though extremely close to unity, that will allow penetration of the scattered radiation to the centre of the atmosphere.

Figs. 46 and 47 are the equivalent graphs of Figs. 44 and 45 for the case of $n = 10^{-2}$. The central temperatures of optically thin atmospheres are low because little incident radiation is absorbed, and the gradient, $dT(0)/d\tilde{\omega}$ is negative because the fraction of the incident radiation that is absorbed is $(1 - \tilde{\omega})$. Nevertheless, the temperature is maintained closer to unity than would be expected from considerations of the thermal radiation field alone, which is the dominant radiation field in determining the temperature when n is small. This is due to the contribution from the stellar radiation field, which though an unimportant term, is still not zero but in fact is far larger than the thermal radiation field in optically thin atmospheres. In the same way as described in Chapter II, a positive gradient, $dT(0)/d\tau_0$ is maintained as τ_0 increases until the term, $[J_s(\tau_0) + J^{inc}_{red}(\tau_0)]$ and the radiation fluxes become effectively zero. Then $dT(0)/d\tau_0$ is almost zero itself. As the albedo approached unity, the range of values of τ_0 in which the gradient $dT(0)/d\tau_0$ is positive, increases and hence, the central temperature increases. In fact, the gradient, $dT(0)/d\tau_0$, though very small, is negative when is very large.

The effect of anisotropy on the central temperatures is shown in Figs. 48 and 49. These figures show the central temperatures plotted as functions of τ_0 for $n = 10^4$ and $n = 10^{-2}$ respectively, and for values of the phase function parameters, $(\alpha, \beta) = (0.0, 0.0)$, $(0.4, 0.0)$, $(\alpha, 0.5)$, $(0.4, 1.0)$ and $(0.0, 1.0)$, where the case $(\alpha, \beta) = (\alpha, 0.5)$ gives the same results for all values of α including unity, the case of isotropic scattering. Both figures show the functions for $\tilde{\omega} = 0.9$, but Fig. 48 also includes the case of $\tilde{\omega} = 0.1$, which shows clearly that anisotropy is unimportant when the albedo is small. Anisotropy is also unimportant in optically thin atmospheres. It is clear from Fig. 48 that the phase function asymmetry parameter is very important in the range of values of τ_0 where the scattered radiation barely penetrates to the centre of the atmosphere. For $n = 10^4$ it is the stellar radiation that is the major contributor to the temperature, and a forward scattering phase function allows a large scattered radiation to penetrate to the centre of the atmosphere. When τ_0 is very large, no scattered radiation reaches the centre of the atmosphere and the temperature depends solely on the thermal radiation field at that point. This is not strongly dependent on the phase function. However, less scattered radiation is lost through the surface with a forward peaked phase function than with an isotropic or backward peaked phase function. Hence, the central temperatures of very thick atmospheres are fractionally higher for forward peaked phase functions than for backward peaked phase functions. The results of Fig. 49 correspond closely to the equivalent results of Chapter II. The gradient of the mean intensity of the thermal radiation field is given by equation (III-39) and is large by virtue of the factor, $1/n$. The greater the asymmetry parameter, the greater is the penetration of the stellar radiation and the greater is the inward flux of thermal radiation. By conservation of energy, the outward thermal flux is then greater so that the temperature increases rapidly as the asymmetry parameter increases.

The temperature profiles are shown in Figs. 50 and 51 for $n = 10^4$ and 10^{-2} respectively. Both show families of curves for which the phase function varies, and also the value of τ_0. The albedo is 0.9 in both cases. The results of Fig. 50 need little discussion. The surface temperatures are dominated by the stellar radiation field whose surface value does not vary greatly with either τ_0 or (α, β). Optically thin atmospheres naturally have a higher surface temperature than optically thick atmospheres on account of the radiation that has traversed the atmosphere. Backward scattering phase functions cause more scattering out of the atmosphere than forward scattering phase functions and consequently yield higher surface temperatures. The central temperatures we have discussed, and the temperature profiles link the central and surface temperatures accordingly. When $n = 10^{-2}$, as in Fig. 51, the surface temperatures increase as τ_0 increases because, as τ_0 increases, so the emergent stellar flux decreases and the emergent thermal flux increases to maintain the condition of zero net flux. Again most of the properties of Fig. 51 stem from the results already discussed for the central temperatures.

Only two values of n have been considered up to this point. Fig. 52 shows the central temperatures as functions of n for the five standard values of τ_0 . Fig. 53 is a cross-section of this showing the central temperatures as functions of τ_0 for several values of n . Both are for the case of no scattering. Fig. 52 shows an intersection point of all five curves. This occurs at n = 1 but is not an exact point of intersection. When n is larger than this value, the stellar radiation dominates the temperature and optically thin atmospheres show the highest central temperatures; whereas when n is lower than this value, the thermal radiation field dominates the temperature and optically thick atmospheres show the highest central temperatures. Had the same diagram been plotted for isotropic scattering of albedo, 0.9 a similar intersection would have been noted but this would have occurred at a value of n close to 4.0. The value of n at which the intersection occurs is controlled by the scattering and the relative importance of the ratios, $(\kappa_s + \sigma_s)/\kappa_p$ and κ_s/κ_p in determining the temperature. Its precise value is determined by a complex balancing of the different radiation fields of the problem. One final point of note is that n is unimportant when τ_0 is large provided that n is greater than unity. This can be seen from Fig. 53, but it will not be strictly true when scattering is present, because, for a very large value of n , even a very small scattered radiation field will make a significant contribution to the temperature.

The results described in this section have been those obtained by the more general method of solution known as method I, in which the Eddington approximation was applied to the entire scattered radiation field. In the previous section it was seen that, for certain phase functions only, an alternative method, known as method II could be developed, in which the Eddington approximation was applied to only part of the scattered radiation field. The results of these two methods are compared in Figs. 54 and 55 for values of n of 10^{4} and 10^{-2} respectively. They show the temperature profiles of atmospheres of certain optical radii in which there is anisotropic scattering. In Fig. 54 the curves are shown for values of τ_0 of 1.0, 5.0 and 50.0, and for values of (α, β) of (0.0, 0.0), (0.0, 0.5) and (0.0, 0.1); and in Fig. 55 for values of τ_0 of 1.0 and 10.0, and values of (α, β) of (0.0, 0.0) and (0.0, 1.0). The first of these figures involves temperatures that are dominated by the stellar radiation and hence the differences between the results from the two methods bear a strong resemblance to the differences between the functions, $J_6(\tau)$, as discussed in the previous section. However, when τ_0 is less than or equal to unity the reduced incident radiation field is greater than the scattered radiation field so that, not only is the temperature almost independent of the phase function, but independent of the method used for its calculation. There is a definite difference between the results from the two methods in those regions of thicker atmospheres where the scattered radiation dominates the temperature. That is, at all optical depths when $\tau_0 = 5.0$, but only in the outer shell for which τ is greater than $0.9\tau_0$ when $= 50.0$. Nevertheless, the discrepancy between the temperatures derived by the two methods is not great. When $n = 10^{-2}$ the thermal radiation field dominates the temperature, so that Fig. 55 essentially shows differences between the results from the two methods for the mean intensity of the thermal radiation field. This was not discussed in the previous section. It can be seen that there is no difference between the temperature profiles of optically thin atmospheres but there is between those of optically thick atmospheres, in which case it is no greater than 2%. This is smaller than the maximum difference for the case, $n = 10^4$, which is 7%. In general, the difference between the results from the two methods are much smaller in the temperatures than they were in the mean intensities of the scattered radiation field.

5. The Emergent Radiation

It was seen in Section I.1 that the intensity of the radiation field in any medium can be found from the integral equation that is the formal solution of the equation of transfer, provided that the source function is known. The emergent radiation from a spherical atmosphere can be found from this equation, equation (I-7), in an analogous manner to that used to derive the approximate solution for the emergent radiation from a plane-parallel atmosphere in Section II.5.2. Fig. 56 shows the geometry of a spherical atmosphere pertinent to this derivation. Measuring optical distances in terms of the extinction coefficient for the visible radiation, and with τ_0 the total optical radius of the atmosphere, OP; τ and t, the optical radii, OS and OT; τ_{x_0}, the optical distance RP; and t_x, the optical distance TP, equation (I-7) becomes

$$I_s(0,\mu) = I_0\, e^{-\tau_{x_0}} + \int_0^{\tau_{x_0}} B_s(t)\, e^{-t_x}\, dt_x\,, \qquad \text{(III-74)}$$

for the emergent stellar radiation and

$$I_p(o,\mu) = \int_0^{\tau_{x_0}} B_p(t)\, e^{-t_x/n}\, dt_x/n\,, \qquad \text{(III-75)}$$

for the emergent thermal radiation. It is more convenient to evaluate the emergent radiation as a function of τ , where τ is the optical distance OS, at intervals of τ/τ_0 of 0.0(0.1)1.0; rather than as a function of μ.

Approximate solutions for the two source functions were obtained in Section III.3 for certain types of scattering. Whilst the mean intensities of the radiation fields were found for all values of α and β, the schematic phase function parameters, the source function for the stellar radiation field was found only when α was zero or unity, the cases of linear and isotropic scattering. The source function for the thermal radiation field is isotropic for all phase functions and therefore was found in Section III.3. for all phase functions. Equations (III-74) and (III-75) can only be usefully applied when the source function is isotropic, as for the scattered radiation field when the scattering is isotropic, and for the thermal radiation field.

However, when the scattering is linear only, the intensity of the scattered radiation field is known exactly everywhere. The emergent radiation is a special case of this solution which is given by equation (II-52), and is

$$I_s(\tau) = C_1 e^{\sigma\tau_{x_0}} + C_2 e^{-\sigma\tau_{x_0}}\,, \qquad \text{(III-76)}$$

where $$C_1 = \frac{I_0\left\{\tilde{\omega}(1-\beta) - [1-\tilde{\omega}\beta-\sigma]e^{-\sigma\tau_{x_0}}\right\}}{\left\{[1-\tilde{\omega}\beta+\sigma]e^{\sigma\tau_{x_0}} - [1-\tilde{\omega}\beta-\sigma]e^{-\sigma\tau_{x_0}}\right\}}\,, \qquad \text{(III-77)}$$

$$C_2 = I_0 - C_1 ,$$

$$\sigma^2 = (1-\tilde{\omega}\beta)^2 - \tilde{\omega}^2(1-\beta)^2 \quad \text{and} \quad \left(\tfrac{1}{2}\tau_{x_0}\right)^2 = \tau_0^2 - \tau^2 . \qquad \text{(III-78)}$$

This exact solution for the emergent stellar radiation is given in Figs. 57 and 58 for linear scattering with $\beta = 0.5$ and 1.0 respectively. Both show families of curves whose parameters are τ_0 and $\tilde{\omega}$ which take the values of 0.1, 1.0, 5.0 and 50.0, and 0.1, 0.5 and 0.9 respectively. The majority of the curves are only shown for values of τ/τ_0 ranging from 0.0 to 0.9. The remaining ten percent is omitted to preserve clarity. All the curves do rise to unity at the limb which means that, at the limb of every atmosphere the intensity seen is that of the incident radiation only. We shall consider first, Fig. 57. In general it can be seen that $I_s(\tau)$ is smallest for small values of $\tilde{\omega}$ and large values of τ_0. This is to be expected. Small albedos give rise to smaller scattered radiation fields and thick atmospheres prevent the passage of radiation across them. In the latter case there is very little increase in $I_s(\tau)$ as a function of τ until τ/τ_0 approaches unity. For example, when τ_0 is 50.0 and τ/τ_0 is 0.9, τ_{x_0} is approximately 45.0 so that even at such a high value of τ/τ_0 there is very little penetration through the atmosphere. As τ_0 becomes large so $I_s(o)$ becomes independent of τ_0 and dependent on $\tilde{\omega}$ only. This happens because the major contribution to the emergent radiation is that reflected from the outer layer of the atmosphere and this will not depend upon τ_0 or τ at all, if τ_{x_0} is sufficiently large to prevent radiation penetrating across the atmosphere.

Fig. 58 for the case of $\beta = 1.0$ shows similar results to those of Fig. 57. The main difference is that the emergent intensities from optically thick atmospheres are considerably smaller than their counterparts in Fig. 57. When the parameter, β is equal to unity then there is no backscattered radiation anywhere. The emergent radiation must have entered the atmosphere at the opposite end of the line of transfer so that the emergent radiation will be zero if the optical length of the line is sufficiently large. The two figures show that the emergent radiation from an optically thin atmosphere does not depend crucially on the phase function of the scattering.

The emergent stellar radiation can be obtained from equation (III-74) in the case of isotropic scattering. The source function is given by a reduced form of equation (III-11) but still involves equation (III-27) for $J_S(\tau)$ which is a very complex function of optical radius. The exponent, t_x, is not an elementary function of optical radius either, so it is necessary to integrate equation (III-74) numerically. This procedure is simple for optically thin atmospheres in which the source function is a slowly varying function of optical radius, but rather more complicated for optically thick atmospheres in which the source function has a large maximum near the surface of the atmosphere and is effectively zero for a large region in the centre of the atmosphere. In these latter cases the range of values of τ that dominates the integral is the outermost few units. Thus, care must be taken in applying the numerical integration to ensure that the important part of the range of integration is treated accurately and also, that the source function is closely tabulated over the range of values of optical radius through which it varies rapidly. In the calculations employed the source function was obtained by interpolation on a table of values of the source function evaluated at values of τ/τ_0 0.0(0.1)1.0 and 0.90(0.01)1.00. The direct evaluation of the source function is a relatively lengthy process so that where many values of the source function are required, as in a numerical integration process it is very much quicker to use an interpolation scheme on a precalculated table of source function values.

The results for the emergent stellar radiation from an isotropically scattering atmosphere are shown in Fig. 59 which is quite analogous to Fig. 57. In fact, the results are very similar. This is because both phase functions have an asymmetry parameter of zero. Certainly the similarity between the two sets of results for optically thin atmospheres is very striking. Three slight differences are evident for optically thick atmospheres. Firstly, the optical thickness of the atmosphere is more important in the isotropic scattering case, particularly when the albedo is high. Whereas the emergent radiation from the centre of the atmosphere, which is the equivalent of saying the emergent radiation normal to the surface of the atmosphere, was independent of τ_0 when $\tilde{\omega}$ was 0.9 and (α. β) was (0.0, 0.5), it is not so when (α, β) = (0.0,1.0). Secondly, the emergent radiation from the centre of the atmosphere is smaller for the isotropic scattering case; and thirdly, the emergent radiation from the outer region of the disc projection of the atmosphere is much greater in comparison with the emergent radiation from the centre of the disc, for the isotropic scattering case. All these phenomena are due to the relative amounts of radiation scattered away from the direction of the incident photons. In the case of α equal to zero there is no radiation scattered "sideways". We have seen that in the case of linear scattering with β equal to 0.5, the emergent radiation is independent of τ_0 and τ if τ_{x_0} is greater than a certain optical distance. The radiation emergent from the centre of the atmosphere is made up of contributions of radiation scattered by elements of matter at all points along the line of sight, which in this case is the diameter of the atmosphere. The contribution from a point at depth τ^* is dependent solely on the values of τ^* and $\tilde{\omega}$ when $(\alpha, \beta) = (0.0, 0.5)$ because it scatters radiation that initially entered the atmosphere at the same point whatever the value of τ_0. However, for isotropic scattering the element of matter at point τ^* scatters radiation incident upon it from all directions into the particular line of sight. As τ_0 changes so the radiation incident upon it in a particular direction must change accordingly. Hence, we observe that $I_s(o)$ depends upon τ_0 even for large values of τ_0. This is an effect due to the changing curvature of the surface nearest to the point τ^*. The second and third differences between the results for the two phase functions are complementary. The isotropic scattering causes more radiation to emerge at high values of the angle of emergence, $\cos^{-1}\mu$, but less at low values of this angle. We have already noted the reason why $I_s(\tau)$ is independent of μ and hence, τ, for $\tau/\tau_0 < 0.9$, $\tau_0 = 50.0$ and $(\alpha, \beta) = (0.0, 0.5)$. For large values of τ_0 and isotropic scattering the dominant contribution to the source function at a particular point is radiation scattered from the radiation incident along the line of shortest distance to the surface of the atmosphere. The source function at all points on a line of sight near the edge of the atmosphere is consequently going to be greater than at points on a parallel line further from the edge. Consequently the former case will yield an emergent radiation that increases towards the limb. Hence we understand the differences between Figs. 57 and 59.

The source function for the thermal radiation is isotropic, and the emergent thermal radiation is evaluated by integrating equation (III-75) numerically. Again, a great deal of care must be exercised in performing this integration. When n is unity the same measures that were necessary for the accurate integration of equation (III-74) must be used. When n is 10^4, the attenuation factor is almost negligible and the dominant term in the source function is the stellar radiation field mean intensity which, for large values of τ_0, is virtually restricted to the outer shell of the atmosphere. However, contributions to the emergent radiation arise from both sides of the shell because the attenuation factor is so small. When n is 10^{-2} the dominant term in the source function is the thermal radiation field mean intensity. However, the attenuation factor is now very large, and contributions to the emergent thermal radiation arise from a very thin shell near the surface. Thus, the choice of the range of application of the numerical integration procedures is a complex function of n, τ_0 and τ. There is no incident thermal radiation. Therefore the emergent radiation from the limb, $\tau = \tau_0$ and $\tau_{x_0} = 0.0$, is zero.

The thermal radiation emerging from a spherical atmosphere is plotted in Figs. 60 to 62 as a function of τ/τ_0, the fractional optical radius of a disc projection of the atmosphere. It is drawn in Fig. 60 for isotropic scattering of albedo, 0.5, for various values of τ_0 and n. For optically thin atmospheres, of $\tau_0 = 0.1$ and to a lesser extent, $\tau_0 = 1.0$, the value of τ_0 is important in determining the emergent thermal radiation, and the value of n is almost immaterial. When the atmosphere is optically thin, only a certain fraction of the incident radiation is absorbed to form thermal radiation, and this fraction increases as τ_0 increases. Hence, $I_p(\tau)$ is very small when τ_0 is 0.1. The mean intensity of the thermal radiation at the surface is independent of n , by virtue of the boundary conditions. When n is large, the emergent radiation is comprised of photons emitted from all elements of matter along the line of transfer because the attenuation coefficient is then very small. However, for large values of n the source function is a very weak function of position and is approximately equal to the boundary value, and hence independent of n . Moreover, it is simple to show that $I_p(\tau) \approx B_p(\tau_0)$ for large values of n . For small values of n the attenuation of the thermal radiation is large, even for small values of τ_0 . Consequently, the emergent radiation is dominated by the emission from the outermost layers which, as we have noted, is virtually independent of n . Hence, the greenhouse parameter has little influence on the emergent thermal radiation from optically thin atmospheres.

For optically thick atmospheres, the emergent thermal radiation does depend critically on n , but less so on τ_0 . When n is small the emergent thermal radiation arises from the outermost layers, and as we have seen, the source function near the surface does not depend significantly on τ_0 , provided it is large enough to prevent the penetration of the stellar radiation right across the atmosphere. In such atmospheres the temperatures rises rapidly with optical depth from the surface. Therefore, $I_p(o)$ contains a greater proportion of radiation emitted by hotter layers than does $I_p(0.9\,\tau_0)$; and the function $I_p(\tau)$ has a maximum at $\tau = 0$. The opposite occurs when τ_0 is large and n is large. The hottest layers are those near the surface so that the optical path through the hot layers is smallest when τ is zero and largest when τ is of the order of 0.8 τ_0 . As τ and $\cos^{-1}\mu$ increase beyond a critical value, the optical path through the atmosphere decreases due to truncation at the surface and hence $I_p(\tau)$ decreases to zero as τ tends to τ_0 . This maximum occurs at values of τ/τ_0 of 0.8, 0.92 and 0.995 for values of τ_0 of 5.0, 10.0 and 50.0 respectively. It is an effect due to the curvature of the atmosphere.

In general, the functions, $I_p(\tau)$, intersect at high values of τ/τ_0 on varying any of the atmospheric parameters. Figs. 61 and 62 show the emergent thermal radiation for $n = 10^4$ and, values of $\tilde{\omega}$ of 0.1 and 0.9. Fig. 61 shows optically thin atmospheres of τ_0 of 0.1 and 1.0; and Fig. 62 shows optically thick atmospheres of τ_0 of 5.0, 10.0 and 50.0. Fig. 61 also includes the case, $\tilde{\omega}$ = 0.5. In each figure it can be seen that $I_p(\tau)$ decreases as $\tilde{\omega}$ increases. The thermal source function is dominated by the term $n(1-\tilde{\omega})\left[J_s(\tau) + J^{inc}_{red}(\tau)\right]$, and this is the cause of the albedo dependence of $I_p(\tau)$. The ratio, κ_s/κ_p is given by $n(1 - \tilde{\omega})$; and the larger the albedo the smaller is the stellar absorption coefficient and the smaller is the thermal radiation field generated. The atmosphere of τ_0 = 5.0 shows both the limiting extremes of behaviour. It is like an optically thick atmosphere when $\tilde{\omega}$ = 0.1 and an optically thin atmosphere when $\tilde{\omega}$ = 0.9. This follows directly from the effect of the increased penetration of the scattered radiation when the albedo is high, which gives rise to scattered radiation fields typical of those of thinner atmospheres with lower albedos. Similar conclusions can be drawn about the effect of anisotropy. When β is unity the atmospheres will behave as though they were optically thinner with isotropic scattering, and vice versa for atmospheres with β equal to zero. The equivalent diagrams to Figs. 61 and 62 for $n = 10^{-2}$ have not been given. The effect of varying the albedo is exactly the same as when $n = 10^{4}$, but the functions $I_{\rho}(\tau)$ are of different structure as shown in Fig. 60.

6. The Effect of a Variable Density

In the preceding sections of this Chapter we have assumed that the density of the atmosphere is constant throughout the atmosphere. The need to assume a particular form for the density function arises in the radiative transfer problem in spherical atmospheres from the form of the total differential of the intensity which is expressed in terms of the partial differentials, $\partial/\partial r$ and $\partial/\partial\mu$. It was not necessary to make such an assumption in the radiative transfer problem in plane-parallel atmospheres because the total differential of the intensity could be expressed in terms of one differential, d/dz, and an optical depth scale could be defined by, $d\tau = -(\kappa_s + \sigma_s)\rho(z)dz$. In this section we shall allow the density to be a function of the position variable, r. We shall still maintain the restriction that there is no variation of the density over a spherical surface concentric with the atmospheric surface, in the same way that we allowed no density variation along a plane parallel to the surface of a plane-parallel atmosphere. It is convenient to speak of a density function, $\rho(r)$ and a constant absorption coefficient, but a variation of the latter will introduce no added complication to the problem. However, we shall still assume the albedo to be constant throughout the atmosphere.

The equation of transfer for the scattered radiation field in a spherical atmosphere is given by equation (III-12). We shall restrict the discussion to the case of isotropic scattering for which the equation of transfer is

$$\mu \frac{\partial I_s(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_s(r,\mu)}{\partial \mu} = -(\kappa_s+\sigma_s)\rho(r)\Big[I_s(r,\mu) - \tilde{\omega} J_s(r) - \tilde{\omega} J^{inc}_{md}(r)\Big]. \qquad \text{(III-79)}$$

To define an optical distance scale it is clear that both $(\kappa_s+\sigma_s)\rho(r)dr$ and $(\kappa_s+\sigma_s)\rho(r)r$ are required as functions of the optical distance. This is only possible if $\rho(r)$ is a pre-defined function and only simple if $\rho(r)$ is constant. Moreover, it is not possible for every function, $\rho(r)$. A density function that has been used extensively is that developed by Chandrasekhar (1960),

$$(\kappa_s+\sigma_s)\rho(r) = ar^{-m}, \qquad \text{(III-80)}$$

where a and m are constants. We define the optical depth in the normal manner by

$$d\tau = -(\kappa_s+\sigma_s)\rho(r)\,dr, \qquad \text{(III-81)}$$

or

$$\tau = \int_r^R (\kappa_s+\sigma_s)\rho(r)\,dr,$$

where R is the radius of the atmosphere. The density function defined by equation (III-80) has an infinite singularity at the origin, which though physically unrealistic will not affect the results to any great extent. Combining equations (III-80) and (III-81) gives

$$(\kappa_s + \sigma_s)\,\rho(r)\,r = (m-1)\,\tau + b\,, \tag{III-82}$$

where $\quad b = \text{constant} = a / R^{m-1}\,.$

The solution of equation (III-79) with the density function, (III-80) and optical depth, (III-81), is solved by Eddington's method and follows closely Huang's (1969b) analysis for circumstellar shells. The first two moment equations derived from equation (III-79) are

$$\frac{dH_s(r)}{dr} + \frac{2H_s(r)}{r} = -(\kappa_s+\sigma_s)\,\rho(r)\left[(1-\tilde{\omega})\,J_s(r) - \tilde{\omega}\,J_{rad}^{inc}(r)\right], \tag{III-83}$$

and

$$\frac{dJ_s(r)}{dr} = -(\kappa_s+\sigma_s)\,\rho(r)\; 3H_s(r)\,, \tag{III-84}$$

where we have used the Eddington approximation in formulating equation (III-84). Differentiating equation (III-84) and using equation (III-83), we obtain

$$\frac{d^2 J_s(r)}{dr^2} + \left[\frac{2}{r} - \frac{1}{\rho(r)}\frac{d}{dr}\rho(r)\right]\frac{dJ_s(r)}{dr} =$$

$$3\,(\kappa_s+\sigma_s)^2\,\rho^2(r)\left[(1-\tilde{\omega})\,J_s(r) - \tilde{\omega}\,J_{rad}^{inc}(r)\right]. \tag{III-85}$$

Now, from equation (III-80) we have

$$\frac{1}{\rho(r)} \frac{d\rho(r)}{dr} = -\frac{m}{r} .$$

so that $$\frac{d^2 J_s(r)}{dr^2} + \frac{(m+2)}{r}\frac{dJ_s(r)}{dr} - 3(\kappa_s+\sigma_s)^2 \rho^2(r)(1-\tilde{\omega}) J_s(r)$$

$$= -3(\kappa_s+\sigma_s)^2 \rho^2(r)\, \tilde{\omega}\, J_{red}^{inc}(r) ,$$

which, in terms of the optical depth as given by equations (III-81) and (III-82) is

$$\frac{d^2 J_s(\tau)}{d\tau^2} - \frac{(m+2)}{[(m-1)\tau + b]} \frac{dJ_s(\tau)}{d\tau} - \epsilon^2 J_s(\tau) = \qquad \text{(III-86)}$$

$$= -3\,\tilde{\omega}\, J_{red}^{inc}(\tau) ,$$

where $$\epsilon^2 = 3(1-\tilde{\omega}) .$$

Equation (III-86) can be simplified by changing the variable τ to x, where x is defined as

$$x = \epsilon\left[\tau + b/(m-1)\right] , \qquad \text{(III-88)}$$

whence, equation (III-86) becomes

$$\frac{d^2 J_s(x)}{dx^2} - \frac{(m+2)}{(m-1)x}\frac{dJ_s(x)}{dx} - J_s(x) = \frac{-\tilde{\omega}}{(1-\tilde{\omega})} J_{red}^{inc}(x) .$$

Again, changing the variable $J_s(x)$ to $f(x)$, where

$$f(x) = x^{-\nu} J_s(x) , \qquad \text{(III-89)}$$

and $$\nu = (2m+1)/2(m-1),$$ (III-90)

we obtain

$$x^2 \frac{d^2 f(x)}{dx^2} + x \frac{df(x)}{dx} - (x^2+\nu^2) f(x) = \frac{-\tilde{\omega} x^{\nu}}{(1-\tilde{\omega})} J_{red}^{inc}(x). \qquad \text{(III-91)}$$

This is an inhomogeneous Bessel equation of purely imaginary argument of order ν. Its solution is found by the method of variation of parameters, and is

$$f(x) = \frac{-\tilde{\omega}}{(1-\tilde{\omega})} \left\{ I_\nu(x) \int_{C_1}^{x} K_\nu(x)\, x^{1-\nu} J_{red}^{inc}(x)\, dx + K_\nu(x) \int_{x}^{C_2} I_\nu(x)\, x^{1-\nu} J_{red}^{inc}(x)\, dx \right\}. \qquad \text{(III-92)}$$

The functions, $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel and modified Hankel functions respectively. Details of these functions and the application of the method of variation of parameters to Bessel equations are given by Watson (1952) in his standard work, "Treatise on Bessel Functions". Further details and useful tables of these functions are available in various works on mathematical functions such as that by Abramowitz and Stegun (1964).

The two constants of integration, C_1 and C_2, are determined by two boundary conditions. At the origin the mean intensity of the scattered radiation must be zero because we have a non-conservative scattering problem. That is, we require $f(\infty) = 0$. Now, $I_\nu(\infty)$ is infinity and $K_\nu(\infty)$ is zero so that C_1 must be infinity in order to satisfy the condition that $f(\infty)$ is zero. Thus, the solution for the mean intensity of the scattered radiation field, given by equation (III-89) and (III-91) can be expressed as

$$J_s(x) = \frac{-\tilde{\omega}\, x^{\nu}}{(1-\tilde{\omega})} \left[A K_\nu(x) - F_\nu(x) \right], \tag{III-93}$$

where
$$F_\nu(x) = K_\nu(x) \int_0^x x^{1-\nu} J_{\rm red}^{\rm inc}(x)\, I_\nu(x)\, dx \; +$$

$$+ I_\nu(x) \int_x^\infty x^{1-\nu} J_{\rm red}^{\rm inc}(x)\, K_\nu(x)\, dx \,, \tag{III-94}$$

and A is the second constant of integration. This constant is found by the use of the Eddington boundary condition, $J_s(x_o) = 2H_s(x_o)$, where x_o is the value of x at the surface, which from equation (III-88), is given by

$$x_o = \epsilon b / (m-1) \,. \tag{III-95}$$

The flux of the scattered radiation can be found from equations (III-81), (III-84) and (III-88), and is

$$H_s(x) = \frac{\epsilon}{3} \frac{d J_s(x)}{dx} \,. \tag{III-96}$$

Hence
$$H_s(x) = - \frac{\epsilon \tilde{\omega} \nu x^{\nu-1}}{3(1-\tilde{\omega})} \left[A K_\nu(x) - F_\nu(x) \right] -$$

$$- \frac{\epsilon \tilde{\omega} x^{\nu}}{3(1-\tilde{\omega})} \left[A K_\nu'(x) - F_\nu'(x) \right], \tag{III-97}$$

where the primed quantities refer to derivatives with respect to x. The derivative of $K_\nu(x)$ is given by Abramowitz and Stegun (1964), and the derivative of $F_\nu(x)$ is

$$F_\nu'(x) = K_\nu'(x)\int_0^x x^{1-\nu}\, J_{med}^{inc}(x)\, I_\nu(x)\, dx \; +$$

$$+\; I_\nu'(x)\int_x^\infty x^{1-\nu}\, J_{med}^{inc}(x)\, K_\nu(x)\, dx\,. \tag{III-98}$$

Thus the constant, A, is given by

$$A = \frac{\{(x_0 - 2\epsilon\nu/3)\, F_\nu(x_0) - 2\epsilon x_0 F_\nu'(x_0)/3\}}{\{(x_0 - 2\epsilon\nu/3)\, K_\nu(x_0) - 2\epsilon x_0 K_\nu'(x_0)/3\}}\,. \tag{III-99}$$

This completes the solution for $J_S(x)$. In Section III.1 we saw that the reduced incident radiation could be found by numerical integration only, when $\rho(r)$ was not a constant. Thus, the necessity of numerical integration for the evaluation of $F_\nu(x)$ introduces no further restrictions to the capability of the solution. However, great care must be taken in the evaluation of this function because the integrand is a rapidly varying function of x in many cases.

The equation of transfer for the thermal radiation is

$$\mu\frac{\partial I_p(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_p(r,\mu)}{\partial \mu} = -K_p\,\rho(r)\, I_p(r,\mu) \; +$$

$$+\, K_p\,\rho(r)\, J_p(r) + (1-\tilde{\omega})(K_S+\sigma_S)\rho(r)\left[J_S(r) + J_{med}^{inc}(r)\right]. \tag{III-100}$$

Integrating this equation in the standard way gives the first two moment equations, which are

$$\frac{dH_p(r)}{dr} + \frac{2H_p(r)}{r} = (1-\tilde{\omega})(\kappa_s+\sigma_s)\rho(r)\left[J_s(r)+J^{inc}_{red}(r)\right], \quad \text{(III-101)}$$

and

$$\frac{dJ_p(r)}{dr} = -3\kappa_p\rho(r)\,H_p(r). \quad \text{(III-102)}$$

We have again used the Eddington approximation in deriving equation (III-102). Changing the variable r to x, using equations (III-80), (III-81) and (III-88), we obtain

$$\frac{dH_p(x)}{dx} - \frac{2}{(m-1)x}H_p(x) = -\frac{\epsilon}{3}\left[J_s(x)+J^{inc}_{red}(x)\right], \quad \text{(III-103)}$$

and

$$\frac{dJ_p(x)}{dx} = \frac{3}{n\epsilon}H_p(x). \quad \text{(III-104)}$$

Equation (III-103) can be solved directly using the relationship

$$x^{\eta}\frac{d}{dx}\left[x^{-\eta}y(x)\right] = \frac{dy(x)}{dx} - \frac{\eta}{x}y(x). \quad \text{(III-105)}$$

The boundary condition to which this equation is subject, is that of conservation of energy. This, when expressed mathematically, involves the flux, $H_s(x)$ which we have found numerically. Consequently, the constant of integration cannot be specified exactly. However, when considering semi-infinite plane-parallel atmospheres it was seen that an alternative approach to the solution for the mean intensity of the thermal radiation field was available. We shall now use this alternative method which will allow the establishment of an exact value for the constant of integration. This alternative method involves the solution of the equation of transfer for the sum of the scattered and thermal radiation fields which will be denoted by the subscript T. Adding equations (III-83) and (III-101), we obtain

$$\frac{dH_T(r)}{dr} + \frac{2H_T(r)}{r} = (\kappa_s + \sigma_s)\rho(r)\, J^{inc}_{red}(r), \qquad \text{(III-106)}$$

which expressed in terms of the variable x, becomes

$$\frac{dH_T(x)}{dx} - \frac{2H_T(x)}{(m-1)x} = \frac{-1}{\epsilon} J^{inc}_{red}(x). \qquad \text{(III-107)}$$

The solution of this equation, using the relationship (III-105), is

$$H_T(x) = \frac{x^{\eta}}{\epsilon}\int_x^c x^{-\eta} J^{inc}_{red}(x)\,dx, \qquad \text{(III-108)}$$

where $\eta = 2/(m-1)$ which is a positive constant.

The intensity of the reduced incident radiation field obeys the equation of transfer also. It is merely the expression of the attenuation of the field because the source function for the reduced incident radiation field is zero. Thus, we have

$$\mu\frac{\partial I^{inc}_{red}(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I^{inc}_{red}(r,\mu)}{\partial \mu} = -(\kappa_s+\sigma_s)\rho(r)\, I^{inc}_{red}(r,\mu). \qquad \text{(III-109)}$$

The first moment integral of this equation of transfer, expressed in terms of the variable, x, is

$$\frac{dH^{inc}_{red}(x)}{dx} - \frac{2H^{inc}_{red}(x)}{(m-1)x} = \frac{1}{\epsilon} J^{inc}_{red}(x),$$

the solution of which is

$$H^{inc}_{red}(x) = \frac{-x^{\eta}}{\epsilon}\int_x^{\infty} x^{-\eta} J^{inc}_{red}(x)\,dx\,, \qquad \text{(III-110)}$$

where $\eta = 2/(m-1)$ as above, and we have used the boundary condition that the limit of $H^{inc}_{red}(x)$ as x tends to infinity, is zero.
Conservation of energy demands that

$$H_T(x) = -H^{inc}_{red}(x)\,,$$

so therefore the constant, C, in equation (III-108) must be infinity.
Expressing equation (III-84) as a function of x and adding to it equation (III-104), we obtain

$$H_T(x) = \frac{\epsilon}{3}\left[\frac{dJ_S(x)}{dx} + n\,\frac{dJ_p(x)}{dx}\right].$$

Assuming the greenhouse parameter to be independent of position in the atmosphere, x, we integrate this equation and obtain

$$n\,J_p(x) + J_S(x) = \frac{3}{\epsilon}\int_{x_0}^{x} H_T(x)\,dx + nC_1\,,$$

or

$$J_p(x) = C_1 - \frac{1}{n}J_S(x) - \frac{3}{\epsilon n}\int_{x_0}^{x} H^{inc}(x)\,dx\,. \qquad \text{(III-111)}$$

The constant, C_1, is found by application of the Eddington approximate boundary condition, $J_p(x_0) = 2H_p(x_0)$ and is given by

$$C_1 = -2H^{inc}_{red}(x_0) - \left[1 - \frac{1}{n}\right]J_S(x_0)\,. \qquad \text{(III-112)}$$

The simple equation for $J_\rho(\tau)$ found in Section III.3, given by equation (III-46) is not valid in this case because the function, $J_\rho^\circ(x)$, the mean intensity of the thermal radiation field in the case of no scattering, is measured in terms of a variable, x, that is not the variable x used above. The variable, x, as defined by equation (III-88) is dependent on $\tilde{\omega}$ so that equation (III-46) written in terms of x is not valid for this problem with a variable density.

It is a simple matter to extend the results of this section to include anisotropic scattering according to the schematic phase function. The equation of transfer for the scattered radiation, (III-12) is now

$$\mu \frac{\partial I_s(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial I_s(r,\mu)}{\partial \mu} = -(\kappa_s+\sigma_s)\rho(r)\Big\{ I_s(r,\mu) -$$

$$- \tilde{\omega}\alpha J_s(r) - \tilde{\omega}(1-\alpha)\left[\beta I_s(r,\mu) + (1-\beta) I_s(r,-\mu)\right] - \tilde{\omega}\alpha J_{red}^{inc}(r) -$$

$$- \tilde{\omega}(1-\alpha)\left[\beta I_{red}^{inc}(r,\mu) + (1-\beta) I_{red}^{inc}(r,-\mu)\right]\Big\}, \qquad \text{(III-113)}$$

where we assume that the parameters, $\tilde{\omega}$, α and β are not functions of position. The analysis given above for the isotropic scattering problem, when applied to equation (III-113) yields

$$J_s(x) = \frac{-\tilde{\omega}x^\nu}{(1-\tilde{\omega})}\left[A K_\nu(x) - F_\nu(x)\right], \qquad \text{(III-114)}$$

where $F_\nu(x)$ is given by equation (III-94), and x by equation (III-88). However, we now have

$$\epsilon^2 = 3\gamma(1-\tilde{\omega}), \tag{III-115}$$

where $\gamma = 1 - \tilde{\omega}(1-\alpha)(2\beta-1).$

The constant A is obtained in the same manner and is

$$A = \frac{\{(x_0 - 2\epsilon\nu/3\gamma)F_\nu(x_0) - 2\epsilon x_0 F_\nu'(x_0)/3\gamma\}}{\{(x_0 - 2\epsilon\nu/3\gamma)K_\nu(x_0) - 2\epsilon x_0 K_\nu'(x_0)/3\gamma\}} \tag{III-116}$$

The analysis for the thermal radiation is exactly the same as before and the mean intensity of the thermal radiation field is given by equations (III-111) and (III-112). Hence, the temperature, as a function of x, is

$$T^4 = J_p(x) + n(1-\tilde{\omega})\left[J_s(x) + J_{red}^{inc}(x)\right]. \tag{III-117}$$

We shall not give a complete account of the results for an atmosphere with the density function of equation (III-80). The results are very similar in a great many respects to those obtained for the constant density atmosphere. This is certainly true of the stellar radiation fields. The mean intensity of the reduced incident radiation simply falls of in an exponential-like curve to a value of 10^{-3} in two or three units of optical depth. Results have been calculated for the case of an atmosphere of a = 1.0; R = 1.0 and 10.0; and m = 2.5, 1.75 and 1.5. In general, it is not possible to say that $J_{red}^{inc}(\tau)$ is a simple function of m. The curves of $J_{red}^{inc}(\tau)$ for different values of m intersect several times though in no circumstances do they deviate very much one from another. The major difference between this density function and the constant density is presence of the central core which is optically very thick and, in fact, infinitely thick at the origin. This core can be defined as the region of density greater than an arbitrary density, ρ_0. For most points at a particular optical depth the radius decreases as m increases. Consequently, the angle subtended by the core of the atmosphere increases as m increases. In fact, the area of the core increases as m increases also, because ρ_0 is usually a high value of the density. Hence, the mean intensity of the reduced incident radiation at a particular value of τ, decreases as m increases. This situation arises for the most part in the outer two or three units of optical distance. Deeper in the atmosphere the converse is true; r increases as m increases; the angle subtended by the core decreases and $J^{inc}_{red}(\tau)$ increases, though is very small for these optical depths. The situation is complicated further in the region within 0.75 units of optical depth from the surface. In this case, $J^{inc}_{red}(\tau)$ is largest when m is 2.5. When R is 10.0 the densities near the surface are far more important than they were when m was 1.0, because the central core is farther away. Hence, as m increases so the surface density decreases and the reduced incident radiation leaving the atmosphere at angle of μ less than 0.7, increases. The whole process is too complicated to discuss qualitatively in any further detail because there are so many new variables several of which are interdependent.

The mean intensity of the scattered radiation is shown in Fig. 63 as a function of optical depth for isotropic scattering of albedo, 0.5; with a = 1.0, m = 2.5, 1.75 and 1.5 and R = 1.0 and 10.0. In general, the smaller atmosphere has a larger inward flux of reduced incident radiation because the densities nearer the surface are larger and therefore less radiation can escape from the atmosphere at large angles of emergence. Consequently, more radiation is absorbed and more radiation is scattered, so that we observe that $J_s(\tau)$ is larger for all values of m and R = 1.0 than for corresponding values of m with R = 10.0. Again, the form of these curves are similar to those of the constant density case. As m increases we have smaller scattered radiation fields at all optical depths. The scattered radiation cannot cross the central core. The larger the value of m the greater is the area of this core and the smaller the radial value of a given optical depth. Thus $J_s(\tau)$ decreases as m increases.

The mean intensity of the thermal radiation and the temperature are plotted as functions of optical depth in Fig. 64 for values of n of 10^4 and 10^{-2} ; a = 1.0, R = 10.0, m = 2.5 and for isotropic scattering of albedo, 0.5. The mean intensities of the reduced incident and scattered radiation fields are also shown. The scale of the ordinate is different for several of the functions but all are shown together for this one case only to show their form in a general comparison with the equivalent graphs of the constant density atmosphere. Whereas the atmosphere was similar to the optically thick constant density atmosphere with regard to the stellar radiation fields this is not true for the thermal radiation fields in such a general manner. The core of the atmosphere is still optically thick and the thermal radiation is of constant mean intensity below a depth of 3 units when n is equal to or less than unity and constant throughout the atmosphere when n is large. This situation is precisely that of the optically thick constant density atmosphere. However, $J_p(\tau_0)$ and hence $J_p(\tau)$ for n = 10^4, is closest to its value in an optically thin constant density atmosphere. This stems from the smaller flux of the reduced incident radiation at the surface which leads to smaller scattered and thermal emergent fluxes and smaller surface values of the mean intensities. The reduced incident radiation flux is smaller than in the optically thick constant density atmosphere because the intensity of the incident radiation traversing the atmosphere at small and grazing angles to the surface is attenuated far less in the variable density atmosphere. Thus the spherical atmosphere whose density function is given by equation (III-80) is like an optically thick constant density atmosphere with regard to the stellar radiation but unlike either an optically thick or optically thin constant density atmosphere with regard to the thermal radiation. This does not indicate a setback to the use of the constant density atmosphere but rather shows the inadequacies of the density function of equation (III-80). However, we have used small values of a and R. With larger values of these parameters the core region would be proportionately smaller with respect to the optical depth scale, and consequently would affect the atmosphere to a lesser extent. In that case the variable density atmosphere would give results closer in character to those of the constant density atmosphere.

The effect of a variable density can be explored in another way, but for a special case only. In Section III.3 it was seen that there were two methods available for the solution of the scattering problem in spherically symmetric systems. The first method involved the application of the Eddington approximation to the whole scattered radiation field whereas the second involved that application to part of the scattered radiation field, the intensity of the remainder being obtained exactly. In the limiting case of linear scattering only the solution for the whole of the scattered radiation field was exact. We have already used the first method in this section, and we shall now employ the second to the case of linear scattering. This will not only yield the exact solution but be valid for any density function. For linear scattering the intensity of the total stellar radiation field at any point in the atmosphere is given by the solution of the equation of transfer in a one-dimensional medium of finite length. The solution of this problem has been obtained for the constant density case in equations (II-49) and following. Equations (II-48), suitably altered to allow a variable density are

$$\frac{dI^+(s)}{ds} = -(\kappa_s+\sigma_s)\rho(s)I^+(s) + \beta\sigma_s\rho(s)I^+(s) + (1-\beta)\sigma_s\rho(s)I^-(s), \qquad \text{(III-118)}$$

and $$-\frac{dI^-(s)}{ds} = -(\kappa_s+\sigma_s)\rho(s)\,I^-(s) + \beta\sigma_s\rho(s)I^-(s) + (1-\beta)\rho(s)\sigma_s I^+(s)$$

where s is the geometrical distance along the line of transfer. The denominators of the left-hand sides of these equations contain the differential, ds , only, so that there need be no restriction on the function $\rho(s)$ if the optical distance along the line is defined as x such that

$$dx = (\kappa_s+\sigma_s)\rho(s)\,ds \;;\quad x = \int_0^s (\kappa_s+\sigma_s)\rho(s)\,ds\,. \qquad \text{(III-119)}$$

With the optical distance scale so defined, equations (III-118) reduce to equations (II-49). The boundary conditions are that there is an intensity I_0 incident on each end of the line of transfer. The solution is thus given by equations (II-49) and (II-51), and the constants of integration by equations (II-55). That is,

$$I(r,\mu) = C_1 e^{\sigma x} + C_2 e^{-\sigma x} , \qquad \text{(III-120)}$$

and $$I(r,-\mu) = \frac{1}{\tilde{\omega}(1-\beta)} \left\{ C_1 e^{\sigma x}[1-\tilde{\omega}\beta+\sigma] + C_2 e^{-\sigma x}[1-\tilde{\omega}\beta-\sigma] \right\} ,$$

where $$C_1 = \frac{I_0 \left[\tilde{\omega}(1-\beta) - (1-\tilde{\omega}\beta-\sigma) e^{-\sigma x_0} \right]}{\left[(1-\tilde{\omega}\beta+\sigma) e^{\sigma x_0} - (1-\tilde{\omega}\beta-\sigma) e^{-\sigma x_0} \right]} ,$$

(III-121)

$$C_2 = I_0 - C_1 ,$$

and $$\sigma^2 = (1-\tilde{\omega}\beta)^2 - \tilde{\omega}^2(1-\beta)^2$$

The optical distances, x and x_0 are given by equations (III-119);

$$x_0 = (\kappa_s + \sigma_s) \int_0^{s_0} \rho(s')\, ds' \qquad \text{(III-122)}$$

and $$x = (\kappa_s + \sigma_s) \int_0^{s} \rho(s')\, ds' .$$

The distances s and s_0 are found from the geometry of the atmosphere and are

$$s_0 = 2\sqrt{R^2 - r^2(1-\mu^2)} \tag{III-123}$$

and $$s = r\mu + \frac{1}{2}s_0 .$$

The mean intensity, J(r) is obtained by a Gaussian quadrature integration of equations (III-120). For each point of the quadrature the optical distances, x and x_0 are found by numerical integration of equations (III-122). The density can be any function of r, either an analytical function or a tabular function. The density $\rho(s')$ is easily found from $\rho(r)$ by use of equations (III-123). The mean intensity of the scattered radiation field is given by

$$J_s(r) = J(r) - J_{red}^{inc}(r) ,$$

(III-124)

where $$J_{red}^{inc}(r) = \frac{1}{2}\int_{-1}^{+1} I_0 \, e^{-x(\mu)} \, d\mu .$$

Thus, we have the exact solution for the scattering problem in a spherical atmosphere with a variable density function of any radial dependence, with the only restriction that the scattering is linear. This is not as severe a restriction as it first appears because the linear scattering problem with a value of β of 0.5 gave the same results as the isotropic scattering problem in a constant density atmosphere solved under Eddington's approximation. Moreover, this special case will give a good insight into the effect of a variable density function on the solutions for the radiation fields.

The equations for the thermal radiation field in Eddington's approximation are simpler than those for the scattered radiation field. They are sufficiently simple to allow the solution of the arbitrary density function case to be continued. The flux of the thermal radiation field is given by equation (III-101) which yields a numerical integral that is easy to evaluate. We have

$$H_p(r) = \frac{(1-\tilde{\omega})(\kappa_s+\sigma_s)}{r^2}\int_0^r r^2\rho(r)\,J(r)\,dr + C_1 \qquad \text{(III-125)}$$

The constant of integration is zero by virtue of the condition of conservation of energy which demands that $H_p(r) = -H(r)$. It can easily be shown that $H(r)$ is equal to $-H_p(r)$ as in equation (III-125) with C_1 equal to zero. The mean intensity of the thermal radiation field is found by integration of equation (III-102). Thus

$$J_p(r) = C_2 - \frac{3(\kappa_s+\sigma_s)}{n}\int_0^r \rho(r)\,H_p(r)\,dr, \qquad \text{(III-126)}$$

where we have assumed that the greenhouse parameter, n, is not a function of the radius, r. Eddington's approximation has been used in obtaining this equation and the constant, C_2, is found by using Eddington's approximate boundary condition. Hence, we obtain

$$C_2 = 2H_p(R) + \frac{3(\kappa_s+\sigma_s)}{n}\int_0^R \rho(r)\,H_p(r)\,dr,$$

so that $$J_p(r) = 2H_p(R) + \frac{3(\kappa_s+\sigma_s)}{n}\int_r^R \rho(r)\,H_p(r)\,dr. \qquad \text{(III-127)}$$

The temperature profiles are given by equation (III-117). Thus we have the complete solution for a general density function, subject to the condition of linear scattering. The solution for the isotropic emission of the thermal radiation is possible in this case even though the isotropic scattering problem introduced the practical requirement that the density function should be of the form given by equation (III-80). This is due to the conservative nature of the transfer. It is the non-conservative nature of the scattering that introduces the term, $(1 - \tilde{\omega})J_s(r)$ into equation (III-83) without which the complex factor, $[2/r - (1/\rho(r))d\rho(r)/dr]$ would be absent from equation (III-85) and a general density function permitted in the subsequent theory.

Again, extensive results are available for this method, but only one special case will be shown. It is sufficient to quote the results from one case only to show the main effects of the variable density function. Fig. 65 shows the mean intensities of the scattered and thermal radiation fields, together with the temperature plotted as a function of fractional optical radius for an atmosphere in which $n = 1.0$, $\tilde{\omega} = 0.9$, $\tau_0 = 5.0$, $\alpha = 0.0$ and $\beta = 0.5$. The continuous curves show the quantities in an atmosphere of density; $\rho(r) = \rho_0[1{\cdot}5 - r]$, and the broken curves show the same quantities in an atmosphere of constant density, $\rho(r) = \rho_0$. In general, the two sets of curves are very similar. Certainly any qualitative conclusions derived from the one will be the same as those derived from the other. Were the fractional radius used as the abscissa there would be less similarity between the sets of results, but the optical distance scale is recognised to be the more important in radiative transfer problems. It is interesting to see how the differences are of opposite sign in the two radiation fields, and hence partly cancel when evaluating the temperature. When n is 10^4 the stellar radiation field dominates the temperature and the differences will be similar to those for the stellar radiation field. Similarly, the temperature differences will be similar to the thermal radiation mean intensity differences when n is 10^{-2}. This density function provides more realistic atmospheres than the one of equation (III-80). As we observed, the latter density function was neither optically thick nor optically thin for the thermal radiation, but could be adjusted to be optically thick. With this method we have a density function that can produce an optically thin atmosphere that has the decrease in density towards the surface. In general, we see that the constant density is a good approximation to the density function chosen earlier.

7. Modification of the Incident Radiation

The incident radiation that has been considered in the foregoing sections of this Chapter has been dilute uniform isotropic starlight. In this section we shall consider modifications of this. The nature of the interstellar radiation field was discussed extensively in Section III.1, and two elementary extensions of the approximate form for the interstellar radiation field were proposed.

The first is the addition of an undilute uniform isotropic thermal radiation field attributed to the universal microwave background radiation. Let the integrated intensity of this radiation be αI_0, where I_0 is the intensity of the dilute starlight. The inclusion of this additional radiation field does not, of course, affect the scattered radiation field whose mean intensity is given by equation (III-27). The source function for the thermal radiation field is the source function for the thermal radiation field of the standard problem, $\alpha = 0$, given by equation (III-31), plus a contribution arising from the absorption and re-radiation of the reduced incident thermal radiation. This process is conservative and isotropic so that the additional term is the mean intensity of the reduced incident thermal radiation, which is simply, $\alpha J_{red}^{inc}(\tau/n)$, the thermal radiation being attenuated by the factor τ/n, which equals the optical distance, $\kappa_p \rho r$. Thus, we have

$$B_p^*(\tau) = J_p^*(\tau) + \alpha J_{red}^{inc}(\tau/n) + n(1-\varpi)\left[J_s(\tau) + J_{red}^{inc}(\tau)\right], \quad \text{(III-128)}$$

where the asterisk superscript refers to those functions of the present problem with the additional incident radiation.

The equation of transfer and its first two moment equations are

$$\mu \frac{\partial I_p^*}{\partial \tau}(\tau,\mu) + \frac{(1-\mu^2)}{\tau}\frac{\partial I_p^*}{\partial \mu}(\tau,\mu) = -\frac{1}{n} I_p^*(\tau,\mu) +$$

$$+\frac{1}{n} J_p^*(\tau) + \frac{\alpha}{n} J_{red}^{inc}(\tau/n) + (1-\varpi)\left[J_s(\tau) + J_{red}^{inc}(\tau)\right], \quad \text{(III-129)}$$

$$\frac{dH_p^*}{d\tau}(\tau) + \frac{2H_p^*(\tau)}{\tau} = \frac{\alpha}{n} J_{red}^{inc}(\tau/n) + (1-\varpi)\left[J_s(\tau) + J_{red}^{inc}(\tau)\right], \quad \text{(III-130)}$$

and

$$\frac{dJ_p^*(\tau)}{d\tau} = -\frac{3}{n} H_p^*(\tau), \quad \text{(III-131)}$$

where the Eddington approximation, $K_p^*(\tau) = (1/3)\,J_p^*(\tau)$, has been used in the derivation of the second of the two moment equations. These two moment equations combine to give

$$\frac{d^2}{d\tau^2}\left[\tau J_p^*(\tau)\right] = -\frac{3(1-\tilde{\omega})\tau}{n}\left[J_s(\tau) + J_{red}^{inc}(\tau)\right] - \frac{3\alpha\tau}{n^2} J_{red}^{inc}(\tau/n), \tag{III-132}$$

the solution of which is

$$J_p^*(\tau) = J_p(\tau) + n\alpha J_p^{\circ}(\tau/n) + C_1 + C_2/\tau \ , \tag{III-133}$$

where $J_p(\tau)$ is the solution for the mean intensity of the thermal radiation field of the standard problem, which is given by equation (III-46), and $J_p^{\circ}(\tau)$ is the solution of the standard problem in which there is no scattering, and is given by equation (III-42). The constants of integration in both equations (III-42) and (III-46) are ignored and absorbed into the two constants of equation (III-133).

The flux of the thermal radiation is given by equation (III-131), which using equation (III-133), gives,

$$H_p^*(\tau) = -\frac{n}{3}\frac{dJ_p(\tau)}{d\tau} - \frac{\alpha}{3}\frac{dJ_p^{\circ}(\tau/n)}{d\tau} - \frac{C_2}{\tau^2} \ , \tag{III-134}$$

which, in turn, on using equations (III-39) and (III-44) becomes

$$H_p^*(\tau) = H_p(\tau) - \alpha H_{red}^{inc}(\tau/n) - C_2/\tau^2 \ . \tag{III-135}$$

In order to conserve energy in the atmosphere it is necessary to have

$$H_p^*(\tau) + H_s(\tau) + H_{red}^{inc}(\tau) + \alpha H_{red}^{inc}(\tau/n) = 0 \ , \tag{III-136}$$

where αH^{inc}_{red} (τ/n) is the flux of the reduced incident thermal radiation. It was a condition of the standard problem, equation (III-45), that

$$H_p(\tau) = -H_s(\tau) - H^{inc}_{red}(\tau) , \tag{III-137}$$

so that equations (III-135) to (III-137) give the result that C_2 equals zero. This condition also ensures that the mean intensity of the thermal radiation field remains finite at the centre of the atmosphere. The other constant, C_1, is found by using the Eddington boundary condition, $J_p(\tau_0) = 2H_p(\tau_0)$, which gives

$$C_1 = 2H_p(\tau_0) - 2\alpha H^{inc}_{red}(\tau_0/n) - J_p(\tau_0) - n\alpha J_p^{0}(\tau_0/n). \tag{III-138}$$

This completes the solution for the mean intensity and flux of the thermal radiation field. The temperature of the atmosphere is given, as before, by the fourth root of the source function, which in this case, is given by equation (III-128).

The effects of the additional incident radiation on the temperature profiles of the atmosphere are shown in Figs. 66 and 67. Fig. 66 shows the temperature profiles and central temperatures of an isotropically scattering atmosphere of albedo, 0.1 and 0.9, and greenhouse parameter, 10^4. The central temperatures are shown as functions of τ_0 by the continuous curves and the temperature profiles of an atmosphere of $\tau_0 = 10.0$ are shown by the broken curves with $(\tau_0 - \tau)$ as the abscissa. These latter curves are truncated at $\tau/\tau_0 = 0.9$ due to the logarithmic scale of the abscissa. When n is 10^4 the stellar radiation field dominates the temperature.

Consequently, an increase in the thermal radiation field will be important only at those positions where the stellar radiation is unimportant. Thus we see that there is no noticeable increase in the temperatures near the surfaces of the atmospheres nor in the centres of optically thin atmospheres, even with $\alpha = 5.0$. At the centres of optically thick atmospheres the stellar radiation is effectively absent. Consequently, the central temperature of an atmosphere of $\tau_0 = 50.0$ when $\alpha = 5.0$ is approximately double that when $\alpha = 0.0$. When $n = 10^4$ the atmospheres are optically thin to the thermal radiation so there is no build up of the thermal radiation. It can be seen that the albedo is unimportant in determining the central temperatures of optically thick atmospheres. It is certainly not very important in the standard problem, $\alpha = 0$, and becomes less important as α increases, and as the temperature depends more and more on the incident thermal radiation rather than the thermal radiation generated by the atmospheric degrading of the stellar radiation. In Section III.1 we saw that a typical value of α lies between 0.5 and 1.0. Thus, Fig. 66 shows that, for $n = 10^4$, any incident thermal radiation will not be important in the heating of an interstellar dust cloud.

The effect of anisotropy of the scattering of the starlight on the results of Fig. 66 can be stated as simply being to change the limiting value of the optical depth, $(\tau_0 - \tau)$, or τ_0 at which the incident thermal radiation affects the temperature of the atmosphere, provided that α is large enough for any change to be detected. As the penetration of the stellar radiation increases so the effect of the extra thermal radiation becomes restricted to greater and greater optical depths.

When n is less than unity the thermal radiation field dominates the temperature and the extra incident radiation will thus be important. Fig. 67 is the analogue of Fig. 66 for $n = 10^{-2}$. However, in Fig. 67 only the graphs for $\tilde{\omega} = 0.9$ are shown. Again the central temperatures are plotted as functions of τ_0, and the temperature profiles as functions of optical depth, $(\tau_0 - \tau)$. The former are shown for values of α of 0.0, 0.1, 1.0 and 5.0 with continuous curves, and the latter for values of α of 0.0 and 5.0 and for values of τ_0 of 1.0 and 10.0, with broken curves. There is a discontinuity in the abscissa scale at the point, unity, due to a change in scale from linear to logarithmic. This permits the inclusion of surface temperatures which are important in this case, but were unimportant when n was 10^4. The temperature increase with α is greatest for optically thin atmospheres and the surface temperatures of all atmospheres are always increased to a greater extent than the central temperatures. The temperature increase with α is greatest at small optical depths. These increases are by no means negligible, as they were when n was 10^4. In fact, the parameter α is more important than τ_0 in controlling the surface temperatures particularly when it is greater than unity. In the centres of optically thick atmospheres the increase in temperature with α, though not very large, does not decrease as τ_0 increases as was the case in the centres of optically thin atmospheres. All these phenomena are features of Fig. 67 and are due to the very small mean free path of the thermal photons. This causes a great many thermal photons to be trapped near the surface and hence increase the surface temperatures rather than the central temperatures. The thermal radiation derived from the additional incident thermal radiation can be considered separately from that derived from the stellar radiation field. It consists of a conservative isotropic scattering problem in an atmosphere whose optical radius is measured in units of $\tau_p = \tau/n$. The second moment equation, in Eddington's approximation, derived from the appropriate equation of transfer is

$$\frac{dJ_p^{(1)}(\tau_p)}{d\tau_p} = 3H_p^{(1)}(\tau_p) ,$$

where the superscript, unity, refers to the thermal radiation derived from the incident thermal radiation. Now, $H_p{}^{inc}_{red}(\tau_p)$ decreases to zero due to attenuation, so that $J_p^{(1)}(\tau_p)$ is constant with depth below the certain optical depth at which the reduced incident radiation field is effectively zero. For $n = 10^{-2}$, this occurs at a very small value of $(\tau_0 - \tau)$. Therefore, the increase in temperature with a is constant except near the surface, where, of course, it is greater. This is observed in Fig. 67 from the parallel nature of the temperature profiles for each value of τ_0 and at most values of τ.

The second modification of the incident radiation that we shall investigate is the addition of radiation from a nearby star. In Section III.1 we saw that about one sixth of the total energy density of the interstellar radiation field in the Sun's vicinity is due to Sirius. Thus a nearby star may be important in controlling the temperature of interstellar dust clouds. When such a star is close enough to contribute a certain fraction, say one tenth, of the total energy of the incident radiation but sufficiently far away from the cloud that the distance between them is very large in comparison with the radius of the star, then the radiation from that star can be represented by a parallel beam of radiation incident with a uniform intensity on all points of a hemispherical surface on one side of the cloud. This geometrical approximation is reasonable if we are to ascertain the effect of the star on the temperature profiles of the cloud. The linearity of the scattering phenomenon allows the parallel incident radiation and the radiation scattered from it to be treated as separate radiation fields. Thus we shall consider the problem of parallel incidence only and then add the results to those obtained for the standard problem.

The geometry of the problem is shown in Fig. 68. Again, we shall consider the absorption coefficients and density to be constant throughout the atmosphere, and the optical radius, OP, to be defined as $(\kappa_s + \sigma_s)\rho r$. We define

$$\tau_{x_0} = 2 \times IN = 2\sqrt{\tau_0^2 + \tau^2(\mu^2 - 1)} \qquad \text{(III-139)}$$

and $$\tau_x = IP = \tau\mu + \sqrt{\tau_0^2 + \tau^2(\mu^2 - 1)} \; ,$$

where τ_0 is the total optical radius of the atmosphere. The incident radiation is a parallel beam of net flux, πF^*, across unit area normal to itself, in the direction, $\mu = +1$. There is still azimuthal symmetry but the complete spherical symmetry of the standard problem has been lost. The equation of transfer is now more complex than that of the standard problem because the intensity at point P is now a function of two position variables,

τ and θ, and two directional co-ordinates θ' and ϕ'. The equation of transfer for a completely general problem in spherical symmetry is given by Uesugi and Tsujita (1969). In our problem it reduces to

$$\cos\theta \frac{\partial I}{\partial \tau}(\tau, \theta, \theta', \phi') + \frac{\sin\theta' \cos\phi'}{\tau} \frac{\partial I}{\partial \theta}(\tau, \theta, \theta', \phi')$$

$$- \frac{\sin\theta'}{\tau} \frac{\partial I}{\partial \theta'}(\tau, \theta, \theta', \phi') - \frac{\sin\theta' \sin\phi'}{\tau \tan\theta} \frac{\partial I}{\partial \phi'}(\tau, \theta, \theta', \phi')$$

$$= -I(\tau, \theta, \theta', \phi') + B(\tau, \theta, \theta', \phi') . \qquad \text{(III-140)}$$

The solution of this equation is very much more complex than that of equation (I-10). This sub-section is included merely to show the effect of a nearby star, so we shall consider the case of pure linear scattering. When the scattering is linear the radiative transfer is linear also and the one-dimensional equation of transfer provides an equation much easier to solve than equation (III-140). The approximation of linear scattering of $\beta = 0.5$ gave results close to that of the isotropic scattering problem, so the approximation will give results of the correct order of magnitude. However, they will not be as good as in the standard problem because that was spherically symmetric and the radiation was incident in all directions giving rise to a normal radiation field within the atmosphere. This is not true in this case because the resulting radiation field will be linear also, which is not physically realistic.

The solution for the total stellar radiation field in a one-dimensional medium is given by equations (II-49) to (II-51) and equations (II-53). Thus, if $I^*(\tau, \mu, \mu')$ is the intensity of the total stellar radiation field at position, (τ, μ), and in the direction, μ', due to the incident radiation, πF^*, then

$$I^*(\tau, \mu, +\mu') = \delta(\mu'-1)\left[C_1 e^{\sigma\tau_x} + C_2 e^{-\sigma\tau_x}\right]$$

and
$$I^*(\tau, \mu, -\mu') = \frac{\delta(\mu'-1)}{\tilde{\omega}(1-\beta)}\left[C_1(1-\tilde{\omega}\beta+\sigma)e^{\sigma\tau_x} + \right.$$
$$\left. + C_2(1-\tilde{\omega}\beta-\sigma)e^{-\sigma\tau_x}\right], \qquad \text{(III-141)}$$

where $\mu' > 0$, and there is no azimuth co-ordinate by virtue of the definition of the co-ordinate axis. In equations (III-141) the constants are given by

$$\sigma^2 = (1-\tilde{\omega}\beta)^2 - \tilde{\omega}^2(1-\beta)^2 ,$$

$$C_1 = \frac{-(1-\tilde{\omega}\beta-\sigma)\pi F^* e^{-\sigma\tau_{x_0}}}{\left[(1-\tilde{\omega}\beta+\sigma)e^{\sigma\tau_{x_0}} - (1-\tilde{\omega}\beta-\sigma)e^{-\sigma\tau_{x_0}}\right]} , \qquad \text{(III-142)}$$

and
$$C_2 = \pi F^* - C_1 .$$

The mean intensity of the stellar radiation field is therefore

$$J^*(\tau, \mu) = \frac{C_1\left[1-\tilde{\omega}(2\beta-1)+\sigma\right]e^{\sigma\tau_x}}{4\pi\tilde{\omega}(1-\beta)} +$$
$$+ \frac{C_2\left[1-\tilde{\omega}(2\beta-1)-\sigma\right]e^{-\sigma\tau_x}}{4\pi\tilde{\omega}(1-\beta)} . \qquad \text{(III-143)}$$

The thermal radiation presents a greater problem because the emission of radiation by the particles in the atmosphere is isotropic. However, the value of n is generally considered to be large for dust clouds, and when n is large the thermal radiation is unimportant in determining the temperature. Consequently, we shall not attempt to solve the problem for the thermal radiation when n is less than or equal to unity but shall adopt a very rough estimate for the mean intensity of the thermal radiation field and calculate the temperature profiles of the atmosphere for the case, $n = 10^4$, only. The mean intensity of the thermal radiation field was constant throughout the atmosphere in the standard problem and was of the order of 0.5 for optically thick atmospheres. We shall assume the same values in this case. Fig. 69 shows the temperatures measured in units of T_e, for an atmosphere illuminated by parallel radiation only. It is for the case of scattering with albedo, 0.9, $\beta = 0.5$, $\tau_0 = 10.0$ and $n = 10^4$. The temperature T_e is the effective temperature of the incident radiation given by the fourth root of $\pi F^*/\sigma$. Fig. 69 shows the temperature contours, with the radiation incident on the lower hemisphere. The temperature is strongly dependent on $J^*(\tau,\mu)$ and clearly shows the attenuation of this quantity across the atmosphere.

Fig. 70 shows the temperature contours of an atmosphere of parameters, $\tau_0 = 10.0$, $n = 10^4$, $\tilde{\omega} = 0.5$, and $\beta = 0.5$, with both isotropic and parallel incident radiation. The temperature is measured in units of T_e which is defined as in Section III.4 as $[\pi I_0/\sigma]^{1/4}$. The incident parallel flux is related to the incident isotropic intensity by the parameter $\lambda^* = F/I_0$. In Fig. 70 the value of λ^* is 2.0, which is large. The conlusion that we draw from this, is that the contours are translated in the direction of the incident parallel radiation so that the centre is no longer the coolest part of the atmosphere. Consequently we see that, for a dust cloud in the vicinity of the Sun, Sirius, for which λ^* is about two thirds, does not alter the magnitude of the temperature contours to any significant extent, but does move them in position relative to the centre of the cloud.

8. Summary

The mean intensities and fluxes of the scattered and thermal radiation fields within a spherical atmosphere of constant density situated in a dilute uniform isotropic stellar radiation field, have been obtained as functions of position in the atmosphere, subject to Eddington's approximation, by means of a simple algebraic analysis analogous to that developed in Chapter II for plane-parallel atmospheres. The scattering was assumed to be non-conservative and anisotropic according to the schematic phase function, (I-29). In the previous section we saw that such an analysis is possible for the case of uniform, isotropic radiation only. This is, however, a good approximation to the radiation incident upon an interstellar dust cloud situated near the galactic plane. The reason for the restriction on the density and the absorption coefficients was seen to be a consequence of the presence of both a length and its differential in the denominator of the total differential of the intensity in the equation of transfer, though it was only effective in the case of non-conservative scattering. Furthermore, the moments of the reduced incident radiation field could only be expressed analytically in constant density atmospheres.

The analysis was executed in two ways. Firstly, Eddington's approximation was assumed to apply to the whole of the scattered radiation field. This assumption could not have been made in the previous Chapter for parallel incident radiation because the radiation fields were dependent on azimuth for anisotropic scattering. There is no azimuth dependence in the spherical atmospheres of the standard problem of this Chapter because we enjoy complete spherical symmetry. Secondly, the fraction of the scattered radiation that was scattered continuously by the delta-function spikes of the schematic phase function was treated exactly and the remainder of the scattered radiation was assumed to obey Eddington's approximation. It was this second method that was used for plane-parallel atmospheres. It possessed the advantage that part of the solution was exact and the disadvantage that simple solutions were available for the phase functions, $(\alpha, \beta) = (\alpha, 1.0)$ and $(0.0, \beta)$ only. The former were completely analytical but the latter partially numerical. The restriction was due to the geometry of the problem and did not arise in the work on plane-parallel atmospheres. A distinct feature of the general method was that it was independent of the shape of the phase function for any particular values of $\tilde{\omega}$ and g. This was a direct consequence of the Eddington approximation combined with the spherical symmetry of the problem.

Another important feature of method II was its solution for $J_s(\tau)$ in the case of linear scattering only. This solution was exact, and hence provided a measure of the validity of the results of method I found by use of the Eddington approximation. It also provided exact values of the ratios that were ascribed certain constant values in the Eddington approximation. Furthermore, it gave information about the exact directional dependence of the intensity of the scattered radiation because the exact solution was a solution of the equation of transfer for the intensity in a one-dimensional medium, and was a solution for the intensity itself. It was found that the intensity distribution could deviate considerably from an isotropic distribution without causing a significant deviation of the ratio, $r(\tau) = J_s(\tau)/K_s(\tau)$, from 3.0; and that $r(\tau) = 3.0$ was a reasonable approximation for all but optically thick atmospheres. It was also seen that the ratio, $r_0(\tau_0) = H_s(\tau_0)/J_s(\tau_0)$, could be satisfactorily approximated to 2.0 for all but optically thin atmospheres. Thus, the results for the scattered radiation field by method I were poorest for very thin and very thick atmospheres. However, in the former case the scattered radiation is negligible in comparison with the reduced incident radiation, and at most optical depths in optically thick atmospheres the scattered radiation is very small so that the temperature profiles derived by the two methods do not differ greatly. It was also seen that the approximate results improved as g decreased because this created radiation fields that were more isotropic.

Both methods yielded the same simple expression for the mean intensity of the thermal radiation field and it was seen that that form was also valid for plane-parallel atmospheres but only for those phase functions for which method II was applicable. It was concluded that this simple expression for $J_p(\tau)$ in terms of $J_s(\tau)$ and $J_p^o(\tau)$, the mean intensity of the thermal radiation field which would be generated in the absence of scattering, arose when the whole scattered radiation field obeyed Eddington's approximation or when any part of the scattered radiation field that did not, was scattered with no change of direction. This latter situation corresponded to the former with a new scale of optical distance. The result was proved by integrating the approximate expression for $J_s(\tau)$. However, it can be proved without resorting to this long and tedious process. The pairs of moment equations for the scattered and thermal radiation fields are equations, (III-15) and (III-17), and, (III-38) and (III-39), respectively, and the pair of moment equations for the thermal radiation field that would be generated in the absence of scattering is equations (III-34) and (III-35). Adding equations (III-15) and (III-38), and using the equation of conservation of energy or constant net flux, equation (III-45), we obtain the result

$$H_s(\tau) + H_p(\tau) = H_p^\circ(\tau) = - H_{red}^{inc}(\tau) \; ; \qquad \text{(III-144)}$$

and, adding equations (III-17) and (III-39) we find

$$\frac{1}{\gamma} J_s(\tau) + n J_p(\tau) = C_1 + 3\int \left[\frac{(1-\gamma)}{\gamma} H_{red}^{inc}(\tau) - H_p^\circ(\tau) \right] d\tau \, .$$

With the aid of equations (III-35) and (III-144), we obtain the desired result;

$$J_p(\tau) = C_1 - \frac{1}{n\gamma} \left[J_s(\tau) - n J_p^\circ(\tau) \right] .$$

In this derivation we see clearly the dependence of the result upon the form of the second moment equations as amended by the Eddington approximation.

The results of the analysis are based on similar principles to those of Chapter II for plane-parallel atmospheres. The scattered radiation increases with optical depth until it reaches a maximum and then decays away almost to zero if τ_0 is large. The mean intensity of the thermal radiation is constant and small throughout the atmosphere when n is large because the thermal photons have a very long mean free path; and increases rapidly with depth, when n is small, in those regions of the atmosphere where there is an inward flux of stellar radiation, to assume a large constant value in the central regions of optically thick atmospheres where the flux of the stellar radiation is approximately zero. The differences between the results of the two chapters are essentially due to the different incident radiation fields so that very little information can be obtained about the effects of the geometry per se. It is in the realm of the emergent radiation fields that differences are most noticeable, for there is very often a radiation field of significant magnitude, which crosses the spherical atmosphere. This is an obvious result for the reduced incident radiation in an optically thin atmosphere but perhaps a more important feature of the sphericity of the atmosphere is inherent in the thermal radiation emerging from an optically thick atmosphere when n is large. This emergent radiation has one source region on the near side of the atmosphere and another on the far side. This is because the main source of the thermal radiation is the stellar radiation, which is restricted to the outer layers of an optically thick atmosphere and once generated suffers very little absorption on passage across the atmosphere. The fact that the main source of the thermal radiation lies in the outer shell in this case also gives rise to the higher intensity of the radiation emerging from the outer limb. We may conclude that a cloud whose infra-red emission is brightened in the limb is optically thick and has a high value of n. A cloud of almost uniform brightness may be either an optically thin cloud of any greenhouse parameter or an optically thick cloud with greenhouse parameter, unity. A cloud with strong limb darkening may be either an optically thick cloud of small greenhouse parameter or a cloud of optical radius of the order of unity with any greenhouse parameter. Thus we may obtain a good idea of the values of τ_0 and n of a cloud by an infra-red surface brightness map. The above conclusions are very general and do not apply very close to the limb because all clouds are limb darkened in the infrared at their extremities.

The density of a real spherical atmosphere decreases towards the surface. It was found that the theory could be modified to cope with a density function that was given by an inverse power of the radius. However, this produced an infinite singularity at the origin and precluded a variability of optical thickness. When n was small the thermal radiation in such an atmosphere was seen to be similar to that in an optically thin constant density atmospheres, as was the stellar radiation; but when n was large the thermal radiation exhibited features typical of both optically thick and optically thin constant density atmospheres. The special case of linear scattering in an atmosphere of a general density function was solvable by method II, and its results showed the precise form of the density function to be unimportant in controlling the results of the problem, and especially the qualitative conclusions.

Finally, additional radiation sources were considered. An additional thermal radiation field, said to represent the microwave background radiation, was found to have little effect on the temperatures of atmospheres with large values of n, which are those typical of interstellar clouds. It did have a significant effect on the surface temperatures of atmospheres with small values of n. Consequently, it will be useful to investigate the possible influence of an incident thermal radiation field on the temperatures of planetary atmospheres. The effect of a nearby star was seen to be both very important and relatively unimportant. It is very important in creating a situation that is not spherically symmetric thus necessitating the use of a more complex equation of transfer. It is relatively unimportant in changing the temperatures of typical interstellar clouds. This was seen by a rather rough approximate technique, and the main effect of a nearby star was to offset the temperature contours from the centre of the atmosphere. The problem is completely different in the case of a reflection nebula whose incident radiation primarily stems from one star, which in most cases is very much nearer than that considered in the context of a nearby star perturbing the temperature profiles of an interstellar dust cloud.

CONCLUDING REMARKS

The primary objective of this study was to gain insight into the role and importance of each atmospheric parameter involved in the problem of radiative heating. This was stated in the introductory Chapter and it was proposed that it could best be achieved by adopting a simple mathematical model for the physical problem and solving the equation of transfer by approximate analytical means. The atmosphere has consequently been treated as grey with respect to any incident radiation derived from stellar sources and grey with respect to infra-red radiation emitted by the atmospheric constituents, the ratio between the extinction coefficients in the two parts of the spectrum being n, the greenhouse parameter. The transfer of the radiation in these two parts of the spectrum has been solved using Eddington's approximation, which is generally the best approximation that will permit an analytical solution. The algebra involved in these simple solutions has been complex in places and, although we have tried to simplify the model sufficiently to reduce the variables to a tractable number, there are still sufficient to make the qualitative results discussed in the main body of the thesis, lengthy and intricate in places. In these concluding remarks we shall endeavour to indicate the tenor of the results as a whole, by making several general statements concerning the roles of the important atmospheric parameters in determining the temperatures.

Firstly, it is clear that the fundamental parameter of the radiative heating problem is the greenhouse parameter, n. It is a measure of the reciprocal of the mean free paths of the photons in the stellar and infra-red parts of the spectrum, so that a large value of n allows the infra-red photons easy escape from the atmosphere, and a small value of n prevents escape of the infra-red photons. We have seen that this stems from the need to maintain an energy density gradient inversely proportional to the mean free path of the photons in order to produce a certain flux through the atmosphere. The flux of the thermal radiation that must be maintained through the atmosphere is determined from the fluxes of the stellar radiation fields via the restriction that energy must be conserved in the atmosphere, which is equivalent to maintaining a condition of zero net flux. Hence, a small value of n gives rise to a large thermal radiation field and a large value of n to a small one, the latter being constant throughout the atmosphere and the former constant at points deeper in the atmosphere than the penetration depth of the stellar radiation. As well as controlling the magnitude of the mean intensity of the thermal radiation field, the greenhouse parameter controls the relative importance of the two fields in determining the temperature of the atmosphere. The source function for the thermal radiation is, in general, proportional to the fourth power of the temperature and equal to the mean intensity of the thermal radiation plus the mean intensity of the stellar radiation multiplied by the factor, $\eta(1 - \tilde{\omega})$. This is the energy balance principle as expressed by equation (I-55), applied to a two part grey atmosphere. A high value of n means that the emitted radiation is emitted poorly, and the temperature, high, and dependent on the main source of absorption, the stellar radiation. For a low value of n the infra-red radiation is emitted easily so that the stellar radiation is unimportant in controlling the temperature. Thus, we see that a spherical atmosphere is brightened in the infra-red near the limb when n is large and darkened near the limb when n is small. The behaviour of the limb-brightness in the infra-red when n is unity depends strongly on the other parameters. Thus, we see the two-fold role of the greenhouse parameter, in controlling the energy balance in the element of matter and in controlling the transfer of the infra-red radiation relative to the visible radiation.

Another parameter whose value is very important is τ_0 , the total optical thickness or radius of the atmosphere. It is extremely critical when it is below a certain limit. At distances sufficiently far from the surface of an atmosphere the stellar fluxes, and hence, the thermal flux and temperature gradient, are all zero. Consequently, τ_0 has little effect on the temperatures of such atmospheres. However, it is most important if it is sufficiently small to permit some radiation to pass through the atmosphere because it then measures the amount of absorbing material in the atmosphere, and hence is proportional in some way to the temperature. It affects the emergent thermal radiation to such an extent that the greenhouse parameter is unimportant when τ_0 is small.

The albedo is a very important parameter, and like the greenhouse parameters, enters into the problem in two ways. Firstly it enters into the energy balance condition. It is the ratio of the absorption coefficients that is important in this context rather than the ratio of the extinction coefficients, and this ratio is $n(1 - \tilde{\omega})$. That is, the albedo controls the relative weights that the two radiation fields exert when controlling the temperature. When n is small its influence in this way is negligible, but when n is large we can say that the fourth power of the temperature is roughly proportional to the fraction, $(1 - \tilde{\omega})$. The other role of the albedo is to enable the stellar radiation to penetrate deeper into the atmosphere. Consequently, between one and ten units of optical distance from the surface, the scattered radiation depends critically on the value of the albedo. Hence, the albedo is very important in controlling the temperature at those optical depths when n is large. It is in these cases that the optical thickness of the atmosphere is very important because the amount of scattered radiation able to penetrate through an atmosphere depends critically on both $\tilde{\omega}$ and τ_0 if τ_0 is one of these intermediate values. In optically thin atmospheres the largest part of the radiation lies in the reduced incident radiation field and it is only in the first mentioned role that the albedo's control of the temperature is manifested.

We have seen that the only other important parameter in the phase function is the asymmetry parameter. This has only one major role and that is to control the penetration of the scattered radiation. A high value of g effectively reduces the optical thickness of an atmosphere. Apart from its most obvious effect in controlling the scattered radiation and the temperature when n is large, the asymmetry parameter is very important in controlling the internal temperatures of optically thick atmospheres when n is small. This occurs through the very steep temperature gradient being maintained throughout a greater depth when the stellar radiation is given greater penetrating power. This has a substantial effect in planetary atmospheres. The asymmetry parameter also plays a minor role in controlling the energy lost through the surface by back-scattering.

A further parameter, this time only applicable to the plane-parallel atmospheres, is $\cos^{-1}\mu_0$, the angle of incidence. We have seen that this merely reduces the temperatures by a factor close to μ_0, due to the change in energy flux entering the atmosphere. Again, there are minor effects which are important in the temperature profiles only in certain limiting cases such as those involving grazing incidence.

The final parameter, again only used in association with plane-parallel atmospheres, is the ground albedo, λ. The ground was seen to behave like a semi-infinite atmosphere of greenhouse parameter, unity and albedo, λ. Consequently its effects were dependent on n and $\tilde{\omega}$ according to whether they were above or below unity and λ respectively. In general, the presence of a ground layer at the lower surface of a finite atmosphere made the temperatures of the atmosphere more akin to those in the equivalent uppermost layers of a semi-infinite atmosphere.

These are the major roles of the various atmospheric parameters of the radiative heating problem. There are many minor aspects of the influence of these parameters on the radiation fields, and these have been described in the body of the thesis. We may conclude that the approximate treatment used, has provided a satisfactory framework for illuminating these aspects of the problem.

We have also furnished useful information concerning the parameters in relation to their future use in accurate model atmospheres. The analytical theory has made it clear for which ranges of values they are at their most critical. The albedo is most critical near unity, and it is close to unity in planetary atmospheres though not in interstellar clouds. Consequently, the albedo must be known exactly for accurate models of the former but less so for models of the latter. The asymmetry parameter is most important when it approaches unity, particularly when n is small. Again, this is the case for planetary atmospheres. Therefore we must establish accurate values of g for these atmospheres. This conclusion is not in conflict with the earlier idea that the phase function is unimportant. The phase function need not be treated accurately in the transfer problems but it must be known accurately to allow an accurate determination of g, which must be made by numerical means for physically realistic models. Thus, we are not excused the evaluation of an accurate phase function, but merely its inclusion as an entity in the radiative transfer calculations. The greenhouse parameter can assume any value in a wide range of possibilities and, in general, it is important in every part of that range. However, in certain limiting circumstances its precise value is not important. If the atmosphere is optically thin, the emergent thermal radiation is virtually independent of n , as is the central temperature of a spherical atmosphere when n is small. Again, the central temperatures of optically thick spherical atmospheres are independent of the exact value of n provided it is large. However, n is very important in most cases.

The final point of consideration must lie in the realm of the possible extensions of the work. A great many approximations and simplifications have been made in order to gain insight into the physics of the problem. It will be necessary to ascertain which of these can be relaxed and made more physically realistic. In Section III.6 the restriction of constant density was relaxed and it was seen that an analytical solution was possible for a density function that was an inverse power of the radius. Such a density function was seen to be less realistic in a number of ways than the initial constant density function. A general density function was used for the special case of linear scattering and it was found that the results were very similar.to those of the constant density atmosphere. Consequently, further efforts on this line will not be particularly fruitful. It would be possible to introduce depth dependent albedos and greenhouse parameters but the Eddington approximation treatment used in the theory would not warrant such an elaborate extension of the model. Again, the penultimate section of the final chapter showed that the inclusion of perturbatory incident radiation fields could be accounted for satisfactorily by approximate techniques. The effect of these radiation fields were minor except in the case of infra-red radiation incident on an atmosphere of low greenhouse parameter. Consideration of this additional feature to the planetary atmosphere problem would be simple by use of the methods of Chapter II and would be a worthwhile enterprise. The present theory would need extensive modification in application to objects with non-axially symmetric radiation fields, such as reflection nebulae. The subject of anisotropy has been treated schematically, but we have shown that a more exact treatment of complex phase functions would be futile in the Eddington approximation.

The main line for future work must lie in tackling the frequency dependence of the radiative transfer, in particularly that of the infra-red radiation. We have already intimated the trivial extension to monochromatic coherent scattering as predicted by the Mie theory. The validity of this type of scattering must be checked for real grains. If it is false, the frequency dependent transfer problem will be very much more complex. The infra-red transfer problem depends very much on the value of n defined in terms of appropriate grey absorption coefficients. If n is very large, then the non-grey problem can be treated in an approximate manner similar to the perturbation method used by Werner and Salpeter (1969). However, estimates of n show that is not large enough for this treatment for many grains; and certainly n is small in the planetary atmosphere problem. Consequently, the frequency dependence of the infra-red radiation is the most important aspect of the problem to investigate. It is of sufficient importance to warrant the continuation of an approximate method of solution of the equation of transfer, which should show its effects on the solution to the greatest extent. This, together with the results presented in this thesis will provide enough background information to enable a more complex, exact method of solution to be formulated, and thus, produce accurate temperature profiles for model planetary atmospheres and interstellar dust clouds. Nevertheless, we have here, an extensive set of approximate solutions for the radiative heating problems of planetary atmospheres and interstellar dust clouds, large sections of which were hitherto unavailable.

APPENDIX

1. The Exponential Integral Function

The exponential integral function occurs very frequently in radiative transfer problems and is defined in all the standard reference texts. The more important properties of this function are quoted in this Appendix for completeness. Further details may be found in those works by Chandrasekhar (1960) and Kourganoff (1952).

The nth exponential integral, $E_n(x)$, for positive real arguments is defined by

$$E_n(x) = \int_1^\infty e^{-xt} \frac{dt}{t^n} = \int_0^1 e^{-x/\mu} \mu^{n-1} \frac{d\mu}{\mu} = x^{n-1} \int_x^\infty \frac{e^{-s} ds}{s^n} \tag{A-1}$$

These functions satisfy the following recurrence relation;

$$n E_{n+1}(x) = e^{-x} - x E_n(x) , \qquad (n \geq 1) , \tag{A-2}$$

so that every exponential integral function can be reduced to the first of the series. Their derivatives are given by

$$E'_{n+1}(x) = -E_n(x) , \quad (n \geq 1) ; \quad E_1'(x) = -e^{-x}/x , \tag{A-3}$$

and their values at zero, by

$$E_n(0) = 1/(n-1) , \qquad (n \geq 1) . \tag{A-4}$$

The first exponential integral can be expressed by a series expansion for small values of x and an asymptotic expansion for large values of x. That is,

$$E_1(x) = -\gamma - \log x + \sum_{n=1}^{\infty} (-1)^{n-1} x^n / n.n! \,, \qquad \text{(A-5)}$$

for small values of x, where $\gamma = 0.5772156......$ is the Euler-Mascheroni constant; and

$$E_1(x) = \frac{e^{-x}}{x} \left[1 - \frac{1}{x} + \frac{2}{x^2} - \frac{6}{x^3} + \cdots \right], \qquad \text{(A-6)}$$

for large values of x. In practice it is easier to use a polynomial or rational approximation for $E_1(x)$. Several such approximations are available; and the ones employed in the calculations used in this thesis were those given by Abramowitz and Stegun (1964). For $0 \leqslant x \leqslant 1$

$$E_1(x) + \log x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \epsilon(x), \qquad \text{(A-7)}$$

where $|\epsilon(x)| < 2 \cdot 10^{-7}$, and

a_0 = -0.57721 566, a_1 = 0.99999 193

a_2 = -0.24991 055, a_3 = 0.05519 968

a_4 = -0.00976 004, and a_5 = 0.00107 857

for $1 \leqslant x \leqslant \infty$

$$x e^{-x} E_1(x) = \frac{x^4 + b_1 x^3 + b_2 x^2 + b_3 x + b_4}{x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4} + \epsilon(x), \qquad \text{(A-8)}$$

where $|\epsilon(x)| < 2 . 10^{-8}$

$b_1 = 8.57332\ 87401$, $b_2 = 18.05901\ 69730$,

$b_3 = 8.63476\ 08925$, $b_4 = 0.26777\ 37343$,

$c_1 = 9.57332\ 23454$, $c_2 = 25.63295\ 61486$,

$c_3 = 21.09965\ 30827$, and $c_4 = 3.95849\ 69228$.

For negative values of x the series and asymptotic expansions given by equations (A-5) and (A-6) were used. Extensive tables of the exponential integral function are given by Abramowitz and Stegun.

2. The F_n-functions

There are many functions that involve the exponential integral function, including many integrals in which it appears in the integrand. One such function is the integral of the product of an exponential integral function and the exponential function. This function has been defined in a variety of ways. We shall use the form and notation of Van de Hulst (1948), who also gives an extensive list of its properties.

We define the set of functions

$$F_n(b,x) = \int_0^x e^{bt} E_n(t)\,dt\,, \qquad \begin{cases} x \geq 0 \\ n \geq 1 \\ b \text{ is real}. \end{cases} \qquad \text{(A-9)}$$

These functions satisfy the recurrence relation

$$b F_{n+1}(b,x) = e^{bx} E_{n+1}(x) - \frac{1}{n} + F_n(b,x)\,, \qquad \text{(A-10)}$$

and their partial derivations are

$$\frac{\partial F_n(b,x)}{\partial x} = e^{bx} E_n(x) \tag{A-11}$$

and
$$\frac{\partial F_n(b,x)}{\partial b} = \frac{1}{(b-1)}\left[e^{(b-1)x} - 1\right] - n F_{n+1}(b,x) . \tag{A-12}$$

The first of these functions can be expressed in terms of the exponential integral function

$$F_1(b,x) = \frac{1}{b}\left\{ e^{bx} E_1(x) - E_1[x(1-b)] - \log(|1-b|)\right\}, \quad \left\{\begin{matrix} b \neq 0 \\ b \neq 1 \end{matrix}\right. ,$$

$$F_1(0,x) = 1 - E_2(x) , \tag{A-13}$$

and
$$F_1(1,x) = e^{x} E_1(x) + \log x + \gamma ,$$

where γ is the Euler-Mascheroni constant. If x is infinity the F_n-integrals coverage only when b is less than unity, whence

$$b F_{n+1}(b,\infty) = -\frac{1}{n} + F_n(b,\infty) ,$$

$$F_1(b,\infty) = -\frac{1}{b}\log(1-b) , \quad b \neq 0 , \tag{A-14}$$

and
$$F_1(0,\infty) = 1 .$$

Another integral of frequence occurence is that of the product of an F_n-function and the exponential function. This can be reduced to the F_n-functions by the following relation:

$$a\int_0^x e^{-at} F_n(b,t)\,dt = F_n\left[(b-a),x\right] - e^{-ax} F_n(b,x). \qquad \text{(A-15)}$$

The F_n-functions have not been tabulated extensively, but their evaluation via equations (A-10) and (A-13) is simple with the aid of a computer. However, considerable care must be employed when the parameter, b, is very small, because a loss of significance can easily arise. This situation developed, for example, in the equations for the approximate solution for the emergent thermal radiation from a finite plane-parallel atmosphere with a ground layer, when the argument, b, was $1/n\mu$ which was very small when n was 10^4.

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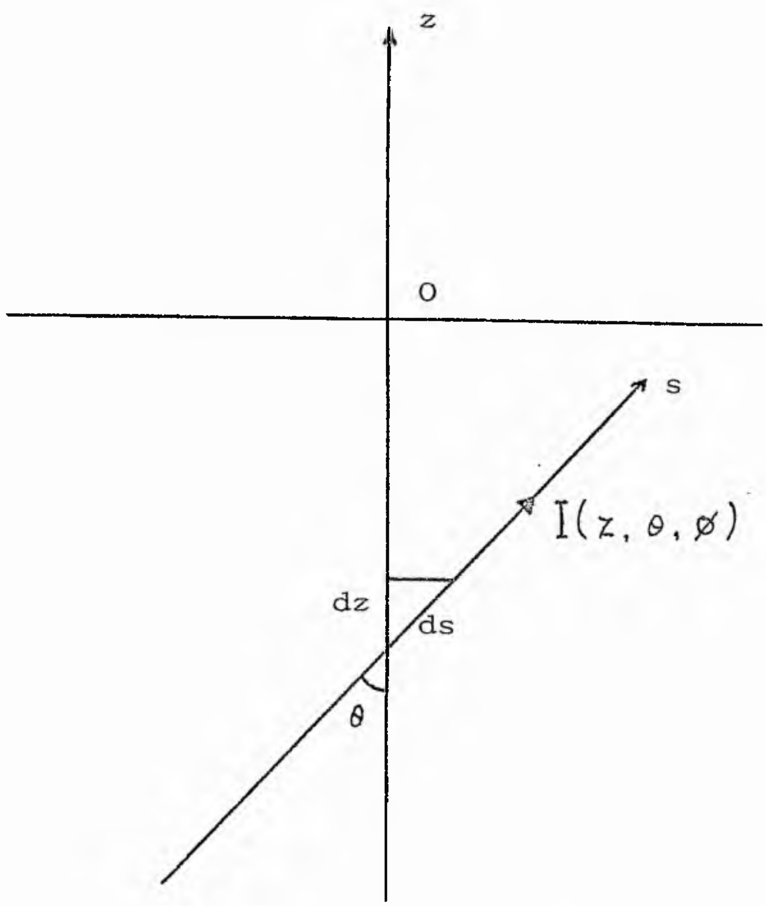


FIG. 1. The geometry of a plane-parallel atmosphere.

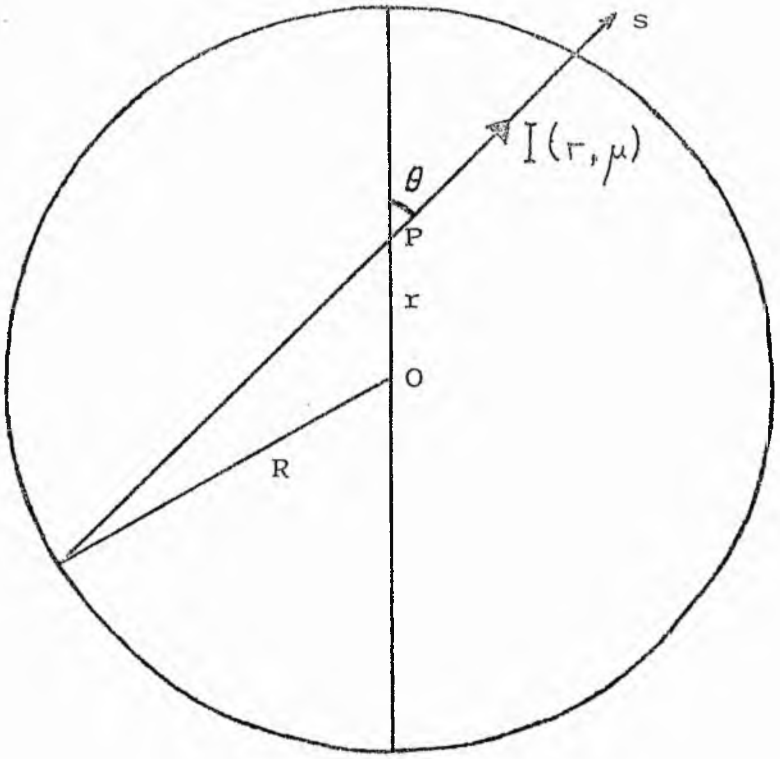


FIG. 2. The geometry of an axially symmetric spherical atmosphere. $\mu = \cos\theta$.

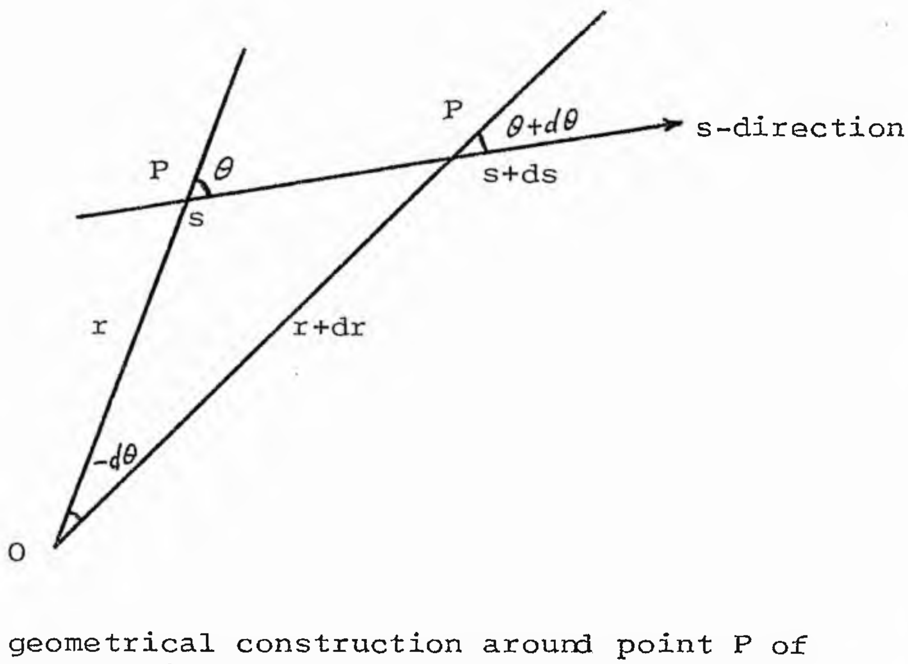


FIG. 3. The geometrical construction around point P of Fig. 2 used in the derivation of the equation of transfer.

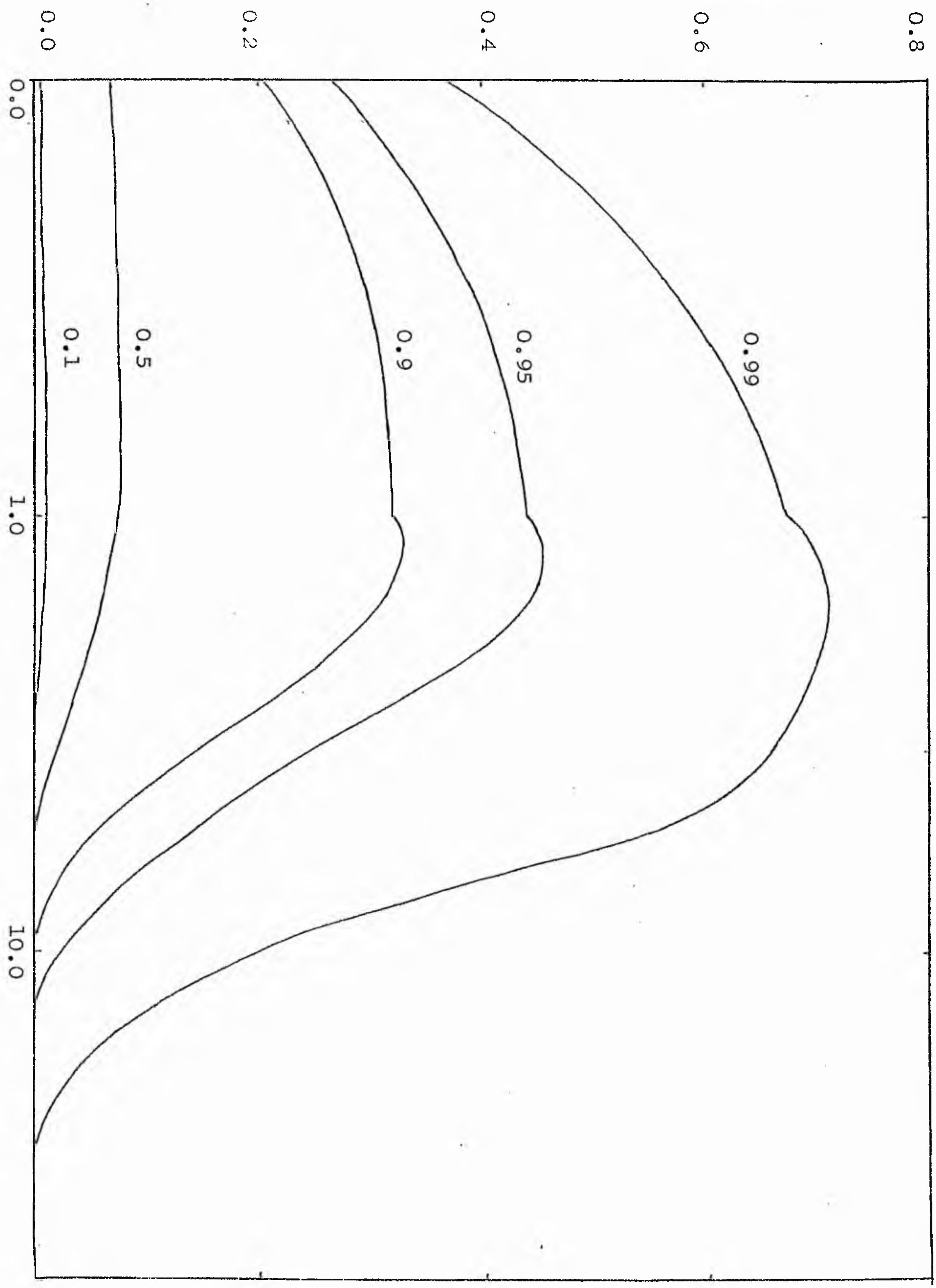


FIG. 4. The mean intensity of the scattered radiation field in a semi-infinite plane-parallel atmosphere as a function of optical depth, τ, for several values of the albedo.

The scattering is isotropic and the incident radiation is normal to the surface of the atmosphere. There is a change from a linear to a logarithmic scale in the abscissa, τ, at $\tau = 1.0$.

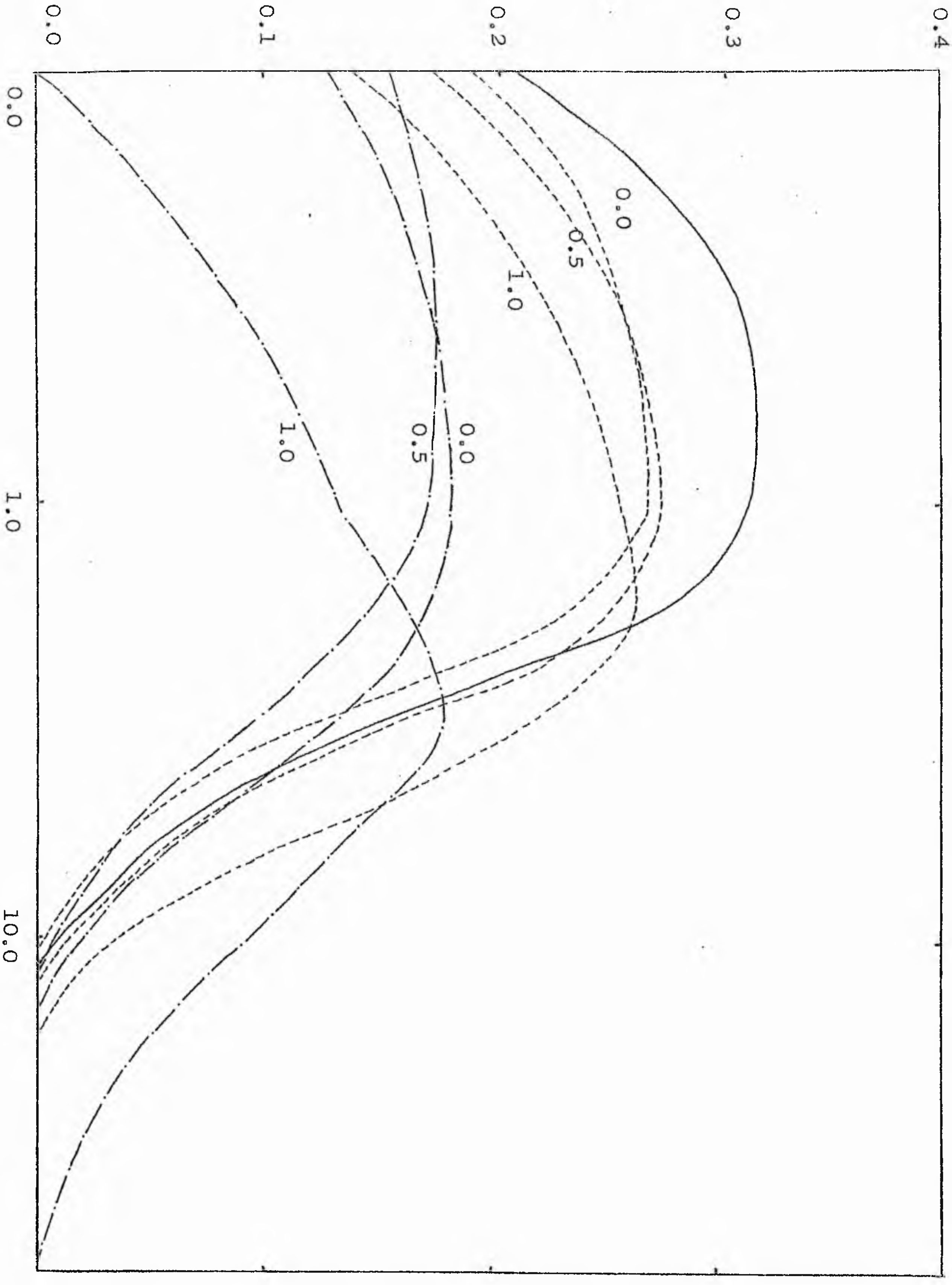


FIG. 5. The mean intensity of the scattered radiation field in a semi-infinite plane-parallel atmosphere as a function of optical depth, for several phase functions.

The albedo is 0.9 and the incidence is normal. The values of β are indicated on the figure and the curves with $\alpha = 1.0$ (isotropic scattering), $\alpha = 0.4$ and $\alpha = 0.0$ (linear scattering) are represented by continuous, broken and dashed curves respectively.

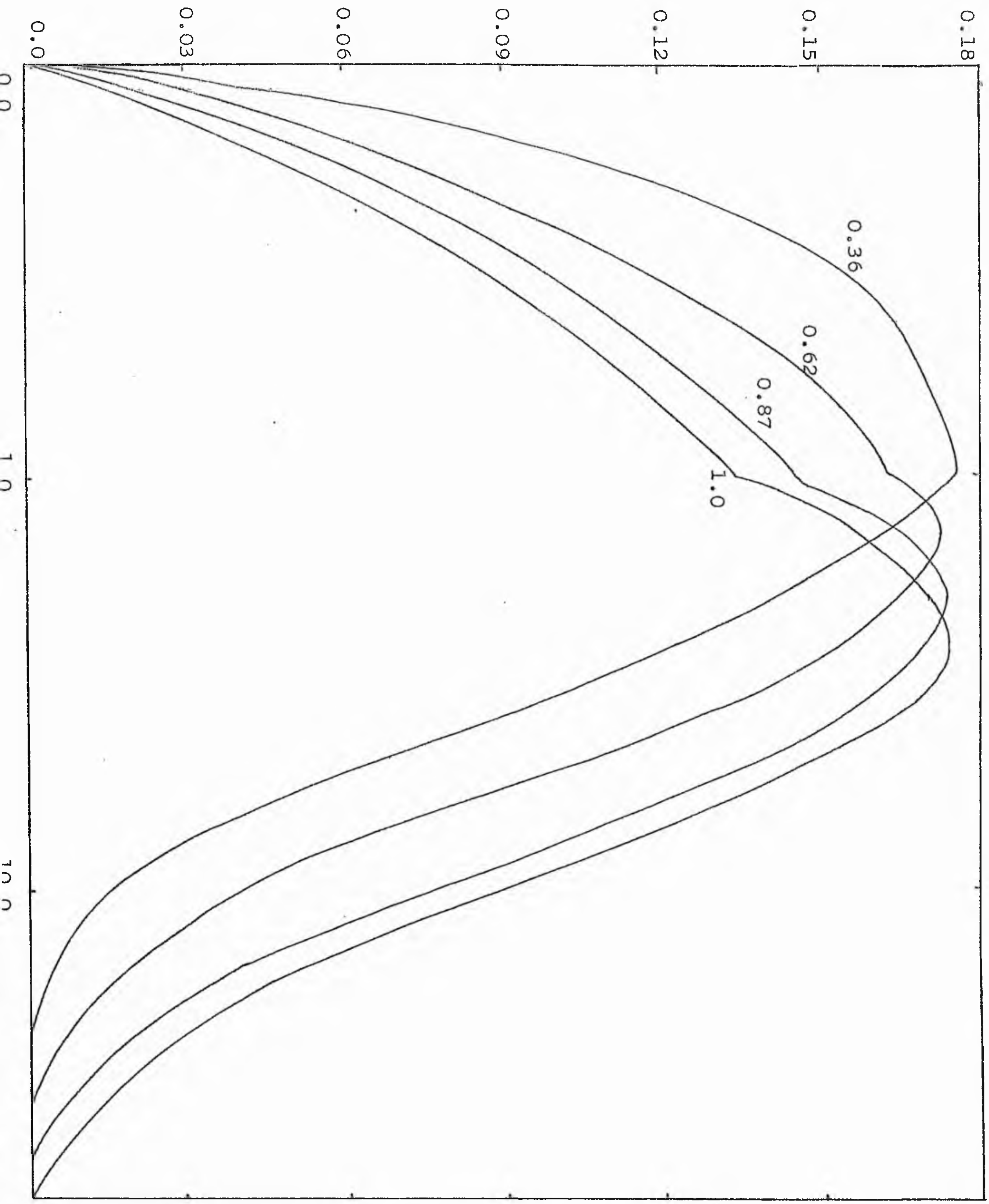


FIG. 6. The mean intensity of the scattered radiation field in a semi-infinite plane-parallel atmosphere as a function of optical depth, for several values of μ_0.

The scattering is isotropic and of albedo, 0.9.

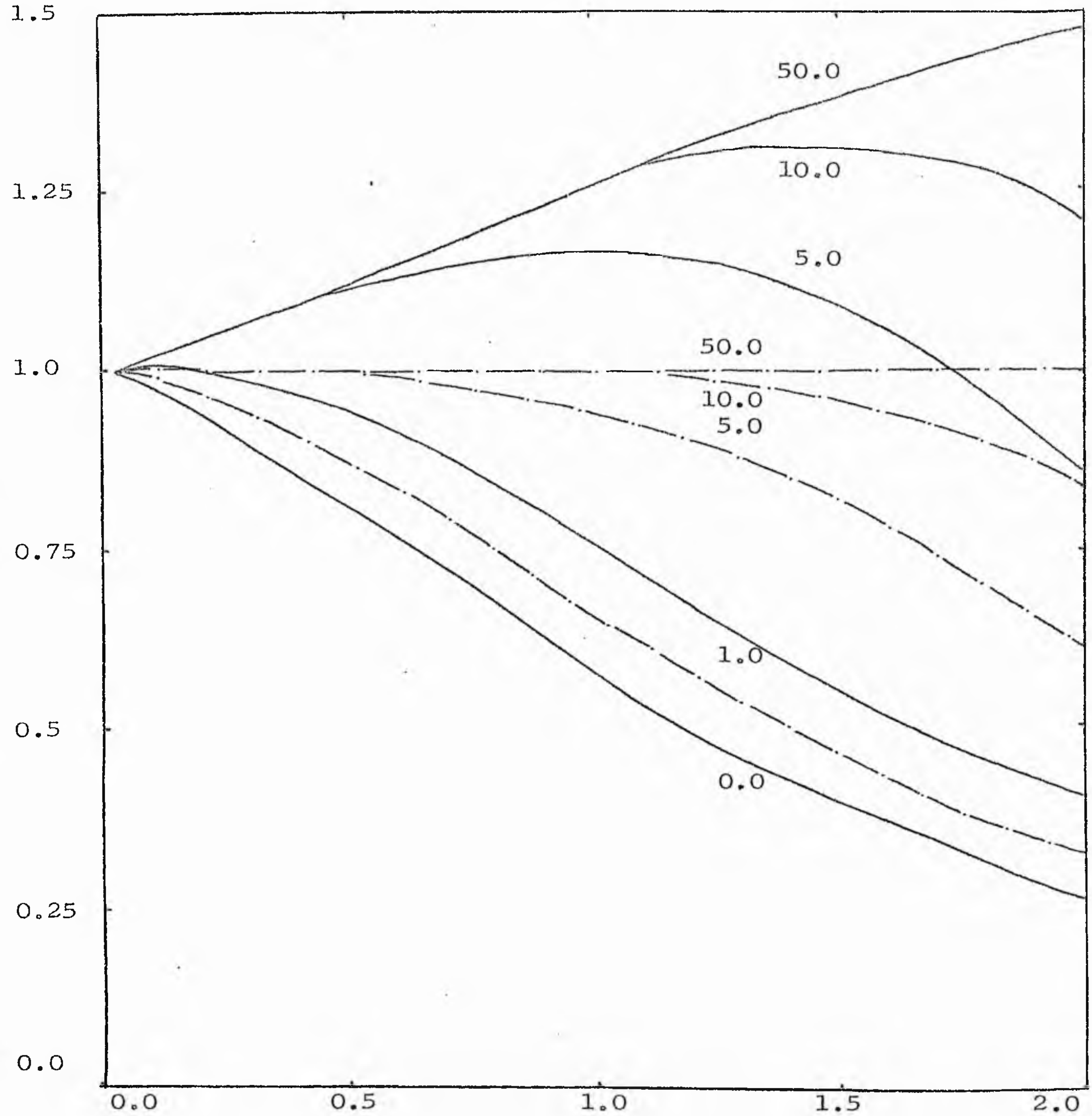


FIG. 7. The variation of the mean intensity of the thermal radiation field in a semi-infinite plane-parallel atmosphere with albedo, for various values of optical depth and three values of the greenhouse parameter, n.

The ordinate is the ratio $J_p(\tau,\tilde{\omega})/J_p(\tau,0.1)$ and the abscissa is $-\log(1-\tilde{\omega})$. The continuous curves represent $n = 10^{-2}$ and the broken curves, $n = 1.0$. The values of τ are indicated on the diagram. When $n = 10^{4}$ the results are the same for all values of τ and the same as those for $\tau = 0$ for the other values of n. The scattering is isotropic and the incident radiation is normal to the surface.

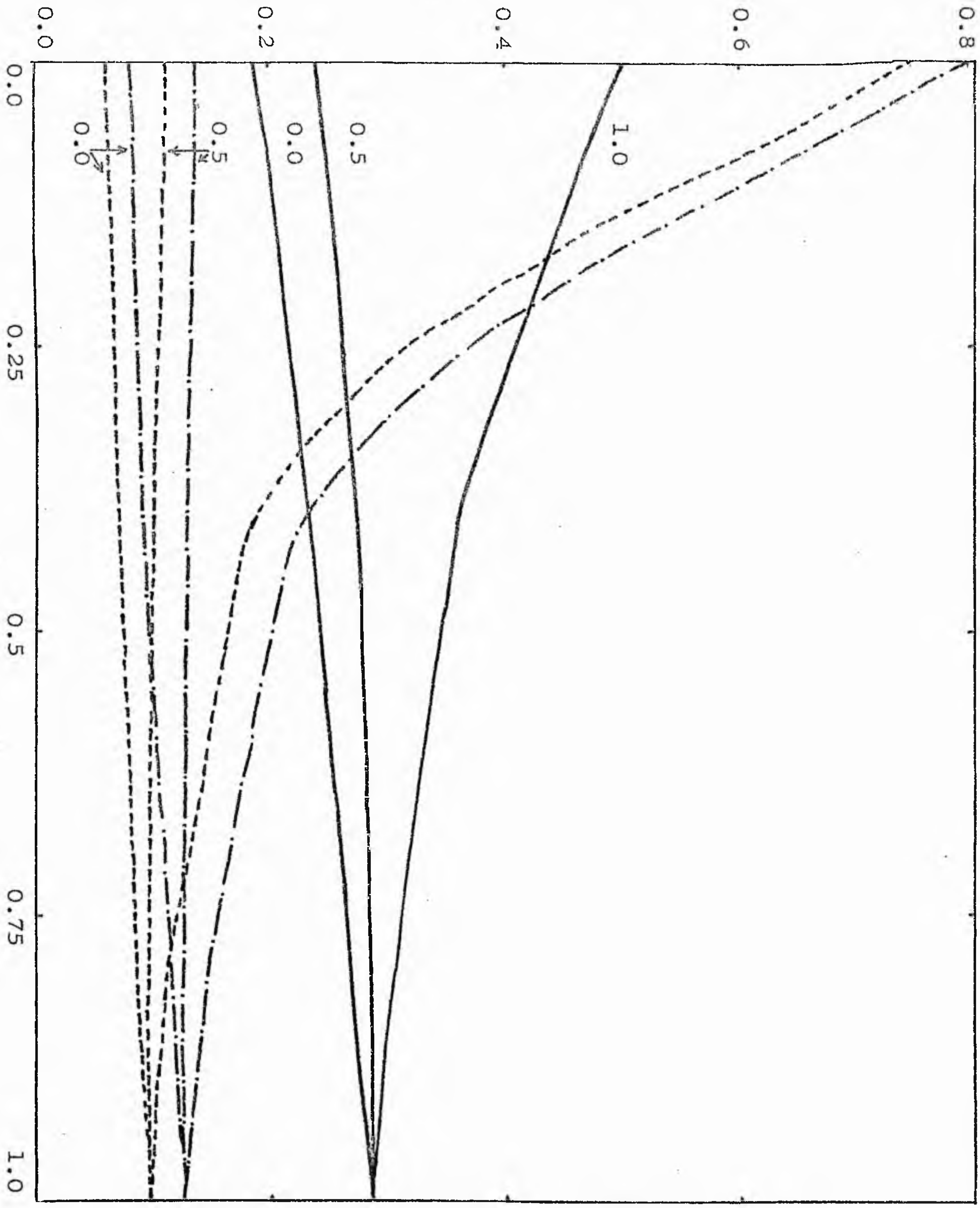


FIG. 8. The mean intensity of the thermal radiation field in a semi-infinite plane-parallel atmosphere as a function of α for several values of β and n.

The albedo is 0.9 and the incidence is normal. The scale of the ordinate should be multiplied by a factor of 10^2 for the case of $n = 10^{-2}$. The values of β are shown on the figure and the values of n of 10^4, 1.0 and 10^{-2} are represented by continuous, broken and dashed lines respectively.

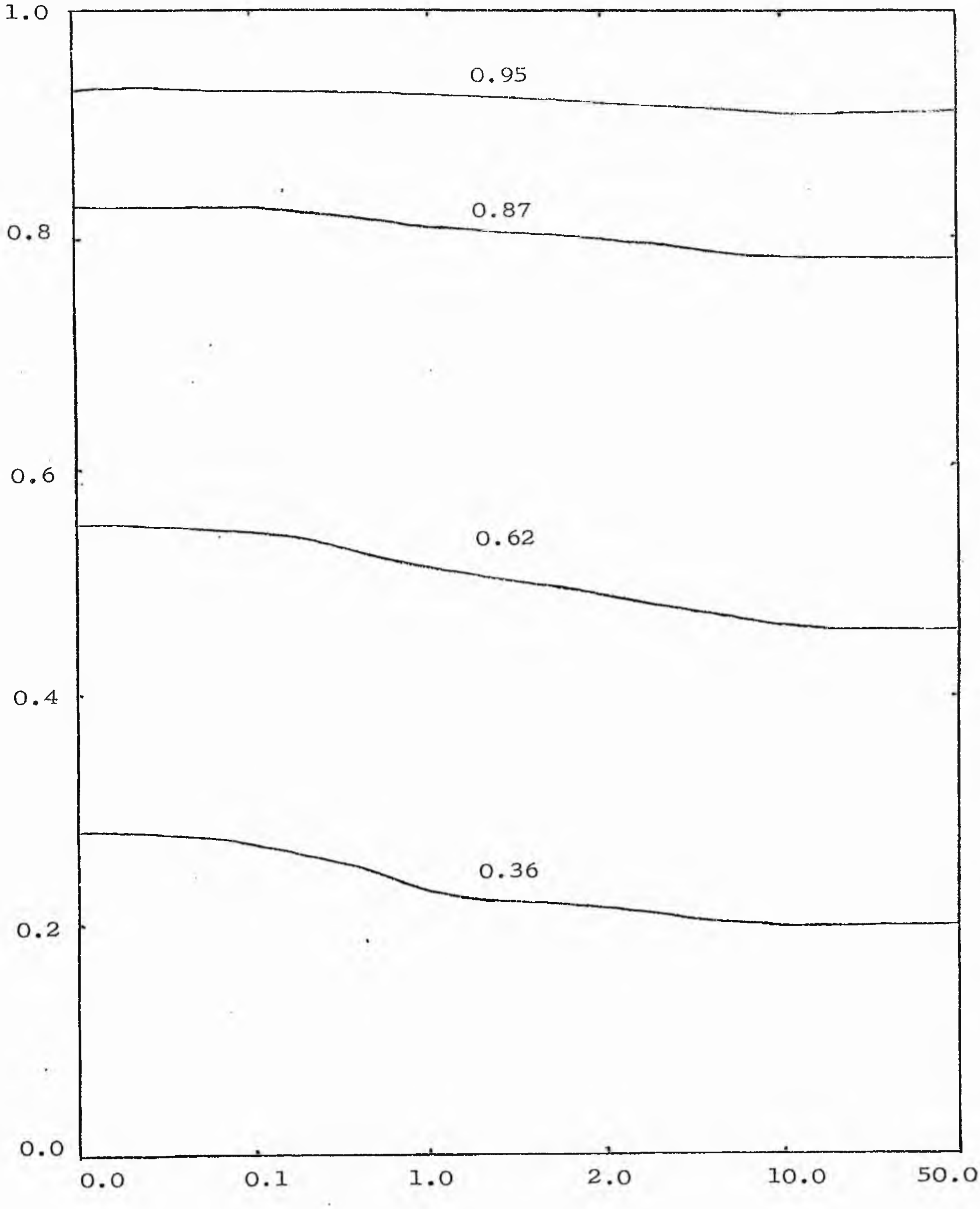


FIG. 9. The mean intensity of the thermal radiation field in a semi-infinite plane-parallel atmosphere as a function of optical depth for various values of μ_o.

The ordinate is the ratio, $J_p(\tau,\mu_o)/J_p(\tau,1.0)$, the scattering is isotropic, of albedo, 0.9, and $n = 10^{-2}$. The scale of the abscissa is arbitrary.

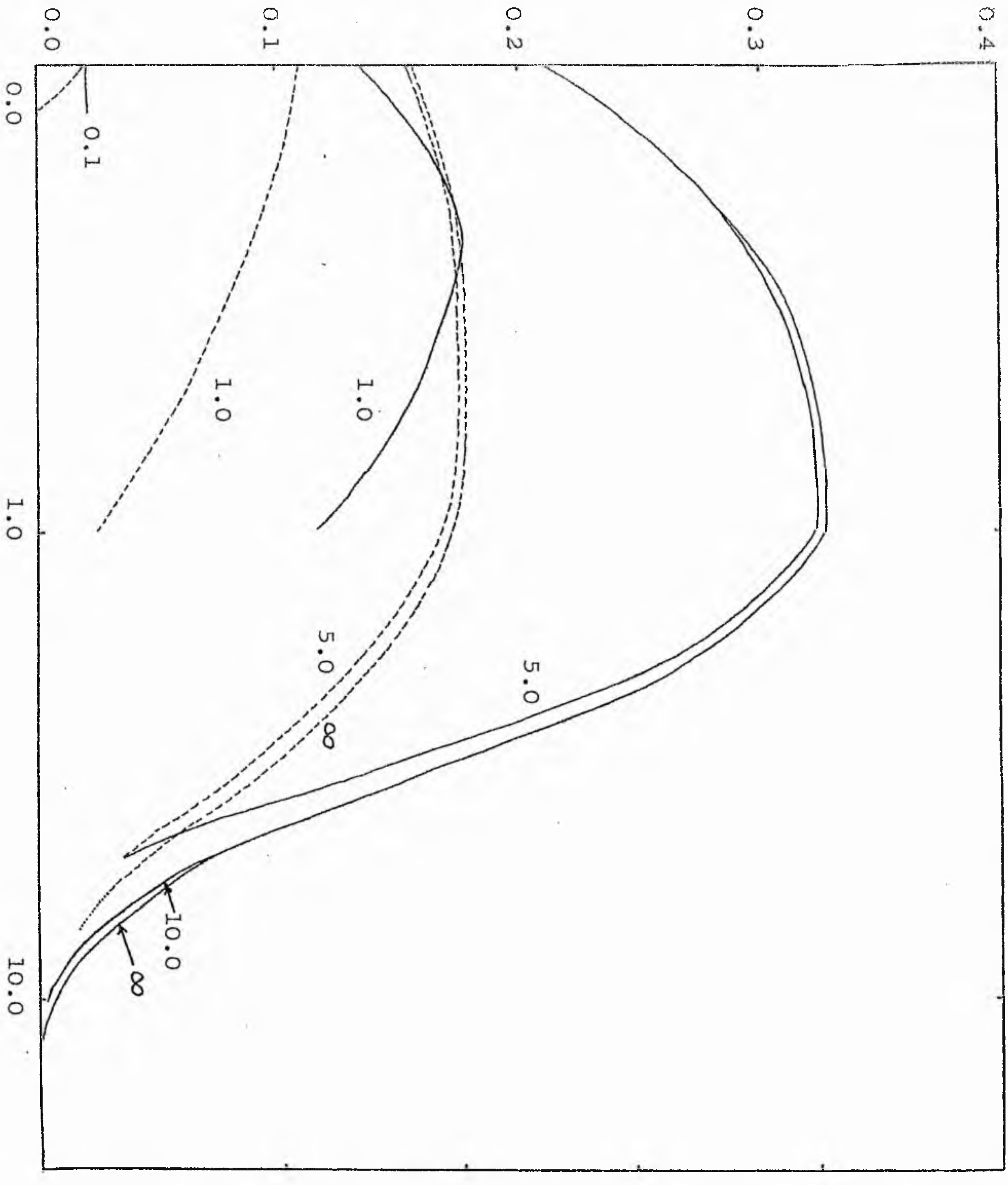


FIG. 10. The mean intensity of the scattered radiation field in a finite plane-parallel atmosphere as a function of optical depth for various values of τ_0.

The continuous curves represent isotropic scattering and the dashed curves, scattering with $(\alpha, \beta) = (0.0, 0.0)$. The albedo is 0.9 and the incidence, normal.

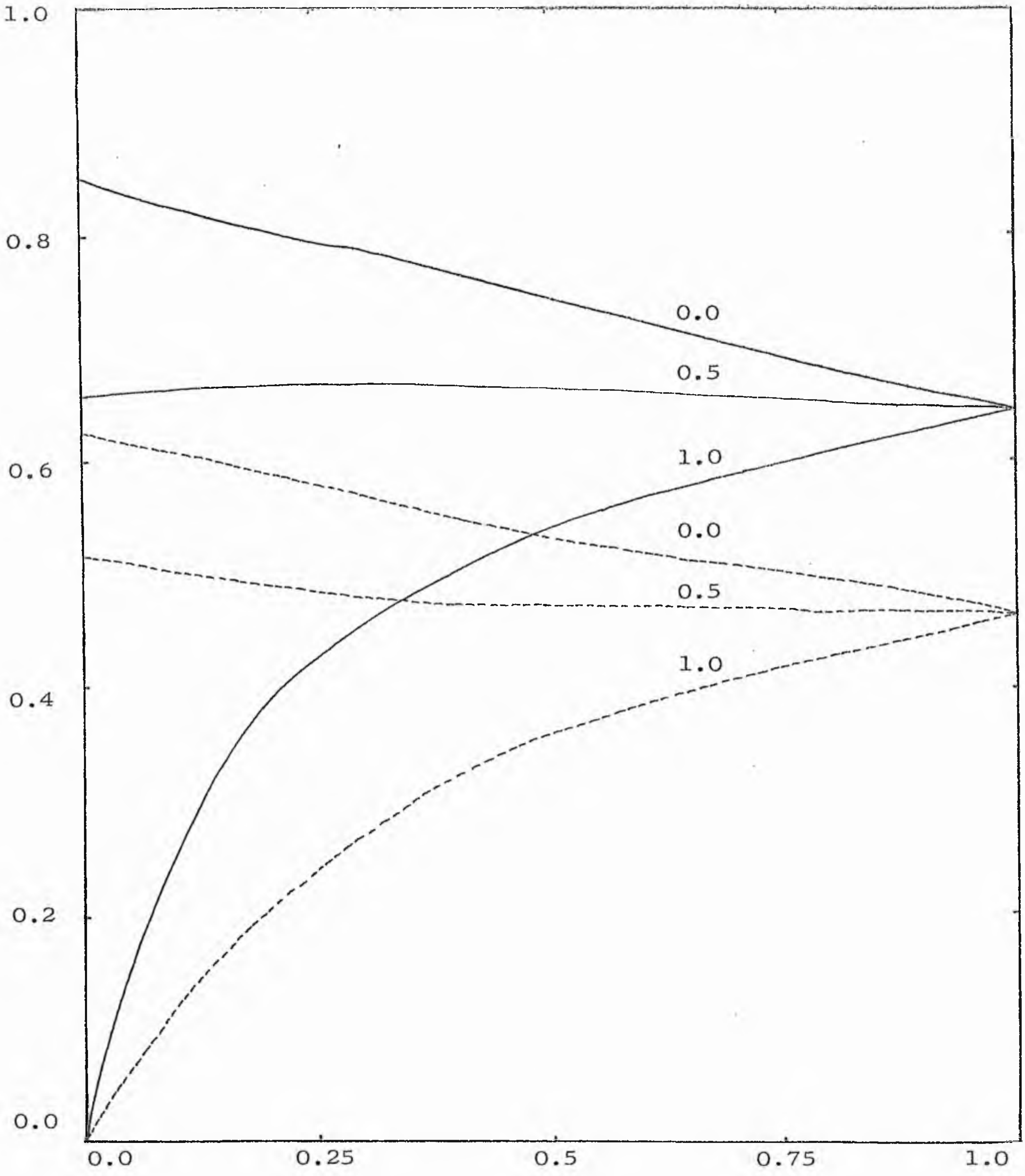


FIG. 11. The effect of anisotropy on the deviation of the mean intensity of the scattered radiation field in a finite plane-parallel atmosphere from that in a semi-infinite atmosphere.

The quantity, $[J^{\infty}_{sc}(\tau_0) - J_{sc}(\tau_0)]/J^{\infty}_{sc}(\tau_0)$ is plotted against a for the three values of β indicated on the figure. The continuous curves are for $\tau_0 = 1.0$ and the dashed curves for $\tau_0 = 10.0$. The albedo is 0.9 and the incidence, normal.

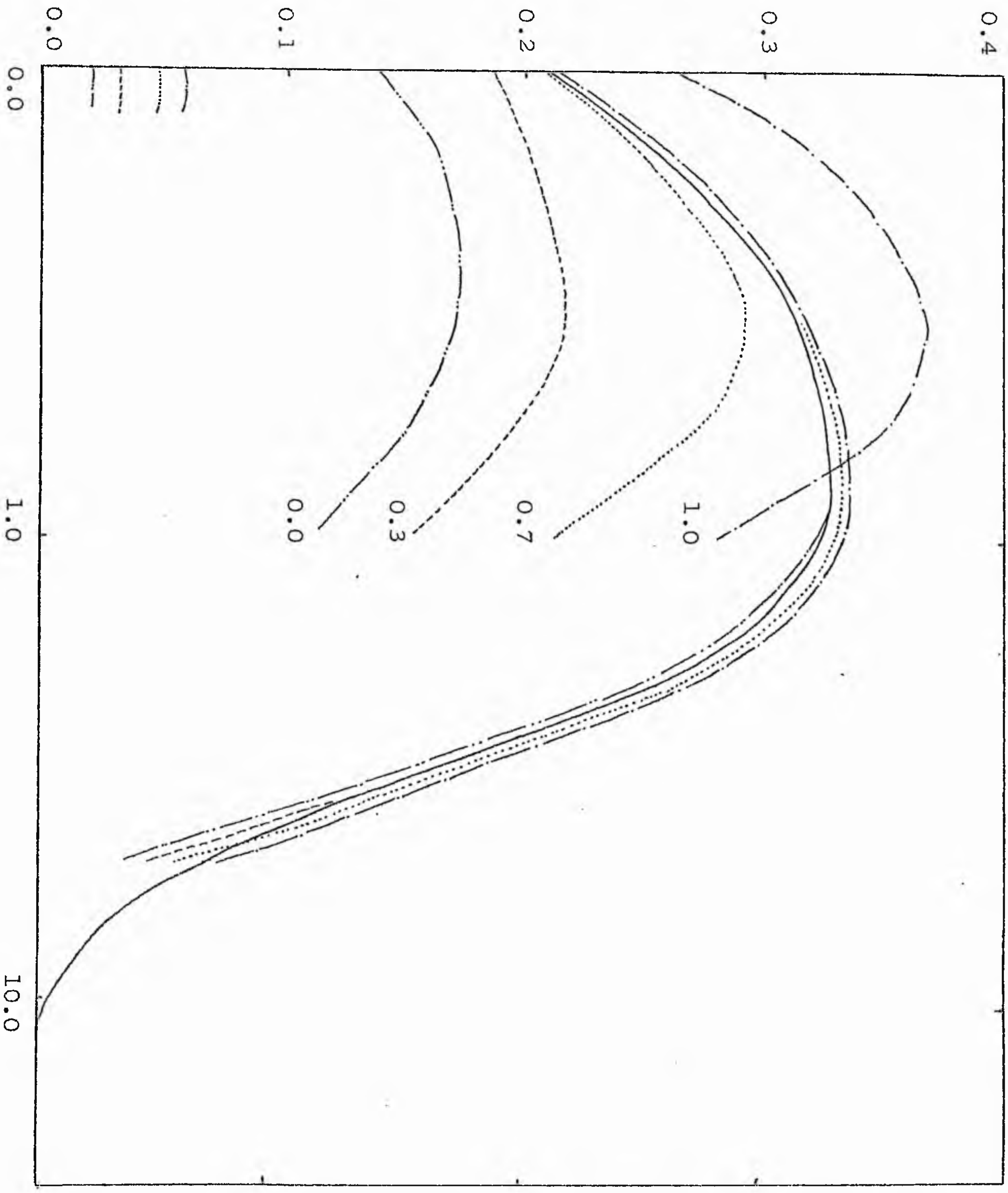


FIG. 12. The mean intensity of the scattered radiation field in a finite plane-parallel atmosphere with a ground layer, as a function of optical depth for several values of τ_0 and λ.

The incidence is normal and the scattering, isotropic with albedo, 0.9. The case of λ = 0.0 is identical to that of an atmosphere with no ground.

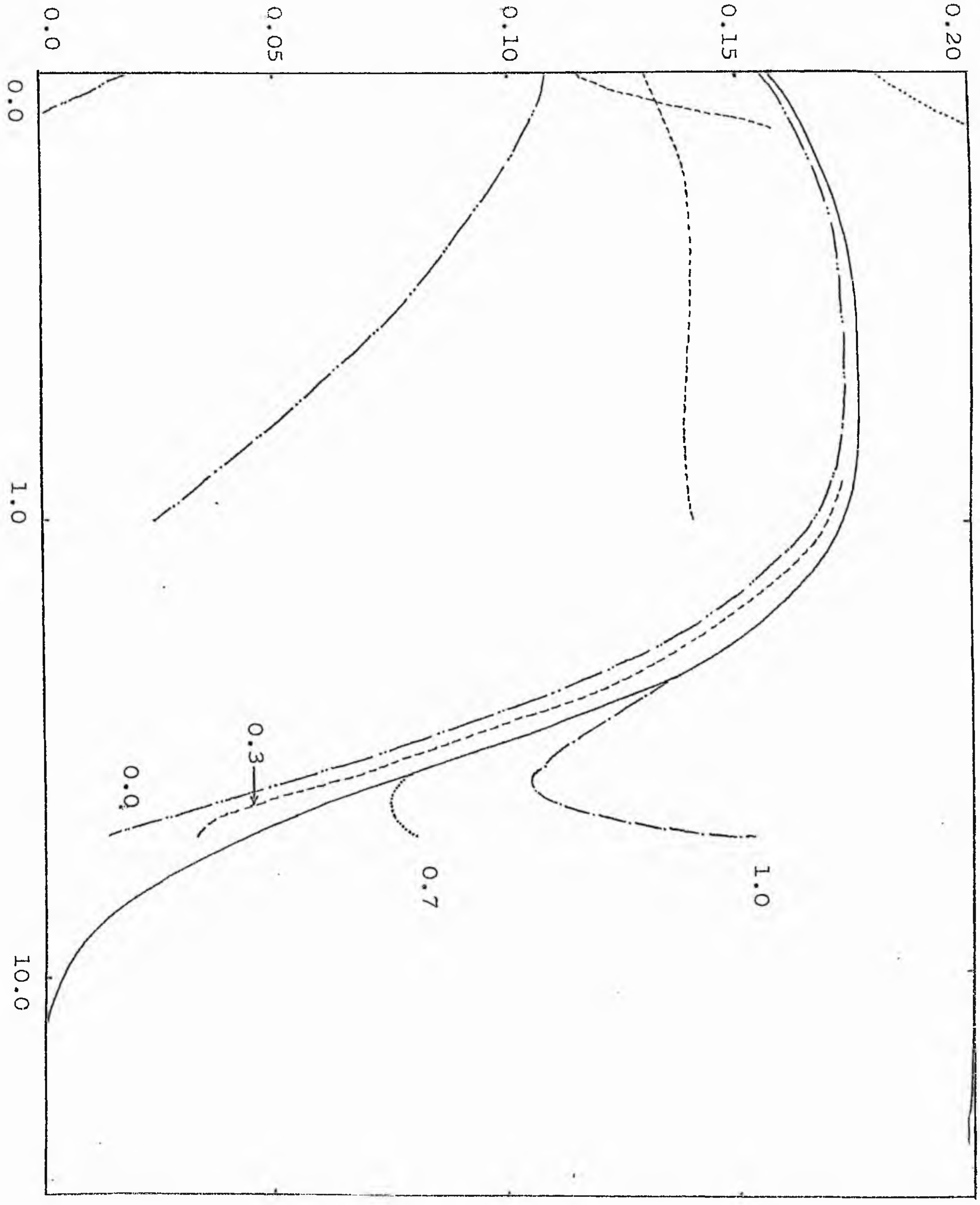


FIG. 13. As Fig. 12 but the ordinate is now the sum of the mean intensities of the scattered and reduced visible ground radiation fields.

Those curves for small values of τ_0 and large values of λ are of too great a magnitude to be included.

FIG. 14. The mean intensity of the thermal radiation field at the midpoint of a finite plane-parallel atmosphere with a ground layer as a function of τ_0 for several phase functions and values of the ground albedo, λ.

The incidence is normal, the scattering albedo, 0.9 and $n = 10^4$; the scattering is linear, $\alpha = 0$ and the values of β are indicated on the figure. The continuous, broken and dashed curves refer to values of λ of 0.9, 0.5 and 0.0 respectively. The results for an atmosphere with no ground are the same as those with $\lambda = 0.0$

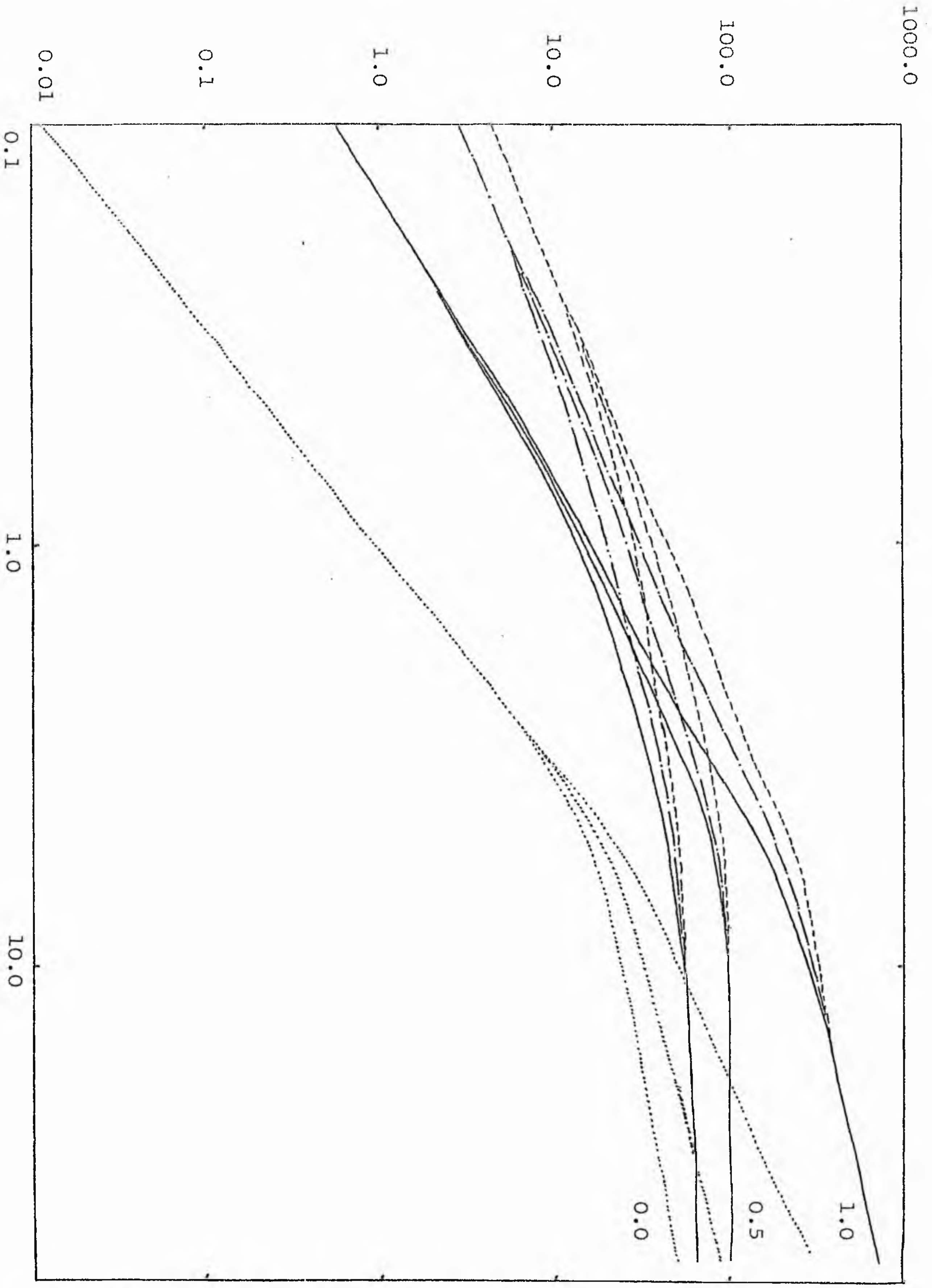


FIG. 15. As Fig. 14 for $n = 10^{-2}$ but with the ordinate now being the maximum value of the mean intensity of the thermal radiation.

The case of the atmosphere with no ground is now different from that of $\lambda = 0.0$ and is shown by the dotted curves. The dashed curves now represent $\lambda = 0.1$ rather than $\lambda = 0.0$ as in the previous figure.

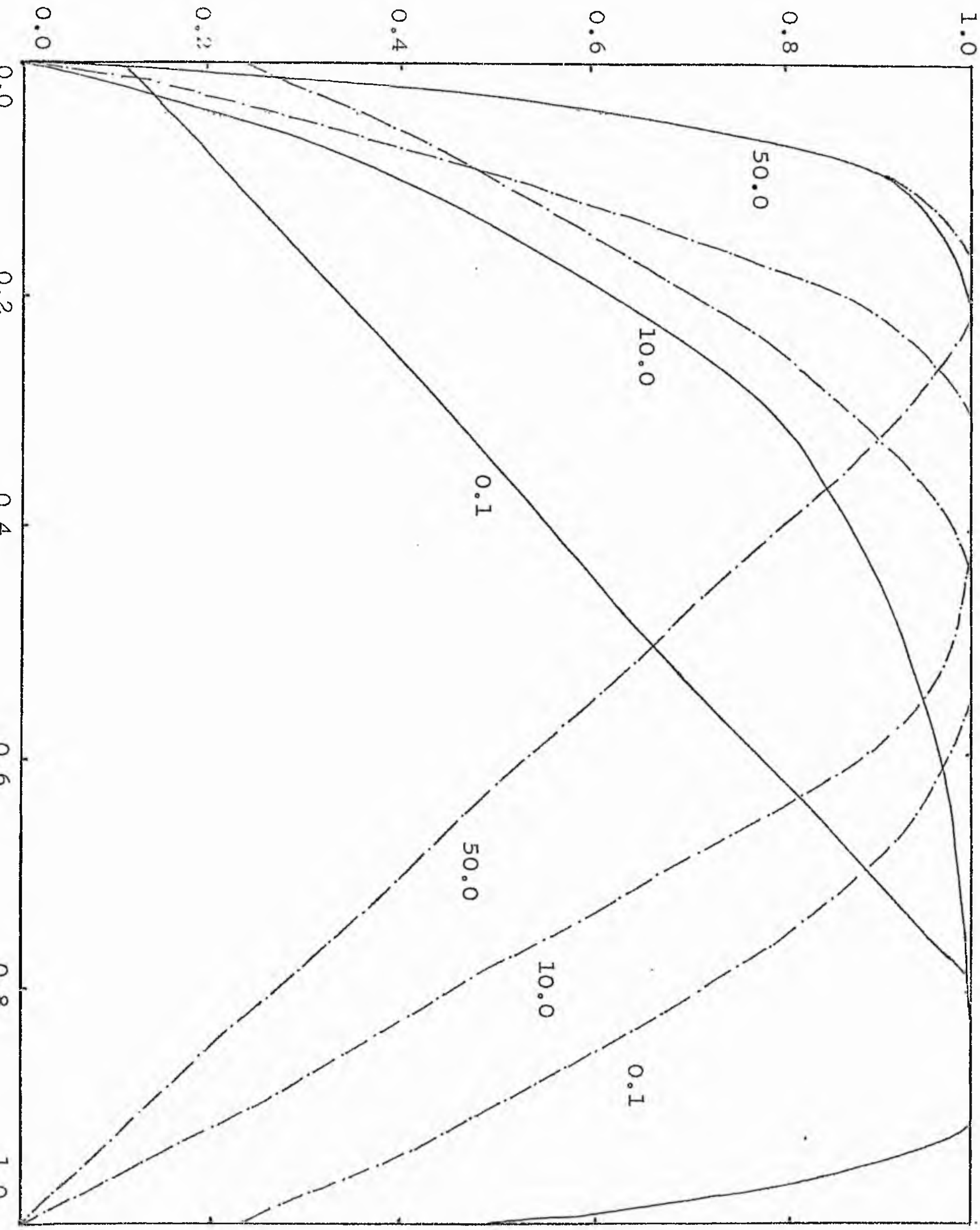


FIG. 16. The profiles of the mean intensity of the thermal radiation field in a finite plane-parallel atmosphere with and without a ground for several values of τ_0.

The ordinate is the fraction $J_p(\tau)/J_p^{max}$ and the abscissa, the fractional optical depth, τ/τ_0. The scattering has $(\alpha,\beta) = (0.0,0.5)$ and albedo, 0.9, the incidence is normal, and n is 10^{-2}. The broken curves represent an atmosphere with no ground and the continuous curves, one with a ground layer of arbitrary albedo.

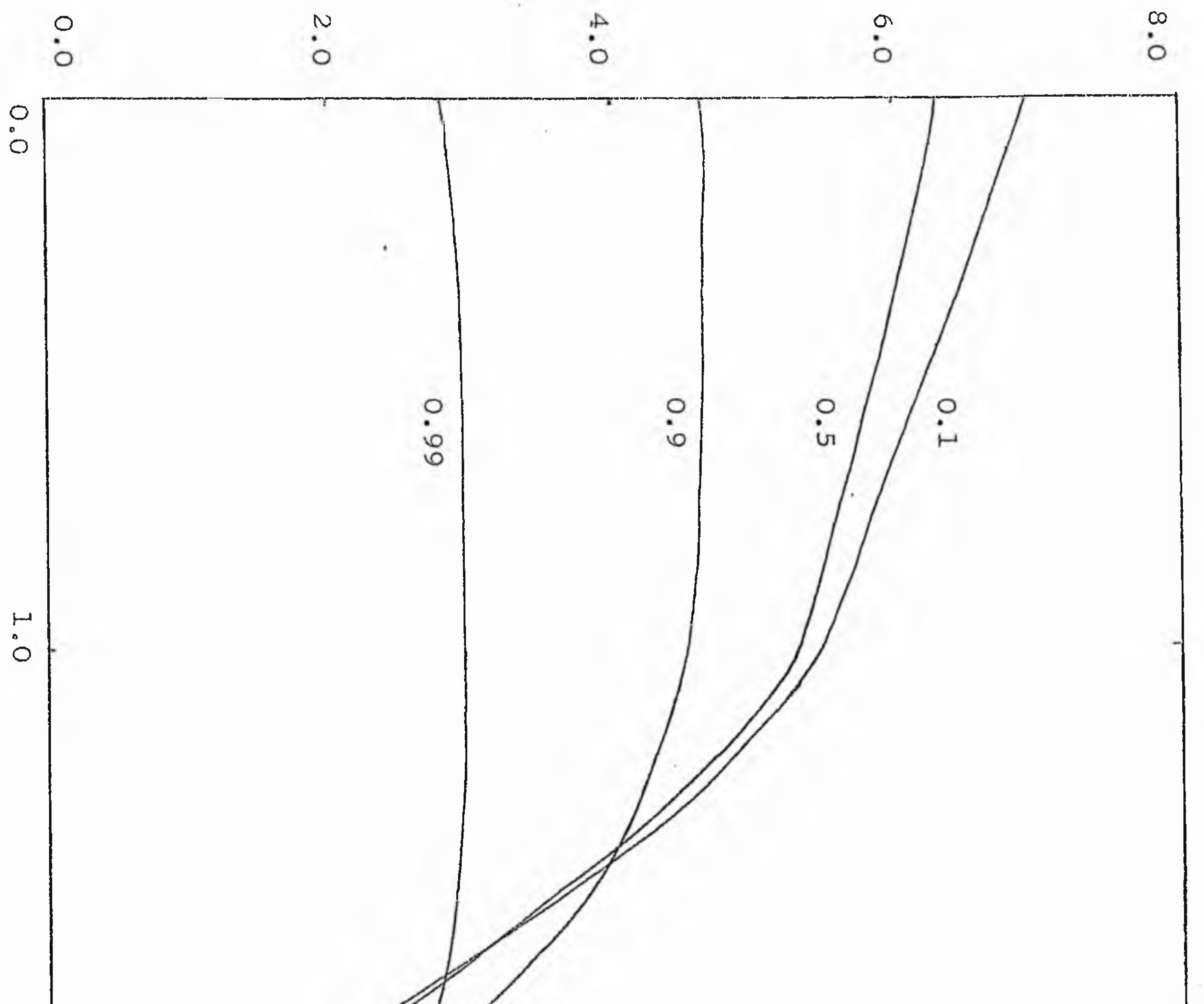


FIG. 17. The temperature of a semi-infinite plane-parallel atmosphere as a function of optical depth for several values of the albedo.

The scattering is isotropic, the incidence, normal and n is 10^4. The temperatures are measured in units of T_e.

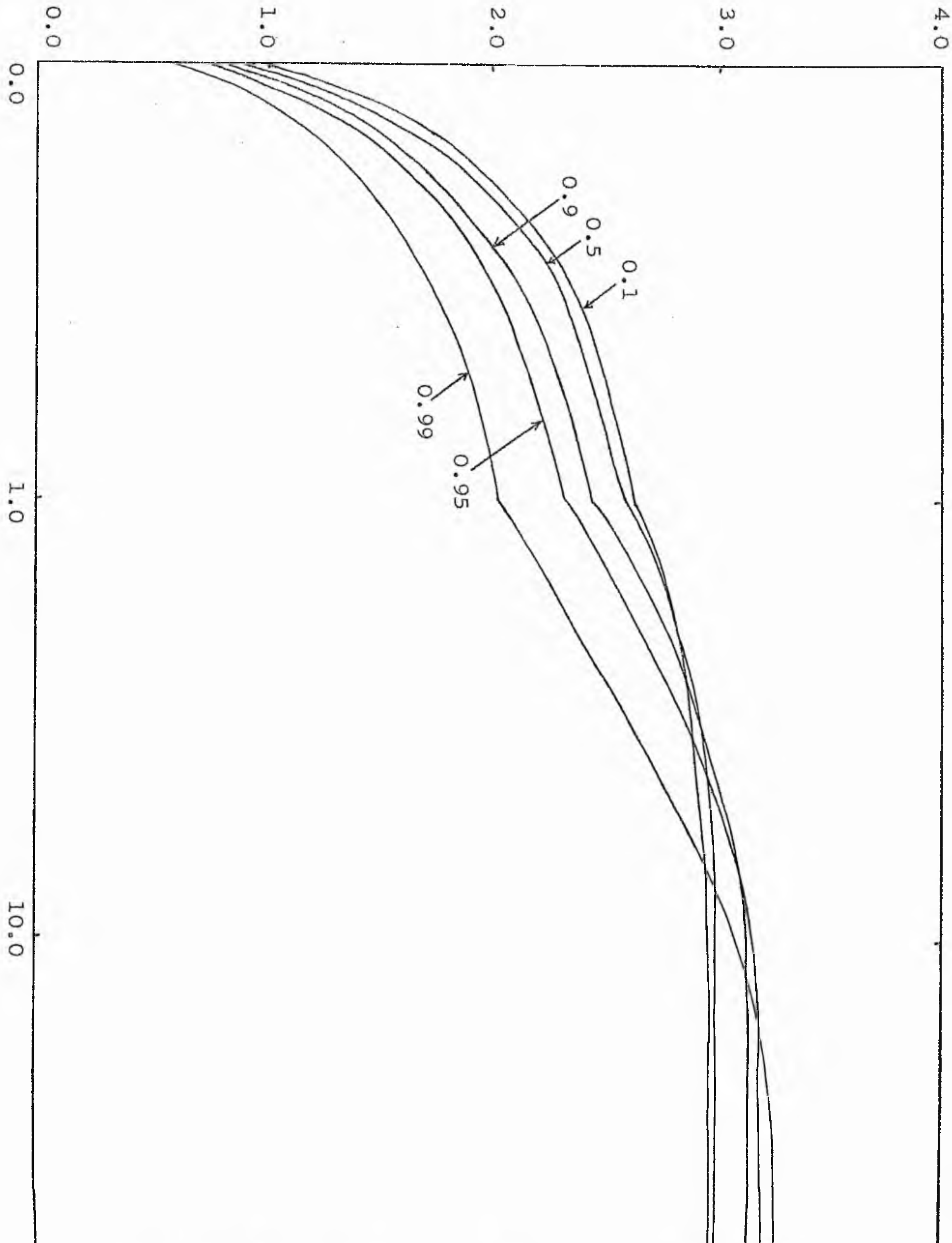


FIG. 18. As Fig. 17 for $n = 10^{-1}$.

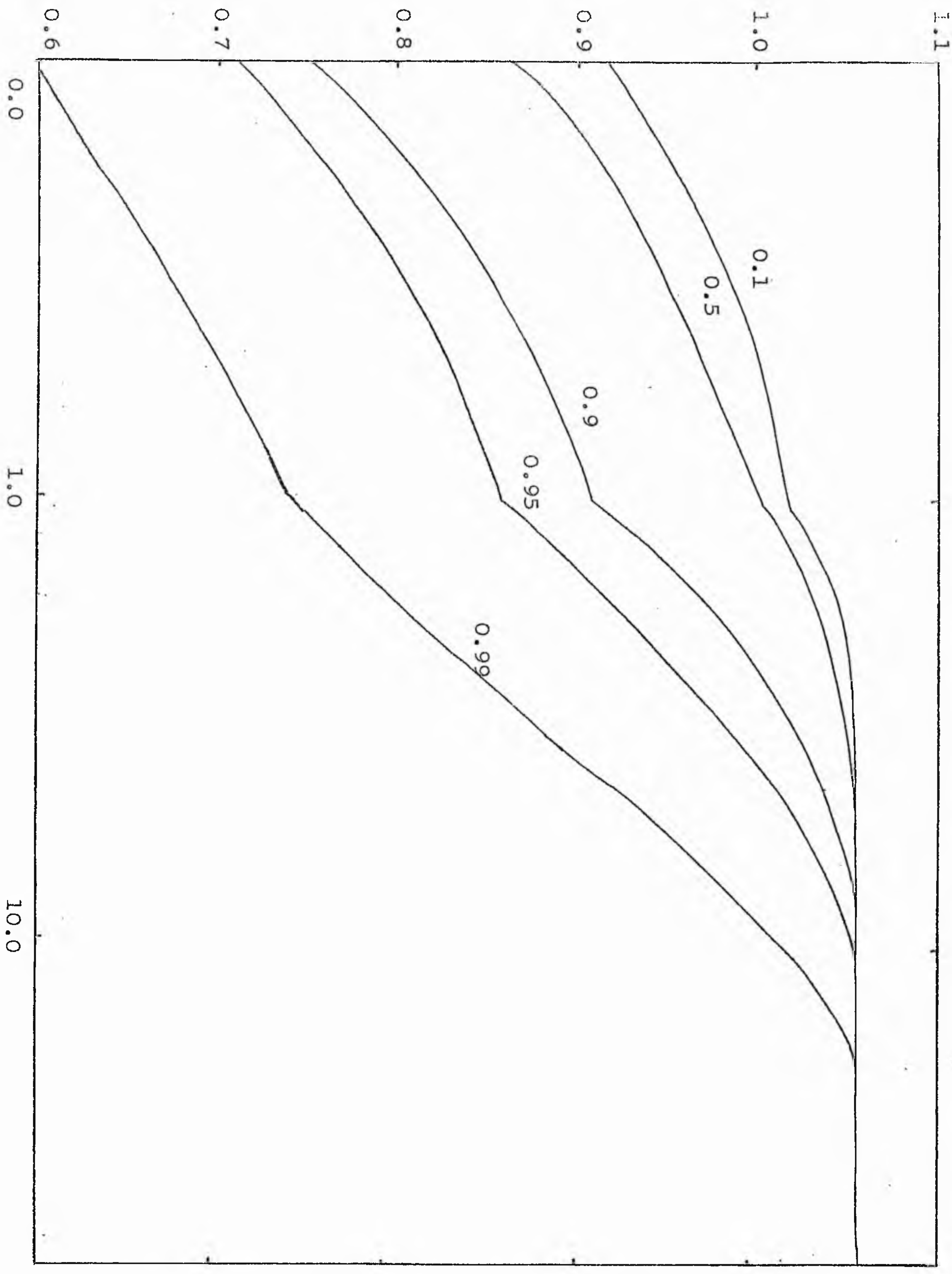


FIG. 19. As Fig. 17 for n = 1.0.

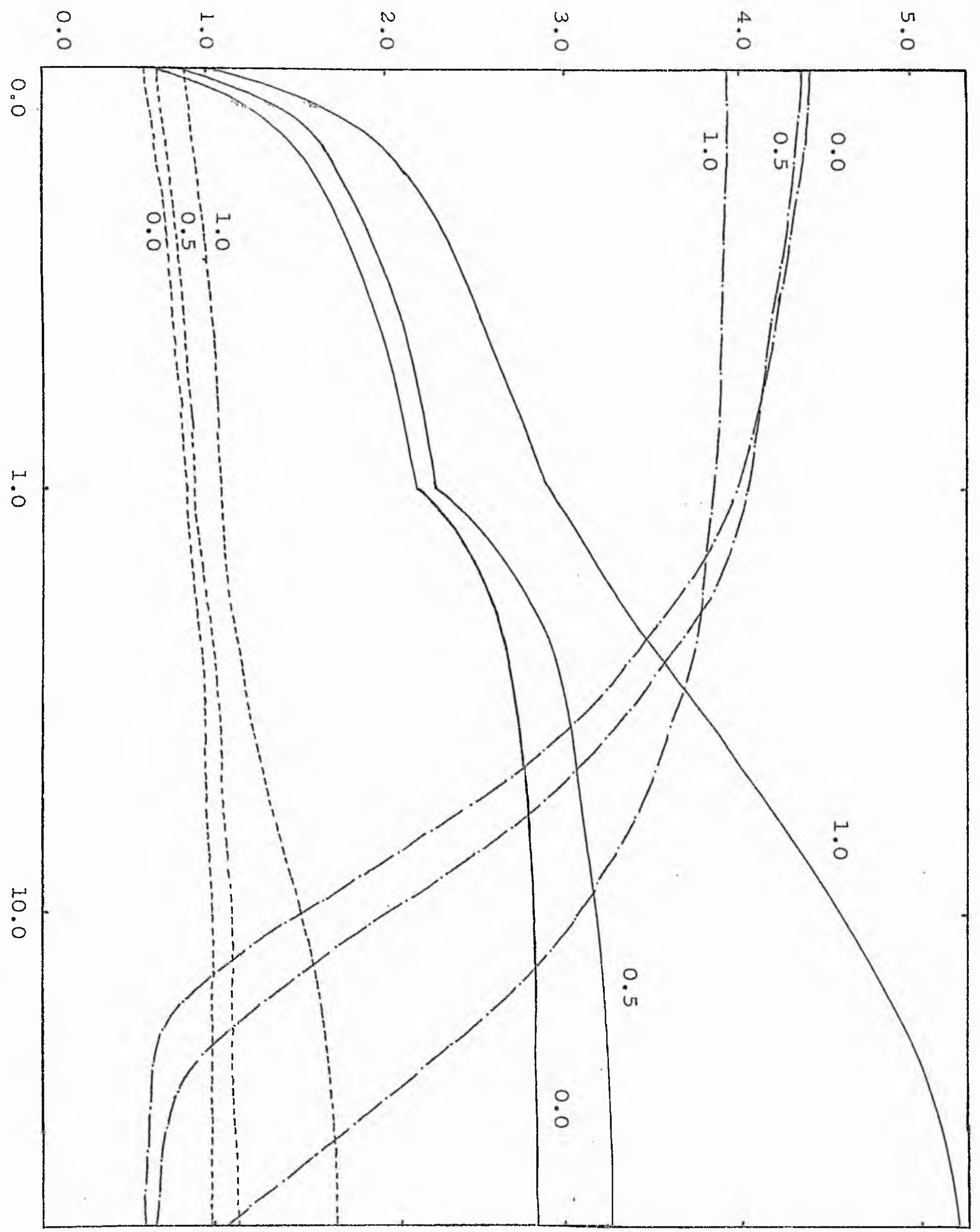


FIG. 20. The temperatures of a semi-infinite atmosphere as a function of optical depth for several phase functions and three values of n.

The phase functions are all of zero α and have β indicated on the figure. The albedo is 0.9 and the incidence, normal. The continuous, dashed and broken curves represent values of n of 10^{-2}, 1.0 and 10^{4} respectively.

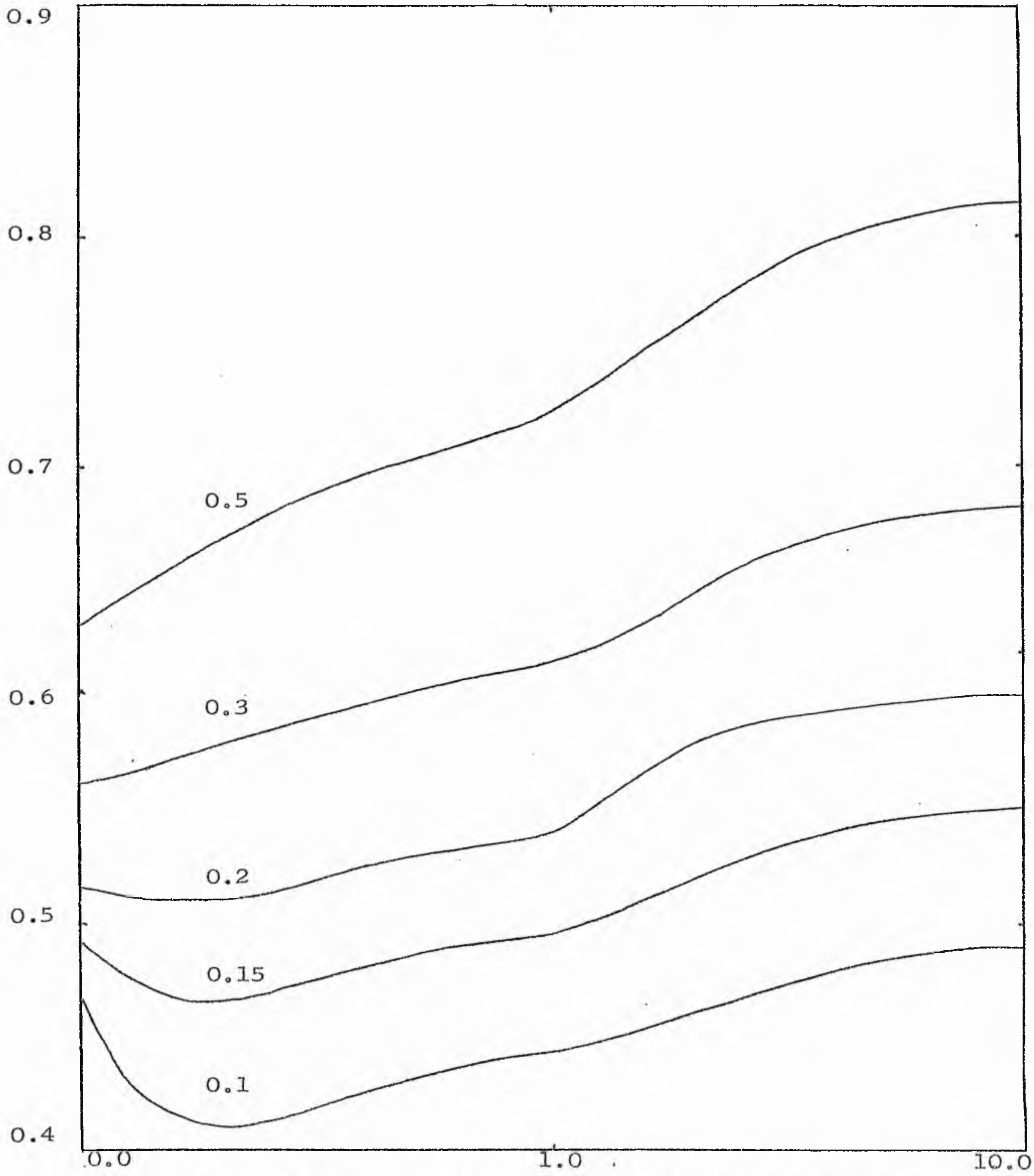


FIG. 21. The temperature of a semi-infinite atmosphere as a function of optical depth for several values of μ_0 when $n = 1.0$.

The scattering is isotropic and of albedo, 0.9. The scale of the abscissa is again partly linear and partly logarithmic.

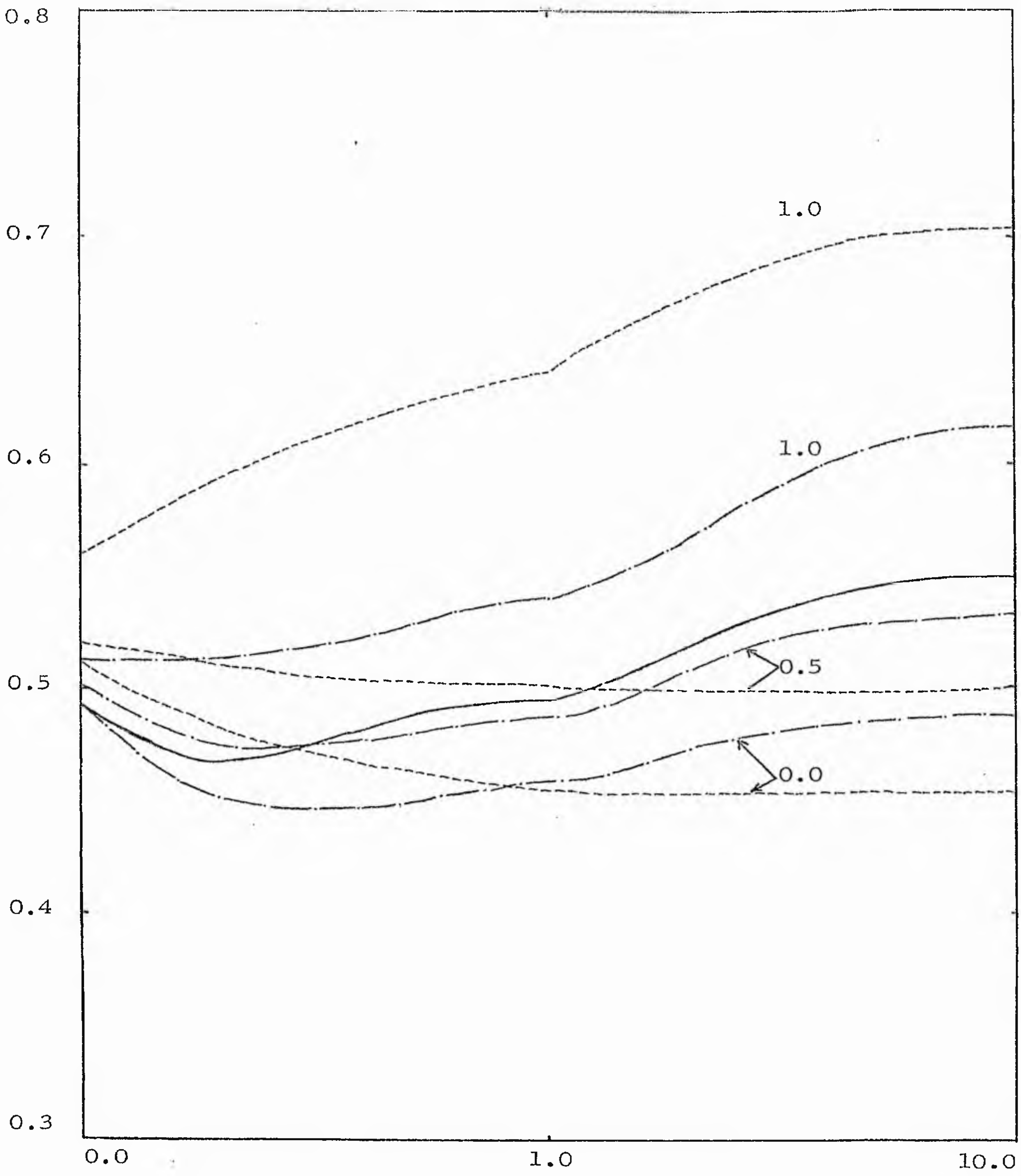


FIG. 22. As Fig. 21 but for $\mu_0 = 0.15$ and several phase functions.

The continuous, broken and dashed curves refer to values of α of 1.0, 0.4 and 0.0 respectively, and the values of β are indicated on the figure.

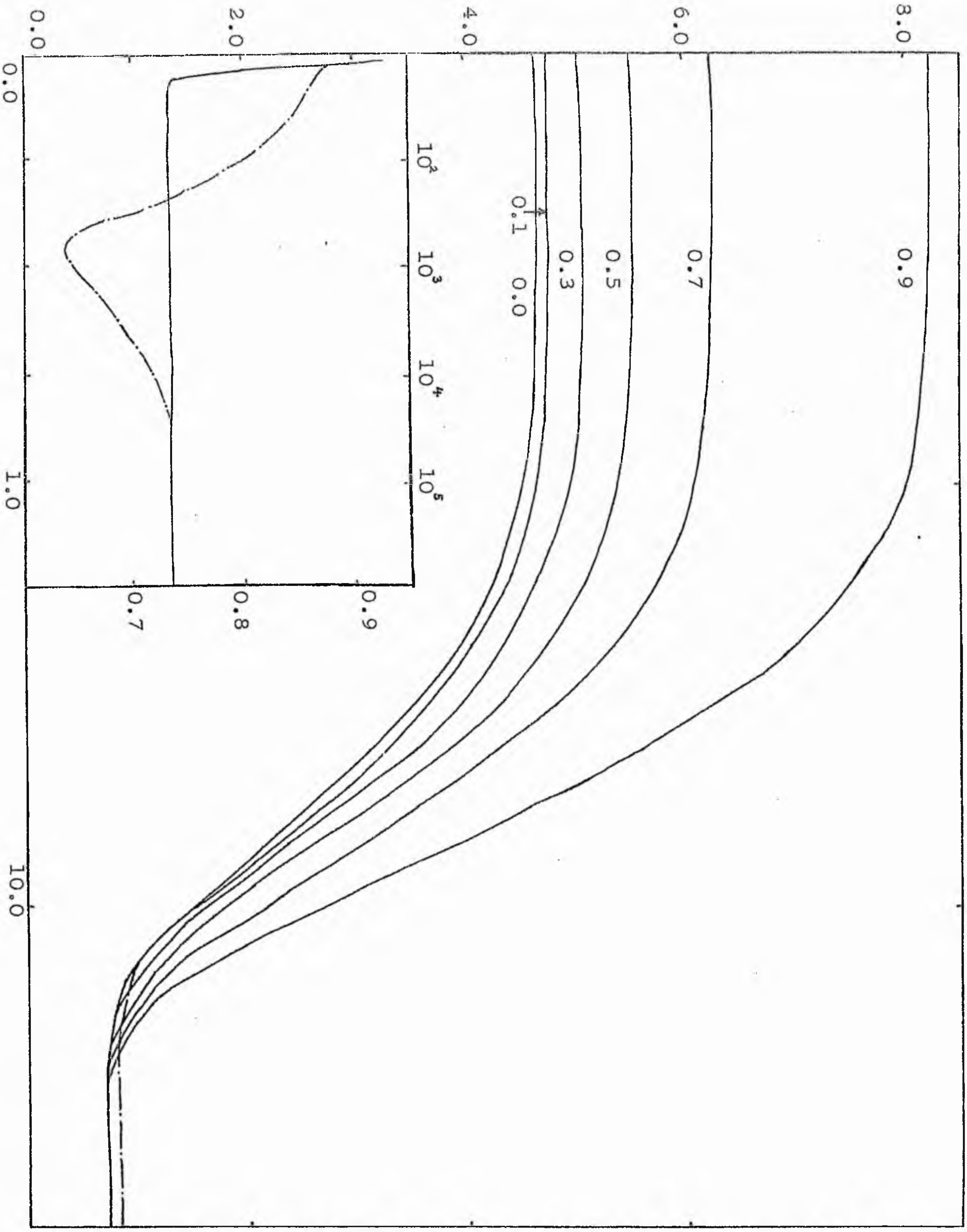


FIG. 23. The temperature of a semi-infinite plane-parallel atmosphere for which n is 10^4 as a function of optical depth for several values of $\tilde{\omega}_p$, the thermal scattering albedo.

The incidence is normal, the stellar scattering isotropic and of albedo 0.9. The values of $\tilde{\omega}_p$ are shown on the figure.

The broken curve is the temperature after one lambda operation, with $\tilde{\omega}_p$ = 0.0. The inset shows the temperature at very large depths for which the behaviour of the lambda operator is most interesting.

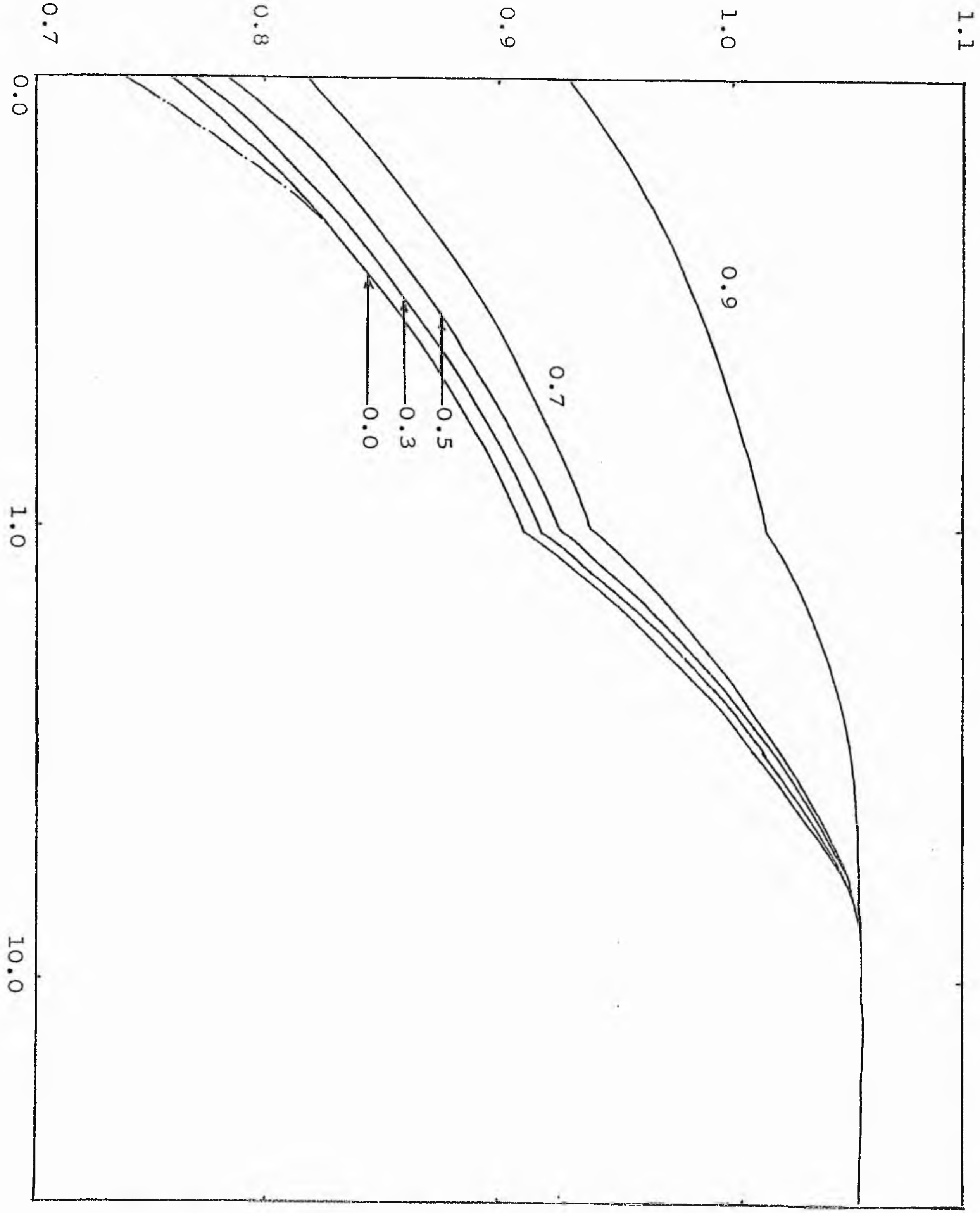


FIG. 24. As Fig. 23 for n = 1.0.

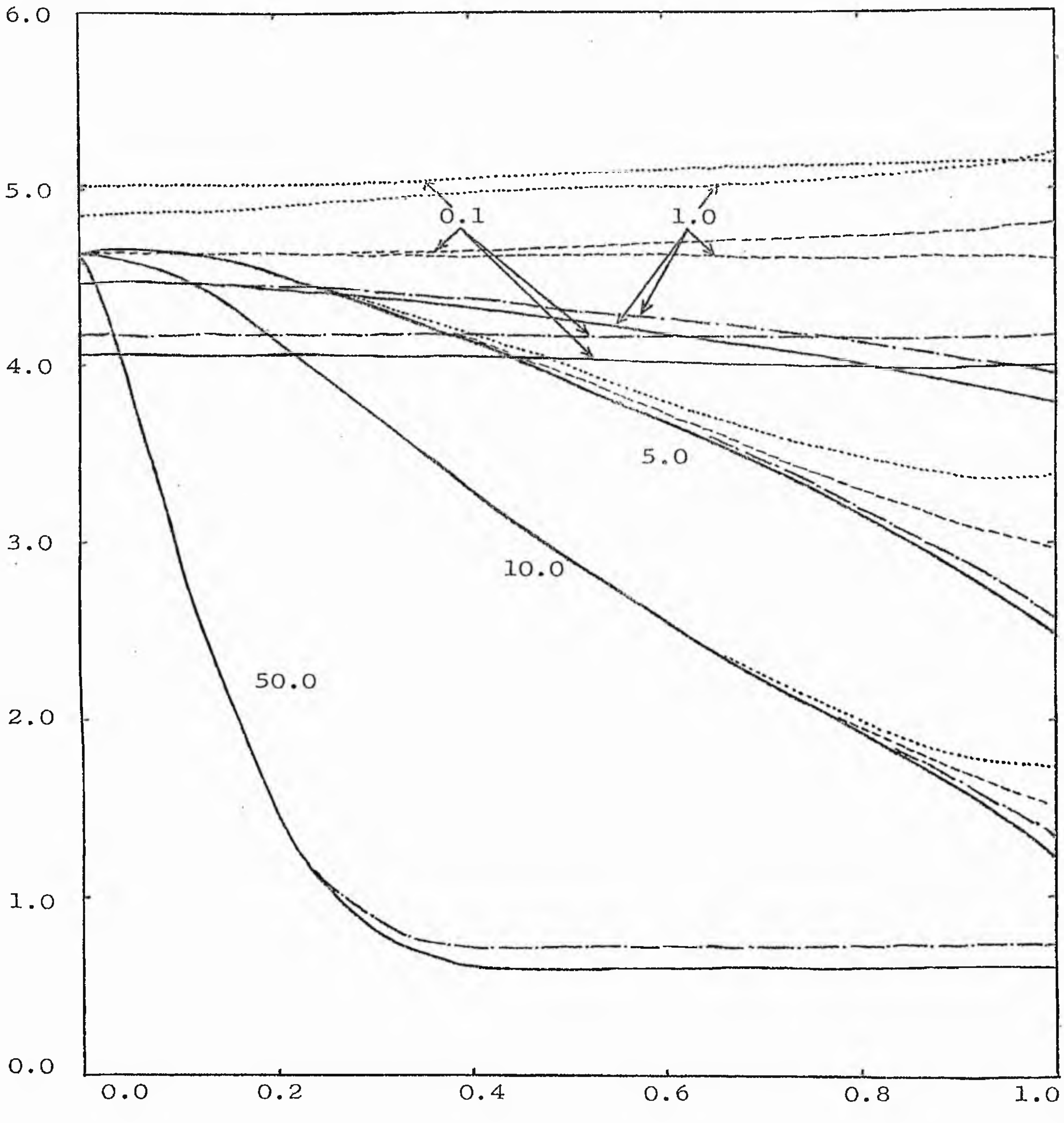


FIG. 25. The temperature profiles of finite plane-parallel atmospheres of various optical thickness for isotropic scattering and $n = 10^4$.

The incidence is normal and the albedo is 0.9. The abscissa is τ/τ_0 and the values of τ_0 are indicated on the figure. The broken, dashed and dotted curves represent values of λ of 0.1, 0.5 and 0.9 respectively and the continuous curve represents the atmosphere with no ground.

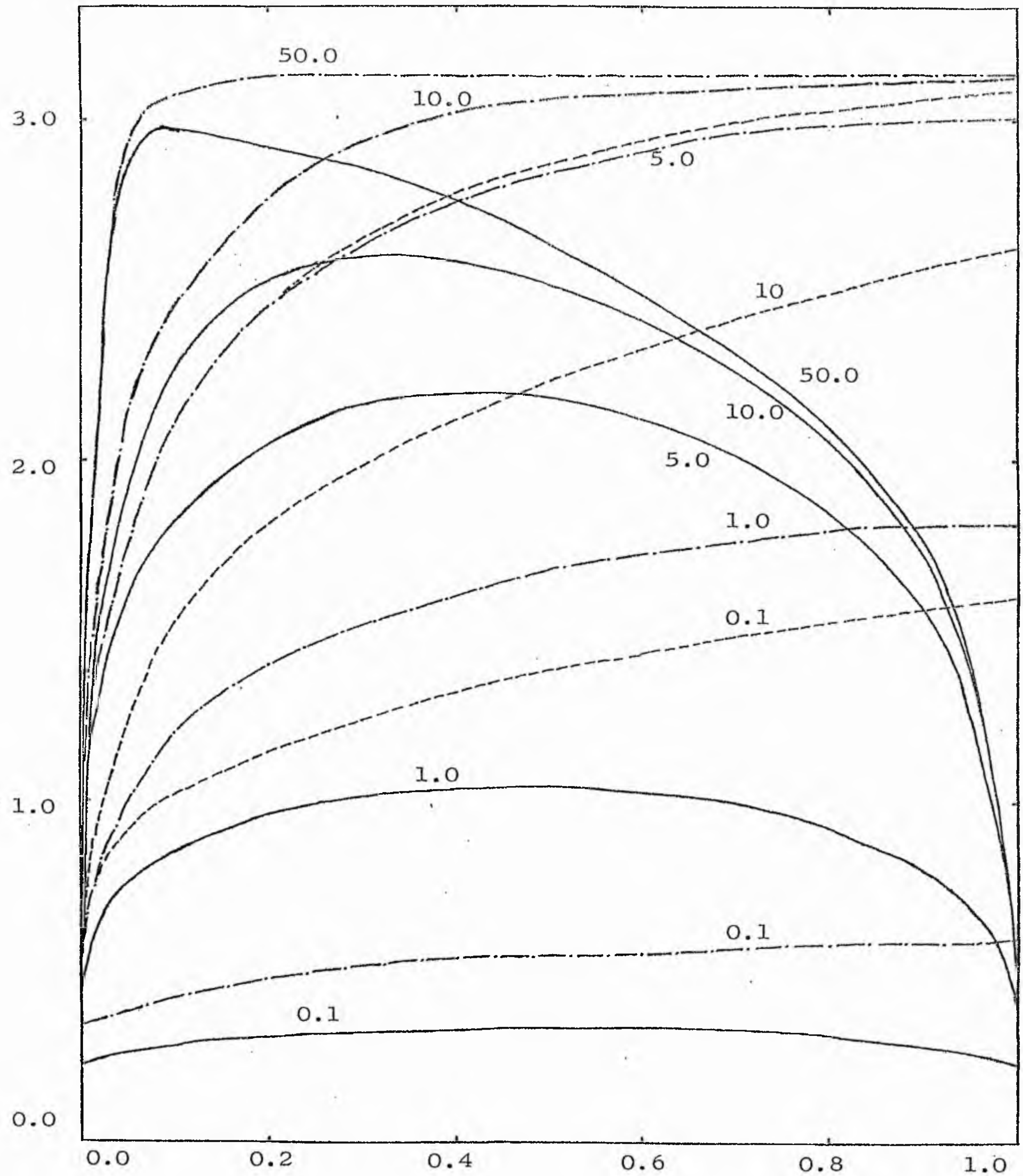


FIG. 26. As Fig. 25 for $n = 10^{-2}$.

The broken and dashed curves now represent values of λ of 1.0 and 0.1 respectively.

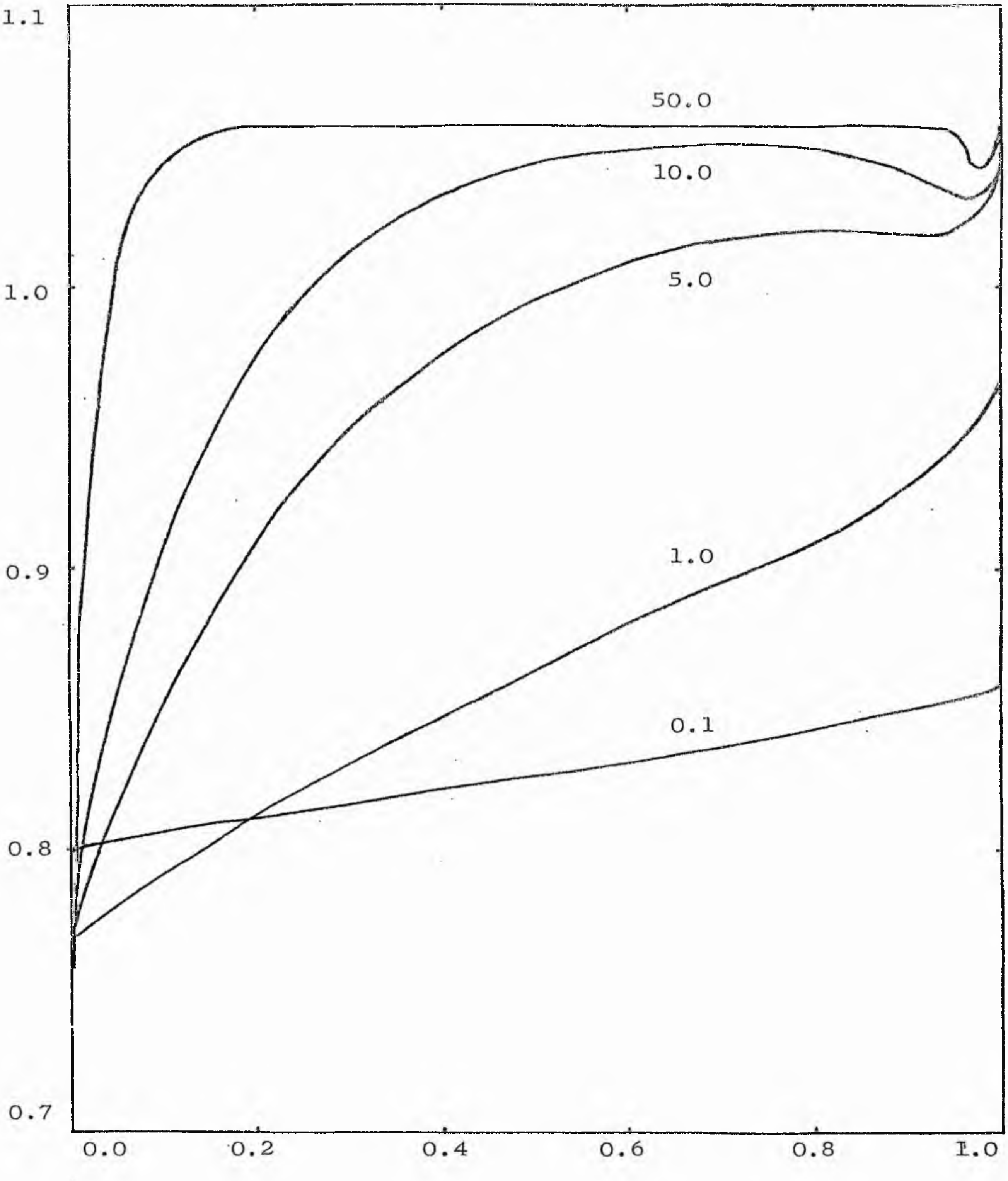


FIG. 27. As Fig. 25 for n = 1.0 but for the case of λ = 0.1 only.

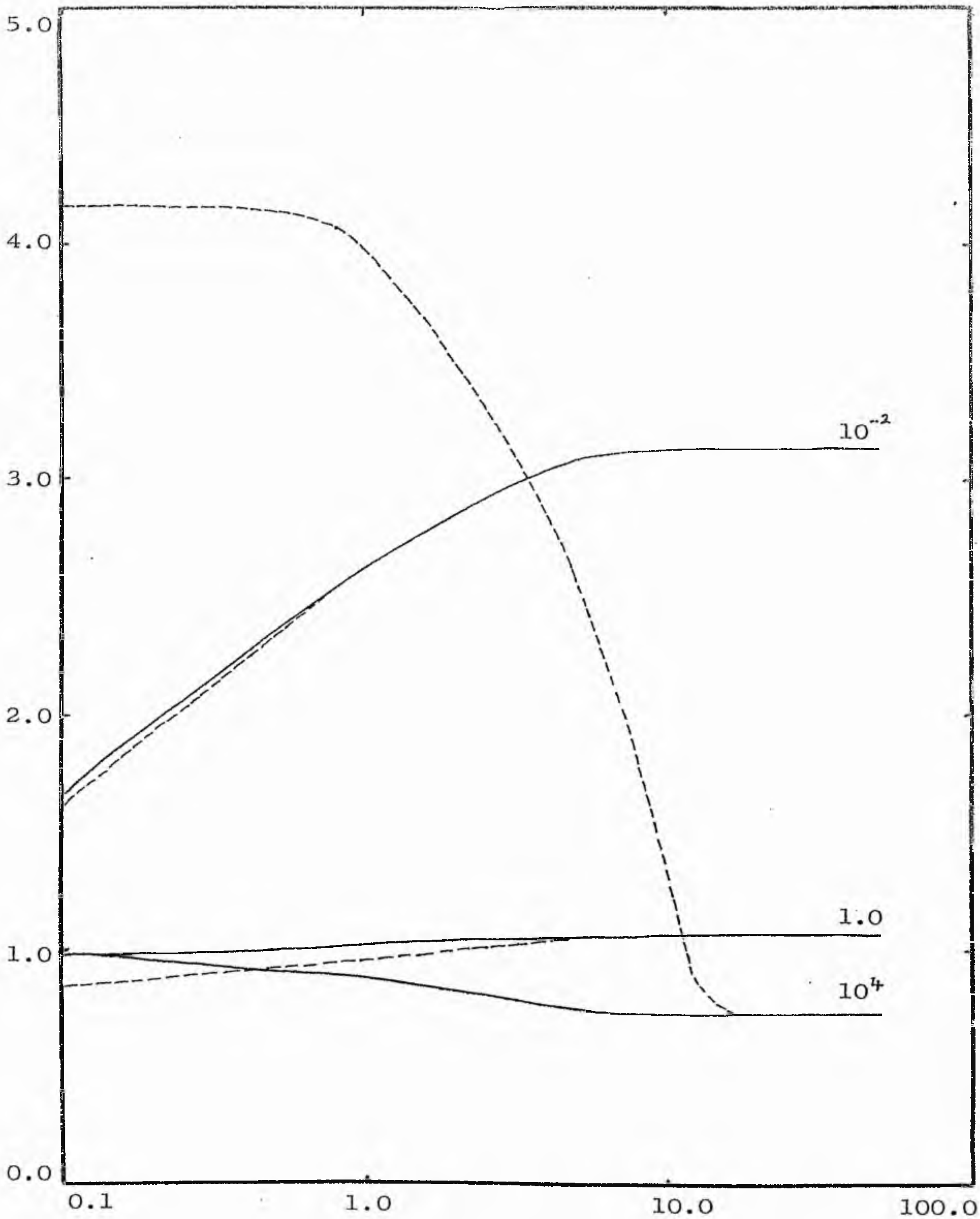


FIG. 28. A comparison of the temperature of the ground and the temperature of the lower surface of the finite plane-parallel atmosphere in contact with it.

The scattering is isotropic of albedo, 0.9 and the incidence, normal. The value of n is shown on the figure and λ is 0.1. The ground temperatures plotted as functions of τ are shown as continuous curves and the atmosphere temperatures as dashed curves.

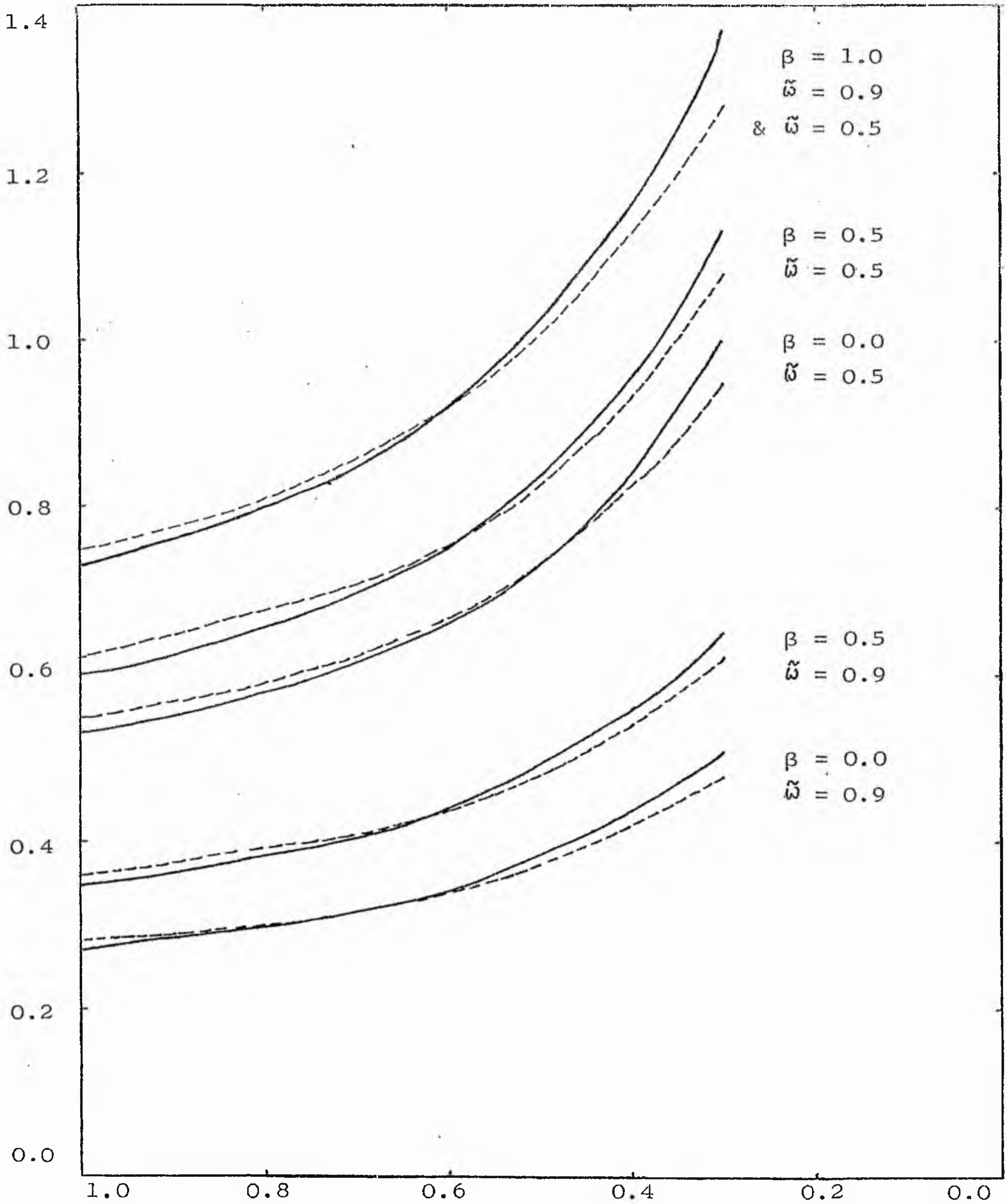


FIG. 29. The emergent thermal radiation from a semi-infinite plane-parallel atmosphere, as a function of μ, the cosine of the angle of emergence.

The continuous curves represent the exact solutions and the dashed curves the approximate solutions. In this case $n = 10^4$ and the incidence is normal. The parameter, $\alpha = 0.0$ and the values of β and $\tilde{\omega}$ are shown on the figure. When $\beta = 1.0$ the value of is irrelevant.

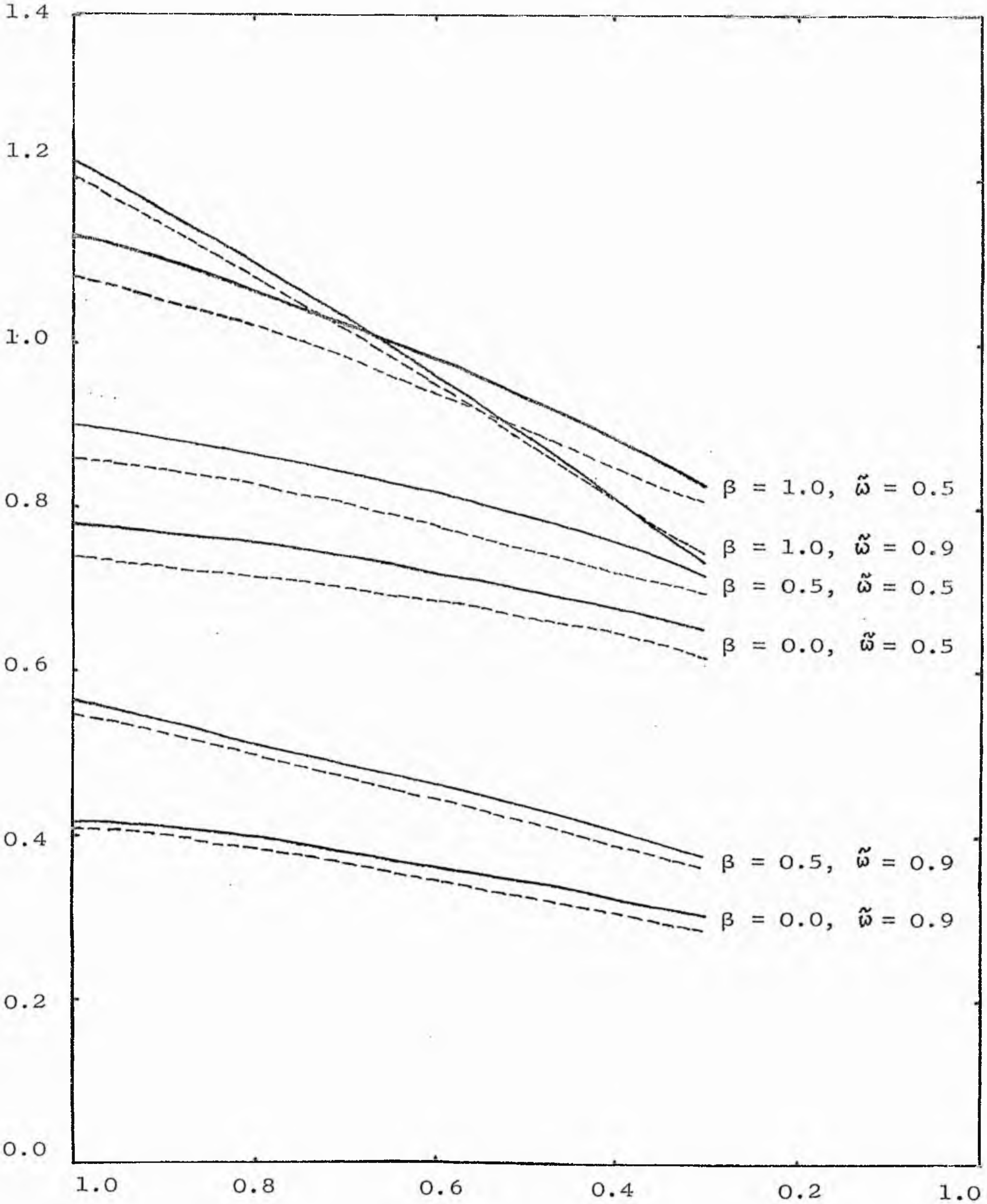


FIG. 30. As Fig. 29 for n = 1.0

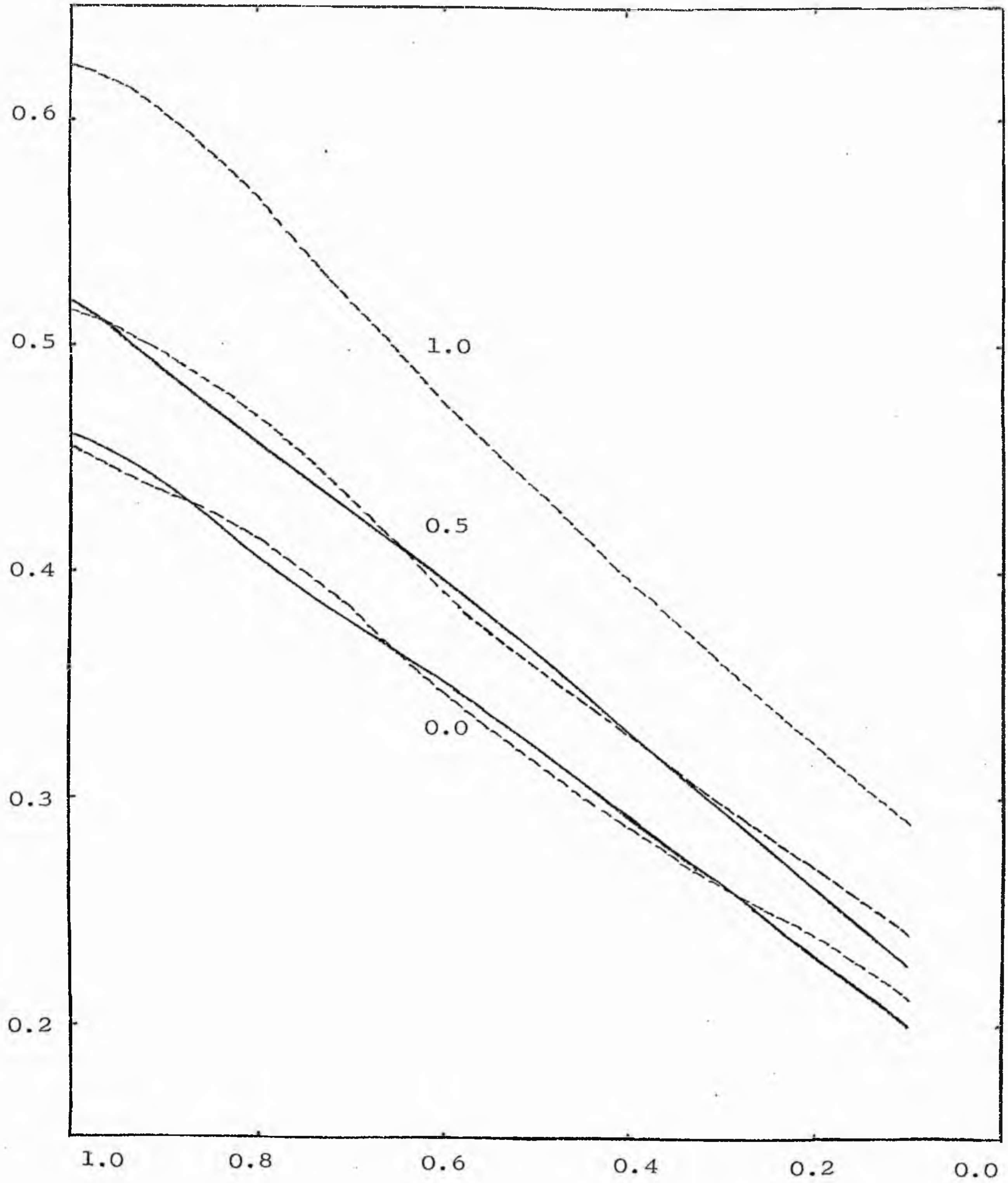


FIG. 31. As Fig. 29 for $n = 10^{-2}$ but for $\tilde{\omega} = 0.5$ and $\mu_0 = 0.5$ only.

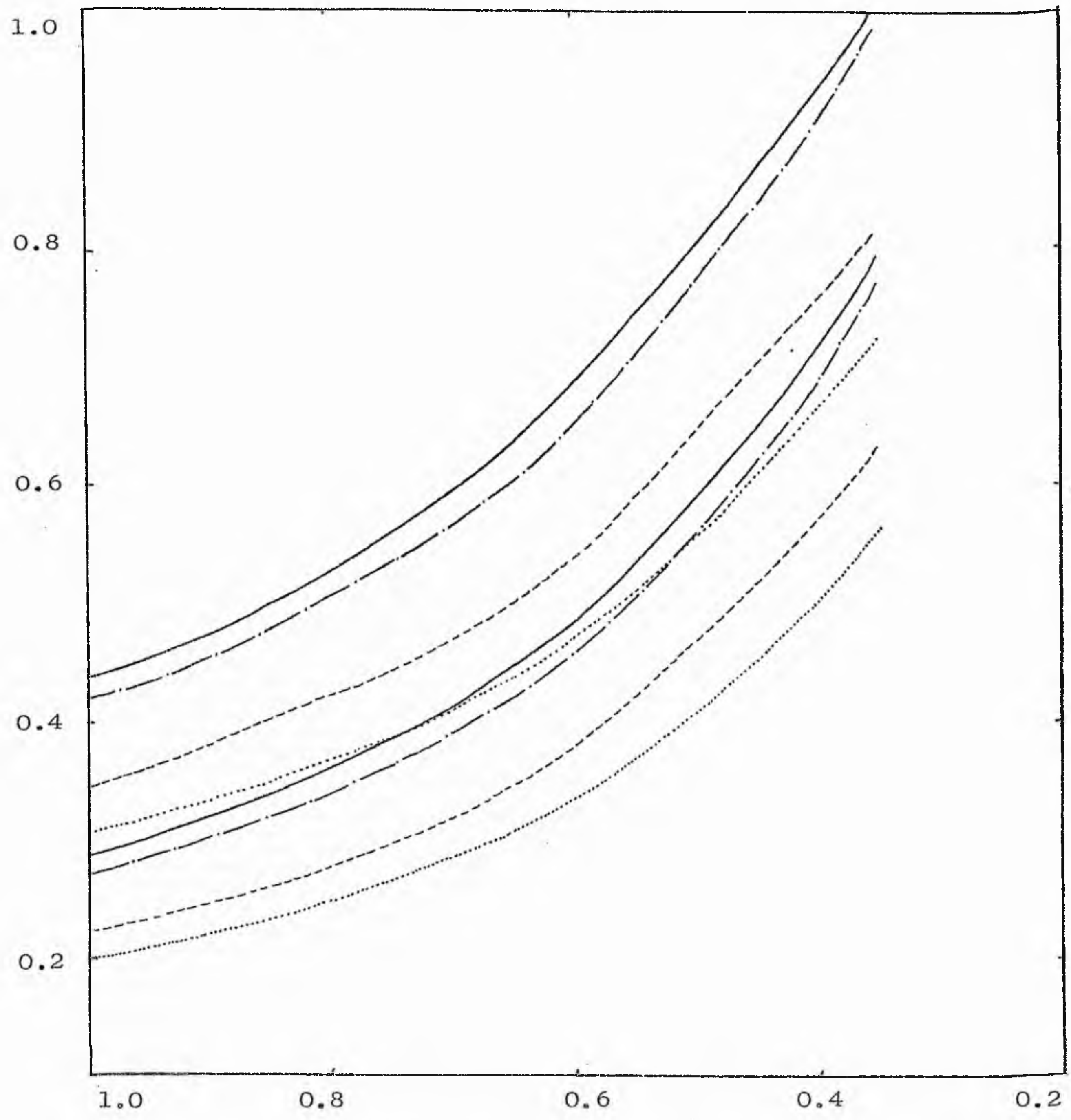


FIG. 32. The angular distribution of the emergent thermal radiation from a finite plane-parallel atmosphere with $n = 10^4$.

The ordinate is the intensity and the abscissa is μ. The scattering has $\alpha = 0.0$ and $\tilde{\omega} = 0.5$ and the incidence is normal. The broken, dashed and dotted curves refer to the approximate solutions for values of β of 1.0, 0.5 and 0.0 respectively, and the continuous curves refer to the exact solutions for the case of $\beta = 1.0$. The upper curve of each type is for $\tau_0 = 1.0$ and the lower for $\tau_0 = 0.1$.

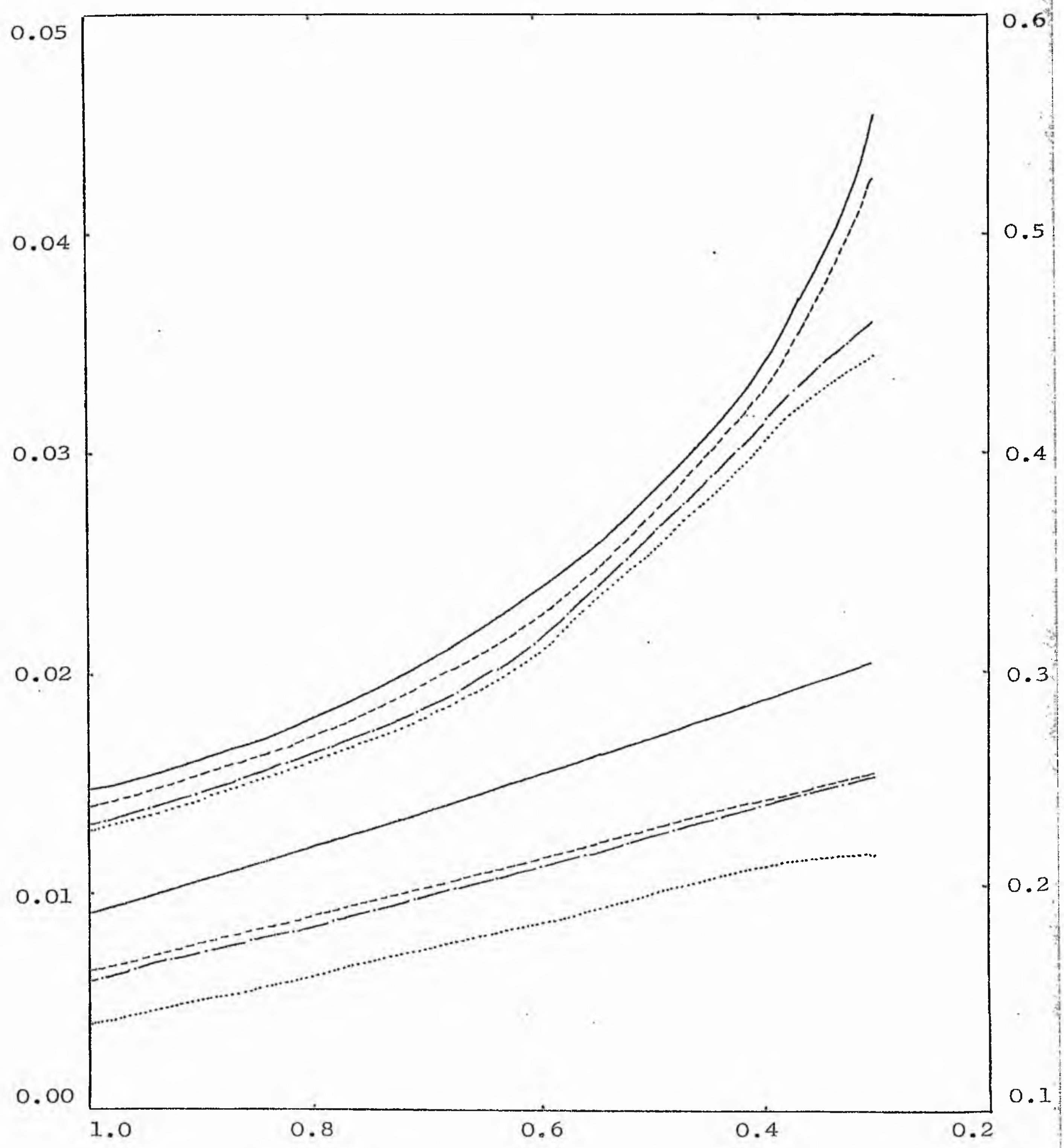


FIG. 33. As Fig. 32 for n = 1.0 and β = 1.0 only.

The dashed and dotted curves represent the exact and approximate solutions for linear scattering (α = 0.0), and the continuous and broken curves represent the exact and approximate solutions for isotropic scattering (α = 1.0). However the lower curve in each case now refers to τ_0 = 1.0 and the upper curve to τ_0 = 0.1. The ordinate scale for the latter is on the left-hand axis and that for the former is on the right-hand axis.

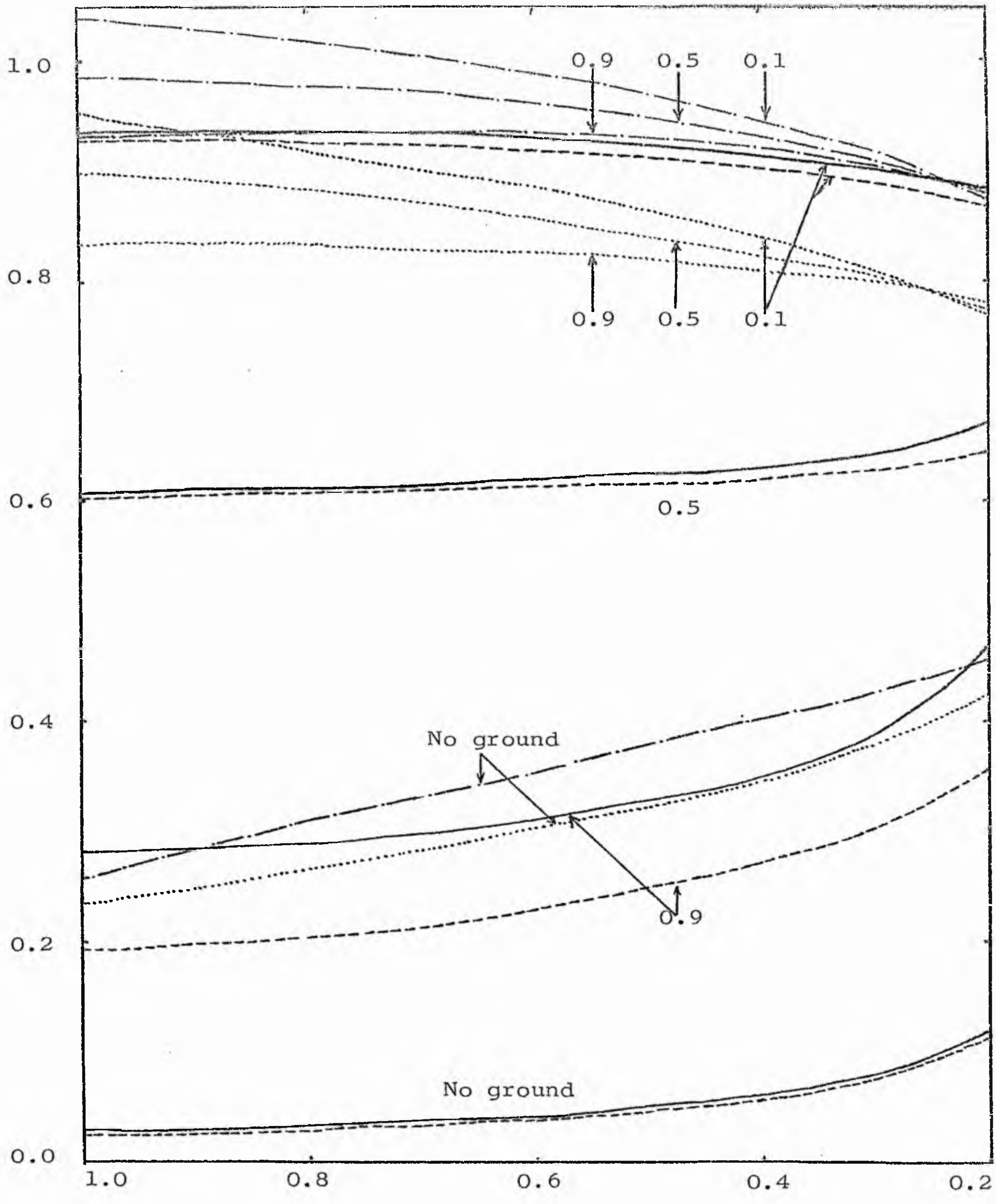


FIG. 34. The angular distribution of the emergent thermal radiation from a finite plane-parallel atmosphere with a ground layer and n = 1.0.

There is no scattering and the incidence is normal. The continuous and dashed lines represent the exact and approximate solutions for atmospheres with $\tau_0 = 0.1$, and the broken and dotted line the same for $\tau_0 = 1.0$. The values of λ are noted on the figure and the case of no ground is included for comparison.

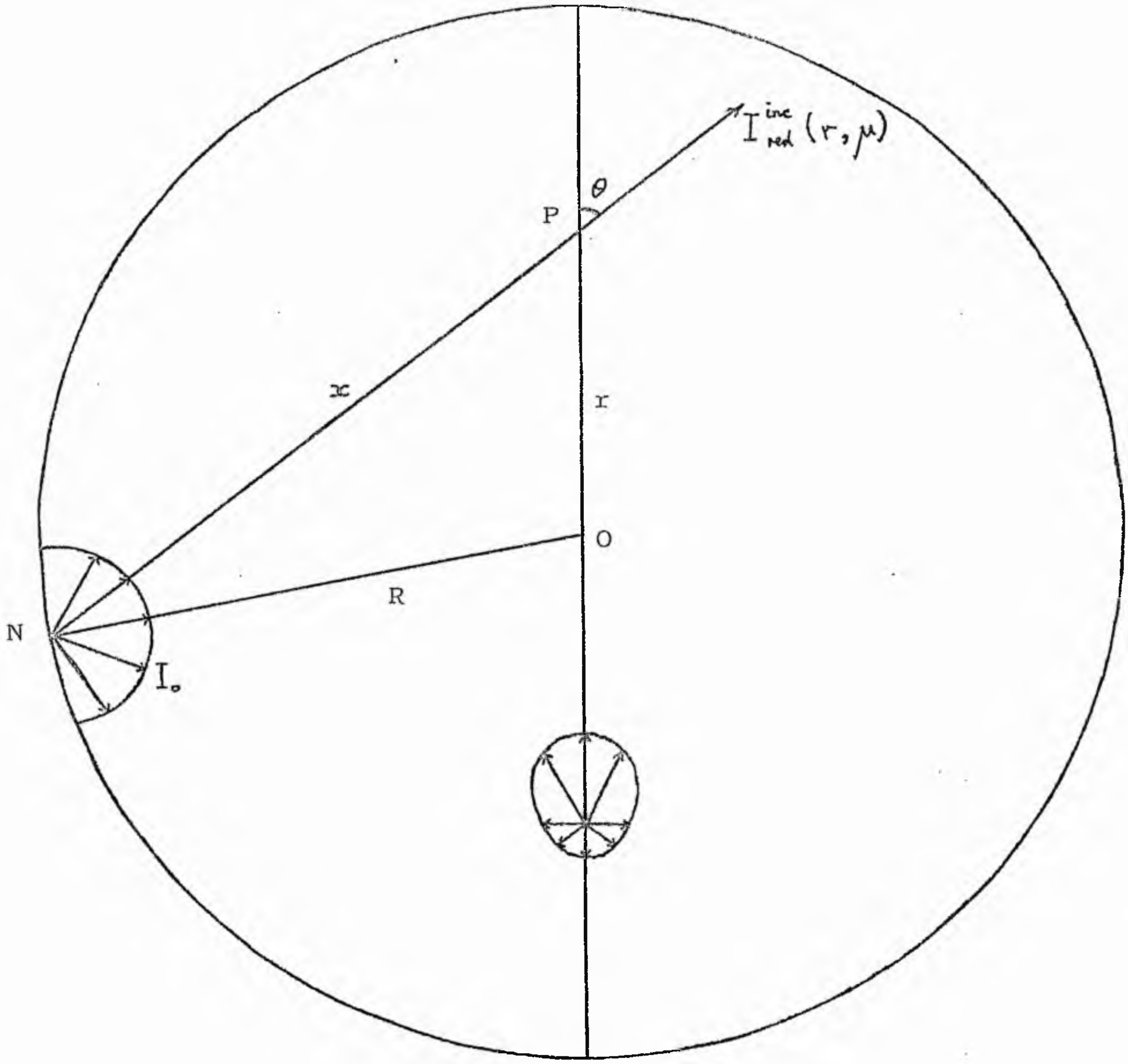


FIG. 35. The reduced incident radiation field in a spherical atmosphere with uniform isotropic incident radiation. $\mu = \cos\theta$.

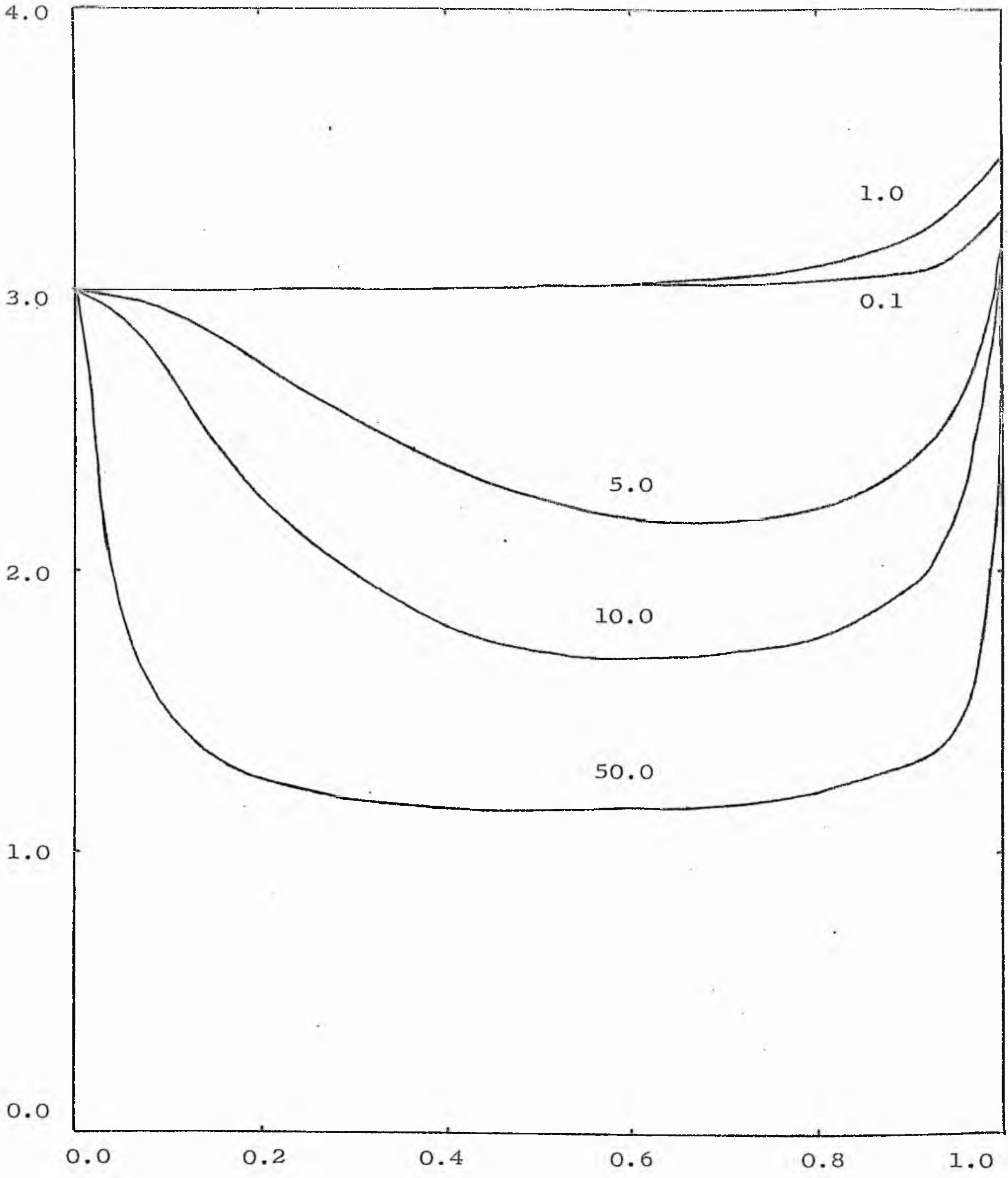
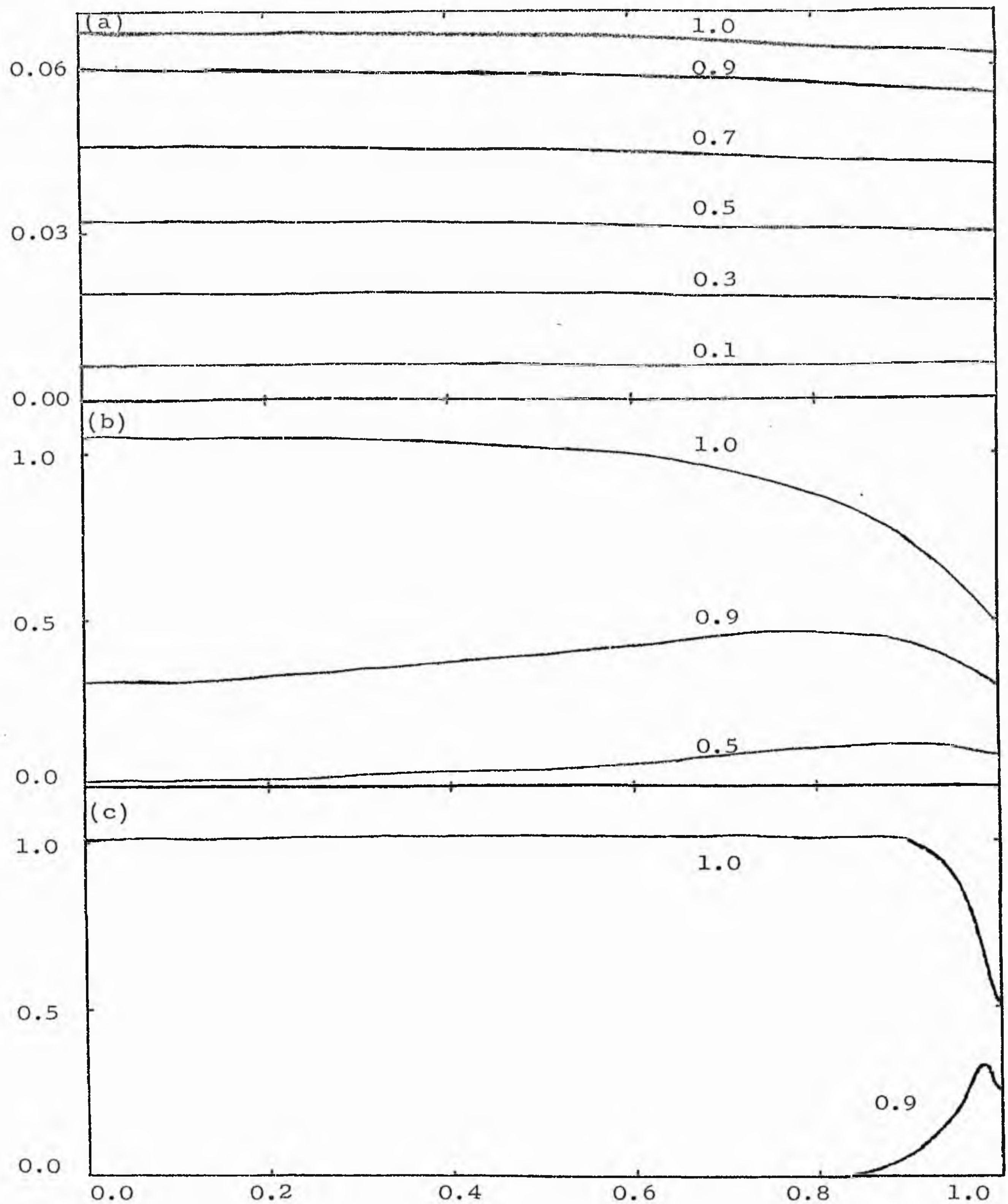


FIG. 36. The ratio, $J^{inc}_{red}(\tau)/K^{inc}_{red}(\tau)$ as a function of τ/τ_0 for several values of τ_0.



FIGS. 37 (a) to (c). The mean intensity of the scattered radiation field in a spherical atmosphere as a function of fractional optical radius.

The three figures are for atmospheres of optical radii, 0.1, 5.0 and 50.0 respectively and are drawn for several values of the albedo. The scattering is isotropic.

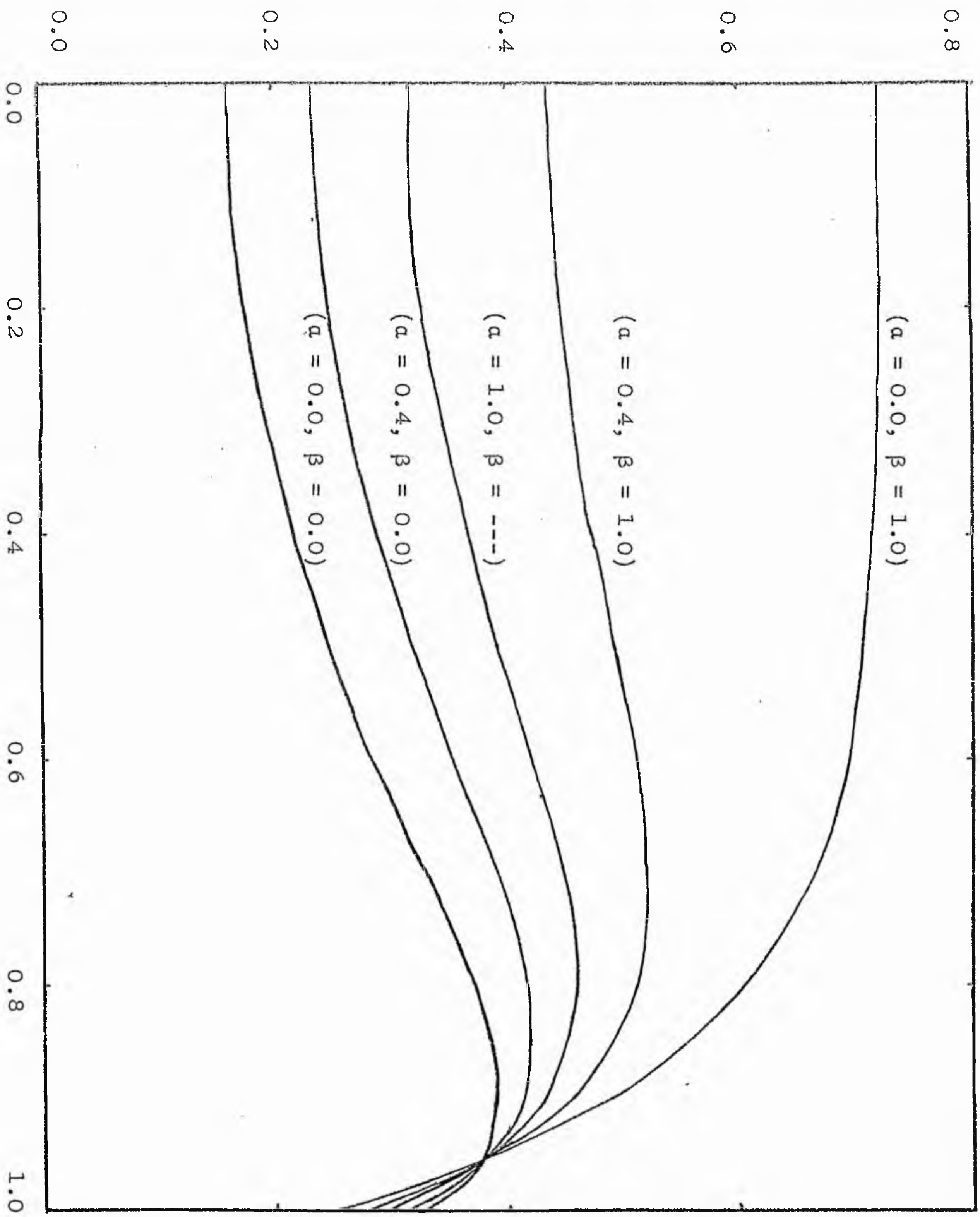


FIG. 38. The mean intensity of the scattered radiation field in a spherical atmosphere as a function of fractional optical radius for several phase functions.

The phase function parameters are shown on the figure, which shows an atmosphere of τ_0 = 5.0 for scattering of albedo, 0.9.

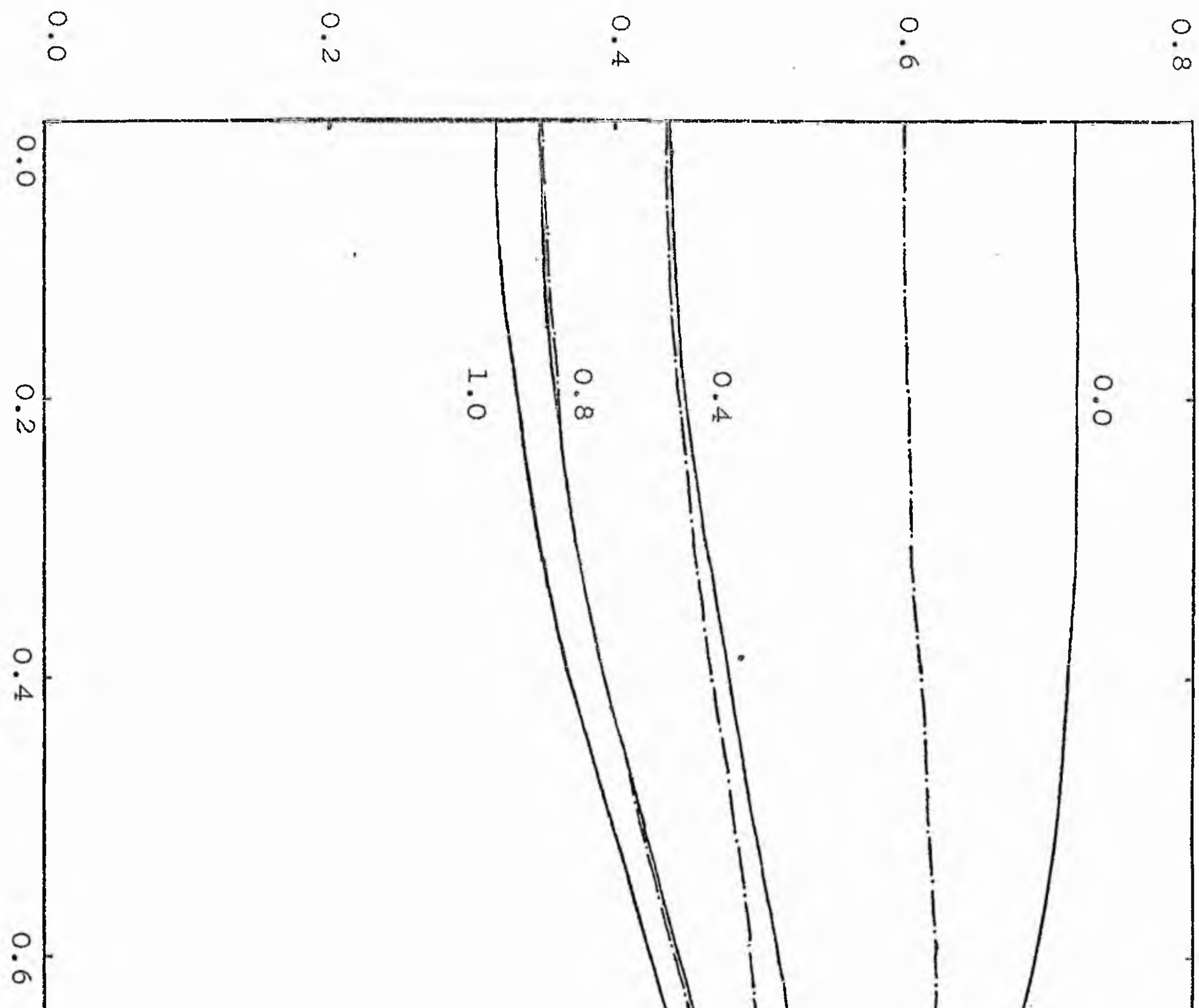


FIG. 39. A comparison of the results for the mean intensity of the scattered radiation field in a spherical atmosphere as obtained by methods I and II.

The continuous and broken curves represent methods I and II respectively. The parameter β is 1.0. Otherwise it is as Fig. 38 with the values of α indicated on the figure.

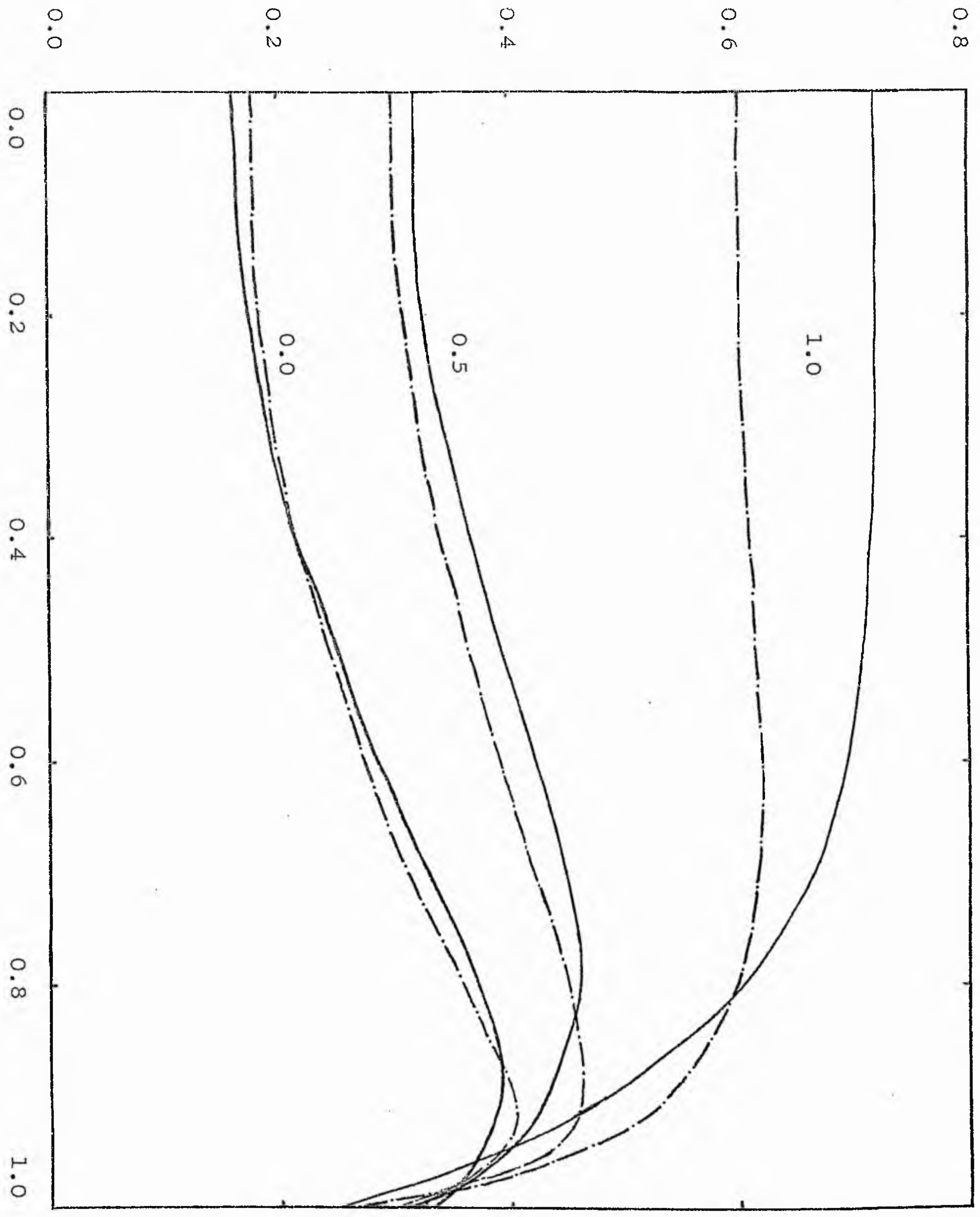
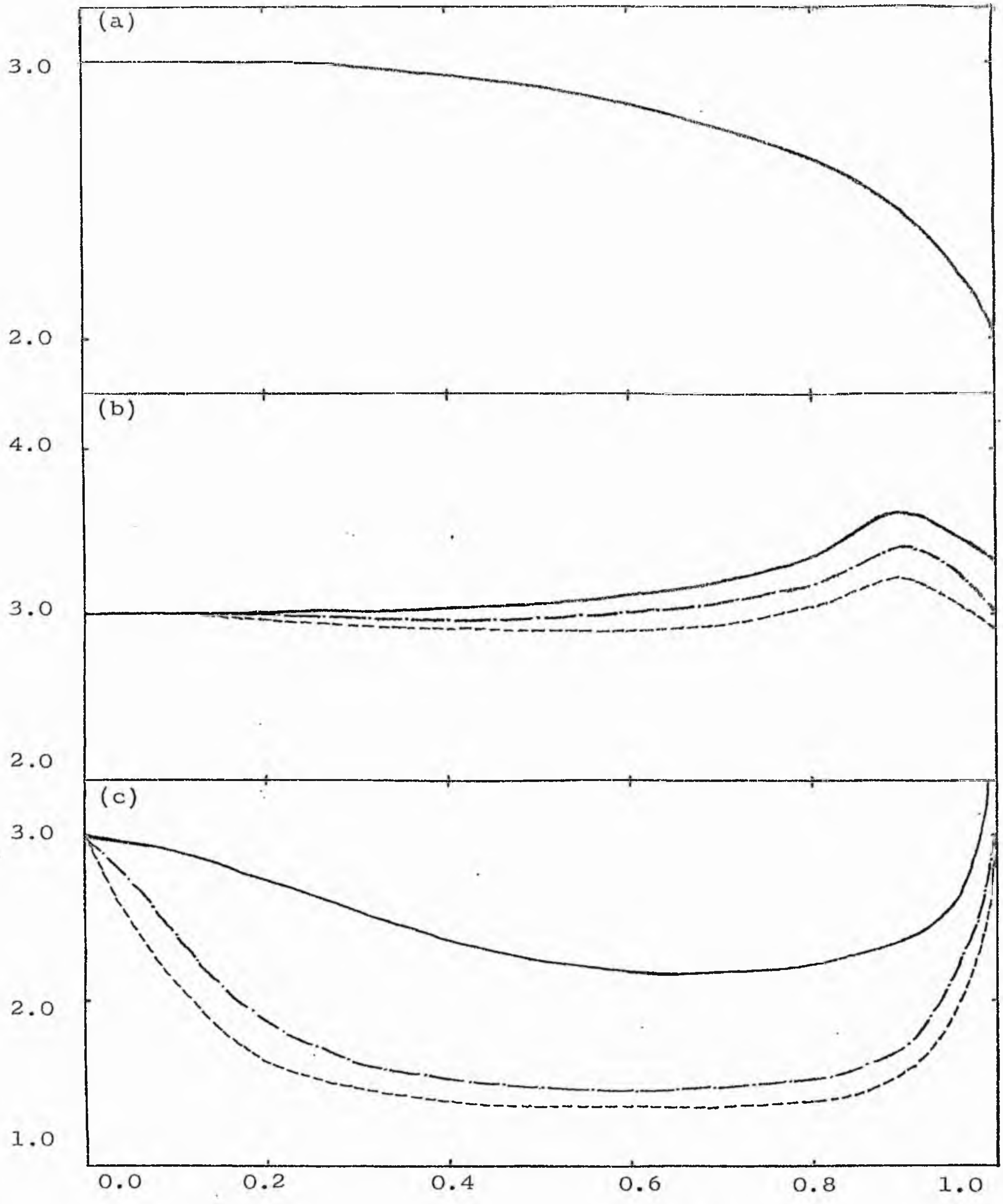
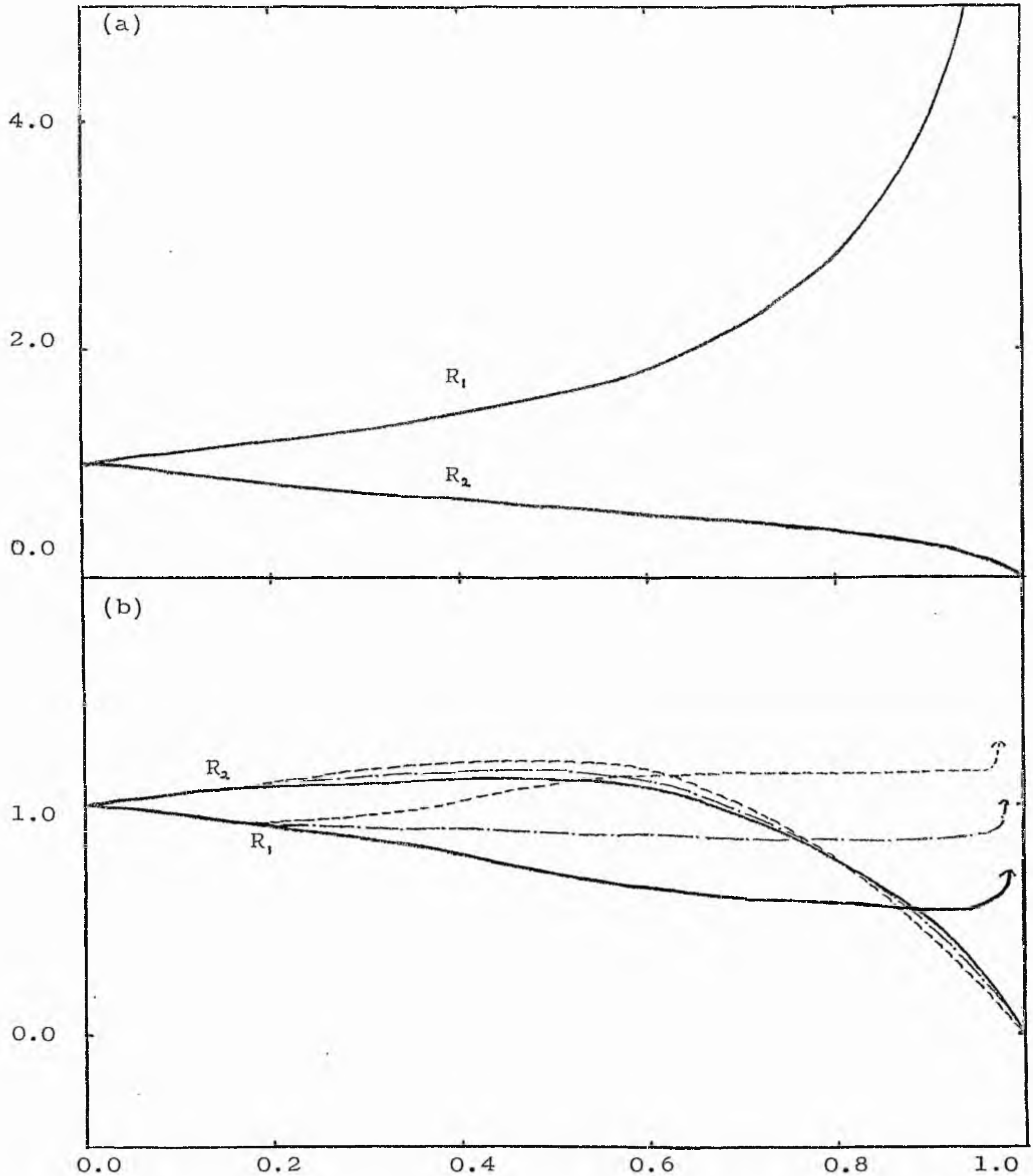


FIG. 40. As Fig. 39 for phase functions with $\alpha = 0$. The values of β are indicated on the figure.



FIGS. 41 (a) to (c). The ratio $r(\tau) = J_{sc}(\tau)/K_{sc}(\tau)$ as a function of the fractional optical radius for spherical atmospheres of optical radii, 0.1, 5.0 and 50.0 respectively.

The parameters $\tilde{\omega}$ and α are 0.9 and 0.0 respectively, and the values of β of 1.0, 0.5 and 0.0 are represented by continuous, broken and dashed lines respectively. In Fig.41(a) all three curves are coincident; and these ratios are exact.



FIGS. 42 (a) and (b). The ratios, $R_1 = I_{sc}(\tau,+1)/I_{sc}(\tau,0)$ and $R_2 = I_{sc}(\tau,-1)/I_{sc}(\tau,0)$, plotted as functions of fractional optical radius for spherical atmospheres of optical radii, 0.1 and 5.0 respectively.

These ratios are exact and are for scattering with α and $\tilde{\omega}$ equal to 0.0 and 0.9 respectively. In Fig. 42(a) the results are independent of β, but in Fig. 42(b) the values of β of 1.0, 0.5 and 0.0 are represented by continuous, broken and dashed curves respectively.

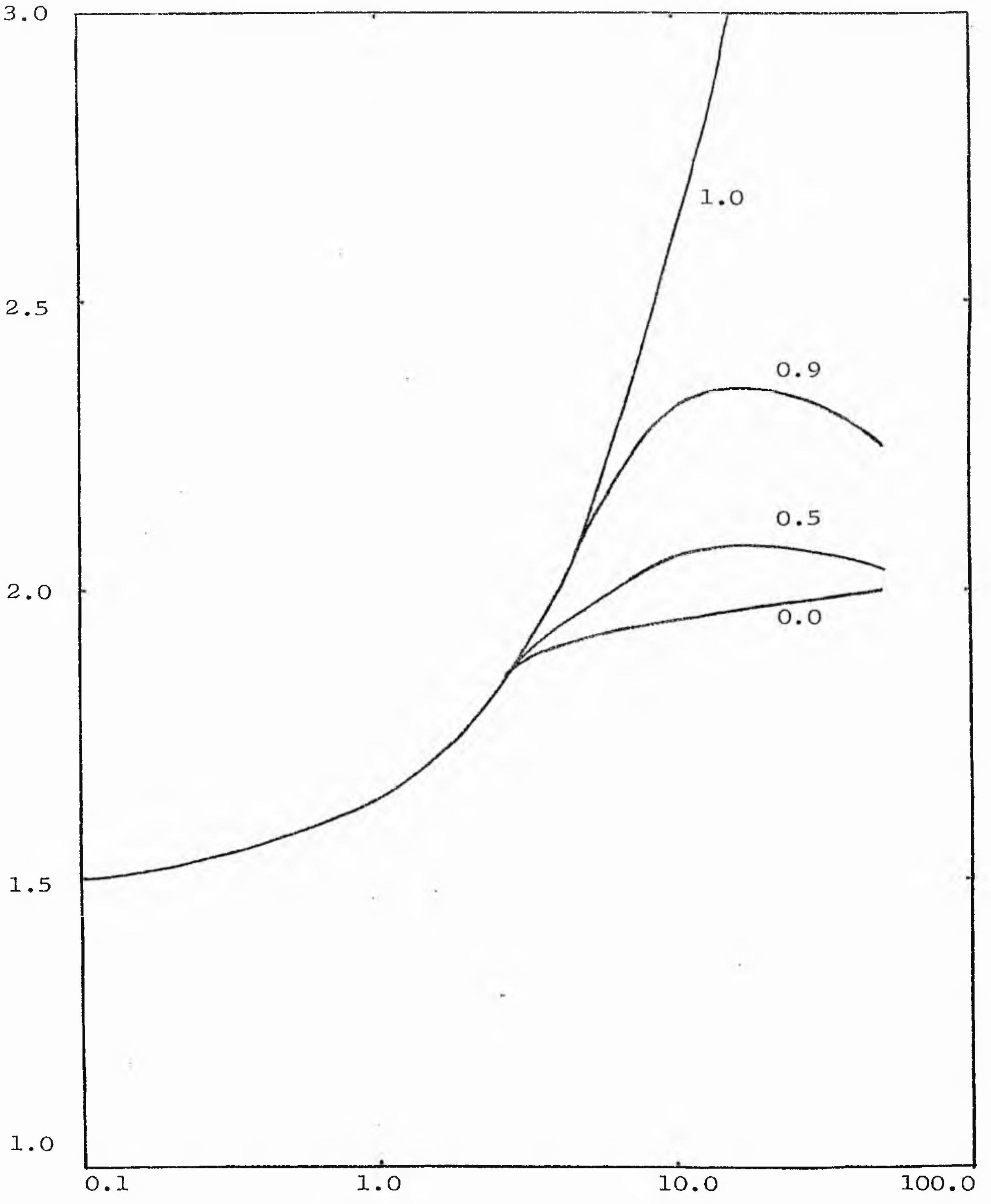


FIG. 43. The ratio $r_o(\tau_o) = J_{sc}(\tau_o) / H_{sc}(\tau_o)$ for a spherical atmosphere plotted against τ_o.

Again the curves are exact and are for scattering with α and $\tilde{\omega}$ equal to 0.0 and 0.9 respectively. The values of β are indicated on the figure.

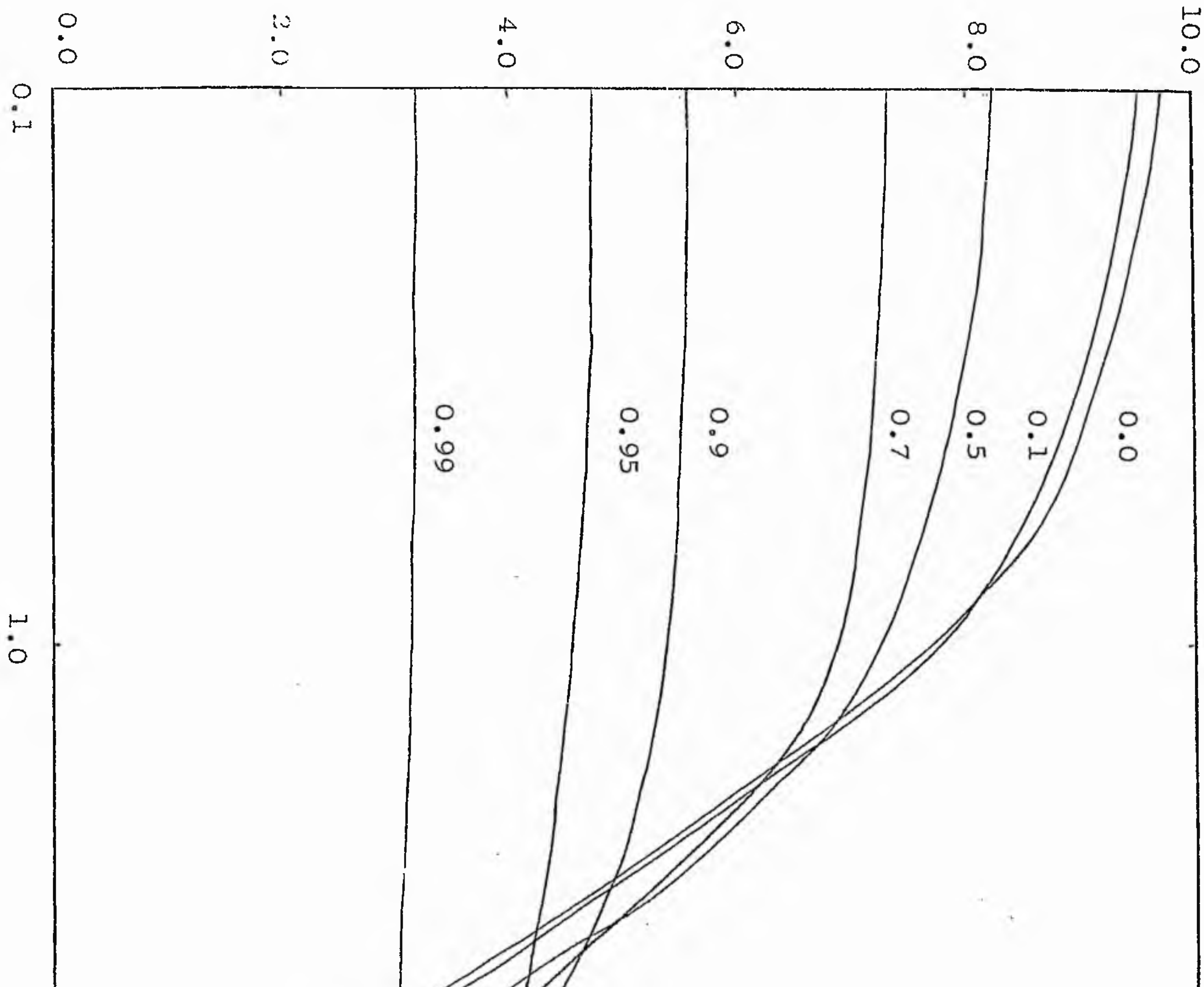


FIG. 44. The central temperature of a spherical atmosphere as a function of optical radius for several values of the albedo, when n is 10^4.

The scattering is isotropic and the values of the albedo are shown on the figure.

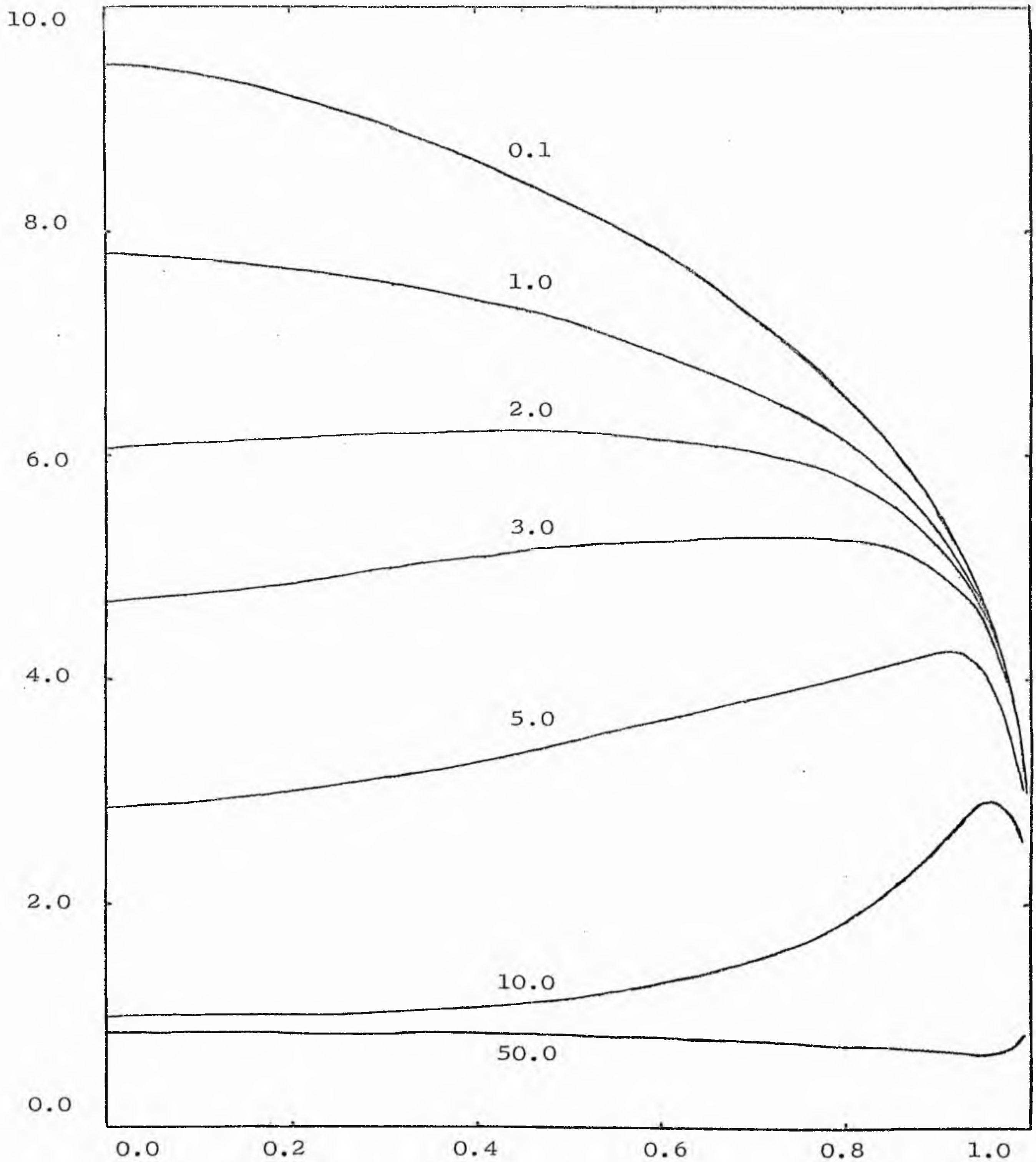


FIG. 45. The central temperature of a spherical atmosphere as a function of albedo for several values of the optical radius, when n is 10^4.

The scattering is isotropic and the values of the optical radius are shown on the figure.

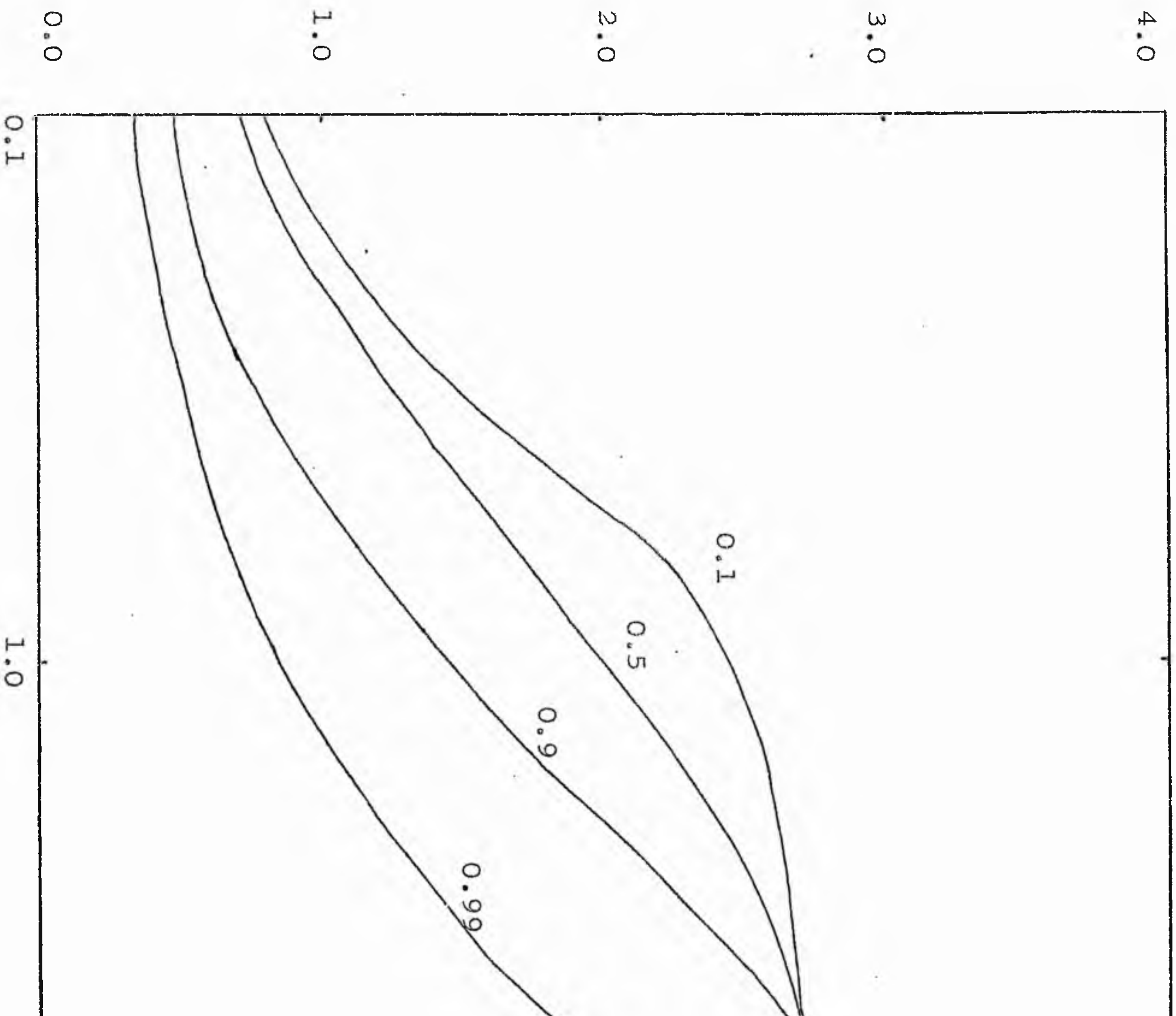


FIG. 46. As Fig. 44 for $\tau = 10^{-2}$. The values of the albedo are shown in the figure.

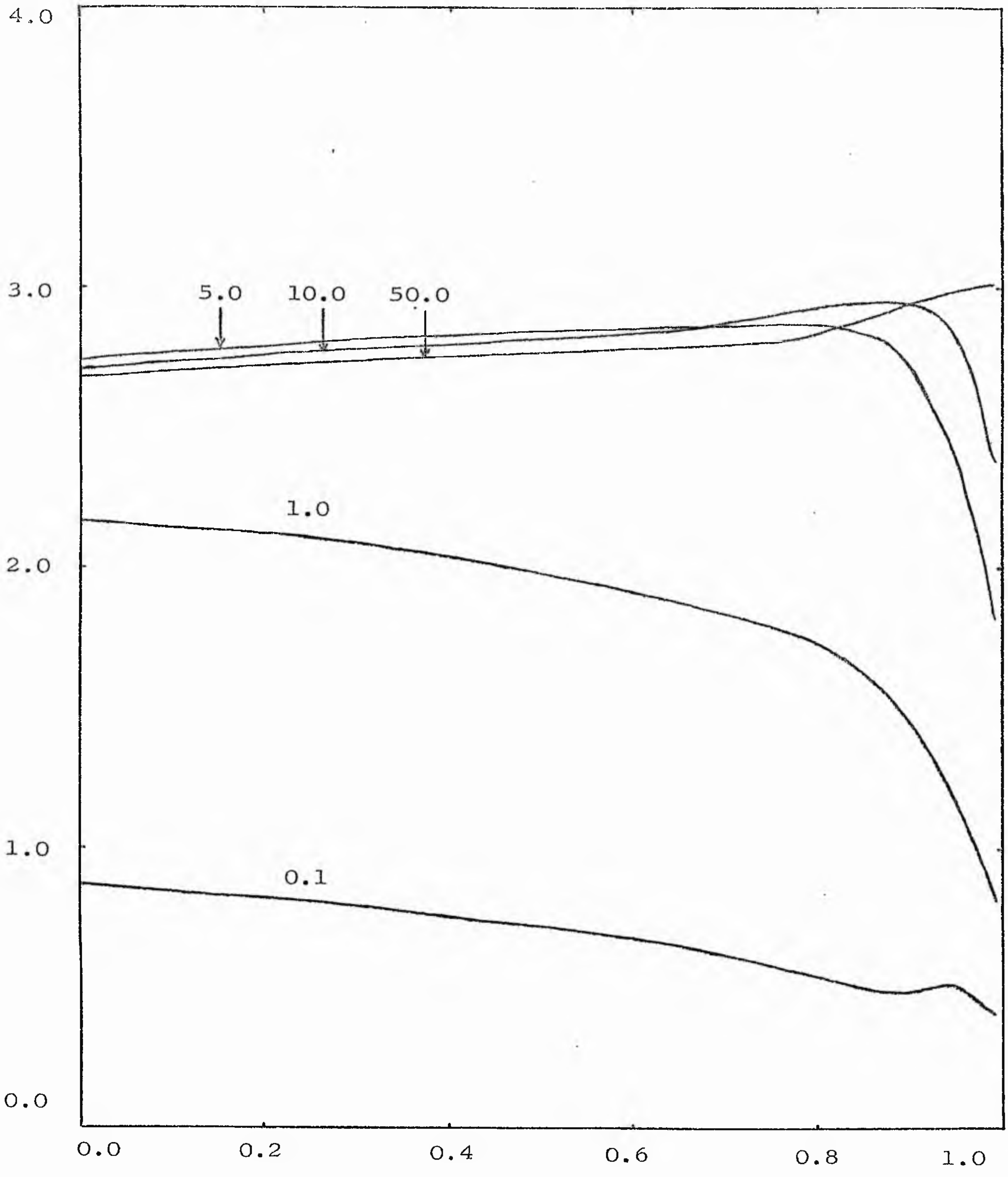


FIG. 47. As Fig. 45 for $n = 10^{-2}$. The values of the optical radius are shown on the figure.

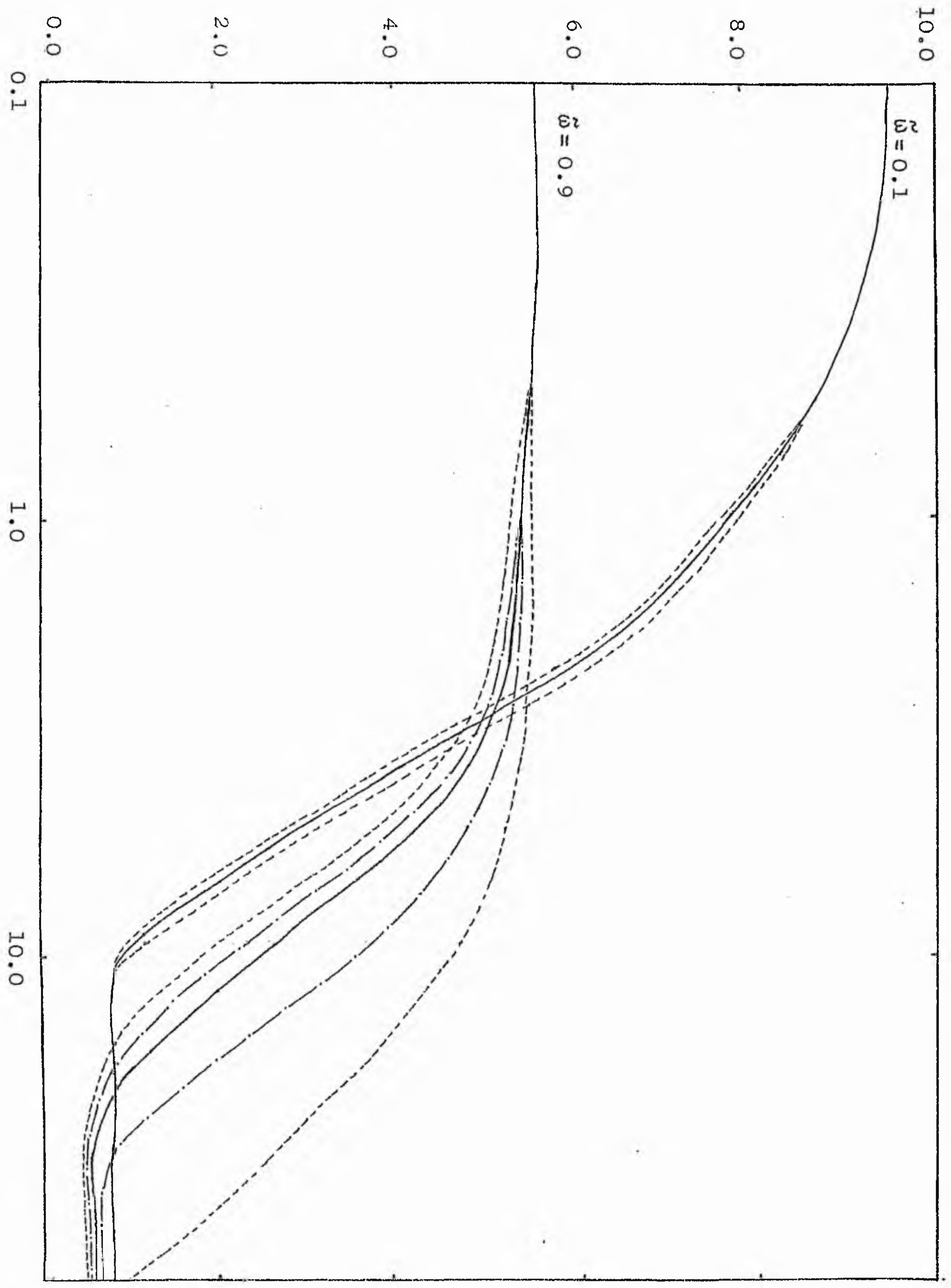


FIG. 48. The central temperature of a spherical atmosphere when n is 10^4 shown as a function of optical radius for several phase functions.

Two values of the albedo are included and are marked on the figure. The dashed lines refer to $\alpha = 0.0$, the broken lines to $\alpha = 0.4$ and the continuous lines to $\alpha = 1.0$. Those curves of T greater than T (isotropic) are of $\beta = 1.0$ and those curves of T less than T (isotropic) are of $\beta = 0.0$.

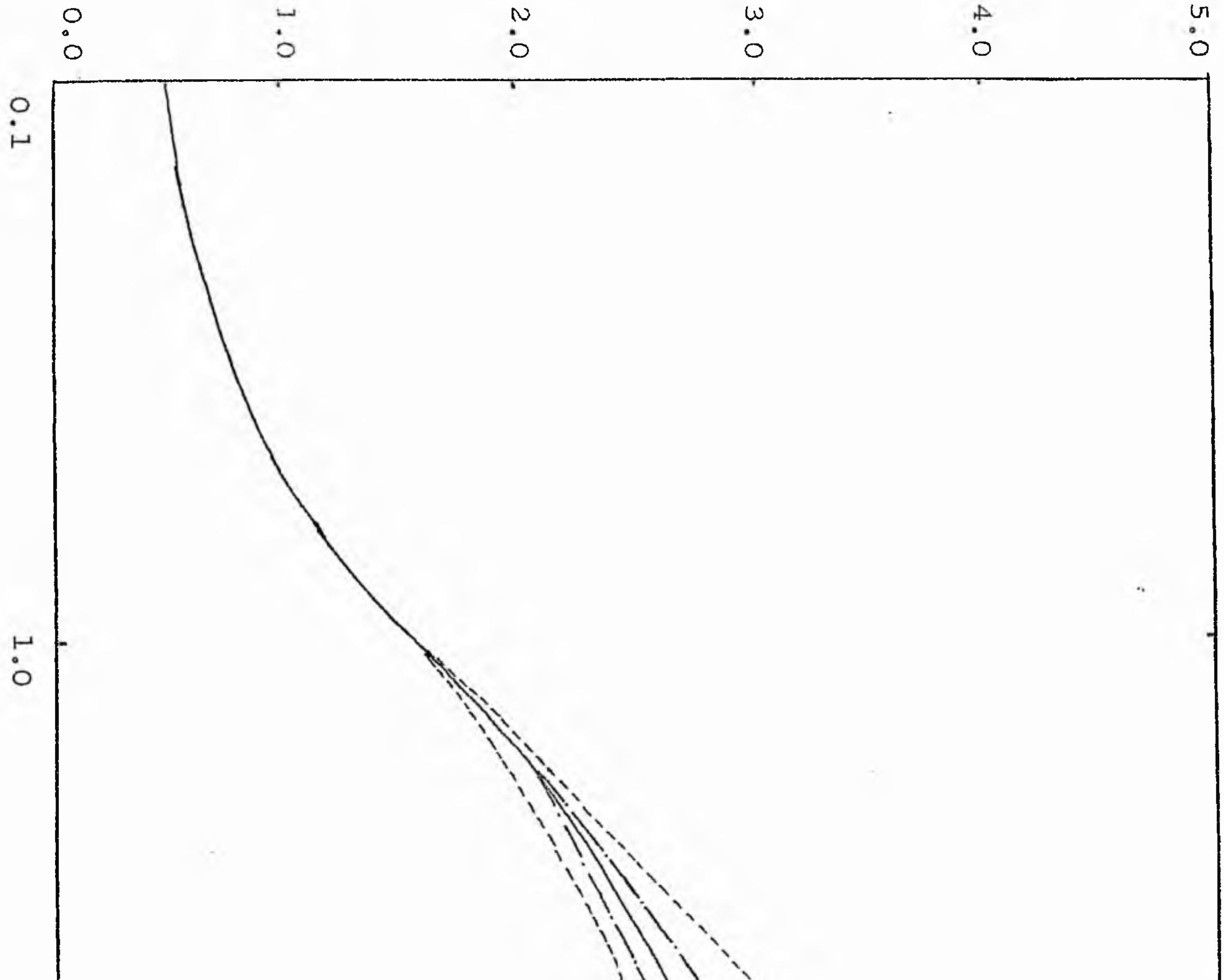


FIG. 49. As Fig. 48 for $n = 10^{-2}$. Only the case of $\tilde{\omega} = 0.9$ is shown and the values of β are indicated on the figure.

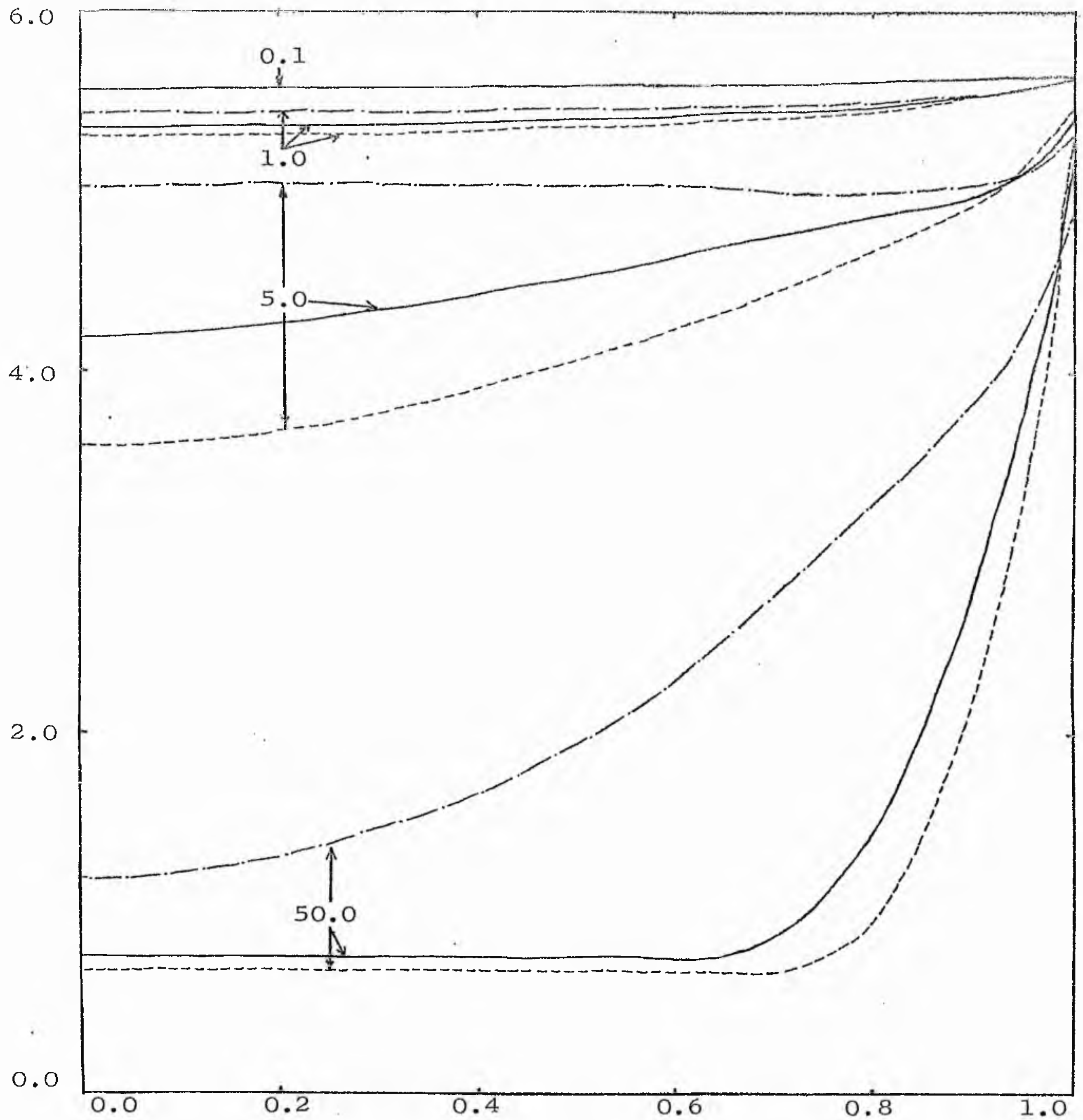


FIG. 50. The temperature profiles of a spherical atmosphere as a function of fractional optical radius for $n = 10^4$.

The albedo is 0.9 and the value of α is 0.0. The values of β of 0.0, 0.5 and 1.0 are represented by dashed, continuous and broken lines respectively. The continuous curves also represent isotropic scattering. The values of τ_0 are shown on the figure.

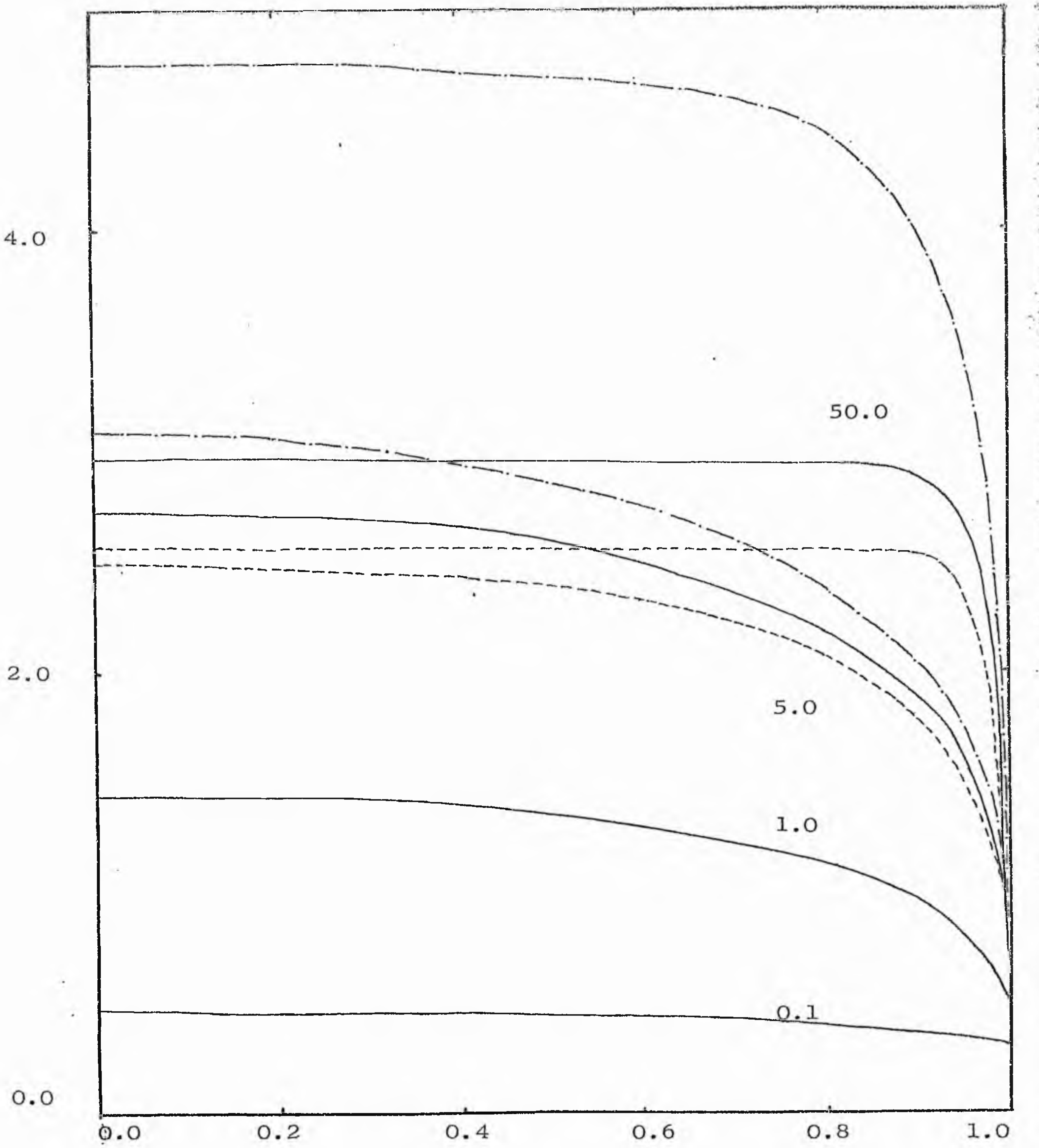


FIG. 51. As Fig. 49 for $n = 10^{-2}$.

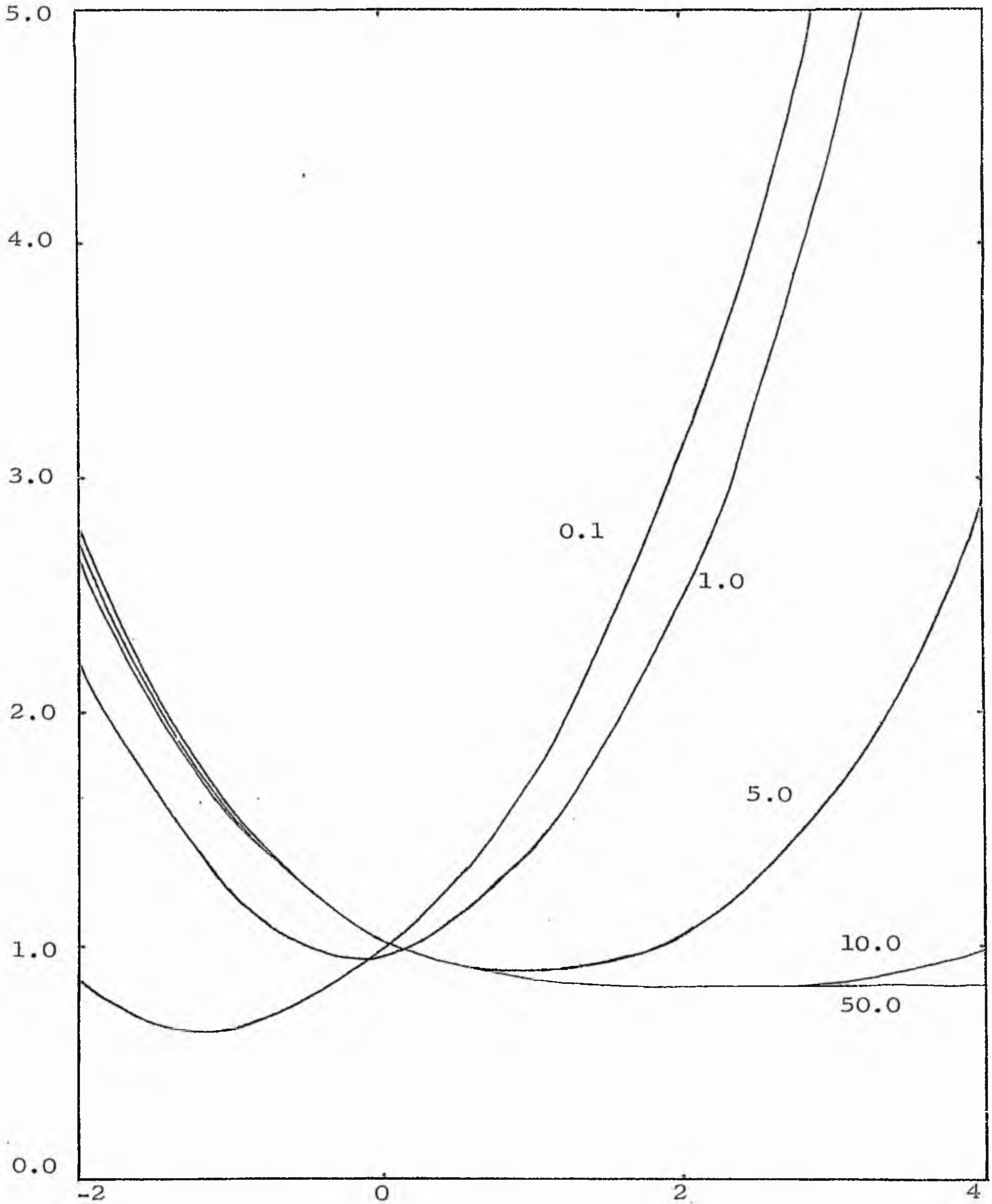


FIG. 52. The central temperatures of a spherical atmosphere as a function of n for several values of n.

The abscissa is $\log_{10} n$. There is no scattering of the stellar radiation and the values of τ_0 are indicated on the figure.

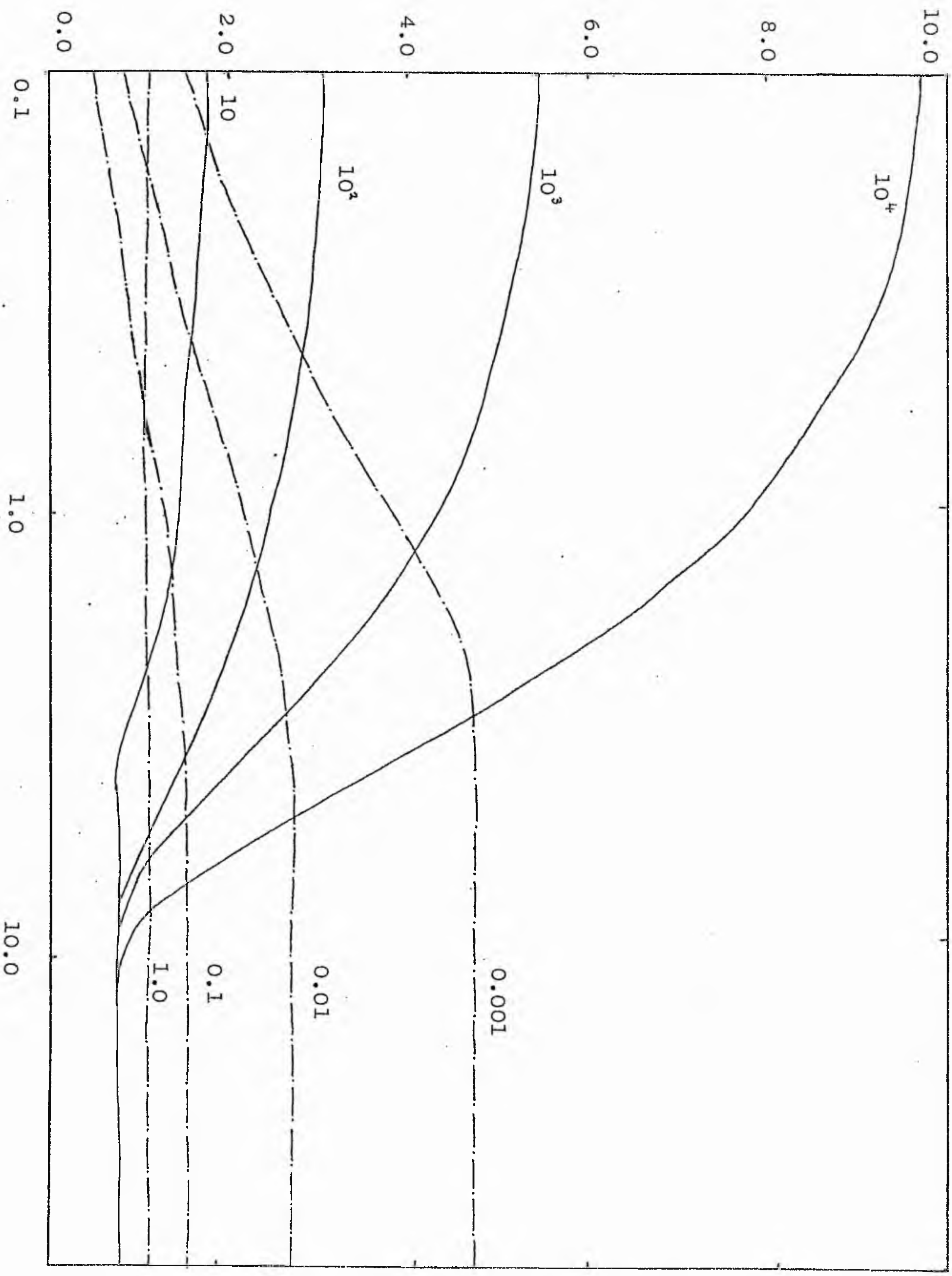


FIG. 53. The central temperatures of a spherical atmosphere as a function of optical radius for several values of n.

There is no scattering of the stellar radiation, and the values of n are indicated on the figure.

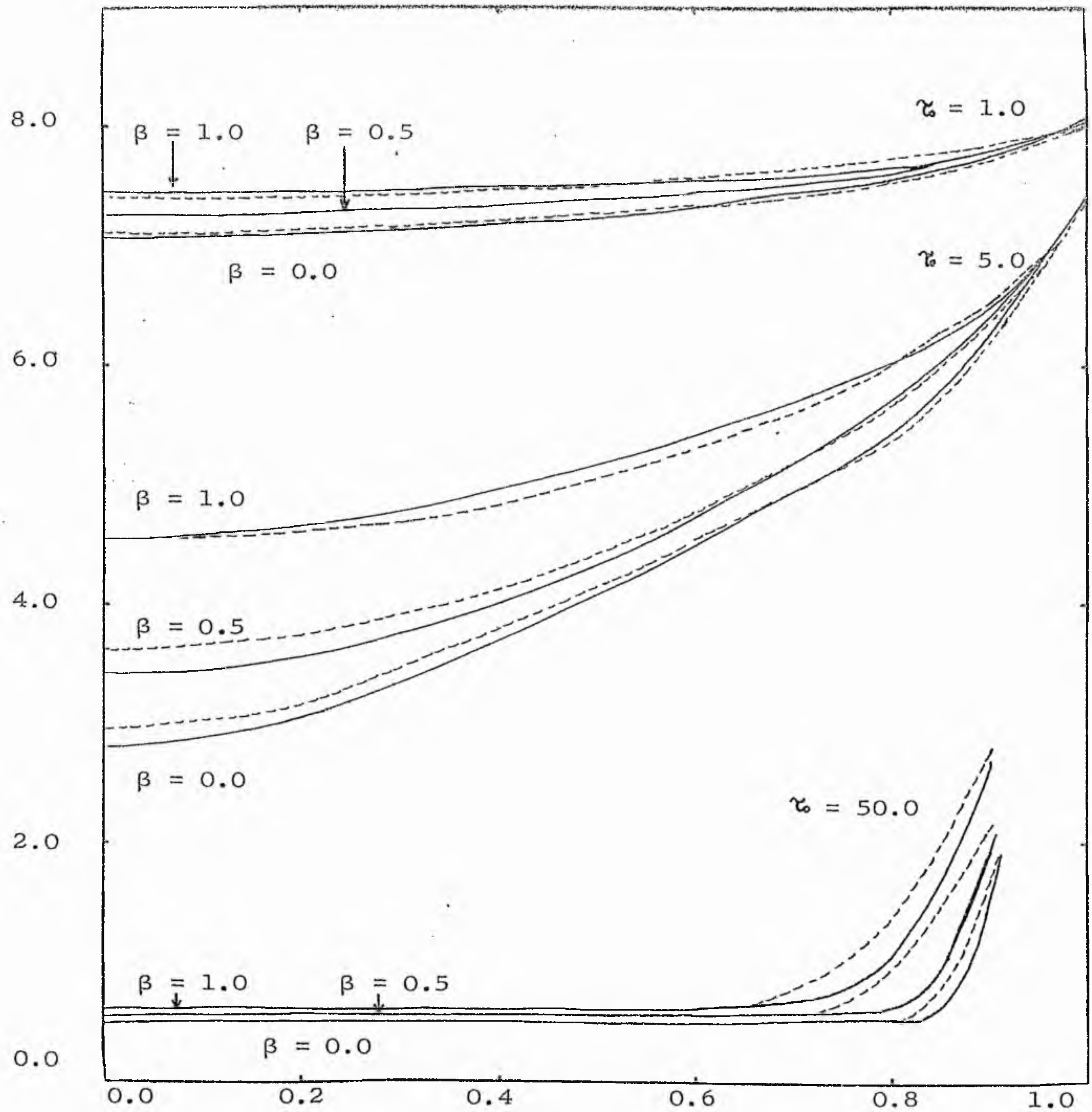


FIG. 54. A comparison of the temperature profiles of a spherical atmosphere as obtained by methods I and II when n is 10^4.

The abscissa is τ/τ_0 and the results from methods I and II are represented by continuous and dashed curves respectively. The scattering has $\tilde{\omega}$ and α equal to 0.9 and 0.0 respectively in all cases, and the values of τ_0 and β are shown on the figure.

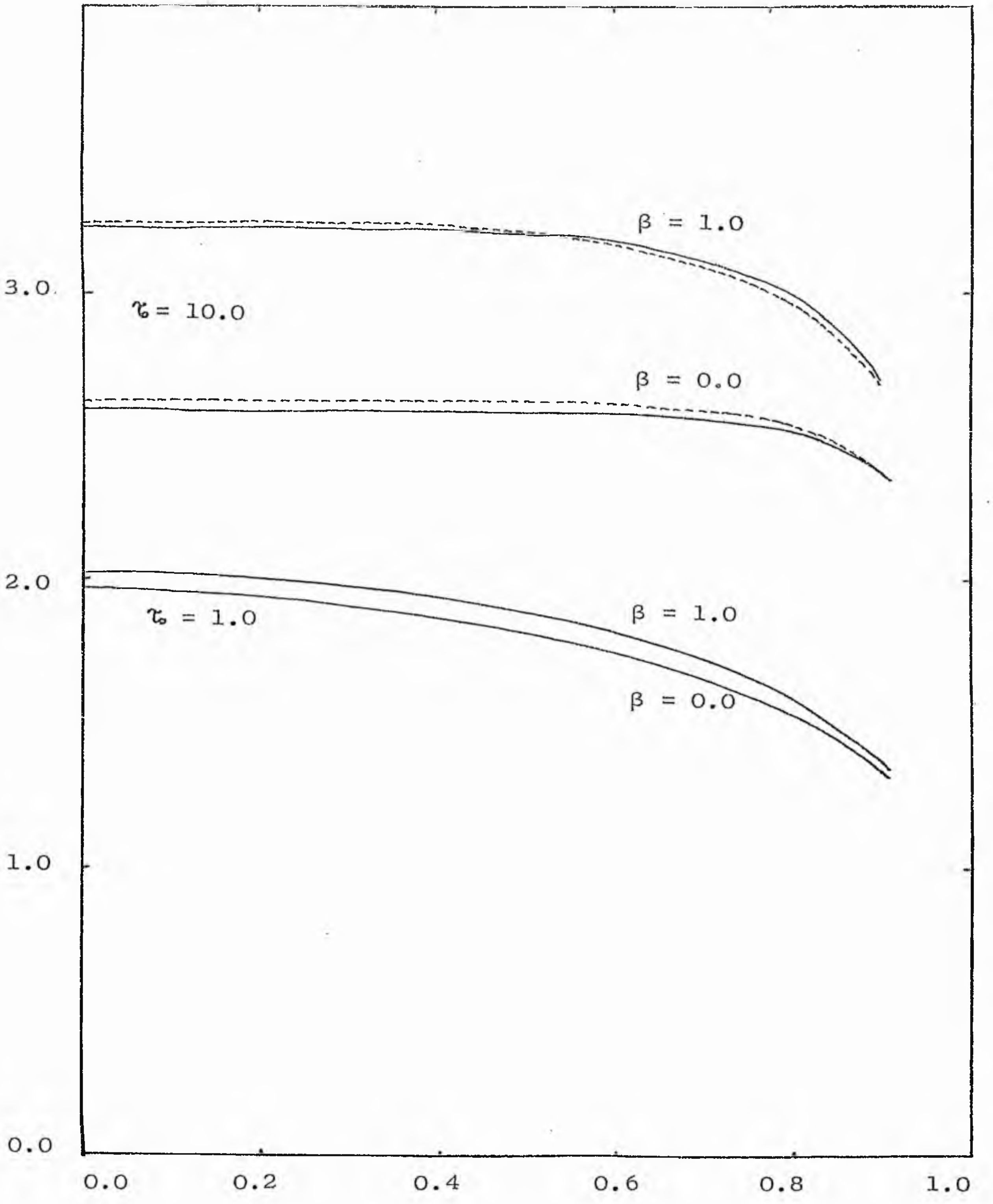


FIG. 55. As Fig. 54 for $n = 10^{-2}$.

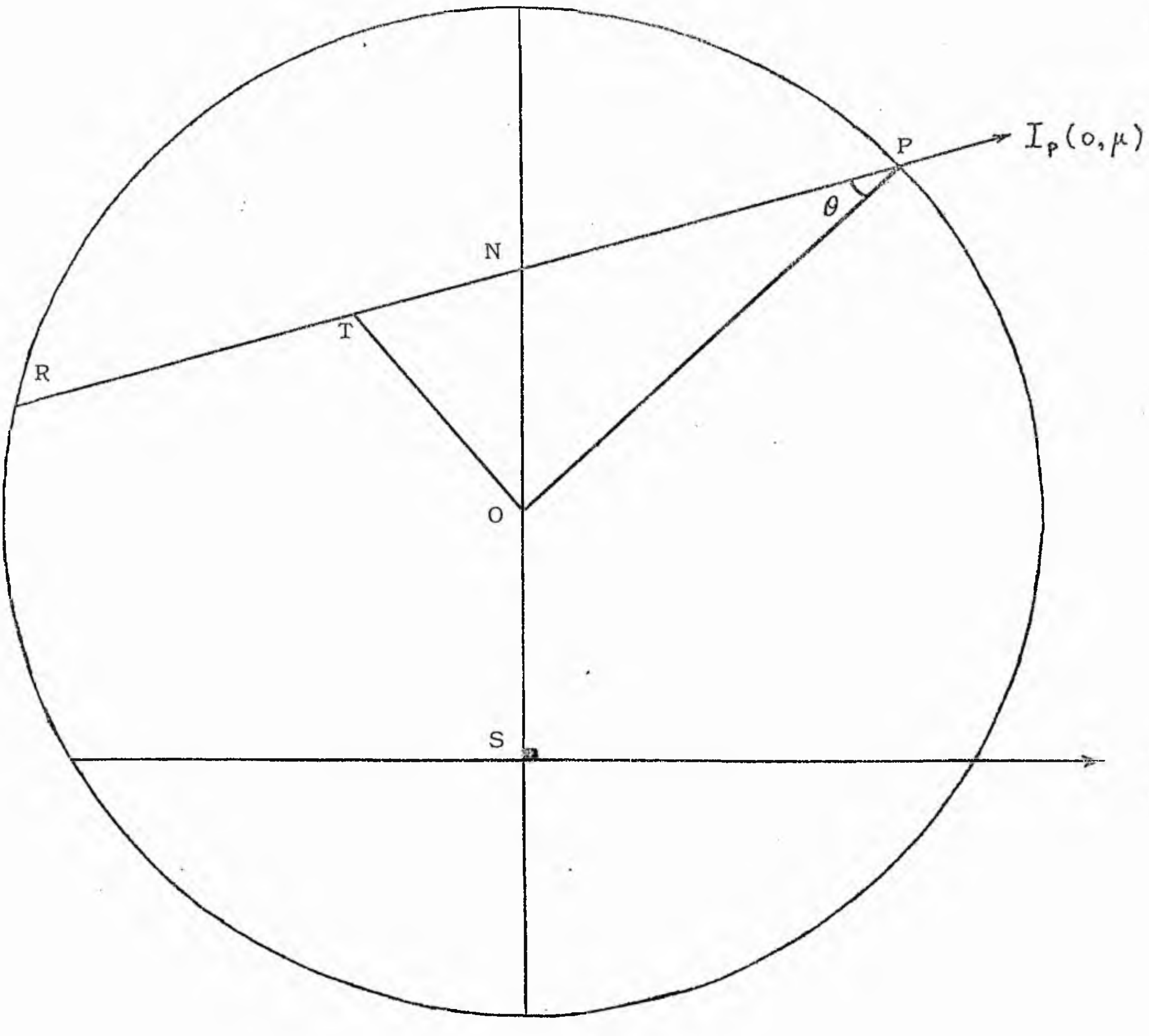


FIG. 56. The geometry of the spherical atmosphere to show the construction of the emergent radiation. $\mu = \cos\theta$.

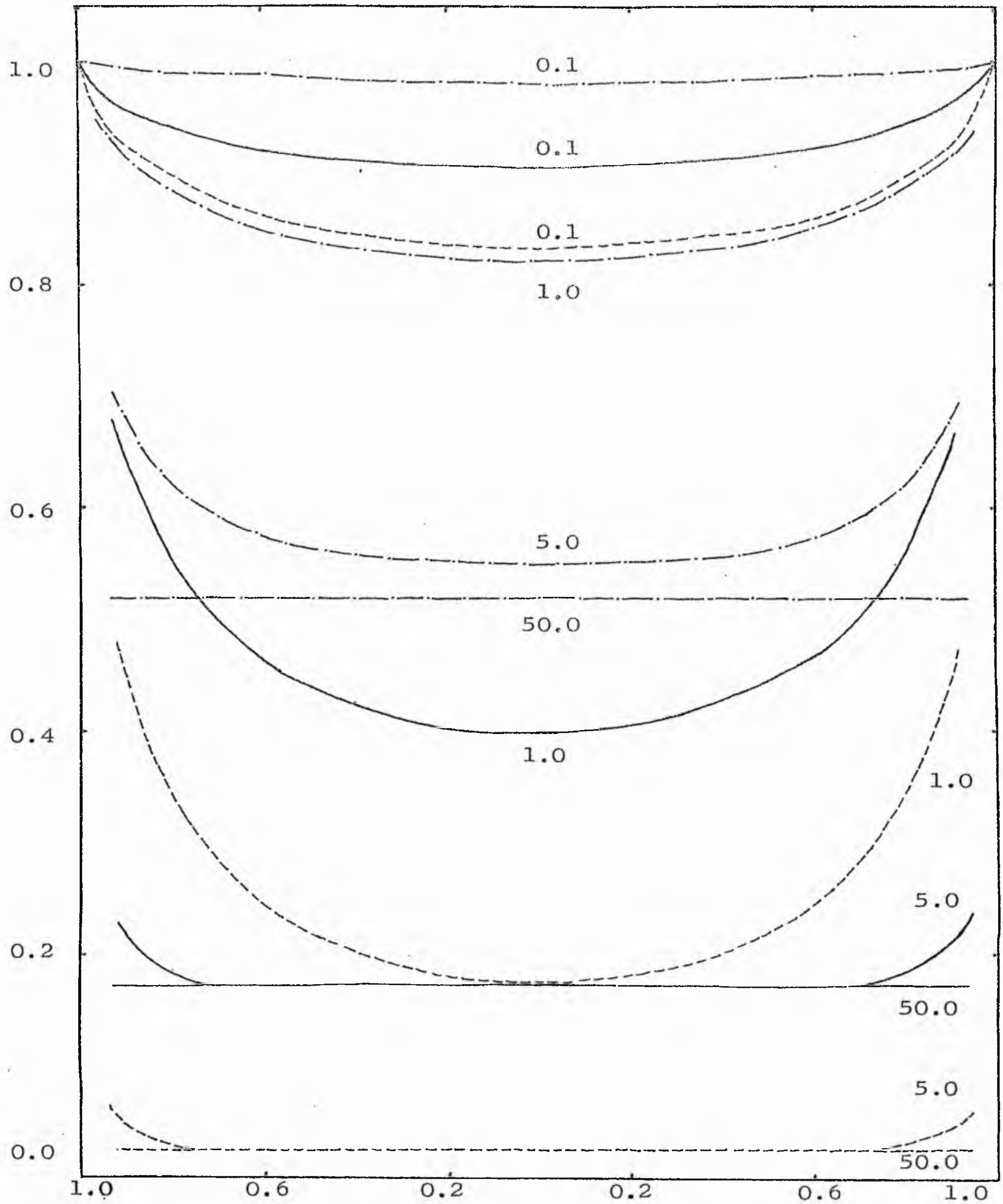


FIG. 57. The angular distribution of the emergent stellar radiation from a spherical atmosphere.

The ordinate is the sum of the intensities of the stellar radiation fields and the abscissa is the fractional optical radius of a disc projection of the atmosphere. The scattering phase function is $(\alpha,\beta) = (0.0,0.5)$. The values of τ_0 are indicated on the figure whilst albedos of 0.1, 0.5 and 0.9 are represented by broken, continuous and dashed curves respectively.

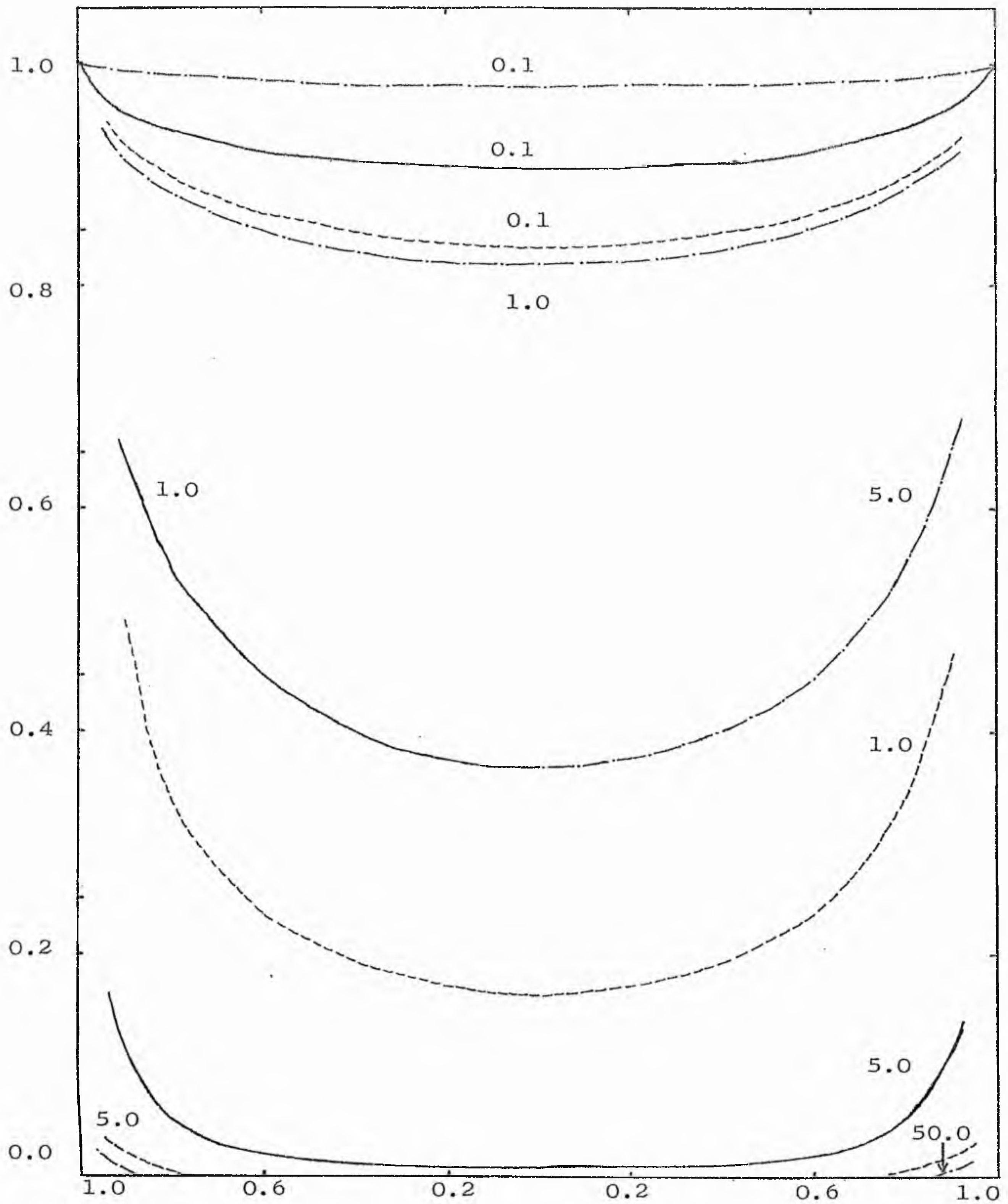


FIG. 58. As Fig. 57 for the phase function, $(\alpha,\beta) = (0.0,1.0)$.

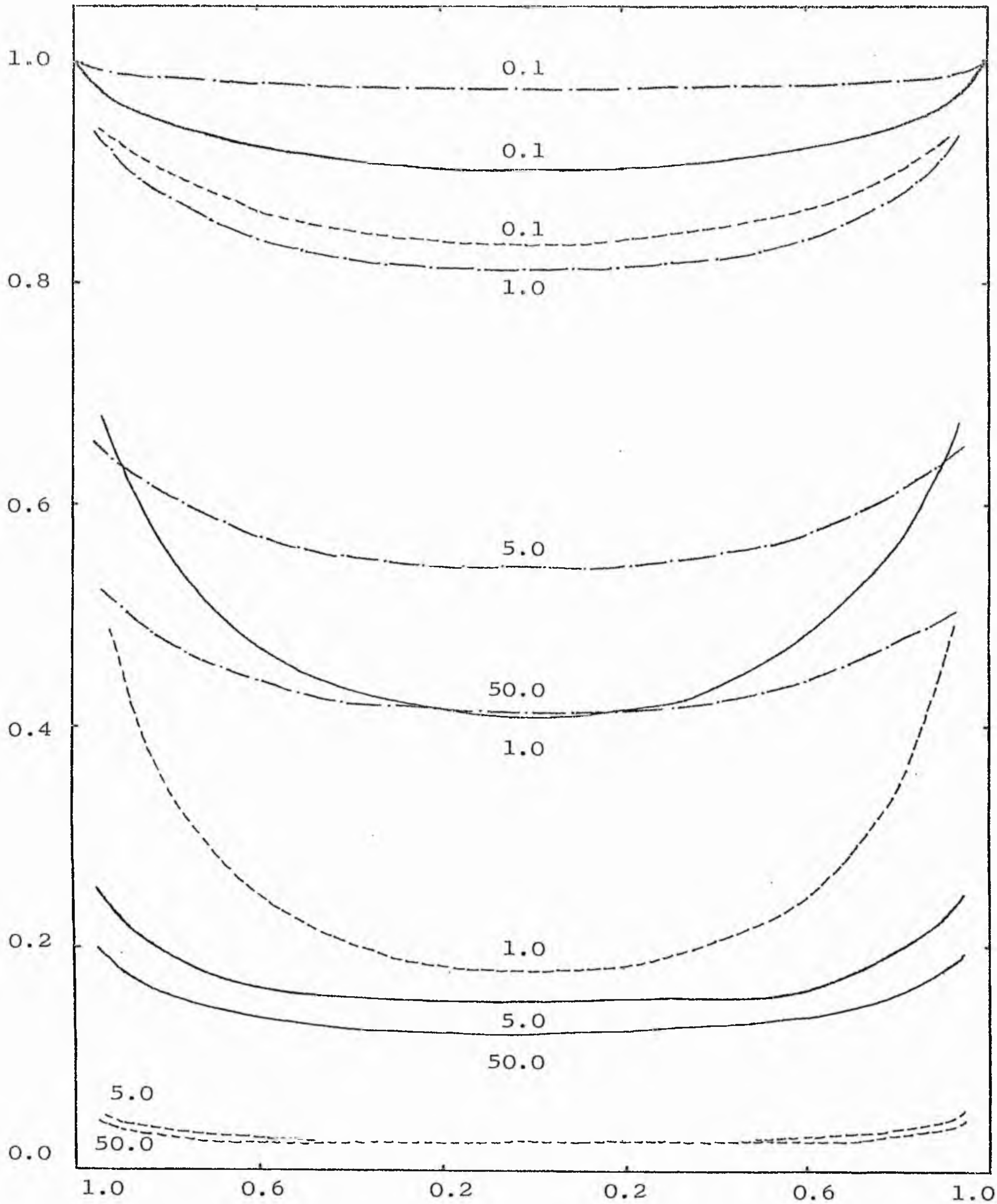


FIG. 59. As Fig. 57 for isotropic scattering.

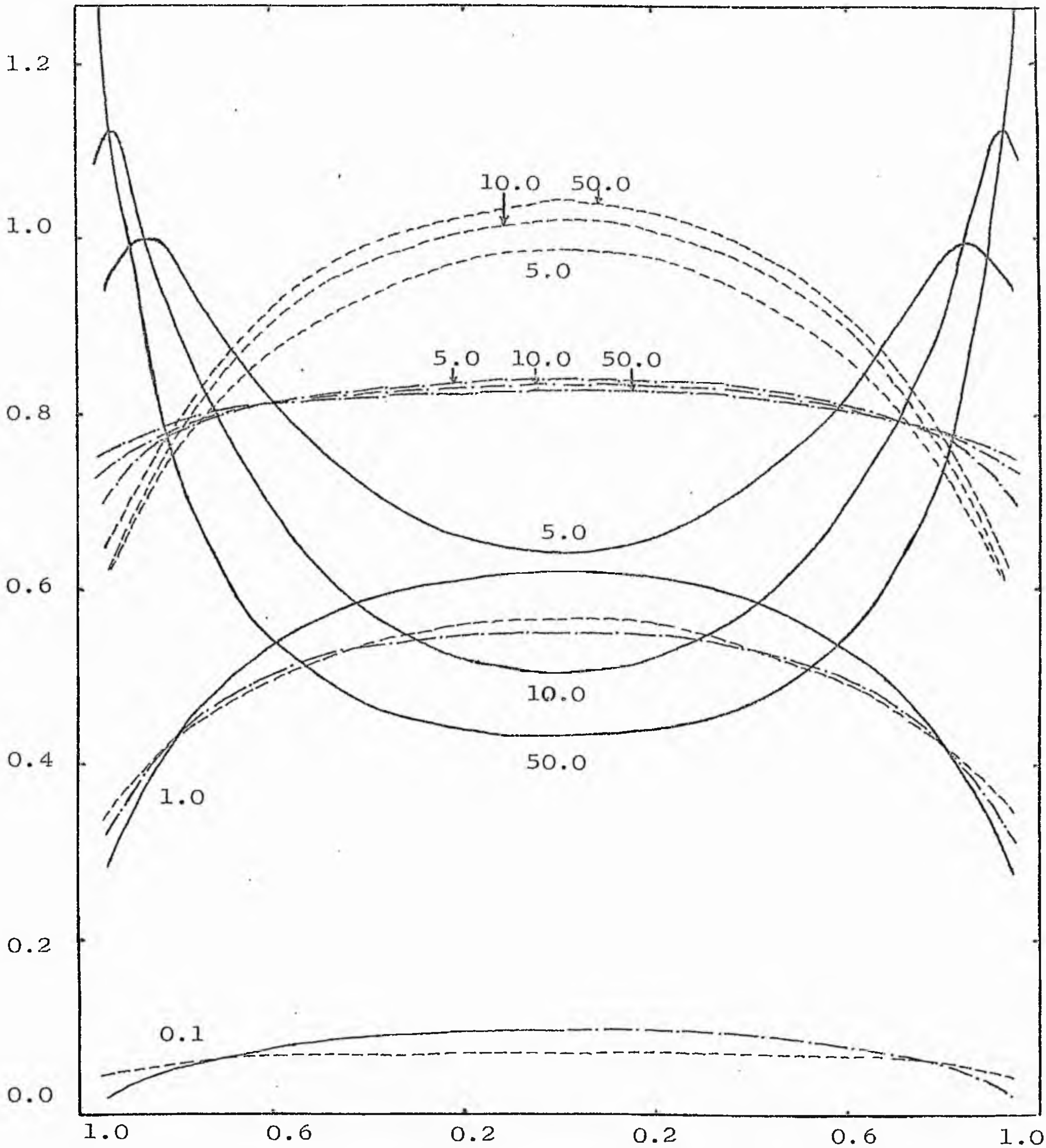


FIG. 60. The angular distribution of the emergent thermal radiation from a spherical atmosphere for three values of n.

The ordinate is the intensity of the emergent thermal radiation and the abscissa is the fractional optical radius of a disc projection of the atmosphere. The values of n of 10^{4}, 1.0 and 10^{-2} are represented by continuous, broken and dashed lines respectively whilst values of τ_0 are shown on the figure. The scattering is isotropic and of albedo, 0.5.

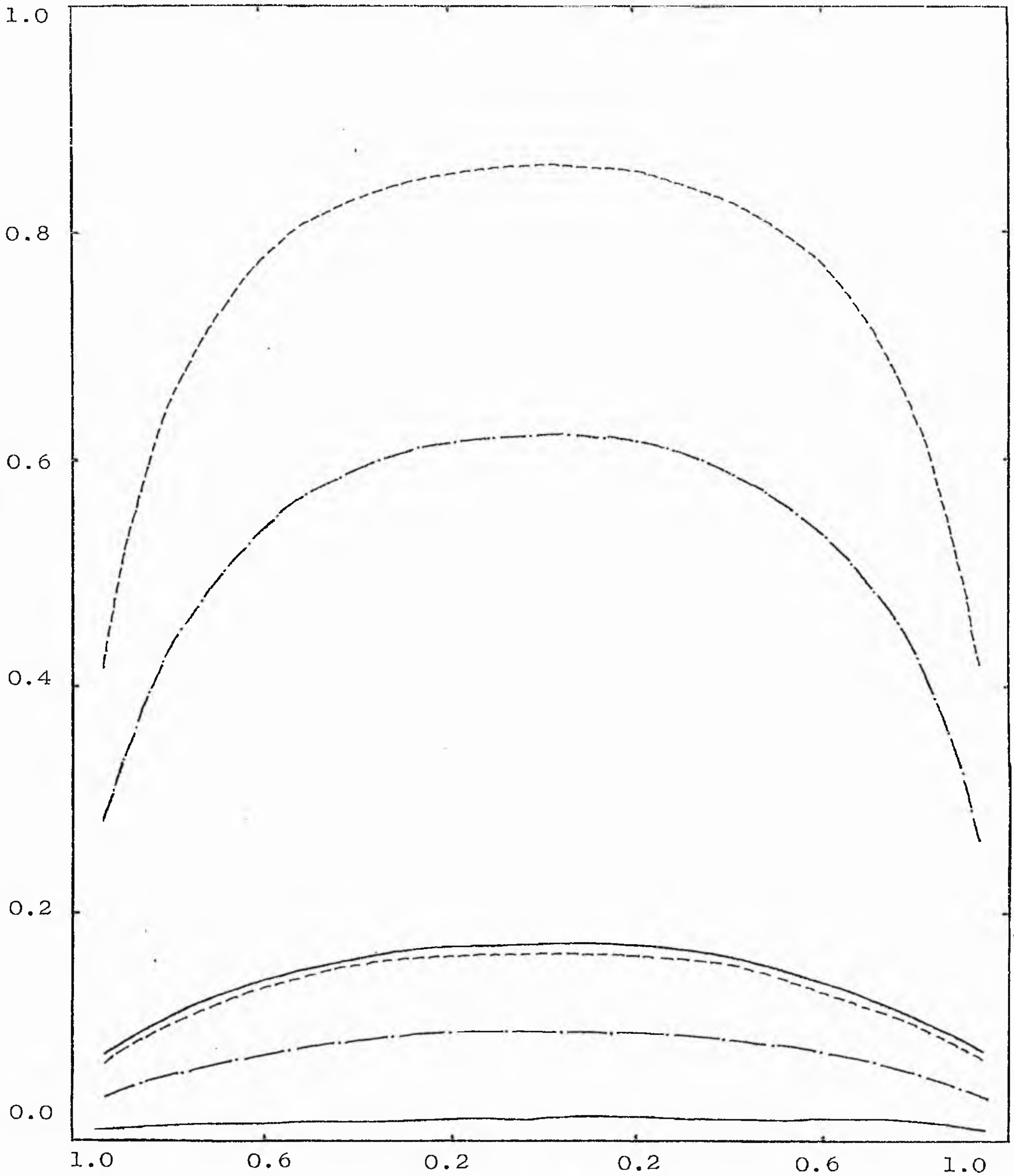


FIG. 61. As Fig. 60 for $n = 10^4$ and optically thin atmospheres of a number of albedos.

The continuous, broken and dashed curves refer to albedos for isotropic scattering of 0.9, 0.5 and 0.1 respectively. The upper curve in each case refers to $\tau_0 = 1.0$ and the lower to $\tau_0 = 0.1$.

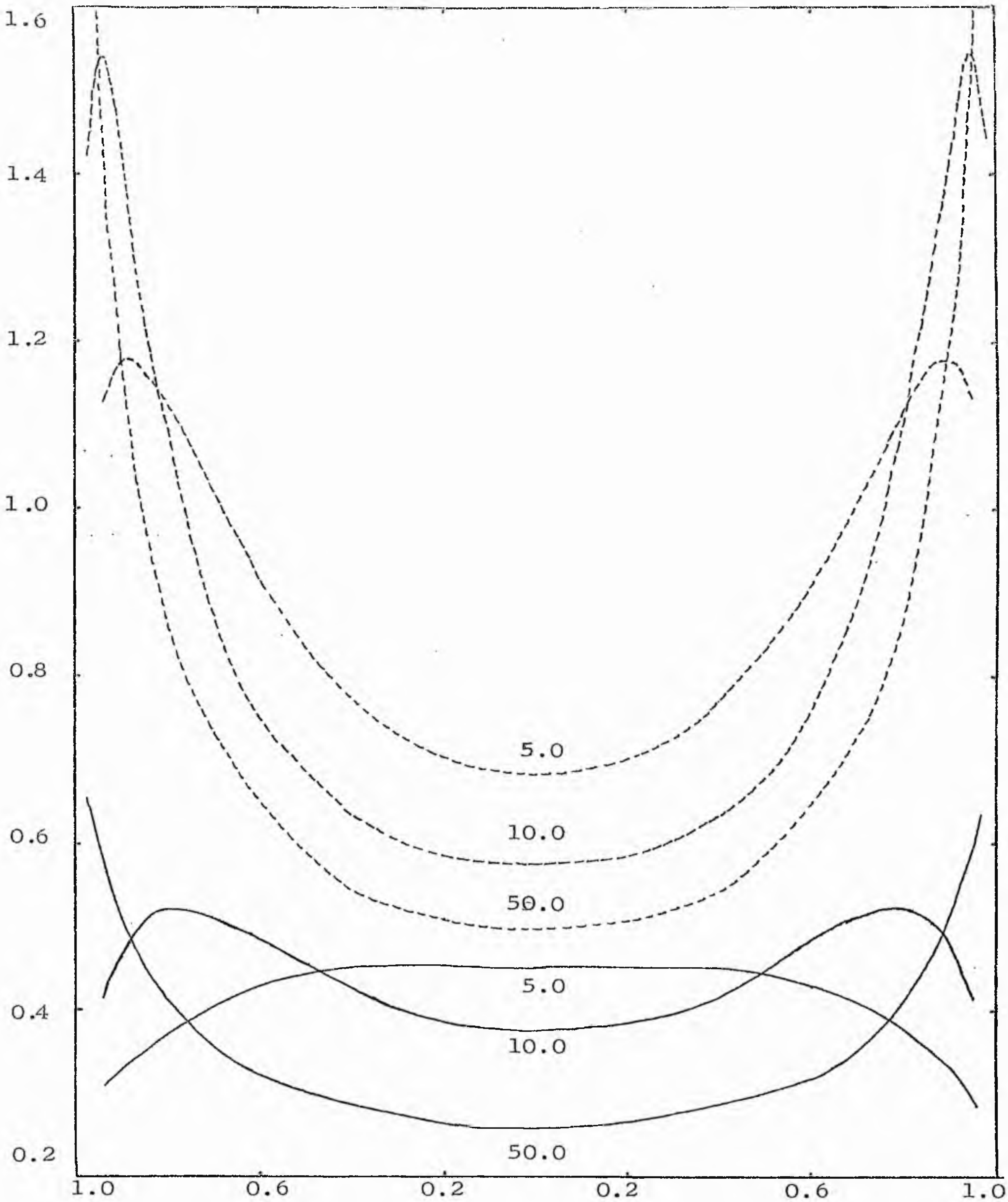


FIG. 62. As Fig. 61 for optically thick atmospheres.
The values of τ_0 are shown on the figure and the curves for albedo, 0.5 are omitted.

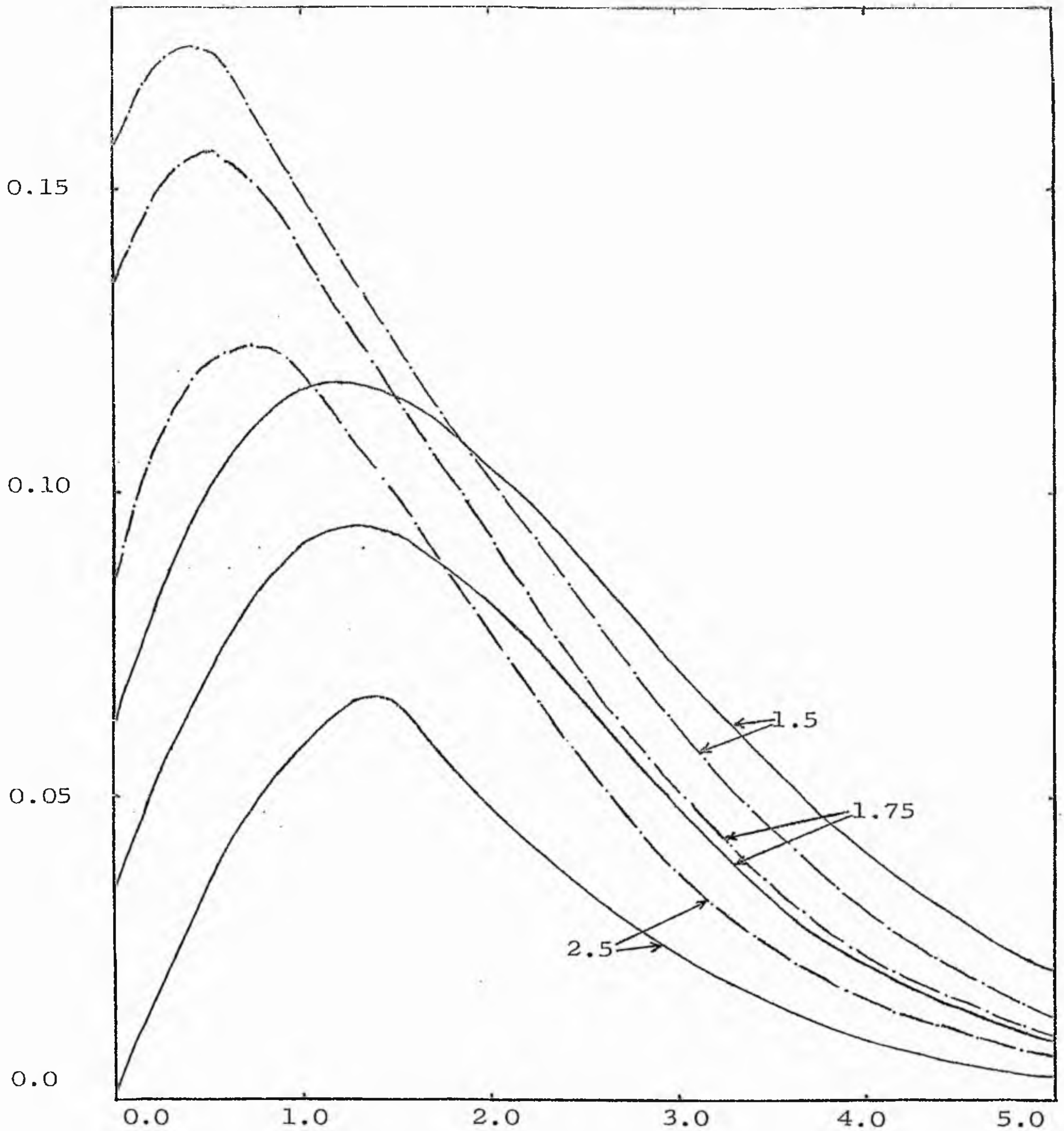
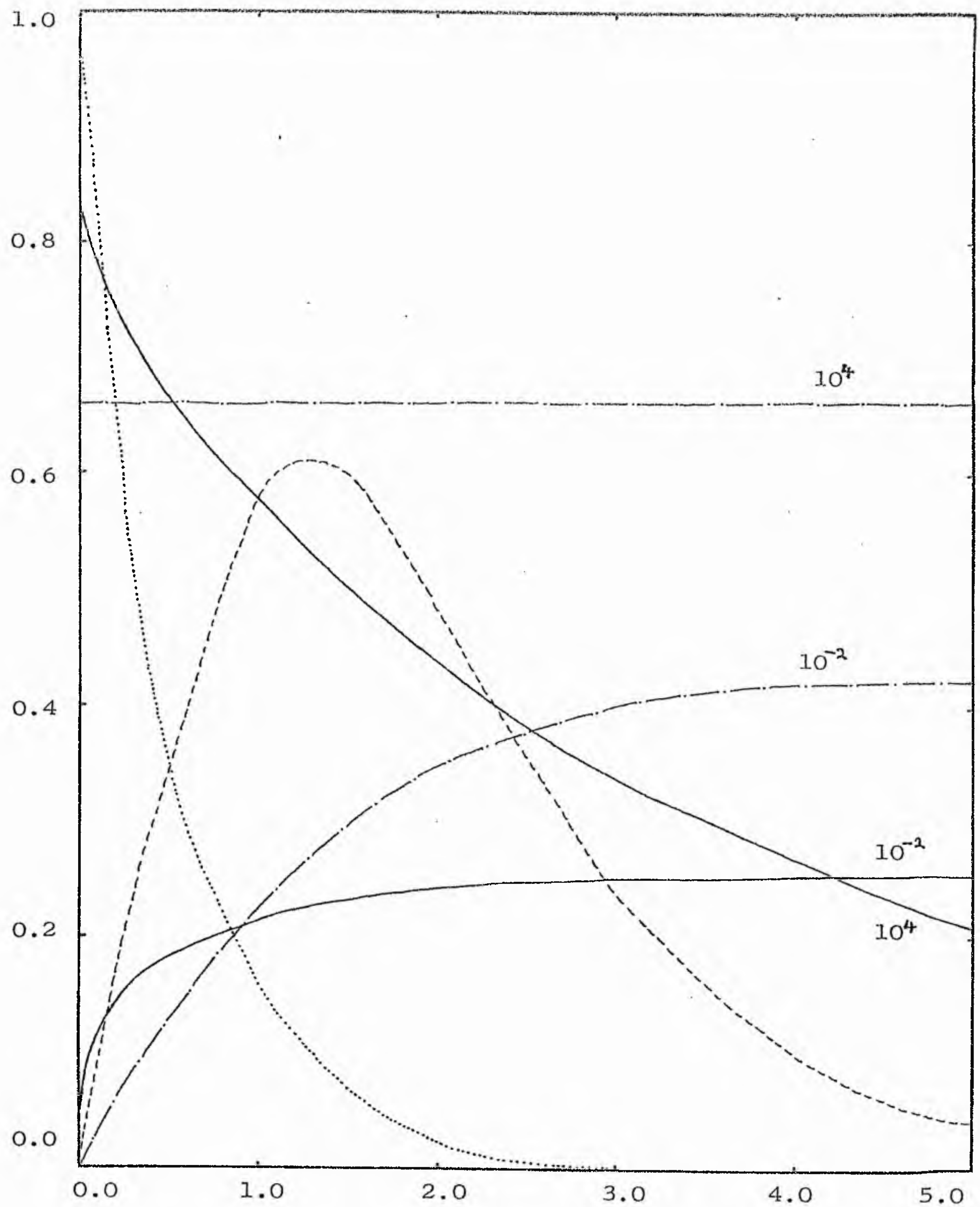


FIG. 63. The mean intensity of the scattered radiation in a spherical atmosphere of variable density function, $\rho(r) = a r^{-m}$ as a function of optical depth.

The scattering is isotropic with albedo, 0.5 and the value of a is 1.0. The continuous curves represent RT = 10 and the broken curves, RT = 1.0 where RT is the geometrical radius of the atmosphere. The values of m are shown on the figure.

FIG. 64. The temperature and mean intensities of the radiation fields in a spherical atmosphere of variable density function, $\rho(r) = ar^{-m}$ as a function of optical depth.

The temperatures are shown as continuous curves, the mean intensities of the thermal radiation fields as broken curves, the mean intensity of the scattered radiation field as a dashed curve and the mean intensity of the reduced incident radiation field as a dotted curve. The values of n are shown on the figure where appropriate. The scattering is isotropic, the albedo, 0.5, RT = 1.0, m = 2.5 and a = 1.0. The ordinate must be multiplied by factors of 0.1 for the mean intensities of the scattered radiation field and the thermal radiation field when n is 10^4, by a factor of 10 for the temperatures, and by a factor of 10^2 for the mean intensity of the thermal radiation field when n is 10^{-2}.


1.0
0.8
0.6
0.4
0.2
0.0
0.0
1.0
2.0
3.0
4.0
5.0
10^4
10^{-2}
10^{-2}
10^4

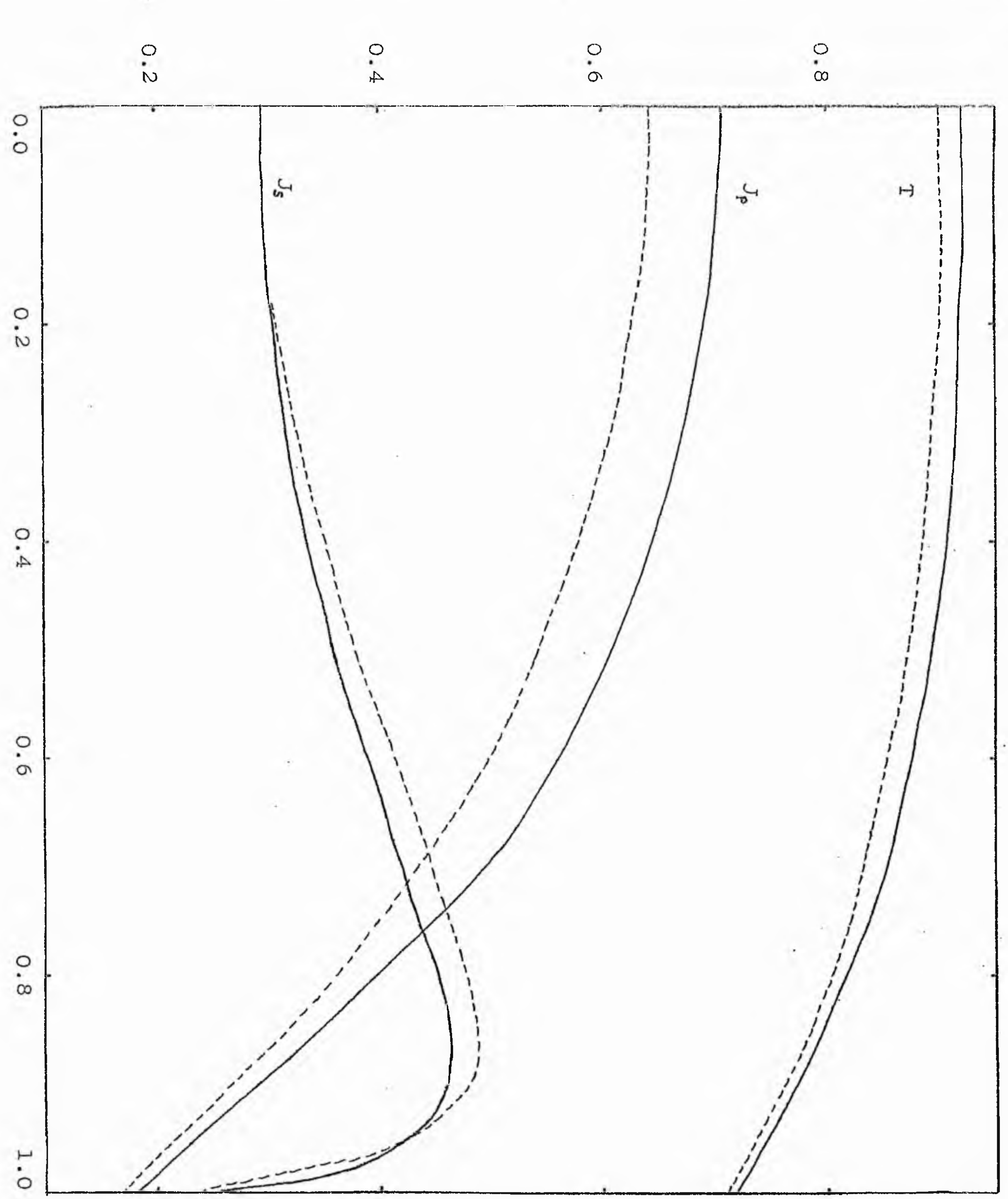


FIG. 65. A comparison of the temperature and mean intensities of the scattered and thermal radiation fields in a spherical atmosphere of variable density, $\rho(r) = \rho_o (1.5 - r)$ with those in a constant density spherical atmosphere.

The continuous curves apply to the variable density atmosphere and the dashed curves to the constant density atmosphere. The abscissa is the fractional optical radius, and the atmospheric parameters are; $n = 1.0$, $\tilde{\omega} = 0.9$, $\alpha = 0.0$, $\beta = 0.5$ and $\tau_o = 5.0$.

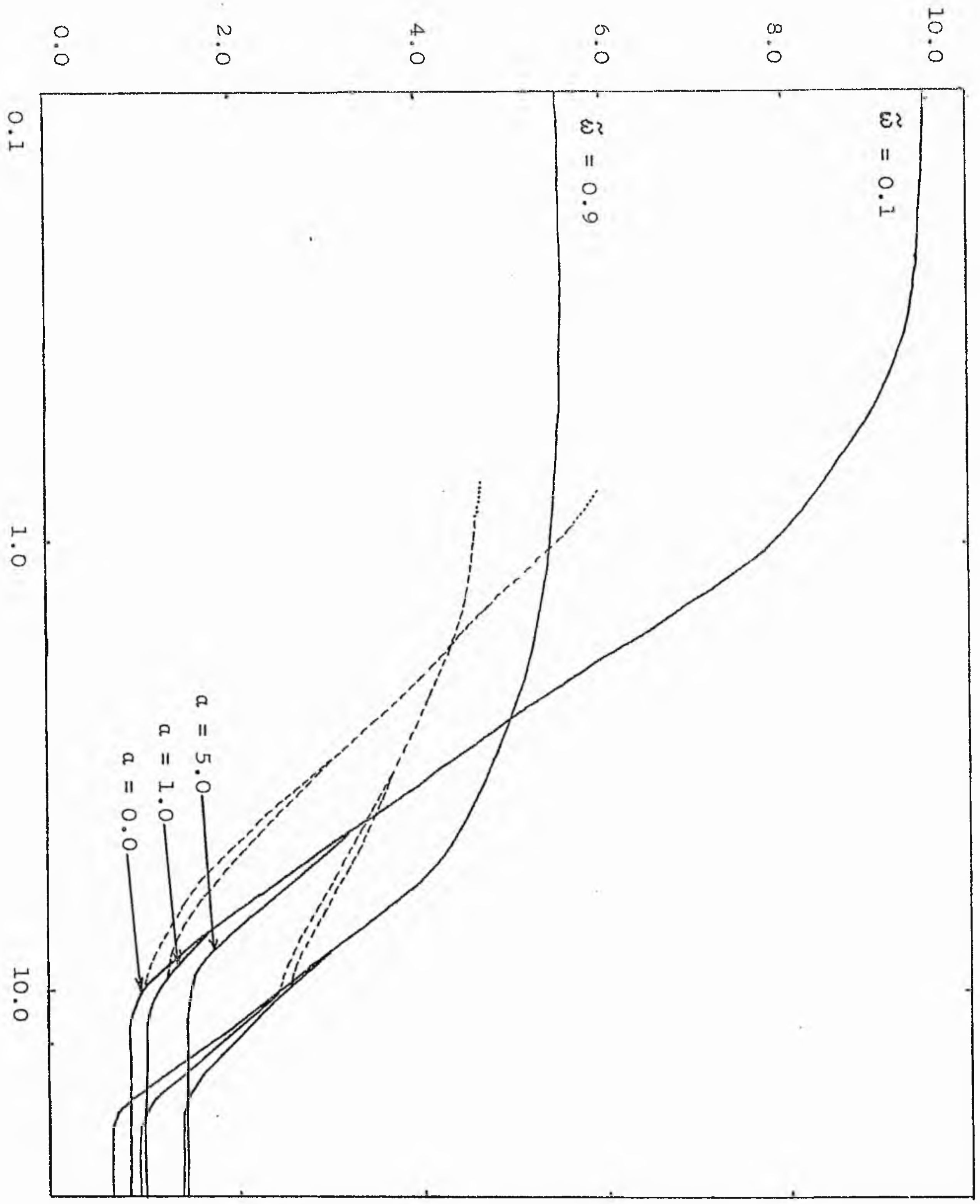


FIG. 66. The effect of an additional incident thermal radiation field of intensity, αI_0 on the temperatures of a spherical atmosphere.

The continuous curves represent the central temperatures as functions of the optical radius of the atmosphere and the dashed curves represent the internal temperatures of atmospheres of $\tau_0 = 10.0$ as functions of optical depth. They are not terminated because the scale of the abscissa is logarithmic. The scattering is isotropic, $n = 10^4$, and the values of the albedo and α are indicated on the figure.

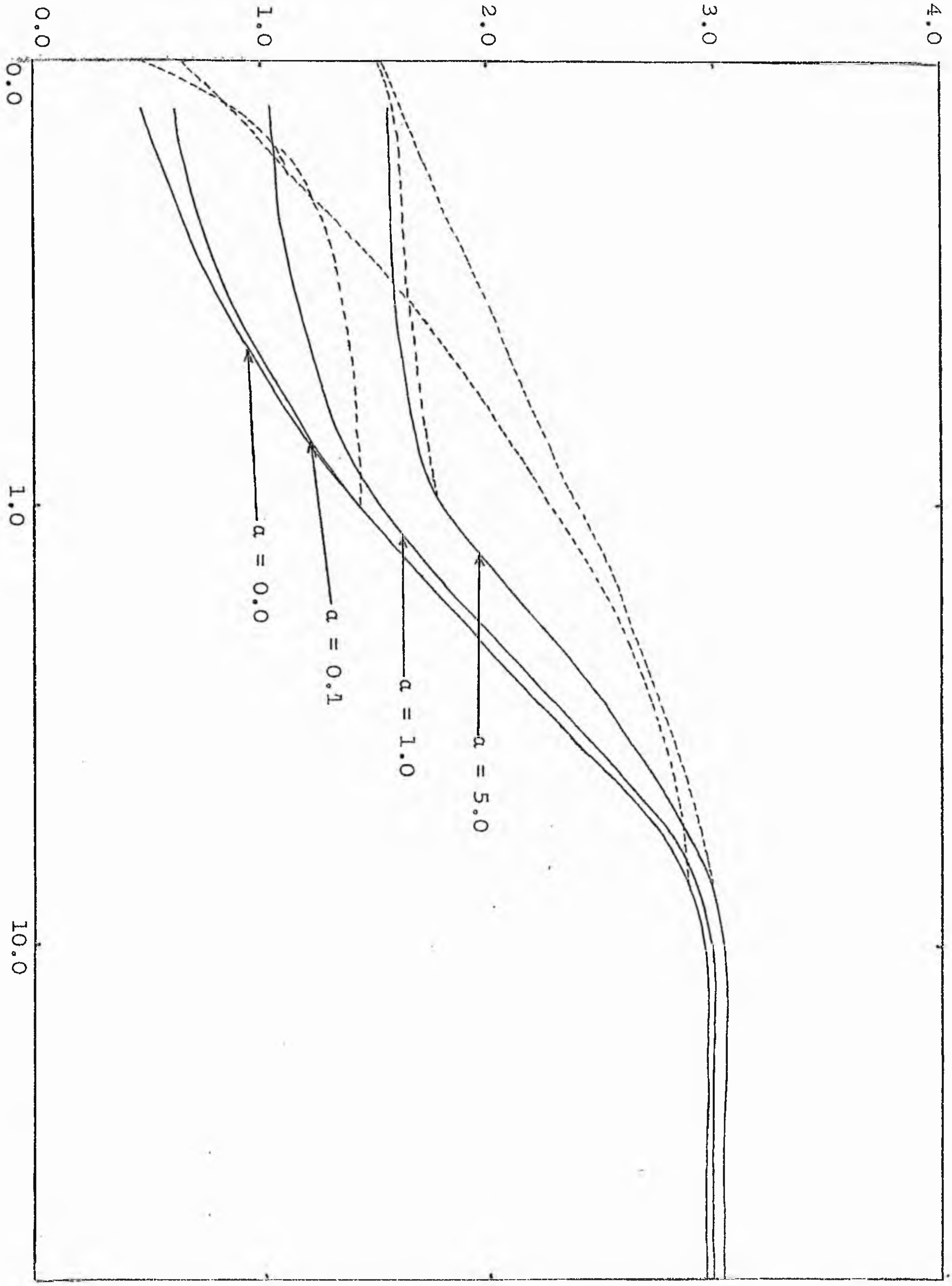


FIG. 67. As Fig. 66 for $n = 10^{-2}$. The scale of the abscissa is now linear below unity and the temperature profiles of atmospheres of $\tau_0 = 1.0$ and 10.0 are shown, but for $\tilde{\omega} = 0.9$ only.

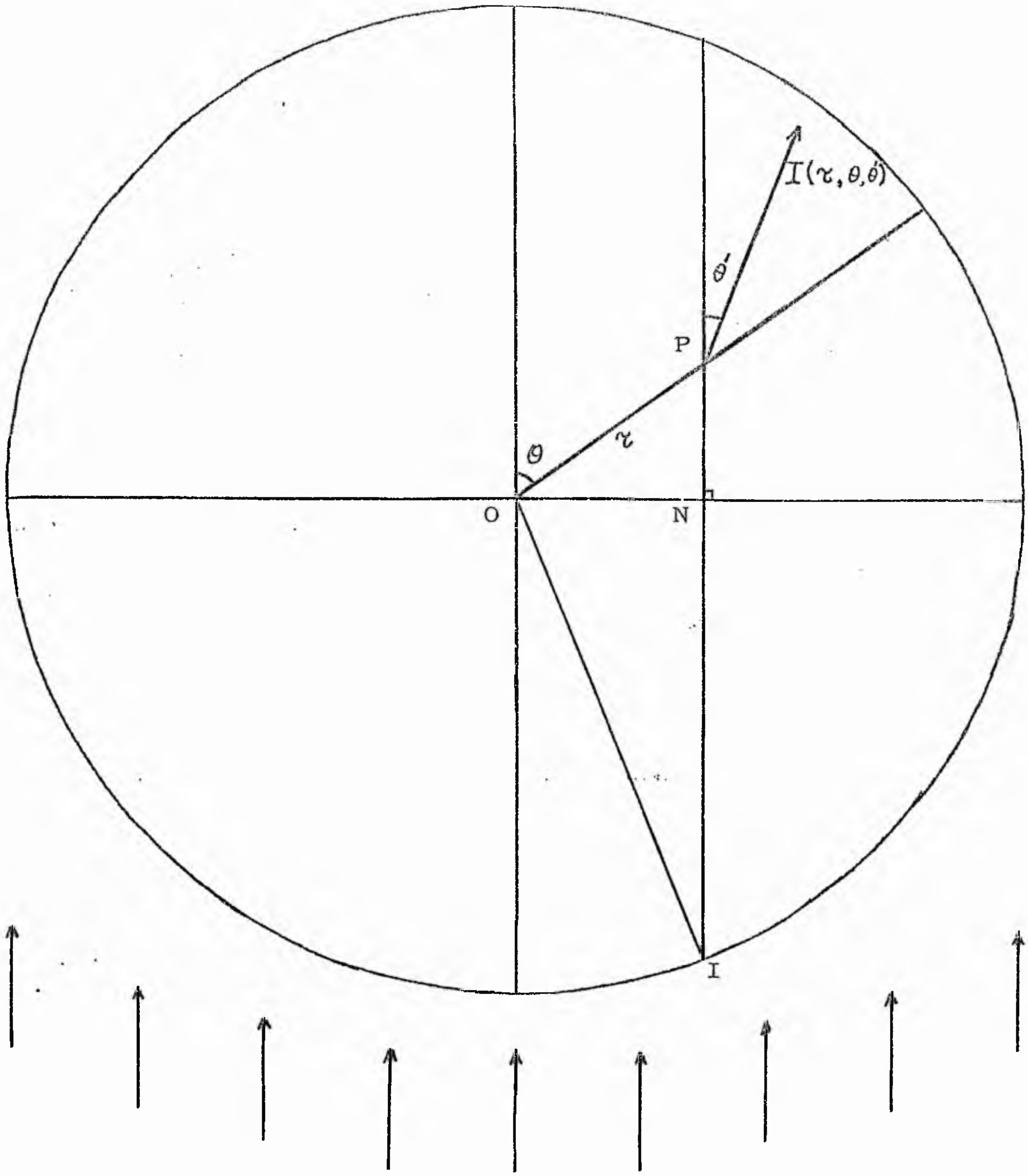


FIG. 68. The geometry of a spherical atmosphere illuminated by parallel radiation.

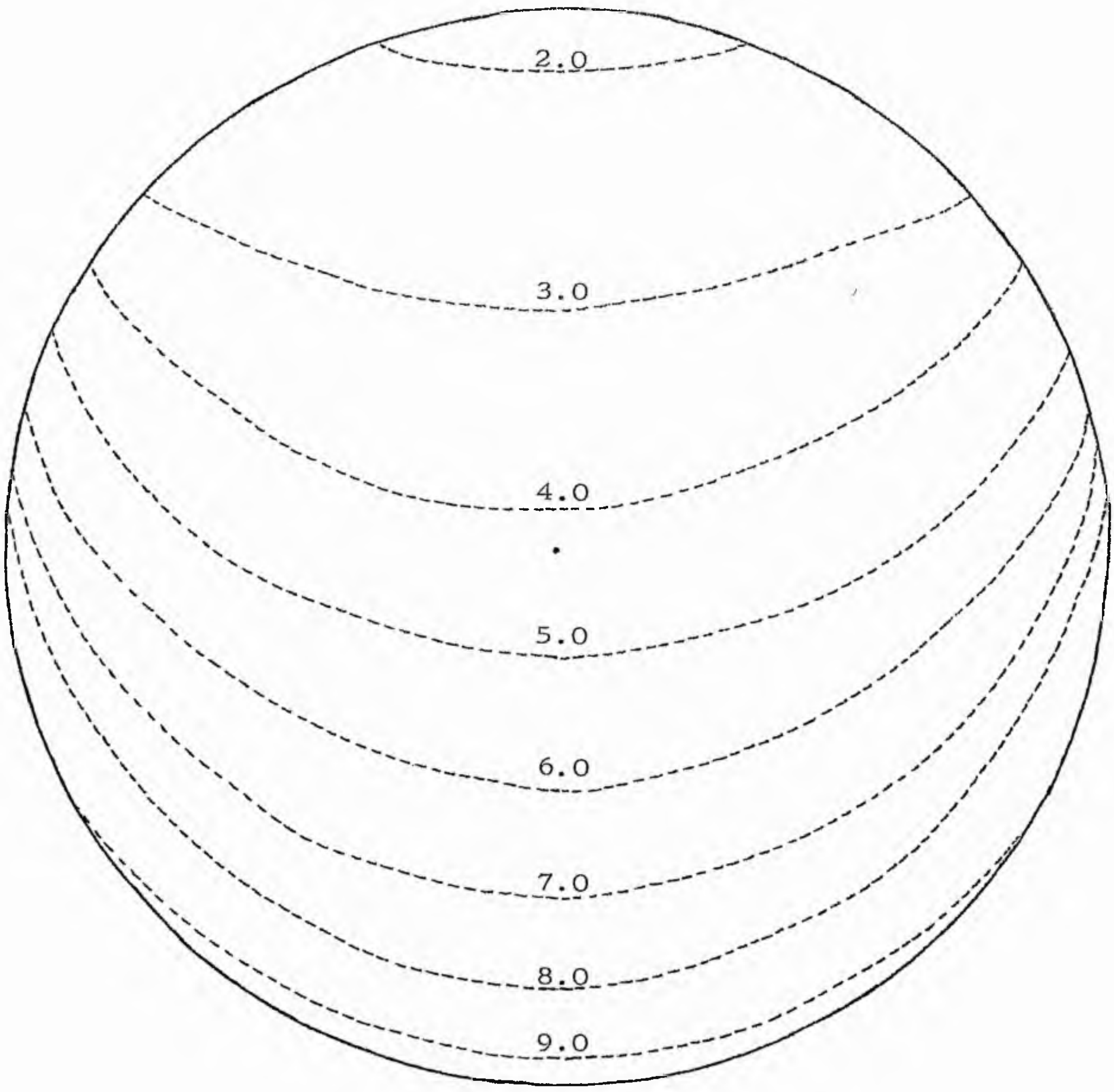


FIG. 69. The temperature contours in a spherical atmosphere illuminated by parallel radiation as in Fig. 68.

The atmospheric parameters are; $n = 10^4$, $\tau_0 = 10.0$, $\tilde{\omega} = 0.9$, $\alpha = 0.0$, and $\beta = 0.5$. The temperatures are indicated on the figure.

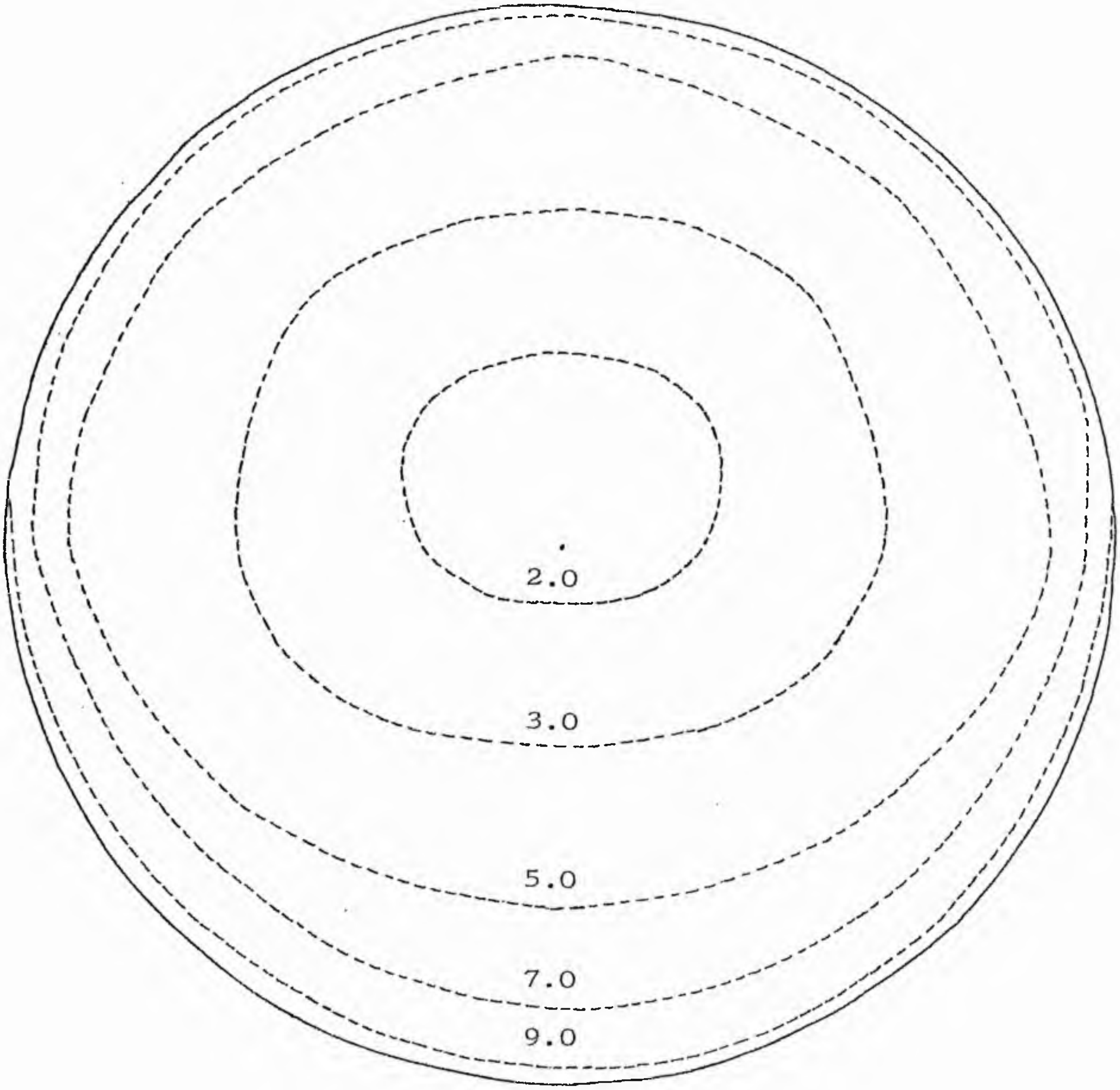


FIG. 70. The temperature contours of a spherical atmosphere illuminated by both parallel and uniform isotropic incident radiation.

The atmospheric parameters are; $n = 10^4$, $\tau_0 = 10.0$, $\tilde{\omega} = 0.9$, $\alpha = 0.0$ and $\beta = 0.5$. The temperatures are indicated on the figure.