GEOMETRIC QUANTISATION AND QUANTUM MECHANICS IN DIRAC'S FRONT FORM

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A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1994

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A thesis submitted for the degree of
Doctor of Philosophy in
the University of St. Andrews by

J.J. Powis

St Leonard's College

December 8, 1993

Declaration

I, J.J. Powis hereby certify that this thesis is a record of the work carried out by me, and that it has not been submitted in any previous application for a higher degree.

December 8th 1993 J.J. Powis

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate to the degree of Doctor of Philosophy in the University of St. Andrews and that he is qualified to submit this thesis in application for that degree.

December 8th 1993 K.K.Wan

Reasearch Supervisor

Declaration

I was admitted as a reasearch student under ordinance No. 12 on 1st October 1990, and as a candidate for the degree of PhD on 1st October 1991; the higher study of which this is a record was carried out in the University of St. Andrews between 1990 and 1993.

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Declaration

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Acknowledgements

I would like to thank my supervisor K.K. Wan for giving me the benefit of his fecund mind and the patience with which he has endured the role of irresistible force to my immovable object. I would also like to thank Fiona Harrison....for her example.

To Mum, Dad and Sam

Forsan et haec olim meminisse juvabit.

VIRGIL

Failure after long perseverance is much grander than never to have a striving good enough to be called a failure.

GEORGE ELIOT; *Middlemarch*

Ibi omnis effusus labor!

THOMAS DE QUINCEY; *Confessions of an English Opium Eater*

Abstract

We give a brief review of geometric quantisation up to and including the Blattner-Kostant-Sternberg kernal. In general this leads to symmetric operators that are not essentially self-adjoint so motivating a study of Hermitian operators as observables in a generalised quantum mechanics. We show that a generalised squaring axiom can reproduce the results of Blattner-Kostant-Sternberg quantisation. We also show that quantisation with respect to polarisations with compact leaves gives results that conflict with the nonlocal nature of quantum mechanics.

We develop a front form quantum mechanics of a free scalar particle using geometric quantisation. The front and instant forms are related via unitary maps derived from the pairing which intertwines quantisations with respect to the these forms. The front form position operator has a maximally symmetric component so we are compelled to work within the framework of a generalised quantum mechanics; this results in there being no Hegerfeldt type instantaneous spreading of initially localised wavefunctions in the front form. Finally we show that this model can be lifted to a many particle free field theory.

Contents

Introduction

This thesis is divided into three Chapters.

In Chapter 1 we give a brief review of the theory of geometric quantisation both the basic scheme, that concerned with the quantisation of classical observables linear in momentum and the formulae of Bao and Zhu and of Tuynman based on the so called Blattner-Kostant-Strenberg Kernal. We show that the methods of Bao and Zhu and Tuynman are inequivalent and in any case often give results that can equally well be obtained using the basic scheme and a generalised squaring axiom. The fact that they lead to operators that are not essentially self-adjoint motivates a study of symmetric operators as observables in a generalised quantum mechanics. Chapter 1 also contains, among other things, a comment on the problems pertaining to quantisation with respect to polarisations with compact leaves which seems to give a theory that is local and therefore in conflict with E.P.R type phenomena. This is essentially the content of a paper published in Algebras, Groups and Geometries [79].

In Chapter 2 we apply the theory of geometric quantisation to obtain a front form quantum mechanics that is surprisingly consistent. The spectra of the operators are contained in the range of the classical observables they represent and the instant and front form pictures are unitarily equivalent via maps derived from the pairing. The front form position operator is maximally symmetric, its role as a well defined observable is assured by the methods of Chapter 1. Finally we show that the front form is free of Hegerfeldt type instantaneous spreading of initially localised wavefunctions. Chapter 2 also contains a critique of the point form and a brief analysis of the dynamic and kinematic subgroup structures of the various forms. In particular we show that the generators of the kinematic subgroups of the instant and front forms, as well as the Hamiltonians, can be quantised in either picture to obtain representations that are unitarily related by the pairing maps. Some of this work will also appear in The International Journal of Theoretical Physics [80] and the Proceedings of The Third Wigner Symposium [93].

Chapter 3 contains a front form field theory of a free scalar particle. Since the Lagrangian is singular we are led to define a Dirac bracket that modifies the usual equal time commutator; the expression we derive in this way appears as an ansatz in other front form field theories. We show that the theory is relativistically invariant despite our breaking of manifest covariance in its formulation.

We conclude with a discussion of the problems that prevent a straightforward generalisation of this work to directly interacting particles.

Chapter 1

..all science aspires to be like physics, and physics aspires to be like mathematics.

LEWIS WOLPERT; *The Unnatural Nature Of Science.*

1.1 The Essentials of Geometric Quantisation

1.1.1 Polarisations and Densities

The phase space of a classical system is a symplectic manifold (M, ω) where M is a 2n-dimensional manifold and ω a non degenerate closed two-form. As a prerequisite the formalism of geometric quatisation requires that we are able to construct a complex Hermitian line bundle B over M, the prequantum bundle, i.e. M must be identifiable as the base space of a fibre bundle where each fibre B_m (m a point in M) is a one-dimensional vector space with an inner product $(\, , \,)$. Let $\Gamma(M)$ denote the set of all sections of B over M and $\Gamma^\infty(M)$ the set of all smooth sections over M. Let $V(M)$ denote the space of real vector fields over M. The inner product of the Hermitian line bundle is required to be smooth, i.e. if $s \in \Gamma^\infty(M)$ then (s, s) is a smooth function over M. We require also a connection ∇ on B with curvature $\omega/\hbar$. The Hermitian structure must be compatible with the connection, i.e. if s and $s' \in \Gamma^\infty(M)$ then

$$X(s, s') = (\nabla_X s, s') + (s, \nabla_X s') \; \forall \; X \in V(M).$$

It is always possible to find a suitable B when M is a cotangent bundle ([5] page 11) and in that case there also exists a global trivialisation of B ([2] page 122) and so a global unit section s_0 of B ([5] page 11). Any section $s \in \Gamma^\infty(M)$ can therefore be written as

$$s = \phi s_0$$

$\phi \in C^\infty(M)$ where $C^\infty(M)$ denotes the set of smooth complex valued functions on M. We have the following

Theorem 1 *On any 2n-dimensional symplectic vector space (V,ω) there exists a natural 2n-form ϵ_ω called the Liouville form where*

$$\epsilon_\omega = \frac{(-1)^{\frac{1}{2}n(n-1)}}{n!}\omega \wedge ... \wedge \omega.$$

The wedge product is repeated n times.

Consider the set $\mathcal{H}$ of all $s \in \Gamma(M)$ such that the integral of $(s,s)\epsilon_\omega$ over M exists and is finite. $\mathcal{H}$ is then a Hilbert space (the prequantum Hilbert space) with inner product $<,>_\mathcal{H}$ where

$$< s_1, s_2 >_\mathcal{H} = \left(\frac{1}{2\pi\hbar}\right) \int_M (s_1, s_2)\epsilon_\omega \quad s_1, s_2 \in \mathcal{H}. \tag{1.1}$$

We could take $\mathcal{H}$ as the state space of our quantum theory. We require a natural identification of classical observables $f \in C^\infty(M)$ with operators $\hat{f}$ on $\mathcal{H}$.

Definition 1 *The vector field X_f defined by*

$$X_f \rfloor \omega + df = 0$$

where $f \in C^\infty(M)$ is called the Hamiltonian vector field generated by f.

In a canonical coordinate system, which shall be available globally in the cases we consider, it is easy to show that

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}.$$

Classical observables $f \in C^\infty(M)$ are associated with operators $\hat{f}$ on $\mathcal{H}$ via the prescription

$$\hat{f}s = -i\hbar\nabla_{X_f}s + fs$$

([4] page 59 and [2] 5.4.1 page 121). It can be shown that if f has a complete Hamiltonian vector field then there exists a subspace of $\mathcal{H}$ on which $\hat{f}$ is self-adjoint. Otherwise $\hat{f}$ is symmetric and whether or not there exists a domain on which $\hat{f}$ is self-adjoint must be established by other means. It has been shown that this pre-quantisation scheme, as it is called, gives rise to highly reducible representations of the algebra of classical observables ([3] page 38 and [2] page 133). It is therefore necessary to modify the formalism. The quantisation is now to be dependent on the choice of a particular real polarisation P.

Definition 2 *A real smooth distribution F on M is a map that assigns to each point $m \in M$ a subspace $F_m \subset T_mM$ such that k=dim F_m is constant and there exists k smooth vector fields that span F_m at each m*

page 290 [2] or page 4 [5]. For a real distribution we can define integral surfaces as follows.

Definition 3 *An integral surface of a real disribution F is a connected submanifold N of M such $T_mN = F_m \; \forall \; m \in N$.*

Definition 4 *A real k-dimensional distribution F is integrable if there exists a coordinate chart $(x^1, ...x^{2n})$ such that the surfaces*

$$x^{k+1} = constant, ...x^{2n} = constant$$

are integral surfaces of F. These coordinates are said to be adapted to F.

Definition 5 *Let (V, ω) be a symplectic vector space. Let F be a subspace of V. Let F^0 denote the annihilator of F i.e.*

$$F^0 = \{X \; \in V : \omega(X, Y) = 0 \; \forall \, Y \; \in F\}.$$

F is said to be a Lagrangian subspace if $F = F^0$.

We can now define the notion of a real polarisation which is fundamental to the theory of geometric quantisation.

Definition 6 *A real polarisation is a smooth distribution P on M which is also integrable and where P_m is a Lagrangian subspace of T_mM.*

Definition 7 *Two polarisations P and P' are said to be transverse if $P_m + P'_m = T_mM$ for all $m \in M$.*

Let $V(M, P)$ denote those elements X of $V(M)$ such that $X_m \in P_m$. Let $\Gamma^\infty(M, P)$ denote the set of polarised sections that are covariantly constant along P i.e.

$$\Gamma^\infty(M, P) = \{s : s \in \Gamma^\infty(M) : \nabla_X s = 0 \;\; \forall X \in V(M, P)\}.$$

Now since we wish to obtain an irreducible representation of the algebra of classical observables from the reducible representation given by pre-quantisation we must reduce the size of the quantum Hilbert space $\mathcal{H}$. We might choose a polarisation P and associate the classical observables with operators on that subspace of $\mathcal{H}$ consisting of sections covariantly constant with respect to P. However we cannot take as our state space the set of polarised sections with inner product given by (1.1) since for polarised sections the integral $(s, s)\epsilon_\omega$ over M is never finite. We could escape this difficulty by defining the inner product as an integral over $Q = M/N$ (sometimes written $Q = M/P$) but there is no natural measure on Q. We shall show that there exists an integral of a canonically defined density on T_qQ and this observation will indicate how we should modify our state space.

Definition 8 *Let V be an n-dimensional real vector space. An r density, $r \in \mathbb{R}$, is a map ν from the set of bases for V to the complex numbers which obeys the following transformation law*

$$\nu\{C_j^i X_i\} = \mid detC \mid^r \nu\{X_i\}$$

where $C \in GL(n, \mathbb{R})$.

The set of all r densities over V is denoted $\Delta_r(V)$ and is a one-dimensional vector space.

Definition 9 *Let*

$$\Delta_r(P) = \bigcup_{m \in M} \Delta_r(P_m)$$

This is a line bundle over M where each fibre is $\Delta_r(P_m)$. The sections of $\Delta_r(P)$ are called r-P-densities.

It is common practice to abuse this notation and denote the set of $r - P$-densities also as $\Delta_r(P)$. It is possible to define a partial covariant derivative that acts on the set of smooth $r - P$-densities [5]. This is achieved via a partial connection.

Definition 10 *A partial connection on P is a map*

$$\overline{\nabla} : V(M, P) \times V(M, P) \to V(M, P)$$

also written as

$$(X, Y) \to \overline{\nabla}_X Y$$

where $\overline{\nabla}_X Y$ is defined by the relation

$$\overline{\nabla}_X Y \rfloor \omega = X \rfloor d(Y \rfloor \omega).$$

Less abstractly take $n = 2$ and suppose that (p, q) is a coordinate system adapted to P then if X and $Y \in V(P, M)$ we know that

$$X = X(p,q)\frac{\partial}{\partial p} \text{ and } Y = Y(p,q)\frac{\partial}{\partial p}$$

and

$$\overline{\nabla}_X Y = X(p,q)\frac{\partial Y}{\partial p}(p,q)\frac{\partial}{\partial p}.$$

Definition 11 *Suppose $\overline{\nabla}$ is a partial connection on P and ν a smooth r-P-density then $\overline{\nabla}_X \nu$, where $X \in V(M, P)$, is a smooth r-P-density defined by*

$$(\overline{\nabla}_X \nu)\{X_i\} = X(\nu\{X_i\}) \tag{1.2}$$

where X_i is any field of bases for P satisfying

$$\overline{\nabla}_X X_i = 0. \tag{1.3}$$

Definition 12 *Those $r - P$-densities that satisfy*

$$\overline{\nabla}_X \nu = 0 \;\; \forall \;\; X \in V(M, P)$$

are said to be polarised with respect to P.

We now indicate how r-densities are related to more familiar geometric objects.

Theorem 2 *Let V be an m-dimensional vector space and ϵ an m-form on V then ϵ determines an r-density $\mid \epsilon \mid^r$ on V defined as follows*

$$\mid \epsilon \mid^r \{X_1, ..X_m\} = (m! \mid \epsilon\{X_1, ..X_m\} \mid)^r$$

where $\{X_1, ..X_m\}$ is a basis for V.

It turns out that the densities on a given vector space are intimately related to those of its subspaces. In particular we shall need the following

Theorem 3 *Let W be an m-dimensional subspace of of a 2n-dimensional vector space V. Let $\{Y_i\}$ be a basis for W, $\{Z_i\}$ a basis for V/W and $\{X_i\}$ a basis for V such that*

$$\{X_1, ...X_m\} = \{Y_1, ...Y_m\}$$

and

$$\{IX_{m+1}, ...IX_{2n}\} = \{Z_1, ...Z_{2n-m}\}$$

where I is the projection $I : V \to V/W$. Let $\rho \in \Delta_r(V)$ and $\nu \in \Delta_r(W)$ then

$$\frac{\rho}{\nu}\{Z_i\} = \rho\{X_i\}\nu^{-1}\{Y_i\}$$

defines an element of $\Delta_r(V/W)$.

Now

$$T_qQ = T_mM/P_m \tag{1.4}$$

where m is any point in M such that $\pi(m) = q$ (π the projection from M to the quotient space Q). Therefore

$$\Delta_1(T_qQ) = \Delta_{-1}(P_m) \otimes \Delta_1(T_mM)$$

$$= \Delta_{-1/2}(P_m) \otimes \Delta_{-1/2}(P_m) \otimes \Delta_1(T_mM).$$

Using this it can be shown that if ν and ν' are $-\frac{1}{2} - P-$densities covariantly constant along P and s and s' are likewise polarised sections of B then

$$(s, s')_m \nu_m \nu'_m \Delta_1(T_mM)$$

is a $\Delta_1(T_qQ)$ valued class function on M i.e. is independent of the particular choice of m so long as $\pi(m) = q$. What is more we clearly have a canonical choice for the $\Delta_1(T_mM)$ i.e. $\mid \epsilon_\omega \mid$.

1.1.2 Half Density Quantisation

We are now in a position to define the state space $\mathcal{H}_P$ for the half density quantisation based on a polarisation P. Elements of $\mathcal{H}_P$ are called P wave functions ([5] page 19). An element $\sigma = s.\nu$ of the vector space $\Gamma^\infty \otimes \Delta_{-1/2}(P) \in \mathcal{H}_P$ if s and ν are smooth polarised sections of B and $\Delta_{-1/2}(P)$ respectively and

$$\left(\frac{1}{2\pi\hbar}\right) \int_Q (s, s)\nu\nu \mid \epsilon_\omega \mid$$

exists and is finite[1]. On this set $\mathcal{H}_P$ we can define an inner product $< , >_{\mathcal{H}_P}$

$$< \sigma, \sigma' >_{\mathcal{H}_P} = \left(\frac{1}{2\pi\hbar}\right) \int_Q (s, s')_m \nu_m \nu'_m \mid \epsilon_\omega \mid .$$

$\mathcal{H}_P$ is actually the completion of this pre-Hilbert space with respect to the inner product $< , >_{\mathcal{H}_P}$. We wish to find a natural correspondance between classical observables and operators in $\mathcal{H}_P$.

[1]We define the notion of integration of densities as follows. Let Q be an n dimensional manifold and suppose $\epsilon \in \Delta_1(TQ)$ then we define the integral of ϵ over Q by

$$\int_Q \epsilon = \int_Q \epsilon \left\{\frac{\partial}{\partial q^i}\right\} d^n q$$

where q^i are coordinates on Q.

Definition 13 *A vector field Y is said to preserve the polarisation P if*

$$[Y, X] \in V(M, P) \ \ \forall \ X \in V(M, P).$$

Let $C^\infty(M, P, 1)$ denote those f in $C^\infty(M)$ such that X_f preserves P. Any $f \in C^\infty(M, P, 1)$ defines an operator $\hat{f}$ in $\mathcal{H}_P$ according to

$$\hat{f}(s.\nu) = (-i\hbar\nabla_{X_f}s + fs).\nu - i\hbar.L_{X_f}\nu$$

where L_{X_f} denotes Lie derivative. Later we shall give a less abstract expression for $\hat{f}$ at which time we shall also discuss the domain of the operator and the conditions under which it is symmetric or essentially self-adjoint. We must have $f \in C^\infty(M, P, 1)$ because only these classical observables give rise to operators $\hat{f}$ that leave $\mathcal{H}_P$ invariant. It is easy to determine the generic form of the elements of $C^\infty(M, P, 1)$. If $f \in C^\infty(M, P, 1)$ then in a coordinate system adapted to P

$$f(p, q) = \zeta(q)p + \eta(q)$$

where ζ and η are smooth functions. A classical observable not directly quantizable in P i.e. not in $C^\infty(M, P, 1)$ may be quantizable in another polarisation P'. A discussion of how we may obtain from this a quantisation of the observable in P must wait until we have discussed pairings.

1.1.3 General Form of Polarised Sections

If we choose a so called connection potential $\beta = pdq$ then we can define a connection with the correct curvature by

$$\nabla_X s_0 = -\frac{i}{\hbar}(X \rfloor \beta)s_0. \tag{1.5}$$

Now any s can be written in the form ϕs_0. Since

$$\nabla_X \phi(p, q)s_0 = X(\phi)s_0 + \phi\nabla_X s_0$$

we see that for $X \in V(M, P)$

$$\nabla_X \phi s_0 = X(\phi)s_0$$

so that, in general, sections polarised with respect to P will be of the form ϕs_0 where ϕ is a polarised function, i.e. a function such that

$$X\phi = 0 \ \forall X \in V(M, P).$$

In a coordinate system adapted P this restriction becomes

$$c(p,q)\frac{\partial}{\partial p}\phi(p,q) = 0$$

so that the polarised functions are those that in a coordinate system adapted to P have the representation $\phi(q)$.

1.1.4 Relationship between Covariantly Constant Sections of Transverse Polarisations

If P' is transverse to P then

$$s_0' = \exp(if/\hbar)s_0$$

is a unit section polarised with respect to P' where $f(q,q')$ is a generating function between the coordinate system (p',q') adapted to P' and the coordinate system (p,q) adapted to P, i.e.

$$p = \frac{\partial f}{\partial q} \quad \text{and} \quad p' = -\frac{\partial f}{\partial q'}.$$

Proof: All vector fields $\in V(M,P')$ are of the form

$$c(p',q')\frac{\partial}{\partial p'}.$$

Now

$$\nabla_{c(p',q')\frac{\partial}{\partial p'}} s_0' = \nabla_{c(p',q')\frac{\partial}{\partial p'}} \exp(if/\hbar)s_0.$$

Using (1.5) this becomes

$$\begin{aligned} c(p',q')\frac{i}{\hbar}\frac{\partial f}{\partial p'}\exp(if/\hbar)s_0 + \exp(if/\hbar)\left(-\frac{i}{\hbar}\right)\left(c(p',q')\frac{\partial}{\partial p'}\rfloor pdq\right)s_0 \\ = c(p',q')\frac{i}{\hbar}\frac{\partial f}{\partial p'}\exp(if/\hbar)s_0 - \exp(if/\hbar)\frac{i}{\hbar}c(p',q')\left(\frac{\partial}{\partial p'}\rfloor pdq\right)s_0. \end{aligned}$$

From the usual transformation rules for 1-forms it is easy to show that

$$pdq = \frac{\partial q}{\partial p'}pdp'.$$

Using this and the fact that

$$\begin{aligned} \frac{\partial f}{\partial p'} &= \frac{\partial q}{\partial p'}\frac{\partial f}{\partial q} + \frac{\partial q'}{\partial p'}\frac{\partial f}{\partial q'} \\ &= \frac{\partial q}{\partial p'}p \end{aligned}$$

we have

$$\begin{aligned} \nabla_{c(p',q')\frac{\partial}{\partial p'}} s_0' &= c(p',q')\frac{i}{\hbar}\frac{\partial q}{\partial p'}p\exp(if/\hbar)s_0 - c(p',q')\frac{i}{\hbar}\frac{\partial q}{\partial p'}p\exp(if/\hbar)s_0 \\ &= 0 \end{aligned}$$

as required.

1.1.5 General Form of Polarised Half Densities

There exists a canonical choice of -$\frac{1}{2}$-density on $T_m M$ namely $\mid \epsilon_\omega \mid^{-\frac{1}{2}}$. Also there exists a natural $\frac{1}{2}$-density on $T_q Q$. Suppose that from Riem(Q) (the set of all Riemannian metrics on Q) we select a metric G then from the volume element on Q i.e.

$$\epsilon = \sqrt{g} dq^1 \wedge ... \wedge dq^n,$$

where g denotes the determinant of the metric ([78] page 14), we can construct a $\frac{1}{2}$-density $\mid \epsilon \mid^{1/2}$ via Th 1. Because of (1.4) we are led to a canonical choice of $-\frac{1}{2} - P$ density on $T_q Q$ i.e.

$$\nu = \mid \epsilon_\omega \mid^{-\frac{1}{2}} \mid \epsilon \mid^{\frac{1}{2}} .$$

Suppose dim $M = 2$ then in a coordinate system adapted to P we have

$$\nu = (g(q))^{1/4} \mid dp \mid^{-1/2}$$

([5] page 49 or 53). We now wish to show that ν is polarised with respect to P. Any basis for P will be of the form $c(p,q)\frac{\partial}{\partial p}$ and of course any $X \in V(M,P)$ will be of the same form. We require the generic form of elements of the set of all bases for P that satisfy the auxilliary condition (1.3) i.e. $Y(p,q)\frac{\partial}{\partial p}$ such that

$$\overline{\nabla}_{X(p,q)\frac{\partial}{\partial p}} Y(p,q)\frac{\partial}{\partial p} = X(p,q)\frac{\partial Y}{\partial p}(p,q)\frac{\partial}{\partial p} = 0.$$

Clearly these are given by $Y(q)\frac{\partial}{\partial p}$. From (1.2) we have

$$\begin{aligned}
\left\{\overline{\nabla}_{c(p,q)\frac{\partial}{\partial p}} \nu\right\}\left\{Y(q)\frac{\partial}{\partial p}\right\} &= c(p,q)\frac{\partial}{\partial p}\nu\left\{Y(q)\frac{\partial}{\partial p}\right\} \\
&= c(p,q)\frac{\partial}{\partial p}\left[(g)^{\frac{1}{4}}\left|dp \rfloor Y(q)\frac{\partial}{\partial p}\right|^{-1/2}\right] \\
&= c(p,q)\frac{\partial}{\partial p}[(g(q))^{1/4}Y(q)] \\
&= 0
\end{aligned}$$

and therefore ν is polarised with respect to P. Now a general smooth half density is of the form $\phi\nu$. Since

$$\begin{aligned}
\overline{\nabla}_{c(p,q)\frac{\partial}{\partial p}} \phi\nu \left\{Y(q)\frac{\partial}{\partial p}\right\} &= c(p,q)\frac{\partial}{\partial p}[\phi(g(q))^{1/4}Y(q)] \\
&= c(p,q)(g(q))^{1/4}Y(q)\frac{\partial \phi}{\partial p}
\end{aligned}$$

we see that $\phi\nu$ will be a half density polarised with respect to P when ϕ is a polarised function i.e. when ν has a representation of the form $\phi(q)\nu$ in a coordinate system adapted to P. This is precisely the condition imposed on ϕ to obtain polarised sections of B. We can therefore write any polarised section of $B \otimes \Delta_{-1/2}(P)$ as $\phi(q)s_0.\nu$ where s_0 and ν are the polarised sections defined above. This follows from [1] equation 5 page 28. Of course not every section of $B \otimes \Delta_{-\frac{1}{2}}(P)$ of this form $\in \mathcal{H}_P$. Only those that are square integrable are P wave functions. This can now be stated as follows. Suppose

$$\sigma = \phi(q)(g(q))^{1/4}s_0. \mid dp \mid^{-1/2} \tag{1.6}$$

then $\sigma \in \mathcal{H}_P$ if

$$\frac{1}{2\pi\hbar}\int_Q \phi(q)\phi^*(q)(g(q))^{1/2} \mid dp \mid^{-1} \mid dp \wedge dq \mid < \infty$$

which by definition is the requirement that

$$\int_Q \phi(q)\phi^*(q)(g(q))^{\frac{1}{2}} \mid dp \mid^{-1} \mid dp \wedge dq \mid \left\{\frac{\partial}{\partial q}\right\} dq < \infty.$$

Taking

$$X = \left\{\frac{\partial}{\partial p}, \frac{\partial}{\partial q}\right\}, \quad Y = \frac{\partial}{\partial p} \quad and \quad Z = \frac{\partial}{\partial q}$$

in Th 3 this implies

$$\int_Q \mid \phi(q) \mid^2 (g(q))^{1/2} \left| \frac{\partial}{\partial p} \rfloor \frac{\partial}{\partial q} \rfloor dp \wedge dq \right| \left| \frac{\partial}{\partial p} \rfloor dp \right|^{-1} dq < \infty$$

or

$$\int_q (g(q))^{1/2} \mid \phi(q) \mid^2 dq < \infty.$$

Similarly we can see that in a coordinate system adapted P the inner product between two P wave functions

$$\sigma_1 = \phi(q)(g(q))^{1/4}s_0 \mid dp \mid^{-1/2}$$

$$\sigma_2 = \psi(q)(g(q))^{1/4}s_0 \mid dp \mid^{-1/2}$$

becomes

$$< \sigma_1, \sigma_2 >_{\mathcal{H}_P} = \int_Q \phi(q)\psi^*(q)(g(q))^{1/2} dq. \tag{1.7}$$

In this way the quantum Hilbert space associated with a given polarisation is identified with a space of square integrable functions on Q.

1.1.6 Pairing

To compare quantisations carried out in different polarisations P and P' we shall require a map between $\mathcal{H}_P$ and $\mathcal{H}_{P'}$. A pairing is a map

$$<\,,\,>_{PP'}: \mathcal{H}_P \times \mathcal{H}_{P'} \to \mathbb{C}$$

and is only defined for compatible polarisations ([5] page 20 definition 1.6.6). We shall restrict ourselves to polarisations that are real and transverse. These are automatically compatible ([2] page 160). Since

$$\begin{aligned} T_mM &= P_m \oplus P'_m \\ &= \frac{T_mM}{P'_m} \oplus \frac{T_mM}{P_m} \end{aligned}$$

we have

$$\begin{aligned} \Delta_{\frac{1}{2}}(T_mM) &= \Delta_{-\frac{1}{2}}(P'_m) \otimes \Delta_{\frac{1}{2}}(T_mM) \otimes \Delta_{-\frac{1}{2}}(P_m) \otimes \Delta_{\frac{1}{2}}(T_mM) \\ \Delta_{\frac{1}{2}}(T_mM) &= \Delta_{-\frac{1}{2}}(P'_m) \otimes \Delta_{-\frac{1}{2}}(P_m) \otimes \Delta(T_mM) \end{aligned}$$

so

$$\Delta_1(T_mM) = \Delta_{-\frac{1}{2}}(P'_m) \otimes \Delta_{-\frac{1}{2}}(P_m) \otimes \Delta_{\frac{3}{2}}(T_mM).$$

Therefore

$$(s, s')\nu\nu' \mid \epsilon_\omega \mid^{3/2}$$

is a well defined 1-density on T_mM and we can define a pairing $<\,,\,>_{PP'}$ between P and P' as follows

$$< \sigma, \sigma' >_{PP'} = \left(\frac{1}{2\pi\hbar}\right) \int_M (s, s')_m \nu_m \nu'_m \mid \epsilon_\omega \mid_m^{3/2} .$$

Using the canonical representation (1.6) of P and P' wave functions we obtain

$$\begin{aligned} < \sigma, \sigma' >_{PP'} &= \int_M \phi\phi'^* \exp(-if/\hbar) \mid \epsilon_\omega \mid^{-1/2} \mid \epsilon \mid^{1/2} \mid \epsilon_\omega \mid^{-1/2} \mid \epsilon' \mid^{1/2} \mid \epsilon_\omega \mid^{3/2} \\ &= \int_M \phi\phi'^* \exp(-if/\hbar) \mid \epsilon_\omega \mid^{1/2} \mid \epsilon \mid^{1/2} \mid \epsilon' \mid^{1/2} . \end{aligned}$$

From the definition of integration of densities the above becomes

$$\begin{aligned} &\int_M \phi\phi'^* \exp(-if/\hbar) \mid \epsilon_\omega \mid^{1/2} \left\{\frac{\partial}{\partial q'}, \frac{\partial}{\partial q}\right\} \mid \epsilon \mid^{1/2} \left\{\frac{\partial}{\partial q}\right\} \mid \epsilon' \mid^{1/2} \left\{\frac{\partial}{\partial q'}\right\} dq dq' \\ &= \int_M \phi\phi'^* \exp(-if/\hbar) \mid \epsilon_\omega \mid^{1/2} \left\{\frac{\partial}{\partial q'}, \frac{\partial}{\partial q}\right\} \mid \frac{\partial}{\partial q} \rfloor dq \mid^{1/2} \mid \frac{\partial}{\partial q'} \rfloor dq' \mid^{1/2} (g(q))^{1/4} (g'(q'))^{1/4} dq dq' \\ &= \int_M \phi\phi'^* \exp(-if/\hbar) \mid \epsilon_\omega \mid^{1/2} \left\{\frac{\partial}{\partial q'}, \frac{\partial}{\partial q}\right\} (g(q))^{1/4} (g'(q'))^{1/4} dq dq'. \qquad (1.8) \end{aligned}$$

Now

$$
\begin{aligned}
| \epsilon_\omega |^{1/2} \left\{ \frac{\partial}{\partial q}, \frac{\partial}{\partial p} \right\} &= \left| \frac{\partial}{\partial p} \rfloor \frac{\partial}{\partial q} \rfloor (dp \wedge dq) \right|^{1/2} \\
&= \left| \frac{\partial}{\partial p} \rfloor \left(\frac{\partial}{\partial q} \rfloor dp \right) dq - \left(\frac{\partial}{\partial q} \rfloor dq \right) dp \right|^{1/2} \\
&= \left| - \frac{\partial}{\partial p} \rfloor dp \right|^{1/2} \\
&= 1.
\end{aligned}
$$

Since

$$
\frac{\partial}{\partial q'} = \frac{\partial p}{\partial q'} \frac{\partial}{\partial p} + \frac{\partial q}{\partial q'} \frac{\partial}{\partial q}
$$

we see that the transformation from the basis natural to (q, p) to that natural to (q, q') is described by

$$
T = \begin{pmatrix} 1 & 0 \\ \frac{\partial q}{\partial q'} & \frac{\partial p}{\partial q'} \end{pmatrix}
$$

so $\det(T) = \frac{\partial p}{\partial q'}$ and since $| \epsilon_\omega |^{1/2}$ is a half density we have by definition that

$$
\begin{aligned}
| \epsilon_\omega |^{1/2} \left\{ \frac{\partial}{\partial q}, \frac{\partial}{\partial q'} \right\} &= \left| \frac{\partial p}{\partial q'} \right|^{1/2} | \epsilon_\omega |^{1/2} \left\{ \frac{\partial}{\partial q}, \frac{\partial}{\partial p} \right\} \\
&= \left| \frac{\partial p}{\partial q'} \right|^{1/2}.
\end{aligned}
$$

Therefore (1.8) becomes

$$
< \sigma, \sigma' >= \int_M \phi \phi'^{*} \exp(-if/\hbar) \left| \frac{\partial p}{\partial q'} \right|^{1/2} (g(q))^{1/4} (g'(q'))^{1/4} dq dq'. \tag{1.9}
$$

Formally the pairing leads us to consider two maps

$$
U_{PP'} : \mathcal{H}_P \to \mathcal{H}_{P'} \quad \text{and} \quad U_{P'P} : \mathcal{H}_{P'} \to \mathcal{H}_P
$$

partially defined by the requirement that

$$
< \sigma, U_{P'P} \sigma' >_{\mathcal{H}_P} = < U_{PP'} \sigma, \sigma' >_{\mathcal{H}_{P'}} = < \sigma, \sigma' >_{PP'} .
$$

Writing these out explicitly when P and P' are transverse using (1.7) and (1.9) it is easy to see by inspection that if we put

$$
U_{PP'} \phi(q) s_0 . \nu = \phi'(q') s_0' . \nu'
$$

and

$$
U_{PP'} \phi'(q') s_0' . \nu' = \phi(q) s_0 . \nu
$$

where

$$\phi'(q') = \left(\frac{1}{2\pi\hbar}\right)^{1/2} \int_Q \phi(q) \exp(-if/\hbar) \left|\frac{\partial^2 f}{\partial q \partial q'}\right|^{1/2} (g(q))^{1/4} dq$$

and

$$\phi(q) = \left(\frac{1}{2\pi\hbar}\right)^{1/2} \int_{Q'} \phi'(q') \exp(if/\hbar) \left|\frac{\partial^2 f}{\partial q \partial q'}\right|^{1/2} (g(q'))^{-1/4} dq'$$

then the relations are satisfied. Sometimes $U_{PP'}$ and $U_{P'P}$ are unitary maps and in that case they provide a means of quantizing classical observables in P when $f \notin C^\infty(M, P, 1)$ but $f \in C^\infty(M, P', 1)$.

1.1.7 Form of Operators

Suppose $f = \zeta(q)p + \eta(q)$ then

$$\hat{f}s.\nu = (-i\hbar\nabla_{X_f}s + fs).\nu - i\hbar s.L_{X_f}\nu$$

$$= -i\hbar\nabla_{X_f}(\phi(q)s_0).\nu + f\phi(q)s_0.\nu - i\hbar\frac{1}{2}(\sqrt{g}\zeta(q))_{,q}\,\phi(q)s_0.\nu$$

$$= \left(-i\hbar(X_f(\phi)s_0 + \phi\nabla_{X_f}s_0) + f\phi(q)s_0\right).\nu - i\hbar\frac{1}{2}(\sqrt{g}\zeta(q))_{,q}\,\phi(q)s_0.\nu$$

Since

$$X_f = \zeta(q)\frac{\partial}{\partial q} - (\zeta'(q)p + \eta'(q))\frac{\partial}{\partial p},$$

where prime denotes differentiation with respect to q, we obtain

$$\hat{f}s.\nu = -i\hbar\left(\zeta(q)\frac{\partial}{\partial q} + (-\frac{i}{\hbar})p\zeta(q)\right)\phi s_0.\nu + (\zeta(q)p + \eta(q))\phi(q)s_0.\nu - i\hbar\frac{1}{2}(\sqrt{g}\zeta(q))_{,q}\,\phi(q)s_0.\nu$$

$$= \left(-i\hbar\zeta(q)\frac{\partial}{\partial q} - p\zeta(q) + p\zeta(q) + \eta(q) - i\hbar\frac{1}{2}(\sqrt{g}\zeta(q))_{,q}\right)\phi(q)s_0.\nu$$

$$= \left[-i\hbar\left(\zeta(q)\frac{\partial}{\partial q} + \frac{1}{2}(\sqrt{g}\zeta(q))_{,q}\right) + \eta(q)\right]\phi(q)s_0.\nu$$

This operator is symmetric on $C_0^\infty(Q)$. If X_f is complete on M then $\hat{f}$ is essentially self-adjoint on $C_0^\infty(Q)$. We can weaken this to prove that $\hat{f}$ is essentially self-adjoint on $C_0^\infty(Q)$ iff $\pi_* X_f$ is complete, where * denotes push forward.

1.1.8 Quantisation on Complex Manifolds

We have described the half density quantisation scheme as it applies to real polarisations and showed that it yields symmetric operators for observables linear in momentum or self-adjoint operators if additional criteria are met. In this section we discuss half density and also half form quantisation as it applies to complex manifolds and complex polarisations.

Our treatment of the half form scheme is very cursory. We only mention it because Tuynman's method, which we shall examine in section 1.3.3, is usually stated in terms of half forms although a half density treatment is also possible. Complex manifold theory on the other hand is mandatory for an understanding of Tuynman's work.

To discuss differential geometry on a complex manifold we shall need to know a little complex linear algebra. Let V be a real vector space. Its complexification $V^{\mathbb{C}}$ is the complex vector space such that

1. $X + iY \in V^{\mathbb{C}}$ iff $X, Y \in V$

2. $(X_1 + iY_1) + X_2 + iY_2 = X_1 + X_2 + i(Y_1 + Y_2)$

3. $(a + ib)(X + iY) = aX - bY + i(bX + aY) \;\; X, Y \in V \;\; a, b \in \mathbb{R}$.

It is easy to show that the above imply that $V^{\mathbb{C}}$ is a vector space over $\mathbb{C}$. On this complex vector space we define the notion of complex conjugation of a vector by

$$\overline{X + iY} = X - iY.$$

The dimension of $V^{\mathbb{C}}$ is the same as V because if $\{e_i\}_{i=1}^{n}$ is a basis for V then it is also a basis for $V^{\mathbb{C}}$ since

$$X + iY = X^k e_k + iY^k e_k = (X^k + iY^k)e_k.$$

Complex Structure on a Real Vector Space

Let V be a vector space of dimension n over $\mathbb{R}$. Then an endomorphism J, i.e. a linear mapping $J : V \to V$ such that $J^2 = -I$ is called a complex structure. Let V be a real vector space with a complex structure J and let $V^{\mathbb{C}}$ denote its complexification. Then define two subsets of $V^{\mathbb{C}}$ as follows

$$V^{1,0} = Z : Z = X - iJ(X)$$

$$V^{0,1} = Z : Z = X + iJ(X) \tag{1.10}$$

$$X \in V.$$

It follows that $V^{1,0}$ and $V^{0,1}$ are subspaces of $V^{\mathbb{C}}$ and if $Z \in V^{1,0}$ then $\overline{Z} \in V^{0,1}$. For this reason $V^{1,0}$ and $V^{0,1}$ are said to be complex conjugates of each other. It turns out that

$$V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

so the existence of a complex structure on V implies that $V^{\mathbb{C}}$ can be written as the direct sum of two subspaces that are complex conjugates of each other. J defined on V can be extended by linearity to act on $V^{\mathbb{C}}$ as follows

$$J(X+iY) = J(X) + iJ(Y), \quad X, Y \in V.$$

With this extension we can define the subspaces $V^{1,0}$ and $V^{0,1}$ of $V^{\mathbb{C}}$ as

$$V^{1,0} = Z : J(Z) = iZ$$

$$V^{0,1} = Z : J(Z) = -iZ$$

$$Z \in V^{\mathbb{C}}.$$

Suppose we have a two dimensional vector space V with basis

$$\left(\frac{\partial}{\partial p}, \frac{\partial}{\partial q}\right).$$

At this stage V is actually arbitrary but the notation suggests that we have in mind that $V = T_m M$ and in fact this will be the case that is of most interest to us. There is a canonical complex structure J_c on V defined by

$$J_c\left(\frac{\partial}{\partial p}\right) = \frac{\partial}{\partial q} \quad \text{and} \quad J_c\left(\frac{\partial}{\partial q}\right) = -\frac{\partial}{\partial p}.$$

Compatible Complex Structures

A compatible complex structure on a symplectic vector space (V, ω) is a complex structure such that

$$\omega(J(X), J(Y)) = \omega(X, Y) \;\; \forall \;\; X, Y \in V.$$

Complex Symplectic Vector Spaces

The complexification of a real symplectic vector space gives rise to a complex symplectic vector space. The two form on the complex space, which we shall also denote by ω, is defined by extending the real two form by linearity as follows

$$\omega(X+iY, X'+iY') = \omega(X, X') - \omega(Y, Y') + i\omega(X, Y') + i\omega(Y, X'). \qquad (1.11)$$

Obviously we need a two form on the complex space to define things like a Lagrangian subspace (definition 5) of a complex vector space. For example we can show that

$$\frac{\partial}{\partial p} - i\frac{\partial}{\partial q} \qquad (1.12)$$

is a Lagrangian subspace of the complexification of the real vector space V spanned by

$$\left(\frac{\partial}{\partial p}, \frac{\partial}{\partial q}\right).$$

To see this we must identify the annihilator F^0 of (1.12). This consists of vectors of the form

$$a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q} + i\left(c\frac{\partial}{\partial p} + d\frac{\partial}{\partial q}\right) \tag{1.13}$$

which satisfy

$$\omega\left(a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q} + i\left(c\frac{\partial}{\partial p} + d\frac{\partial}{\partial q}\right), \frac{\partial}{\partial p} - i\frac{\partial}{\partial q}\right) = 0.$$

Using (1.11) this becomes

$$\omega\left(a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right) - \omega\left(c\frac{\partial}{\partial p} + d\frac{\partial}{\partial q}, -\frac{\partial}{\partial q}\right) + i\omega\left(a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q}, -\frac{\partial}{\partial q}\right) + i\omega\left(c\frac{\partial}{\partial p} + d\frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right) = 0$$

which is only true if

$$b - c + ia + id = 0$$

i.e. if $b = c$ and $a = -d$. Therefore from (1.13) F^0 consists of vectors of the form

$$\begin{aligned} &d\left(-\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right) + b\left(\frac{\partial}{\partial q} + i\frac{\partial}{\partial p}\right) \\ &= -d\left(\frac{\partial}{\partial p} - i\frac{\partial}{\partial q}\right) + bi\left(-i\frac{\partial}{\partial q} + \frac{\partial}{\partial p}\right) \end{aligned}$$

which is just a complex mulitiple of (1.12) so $F^0 = F$ and the space is Lagrangian.

Kahler Lagrangian Subspace

For a complex symplectic vector space $(V^{\mathbb{C}}, \omega)$ we can define a special kind of complex Lagrangian suspace called a Kahler Lagrangian subspace. We shall not give the general definition of a Kahler Lagrangian subspace but merely show how to construct them . Let J be any compatible complex structure on V extended to $V^{\mathbb{C}}$ by linearity then the Lagrangian subspace K of $V^{\mathbb{C}}$ defined by

$$K = \{X \in V^{\mathbb{C}} : X = (J + i)Y,\ \ Y \in V^{\mathbb{C}}\}$$

is a Kahler Lagrangian subspace. If F is any Lagrangian subspace of V then

$$K = \{X : X = \frac{1}{2}c^i(X_i - iJX_i) : X_i \text{ a basis for } F,\ \ c^i \in \mathbb{C}\}.$$

Suppose

$$F = \frac{\partial}{\partial p}$$

then F is a Lagrangian subspace of V. Choose $J = J_c$. In this case

$$K = \frac{1}{2}c\left(\frac{\partial}{\partial p} - iJ_c\left(\frac{\partial}{\partial p}\right)\right)$$
$$= \frac{1}{2}c\left(\frac{\partial}{\partial p} - i\frac{\partial}{\partial q}\right)$$

so this choice of Lagrangian subspace of V generates the Lagrangian subspace of $V^{\mathbb{C}}$ discussed above which has now been shown to be Kahler. If $X = (J+i)Y$ then $JX = J(J+i)Y = -Y + iJY = i(i+J)Y = iX$ so clearly we have

$$K = V^{1,0}$$

so that

$$K \cap \overline{K} = 0 \text{ i.e. } V^{\mathbb{C}} = K \oplus \overline{K}.$$

Complex Polarisation

Let (M,ω) be a 2n-dimensional symplectic manifold. A complex polarisation is a smooth complex distribution F on M such that

1. F is integrable
2. F_m is a complex Lagrangian subspace of $T^{\mathbb{C}}_m M$ $m \in M$
3. $F_m \cap \overline{F_m} \cap T_m M$ has constant dimension.

If in addition F_m is a Kahler Lagrangian subspace of $T^{\mathbb{C}}_m M$ for each $m \in M$ then F is Kahler polarisation. Of course when F is Kahler $F \cap \overline{F} = 0$ so $F_m \cap \overline{F_m} \cap T_m M$ has constant dimension zero.

Much of the structure we have described on the complexified tangent space to a symplectic manifold can be examined in a more straightforward way. Suppose we perform the coordinate tranformation

$$z = p + iq \qquad \overline{z} = p - iq.$$

We may transform the symplectic two form $\omega = dp \wedge dq$ in the usual way. The inverse coordinate transformations are

$$p = \frac{z + \overline{z}}{2} \;\; q = \frac{z - \overline{z}}{2i}.$$

Therefore

$$dp = \frac{\partial p}{\partial z}dz + \frac{\partial p}{\partial \overline{z}}d\overline{z} = \frac{1}{2}dz + \frac{1}{2}d\overline{z}$$

$$dq = \frac{\partial q}{\partial z}dz + \frac{\partial q}{\partial \bar{z}}d\bar{z} = \frac{1}{2i}dz - \frac{1}{2i}d\bar{z}$$

so

$$\begin{aligned} dp \wedge dq &= \frac{1}{2}dz + \frac{1}{2}d\bar{z} \wedge \frac{1}{2i}dz - \frac{1}{2i}d\bar{z} \\ &= -\frac{1}{4i}dz \wedge d\bar{z} + \frac{1}{4i}d\bar{z} \wedge dz \\ &= -\frac{1}{2i}dz \wedge d\bar{z} \\ &= \frac{i}{2}dz \wedge d\bar{z} \end{aligned}$$

which describes the extended two form. Clearly if

$$\theta = -\frac{i}{2}\bar{z}dz \tag{1.14}$$

then $\omega = d\theta$. Using the usual transformation rules for vectors it is easy to show that

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial p} + i\frac{\partial}{\partial q}\right)$$

and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial p} - i\frac{\partial}{\partial q}\right).$$

From what has gone before we know that $\frac{\partial}{\partial z}$ is a Kahler polarisation. $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are a basis for $V^{\mathbb{C}}$. Using this basis makes manifest the direct sum decomposition discussed above.

Half Density and Half Form Quantisation on a Complex Manifold

The half density scheme gives untenable results when used to quantise some systems[2]. The half form scheme often succeeds where the half density scheme fails but is much more complicated although superficially the two schemes are quite similar. In the half density scheme the quantum Hilbert space comprises sections of the line bundle $B \otimes \Delta_{-1/2}$ that are covariently constant along some polarisation F while in the half form scheme it consists of polarised sections of a new line bundle usually denoted $B \otimes \overline{\delta_{-\frac{1}{2}}}$. The precise nature of

[2]Physicists motivate half form quantisation by pointing out that half densities lead to the incorrect spectrum of the simple harmonic oscillator while mathematicians observe that half densities give only projective representations of Lie algebras of classical observables. In fact Wan and McKenna have shown that the half density scheme can be used to quantise the harmonic oscillator if we use a modified connection potential and it turns out that the same method can be used to obtain proper i.e. non projective representations (see Appendix 1).

the objects in the vector space $\overline{\delta_{-\frac{1}{2}}}$ need not concern us. Suffice to say they are the half forms from which the scheme derives its name . We shall only be concerned with the case

$$F = \frac{\partial}{\partial z}. \tag{1.15}$$

It turns out that as for real polarisations, arbitrary polarised sections of $B \otimes \Delta_{-\frac{1}{2}}$ and $B \otimes \overline{\delta_{-\frac{1}{2}}}$ can be written $\phi(z)s_0'.\nu$ where here, and in what follows, ν will represent either the canonoical half density $\frac{1}{4} | dz |^{-\frac{1}{2}}$ or the canonical half form $2^{\frac{1}{4}}(d\overline{z})^{-\frac{1}{2}}$. No confusion should arise. The nature of ν in a particular expression should be obvious from the context. The state space $\mathcal{H}_F$ can be identified with the set of square integrable holomorphic functions i.e.

$$\mathcal{H}_F = \left\{ \phi(z) : \frac{1}{2\pi\hbar} \int\int | \phi(z) |^2 \exp\left(-\frac{z\overline{z}}{2\hbar}\right) dpdq < \infty \right\}.$$

It can be shown that classical observables $c \in C^\infty(M, F, 1)$ are of the form

$$c = A\overline{z} + \overline{A}z + Bz\overline{z} + D$$

and

$$X_c = -2i(A + B\overline{z})\frac{\partial}{\partial \overline{z}} + 2i(A + Bz)\frac{\partial}{\partial z} \tag{1.16}$$

([5] page 72). For the canonical choice of connection potential (1.14) the expression for the covarient derivative of a section s_0 is

$$\nabla_X s_0 = -\frac{i}{\hbar}(X\rfloor - \frac{i}{2}\overline{z}dz)s_0. \tag{1.17}$$

In both the half form and half density schemes the classical observable c is quantised to give $\hat{c}$ where

$$\hat{c}\phi = (-i\hbar\nabla_{X_c}s + cs).\nu - i\hbar s.L_{X_c}\nu.$$

The difference between the two schemes comes about because of the difference between the Lie derivative of the canonical half density and the canonical half form. It can be shown that

$$L_{X_c}2^{\frac{1}{4}} \mid dz \mid^{-\frac{1}{2}} = 0$$

([5] page 114). Using (1.17) we obtain

$$\hat{c}\phi = (2\hbar(A + Bz)\frac{\partial\phi}{\partial z} + (\overline{A}z + D)\phi)s_0.\nu \tag{1.18}$$

([2] page 139). However

$$L_{X_c}2^{\frac{1}{4}}(d\overline{z})^{-\frac{1}{4}} = iB2^{\frac{1}{2}}(d\overline{z})^{-\frac{1}{2}}$$

so for the half form scheme we have

$$\hat{c}\phi = [2\hbar(A+Bz)\frac{\partial \phi}{\partial z} + (\overline{A}z + D + \hbar B)\phi]s_0.\nu \tag{1.19}$$

([2] page 212).

1.1.9 An Application of Geometric Quantisation and the Pairing Construction

Consider the configuration space $\mathbb{R}$ with global coordinate q and metric $g(q) = 1$. The cotangent bundle $M = \mathbb{R}^2$ has the canonical coordinate system (p, q). Wan and Sumner have demonstrated that every classical observable of the form

$$A_n = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} q^i p^j \tag{1.20}$$

can be quantised locally if the leading coefficient

$$C_n = \sum_{i=0}^{m} a_{in} q^i$$

is nowhere zero [10]. In this section we seek to generalize this result to the case where the leading coefficient can vanish. We denote by P the vertical polarisation, the distribution that at each point (p, q) of M assigns a vector in the subspace of $T_{(p,q)}(M)$ spanned by $\frac{\partial}{\partial p}$. $\mathcal{H}_P$ is then the space of all functions ϕ over $\mathbb{R}$ such that

$$\frac{1}{2\pi\hbar}\int_{\mathbb{R}} |\ \phi(q)\ |^2\, dq < \infty$$

([5] page 84). Consider the canonical coordinate transformation

$$p' = ap + bq \qquad q' = cp + dq \tag{1.21}$$

where a, b, c and d are constants, $c \neq 0$ and $d \neq 0$ and $ad - bc = 1$ ([5] page 82). The inverse transformation is

$$q = aq' - cp' \qquad p = dp' - bq'.$$

Let P' denote the polarisation for which (p', q') are the adapted coordinates, i.e. P' assigns to each point in M a vector in the subspace of $T_{(p,q)}(M)$ spanned by $\frac{\partial}{\partial p'}$. $\mathcal{H}_{P'}$ consists of functions ψ over $Q' = M/P'$ such that

$$\frac{1}{2\pi\hbar}\int_{\mathbb{R}} |\ \psi(q')\ |^2\, dq' < \infty.$$

As a result of the transformation (1.21) a classical observable of the form (1.20) acquires a coordinate representation

$$\sum_{i=0}^{m'}\sum_{j=0}^{n'} a'_{ij} q'^{i} p'^{j}$$

or more explicitly

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{s=0}^{i}\sum_{r=0}^{j} a_{ij}\frac{i!}{(i-s)!s!}\frac{j!}{(j-r)!r!}a^{i-s}d^{j-r}(-c)^{s}b^{r}q'^{i+r-s}p'^{s+j-r}.$$

Clearly the leading coefficient is

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{s=0}^{i}\sum_{r=0}^{j} a_{ij}\frac{i!}{(i-s)!s!}\frac{j!}{(j-r)!r!}a^{i-s}d^{j-r}(-c)^{s}b^{r}q'^{i+r-s}$$

where $s+j-r=n+m$. It is easy to see that the solution of this equation is unique and the leading coefficient is $a_{mn}(-c)^{m}(d)^{n}$ i.e. a constant $\neq 0$. Let $\zeta'(q')$ be a localizing function with support consisting of any interval in the q' coordinate curve [10]. Applying the local polynomial quantisation scheme of Wan and Sumner we have

$$\hat{A}'_{n'} = \sum_{j=0}^{n'} C'_j(q')(-i\hbar)^j \left(\zeta'(q')\frac{\partial}{\partial q'} + \frac{1}{2}\frac{d\zeta'(q')}{dq'}\right)^j.$$

This operator is essentially self-adjoint on the domain $C_0^{\infty}(\mathbb{R}) \in \mathcal{H}_{P'}$. The pairing construction leads to a unitary map $U:\mathcal{H}_P \rightarrow \mathcal{H}_{P'}$ given by

$$(U\phi)(q') = \frac{1}{(2\pi\hbar)}\int_{-\infty}^{\infty}\phi(q)\exp\left(i\frac{(\frac{1}{2}aq'^2 - qq' + \frac{d}{2}q^2)}{c\hbar}\right) \mid \mathrm{c} \mid^{-\frac{1}{2}} dq$$

$$(U^{-1}\psi)(q) = \int_{-\infty}^{\infty}\psi(q')\exp\left(-i\frac{(\frac{1}{2}aq'^2 - qq' + \frac{d}{2}q^2)}{c\hbar}\right) \mid \mathrm{c} \mid^{-\frac{1}{2}} dq'$$

[5]. It follows that the observable in the (p,q) coordinate system is

$$\hat{A}_n = U^{-1}\hat{A}'_{n'}U = U^{-1}\sum_{j=0}^{n'} C'_j(q')(-i\hbar)^j\left(\zeta'(q')\frac{\partial}{\partial q'} + \frac{1}{2}\frac{d\zeta'(q')}{dq'}\right)^j U$$

$$= \sum_{j=0}^{n'} U^{-1}C'_j(\tilde{q}')U\left[U^{-1}(-i\hbar)\left(\zeta'(q')\frac{\partial}{\partial q'} + \frac{1}{2}\frac{d\zeta'(q')}{dq'}\right)U\right]^j.$$

We have

$$\hat{q}' \equiv U^{-1}\tilde{q}'U = -i\hbar c\frac{\partial}{\partial q} + dq,$$

where $\tilde{q}'$ denotes the self-adjoint multiplication operator in $\mathcal{H}_{P'}$, and

$$U^{-1}(-i\hbar)\left(\zeta'(q')\frac{\partial}{\partial q'} + \frac{1}{2}\frac{d\zeta'}{dq'}(q')\right)U = \frac{i\hbar}{2}\left(\frac{d\zeta'}{dq'}(q')\hat{q}' + \frac{a}{c}\hat{q}'\zeta'(\hat{q}') - \frac{q}{c}\zeta'(\hat{q}')\right)$$

([5] page 119) so that

$$\hat{A}_n = \sum_{j=0}^{n'} C'_j\left(\hat{q'}\right)\left[\frac{i\hbar}{2}(\zeta'_{,q'}(\hat{q'}) + \frac{a}{c}\hat{q'}\zeta'(\hat{q'}) - \frac{q}{c}\zeta'(\hat{q'})\right]^j .$$

The physical interpretation of localisation in Q' is particularly straightforward when $c = 1$ since in that case (q, q') is a canonical coordinate system so we are simply localising in momentum space.

1.2 Symmetric Observables

It is generally assumed that a quantisation rule must consist of an algorithm associating with each classial observable a unique self-adjoint operator, unique so as to avoid any arbitrariness in the resulting quantum theory and self-adjoint so that the statistical interpretation be straight forward via the projector valued measure. This idee fixe has caused the subject to founder somewhat but recently a new scheme has been proposed which promises to overcome the present impasse. In a sense it might be said that the new scheme admits symmetric operators as observables. The form of these symmetric operators will usually be suggested quite strongly when formal quantisation rules are applied to the algebraic functions representing classical observables. It is hoped that the statistcal interpretation can be carried out because an entity very similar to the projector valued measure defining a self-adjoint operator can also be associated with a symmetic operator. We begin with a brief outline of the relevant mathematics in abstract. These results are taken from [27] and [9].

Symmetric Operators and Their Extensions

We shall often be concerned with the existence and classification of symmetric extensions of a symmetric operator A with dense domain in a seperable Hilbert space $\mathcal{H}$. The theory of deficiency indices will therefore play an important part in what follows. If R denotes the range of an operator then we have the following

Definition 14 *Let λ be an arbitrary non real number. Denote by M_λ and $M_{\bar{\lambda}}$ $R(A-\lambda)$ and $R(A-\bar{\lambda})$ respectively then the deficiency or defect spaces N_λ and $N_{\bar{\lambda}}$ are given by*

$$N_\lambda = \mathcal{H} \ominus M_\lambda \qquad N_{\bar{\lambda}} = \mathcal{H} \ominus M_{\bar{\lambda}}.$$

If m=dimN_λ and n=dim$N_{\bar{\lambda}}$ then we say that m and n are the deficiency or defect indices of A.

Alternatively

Definition 15

$$N_\lambda = \left\{\phi \in D(A^*) : (A^* - \overline{\lambda})\phi = 0\right\}$$

and

$$N_{\overline{\lambda}} = \{\phi \in D(A^*) : (A^* - \lambda)\phi = 0\}.$$

There exists a general method for constructing all the closed extensions of a closed symmetric operator A.

Theorem 4 *Every closed symmetric extension A' of the closed symmetric operator A is determined by an isometric operator U with domain D_U (a closed subspace of $N_{\bar{\lambda}}$) and range R_U (a closed subspace of N_λ). More precisely*

$$D_{A'} = \{\phi' \in \mathcal{H} : \phi' = \phi + \psi - U\psi, \phi \in D_A, \psi \in D_U\} \tag{1.22}$$

and

$$A'\phi' = A\phi + \lambda\psi - \bar{\lambda}U\psi. \tag{1.23}$$

The deficiency spaces of A' denoted N'_λ and $N'_{\bar{\lambda}}$ are related to the defect spaces of A by

$$N'_\lambda = N_\lambda \ominus R_U \quad \text{and} \quad N'_{\bar{\lambda}} = N_{\bar{\lambda}} \ominus D_U.$$

Clearly then with an obvious notation

$$m' = m - \dim R_U \tag{1.24}$$

$$n' = n - \dim D_U. \tag{1.25}$$

For the case of m and n finite we can obtain a less abstract expression for the generic form of the extensions A' of A. We illustrate the method for the case $m = n$. Choose an orthonormal basis ϕ_i for $N_{\bar{\lambda}}$ and an orthonormal basis ϕ'_i for N_λ where $i = 1 \to n$. Now every isometric operator between finite dimensional spaces is unitary and can be associated with a unitary matrix W_{ij} which determines its action according to the following equations where we have used the Einstein summation convention

$$U\phi_i = W_{ik}\phi'_k.$$

If $\psi = \epsilon_i \phi_i$ then $U\psi = U\epsilon_i\phi_i = \epsilon_i U\phi_i = \epsilon_i W_{ik}\phi'_k$ and equations (1.22) and (1.23) become

$$\phi' = \phi + \epsilon_i\phi_i - \epsilon_i W_{ik}\phi'_k$$

$$A'\phi' = A\phi + \lambda\epsilon_i\phi_i - \bar{\lambda}\epsilon_i W_{ik}\phi'_k.$$

Equations (1.24) and (1.25) reduce to

$$m' = m - \text{dim}D_U \quad \text{and} \quad n' = n - \text{dim}D_U.$$

Perhaps one the most familiar results from the theory of deficiency indices is the following

Theorem 5 *A closed operator A is self-adjoint iff both its defect indices are zero.*

In fact the self-adjoint operators are a subset of the maximally symmetric operators. These will play an important part in what follows.

Definition 16 *A closed symmetric operator A is maximally symmetric iff at least one of its defect indices is zero.*

Th 4 serves to make the following results immediatly obvious.

Theorem 6 *Maximally symmetric operators admit no proper symmetric extensions.*

The construction given in Th 4 trivially fails to provide any proper symmetric extensions of a maximally symmetric operator.

Theorem 7 *A has a maximally symmetric extension.*

To see this take $D_U = N_{\bar{\lambda}}$ if $n < m$ and $R_U = N_\lambda$ if $n > m$. Of course if A is maximally symmetric this gives A as its own maximally symmetric extension. The definition of U is still largely arbitrary so in general there exists no unique maximally symmetric extension of a symmetric operator A. Trivially the maximally symmetric extension of a maximally symmetric operator is unique.

Theorem 8 *If $m \neq n$ then none of the extensions A' of A is self-ajoint.*

Suppose the contrary. Then, from Th 1, $m' = 0$ and $n' = 0$ so that from (1.24) and (1.25)

$$m - \text{dim}R_U = 0 \qquad n - \text{dim}R_U = 0$$

with $m \neq n, m > 0, n > 0$ which clearly leads to a contradiction.

Until now we have tacitly assumed that A and its symmetric extensions A' (if they exist) are operators on the same Hilbert space $\mathcal{H}$. We propose the following generalization.

Definition 17 *Let A be a symmetric operator on H and let $\mathcal{H}^+$ be a be a Hilbert space such that $\mathcal{H} \subset \mathcal{H}^+$ then every symmetric operator B^+ on $\mathcal{H}^+$ such that $A \subset B^+$ is called a symmetric extenion of A.*

Clearly

$$D_A \subset D_{B+} \cap \mathcal{H} \subset D_{B+}.$$

This gives rise to three distinct cases depending on which of these inclusions is proper and which improper.

$$D_A \subset D_{B+} \cap \mathcal{H} = D_{B+} \qquad \text{Type 1}$$

This corresponds to the case of extension without exit described above. The genuinely new situations occur when

$$D_A = D_{B+} \cap \mathcal{H} \subset D_{B+} \qquad \text{Type 2}$$

and

$$D_A \subset D_{B+} \cap \mathcal{H} \subset D_{B+}. \qquad \text{Type 3}$$

It is not difficult to see that

Theorem 9 *A maximally symmetric operator admits extensions of type 2 only.*

By definition such an operator has no extensions of type 1. Suppose an extension B^+ of type 3 exists then by restricting B^+ to $\mathcal{H}$ we obtain an extension of type 1 and derive a contradiction.

Theorem 10 *Every symmetric operator A defined on an Hilbert space $\mathcal{H}$ with arbitrary defect indices (m,n) can be extended to a self-adjoint operator B^+ on $\mathcal{H}^+ \supset \mathcal{H}$.*

The following results are very important in that they demonstrate to what extent general symmetric operators resemble self-adjoint operators.

Definition 18 *A resolution of the identity is an operator valued function F on $\mathbb{R}$ such that F_λ is a bounded positive hermitian operator where*

$$F_{\lambda_1} \geq F_{\lambda_2} \quad \lambda_1 \geq \lambda_2$$

$$F_\lambda = \lim_{\lambda' \to \lambda - 0} F_{\lambda'} \ \forall \ \lambda \in \mathbb{R}$$

$$\lim_{\lambda \to -\infty} F_\lambda = 0 \ \textit{and} \ \lim_{\lambda \to +\infty} F_\lambda = I.$$

Following Naimark [9] we use the term Hermitian to denote a symmetric operator whose domain is not dense. This is not an essential distinction but it is conceptually usefull in dealing with symmetric operators with extensions with exit. Notice also that positive operators are understood to be symmetric but again we follow Naimark and say positive symmetric operator. The resolution of the identity is clearly a generalisation of the orthogonal resolution of the identity or orthogonal spectral function one meets in connection with self-adjoint operators. Similarly the positive operator valued measure defined next generalizes the concept of spectral measure.

Definition 19 *A positive operator valued measure (or P.O.V measure) is a map M which assigns an element $M(b)$ of $\{P\}$ (the set of positive symmetric operators) to each element b of the Boolean σ algebra B of the measure space $(\mathbf{R},B)$ (where B denotes Borel sets) such that*

$$M(\mathbf{R}) = I$$

$$M(\cup_{j=1}^{\infty} b_j) = \sum_{j=1}^{\infty} M(b_j), \quad b_i \cap b_j = \phi$$

where $\sum_{j=1}^{\infty} M(b_j) = w\text{-}lim \sum_{j=1}^{n} M(b_j)$.

If $\Delta = (t_1, t_2)$ the operator $F(\Delta) = F_{t_1} - F_{t_2}$ is a P.O.V measure on $\mathbf{R}$. We have the following integral type representation of symmetric operators

Theorem 11 *Let A be a symmetric operator in H with a self-adjoint extension B^+ in $\mathcal{H}^+$. Let F_t^{0+} be the orthogonal spectral function of B^+ and P^+ the projection operator of $\mathcal{H}^+$ on $\mathcal{H}$. Put*

$$F_t = P^+ F_t^{0+} \tag{1.26}$$

then for $f \in D_A$ and $g \in \mathcal{H}$ we have

$$(Af, g) = \int_{-\infty}^{\infty} t d(F_t f, g) \tag{1.27}$$

and

$$\| Af \|^2 = \int_{-\infty}^{\infty} t^2 d(F_t f, f). \tag{1.28}$$

We make the following

Definition 20 *If A is a symmetric operator and F_t a resolution of the identity such that (1.27) and (1.28) are true $\forall f \in D_A$ and $g \in \mathcal{H}$ then F_t is said to be a spectral function of A.*

Notice that a general symmetric operator is not associated with a unique resolution of the identity. Theorem 11 shows that every self-adjoint extension of A generates a spectral function of A via (1.26). That all the spectral functions of a symmetric operator A are generated by its self-adjoint extensions in this way is the content of the next

Theorem 12 *Every spectral function F_t of a symmetric operator A on $\mathcal{H}$ has the form $P^+F_t^{0+}$ where F_t^{0+} is the orthogonal spectral function of some self-adjoint extension B^+ of A in $\mathcal{H}^+$ and P^+ is the projection operator of $\mathcal{H}^+$ on $\mathcal{H}$.*

For maximally symmetric operators the situation is somewhat different in that

Theorem 13 *A symmetric operator has a unique spectral function iff it is maximal. This spectral function is orthogonal iff the operator is self-adjoint.*

Note that a maximally symmetric operator that is not self-adjoint has many self-adjoint extensions of type 2 each associated with a distinct F_t^{0+}. The remarkable fact is that all the operators $P^+F_t^{0+}$ on $\mathcal{H}$ are identical. Therefore any self-adjoint extension of a maximally symmetric operator will generate its unique generalised spectral function. We should also notice that in contrast to self-adjoint operators the domain of a symmetric operator generally defies description in terms of an integral representation with respect to an arbitrary resolution of the identity which will natually tend to overestimate it. Certainly

$$\int_{-\infty}^{\infty} t^2 d(F_t f, f) < \infty \ \forall \ f \in D_A$$

but, in general, also for many more f besides. In fact it can be shown that

$$\int_{-\infty}^{\infty} t^2 d(F_t f, f) < \infty \text{ iff } f \in D_{B^+} \cap \mathcal{H}$$

where of course B^+ is the self-adjoint extension of A generating F_t. Since $D_A \subset D_{B^+}$ we have $D_A \cap \mathcal{H} \subset D_{B^+} \cap \mathcal{H}$. Since $D_A \cap \mathcal{H} = D_A$ this implies $D_A \subset D_{B^+} \cap \mathcal{H}$ with, in general, proper inclusion. However for a maximally symmetric operator A each self-adjoint extension will be of type 2 so that $D_A = D_{B^+} \cap \mathcal{H}$. We therefore have the following

Theorem 14 *For any maximally symmetric operator A there exists a unique spectral function F_t such that (1.27) and (1.28) hold and*

$$\int_{-\infty}^{\infty} t^2 d(F_t f, f) < \infty \ \textit{iff} \ f \in D_A. \tag{1.29}$$

For an arbitrary symmetric operator the situation is not entirely hopeless since

Theorem 15 *For a general symmetric operator there exist spectral functions satisfying (1.27), (1.28) and (1.29).*

For any symmetric operator there exist self- adjoint extensions of type 2. Any such extension will generate a specral function satisfying the required conditions.

Symmetric Observables and Generalised Quantum Mechanics

Quantisation schemes generally lead, in the first instance, to symmetric operators that are not essentially self-adjoint. If the operator has equal (non zero) defect indicies it will have many self-adjoint extensions. If the symmetric operator is not maximally symmetric and has unequal defect indicies then it will have no self-adjoint extensions at all but it will have many maximally symmetric extensions. We have seen that maximally symmetric operators resemble self-adjoint operators in that they are associated to a unique resolution of the identity that weakly determines the operator and describes its domain. This resolution of the identity or the concomitant P.O.V measure allows us to define what it means to measure a symmetric observable. Following Born we say that

$$\frac{< \phi, F(\Delta)\phi >}{|| \phi ||}$$

represents the probability that the measurement of a prepared state ϕ will return a value in Δ. It would appear that we can regard maximally symmetric operators as quantum observables so that there is no more difficulty in handling a symmetric operator with no self-adjoint extensions but many maximally symmetric extensions than there is in dealing with a symmetric operator with many self-adjoint extensions. To quantise an observable we apply a quantisation rule. If the resulting operator is not essentially self-adjoint or maximally symmetric then we determine its maximal symmetric or self-adjoint extensions. In general there will be no way of deciding which extension should actually represent the observable. The expectation values of all the extensions will agree on most states anyway. Of course if we allow for extensions with exit and are prepared for an even more arbitrary quantisation scheme then Th 10 tells us that we can always obtain a self- adjoint operator. However it is probably best to take the view that extensions with exit should only be used as a means of realising P.O.V measures of maximally symmetric operators and that vectors in the larger Hilbert space are, as it were, non physical.

1.2.1 Symmetric Differential Operators

The application of formal quantisation rules to classical observables yields only formal differential operators. Fortunately there is a proceedure for constructing well defined operators from formal differential expressions. The main reference for this section is [6].

Consider the formal differential operator

$$\tau = \sum_{i=0}^{n} a_i(x) \frac{d^i}{dx^i}$$

of order n where $a_i(\ ) \in C^\infty(I)$ for some interval I of $\mathbf{R}$. If $a_n(x) > 0 \ \forall \quad x \in I$ we say τ is regular. Suppose that $A^n(I)$ represents the set of functions f which have $n-1$ continuous derivatives in I and for which f^{n-1} is absolutely continuous. Let $H^n_\tau(I)$ denote functions in $A^n(I)$ such that f and τf are in $L^2(I)$ and let $H^n(I)$ denote those functions f in $A^n(I)$ such that f and f^n are in $L^2(I)$. Let $H^n_0(I)$ denote the set of all functions in $H^n(I)$ which vanish outside some compact subset of the interior of I. From τ we define two operators $T_0(\tau)$ and $T_1(\tau)$ as follows

$$D(T_0(\tau)) = H^n_0(I), \quad T_0(\tau)f = \tau f, \quad f \in D(T_0(\tau))$$

and

$$D(T_1(\tau)) = H^n_\tau(I), \quad T_1(\tau)f = \tau f, \quad f \in D(T_1(\tau)).$$

The formal adjoint τ^* of τ is defined to be

$$\tau^* = \sum_{j=0}^{n} b_j(t) \frac{d^j}{dt^j}$$

where

$$b_j(t) = \sum_{k=j}^{n} (-1)^k \binom{k}{j} \left(\frac{d^{k-j}}{dt^{k-j}} \right) \overline{a_k(t)}.$$

If $\tau = \tau^*$ then τ is said to be formally self-adjoint. When τ is regular and formally self-adjoint we have that

$$T_0(\tau) \subset T_1(\tau) = T_0^*(\tau)$$

so $T_0(\tau)$ is symmetric.

Theorem 16 *If τ is formally self-adjoint the spaces N_λ and $N_{\bar\lambda}$ consist precisely of those solutions of $(\tau - i)f = 0$ and $(\tau + i)f = 0$ which belong to $L^2(I)$.*

A boundary value for a symmetric operator T is a continuous linear functional on $D(T^*)$. In the case of a formally self-adjoint formal differential operator this can be expressed as follows.

Definition 21 *Let τ be a formal differential operator on an interval I with end points c and d. A boundary value for τ is a continuous linear functional A on $D(T_1(\tau))$ which vanishes on $D(T_0(\tau))$. If $Af = 0 \; \forall f \in D(T_1(\tau))$ which vanish in a neighbourhood of c then A will be called a boundary value at c. Similarly boundary value at d.*

Theorem 17 *The space of boundary values for a formal differential operator τ is the direct sum of the boundary values for τ at a and the boundary values for τ at b.*

Theorem 18 *Let τ be a formal differential operator of order n on an interval I with end points c and d and suppose the end point c is fixed (i.e. $< \infty$). Then the functionals $A_i f = f^i(c)$ $i = 0 \rightarrow n-1$ form a complete set of boundary values at c.*

Definition 22 *An equation Bf=0 where B is a boundary value for τ is called a boundary condition for τ.*

Theorem 19 *Consider a formally self-adjoint formal differential operator τ. Suppose $T_0(\tau)$ has finite defect indices. Let $A_1 ... A_p$ be any complete set of boundary values for τ. If we introduce on $D(T_1(\tau))$ the following bilinear form*

$$[\psi_1, \psi_2] = -i(T_1(\tau)\psi_1, \psi_2) + i(\psi_1, T_1(\tau)\psi_2) \tag{1.30}$$

then it turns out that under these conditions

$$[\psi_1, \psi_2] = \sum_{i,j=1}^{p} \alpha_{ij} A_i \psi_1 \bar{A_j} \psi_2$$

where $\alpha_{ij} = \overline{\alpha_{ji}}$.

Theorem 20 *Given a complete set $A_1 ... A_p$ of boundary values for a formally self-adjoint formal differential operator τ we find that the α_{ij} appearing above are uniquely determined by any set of elements $\psi_1 ... \psi_p$ of $D(T_1(\tau))$ satisfying $det(A_i \psi_j) \neq 0$ according to the equations*

$$\alpha_{ij} = \sum_{k,l=1}^{p} [\psi_k, \psi_l] b_{ki} \bar{b_{lj}}$$

where b_{ij} is the matrix inverse to $A_i \psi_j$.

Definition 23 *A set of boundary conditions $B_i = 0$ $i = 1 \rightarrow k$ is said to be symmetric if the equations $B_i \psi_1 = B_i \psi_2 = 0 \Rightarrow [\psi_1, \psi_2] = 0$.*

Theorem 21 *Let T be a symmetric operator with equal finite defect indices (= n say). Then the restriction of T^* to the subspace of $D(T^*)$ determined by any symmetric family of n linearly independent boundary conditions is a self-adjoint extension of T and ANY self-adjoint extension of T is of this form.*

Example

We can illustrate these results if we consider the special case

$$\tau = i\frac{d}{dx} \quad I = [a, b]. \tag{1.31}$$

τ is then formally self-adjoint and regular. We can therefore define a symmetric operator $T_0(\tau)$ as follows

$$D(T_0(\tau)) = H_0^1(I), \quad T_0(\tau)f = i\frac{d}{dx}f, \quad f \in D(T_0(\tau))$$

and its adjoint

$$D(T_1(\tau)) = H_\tau^1(I), \quad T_1(\tau)f = i\frac{d}{dx}f \quad f \in D(T_1(\tau)).$$

We wish to determine the defect indices of $T_0(\tau)$. With τ and I as given in (1.31) it is trivial to show, using Th 16, that the defect spaces are spanned by the vectors

$$\left.\begin{array}{c} \psi_1 = e^x \\ \psi_2 = e^{-x} \end{array}\right\} \tag{1.32}$$

so that $T_0(\tau)$ has defect indices (1,1). We wish to determine all the self-adjoint extensions of $T_0(\tau)$. Of course the result is well known but the methods imployed in text book treatments are add hoc and require familiarity with obscure properties of various function spaces. Almost certainly more complicated examples could not be handled in this fashion. Our derivation will be essentially algebraic and more generally applicable. Theorems 17 and 18 tell us that for $\tau = i\frac{d}{dx}$ and I=[a,b] a complete set of boundary values are

$$A_1 f = f(a) \quad \text{and} \quad A_2 f = f(b).$$

We require the most general symmetric boundary condition that we can construct from these boundary values. The ψ_1 and ψ_2 appearing in (20) can be chosen to be the functions in (1.32). To see this notice that ψ_1 and ψ_2 are in the defect spaces of $T_0(\tau)$ so obviously ψ_1 and $\psi_2 \subset D(T_0^*(\tau))$ by definition 15. But $D(T_0^*(\tau)) = D(T_1(\tau))$ since τ is formally self-adjoint so ψ_1 and ψ_2 satisfy the first condition of theorem 20 i.e. that they should belong to $D(T_1(\tau))$. Now

$$A_i\psi_j = \begin{pmatrix} A_1\psi_1 & A_1\psi_2 \\ A_2\psi_1 & A_2\psi_1 \end{pmatrix} = \begin{pmatrix} e^a & e^{-a} \\ e^b & e^{-b} \end{pmatrix}$$

so

$$\det(A_i\psi_j) = \exp(a - b) - \exp(b - a).$$

Clearly then $A_i\psi_j \neq 0$, $A_i\psi_j$ is singular only in the degenerate case $a = b$. The inverse matrix is

$$b_{ij} = \frac{1}{(\exp(a-b) - \exp(b-a))} \begin{pmatrix} e^{-b} & e^{-a} \\ -e^{b} & e^{a} \end{pmatrix}.$$

We can proceed to calculate the α_{ij}. We require the terms $[\psi_k, \psi_l]$. For example

$$\begin{aligned}
[\psi_1, \psi_1] &= -i(i\frac{de^x}{dx}, e^x) + i(e^x, i\frac{de^x}{dx}) \\
&= -i\int_b^a (-i)e^x e^x dx + i\int_b^a e^x i e^x dx \\
&= -2\int_b^a e^{2x} dx \\
&= e^{2a} - e^{2b}.
\end{aligned}$$

Similarly we find that

$$[\psi_2, \psi_2] = e^{-2a} - e^{-2b}, \quad [\psi_2, \psi_1] = 0 \quad \text{and} \quad [\psi_1, \psi_2] = 0.$$

Then

$$\begin{aligned}
\alpha_{11} &= \frac{e^{2a} - e^{2b}}{(\exp(a-b) - \exp(b-a))^2} e^{-b}e^{-b} + \frac{e^{-2a} - e^{-2b}}{(\exp(a-b) - \exp(b-a))^2}(-e^b)(-e^b) \\
&= \frac{1}{(\exp(a-b) - \exp(b-a))^2}(\exp 2(a-b) - 2 - \exp 2(b-a)) \\
&= 1.
\end{aligned}$$

Similarly we can show that

$$\alpha_{22} = -1 \quad \text{and} \quad \alpha_{21} = \alpha_{12} = 0.$$

A general boundary condition is of the form

$$A_2 + \beta A_1 = 0.$$

We wish to determine the conditions on β for this boundary condition to be symmetric. We have

$$\begin{aligned}
[f, g] &= \alpha_{11}A_1 f \bar{A_1} g + \alpha_{12}A_1 f \bar{A_2} g + \alpha_{21}A_2 f \bar{A_1} g + \alpha_{22}A_2 f \bar{A_2} g \\
&= \alpha_{11} f(a)\overline{g(a)} + \alpha_{12} f(a) g(\bar{b}) + \alpha_{21} f(b)\overline{g(a)} + \alpha_{22} f(b)\overline{g(b)}. \qquad (1.33)
\end{aligned}$$

From definition 23 we require that if

$$A_2 f + \beta A_1 f = 0$$

i.e.

$$f(b) + \beta f(a) = 0$$

and

$$A_2 g + \beta A_1 g = 0$$

i.e.

$$g(b) + \beta g(a) = 0$$

then (1.33) should be zero. Clearly this is true if

$$\alpha_{11} f(a)\overline{g(a)} + \alpha_{12} f(a)\overline{(-\beta g(a))} + \alpha_{21}\beta f(a)\overline{g(a)} + \alpha_{22}(-\beta f(a))\overline{(-\beta g(a))} = 0 \qquad (1.34)$$

$$(\alpha_{11} - \alpha_{12}\bar{\beta} + \overline{\alpha_{12}\bar{\beta}} + \mid \beta \mid^2 \alpha_{22}) f(a)\overline{g(a)} = 0$$

$$(\alpha_{11} - \alpha_{12}\bar{\beta} + \overline{\alpha_{12}\bar{\beta}} + \mid \beta \mid^2 \alpha_{22}) = 0.$$

Substituting for α_{ij} from above gives

$$\mid \beta \mid^2 = 1.$$

1.2.2 Quantisation: Applications

Consider the two dimensional phase space $\mathbb{R}^2$ with coordinates (p, x) and the classical observable

$$x^{\frac{k}{l}} p. \qquad (1.35)$$

Deceptively simple as they may appear these observables have attracted much attention recently since they serve to illustrate a number of problems that arise in passing from classical to quantum mechanics (see Zhu and Klauder [77]). We wish to determine the quantum analogue of this observable. Geometric quantisation or the symmetrisation rule suggests that this is in some way related to the formal differential operator

$$\tau = -\frac{i\hbar}{2}\left(x^{\frac{k}{l}}\frac{d}{dx} + \frac{d}{dx}x^{\frac{k}{l}}\right)$$

or

$$\tau = -i\hbar\left(x^{\frac{k}{l}}\frac{d}{dx} + \frac{k}{2l}x^{\frac{k}{l}-1}\right). \qquad (1.36)$$

The properties of this formal expression were investigated by Wan and Sumner for the case l=1 and $k > 1$, i.e. $\frac{k}{l}$ an integer > 1 [28, 10]. They showed that in this particular case (1.36) did not lead to a unique self-adjoint operator. We shall show that this remains true

[3] even when $\frac{k}{l} \notin \boldsymbol{N}$ but that in the context of generalised quantum mechanics discussed in section 1.2 the expression (1.36) can give rise to sensible quantum observables. We shall consider only the case k and $l > 0$. Now (1.36) is not a formal differential operator in the sense of section 1.2.1 if $I = \boldsymbol{R}$ since then $x^{\frac{k}{l}} \notin C^{\infty}(\boldsymbol{R})$. In any case τ is not regular when $I = \boldsymbol{R}$ and we would like to avoid this if possible. We therefore remove the origin from the configuration space and consider (1.36) restricted to the positive and negative real axis. The quantum observable then emerges as a direct sum of two operators. We find that in the case k even and l odd we are led to associate with (1.35) the operator

$$-i\hbar\left(\mid x\mid^{\frac{k}{l}}\frac{d}{dx}-\frac{k}{2l}\mid x\mid^{\frac{k}{l}-1}\right)\oplus -i\hbar\left(x^{\frac{k}{l}}\frac{d}{dx}+\frac{k}{2l}x^{\frac{k}{l}-1}\right)$$

which has defect indicies (1,1). We shall show how to determine the self-adjoint extensions of this opertor using the Neumann formulas. When k is odd we quantise (1.35) as

$$-i\hbar\left(-\mid x\mid^{\frac{k}{l}}\frac{d}{dx}+\frac{k}{2l}\mid x\mid^{\frac{k}{l}-1}\right)\oplus -i\hbar\left(x^{\frac{k}{l}}\frac{d}{dx}+\frac{k}{2l}x^{\frac{k}{l}-1}\right).$$

When $k > l$ the defect indicies of this operator are (2,0) and when $k < l$ they are (0,2). In this case we obtain maximally symmetric observables . We shall show how we can derive the associated P.O.V measures in a simplified case. We now turn to the detailed analysis. First consider the case $I = (0, \infty)$ i.e. τ on $\boldsymbol{R}^{+}$. It is a trivial matter to check that τ is formally symmetric. We associate with τ the symmetric operator $T_0(\tau)$. We can use Th 16 of section 1.2.1 to calculate the defect indices of $T_0(\tau)$. N_{-i} consists of those functions ϕ such that $\phi \in L^2(\boldsymbol{R}^{+})$ and

$$-i\hbar\left(x^{\frac{k}{l}}\frac{d}{dx}+\frac{k}{2l}x^{\frac{k}{l}-1}\right)\phi - i\phi = 0.$$

The general solution of this equation is

$$\phi = Ax^{-\frac{k}{2l}}\exp\left(\frac{lx^{1-\frac{k}{l}}}{\hbar(k-l)}\right).$$

In Appendix 2.a we show that for $k > l$, $\phi \notin L^2(\boldsymbol{R}^{+})$ while for $k < l$, $\phi \in L^2(\boldsymbol{R}^{+})$. N_i consists of functions $\phi \in L^2(\boldsymbol{R}^{+})$ such that

$$-i\hbar\left(x^{\frac{k}{l}}\frac{d}{dx}+\frac{k}{2l}x^{\frac{k}{l}-1}\right)\phi + i\phi = 0.$$

[3] The geometric reason for this is obvious. Suppose $m \neq 1$ then the integral curve of $x^m d/dx$ is $x = [(t+c)(1-m)]^{\frac{1}{1-m}}$ where c is a constant. Clealy the vector field is incomplete. When m is even the solution exists for all t but reaches the origin at finite time. We shall see later that we are forced to exclude the origin to ensure regularity. When m is odd x becomes complex for some t.

The general solution of this equation is

$$\phi = Ax^{-\frac{k}{2l}} \exp\left(\frac{lx^{1-\frac{k}{l}}}{\hbar(l-k)}\right).$$

It is easy to show that for $k > l$, $\phi \in L^2(\mathbf{R}^+)$ but for $k < l$, $\phi \notin L^2(\mathbf{R}^+)$. Therefore we have that for $k > l$ the defect indices of $T_0(\tau)$ are (1,0) and for $k < l$ the defect indices are (0,1).

If we now consider the formal differential operator formed by associating with τ the interval $I = (-\infty, 0)$ then in general this operator will not be formally self-adjoint since $x^{\frac{k}{l}}$ need not be real. However for the case l odd we can choose the real root of this expression, i.e. we define

$$x^{\frac{k}{l}} = \mid x \mid^{\frac{k}{l}} (-1)^k. \tag{1.37}$$

Case 1: For k even $x^{\frac{k}{l}} = \mid x \mid^{\frac{k}{l}}$ and we are led to consider the formal differential operator

$$\tau_1 = -\frac{i\hbar}{2}\left(\mid x \mid^{\frac{k}{l}} \frac{d}{dx} + \frac{d \mid x \mid^{\frac{k}{l}}}{dx}\right)$$

on $\mathbf{R}^-$. Since

$$\frac{d \mid x \mid^{\frac{k}{l}}}{dx} = -\frac{k}{l} \mid x \mid^{\frac{k}{l}-1}$$

this becomes

$$\tau_1 = -i\hbar\left(\mid x \mid^{\frac{k}{l}} \frac{d}{dx} - \frac{k}{2l} \mid x \mid^{\frac{k}{l}-1}\right).$$

This operator is formally symmetric. We have

$$a_1 = -i\hbar \mid x \mid^{\frac{k}{l}} \quad \text{and} \quad a_0 = i\frac{\hbar k}{2l} \mid x \mid^{\frac{k}{l}-1}.$$

Therefore

$$\begin{aligned} b_0 &= \overline{a}_0 - \frac{d}{dx}\overline{a}_1 \\ &= -\frac{i\hbar k}{2l} \mid x \mid^{\frac{k}{l}-1} - \frac{d}{dx} i\hbar \mid x \mid^{\frac{k}{l}} \\ &= -\frac{i\hbar k}{2l} \mid x \mid^{\frac{k}{l}-1} - i\hbar\frac{k}{l}(-1) \mid x \mid^{\frac{k}{l}-1} \\ &= \frac{i\hbar k \mid x \mid^{\frac{k}{l}-1}}{2l} \end{aligned}$$

and

$$b_1 = -\overline{a_1} = -i\hbar \mid x \mid^{\frac{k}{l}}$$

so $\tau^* = \tau$.

We seek the defect indices of the associated symmetric operator $T_0(\tau_1)$ on $D(T_0(\tau_1)) = H_0^1(\mathbb{R}^-)$. N_{-i}^1 consists of those functions $\phi \in L^2(\mathbb{R}^-)$ such that

$$- i\hbar \left(| x |^{\frac{k}{l}} \frac{d}{dx} - \frac{k}{2l} | x |^{\frac{k}{l}-1} \right) \phi - i\phi = 0. \tag{1.38}$$

The general solution of this equation is

$$\phi = A | x |^{-\frac{k}{2l}} \exp \left(\frac{lx | x |^{-\frac{k}{l}}}{\hbar(k-l)} \right). \tag{1.39}$$

We can easily verify this. Since

$$\begin{aligned} \frac{d}{dx} | x |^{-\frac{k}{l}} &= - \left(-\frac{k}{l} \right) | x |^{-\frac{k}{l}-1} \\ &= \frac{k}{l} | x |^{-\frac{k}{l}-1} \end{aligned}$$

and

$$\frac{d}{dx} | x |^{-\frac{k}{2l}} = \frac{k}{2l} | x |^{-\frac{k}{2l}-1}$$

from (1.39) we obtain

$$\frac{d\phi}{dx} = A | x |^{-\frac{k}{2l}} \left\{ \frac{l}{\hbar(k-l)} \left(x\frac{k}{l} | x |^{-\frac{k}{l}-1} + | x |^{-\frac{k}{l}} \right) \right\} \exp \left(\frac{lx | x |^{-\frac{k}{l}}}{\hbar(k-l)} \right) +$$

$$A\frac{k}{2l} | x |^{-\frac{k}{2l}-1} \exp \left(\frac{lx | x |^{-\frac{k}{l}}}{\hbar(k-l)} \right).$$

Let $W = A \exp \left(\frac{lx|x|^{-\frac{k}{l}}}{\hbar(k-l)} \right)$. Using $x | x |^{-\frac{k}{2l}-1} = - | x |^{-\frac{k}{2l}}$ we have

$$\begin{aligned} \frac{d\phi}{dx} &= W | x |^{-\frac{k}{2l}} \frac{(-1)}{\hbar} | x |^{-\frac{k}{l}} + W\frac{k}{2l} | x |^{-\frac{k}{l}-1} \\ &= -\frac{W}{\hbar} | x |^{-\frac{3k}{2l}} + \frac{Wk}{2l} | x |^{-\frac{k}{2l}-1}. \end{aligned}$$

Substitute in (1.38)

$$-\frac{W}{\hbar} | x |^{-\frac{3k}{2l}} + \frac{Wk}{2l} | x |^{-\frac{k}{2l}-1} - \frac{k}{2l} | x |^{-1} W | x |^{-\frac{k}{2l}} + | x |^{-\frac{k}{l}} W \frac{| x |^{-\frac{k}{2l}}}{\hbar}$$

$$= 0$$

as required. In Appendix 2.b we show that for k even and $> l$, $\phi \in L^2(\mathbb{R}^-)$ and for k even and $< l$, $\phi \notin \mathbf{R}^-$.

N_i^1 consists of those functions $\phi \in L^2(\mathbb{R}^-)$ such that

$$-i\hbar \left(| x |^{\frac{k}{l}} \frac{d}{dx} - \frac{k}{2l} | x |^{\frac{k}{l}-1} \right) \phi + i\phi = 0.$$

The general solution of this equation is

$$\phi = A \mid x \mid^{-\frac{k}{2l}} \exp\left(\frac{lx \mid x \mid^{-\frac{k}{l}}}{\hbar(l-k)}\right).$$

When k even and $> l$, $\phi \notin L^2(\mathbb{R}^-)$ and for k even and $< l$, $\phi \in L^2(\mathbb{R}^-)$. Therefore when k even $< l$ the defect indices of the symmetric operator $T_0(\tau_1)$ are (1,0) and when $k > l$ they are (0,1).

Case 2: When k is odd we have from (1.37) that

$$x^{\frac{k}{l}} = - \mid x \mid^{\frac{k}{l}}$$

so the symmetrization rule leads us to consider the operator

$$\tau_2 = -\frac{i\hbar}{2}\left(- \mid x \mid^{\frac{k}{l}} \frac{d}{dx} - \frac{d \mid x \mid^{\frac{k}{l}}}{dx}\right)$$

$$= -i\hbar\left(- \mid x \mid^{\frac{k}{l}} \frac{d}{dx} + \frac{k}{2l} \mid x \mid^{\frac{k}{l}-1}\right).$$

which is formally symmetric. The proof is exactly the same as that given for τ_1.

We now determine the defect indices of the symmetric operator $T_0(\tau_2)$ on $H_0^1(\mathbb{R}^-)$. N_{-i}^2 consists of those functions $\phi \in L^2(\mathbb{R}^-)$ such that

$$-i\hbar\left(- \mid x \mid^{\frac{k}{l}} \frac{d}{dx} + \frac{k}{2l} \mid x \mid^{\frac{k}{l}-1}\right)\phi - i\phi = 0.$$

The general solutions are

$$\phi = A \mid x \mid^{-\frac{k}{2l}} \exp\left(\frac{lx \mid x \mid^{-\frac{k}{l}}}{\hbar(l-k)}\right).$$

For k odd and $> l$, $\phi \notin L^2(\mathbb{R}^-)$ while for k odd and $< l$, $\phi \in L^2(\mathbb{R}^-)$. N_i^2 consists of those functions $\phi \in L^2(\mathbb{R}^-)$ such that

$$-i\hbar\left(- \mid x \mid^{\frac{k}{l}} \frac{d}{dx} + \frac{k}{2l} \mid x \mid^{\frac{k}{l}-1}\right)\phi + i\phi = 0.$$

The general solution is

$$\phi = A \mid x \mid^{-\frac{k}{2l}} \exp\left(-\frac{lx \mid x \mid^{-\frac{k}{l}}}{\hbar(l-k)}\right).$$

For k odd and $> l$, $\phi \in L^2(\mathbb{R}^-)$ while for k odd and $< l$, $\phi \notin L^2(\mathbb{R}^-)$. Therefore for the case k odd and $> l$ the defect indicies of $T_0(\tau_2)$ are (1,0) and for k odd and $< l$ the defect indicies are (0,1).

We can use these results to classify the quantum observables representing (1.35). For the case of k even and $> l$ we are led to associate with (1.35) the symmetric operator

$$T = T_0(\tau_1) \oplus T_0(\tau) = -i\hbar\left(\mid x\mid^{\frac{k}{l}}\frac{d}{dx} - \frac{k}{2l}\mid x\mid^{\frac{k}{l}-1}\right) \oplus -i\hbar\left(x^{\frac{k}{l}}\frac{d}{dx} + \frac{k}{2l}x^{\frac{k}{l}-1}\right)$$

with dense domain $H_0(\mathbb{R}^-) \oplus H_0(\mathbb{R}^+) \subset L^2(\mathbb{R}^-) \oplus L^2(\mathbb{R}^+)$. The defect indices of this operator are (1,1). The defect spaces N_i and N_{-i} are spanned by the normalized basis vectors

$$0 \oplus \sqrt{\frac{2}{\hbar}}x^{-\frac{k}{2l}}\exp\left(\frac{lx^{1-\frac{k}{l}}}{\hbar(l-k)}\right)$$

and

$$\sqrt{\frac{2}{\hbar}}\mid x\mid^{-\frac{k}{2l}}\exp\left(\frac{lx\mid x\mid^{-\frac{k}{l}}}{\hbar(k-l)}\right) \oplus 0$$

respectively. We would like to use the Neumann formulas (Th 4) to determine the form of all the self-adjoint extensions of T; however that requires the determination of $\overline{T}$, i.e. the action and domain of the closure of T. Of course

$$D(\overline{T}) = D(\overline{T_0(\tau_1)}) \oplus D(\overline{T_0(\tau)})$$

([9] page 209). We shall show how we can determine $\overline{T_0(\tau)}$. Notice that $\overline{T_0(\tau)}$ exists since $T_0(\tau)$ is a densely defined symmetric operator. $\overline{T_0(\tau)}$ will be a symmetric extension of $T_0(\tau)$ and so will be a restriction of $T_1(\tau)$ to some domain determined by a set of symmetric boundary conditions on $D(T_1(\tau))$ (lemma 26 page 1236 [6]). This defines the action of $\overline{T_0(\tau)}$. Consider now the operator τ restricted to the intervals (0,c] and $[c,\infty)$. The defect indicies of these operators are (1,0) and (1,1) respectively. Denote by τ' the operator formed by restricting τ to $[c,\infty)$. Using Th 20 page 1299 [6] we see that τ has a boundary value at ∞ since τ' has a boundary value there. We know this because the set of boundary values of τ' are a 1 dimensional vector space consisting of the direct sum of a complete set of boundary values at c and a complete set at ∞ (see Th 17). τ' has a boundary value say R at c, i.e. $R(f) = f(c)$ and so one boundary value at ∞. Therefore τ has a complete set of boundary values consisting of a single boundary value B say at ∞. It has no more boundary values since the complete set of boundary values of τ is a 1-dimensinal vector space its defect indices being (1,0). Clearly there is only one symmetric set of boundary conditions (up to equivalence) namely $B(f) = 0$ and so the symmetric extension this induces must be $\overline{T_0(\tau)}$. We can give a marginally less abstract characterisation of $D(\overline{T_0(\tau)})$ since the precise form of B is known ([6] pages 1287 and 1303). In general one associates with any formal differential operator τ as defined in section 1.2.1 the $n \times n$ matrix

$$F_t^{lj}(\tau) = \sum_{i=j}^{n-l-1} (-1)^i \frac{i!}{(i-j)!j!} \frac{d^{i-j}}{dt^{i-j}} a_{l+i+1}(t) \ \ j+1 \leq n-1$$

$$F_t^{lj}(\tau) = 0 \ \ j+1 > n-1$$

where $0 \leq l, \ \ j \leq n-1$. If we define the bilinear expression

$$F_t(f,g) = \sum_{l,j=0}^{n-1} F_t^{lj}(\tau) f^l(t) \overline{g^j(t)}$$

then all boundary values B at an end point a of I, not nessarsarilly fixed, are of the form

$$B(f) = \lim_{t \to a} F_t(f, v) \tag{1.40}$$

where $v = -\tau g$, g being a solution of $\tau^* \tau g + g = 0$ such that $g \in D(T_1(\tau))$. In our particular case it is easy to verify that we must have

$$g = A x^{-\frac{k}{l}} \exp\left(\frac{l x^{1-\frac{k}{l}}}{\hbar(l-k)} \right)$$

then

$$v = -iAx^{-\frac{k}{l}} \exp\left(\frac{l x^{1-\frac{k}{l}}}{\hbar(l-k)} \right). \tag{1.41}$$

Therefore $\overline{T_0(\tau)} = T_1(\tau)$ restricted to that subset of its domain consisting of those funtions f satisfying (1.40) with ν given by (1.41) and $a = \infty$. $\overline{T_0(\tau_1)}$ can be determined in a similar fashion. We shall let $\psi_1 \oplus \psi$ denote a vector in $D(\overline{T})$ where $\psi_1 \in D(\overline{T_0(\tau_1)})$ and $\psi \in D(\overline{T_0(\tau)})$ then a vector in the domain of an arbitrary self-adjoint extension of T will be of the form

$$\psi_1 \oplus \psi + \alpha \left\{ \sqrt{\frac{2}{\hbar}} \mid x \mid^{-\frac{k}{2l}} \exp\left(\frac{l x \mid x \mid^{-\frac{k}{l}}}{\hbar(k-l)} \right) \oplus 0 + \beta \left(0 \oplus \sqrt{\frac{2}{\hbar}} x^{-\frac{k}{2l}} \exp\left(\frac{l x^{1-\frac{k}{l}}}{\hbar(l-k)} \right) \right) \right\}$$

$$= \left\{ \psi_1 + \alpha \sqrt{\frac{2}{\hbar}} \mid x \mid^{-\frac{k}{2l}} \exp\left(\frac{l x \mid x \mid^{-\frac{k}{l}}}{\hbar(k-l)} \right) \right\} \oplus \left\{ \psi + \beta\alpha \sqrt{\frac{2}{\hbar}} x^{-\frac{k}{2l}} \exp\left(\frac{l x^{1-\frac{k}{l}}}{\hbar(l-k)} \right) \right\}$$

where α is an arbitrary complex number and β is a complex number such that $\mid \beta \mid = 1$.

When k even and $< l$ the defect spaces of T i.e. N_i and N_{-i} will be spanned by the vectors

$$0 \oplus \sqrt{\frac{2}{\hbar}} x^{-\frac{k}{2l}} \exp\left(\frac{l x^{1-\frac{k}{l}}}{\hbar(k-l)} \right)$$

and

$$\sqrt{\frac{2}{\hbar}} \mid x \mid^{-\frac{k}{2l}} \exp\left(\frac{l x \mid x \mid^{-\frac{k}{l}}}{\hbar(l-k)} \right) \oplus 0.$$

If we let $\phi_2 \oplus \psi$ deonte an arbitrary vector in $D(\overline{T}) = D(\overline{T_0(\tau_1)}) \oplus D(\overline{T_0(\tau)})$ where of course $\phi_2 \in D(\overline{T_0(\tau_1)})$ then the domain of an arbitrary self-adjoint extension of T consists of vectors of the form

$$\left\{\phi_2 + \alpha\beta\sqrt{\frac{2}{\hbar}} \mid x \mid^{-\frac{k}{2l}} \exp\left(\frac{lx \mid x \mid^{-\frac{k}{l}}}{\hbar(l-k)}\right)\right\} \oplus \left\{\psi + \alpha\sqrt{\frac{2}{\hbar}} x^{-\frac{k}{2l}} \exp\left(\frac{lx^{1-\frac{k}{l}}}{\hbar(k-l)}\right)\right\}$$

where $\mid \beta \mid = 1$.

When k is odd we are led to associate with the classical observable 1.35 the symmetric operator

$$T_1 = T_0(\tau_2) \oplus T_0(\tau) = -i\hbar\left(- \mid x \mid^{\frac{k}{l}} \frac{d}{dx} + \frac{k}{2l} \mid x \mid^{\frac{k}{l}-1}\right) \oplus -i\hbar\left(x^{\frac{k}{l}} \frac{d}{dx} + \frac{k}{2l} x^{\frac{k}{l}-1}\right).$$

When k is odd and $> l$ the defect indices of this operator are (2,0). When k odd and $< l$ the defect indices are (0,2). To carry the theory through we should calculate the P.O.V measures of these operators but that seems to be very difficult. If we restrict the configuration space to $\mathbf{R}^+$ we can determine the P.O.V measures of the maximally symmetric observables. For example, consider the classical observable x^2p whence (1.36) becomes

$$\begin{aligned} &-i\hbar\left(x^2\frac{d}{dx} + x\right) \\ &= -i\hbar x^2 \frac{d}{dx} - i\hbar x \end{aligned} \tag{1.42}$$

on $(0,\infty)$. Now define the map $U : L^2((0,\infty), dx) \to L^2((-\infty, 0), ds)$

$$(Uf)(s) = f\left(-\frac{1}{\hbar s}\right)\left[\frac{1}{\sqrt{\hbar s}}\right].$$

This map is invertible. Its inverse is given by

$$(U^{-1}g)(x) = -\frac{1}{\sqrt{\hbar x}} g\left(-\frac{1}{\hbar x}\right).$$

This is easy to check

$$\begin{aligned} (U^{-1}Uf) &= U^{-1}\left(\frac{1}{\sqrt{\hbar s}} f\left(-\frac{1}{\hbar s}\right)\right) \\ &= -\frac{1}{\sqrt{\hbar x}} \frac{1}{\hbar\left(-\frac{1}{\hbar x}\right)} f\left(-\frac{1}{\hbar\left(-\frac{1}{\hbar x}\right)}\right) \\ &= f(x). \end{aligned}$$

U is also an isometry since if in

$$\int_0^\infty f(x)g^*(x)dx$$

we make the change of variable $x = -1/\hbar s$ we have

$$\int_{-\infty}^{0} f\left(-\frac{1}{\hbar s}\right) g^*\left(-\frac{1}{\hbar s}\right) \frac{1}{s^2} ds$$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{\hbar} s} f\left(-\frac{1}{\hbar s}\right) \frac{1}{\sqrt{\hbar} s} g^*\left(-\frac{1}{\hbar s}\right)$$

$$= \int_{-\infty}^{0} (Uf)(Ug)^* ds.$$

Therefore U is unitary. Now

$$UDU^{-1} = U\left(-i\hbar x^2 \frac{d}{dx} - i\hbar x\right)\left(-\frac{1}{\sqrt{\hbar} x} g\left(-\frac{1}{\hbar x}\right)\right)$$

$$= U\left[-i\hbar x^2 \frac{d}{dx}\left(\frac{(-1)}{\sqrt{\hbar} x} g\left(-\frac{1}{\hbar x}\right)\right) + \frac{i\hbar x}{\sqrt{\hbar} x} g\left(-\frac{1}{\hbar x}\right)\right]$$

$$= i\sqrt{\hbar} U\left[x^2 \frac{d}{dx}\left(\frac{1}{x} g\left(-\frac{1}{\hbar x}\right)\right)\right] + i\sqrt{\hbar} U\left[g\left(-\frac{1}{\hbar x}\right)\right]$$

$$= i\sqrt{\hbar}\frac{1}{\sqrt{\hbar} s}\left[\left(-\frac{1}{\hbar s}\right)^2 \frac{d}{d\left(-\frac{1}{\hbar s}\right)}\left(\frac{1}{\left(-\frac{1}{\hbar s}\right)} g\left(-\frac{1}{\hbar\left(-\frac{1}{\hbar s}\right)}\right)\right)\right] + i\sqrt{\hbar}\frac{1}{\sqrt{\hbar} s} g\left(-\frac{1}{\hbar\left(-\frac{1}{\hbar s}\right)}\right)$$

$$= \frac{i}{\hbar^2 s^3}\frac{d}{d(-\frac{1}{\hbar s})}(-\hbar s g(s)) + \frac{i}{s} g(s).$$

Put $\nu = -1/\hbar s$

$$= -\frac{i}{\hbar s^3}\frac{ds}{dv}\frac{d}{ds}(s g(s)) + \frac{i}{s} g(s).$$

Since

$$\frac{ds}{dv} = \hbar s^2$$

the above becomes

$$-\frac{i}{\hbar s^3}\hbar s^2 \frac{d}{ds}(s g(s)) + \frac{i}{s} g(s)$$

$$= -\frac{i}{s}\left(s\frac{dg}{ds} + g(s)\right) + \frac{i}{s} g(s)$$

$$= -i\frac{dg}{ds}.$$

UDU^{-1} is still maximally symmetric of course but it has an natural self-adjoint extension i.e. $-id/ds$ on $\boldsymbol{R}$. The Weyl Kodeira theorem shows that the spectral function of this operator is precisely what we might have guessed from a formal generalized eigenfunction expansion so that by Naimarks theorem (Th 12) the P.O.V measure of UDU^{-1} is

$$(F(\lambda)f)(s) = \int_{-\infty}^{\lambda} e(\lambda, s) < e(\lambda, s'), f(s') > d\lambda$$

where $s \in (-\infty, 0)$, $f \in L^2(-\infty, 0)$ and

$$< f, g >= \int_{-\infty}^{0} f^* g ds.$$

The P.O.V of (1.42) is therefore

$$((U^{-1}F(\lambda)U)f)(x) = -\frac{1}{\sqrt{\hbar x}} \int_{-\infty}^{\lambda} \exp\left(\lambda, -\frac{1}{\hbar x}\right) < \exp(\lambda, s'), \frac{1}{\sqrt{\hbar s'}} f(-\frac{1}{\hbar s'}) > d\lambda$$

where $x \in (0, \infty)$ otherwise symbols as above. We could also have considered the classical observable $x^{\frac{1}{2}}p$ on $\mathbb{R}^+$ in which case we would require the P.O.V measure of the maximally symmetric operator

$$\begin{aligned} &-i\hbar\left(x^{\frac{1}{2}}\frac{d}{dx} + \frac{1}{4}x^{-\frac{1}{2}}\right) \\ &= -i\hbar x^{\frac{1}{2}}\frac{d}{dx} - \frac{i\hbar}{4}x^{-\frac{1}{2}} \end{aligned} \tag{1.43}$$

on $(0, \infty)$. Define the unitary map $U : L^2((0, \infty), dx) \to L^2((0, \infty), ds)$ by

$$(Uf)(s) = f\left(\frac{\hbar^2 s^2}{4}\right)\left(\frac{\hbar^2 s}{2}\right)^{\frac{1}{2}}.$$

Again we find that the image of the operator (1.43) is

$$-i\frac{d}{ds}$$

so we can proceed to get the P.O.V measure as before.

1.3 The B.K.S Method

In this section we discuss the Blattner-Kostant-Sternberg (B.K.S) method which can be used to quantise observables f in a polarisation F even when $f \notin C^\infty(M, F, 1)$. We shall see that the scheme is unsatisfactory in many respects.

It is quite difficult to obtain an explicit expression for the operator representing a given classical observable using the B.K.S method since this requires that we are able to evaluate certain limits that are non trivial in general. It has been known for some time that the B.K.S method can give explicit results when used to quantise observables of the form $p^2 + V(q)$. In this case the operator obtained formally agrees with the Schrodinger operator derived using canonical quantisation. Until recently this was the most conspicuous success of the B.K.S method. The paucity of examples that could be handled using the scheme has meant that its significance has been difficult to access. However, recently, Bao and Zhu have shown that the B.K.S method can be carried through to obtain operator representations of classical observables of the form $f(q)p^2$ where $f(q) > 0$ and Tuynman's work has essentially completed the task of quantising observables using the B.K.S scheme. It is therefore apposite that we should review the B.K.S method and indicate some of its demerits.

First we should observe that there is no geometric criterion, analogous to the complete Hamiltonian vector field result described in section 1.1.7, that allows one to assertain if a classical observable will yield a self-adjoint operator when quantised using the B.K.S method. We shall show in sections 1.3.2 and 1.3.3 that such a result would be useful because in general the operators produced by quantising classical observables using the methods of Bao and Zhu and Tuynman are not essentially self-adjoint.

The quantisation scheme of Bao and Zhu is derived using elaborate geometric arguments but it turns out that it is nothing more than a special case of a generalised squaring axiom which we shall describe in section 1.3.4. This generalised squaring axiom has the added advantage that it allways produces self-adjoint operators. In some cases we can also recover the results of Bao and Zhu using a modified pairing map. Finally we show that Tuynman's method and that of Bao and Zhu produce different operators when applied to the same classical observable i.e. they are contradictory.

We begin by recalling the definition of the pairing or B.K.S kernal as it is often called in this context.

1.3.1 The B.K.S Kernal

Let F_1 and F_2 be a pair of compatible polarisations on a phase space M. Let $\mathcal{H}_{F_1}$ and $\mathcal{H}_{F_2}$ be the state spaces corresponding to F_1 and F_2 respectivly, i.e. those subsets of the set of all sections of the usual line bundle over M that are covariantly constant along the relevant polarisation and square integrable. There exists an intrinsically defined map

$$<\,,\,>_{F_1F_2} : \mathcal{H}_{F_1} \times \mathcal{H}_{F_2} \to \mathbb{C}$$

called the B.K.S kernal or pairing. This map $<\,,\,>_{F_1F_2}$ induces a linear map

$$U_{F_2F_1} : \mathcal{H}_{F_2} \to \mathcal{H}_{F_1}$$

via

$$<\sigma_1, \sigma_2>_{F_1F_2} = <\sigma_1, U_{F_2F_1}\sigma_2>_{\mathcal{H}_{F_1}}$$

where $\sigma_1 \in \mathcal{H}_{F_1}$ and $\sigma_2 \in \mathcal{H}_{F_2}$.

Quantisation

Recall that when we quantise a classical phase space we choose a polarisation F. This determines a state space $\mathcal{H}_F$. Consider a classical observable g. Suppose g generates a complete Hamiltonian vector field and so gives rise to one parameter group of diffeomorphisms of the phase space denoted ϕ_g^t. Let $\mathcal{T}\phi_g^t$ denote the derived map of ϕ_g^t. We have that g preserves F, i.e. $g \in C^\infty(M, F, 1)$ if

$$F_t = \mathcal{T}\phi_g^t F = F.$$

If g preserves F then we define the operator $\hat{g}$ on $\mathcal{H}_F$ by

$$\hat{g}\sigma = i\hbar \frac{d}{dt}(\phi_g^t \sigma) \mid_{t=0} \tag{1.44}$$

$\sigma \in \mathcal{H}_F$. This is nothing more than the intrinsic description of $\hat{g}$ given in section 1.1. Notice that here we have abused our notation somewhat since in the above ϕ_g^t actually denotes the lift of the action of ϕ_g^t to $\mathcal{H}_F$. This lifted ϕ_g^t is a vector space isomorphism ([4] page 103) i.e. ϕ_g^t is a one parameter family of unitary maps of $\mathcal{H}_F$ onto itself. Clearly $\hat{g}$ is self-adjoint since it is the infinitesimal generator of this one parameter family of unitary maps.

Now suppose that $F_t \neq F$, i.e. g does not preserve the polarisation F. Let $\mathcal{H}_{F_t}$ be the Hilbert space associated with the polarisation F_t. Suppose that F and F_t are compatible for (0 < t < ϵ). Then, as described above, we can introduce a B.K.S kernal

$$< , >_{F_tF} : \mathcal{H}_{F_t} \times \mathcal{H}_F \to \mathbb{C}$$

which induces a linear map $U_{F_tF} : \mathcal{H}_{F_t} \to \mathcal{H}_F$. For each $t \in (0, \epsilon)$ define $\Phi_t : \mathcal{H}_F \to \mathcal{H}_F$ by

$$\Phi_t = U_{F_tF}\phi_g^t \tag{1.45}$$

Now ϕ_g^t is a unitary map $\mathcal{H}_F \to \mathcal{H}_{F_t}$ if g generates a complete Hamiltonian vector field. Φ_t will be a one parameter family of unitary maps on F when U_{F_tF} is unitary. In that case the operator defined by

$$\hat{g} = i\hbar \frac{d}{dt}\Phi_t \mid_{t=0} \tag{1.46}$$

will be self-adjoint. However in most cases the operator U_{F_tF} derived from the B.K.S kernal will not be unitary and $\hat{g}$ will not be self-adjoint. Of course we could use any unitary map $\mathcal{H}_{F_t} \to \mathcal{H}_F$ in place of U_{F_tF} in (1.45) and this would give a self-adjoint operator but the point is that U_{F_tF} is derived intrinsically using geometric objects [4]. In general it is very difficult to prove that U_{F_tF} is unitary. Usually we use (1.46) to define the operator $\hat{g}$ on some dense domain and then attempt to find a self-adjoint extension. We shall see an example of this approach later.

1.3.2 The B.K.S Method a la Bao and Zhu

Suppose that F is the vertical polarisation P. If the function g is such that P and $\mathcal{T}\phi_g^t P$ are transverse then g may be quantised in P according to the prescription described above and what is more in this special case an explicit expression may be derived for the operator $\hat{g}$ [4]. It turns out that

$$\hat{g}\psi = (i\hbar)\lim_{t\to 0+}(ih)^{-\frac{1}{2}}\int dp \exp\left[\frac{i}{\hbar}\int_0^t (\theta\rfloor X_g - g)\phi_g^{-s}ds\right][\det\omega(X_q, \phi_g^t X_q)]^{\frac{1}{2}}\psi(\phi_g^t q) \quad (1.47)$$

(See equations (6.49) and (6.50) [4]). We now restrict ourselves to a discussion of observables of the form

$$g = f(q)p^2$$

where $f(q) > 0$. In Appendix 3 we show that in this case

$$\hat{g}\psi = (i\hbar)^2\left(f(q)\psi'' + f'(q)\psi'(q) + \left[\frac{f''(q)}{4} - \frac{(f'(q))^2}{16f(q)}\right]\psi\right) \quad (1.48)$$

as given in equation 21 of a paper by Bao and Zhu [14]. Using this we may formally quantise the classical observable qp^2 to obtain the operator

$$\tau = (i\hbar)^2\left(q\frac{d^2}{dq^2} + \frac{d}{dq} - \frac{k^2}{q}\right) \quad (1.49)$$

where $k = 1/4$. This formal differential operator is regular on $(0, \infty)$. We can show that $T_0(\tau)$ is not essentially self-adjoint by calculating its defect indicies. To this end we require the solutions of the following equation

$$\left((i\hbar)^2\left(q\frac{d^2}{dq^2} + \frac{d}{dq} - \frac{k^2}{q}\right) - \lambda\right)\phi = 0. \quad (1.50)$$

Notice that (1.50) is a Bessel equation since the most general form of Bessel's equation is

$$\frac{d}{dx^2} + \frac{1-2a}{x}\frac{d}{dx} + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right] = 0 \quad (1.51)$$

[4]Notice that when g preserves the polarisation the pairing becomes equal to the inner product on $\mathcal{H}_F$ i.e

$$<\sigma_1, \sigma_2>_{F_tF} = (\sigma_1, \sigma_2) = (\sigma_1, U_{FF}\sigma_2)$$

so then $U_{FF} = I$ and (1.44) is the same as (1.46).

(Eq 16.1 page 516 [8]) and putting $a = 0$, $c = 1/2$, $b = 2\sqrt{\lambda}/\hbar$ and $p = 2k$ we recover (1.50). The most general solution of (1.50) is therefore a linear combination of

$$J_{\frac{1}{2}}\left(\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right) \quad \text{and} \quad N_{\frac{1}{2}}\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right).$$

Suppose we consider the operator (1.49) restricted to $(0, a]$. As $q \to 0$ we have

$$J_{\frac{1}{2}}\left(\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right) \sim \frac{1}{\Gamma(\frac{3}{2})}\left(\frac{\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right)^{\frac{1}{2}} + R\left(\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right)^{\frac{5}{2}}$$

where R is a complex constant. This can be written as

$$J_{\frac{1}{2}}\left(\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right) \sim \frac{1}{\Gamma(\frac{3}{2})}\frac{\lambda^{\frac{1}{4}}}{\sqrt{\hbar}}q^{\frac{1}{4}} + R\frac{2^{\frac{5}{2}}}{\hbar^{\frac{5}{2}}}\lambda^{\frac{5}{4}}q^{\frac{5}{4}}$$
$$= r_1\lambda^{\frac{1}{4}}q^{\frac{1}{4}} + r_2R\lambda^{\frac{5}{4}}q^{\frac{5}{4}}$$

where r_1 and r_2 are real constants. Therefore

$$\left|J_{\frac{1}{2}}\left(\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right)\right|^2 \sim (r_1\lambda^{\frac{1}{4}}q^{\frac{1}{4}} + r_2R\lambda^{\frac{5}{4}}q^{\frac{5}{4}})(r_1\overline{\lambda}^{\frac{1}{4}}q^{\frac{1}{4}} + r_2\overline{R}\overline{\lambda}^{\frac{5}{4}}q^{\frac{5}{4}})$$

$$= r_1^2 \mid \lambda \mid^{\frac{1}{2}} q^{\frac{1}{2}} + r_1r_2 2Re(\lambda^{\frac{1}{4}}\overline{\lambda}^{\frac{5}{4}}\overline{R})q^{\frac{3}{2}} + r_2^2 \mid R \mid^2 \mid \lambda \mid^{\frac{5}{2}} q^{\frac{5}{2}}.$$

Clearly

$$\int_0^a \left|J_{\frac{1}{2}}\left(\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right)\right|^2 dq < \infty.$$

Also as $q \to 0$

$$N_{\frac{1}{2}}\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right) \sim -\frac{\Gamma(\frac{1}{2})}{\pi}\left(\frac{2}{\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}}\right)^{\frac{1}{2}} + R\frac{1}{\left(\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right)^{-\frac{3}{2}}}$$

$$\sim -\frac{\Gamma(\frac{1}{2})}{\pi}\sqrt{\hbar}\lambda^{-\frac{1}{4}}q^{-\frac{1}{4}} + R\frac{2^{\frac{3}{2}}\lambda^{\frac{3}{4}}}{\hbar^{\frac{3}{2}}}q^{\frac{3}{4}}$$
$$\sim r_1\lambda^{-\frac{1}{4}}q^{-\frac{1}{4}} + r_2R\lambda^{\frac{3}{4}}q^{\frac{3}{4}}.$$

Therefore

$$\left|N_{\frac{1}{2}}\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right)\right|^2 \sim r_1^2 \mid \lambda \mid^{-\frac{1}{2}} q^{-\frac{1}{2}} + r_1r_2 2Re(\overline{R}\lambda^{-\frac{1}{4}}\overline{\lambda}^{\frac{3}{4}})q^{\frac{1}{2}} + r_2^2 \mid R \mid^2 \mid \lambda \mid^{\frac{3}{2}} q^{\frac{3}{2}}.$$

Since

$$\int_0^a q^{-\frac{1}{2}}dq < \infty$$

we have

$$\int_0^a \left|N_{\frac{1}{2}}\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right)\right|^2 dq < \infty.$$

Therefore the defect indices of (1.49) on $(0, a]$ are (2,2). Now (1.49) on $(0, a]$ must have two boundary values at a (see cor 23 page 1301 and Th 19 page 1298 [6]), therefore it must have two boundary values at 0 (lemma 21 page 1234 [6]). Hence (1.49) on $(0, \infty)$ has two boundary values at 0 (Th 20 page 1299 [6]). Now consider the operator (1.49) on $[a, \infty)$. We could express any solution of (1.50) as a linear combination of the Hankel functions

$$H_{\frac{1}{2}}^{(+)}\left(\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right) \quad \text{and} \quad H_{\frac{1}{2}}^{(-)}\left(\frac{2\sqrt{\lambda}q^{\frac{1}{2}}}{\hbar}\right).$$

Let $\nu_{\pm,\lambda}$ denote the asymptotic approximations of the Hankel functions given above as $q \to \infty$. We have

$$\nu_{\pm,\lambda} \propto q^{-\frac{1}{4}} \exp\left[\pm i \frac{2\sqrt{\lambda}}{\hbar} q^{\frac{1}{2}}\right]$$

(see table page 525 [8]). Suppose $\lambda = i = \exp[\frac{i\pi}{2}]$ so that $\sqrt{\lambda} = \exp[i\frac{\pi}{4}] = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$. Then

$$\nu_{\pm,i} \propto q^{-\frac{1}{4}} \exp\left[\pm \frac{2q^{\frac{1}{2}}}{\hbar}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right].$$

Now

$$\int_a^\infty |\ \nu_{+,i}\ |^2\, dq = \int_a^\infty q^{-\frac{1}{4}} \exp\left[\frac{\sqrt{2}}{\hbar} q^{\frac{1}{2}}\right] dq.$$

Put $u = q^{\frac{1}{2}}$ so that $dq = 2q^{\frac{1}{2}} du$ and the integral becomes

$$\int_{\sqrt{a}}^\infty 2q^{-\frac{1}{4}} q^{\frac{1}{2}} \exp\left[\sqrt{\frac{2}{\hbar}} u\right] du$$

$$= \int_{\sqrt{a}}^\infty 2q^{\frac{1}{4}} \exp\left[\sqrt{\frac{2}{\hbar}} u\right] du$$

$$= \int_{\sqrt{a}}^\infty 2u^{\frac{1}{2}} \exp\left[\sqrt{\frac{2}{\hbar}} u\right] du$$

which clearly diverges. On the other hand

$$\int_a^\infty |\ \nu_{-,i}\ |^2\, dq = \int_{\sqrt{a}}^\infty 2u^{\frac{1}{2}} \exp\left[-\sqrt{\frac{2}{\hbar}} u\right] du.$$

Put

$$u = \left(\frac{\hbar}{2}\right)^{\frac{1}{4}} x$$

so

$$du = \sqrt{\frac{\hbar}{2}} 2x dx$$

and the integral becomes

$$\left(\frac{\hbar}{2}\right)^{\frac{3}{4}} 4 \int_{(\frac{2a}{\hbar})^{\frac{1}{4}}}^\infty x^2 \exp[-x^2] dx$$

$$< \left(\frac{\hbar}{2}\right)^{\frac{3}{4}} 4 \int_0^\infty x^2 \exp[-x^2] dx$$
$$= \left(\frac{\hbar}{2}\right)^{\frac{3}{4}} 4 \frac{\pi^{\frac{3}{2}}}{2\pi}$$
$$< \infty.$$

When $\lambda = -i$ we have

$$\nu_{\pm,-i} = q^{-\frac{1}{4}} \exp\left[\pm \frac{2q^{\frac{1}{2}}}{\hbar}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right].$$

We can investigate the square integrability of these solutions as we did above for the case $\lambda = i$. In this way we deduce that the defect indices of (1.49) on $[a, \infty)$ are (1,1). Now (1.49) on $[a, \infty)$ has two boundary values at a so no boundary values at ∞. By lemma 10.5.13 page 273 [7] the defect indices of (1.49) on $(0, \infty)$ are

$$(2 + 1 - 2, 2 + 1 - 2)$$

$$= (1, 1)$$

Therefore (1.49) is not essentially self-adjoint. It has a one parameter family of self-adjoint extensions and the quantisation is not unique.

We can give an explicit expression for the domain of the self-adjoint extensions. This will follow from standard Sturm Lioville theory. Define the bilinear form $< \, , \, >$ on $D(T_1(\tau))$ as follows

$$< f, g >= \lim_{q \to 0} -\hbar^2 q (f' \overline{g} - f \overline{g}').$$

We have the limit circle case at 0 and the limit point case at ∞ so the self-adjoint extensions are obtained by restricting $D(T_1(\tau))$ to those f such that

$$< f, g >= 0$$

where $g \in T_1(\tau)$,

$$< g, g >= 0 \tag{1.52}$$

and g linearly independent relative to $D(T_0(\tau))$ (Th 10.5.2 page 268 and Th 10.2.18 page 260 [7]). Notice that following example 10.5.12 page 272 [7] we can take

$$g = c_1 h_1 + c_2 h_2$$

where $h_1 = q^{\frac{1}{4}}$ and $h_2 = q^{-\frac{1}{4}}$ near 0, h_1 and h_2 vanish at infinity. It is easy to show that $\tau h_1 = \tau h_2 = 0$ so h_1 and h_2 are in $D(T_1(\tau))$. From (1.52) we obtain

$$-\hbar^2 q \left(\left(\frac{c_1}{4} q^{-\frac{3}{4}} - \frac{c_2}{4} q^{-\frac{5}{4}} \right) (\overline{c}_1 q^{\frac{1}{4}} + \overline{c}_2 q^{-\frac{1}{4}}) - (c_1 q^{\frac{1}{4}} + c_2 q^{-\frac{1}{4}}) \left(\frac{\overline{c}_1}{4} q^{-\frac{3}{4}} - \frac{\overline{c}_2}{4} q^{-\frac{5}{4}} \right) \right) = 0$$

$$-\hbar^2 q\left(\frac{c_1}{4}\overline{c}_1 q^{-\frac{1}{2}}+\frac{c_1\overline{c}_2}{4}q^{-1}-\frac{c_2\overline{c}_1}{4}q^{-1}-\frac{c_2\overline{c}_2}{4}q^{-\frac{3}{2}}-\frac{c_1\overline{c}_1}{4}q^{-\frac{1}{2}}+c_1\frac{\overline{c}_2}{4}q^{-1}-\frac{c_2\overline{c}_1}{4}q^{-1}+\frac{c_2\overline{c}_2}{4}q^{-\frac{3}{2}}\right)=0$$

$$-\frac{\hbar^2}{2}(c_1\overline{c}_2 - c_2\overline{c}_1) = 0.$$

This is satisfied if

$$c_1\overline{c}_2 = c_2\overline{c}_1 = \overline{(c_1\overline{c}_2)}$$

i.e. $c_1\overline{c}_2 = R$ where R is a real parameter. It is easy to show that this is equivalent to $(c_2/c_1) = | c_2 |^2 /R = 1/\tan\theta$ say. The domains of the one parameter family of self-adjoint extensions comprise $f \in D(T_1(\tau))$ such that

$$\lim_{q\to 0} q\left(\left(\frac{\overline{c}_1}{4}q^{-\frac{3}{4}} - \frac{\overline{c}_2}{4}q^{-\frac{5}{4}}\right) f - (\overline{c}_1 q^{\frac{1}{4}} + \overline{c}_2 q^{-\frac{1}{4}})f'\right) = 0$$

or

$$\lim_{q\to 0} q\left(\left(\tan\theta\frac{q^{-\frac{3}{4}}}{4} - \frac{1}{4}q^{-\frac{5}{4}}\right) f - (\tan\theta q^{\frac{1}{4}} + q^{-\frac{1}{4}})f'\right) = 0.$$

In Appendix 4 we show how we can quantise qp^2 using a modified pairing map. The operator we obtain formally agrees with that derived by Bao and Zhu but is positive definite and essentially self-adjoint.

1.3.3 Quantisation a la Tuynman

In section 1.1 we gave explicit formulae for the operators representing classical observables $\in C^\infty(M, F, 1)$. These were derived using the notion of a partial connection although they could have been obtained using the B.K.S method. The quantisation of observables that preserve the polarisation is a rather trivial example of a situation where the B.K.S scheme can be carried through to yield explicit expressions. Bao and Zhu identified a larger class of observables for which the kernal could be found in closed form but recently Tuynman has shown that the B.K.S scheme can in fact be used to quantise any classical observable with respect to a Kahler polarisation. Tuynman's quantisation of an arbitrary classical observable f is in two parts. The B.K.S method associates with each f an operator L_f that acts on the space of holomorphic sections, the elements of the quantum Hilbert space of the canonical Kahler polarisation F (1.15), according to

$$L_f(\phi(z)s_0.\nu) = \left(2\hbar\frac{\partial f}{\partial \overline{z}}\frac{d}{dz} + \left(f - \overline{z}\frac{\partial f}{\partial \overline{z}} + \hbar\frac{\partial^2 f}{\partial z \partial \overline{z}}\right)\right)\phi(z)s_0.\nu. \qquad (1.53)$$

Notice that this reduces to (1.19) when $f \in C^\infty(M, F, I)$, i.e. $f = A\overline{z} + \overline{A}z + Bz\overline{z} + D$. Since this at least makes the result plausible we shall omit the proof. A second stage in the quantisation proceedure is necessary because in general $L_f(g(z)s_0.\nu)$ is not a holomorphic section since the factor multiplying $\phi(z)$ on the right hand side of (1.53) may not be independent of $\overline{z}$. Tuynman proposes the quantisation $f \to \hat{f}$ where

$$\hat{f}\phi(z)s_0.\nu = EL_f E\phi(z)s_0.\nu = EL_f\phi(z)s_0.\nu.$$

E is the projection onto the holomorphic part of a function, i.e.

$$(Eg)(\omega) = h^{-1}\int g(z,\overline{z})(2h)^{-\frac{1}{2}}\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)\exp\left(-\frac{\overline{z}z}{2\hbar}\right)dpdq.$$

Therefore, droping the $s_0.\nu$ factor which shall be understood, we have

$$\hat{f}\phi(\omega) = \left(\frac{2}{h}\right)^{\frac{1}{2}}\int\left[2\hbar\frac{\partial f}{\partial\overline{z}}\frac{d\phi}{dz} + \left(f - \overline{z}\frac{\partial f}{\partial\overline{z}} + \hbar\frac{\partial^2 f}{\partial\overline{z}\partial z}\right)\phi\right]\exp\left(-\frac{\overline{z}z}{2\hbar}\right)(2h)^{-\frac{1}{2}}\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq \tag{1.54}$$

$$= h^{-1}\int\left(f - \hbar\frac{\partial^2 f}{\partial\overline{z}\partial z}\right)\phi(z)\exp\left(-\frac{z\overline{z}}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq \tag{1.55}$$

(Appendix 5). We note that the transition to complex coordinates given in section 1.1.8 is a canonical coordinate transformation (page 40 [5]) and the generating function for this canonical transformation can be used in the pairing construction to give a unitary map $U : \mathcal{H}_F \to \mathcal{H}_P$ where P is the vertical polarisation. Suppose $z = p + iq$. We have

$$(U\Phi)(q) = \left(e^{\frac{\pi i}{4}}\left(\frac{\gamma}{2\pi}\right)^{\frac{1}{2}}\right)\int\Phi(p+iq)\exp\left(\frac{-\gamma}{4}(p^2+q^2-2ipq)\right)dp \tag{1.56}$$

and

$$(U^{-1}\phi)(z) = \left(e^{-\frac{\pi i}{4}}\left(\frac{\gamma}{2\pi}\right)^{\frac{1}{2}}\right)\int\phi(t)\exp\left(-\frac{\gamma}{4}(-z^2+4itz+2t^2)\right)dt$$

where $\gamma = 1/\hbar$. This allows us to find the quantisation of f in the vertical polarisation. This is given by

$$U\hat{f}U^{-1}\phi(q) = \frac{1}{2(\pi\hbar)^{\frac{3}{2}}}\int dz_1dz_2dt\left[f(z_1,z_2) - \frac{\hbar}{4}(\Delta_{z_1}+\Delta_{z_2})f\right]\exp\left[\frac{i}{\hbar}z_1(q-t)\right]$$
$$\exp\left[-\frac{1}{2\hbar}((q-z_2)^2+(t-z_2)^2)\right]\phi(t) \tag{1.57}$$

(Appendix 6) where $z = z_1 + iz_2$ and for example

$$\Delta_{z_1} = \frac{\partial^2}{\partial {z_1}^2}.$$

Quantisation of qp^2 a la Tuynman

We shall see that Tuynman's scheme leads us to associate the following formal differential operator with the classical observable qp^2

$$-\hbar^2\left(q\frac{d^2}{dq^2} + \frac{d}{dq}\right). \tag{1.58}$$

We show that the defect indicies of this operator are $(1,1)$ and find the one parameter familly of self-adjoint extensions. We show that Tuynman's result can be obtained more easilly by quantising qp^2 in the horizontal polarisation and using the pairing map to transform the operator back to the vertical polarisation. We observe that the quantisation of qp^2 obtained using Tuynman's method differs from that obtained using the formula of Bao and Zhu.

The derivation of (1.58) is rather involved and appears in Appendix 7. As yet we have not specified a domain for this operator. Tuynman does give a description of the maximal domain of the operators generated by his quantisation scheme but in general they are rather difficult to handle except for the special case of a compact Kahler manifold without boundary. For this reason we prefer to use Tuynman's method to obtain a formal quantisation and analyse the domain later. This is probably the best way to proceed here since the phase space has a boundary. This arises because of the restriction $q > 0$. We can show that the operator (1.58) does have self-adjoint extensions. We shall do this by finding its defect indicies. We require the solutions of

$$-\hbar^2 q\frac{d^2}{dq^2} - \hbar^2\frac{d}{dq} - \lambda = 0$$

or

$$\frac{d^2}{dq^2} + \frac{1}{q}\frac{d}{dq} + \frac{\lambda}{q\hbar^2} = 0. \tag{1.59}$$

This is a Bessel equation. To see this take $c = 1/2$, $p = 0$, $a = 0$ and $b = \sqrt{\lambda}2/\hbar$ in (1.51) and recover (1.59). The linearly independent solutions are therefore

$$H_0^{(+)}\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right) \quad \text{and} \quad H_0^{(-)}\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right)$$

which at large q are proportional to

$$q^{-\frac{1}{4}}\exp\left(\pm i\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right).$$

These are identical to the functions we considered in connection with the operator of Bao and Zhu and so we know that the defect indices of (1.58) on $[\frac{\hbar^2}{4},\infty)$ are (1,1) and therefore (1.58) has no boundary values at infinity. We could also express the solution as a linear combination of

$$J_0\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right) \quad \text{and} \quad N_0\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right).$$

We can analyse the J_0 solution in the same way we did in section (1.3.2) for $J_{\frac{1}{2}}$. This shows that $J_0\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right)$ is square integrable at 0. Now as $q \to 0$ we have

$$N_0 \propto \ln\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right)$$

i.e.

$$N_0 \propto \ln\left(\frac{2q^{\frac{1}{2}}}{\hbar}\right) \pm i\frac{\pi}{4}$$

according to whether $\lambda = \pm i$. $N_0\left(\frac{2\sqrt{\lambda}}{\hbar}q^{\frac{1}{2}}\right)$ is square integrable at 0 since

$$\int_0^{\frac{\hbar^2}{4}} \ln^2\left(\frac{2q^{\frac{1}{2}}}{\hbar}\right) dq < \infty.$$

To see this put

$$u = \ln^2\left(\frac{2q^{\frac{1}{2}}}{\hbar}\right). \tag{1.60}$$

When $q = 0$, $u = \infty$ and when $q = \frac{\hbar^2}{4}$, $u = 0$. Also

$$du = \frac{1}{q}\ln\left(\frac{2q^{\frac{1}{2}}}{\hbar}\right) dq.$$

Now

$$\exp(-\sqrt{u}) = \frac{2q^{\frac{1}{2}}}{\hbar}.$$

Notice that we have chosen this branch for the inverse of (1.60) so that $q < \frac{\hbar^2}{4}$. Therefore

$$\int_0^{\frac{\hbar^2}{4}} \ln^2\left(\frac{2q^{\frac{1}{2}}}{\hbar}\right) dq = \int_\infty^0 \frac{(-\sqrt{u})^2}{(-\sqrt{u})}\frac{\hbar^2}{4}\exp(-2\sqrt{u})du$$

$$= \frac{\hbar^2}{4}\int_0^\infty \sqrt{u}\exp(-2\sqrt{u})du.$$

If we let $v = 2\sqrt{u}$ this becomes

$$\int_0^{\frac{\hbar^2}{4}} \ln^2\left(\frac{2q^{\frac{1}{2}}}{\hbar}\right) dq = \frac{\hbar^2}{16}\int_0^\infty v^2\exp(-v)dv.$$

The integral on the r.h.s of the above is finite because

$$\int_0^\infty \exp(-v)dv < \infty.$$

This is sufficient as follows easily by repeated integration by parts, the boundary terms vanishing because

$$\lim_{v\to\infty} v^n \exp(-v) = 0$$

([8] page 39). Therefore the defect indicies of (1.58) on $(0, \hbar^2/4]$ are (2,2). Since there are 2 boundary values at $\hbar^2/4$ (1.58) has 2 boundary values at 0. The defect indicies of the minimal operator associated with (1.58) are (1,1). Tuynmans method leads to a 1 parameter family of possible quantisations of qp^2 just as we had with the method of Bao and Zhu. Also it would appear that Tuynman's operators will have the correct i.e.

positive spectra. (1.58) is formally positive ([6] definition 6 page 1439) so at least the essential spectrum of the self-adjoint extensions will be positive (cor 7 page 1439 and cor 3 page 1437 [6]).

We can obtain the explicit form of the self adjoint extensions as follows. If we define the bilinear form

$$< f, g >= -\lim_{q\to 0} \hbar^2 q(f'\overline{g} - f\overline{g}')$$

then the self-adjoint extensions of $T_0(\tau)$ are given by resricting $T_1(\tau)$ to that subset of its domain consisting of those f such that

$$< f, g >= 0$$

where g is any vector in $D(T_1(\tau))$ such that $< g, g >= 0$ and g is linearly independent relative to $D(T_0(\tau))$. Suppose we put

$$g = c_1 h_1 + c_2 h_2$$

where $h_1 = c$ near the origin, c a real number $\neq 0$ and $h_2 = \ln q^{\frac{1}{2}}$ near the origin. h_1 and h_2 vanish at infinity. It is easy to show that $\tau h_1 = \tau h_2 = 0$ near the origin so that h_1, h_2 and g are in $D(T_1(\tau))$. It is also easy to show that

$$< h_1, h_2 >= \frac{\hbar^2 c}{2}.$$

Since h_1 and h_2 real we can immediately infer from this that $< h_2, h_1 >= -\hbar^2 c/2$. Using lemma 10.2.17 page 259 [7] it is now a simple matter to show that g is linearly independent relative to $D(T_0(\tau))$ if just one of c_1 or c_2 is nonzero since if $c_1 = 0$ we still have

$$< g, h_1 >= -\frac{c_2\hbar^2 c}{2} \neq 0$$

and if $c_2 = 0$ then

$$< g, h_2 >= \frac{c_1\hbar^2 c}{2} \neq 0.$$

$< g, g >= 0$ is equivalent to

$$-\lim_{q\to 0} \hbar^2 q\left[\frac{c_2}{2}q^{-1}(\overline{c}_1 c + \overline{c}_2 \ln q^{\frac{1}{2}}) - (c_1 c + c_2 \ln q^{\frac{1}{2}})\frac{\overline{c}_2}{2}q^{-1}\right] = 0$$

$$\lim_{q\to 0}[c_2\overline{c}_1 c + \mid c_2 \mid^2 \ln q^{\frac{1}{2}} - c_1 c\overline{c}_2 - \mid c_2 \mid^2 \ln q^{\frac{1}{2}}] = 0$$

$$\lim_{q\to 0}(c_2\overline{c}_1 c - c_1 c\overline{c}_2) = 0$$

i.e

$$\overline{c}_1 c_2 = c_1\overline{c}_2 = \overline{\overline{c}_1 c_2}$$

so we must have $c_1/c_2 = \tan\theta$. The domain of the self-adjoint extensions of $T_0(\tau)$ then comprises $f \in D(T_1(\tau))$ such that

$$-\lim_{q\to 0} \hbar^2 q \left[f'(\overline{c}_1 c + \overline{c}_2 \ln q^{\frac{1}{2}}) - f\frac{\overline{c}_2}{2} q^{-1} \right] = 0$$

or

$$\lim_{q\to 0} q \left[f'(\tan\theta c + \ln q^{\frac{1}{2}}) - \frac{f}{2} q^{-1} \right] = 0.$$

We can see that (1.58) differs from the quantisation of qp^2 derived by Bao and Zhu. The two schemes are contradictory.

The quantisation of qp^2 obtained by Bao and Zhu was lent support by the fact that it could also be obtained by working in a polarisation transverse to the vertical polarisation and using a modified pairing construction. Tuynman's result can also be obtained using the pairing construction. Consider the coordinate transformation

$$p = -q' \quad \text{and} \quad q = p'.$$

This is a canonical coordinate transformation with generating function

$$f = -q'q.$$

For the time being we drop the constraint $q > 0$. Strictly speaking (1.58) does not apply when $q > 0$ anyway since in deriving it we set integrals of odd functions over $\mathbb{R}$ to zero. Now $qp^2 = q'^2 p'$ and so $\in C^\infty(M, P', 1)$. Therefore

$$\widehat{q'^2 p'} = -i\hbar \left(q'^2 \frac{d}{dq'} + q' \right). \tag{1.61}$$

The pairing gives a unitary map $U_{P'P} : \mathcal{H}_{P'} \to \mathcal{H}_P$. $U_{P'P}$ is unitary since is just the Fourier transform. The form of the operator (1.61) in $\mathcal{H}_P$ is

$$U_{P'P} - i\hbar \left(q'^2 \frac{d}{dq'} + q' \right) \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(q) \exp(iqq') dq$$

$$= U_{P'P} - \frac{i\hbar}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(q) \left(q'^2 \frac{d}{dq'} + q' \right) \exp(iqq') dq$$

$$= U_{P'P} - \frac{i\hbar}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(q) (q'^2 iq + q') \exp(iqq') dq$$

$$= U_{P'P} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(q) \left(q'^2 (-i)\hbar \frac{i}{\hbar} q - i\hbar q' \right) \exp(iqq') dq$$

$$= U_{P'P} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(q) \left(-\hbar^2 q \frac{d^2}{dq^2} - \hbar^2 \frac{d}{dq} \right) \exp(iqq') dq$$

$$= U_{P'P}\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\left(-\hbar^2\frac{d^2q\phi}{dq^2}+\hbar^2\frac{d\phi}{dq}\right)\exp(iqq')dq$$
$$= \left(-\hbar^2\frac{d^2q\phi}{dq^2}+\hbar^2\frac{d\phi}{dq}\right)$$
$$= -\hbar^2\left(\frac{d}{dq}\left(q\frac{d\phi}{dq}+\phi(q)\right)-\frac{d\phi}{dq}\right)$$
$$= -\hbar^2\left(q\frac{d^2}{dq^2}+\frac{d}{dq}\right)\phi$$

which is the same as (1.58).

1.3.4 Squaring Axiom for Symmetric Operators

In conventional quantum mechanics, where observables are identified exclusively with self-adjoint operators, it is taken for granted that if the classical observable g is quantised to give the operator $\hat{g}$ i.e. $g \to \hat{g}$ then

$$\mathrm{G}(g) \to \mathrm{G}(\hat{g}) \tag{1.62}$$

This is sensible because $\mathrm{G}(\hat{g})$ is a well defined self-adjoint operator. However in quantum mechanical theories that admit symmetric operators as observables the situation is more complicated because there is no operator function calculus for symmetric operators to compare with that for self adjoint operators. However, if we consider the special case $\mathrm{G}(g) = g^2$, whence (1.62) corresponds to the squaring axiom, there does appear to be a natural generalisation to the case of symmetric operators as follows: if g is quantised as the closed symmetric operator $\hat{g}$ then we propose $g^2 \to \hat{g}^*\hat{g}$. This operator is self-adjoint (Th 5.39 page 124 [11]). Notice that when $\hat{g}$ is self- adjoint we recover the squaring axiom of conventional quantum mechanics. The restriction that $\hat{g}$ be a closed symmetric operator is of no consequence since a symmetric operator has a unique minimal closure that we can identify as the quantum observable. We shall see how this works in the following example.

Consider the case of a particle in an infinite potential well, over an interval $I = [a, b]$ of $\mathbb{R}$. Suppose we want to determine the allowed energy states. Clearly we require the Hamiltonian. The traditional treatment of this problem would be as follows. Let $\tau_1 = -i\hbar d/dx$. The momentum p is quantised as $T_0(\tau_1)$ then since classically $H = p^2$ we expect $\hat{H}$ to be related to the operator $\tau_2 = \tau_1^2$ on $C_0^\infty(I)$ i.e. $T_0(\tau_2)$. However $T_0(\tau_2)$ is not essentially self-adjoint. Its self-adjoint extensions are obtained by restricting $T_1(\tau_2)$ with a set of symmetric boundary conditions. Physical arguments are brought to bear to determine the most sensible restriction. We require that the wavefunction be continuous. Since the wave function vanishes outside I we are led to the following quantisation

$$\hat{H} = \tau_2, \quad D(\hat{H}) = \{f : f \in D(T_1(\tau_2)),\ f(a) = f(b) = 0\}.$$

The question now arises as to whether or not we can obtain this result without using the nature of the wavefunction outside I i.e. without involving the environment of the infinite well. It so happens that the modified squaring axiom leads us naturally to the operator $\hat{H}$ given above. $T_0(\tau_1)$ is a symmetric operator. We know that

$$T_0(\tau_1)^* = T_1(\tau_1). \tag{1.63}$$

Now an operator and its closure have the same adjoint so

$$\overline{T_0(\tau_1)}^* = T_1(\tau_1). \tag{1.64}$$

All symmetric extensions of $T_0(\tau_1)$ are obtained by imposing symmetric boundary conditions on $T_1(\tau_1)$ therefore

$$\overline{T_0(\tau_1)} \subset T_1(\tau_1). \tag{1.65}$$

Consider

$$\hat{H}_{sq} = \overline{T_0(\tau_1)}^* \, \overline{T_0(\tau_1)} = T_1(\tau_1)\overline{T_0(\tau_1)}.$$

As we have seen this is self-adjoint. We have

$$D(\overline{T_0(\tau_1)}) = \{f \in D(T_1(\tau_1)),\ f(a) = f(b) = 0\} \tag{1.66}$$

Th 6.31 page 162 [11]. Now

$$D(\hat{H}_{sq}) = \{f \in D(\overline{T_0(\tau_1)}),\ \overline{T_0(\tau_1)}f \in D(T_1(\tau_1))\}$$

i.e. using (1.65) and (1.66)

$$D(\hat{H}_{sq}) = \{f : f \in D(T_1(\tau)),\ f(a) = f(b) = 0,\ T_1(\tau_1)f \in D(T_1(\tau_1))\} \tag{1.67}$$

so that

$$\hat{H}_{sq} = \tau_2$$

$$D(\hat{H}_{sq}) = \{f : f \in D(T_1(\tau)),\ f(a) = f(b) = 0,\ f' \in D(T_1(\tau_1))\}.$$

Now recall $D(T_1(\tau_1)) = H^1_{\tau_1} = H^1$ where the last equality follows from the top of page 1288 [6]. So we can write the domain of $\hat{H}_{sq}$ in a more explicit form namely $D(\hat{H}_{sq})$ consists of functions f such that

1. f is absolutely continous and f and $f' \in L^2(I)$,

2. $f(a)=f(b)=0$,

3. f' is absolutely continuous and f' and $f'' \in L^2(I)$.

Obviously not all of these properties are independent. Simillarly we can give a less implicit definition for $D(\hat{H})$. In fact $D(\hat{H})$ consists of functions f such that

1. f is continuously differentiable and $f \in L^2(I)$,

2. f' is absolutely continuous,

3. $f'' \in L^2(I)$,

4. $f(a)=f(b)=0$.

We can show that $D(\hat{H}) \supset D(\hat{H}_{sq})$. This is obvious since f absolutely continuous and so continuous. The rest are stated explictly. We can also show that $D(\hat{H}_{sq}) \supset D(\hat{H})$. f' is absolutely continuous so continuous. Every continuous function on a closed interval is bounded therefore f' is bounded and f is absolutely continuous (page 78 [12] and page 299 part d [29]). Also we clearly have $f \in A^2(I) \cap L^2(I)$ so by Th 6.26 [11] $f' \in L^2(I)$. The others are stated explicitly. Therefore $\hat{H}_{sq} = \hat{H}$.

The Generalised Squaring Axiom and the Formula of Bao and Zhu

Using this generalisation of the squaring axiom we can quantise observables of the form

$$\zeta(q)p^2 \tag{1.68}$$

$\zeta(q) > 0$. This is the classical observable considered by Bao and Zhu. Formally the operator we obtain corresponds to that given by Bao and Zhu but is now a well defined self-adjoint operator. What is more the operator is positive so its spectrum is identical to the range of the classical observable.

Since $\zeta(q)$ is positive we can write (1.68) as

$$(\zeta^{\frac{1}{2}}(q)p)^2.$$

Consider the observable $\zeta^{\frac{1}{2}}(q)p$. If we assume that $\zeta(q)$ is sufficiently smooth then $\zeta^{\frac{1}{2}}(q)p \in C^{\infty}(M, P, 1)$ so that

$$\widehat{\zeta^{\frac{1}{2}}p} = -i\hbar\left(\zeta^{\frac{1}{2}}(q)\frac{\partial}{\partial q} + \frac{1}{2}\frac{\partial\zeta^{\frac{1}{2}}}{\partial q}\right)$$

is a symmetric operator. According to the modified squaring axiom

$$\widehat{\zeta p^2} = \widehat{\zeta^{\frac{1}{2}} p}^* \, \widehat{\zeta^{\frac{1}{2}} p}$$

where we have taken the closure of the symmetric operator $\widehat{\zeta^{\frac{1}{2}} p}$. More specifically this operator is

$$-\hbar^2 \left(\zeta^{\frac{1}{2}} \frac{d}{dq} + \frac{1}{4}\zeta^{-\frac{1}{2}}\zeta'\right)\left(\zeta^{\frac{1}{2}} \frac{d}{dq} + \frac{1}{4}\zeta^{-\frac{1}{2}}\zeta'\right) = \hbar^2 \left(-\zeta^{\frac{1}{2}} \frac{d}{dq} - \frac{1}{4}\zeta^{-\frac{1}{2}}\zeta'\right)\left(\zeta^{\frac{1}{2}} \frac{d}{dq} + \frac{1}{4}\zeta^{-\frac{1}{2}}\zeta'\right)$$

$$= \hbar^2 \left[-\zeta^{\frac{1}{2}}\left(\zeta^{\frac{1}{2}}\frac{d^2}{d^2} + \frac{1}{2}\zeta^{-\frac{1}{2}}\zeta'\frac{d}{dq} + \frac{1}{4}\zeta^{-\frac{1}{2}}\zeta'\frac{\partial}{\partial q} + \frac{1}{4}\left(\zeta^{-\frac{1}{2}}\zeta'' + \zeta'\left(-\frac{1}{2}\right)\zeta^{-\frac{3}{2}}\zeta'\right)\right) - \frac{1}{4}\zeta'\frac{d}{dq} - \frac{1}{16}\zeta^{-1}\zeta'^2\right]$$

$$= \hbar^2 \left[-\zeta\frac{d^2}{dq^2} - \frac{\zeta'}{2}\frac{d}{dq} - \frac{1}{4}\zeta'\frac{d}{dq} - \frac{1}{4}\zeta'' + \frac{1}{8}\zeta^{-1}\zeta'^2 - \frac{1}{4}\zeta'\frac{\partial}{\partial q} - \frac{1}{16}\zeta^{-1}\zeta'^2\right]$$

$$= \hbar^2 \left[-\zeta(q)\frac{d}{dq^2} - (\frac{1}{2} + \frac{1}{4} + \frac{1}{4})\zeta'\frac{d}{dq} - \frac{1}{4}\zeta'' + (\frac{1}{8} - \frac{1}{16})\zeta^{-1}\zeta'^2\right]$$

$$= \hbar^2 \left[-\zeta\frac{d^2}{dq^2} - \zeta'\frac{d}{dq} - \frac{1}{4}\zeta'' + \frac{1}{16}\zeta^{-1}\zeta'^2\right]$$

which is identical to (1.48). The generalised squaring axiom associates with the positive classical observable qp^2 a well defined positive self adjoint operator that is formally equivalent to that obtained by Bao and Zhu.

It should be noted that it is not difficult to obtain the closure of a formal symmetric differential operator τ. We can simply use the boundary matrix discussed in section 1.2.2 to obtain the boundary values B_i of τ then restrict $T_1(\tau)$ to that subset of its natural domain determined by the strongest ([6] page 1236) set of symmetric boundary conditions i.e. $B_i f = 0$. This determines the minimal closure of $T_0(\tau)$.

1.4 Polarisations with Compact Leaves and Quantum Nonlocality

In this section we discuss the relationship between quantisation with respect to polarisations with compact leaves and nonlocality. This work also appears in [79].

Sometimes a polarisation F will be associated with a quantum Hilbert space $\mathcal{H}_F$ that cannot be represented as square integrable functions over a simply connected manifold. To elucidate the nature of the quantum Hilbert space associated with a general polarisation F we must study the topology of the Bohr Sommerfeld variety of F. We say that a point

$x \in M$ belongs to the Bohr Sommerfeld variety of a polarisation F if the integral surface of F through x can support smooth sections of $B \otimes \Delta_{-\frac{1}{2}}$. Then

$$\mathcal{H}_F = \oplus_{n=1} \mathcal{H}_n$$

where $\mathcal{H}_n$ consists of smooth polarised sections of F with support in the connected component c_n of the Bohr Sommerfeld variety of F. When the Bohr Sommerfeld variety of F is simply connected there is just one component in the direct sum and we recover the usual representation of $\mathcal{H}_F$ in terms of square integrable functions.

Consider the case of the classical observable H_0 representing the Hamiltonian of the simple harmonic oscillator i.e.

$$H_0 = \frac{1}{2}(p^2 + q^2).$$

Introduce a new canonical coordinate system

$$H_0 = \frac{1}{2}(p^2 + q^2) \qquad \theta = \tan^{-1}\left(\frac{q}{p}\right)$$

([5] page 148) and consider the polarisation

$$\Theta = \frac{\partial}{\partial \theta}.$$

Suppose we wish to quantise this classical system with respect to the polarisation Θ. We must identify the quantum Hilbert space $\mathcal{H}_\Theta$. The Bohr Sommerfeld variety of Θ is not simply connected. First notice that the integral surface (definition 3) or curve of Θ passing through any $x \in M$ will be a circle with centre the origin. This follows immediately from the fact that H_0 is the coordinate adapted to Θ so the integral surface is given by $H_0 = c$. Not all of these circles will be able to support smooth polarised sections of Θ. The circles c_n that can support smooth sections are those that satisfy the Bohr Wilson Sommerfeldt (B.W.S) condition

$$\oint_{c_n} H_0 d\theta = 2\pi\hbar n.$$

Since the integral surfaces are all level sets of H_0 the B.W.S condition is

$$H_0(c_n)2\pi = 2\pi\hbar n$$

i.e.

$$H_0(c_n) = \hbar n$$

Clearly

$$\text{Bohr Sommerfeld variety of } \Theta = \cup_{n=1}^{\infty} c_n$$

and this is not a simply connected set. The connected components are the circles c_n and

$$\mathcal{H}_\Theta = \oplus_{n=1}^{\infty} \mathcal{H}_n.$$

We would like a concrete realisation of $\mathcal{H}_\Theta$. Unfortunately a rigorous description would require distributional wavefunctions and we wish to avoid any more technicalities. We can give a heuristic argument to support the idea that $\mathcal{H}_\Theta$ is the space of square summable sequences denoted l^2. Suppose we were to quantise our system in the polarisation

$$\mathcal{H} = \frac{\partial}{\partial H_0}.$$

The Bohr Sommerfeld variety of $\mathcal{H}$ is M since the integral surfaces of $\mathcal{H}$ are given by $q = kp$. We therefore have the usual representation of $\mathcal{H}_\mathcal{H}$ as functions ϕ such that

$$\int_0^{2\pi} \mid \phi(\theta) \mid^2 d\theta < \infty.$$

From standard harmonic analysis we know that we can write

$$\phi(\theta) = \sum_n a_n \exp(in\theta)$$

where

$$a_n = \int_0^{2\pi} \phi(\theta) \exp(-in\theta) d\theta$$

and

$$\sum_0^\infty \mid a_n \mid^2 < \infty.$$

If we make the identification

$$\mathcal{H}_\Theta = l^2$$

then the above represent a sort of discrete version of the pairing map V between $\mathcal{H}_\Theta$ and $\mathcal{H}_\mathcal{H}$ (which we would expect to be a Fourier transform if there were no B.W.S condition) i.e. V:$\mathcal{H}_\Theta \rightarrow \mathcal{H}_\mathcal{H}$ where

$$V^{-1}\phi = \{a_n\} \quad \text{and} \quad V\{a_n\} = \sum_n a_n \exp(in\theta).$$

Hence the identification $\mathcal{H}_\Theta = l^2$. We must now identify $C^\infty(M, \Theta, 1)$. Naively we might expect that $C^\infty(M, \Theta, 1)$ would consist of functions of the form $\zeta(H_0)\theta + \eta(H_0)$ but θ is not a continuous function so in fact a classical observable is in $C^\infty(M, \Theta, 1)$ iff it is of the form $\eta(H_0)$. How should we quantise these classical observables? Recall that the c_n are level sets of H_0, in fact on c_n we have $H_0 = n\hbar$. Since $\mathcal{H}_\Theta$ is really the spectral representation space of $\hat{H}_0$ we expect $\hat{H}_0$ to act as a multiplication operator on l^2. The obvious candidate for $\hat{H}_0$ is

$$\hat{H}_0\{a_n\} = \{n\hbar a_n\}$$

so that $\hat{H}_0$ has a pure point spectrum consisting of the numbers $n\hbar$. Obviously

$$\eta\widehat{(H_0)}\{a_n\} = \{\eta(n\hbar)a_n\}.$$

Suppose that $\eta(H_0) = H_0$ in some disc D around the origin containing the connected components c_n, $1 \leq n \leq m$, of the Bohr Sommerfeld variety of Θ and is merely some arbitrary smooth function of H_0 on $M - D$. Clearly the values a_n, $1 \leq n \leq m$, are in the spectrum of $\eta\widehat{(H_0)}$ and will be unaffected by arbitrary alterations of η on $M - D$. This stability of the spectrum of $\hat{H}_0$ under localized and potentially remote perturbations of H_0 is disconcerting since quantum mechanics is well known to be non-local [25] . It is easy to see that this situation will occur quite generally when the Bohr Sommerfeld variety of the polarisation with respect to which we quantise the system is not simply connected. On the other hand the following example will serve to show that classical observables that are locally equivalent can give rise to very different quantum observables when quantised in polarisations with simply connected Bohr Sommerfeld varieties.

Example

Suppose H_1 is given by

$$H_1 = \frac{1}{2}(p^2 + \epsilon(x))$$

where $\epsilon(x)$ is a smooth function equal to x^2 for $\mid x \mid$ less than or equal to some positive number a and decreases monotonically to zero for $|x| \in [a, b]$, remaining zero $\forall x$ such that $|x| > b$. Clearly we have $H_1 = H_0$ for $|x| \leq a$. Using the B.K.S method we can quantise H_0 and H_1 in the vertical polarisation to obtain the following formal differential operators

$$\hat{H}_0 = \frac{1}{2}(\hat{p}^2 + x^2) \quad \text{and} \quad \hat{H}_1 = (\hat{p}^2 + \epsilon(x)).$$

When acting on their respective natural domains these formal differential expressions define two self-adjoint operators in the Hilbert space $L^2(\mathbb{R})$. The classical observable H_1 is identical to H_0 for $|x| \leq a$ but $\hat{H}_0$ and $\hat{H}_1$ are very different. $\hat{H}_0$ has a purely discrete spectrum whereas, because ϵ is bounded, positive and of compact support, the spectrum of $\hat{H}_1$ has no discrete part (page 119 and 226 [26]).

We could quantise H_1 and H_0 with respect to the polarisation Θ. In both cases the Bohr Sommerfeld variety of Θ will contain, as connected components, the circles c_n where

n now labels just those integral surfaces of Θ in the region $R \in M$ such that $|p| \leq a, |x| \leq a$. $\hat{H}_0$ and $\hat{H}_1$ both have the values $n\hbar$ in their point spectra. This will be the case regardless of the form of ϵ outside R.

This example shows that quantisation with respect to a polarisation with a simply connected Bohr Sommerfeld variety reflects quantum nonlocality whereas the remarks in the previous section indicate that quantisation in a polarisation with a non simply connected Bohr Sommerfeld variety is essentially local. The problem arises because in quantising with respect to a polarisation with a non simply connected Bohr Sommerfeld variety only the nature of the classical observable on the isolated connected components contributes; the global properties of the observable are largely irrelevant. Contrariwise conventional geometric quantisation is more holistic; the requirement that the classical observable f generate a complete Hamiltonian vector field probes the nature of f on all of phase space.

1.5 Appendices

Appendix 1: Group Representations

Geometric quantisation is quite adept at providing operator representations of certain groups. It turns out that the half density scheme is not sophisticated enough to generate proper representations. In general it leads only to projective representations. The half form scheme on the other hand does give proper representations. We shall illustrate this with a discussion of the representations of the group of all linear canonical transformations i.e. $\rho : V \to V$ such that

$$\omega(\rho(X), \rho(Y)) = \omega(X, Y)$$

(pages 2 and 172 [2]). It possible to regard V as a flat Kahler manifold with canonical coordinates p, q. The tangent space at each point of this manifold can also be identified with V. A vector in V can then be written as

$$X = x_1 \frac{\partial}{\partial p} + x_2 \frac{\partial}{\partial q}.$$

It can be shown that

$$f = \frac{F}{2} p^2 - Dpq - \frac{E}{2} q^2 \tag{1.69}$$

(F, D and E constants) generates the linear canonical transformations. For example take $f = F/2p^2$ then

$$X_f = Fp \frac{\partial}{\partial q}.$$

The integral curves are solutions of

$$\frac{dp}{dt} = 0 \text{ and } \frac{dq}{dt} = Fp$$

so this observable generates the following 1 parameter group of diffeomorphisms

$$p \to p, \;\; q \to Fpt$$

which can also be written as

$$p \to p + t\left\{\frac{F}{2}p^2, p\right\} \text{ and } q \to q + t\left\{\frac{F}{2}p^2, q\right\}.$$

Let

$$X = x^1 \frac{\partial}{\partial p} + x^2 \frac{\partial}{\partial q}$$
$$Y = y^1 \frac{\partial}{\partial p} + y^2 \frac{\partial}{\partial p}$$

then

$$\omega(X,Y) = x^1\frac{\partial}{\partial p} + x^2\frac{\partial}{\partial q}\rfloor y^1\frac{\partial}{\partial p} + y^2\frac{\partial}{\partial p}\rfloor(dp\otimes dq - dq\otimes dp)$$
$$= x^1\frac{\partial}{\partial p} + x^2\frac{\partial}{\partial q}\rfloor y^1 dq - y^2 dp$$
$$= x^2y^1 - y^2x^1.$$

Using the usual expression for the push forward of a vector field we have

$$\omega(\rho(X),\rho(Y)) = x^1\frac{\partial}{\partial p} + (x^1Ft + x^2)\frac{\partial}{\partial q}\rfloor y^1\frac{\partial}{\partial p} + (y^1Ft + x^2)\frac{\partial}{\partial q}(dp\otimes dq - dq\otimes dp)$$
$$= x^1\frac{\partial}{\partial p} + (x^1Ft + x^2)\frac{\partial}{\partial q}\rfloor y^1 dq - (y^1Ft + y^2)dp$$
$$= -x^1(y^1Ft + y^2) + (x^1Ft + x^2)y^1$$
$$= x^2y^1 - y^2x^1$$

as required. We cannot quantise functions like f directly but it turns out that if we work in the half density scheme and use the B.K.S method then we must put

$$\hat{f} = \frac{\overline{k}}{4}z^2 + \hbar^2 K\frac{\partial^2}{\partial z^2} + \hbar U z\frac{\partial}{\partial z} \tag{1.70}$$

where

$$K = \frac{1}{2}(E + F - 2iD)$$

and

$$U = \frac{1}{2}(F - E).$$

We can show that this gives a projective representation of the symplectic group algebra. We have

$$(\hat{f}'\hat{f} - \hat{f}\hat{f}')\phi = \left(\frac{1}{4}\overline{k'}z^2 + \hbar^2 k'\frac{\partial^2}{\partial z^2} + \hbar U' z\frac{\partial}{\partial z}\right)\left(\frac{1}{4}\overline{k}z^2\phi + \hbar^2 k\frac{\partial^2\phi}{\partial z^2} + \hbar U z\frac{\partial\phi}{\partial z}\right) - \hat{f}\hat{f}'$$

$$= \frac{1}{4}\overline{k'}z^2\left(\frac{1}{4}\overline{k}z^2\phi + \hbar^2 k\frac{\partial^2\phi}{\partial z^2} + \hbar U z\frac{\partial\phi}{\partial z}\right) + \hbar^2 k'\frac{\partial}{\partial z}\left(\frac{\overline{k}}{4}\left(z^2\frac{\partial\phi}{\partial z} + 2z\phi\right) + \hbar^2 k\frac{\partial^3\phi}{\partial z^3} + \hbar U\left(z\frac{\partial^2\phi}{\partial z^2} + \frac{\partial\phi}{\partial z}\right)\right)$$
$$+\hbar U' z\left(\frac{\overline{k}}{4}\left(z^2\frac{\partial\phi}{\partial z} + 2z\phi\right) + \hbar^2 k\frac{\partial^3\phi}{\partial z^3} + \hbar U\left(z\frac{\partial^2\phi}{\partial z^2} + \frac{\partial\phi}{\partial z}\right)\right) - \hat{f}\hat{f}'$$
$$= \frac{\overline{k'}z^2}{4}\left(\frac{\overline{k}}{4}z^2\phi + \hbar^2 k\frac{\partial^2\phi}{\partial z^2} + \hbar U z\frac{\partial\phi}{\partial z}\right) +$$
$$\hbar^2 k'\left(\frac{\overline{k}}{4}\left(z^2\frac{\partial^2\phi}{\partial z^2} + 2z\frac{\partial\phi}{\partial z} + 2z\frac{\partial\phi}{\partial z} + 2\phi\right) + \hbar^2 k\frac{\partial^4\phi}{\partial z^4} + \hbar U\left(z\frac{\partial^3\phi}{\partial z^3} + \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial z^2}\right)\right)$$
$$+\hbar U' z\left(\frac{\overline{k}}{4}\left(z^2\frac{\partial\phi}{\partial z} + 2z\phi\right) + \hbar^2 k\frac{\partial^3\phi}{\partial z^3} + \hbar U\left(z\frac{\partial^2\phi}{\partial z^2} + \frac{\partial\phi}{\partial z}\right)\right) - \hat{f}\hat{f}'$$

$$= \frac{\overline{k'}z^2}{4}\left(\frac{\overline{k}}{4}z^2\phi + \hbar^2 k\frac{\partial^2\phi}{\partial z^2} + \hbar U z\frac{\partial\phi}{\partial z}\right) +$$

$$\hbar^2 k'\left[\frac{\overline{k}}{4}\left(z^2\frac{\partial^2\phi}{\partial z^2} + 4z\frac{\partial\phi}{\partial z} + 2\phi\right) + \hbar^2 k\frac{\partial^4\phi}{\partial z^4} + \hbar U\left(z\frac{\partial^3\phi}{\partial z^3} + 2\frac{\partial^2\phi}{\partial z^2}\right)\right]$$

$$+\hbar U' z\left(\frac{\overline{k}}{4}\left(z^2\frac{\partial\phi}{\partial z} + 2z\phi\right) + \hbar^2 k\frac{\partial^3\phi}{\partial z^3} + \hbar U\left(z\frac{\partial^2\phi}{\partial z^2} + \frac{\partial\phi}{\partial z}\right)\right) - \hat{f}\hat{f}'$$

$$= \left(\frac{\overline{k'}z^2}{4}\hbar^2 k + \hbar^2 k'\frac{\overline{k}}{4}z^2 + \hbar^2 k'\hbar U 2 + \hbar U' z\hbar U z\right)\frac{\partial^2\phi}{\partial z^2} +$$

$$\left(\frac{\overline{k'}z^2}{4}\hbar U z + \hbar^2 k'\frac{\overline{k}}{4}4z + \hbar U' z\frac{\overline{k}}{4}z^2 + \hbar U' z\hbar U\right)\frac{\partial\phi}{\partial z} +$$

$$(\hbar^2 k'\hbar U z + \hbar U' z\hbar^2 k)\frac{\partial^3\phi}{\partial z^3} + \hbar^2 k'\hbar^2 k\frac{\partial^4\phi}{\partial z^4} + \left(\frac{\overline{k'}z^2}{4}\frac{\overline{k}}{4}z^2 + \hbar^2 k'\frac{\overline{k}}{4}2 + \hbar U' z\frac{\overline{k}}{4}2z\right) - \hat{f}\hat{f}'$$

$$= 2\hbar^3 U k'\frac{\partial^2\phi}{\partial z^2} + \hbar^2 k'\overline{k}z\frac{\partial\phi}{\partial z} + \left(\hbar^2 k'\frac{\overline{k}}{2} + \hbar U'\frac{\overline{k}}{2}z^2\right)\phi + \left\{\left(\frac{\overline{k'}kz^2}{4}\hbar^2 + \frac{k'}{4}\overline{k}z^2\hbar^2 + \hbar^2 z^2 UU'\right)\frac{\partial^2\phi}{\partial z^2} +\right.$$

$$\left(\frac{\hbar}{4}z^3\overline{k'}U + \frac{\hbar}{4}z^3\overline{k}U' + \hbar^2 zU'U\right)\frac{\partial\phi}{\partial z} + (\hbar^3 k'Uz + \hbar^3 kU'z)\frac{\partial^3\phi}{\partial z^3} +$$

$$\left.\hbar^4 k'k\frac{\partial^4\phi}{\partial z^4} + \frac{z^4}{16}\overline{k}\overline{k'}\right\} - \hat{f}\hat{f}'.$$

Clearly $\hat{f}\hat{f}'$ can be obtained by interchanging primes in the preceeding factors. The terms in the curly brackets all have coefficients that are symmetric under this operation and so vanish. Hence the above becomes

$$(2\hbar^3 Uk' - 2\hbar^3 kU')\frac{\partial^2\phi}{\partial z^2} + (\hbar^2 k'\overline{k}z - \hbar^2 k\overline{k'}z)\frac{\partial\phi}{\partial z} + \left(\hbar^2 k'\frac{\overline{k}}{2} + \hbar U'\frac{\overline{k}}{2}z^2 - \hbar^2 k\frac{\overline{k'}}{2} - \hbar U\overline{k'}\frac{z^2}{2}\right)\phi$$

$$= 2\hbar^3(Uk' - kU')\frac{\partial^2\phi}{\partial z^2} + z\hbar^2(k'\overline{k} - k\overline{k'})\frac{\partial\phi}{\partial z} + \left(\frac{\hbar^2}{2}(k'\overline{k} - k\overline{k'}) + \frac{\hbar}{2}z^2(U'\overline{k} - U\overline{k'})\right)\phi. \quad (1.71)$$

Now

$$\left\{\frac{1}{2}F'p^2 - D'pq - \frac{1}{2}E'q^2, \frac{1}{2}Fp^2 - Dpq - \frac{1}{2}Eq^2\right\}$$

$$= (F'p - D'q)(-Dp - Eq) - (-D'p - E'q)(Fp - Dq)$$

$$= -F'Dp^2 - F'Epq + D'Dqp + D'Eq^2 - (-D'Fp^2 + D'Dpq - E'Fqp + E'Dq^2)$$

$$= (D'F - F'D)p^2 + (E'F - F'E)qp + (D'E - E'D)q^2$$

$$= \frac{\mathcal{F}^2}{2}p^2 - \mathcal{D}pq - \frac{\mathcal{E}}{2}q^2. \quad (1.72)$$

Quantising this operator according to (1.70) we obtain as the coefficient of ϕ'

$$\hbar\mathcal{U}z = \frac{\hbar}{2}z(\mathcal{F} - \mathcal{E})$$

$$= \frac{\hbar}{2} z[2(D'F - F'D) + 2(D'E - E'D)]$$

$$= \hbar z(D'F - F'D + D'E - E'D). \tag{1.73}$$

According to (1.71) the coefficient of ϕ' is given by

$$z\hbar^2(k'\overline{k} - k\overline{k'})$$

$$= z\hbar^2 \left[\frac{1}{2}(E' + F' - 2iD')\frac{1}{2}(E + F + 2iD) - \frac{1}{2}(E + F - 2iD)\frac{1}{2}(E' + F' + 2iD')\right]$$

$$= \frac{z\hbar^2}{4}(E'E + E'F + 2iDE' + F'E + F'F + 2iF'D - 2iD'E - 2iD'F + 4D'D -$$

$$EE' - EF' - 2iD'E - FE' - FF' - 2iD'F + 2iDE' + 2iDF' - 4D'D)$$

$$= \frac{z\hbar^2}{4}(4iDE' + 4iF'D - 4iD'E - 4iD'F)$$

$$= z\hbar^2 i(DE' + F'D - D'E - D'F)$$

and this is equal to $-i\hbar \times$(1.73). Also we see that (1.71) gives the coefficient of ϕ'' as

$$2\hbar^3(Uk' - kU')$$

$$= 2\hbar^3 \left[\frac{1}{2}(F - E)\frac{1}{2}(E' + F' - 2iD') - \frac{1}{2}(F' - E')\frac{1}{2}(E + F - 2iD)\right]$$

$$= \frac{\hbar^3}{2}(FE' + FF' - 2iFD' - EE' - EF' + 2iED' - F'E - F'F + 2iF'D + E'E + E'F - 2iE'D)$$

$$= \hbar^3[(FE' - F'E) + i(ED' - FD' + F'D - E'D)]. \tag{1.74}$$

When you quantise (1.72) on the other hand you get

$$\hbar^2\mathcal{K} = \frac{\hbar^2}{2}(\mathcal{E} + \mathcal{F} - 2i\mathcal{D})$$

$$= \frac{\hbar^2}{2}[-2(D'E - E'D) + 2(D'F - F'D) + 2i(E'F - F'E)]$$

$$= \hbar^2[-D'E + E'D + D'F - F'D + i(E'F - F'E)].$$

Multiply this by $-i\hbar$ and it becomes the same as (1.74). According to (1.71) the coefficient of ϕ is

$$\frac{\hbar}{2}z^2(U'\overline{k} - U\overline{k'}) + \frac{\hbar^2}{2}(k'\overline{k} - k\overline{k'})$$

$$= \frac{\hbar}{2}z^2 \left[\frac{1}{2}(F' - E')\frac{1}{2}(E - F + 2iD) - \frac{1}{2}(F - E)\frac{1}{2}(E' + F' + 2iD')\right] + \frac{\hbar^2}{2}(k'\overline{k} - k\overline{k'})$$

$$= \frac{\hbar z^2}{8}[F'E + F'F + 2iF'D - E'E - E'F - 2iE'D - (FE' + FF' + 2iFD' - EE' - EF' - 2iED')] +$$

$$\frac{\hbar^2}{2}(K'\overline{K} - K\overline{K'})$$

$$= \frac{\hbar z^2}{8}[2F'E + 2iF'D - 2E'F - 2iE'D - 2iFD' + 2iED'] + \frac{\hbar^2}{2}(K'\overline{K} - K\overline{K'})$$
$$= \frac{\hbar z^2}{4}[(F'E - E'F) + i(F'D - E'D - FD' + ED')] + \frac{\hbar}{2}(K'\overline{K} - K\overline{K'}).$$

On the other hand quantisation of (1.72) gives as the coefficient of ϕ

$$\frac{1}{4}\mathcal{K}z^2 = \frac{z^2}{4}\frac{1}{2}(\mathcal{E} + \mathcal{F} + 2i\mathcal{D})$$
$$= \frac{z^2}{4}\frac{1}{2}[-2(D'E - E'D) + 2(D'F - F'D) - 2i(E'F - F'E)]$$
$$= \frac{z^2}{4}[-D'E + E'D + D'F - F'D - i(E'F - F'E)].$$

Multiply this by $-i\hbar$ and it becomes

$$\frac{\hbar}{4}z^2[(F'E - E'F) + i(D'E - E'D - D'F + F'D)].$$

All this goes to show that we have

$$[\hat{f}, \hat{f'}] - \frac{\hbar^2}{2}(K'\overline{K} - K\overline{K'}) = -i\hbar[f, f']\hat{}$$

or

$$i[\hat{f}, \hat{f'}] = \hbar[f, f']\hat{} + \frac{i\hbar^2}{2}(K'\overline{K} - K\overline{K'}) \tag{1.75}$$

and since the last term in the above is clearly real we see by 12.82 page 475 [30] that we have obtained a projective representation of the algebra of the symplectic group.

Now it turns out that if we work with the half form scheme and apply the B.K.S method then we are compelled to quantise the classical observable f (1.69) as

$$\hat{\hat{f}} = \frac{\overline{K}}{4}z^2 + \hbar^2 K\frac{\partial^2}{\partial z^2} + \hbar U z\frac{\partial}{\partial z} + \frac{\hbar}{2}U$$
$$= \hat{f} + \frac{\hbar}{2}U.$$

With this new quantisation scheme we have

$$[\hat{\hat{f}}, \hat{\hat{f'}}] = \left(\frac{\overline{K'}}{4}z^2 + \hbar^2 K'\frac{\partial^2}{\partial z^2} + \hbar U' z\frac{\partial}{\partial z} + \frac{\hbar}{2}U'\right)\left(\frac{\overline{K}}{4}z^2 + \hbar^2 K\frac{\partial^2}{\partial z^2} + \hbar U z\frac{\partial}{\partial z} + \frac{\hbar}{2}U\right) - \hat{f}\hat{f'}$$
$$= [\hat{f}, \hat{f'}] + \frac{\overline{K'}}{4}z^2\frac{\hbar}{2}U\phi + \hbar^2 K'\frac{\hbar}{2}U\frac{\partial^2\phi}{\partial z^2} + \hbar U' z\frac{\hbar}{2}U\frac{\partial\phi}{\partial z} + U'\frac{\hbar}{2}\frac{\overline{k}}{4}z^2\phi + U'\frac{\hbar}{2}\hbar^2 K\frac{\partial^2\phi}{\partial z^2} +$$
$$U'\frac{\hbar}{2}\hbar U z\frac{\partial\phi}{\partial z} + U'\frac{\hbar}{2}\frac{\hbar}{2}U\phi - \text{symmetric terms}$$
$$= [\hat{f}, \hat{f'}] + \frac{\overline{K'}}{8}z^2\hbar U\phi + \frac{\hbar^3}{2}K'U\frac{\partial^2\phi}{\partial z^2} + \frac{\hbar^2}{2}zUU'\frac{\partial\phi}{\partial z} + \frac{\hbar}{8}U'\overline{K}z^2\phi +$$
$$U'\frac{\hbar^3}{2}K\frac{\partial^2\phi}{\partial z^2} + U'\frac{\hbar^2}{2}Uz\frac{\partial\phi}{\partial z} + \frac{\hbar^2}{4}U'U\phi - \text{symmetric terms.}$$

In fact the extra terms are symmetric under interchange of primes so

$$[\hat{\hat{f}}, \hat{\hat{f}}'] = [\hat{f}, \hat{f}'].$$

Applying the new quantisation rule to (1.72)

$$\begin{aligned}
[f,\hat{\hat{}} f'] &= [f,\hat{} f'] + \frac{\hbar}{2}\mathcal{U} \\
&= [f,\hat{} f'] + \frac{\hbar}{4}(\mathcal{F} - \mathcal{E}) \\
&= [f,\hat{} f'] + \frac{\hbar}{2}(D'F - F'D + D'E - E'D) \\
&= [f,\hat{} f'] - \frac{\hbar}{2}\left(\frac{K'\overline{K} - K\overline{K'}}{i}\right) \\
&= [f,\hat{} f'] + \frac{\hbar}{2}i(K'\overline{K} - K\overline{K'})
\end{aligned}$$

so

$$\begin{aligned}
\hbar[f,\hat{\hat{}} f'] &= \hbar[f,\hat{} f'] + \frac{\hbar^2}{2}i(K'\overline{K} - K\overline{K'}) \\
&= i[\hat{f}, \hat{f}'] \\
&= i[\hat{\hat{f}}, \hat{\hat{f}}']
\end{aligned}$$

where we have used (1.75). This shows that the half form scheme leads to a proper i.e. non projective representation of the symplectic group algebra.

We now investigate the effect of changing connection potential. The half density quantisation scheme is well known to predict incorrect values for the spectrum of the Hamiltonian of the simple harmonic oscillator. The half form scheme gives the correct values and this was taken to indicate its superiority over the half density scheme. However Wan has shown that a modification of the half density scheme ensures that it also gives the correct spectrum, all that is required is to employ a different connection potential. In fact the same method can be used to ensure that the half density scheme gives a proper i.e. non projective representation of the symplectic group algebra. Suppose we change the connection potential (1.14) to

$$-\frac{i}{2}\overline{z}dz + f(z)dz.$$

This is clearly admissible since taking the exterior derivative of the above gives the symmplectic form. We now have

$$\nabla_X s_0 = -\frac{i}{\hbar}(X \rfloor - \frac{i}{2}\overline{z}dz + f(z)dz)s_0.$$

What effect will this have on the form for operators? Well

$$\hat{c}\phi = (-i\hbar\nabla_{X_c}s + cs).\nu - i\hbar s.L_{X_c}\nu.$$

The Lie derivative of the half density ν still vanishes so

$$\hat{c}\phi = (-i\hbar\nabla_{X_c}s + cs).\nu$$

$$= (-i\hbar\nabla_{X_c}(\phi s_0) + c\phi s_0).\nu$$

$$= (-i\hbar X_c(\phi)s_0 - i\hbar\phi\nabla_{X_c}s_0 + c\phi s_0).\nu.$$

Now remembering that ϕ is holomorphic (independent of $\overline{z}$) and using (1.16) this becomes

$$\hat{c}\phi = \left((-i\hbar)(2i)(A+Bz)\frac{\partial\phi}{\partial z}s_0 - i\hbar\phi\left(-\frac{i}{\hbar}\right)(2i(A+Bz))\left(-\frac{i}{2}\overline{z}+f(z)\right)s_0 + (A\overline{z}+\overline{A}z+Bz\overline{z}+D)\phi s_0\right)\nu$$

$$= \left[2\hbar(A+Bz)\frac{\partial\phi}{\partial z}s_0 - \phi 2i(A+Bz)\left(f(z)-\frac{i\overline{z}}{2}\right)s_0 + (A\overline{z}+\overline{A}z+Bz\overline{z}+D)\phi s_0\right].\nu$$

$$= \left(2\hbar(A+Bz)\frac{\partial\phi}{\partial z} + \phi(\overline{A}z+D) - 2i(A+Bz)\frac{\partial f}{\partial z}\right)s_0.\nu.$$

As a check we can see that if $f(z) = 0$ we obtain (1.18). If we are to recover from this modification of the half density scheme the irreducible representation we obtained using half forms we need only choose f such that

$$-2i(A+Bz)\frac{\partial f}{\partial z} = B\hbar$$

i.e.

$$f = -\frac{\hbar}{2i}\ln(A+Bz).$$

It appears that the use of half forms is inessential in obtaining non projective group representations.

Appendix 2.a

Consider

$$\int_0^\infty x^{-\frac{k}{l}}\exp\left(\frac{2lx^{1-\frac{k}{l}}}{\hbar(k-l)}\right)dx. \tag{1.76}$$

Let $u = x^{\frac{l-k}{l}}$ then $x^{-\frac{k}{l}}dx = \frac{l}{l-k}du$. Now suppose $k > l$ then

$$x = \infty \Rightarrow u = 0$$

and

$$x = 0 \Rightarrow u = \infty$$

so (1.76) becomes

$$\int_{\infty}^{0} \frac{l}{(l-k)} \exp\left(\frac{2lu}{\hbar(k-l)}\right) du$$
$$= \frac{l}{l-k} \int_{\infty}^{0} \exp\left(\frac{2lu}{\hbar(k-l)}\right) du$$
$$= \frac{l}{l-k} \left[\frac{\hbar(k-l)}{2l} \exp\left(\frac{2lu}{\hbar(k-l)}\right)\right]_{\infty}^{0}$$
$$= -\frac{\hbar}{2} \left[\exp\left(\frac{2lu}{\hbar(k-l)}\right)\right].$$

Since $\frac{2l}{\hbar(k-l)} > 0$ the integral diverges. Suppose $k < l$ then

$$x = \infty \Rightarrow u = \infty$$

$$x = 0 \Rightarrow u = 0$$

so that (1.76) becomes

$$\int_{0}^{\infty} \frac{l}{l-k} \exp\left(\frac{2lu}{\hbar(k-l)}\right) du = -\frac{\hbar}{2} \left[\exp\left(\frac{2lu}{\hbar(k-l)}\right)\right]_{0}^{\infty}.$$

Since $\frac{2l}{\hbar(k-l)} < 0$ integral converges.

Appendix 2.b

Consider

$$\int_{-\infty}^{0} |x|^{-\frac{k}{l}} \exp\left(\frac{2lx\,|x|^{-\frac{k}{l}}}{\hbar(k-l)}\right) dx.$$

Put $u = -x$ so above becomes

$$-\int_{\infty}^{0} u^{-\frac{k}{l}} \exp\left(-\frac{2lu^{1-\frac{k}{l}}}{\hbar(k-l)}\right) du$$
$$= \int_{0}^{\infty} u^{-\frac{k}{l}} \exp\left(\frac{2lu^{1-\frac{k}{l}}}{\hbar(l-k)}\right) du.$$

This is almost the same integral as we considered in Appendix 2.a. By symmetry we deduce that integral converges for $k > l$ and diverges for $k < l$.

Appendix 3: Derivation of 1.48

Since $\theta = pdq$ and

$$X_g = \frac{\partial g}{\partial p}\frac{\partial}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial}{\partial p}$$

we have

$$\theta \rfloor X_g = \frac{\partial g}{\partial p} p.$$

We consider only observables of the form

$$g = p^2 f(q)$$

$f(q) > 0$. In this case $\theta \rfloor X_g = g$ and since g is invarient under ϕ_g^{-s} it is constant as far as the integration in (1.47) is concerned so that

$$\hat{g}\psi = (i\hbar) \lim_{t \to 0^+} (ih)^{-\frac{1}{2}} \int dp \exp\left[\frac{i}{\hbar} p^2 f(q) t\right] [\det\omega(X_q, \phi_g^t X_q)]^{\frac{1}{2}} \psi(\phi_g^t q). \qquad (1.77)$$

Now using a Taylor series expansion we have

$$\phi_g^t q = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} q^{(n)} t^n.$$

The differential equations describing the integral curves of the Hamiltonian vector field of g are

$$\frac{dq}{dt} = \frac{\partial g}{\partial p} = 2pf(q) \text{ and } \frac{dp}{dt} = -p^2 f'(q). \qquad (1.78)$$

By repeated use of these equations it is easy to show that we can always write

$$q^{(n)} = \beta_n(q) p^n \qquad (1.79)$$

for some $\beta_n(q)$. Therefore

$$\phi_g^t q = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \beta_n(q) p^n t^n. \qquad (1.80)$$

Now

$$\phi_g^t X_q = X_{\phi_g^t q}.$$

The Hamiltonian vector fields of functions q and $\phi_g^t q$ are easily calculated using the standard formula and (1.80) so that

$$\omega(X_q, X_{\phi_g^t q}) = \omega\left(-\frac{\partial}{\partial p}, \frac{\partial}{\partial p}\left(\sum_0^{\infty} \frac{(-1)^n}{n!} \beta_n(q) p^n t^n\right) \frac{\partial}{\partial q} - \frac{\partial}{\partial q}\left(\sum_0^{\infty} \frac{(-1)^n}{n!} \beta_n(q) p^n t^n\right) \frac{\partial}{\partial p}\right)$$

$$= \omega\left(-\frac{\partial}{\partial p}, \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \beta_n(q) n p^{n-1} t^n\right) \frac{\partial}{\partial q} - \frac{\partial}{\partial q}\left(\sum_0^{\infty} \frac{(-1)^n}{n!} \beta_n(q) p^n t^n\right) \frac{\partial}{\partial p}\right)$$

which we can write as

$$\omega\left(-\frac{\partial}{\partial p}, A\frac{\partial}{\partial q} - \frac{\partial B}{\partial q}\frac{\partial}{\partial p}\right)$$

with the obvious definitions of A and B. By definition this is

$$(dp \otimes dq - dq \otimes dp) \rfloor - \frac{\partial}{\partial p} \rfloor A \frac{\partial}{\partial q} - B \frac{\partial}{\partial p}$$

$$= dq \rfloor A \frac{\partial}{\partial q} - B \frac{\partial}{\partial p}$$

$$= A.$$

Notice that A can also be written as

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)p^{n-1}t^n.$$

Substituting this in (1.77) and putting $u = tp$ ([4] eq 7.24 page 118) we obtain

$$= i\hbar \lim_{t\to 0} \frac{d}{dt}(ih)^{-\frac{1}{2}} \int \frac{du}{t} \exp\left[\frac{i}{\hbar}\frac{f(q)}{t}u^2\right] \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)p^{n-1}t^n\right]^{\frac{1}{2}} \psi\left(\sum_{0}^{\infty} \frac{(-1)^n}{n!}\beta_n u^n\right)$$

$$= (i\hbar)(ih)^{-\frac{1}{2}} \lim_{t\to 0} \frac{d}{dt} \int \frac{du}{t} \exp\left[\frac{i}{\hbar}\frac{f(q)}{t}u^2\right] \left[\sum_{1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)p^{n-1}t^{n-1}t\right]^{\frac{1}{2}} \psi(B).$$

Now writing $p^{n-1}t^{n-1} = (pt)^{n-1} = u^{n-1}$ and taking out the factor of t from the second square bracket (which will aquire a factor of $1/2$) we obtain

$$(i\hbar)(ih)^{-\frac{1}{2}} \lim_{t\to 0} \frac{d}{dt} \int du t^{-\frac{1}{2}} \exp\left[\frac{i}{\hbar}\frac{f(q)}{t}u^2\right] \left[\sum_{1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}} \psi(B)$$

$$= (i\hbar)(ih)^{-\frac{1}{2}} \lim_{t\to 0} \int \left(\frac{d}{dt}t^{-\frac{1}{2}} \exp\left[\frac{i}{\hbar}\frac{f(q)}{t}u^2\right]\right) \left[\sum_{1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}} \psi(B)du. \quad (1.81)$$

We have

$$\frac{d}{dt}t^{-\frac{1}{2}} \exp\left[ia\frac{s^2}{t}\right] = -\frac{1}{4ia}\frac{\partial^2}{\partial s^2}t^{-\frac{1}{2}} \exp\left(\frac{ias^2}{t}\right).$$

Put $s = u$ and $a = f(q)/\hbar$ and (1.81) becomes

$$-(i\hbar)(ih)^{-\frac{1}{2}} \lim_{t\to 0} \int du \frac{1}{\frac{4if(q)}{\hbar}} \frac{d^2}{du^2} t^{-\frac{1}{2}} \exp\left[i\frac{f(q)}{\hbar t}u^2\right] \left[\sum_{1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}} \psi\left(\sum_{0}^{\infty} \frac{(-1)^n}{n!}\beta_n u^n\right).$$

Now using

$$\lim_{t\to 0^+} t^{-\frac{1}{2}} \exp\left(ia\frac{s^2}{t}\right) = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} e^{-\frac{i\pi}{4}}\delta(s)$$

this becomes

$$-(i\hbar)(ih)^{-\frac{1}{2}} \int du \frac{\hbar}{4if(q)} \frac{d^2}{du^2} \left(\left(\frac{\pi}{\frac{f(q)}{\hbar}}\right)^{\frac{1}{2}} e^{-\frac{i\pi}{4}}\delta(u)\right) \left[\sum_{1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}} \psi(B)$$

$$= (i\hbar)(ih)^{-\frac{1}{2}} \int du \frac{\hbar}{4if(q)} \frac{d^2}{du^2} \left(\left(\frac{\pi}{\frac{f(q)}{\hbar}}\right)^{\frac{1}{2}} e^{-\frac{i\pi}{4}}\delta(u)\right) \left[\sum_{1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}} \psi(B)$$

$$= -(i\hbar)(ih)^{-\frac{1}{2}} \frac{\hbar}{4if(q)} \frac{(\pi\hbar)^{\frac{1}{2}}}{f^{\frac{1}{2}}(q)} e^{-\frac{i\pi}{4}} \int du\delta''(u) \left[\sum_{1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}} \psi(B)$$

$$= -\frac{(i\hbar)(i\hbar 2\pi)^{-\frac{1}{2}}\hbar(\pi\hbar)^{\frac{1}{2}}(e^{\frac{i\pi}{2}})^{-\frac{1}{2}}}{4if^{\frac{3}{2}}(q)} \int du\delta'' \left[\sum_{1}^{\infty} \frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}} \psi(B)$$

$$-\frac{i\hbar i^{-\frac{1}{2}}\hbar^{-\frac{1}{2}}2^{-\frac{1}{2}}\pi^{\frac{1}{2}}\hbar\pi^{\frac{1}{2}}\hbar^{\frac{1}{2}}i^{-\frac{1}{2}}}{4if^{\frac{3}{2}}(q)}\int du\delta''\left[\sum_1^\infty\frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}}\psi(B)$$

$$=-\frac{\hbar^2}{i\sqrt{2}4f^{\frac{3}{2}}}\int du\delta''\left[\sum_1^\infty\frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}}\psi(B)$$

$$=\frac{(i\hbar)^2}{4f}\left(\frac{1}{if^{\frac{1}{2}}\sqrt{2}}\right)\int du\delta''\left[\sum_1^\infty\frac{(-1)^n}{(n-1)!}\beta_n(q)u^{n-1}\right]^{\frac{1}{2}}\psi(B)$$

$$=\frac{(i\hbar)^2}{4f}\int du\delta''(u)\left[\sum_1^\infty\frac{(-1)^n\beta_n(q)u^{n-1}}{(n-1)!(-1)f2}\right]^{\frac{1}{2}}\psi(B)$$

$$=\frac{(i\hbar)^2}{4f}\int\delta''(u)\left[\sum_1^\infty\frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f}u^{n-1}\right]^{\frac{1}{2}}\psi(B)du.$$

We can write this as

$$\frac{(i\hbar)^2}{4f(q)}\int du\left[D\psi(B)\frac{d^2\delta(u)}{du^2}\right]$$

$$=\frac{(i\hbar)^2}{4f(q)}\frac{d^2}{du^2}[D\psi(B)]\mid_{u=0}$$

$$=\frac{(i\hbar)^2}{4f(q)}\frac{d}{du}\left(D\frac{d\psi(B)}{du}+\frac{dD}{du}\psi(B)\right)\mid_{u=0}$$

$$=\frac{(i\hbar)^2}{4f(q)}\left(D\frac{d^2\psi(B)}{du^2}+2\frac{dD}{du}\frac{d\psi(B)}{du}+\frac{d^2D}{du^2}\psi(B)\right)\mid_{u=0}$$

$$=\frac{(i\hbar)^2}{4f(q)}\left(D\frac{d}{du}\left(\frac{dB}{du}\frac{d\psi}{dB}\right)+2\frac{dD}{du}\frac{dB}{du}\frac{d\psi}{dB}+\frac{d^2D}{du^2}\psi(B)\right)\mid_{u=0}$$

$$=\frac{(i\hbar)^2}{4f(q)}\left(D\left(\frac{dB}{du}\frac{d}{du}\frac{d\psi}{dB}+\frac{d^2B}{du^2}\frac{d\psi}{dB}\right)+2\frac{dD}{du}\frac{dB}{du}\frac{d\psi}{dB}+\frac{d^2D}{du^2}\psi(B)\right)\mid_{u=0}$$

$$=\frac{(i\hbar)^2}{4f(q)}\left(D\left(\frac{dB}{du}\right)^2\frac{d^2\psi}{dB^2}+\left(D\frac{d^2B}{du^2}+2\frac{dD}{du}\frac{dB}{du}\right)\frac{d\psi}{dB}+\frac{d^2D}{du^2}\psi(B)\right)\mid_{u=0}. \qquad (1.82)$$

Now

$$B=\sum_0^\infty\frac{(-1)^n}{n!}\beta_nu^n=\beta_0+\sum_{n=1}^\infty\frac{(-1)^n}{n!}\beta_nu^n$$

so

$$\frac{dB}{du}=\sum_{n=1}^\infty\frac{(-1)^n}{n!}n\beta_nu^{n-1}. \qquad (1.83)$$

Therefore

$$\frac{dB}{du}\mid_{u=0}=-\beta_1$$

and

$$\left(\frac{dB}{du}\right)^2_{u=0}=\beta_1^2.$$

Also we can easily see that

$$D\mid_{u=0}=\left(\frac{\beta_1}{2f(q)}\right)^{\frac{1}{2}}.$$

Therefore

$$D\left(\frac{dB}{du}\right)^2\frac{d^2\psi}{dB^2}|_{u=0}=\beta_1^2\left(\frac{\beta_1}{2f(q)}\right)^{\frac{1}{2}}\psi''(q)$$

(since $B(0)=\beta_0=q$)

$$=\beta_1^2\left(\frac{\beta_1}{2f}\right)^{\frac{1}{2}}$$

$$=\beta_1^2\frac{\beta_1^{\frac{1}{2}}}{(2f)^{\frac{3}{2}-1}}$$

$$=\frac{2f\beta_1\beta_1\beta_1^{\frac{1}{2}}}{(2f)^{\frac{3}{2}}}$$

$$=2f\beta_1\left(\frac{\beta_1}{2f}\right)^{\frac{3}{2}}.$$

Now

$$\frac{dD}{du}=\frac{1}{2}\frac{1}{D}\sum_1^{\infty}\frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}u^{n-1}$$

$$=\frac{1}{2D}\frac{d}{du}\sum_2^{\infty}\frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}u^{n-1}$$

$$=\frac{1}{2D}\sum_2\frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}(n-1)u^{n-2}$$

$$\frac{dD}{du}\mid_{u=0}=\frac{1}{2}\left(\frac{2f(q)}{\beta_1}\right)^{\frac{1}{2}}(-1)\frac{\beta_2(q)}{2f(q)}$$

$$=-\frac{1}{2}\left(\frac{2f(q)}{\beta_1}\right)^{\frac{1}{2}}\frac{\beta_2(q)}{2f(q)}.$$

Now from (1.83)

$$\frac{d}{du}\left(\frac{dB}{du}\right)=\frac{d}{du}\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}n\beta_nu^{n-1}$$

$$=\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}n(n-1)\beta_nu^{n-2}$$

so

$$\frac{d^2B}{du^2}|_{u=0}=\beta_2.$$

Combining these results we see that

$$\left(D\frac{d^2B}{du^2}+2\frac{dD}{du}\frac{dB}{du}\right)|_{u=0}=\left(\frac{\beta_1}{2f(q)}\right)^{\frac{1}{2}}\beta_2+2\left(-\frac{1}{2}\right)\left(\frac{2f(q)}{\beta_1}\right)^{\frac{1}{2}}\frac{\beta_2(q)}{2f(q)}(-\beta_1)$$

$$=\left(\frac{\beta_1}{2f(q)}\right)^{\frac{1}{2}}\beta_2+\left(\frac{2f(q)}{\beta_1}\right)^{\frac{1}{2}}\frac{\beta_2(q)}{2f(q)}\beta_1$$

$$= \left(\frac{\beta_1}{2f(q)}\right)^{\frac{1}{2}} \beta_2 + \beta_2 \left(\frac{\beta_1}{2f(q)}\right)^{\frac{1}{2}}$$
$$= 2\beta_2 \left(\frac{\beta_1}{2f(q)}\right)^{\frac{1}{2}}.$$

Now

$$\frac{d^2D}{du^2} = \frac{d}{du}\left(\frac{1}{2}D^{-1}\sum_2 \frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}(n-1)u^{n-2}\right)$$
$$= \frac{1}{2}D^{-1}\frac{d}{du}\left(\sum_2 \frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}(n-1)u^{n-2}\right)+\frac{1}{2}\left(\sum_2 \frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}(n-1)u^{n-2}\right)(-1)D^{-2}\frac{dD}{du}$$
$$= \frac{1}{2}D^{-1}\frac{d}{du}\left(\sum_3 \frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}(n-1)u^{n-2}\right)+\frac{1}{2}\left(\sum_2 \frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}(n-1)u^{n-2}\right)(-1)D^{-2}\frac{dD}{du}$$
$$= \frac{1}{2}D^{-1}\left(\sum_3 \frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}(n-1)(n-2)u^{n-3}\right)-\frac{1}{2}D^{-2}\frac{dD}{du}\sum_2 \frac{(-1)^{n-1}}{(n-1)!}\frac{\beta_n(q)}{2f(q)}(n-1)u^{n-2}$$

so that

$$\frac{d^2D}{du^2}|_{u=0} = \frac{1}{2}\left(\frac{2f(q)}{\beta_1}\right)^{\frac{1}{2}}\frac{(-1)^2}{2!}\frac{\beta_3(q)}{2f(q)}2 - \frac{1}{2}\left(\frac{2f(q)}{\beta_1}\right)\left(-\frac{1}{2}\right)\left(\frac{2f(q)}{\beta_1}\right)^{\frac{1}{2}}\frac{\beta_2(q)}{2f(q)}\left((-1)\frac{\beta_2(q)}{2f(q)}\right)$$
$$= \frac{1}{4}\frac{\beta_3}{f(q)}\left(\frac{2f(q)}{\beta_1}\right)^{\frac{1}{2}} - \frac{1}{16}\frac{\beta_2^2}{f^2(q)}\left(\frac{2f(q)}{\beta_1}\right)^{\frac{3}{2}}$$
$$= \frac{\beta_3}{4f}\left(\frac{\beta_1}{2f}\right)^{-\frac{1}{2}} - \frac{1}{16}\frac{\beta_2^2}{f^2(q)}\left(\frac{\beta_1}{2f}\right)^{-\frac{3}{2}}.$$

Because

$$\frac{d\psi(B)}{dB}|_{u=0} = \psi'(q)$$

e.t.c we see that (1.82) becomes

$$= \frac{(i\hbar)^2}{4f}\left(2f\beta_1\left(\frac{\beta_1}{2f}\right)^{\frac{3}{2}}\psi'' + 2\beta_2\left(\frac{\beta_1}{2f}\right)^{\frac{1}{2}}\psi' + \left[\frac{\beta_3}{4f}\left(\frac{\beta_1}{2f}\right)^{-\frac{1}{2}} - \frac{\beta_2^2}{16f^2}\left(\frac{\beta_1}{2f}\right)^{-\frac{3}{2}}\right]\psi\right).$$

It is easy to show from (1.79) and (1.78) that $\beta_1 = 2f$, $\beta_2 = 2f'f$ and $\beta_3 = 4f''f^2$. Substituting this in the above gives (1.48).

Appendix 4: Quantisation of qp^2 Using Modified Pairing Construction

We can quantise qp^2 by working in a new polarisation and transforming back to the vertical polarisation using a modified pairing construction. We seek a canononically conjugate coordinate system of the form

$$q' = \zeta(q)p^2 \ \text{ and } \ p' = g(p,q).$$

By Cartans criterion p' and q' are canonical coordinates iff

$$pdq - p'dq' \tag{1.84}$$

is exact. Well

$$\begin{aligned} d(pdq - p'dq') &= dp \wedge dq - dp' \wedge dq' \\ &= dp \wedge dq - \left(\frac{\partial p'}{\partial p}dp + \frac{\partial p'}{\partial q}dq\right) \wedge \left(\frac{\partial q'}{\partial p}dp + \frac{\partial q'}{\partial q}dq\right) \\ &= dp \wedge dq - \left(\frac{\partial p'}{\partial p}dp + \frac{\partial p'}{\partial q}dq\right) \wedge \left(2\epsilon(q)pdp + \frac{\partial \epsilon}{\partial q}p^2 dq\right) \\ &= dp \wedge dq - \frac{\partial p'}{\partial p}p^2\frac{\partial \epsilon}{\partial q}dp \wedge dq - \frac{\partial p'}{\partial q}2\epsilon(q)pdq \wedge dp \end{aligned}$$

so (1.84) is exact if

$$1 - \frac{\partial p'}{\partial p}p^2\epsilon'(q) + \frac{\partial p'}{\partial q}2\epsilon(q)p = 0. \tag{1.85}$$

Suppose we consider the special case $\zeta(q) = q$ i.e. the classical observable qp^2, then (1.85) becomes

$$1 - p^2\frac{\partial p'}{\partial p} + 2pq\frac{\partial p'}{\partial q} = 0.$$

We can get a particular solution for this if we assume that p' is independent of q since in that case the above becomes

$$1 = p^2\frac{dp'}{dp}$$

for which $p' = -p^{-1}$ is a solution. We therefore have new canonical coordinates

$$q' = qp^2 \text{ and } p' = -\frac{1}{p}.$$

The generating function f for this transformation is

$$f = 2q'^{\frac{1}{2}}q^{\frac{1}{2}}$$

i.e.

$$p = \frac{\partial f}{\partial q} \text{ and } p' = -\frac{\partial f}{\partial q'}.$$

Since

$$\frac{\partial^2 f}{\partial q \partial q'} = \frac{q'^{-\frac{1}{2}}q^{-\frac{1}{2}}}{2}$$

we see that the pairing between the polarisations P' and P is

$$\phi(q) = \frac{1}{\sqrt{2\pi\hbar}}\int_0^\infty \phi'(q')\exp[i2q'^{\frac{1}{2}}q^{\frac{1}{2}}]\frac{q^{-\frac{1}{4}}q'^{-\frac{1}{4}}}{\sqrt{2}}dq'$$

with the corresponding expression for the inverse. Unfortunately in this case the pairing construction does not lead to a unitary map. However there exists a natural amendment to the pairing construction that does give a unitary map. Suppose we add to the kernal of the integral operator its complex conjugate. The new map is

$$(V^{-1}\phi')(q) = \phi(q) = \frac{1}{\sqrt{\pi\hbar}} \int_0^\infty \Phi(q') q^{-\frac{1}{4}} q'^{-\frac{1}{4}} \sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) dq'$$

and

$$(V\phi)(q') = \Phi(q') = \frac{1}{\sqrt{\pi\hbar}} \int_0^\infty \phi(q) q^{-\frac{1}{4}} q'^{-\frac{1}{4}} \sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) dq.$$

There does not appear to be any compelling geometric justification for this ansatz but it does lead to sensible results as we shall see. In the polarisation P' we quantise $q' = qp^2$ as the multiplication operator on $L^2(0,\infty)$. We cannot quantise qp^2 directly in the vertical polarisation but we can infer its quantisation from the modified pairing. It is easy to see that the operator so obtained is formally equivalent to that obtained by Bao and Zhu. Proof:

$$V - \hbar^2\left(q\frac{d^2}{dq^2} + \frac{d}{dq} - \frac{1}{16q}\right)V^{-1}\phi' = V - \hbar^2\left(q\frac{d^2}{dq^2} + \frac{d}{dq} - \frac{1}{16q}\right)\frac{1}{\sqrt{\pi\hbar}}\int_0^\infty \phi'(q') q^{-\frac{1}{4}} q'^{-\frac{1}{4}} \sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) dq'$$

$$= V\left(-\frac{\hbar^2}{\sqrt{\pi\hbar}}\right)\int_0^\infty \phi'(q') q'^{-\frac{1}{4}} \left(q\frac{d^2}{dq^2} + \frac{d}{dq} - \frac{1}{16q}\right) q^{-\frac{1}{4}} \sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) dq'$$

$$= V\left(-\frac{\hbar^2}{\sqrt{\pi\hbar}}\right)\int_0^\infty \phi(q') q'^{-\frac{1}{4}} \left(q\frac{d}{dq}\left[q^{-\frac{1}{4}}\frac{2}{\hbar}q'^{\frac{1}{2}}\frac{1}{2}q^{-\frac{1}{2}}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \frac{1}{4}q^{-\frac{5}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\right]\right.$$

$$\left. + \left[q^{-\frac{1}{4}}\frac{2}{\hbar}q'^{\frac{1}{2}}\frac{1}{2}q^{-\frac{1}{2}}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \frac{1}{4}q^{-\frac{5}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\right] - \frac{1}{16q}q^{-\frac{1}{4}}\sin(2q'^{\frac{1}{2}} q^{\frac{1}{2}})\right) dq'$$

$$= V\left(-\frac{\hbar^2}{\sqrt{\pi\hbar}}\right)\int_0^\infty \phi'(q') q'^{-\frac{1}{4}} \left(q\frac{d}{dq}\left[\frac{q'^{\frac{1}{2}}q^{-\frac{3}{4}}}{\hbar}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \frac{1}{4}q^{-\frac{5}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\right] + \right.$$

$$\left. \frac{q'^{\frac{1}{2}}q^{-\frac{3}{4}}}{\hbar}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \frac{1}{4}q^{-\frac{5}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \frac{1}{16}q^{-\frac{5}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\right) dq'$$

$$= V\left(-\frac{\hbar^2}{\sqrt{\pi\hbar}}\right)\int_0^\infty \phi'(q') q'^{-\frac{1}{4}} \left(q\left[-\frac{1}{\hbar}q'^{\frac{1}{2}}q^{-\frac{3}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\frac{2}{\hbar}q'^{\frac{1}{2}}\frac{1}{2}q^{-\frac{1}{2}} - \frac{3}{4}\frac{1}{\hbar}q'^{\frac{1}{2}}q^{-\frac{7}{4}}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \right.\right.$$

$$\frac{1}{4}q^{-\frac{5}{4}}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\frac{1}{\hbar}2q'^{\frac{1}{2}}\frac{1}{2}q^{-\frac{1}{2}} - \frac{1}{4}\left(-\frac{5}{4}\right)q^{-\frac{9}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\right] + \frac{1}{\hbar}q'^{\frac{1}{2}}q^{-\frac{3}{4}}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) -$$

$$\left. \frac{1}{4}q^{-\frac{5}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \frac{1}{16}q^{-\frac{5}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\right) dq'$$

$$= V\left(-\frac{\hbar^2}{\sqrt{\pi\hbar}}\right)\int_0^\infty \phi'(q') q'^{-\frac{1}{4}} \left(q\left[-\frac{q'q^{-\frac{5}{4}}}{\hbar^2}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \frac{3}{4\hbar}q'^{\frac{1}{2}}q^{-\frac{7}{4}}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \right.\right.$$

$$\frac{1}{4\hbar}q^{-\frac{7}{4}}q'^{\frac{1}{2}}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) + \frac{5}{16}q^{-\frac{9}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\right] + \frac{1}{\hbar}q'^{\frac{1}{2}}q^{-\frac{3}{4}}\cos\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) -$$

$$\left. \frac{1}{4}q^{-\frac{5}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right) - \frac{1}{16}q^{-\frac{5}{4}}\sin\left(\frac{2}{\hbar} q'^{\frac{1}{2}} q^{\frac{1}{2}}\right)\right) dq'$$

$$= V\left(-\frac{\hbar^2}{\sqrt{\pi\hbar}}\right)\int_0^\infty \phi'(q'){q'}^{-\frac{1}{4}}\left(-\frac{q^{-\frac{1}{4}}q'\sin\left(\frac{2}{\hbar}{q'}^{\frac{1}{2}}q^{\frac{1}{2}}\right)}{\hbar^2}\right)dq'$$

$$= V\frac{1}{\sqrt{\pi\hbar}}\int_0^\infty \phi'(q'){q'}^{\frac{3}{4}}q^{-\frac{1}{4}}\sin\left(\frac{2}{\hbar}{q'}^{\frac{1}{2}}q^{\frac{1}{2}}\right)dq'$$

$$= \frac{1}{\sqrt{\pi\hbar}}\int_0^\infty\left(\frac{1}{\sqrt{\pi\hbar}}\int_0^\infty \phi'(q'){q'}^{\frac{3}{4}}q^{-\frac{1}{4}}\sin\left(\frac{2}{\hbar}{q'}^{\frac{1}{2}}q^{\frac{1}{2}}\right)dq'\right)q^{-\frac{1}{4}}{q''}^{-\frac{1}{4}}\sin\left(\frac{2}{\hbar}{q''}^{\frac{1}{2}}q^{\frac{1}{2}}\right)dq$$

$$= \frac{1}{\pi\hbar}\int_0^\infty\int_0^\infty \phi'(q'){q'}^{\frac{3}{4}}q^{-\frac{1}{2}}{q''}^{-\frac{1}{4}}\sin\left(\left(\frac{2q'}{\hbar}\right)^{\frac{1}{2}}\left(\frac{2q}{\hbar}\right)^{\frac{1}{2}}\right)\sin\left(\left(\frac{2q''}{\hbar}\right)^{\frac{1}{2}}\left(\frac{2q}{\hbar}\right)^{\frac{1}{2}}\right)dq'dq.$$

Now put

$$q = \frac{\hbar u^2}{2} \text{ so } dq = \sqrt{2\hbar}\left(\frac{\hbar u^2}{2}\right)^{\frac{1}{2}}du$$

then the above becomes

$$\frac{1}{\pi\hbar}\int_0^\infty\int_0^\infty\phi'\left(\frac{\hbar {u'}^2}{2}\right)\left(\frac{\hbar {u'}^2}{2}\right)^{\frac{3}{4}}\left(\frac{\hbar u^2}{2}\right)^{-\frac{1}{2}}\left(\frac{\hbar {u''}^2}{2}\right)^{-\frac{1}{4}}\sin(u'u)\sin(u''u)\sqrt{2\hbar}\left(\frac{\hbar {u'}^2}{2}\right)^{\frac{1}{2}}\sqrt{2\hbar}\left(\frac{\hbar u^2}{2}\right)^{\frac{1}{2}}dudu'$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty\phi'\left(\frac{\hbar {u'}^2}{2}\right)\left(\frac{\hbar {u'}^2}{2}\right)^{\frac{5}{4}}\left(\frac{\hbar {u''}^2}{2}\right)^{-\frac{1}{4}}\sin(u'u)\sin(u''u)dudu'$$

$$= \frac{2}{\pi}\left(\frac{\hbar}{2}\right)\int_0^\infty\int_0^\infty\phi'\left(\frac{\hbar {u'}^2}{2}\right){u'}^{\frac{5}{2}}{u''}^{-\frac{1}{2}}\sin(u'u)\sin(u''u)dudu'$$

$$= \frac{\hbar}{\pi}\int_0^\infty\int_0^\infty\phi'\left(\frac{\hbar {u'}^2}{2}\right){u'}^{\frac{5}{2}}{u''}^{-\frac{1}{2}}J_{\frac{1}{2}}(u'u)\left(\frac{\pi u'u}{2}\right)^{\frac{1}{2}}J_{\frac{1}{2}}(u''u)(\frac{\pi u''u}{2})^{\frac{1}{2}}dudu'$$

$$= \frac{\hbar}{\pi}\frac{\pi}{2}\int_0^\infty\int_0^\infty\phi'\left(\frac{\hbar {u'}^2}{2}\right){u'}^3 uJ_{\frac{1}{2}}(u'u)J_{\frac{1}{2}}(u''u)dudu' \qquad (1.86)$$

Now from 16 and 17 of [81] page 6 we have

$$\overline{f}(\alpha) = \int_0^\infty x\left(\int_0^\infty \alpha'\overline{f}(\alpha')J_\nu(\alpha' x)d\alpha'\right)J_\nu(\alpha x)dx$$

so that (1.86) becomes

$$\frac{\hbar}{2}{u'}^2\phi'\left(\frac{\hbar {u'}^2}{2}\right)$$
$$= \frac{\hbar}{2}\frac{2}{\hbar}q'\phi'(q')$$
$$= q'\phi'(q').$$

In fact it is easier to obtain this result by noticing that the modified pairing map is simply a Hankel transform which is well known to be the spectral transform of the Bessel operator (1.49). Notice that this method yields a well defined self-adjoint operator whereas the B.K.S scheme gives only a formal differential expression that on its natural domain is not essentially self-adjoint. Notice also that the operator is positive so its spectrum matches the domain of the classical observable.

Appendix 5: Derivation of 1.55

Splitting up the integral (1.54) we obtain

$$h^{-1}\int(2\hbar\frac{\partial f}{\partial \overline{z}}\frac{d\phi}{dz}-\overline{z}\frac{\partial f}{\partial \overline{z}}\phi)\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq+h^{-1}\int\left(f+\hbar\frac{\partial^2 f}{\partial\overline{z}\partial z}\right)\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq.$$

Since

$$dpdq = \left(\frac{1}{2i}\right) dz d\overline{z}$$

this can be written as

$$\frac{h^{-1}}{2i}\int 2\hbar\frac{\partial f}{\partial \overline{z}}\frac{d\phi}{dz}\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dzd\overline{z}-\frac{h^{-1}}{2i}\int\overline{z}\frac{\partial f}{\partial\overline{z}}\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dzd\overline{z}+$$

$$h^{-1}\int\left(f+\hbar\frac{\partial^2 f}{\partial\overline{z}\partial z}\right)\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq.$$

Integrating the first term by parts with respect to z this becomes

$$-\frac{h^{-1}}{2i}\int 2\hbar\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial\overline{z}}\exp\left(-\frac{-\overline{z}z}{2\hbar}\right)\right)\phi\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dzd\overline{z}-\frac{h^{-1}}{2i}\int\overline{z}\frac{\partial f}{\partial\overline{z}}\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dzd\overline{z}+$$

$$h^{-1}\int\left(f+\hbar\frac{\partial^2 f}{\partial\overline{z}\partial z}\right)\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq$$

$$=-\frac{h^{-1}}{2i}\int 2\hbar\left(\frac{\partial f}{\partial\overline{z}}\left(-\frac{\overline{z}}{2\hbar}\right)\exp\left(-\frac{\overline{z}z}{2\hbar}\right)+\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\frac{\partial^2 f}{\partial z\partial\overline{z}}\right)\phi\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dzd\overline{z}-$$

$$\frac{h^{-1}}{2i}\int\overline{z}\frac{\partial f}{\partial\overline{z}}g\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dzd\overline{z}+h^{-1}\int\left(f+\hbar\frac{\partial^2 f}{\partial\overline{z}\partial z}\right)g\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq$$

$$=h^{-1}\int\overline{z}\frac{\partial f}{\partial\overline{z}}\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq-2h^{-1}\int\hbar\frac{\partial^2 f}{\partial z\partial\overline{z}}g\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq-$$

$$h^{-1}\int\overline{z}\frac{\partial f}{\partial\overline{z}}\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq+h^{-1}\int\left(f+\hbar\frac{\partial^2 f}{\partial\overline{z}\partial z}\right)g\phi\exp\left(-\frac{\overline{z}z}{2\hbar}\right)\exp\left(\frac{\overline{z}\omega}{2\hbar}\right)dpdq$$

The first and the third terms cancel and the second and the fourth terms combine to give 1.55.

Appendix 6: Derivation of 1.57

$$(\hat{f}(U^{-1}\phi))(\omega)=h^{-1}\int dz_1dz_2\exp\left(\frac{\overline{z}(\omega-z)}{2\hbar}\right)\left(f-\hbar\frac{\partial^2 f}{\partial z\partial\overline{z}}\right)\times$$

$$e^{-\frac{\pi i}{4}}\left(\frac{\gamma}{2\pi}\right)^{\frac{1}{2}}\int\phi(t)\exp\left(-\frac{\gamma}{4}(-z^2+4itz+2t^2)\right)dt$$

where $\gamma = 1/\hbar$. Using (1.56) we obtain

$$U(\hat{f}(U^{-1}\phi)(\omega_2)=e^{\frac{\pi i}{4}}\left(\frac{\gamma}{2\pi}\right)^{\frac{1}{2}}\int d\omega_1\exp\left(-\frac{\gamma}{4}(\omega_1^2+\omega_2^2-2i\omega_1\omega_2)\right)h^{-1}\times$$

$$\int dz_1 dz_2 \exp\left(\frac{\overline{z}(\omega_1+i\omega_2-z)}{2\hbar}\right)\left(f(z,\overline{z})-\hbar\frac{\partial^2 f}{\partial z\partial\overline{z}}\right)e^{-\frac{\pi i}{4}}\left(\frac{\gamma}{2\pi}\right)^{\frac{1}{2}}\int\phi(t)\exp\left(-\frac{\gamma}{4}(-z^2+4itz+2t^2)\right)dt. \tag{1.87}$$

Now

$$\begin{aligned}\frac{\partial^2 f}{\partial z\partial\overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial z_1}-i\frac{\partial}{\partial z_2}\right)\frac{1}{2}\left(\frac{\partial}{\partial z_1}+i\frac{\partial}{\partial z_2}\right)f\\ &= \frac{1}{4}\left(\frac{\partial}{\partial z_1}-i\frac{\partial}{\partial z_2}\right)\left(\frac{\partial f}{\partial z_1}+i\frac{\partial f}{\partial z_2}\right)\\ &= \frac{1}{4}\left(\frac{\partial^2 f}{\partial {z_1}^2}+i\frac{\partial^2 f}{\partial z_1\partial z_2}-i\frac{\partial^2 f}{\partial z_2\partial z_1}+\frac{\partial^2 f}{\partial {z_2}^2}\right)\\ &= \frac{1}{4}\left(\frac{\partial^2 f}{\partial {z_1}^2}+\frac{\partial^2 f}{\partial {z_2}^2}\right)\end{aligned}$$

so

$$f(z,\overline{z})-\hbar\frac{\partial^2 f}{\partial z\partial\overline{z}} = f(z,\overline{z})-\frac{\hbar}{4}(\Delta_{z_1}+\Delta_{z_2})f.$$

With this (1.87) becomes

$$\begin{aligned}&\left(\frac{\gamma}{2\pi}\right)h^{-1}\int d\omega_1\int dz_1\int dz_2\int dt\left[f(z,\overline{z})-\frac{\hbar}{4}(\Delta_{z_1}+\Delta_{z_2})f\right]\exp\left(-\frac{\gamma}{4}(\omega_1^2+\omega_2^2-2i\omega_1\omega_2)\right)\\ &\qquad\exp\left(\frac{\overline{z}(\omega_1+i\omega_2-z)}{2\hbar}\right)\exp\left(-\frac{\gamma}{4}(-z^2+4itz+2t^2)\right)\phi(t)\\ &=\left(\frac{\gamma}{2\pi}\right)h^{-1}\int d\omega_1\int dz_1\int dz_2\int dt\left[f(z,\overline{z})-\frac{\hbar}{4}(\Delta_{z_1}+\Delta_{z_2})f\right]\exp\left(-\frac{\gamma}{4}\omega_1^2+\frac{\gamma}{4}2i\omega_2\omega_1-\frac{\gamma\omega_2^2}{4}\right)\\ &\qquad\exp\left(\frac{\overline{z}\omega_1}{2\hbar}+\frac{\overline{z}}{2\hbar}(i\omega_2-z)\right)\exp\left(-\frac{\gamma}{4}(-z^2+4itz+2t^2)\right)\phi(t)\\ &=\left(\frac{\gamma}{2\pi}\right)h^{-1}\int d\omega_1\int dz_1\int dz_2\int dt\left[f(z,\overline{z})-\frac{\hbar}{4}(\Delta_{z_1}+\Delta_{z_2})f\right]\\ &\exp\left(-\frac{\gamma}{4}\omega_1^2+\frac{\gamma}{4}2i\omega_2\omega_1+\frac{\overline{z}\omega}{2\hbar}\right)\exp\left(-\frac{\gamma}{4}\omega_2^2+\frac{\overline{z}}{2\hbar}(i\omega_2-z)-\frac{\gamma}{4}(-z^2+4itz+2t^2)\right)\phi(t).\end{aligned}$$

Completing the square in the argument of the first exponential this becomes

$$\begin{aligned}&\left(\frac{\gamma}{2\pi}\right)h^{-1}\int d\omega_1\int dz_1\int dz_2\int dt\left[f(z,\overline{z})-\frac{\hbar}{4}(\Delta_{z_1}+\Delta_{z_2})f\right]\\ &\exp\left[-\frac{\gamma}{4}\left(\left(\omega_1-\left(i\omega_2+\frac{\overline{z}}{\gamma\hbar}\right)\right)^2-\left(i\omega_2+\frac{\overline{z}}{\gamma\hbar}\right)^2\right)\right]\exp\left(-\frac{\gamma}{4}\omega_2^2+\frac{\overline{z}}{2\hbar}(\omega_2 i-z)-\frac{\gamma}{4}(-z^2+4itz+2t^2)\right)\phi(t)\\ &=\left(\frac{\gamma}{2\pi}\right)h^{-1}\int d\omega_1\int dz_1\int dz_2\int dt\left[f(z,\overline{z})-\frac{\hbar}{4}(\Delta_{z_1}+\Delta_{z_2})f\right]\\ &\exp\left[-\frac{\gamma}{4}\left(\omega_1-\left(i\omega_2+\frac{\overline{z}}{\gamma\hbar}\right)\right)^2\right]\exp\left[-\frac{\gamma}{4}\omega_2^2+\frac{\overline{z}}{2\hbar}(\omega_2 i-z)-\frac{\gamma}{4}(-z^2+4itz+2t^2)+\frac{\gamma}{4}\left(i\omega_2+\frac{\overline{z}}{\gamma\hbar}\right)^2\right]\phi(t).\end{aligned} \tag{1.88}$$

Now if we put

$$W=\omega_1-\left(i\omega_2+\frac{\overline{z}}{\gamma\hbar}\right)$$

then

$$\int d\omega_1 \exp\left[-\frac{\gamma}{4}\left(\omega_1 - \left(i\omega_2 + \frac{\overline{z}}{\gamma\hbar}\right)\right)^2\right] = \int \exp\left[-\frac{\gamma}{4}W^2\right] dW$$
$$= \int \exp\left(-\frac{\pi W^2}{\frac{4\pi}{\gamma}}\right) dW$$
$$= \left(\frac{4\pi}{\gamma}\right)^{\frac{1}{2}}$$

using (1.98). Therefore (1.88) becomes

$$\left(\frac{\gamma}{2\pi}\right)\left(\frac{4\pi}{\gamma}\right)^{\frac{1}{2}} h^{-1} \int dz_1 \int dz_2 \int dt \left[f(z,\overline{z}) - \frac{\hbar}{4}(\Delta_{z_1} + \Delta_{z_2})f\right]$$
$$\exp\left[-\frac{\gamma}{4}\omega_2^2 + \frac{\overline{z}}{2\hbar}(\omega_2 i - z) - \frac{\gamma}{4}(-z^2 + 4itz + 2t^2) + \frac{\gamma}{4}\left(i\omega_2 + \frac{\overline{z}}{\gamma\hbar}\right)^2\right]\phi(t). \qquad (1.89)$$

Now the argument in the exponential is

$$-\frac{\gamma}{4}\omega_2^2 + \frac{z_1 - iz_2}{2\hbar}(\omega_2 i - z_1 - iz_2) - \frac{\gamma}{4}(-(z_1 + iz_2)^2 + 4it(z_1 + iz_2) + 2t^2) +$$
$$\frac{\gamma}{4}\left(i\omega_2 + \frac{1}{\gamma\hbar}(z_1 - iz_2)\right)^2$$
$$= -\frac{\gamma}{4}\omega_2^2 + \frac{z_1 - iz_2}{2\hbar}(\omega_2 i - z_1 - iz_2) + \frac{\gamma}{4}(z_1 + iz_2)^2 - \frac{\gamma}{4}4it(z_1 + iz_2) - \frac{\gamma}{4}2t^2 +$$
$$\frac{\gamma}{4}\left(-\omega_2^2 + \frac{2i}{\gamma\hbar}\omega_2(z_1 - iz_2) + \frac{1}{\gamma^2\hbar^2}(z_1 - iz_2)^2\right)$$
$$= -\frac{\gamma\omega_2^2}{4} + \frac{1}{2\hbar}(i\omega_2 z_1 - z_1^2 - iz_1 z_2 + z_2\omega_2 + iz_1 z_2 - z_2^2) + \frac{\gamma}{4}(z_1^2 + 2iz_1 z_2 - z_2^2) - \gamma t(iz_1 - z_2) - \frac{\gamma}{4}2t^2 +$$
$$\frac{\gamma}{4}\left(-\omega_2^2 + \frac{2\omega_2}{\gamma\hbar}(iz_1 + z_2) + \frac{1}{\gamma^2\hbar^2}(z_1^2 - i2z_1 z_2 - z_2^2)\right).$$

In the first bracket the terms involving iz_1z_2 cancel and we are left with

$$-\frac{\gamma\omega_2^2}{4} + i\frac{\omega_2 z_1}{2\hbar} - \frac{z_1^2}{2\hbar} + \frac{z_2\omega_2}{2\hbar} - \frac{z_2^2}{2\hbar} + \frac{\gamma z_1^2}{4} + \frac{i\gamma z_1 z_2}{2} - \frac{\gamma z_2^2}{4} - i\gamma z_1 t + \gamma t z_2 - \frac{\gamma}{4}2t^2 +$$
$$\frac{\gamma}{4}\left(-\omega_2^2 + \frac{i2\omega_2 z_1}{\gamma\hbar} + \frac{2\omega_2 z_2}{\gamma\hbar} + \frac{z_1^2}{\gamma^2\hbar^2} - \frac{i2z_1 z_2}{\gamma^2\hbar^2} - \frac{z_2^2}{\gamma^2\hbar^2}\right). \qquad (1.90)$$

The imaginary part of this expression is

$$\frac{\omega_2 z_1}{2\hbar} + \frac{\gamma z_1 z_2}{2} - \gamma z_1 t + \frac{\omega_2 z_1}{\hbar 2} - \frac{z_1 z_2}{2\gamma\hbar^2}$$
$$= \frac{\omega_2 z_1}{\hbar} - \gamma z_1 t + \left(\frac{\gamma}{2} - \frac{1}{2\gamma\hbar^2}\right) z_1 z_2$$

Remembering that $\gamma = 1/\hbar$ we see that the last term vanishes and the above becomes

$$\frac{z_1}{\hbar}(\omega_2 - t).$$

The real part of (1.90) is

$$-\frac{\gamma\omega_2^2}{4}-\frac{z_1^2}{2\hbar}+\frac{z_2\omega_2}{2\hbar}-\frac{z_2^2}{2\hbar}+\frac{\gamma z_1^2}{4}-\frac{\gamma z_2^2}{4}+\gamma t z_2-\frac{\gamma 2t^2}{4}-\frac{\gamma\omega_2^2}{4}+\frac{2\gamma\omega_2 z_2}{4\gamma\hbar}+\frac{\gamma z_1^2}{4\gamma^2\hbar^2}-\frac{\gamma z_2^2}{4\gamma^2\hbar^2}$$
$$=-\frac{\omega_2^2}{4\hbar}-\frac{z_1^2}{2\hbar}+\frac{z_2\omega_2}{2\hbar}-\frac{z_2^2}{2\hbar}+\frac{z_1^2}{4\hbar}-\frac{z_2^2}{4\hbar}+\frac{tz_2}{\hbar}-\frac{t^2}{2\hbar}-\frac{\omega_2^2}{4\hbar}+\frac{\omega_2 z_2}{2\hbar}+\frac{z_1^2}{\hbar 4}-\frac{z_2^2}{4\hbar}.$$

The 1st, 9th. 2nd, 5th and 11th. 3rd and 10th. 4th, 6th and 12th combine and this expression reduces to

$$-\frac{\omega_2^2}{2\hbar}+\frac{z_2\omega_2}{\hbar}-\frac{z_2^2}{\hbar}-\frac{t^2}{2\hbar}+\frac{tz_2}{\hbar}$$
$$=-\frac{\omega_2^2}{2\hbar}+\frac{z_2\omega_2}{\hbar}-\frac{z_2^2}{2\hbar}-\frac{t^2}{2\hbar}+\frac{tz_2}{\hbar}-\frac{z_2^2}{2\hbar}$$
$$=-\frac{1}{2\hbar}[\omega_2^2-2\omega_2 z_2+z_2^2+t^2-2tz_2+z_2^2]$$
$$=-\frac{1}{2\hbar}[(\omega_2-z_2)^2+(t-z_2)^2]$$

so we see that (1.89) can be written as

$$=\left(\frac{\gamma}{2\pi}\right)\left(\frac{4\pi}{\gamma}\right)^{\frac{1}{2}}h^{-1}\int dz_1dz_2dt\left[f(z_1z_2)-\frac{\hbar}{4}(\Delta_{z_1}+\Delta_{z_2})f\right]\exp\left[\frac{i}{\hbar}z_1(\omega_2-t)\right]$$
$$\exp\left[-\frac{1}{2\hbar}((\omega_2-z_2)^2+(t-z_2)^2)\right]\phi(t).$$

Put $h=2\pi\hbar$ and $\omega_2=q$ and we obtain (1.57).

Appendix 7: Derivation of 1.58

The observable we wish to quantise is of the form $f(r,s)=g(s)r^2$. Substituting this in (1.57) we obtain

$$=\frac{1}{2(\pi\hbar)^{\frac{3}{2}}}\iiint\exp\left(ir(q-t)\right)\exp(-\frac{1}{2\hbar}((t-s)^2+(q-s)^2)\Big)\Big(r^2g(s)-\frac{\hbar}{4}(2g(s)+r^2\Delta_s g)\Big)\phi(t)drdsdt$$
$$=\frac{1}{2(\pi\hbar)^{\frac{3}{2}}}\iiint\exp(ir(q-t))\exp\Big(-\frac{1}{2\hbar}((t-s)^2+(q-s)^2)\Big)r^2\left(g(s)-\frac{\hbar}{4}\Delta_s g\right)\phi(t)drdsdt-$$
$$\frac{1}{2(\pi\hbar)^{\frac{3}{2}}}\frac{\hbar 2}{4}\iiint\exp(ir(q-t))\exp\Big(-\frac{1}{2\hbar}((t-s)^2+(q-s)^2)\Big)g(s)\phi(t)drdsdt. \qquad (1.91)$$

The first term can be written as

$$=-\frac{1}{2(\pi\hbar)^{\frac{3}{2}}}\iiint\hbar^2\frac{d^2}{dt^2}\exp(ir(q-t))\exp\left(-\frac{1}{2\hbar}((t-s)^2+(q-s)^2)\right)\left(g(s)-\frac{\hbar}{4}\Delta_s g\right)\phi(t)drdsdt$$
$$=-\frac{\hbar^2}{2(\pi\hbar)^{\frac{3}{2}}}\iiint\frac{d^2}{dt^2}\exp(ir(q-t))\exp\Big(-\frac{1}{2\hbar}(t-s)^2\Big)\exp\Big(\frac{1}{2\hbar}(q-s)^2\Big)\left(g(s)-\frac{\hbar}{4}\Delta_s g\right)\phi(t)drdsdt$$
$$=-\frac{\hbar^2}{2(\pi\hbar)^{\frac{3}{2}}}\iiint\exp(ir(q-t))\exp\Big(-\frac{1}{2\hbar}(q-s)^2\Big)\Big(g(s)-\frac{\hbar}{4}\Delta_s g\Big)\frac{d^2}{dt^2}\Big[\exp(-\frac{1}{2\hbar}(t-s)^2)\phi(t)\Big]drdsdt$$

$$= -\frac{\hbar^2}{2(\pi\hbar)^{\frac{3}{2}}} \iiint \exp(ir(q-t)) \exp\Big(-\frac{1}{2\hbar}(q-s)^2\Big)\Big(g(s) - \frac{\hbar}{4}\Delta_s g\Big)$$

$$\frac{d}{dt}[\exp(-\frac{1}{2\hbar}(t-s)^2)\frac{d\phi}{dt} + \phi(t)\frac{d}{dt}\exp(-\frac{1}{2\hbar}(t-s)^2)]drdsdt$$

$$= -\frac{\hbar^2}{2(\pi\hbar)^{\frac{3}{2}}} \iiint \exp(ir(q-t)) \exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\left(g(s) - \frac{\hbar}{4}\Delta_s g\right)$$

$$\left[\exp\Big(-\frac{1}{2\hbar}(t-s)^2\Big)\frac{d^2}{dt^2}\phi(t) + 2\frac{d}{dt}\exp\Big(-\frac{1}{2\hbar}(t-s)^2\Big)\frac{d\phi}{dt} + \phi(t)\frac{d^2}{dt^2}\exp\Big(-\frac{1}{2\hbar}(t-s)^2\Big)\right]drdsdt. \tag{1.92}$$

If we let

$$z = r/\hbar \tag{1.93}$$

then the second term in the above becomes

$$-\frac{\hbar^2(2\pi)\hbar}{2(\pi\hbar)^{\frac{3}{2}}(2\pi)} \iiint \exp(iz(q-t)) \exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\left(g(s) - \frac{\hbar}{4}\Delta_s g\right)$$

$$2\frac{d}{dt}\exp\left(-\frac{1}{2\hbar}(t-s)^2\right)\frac{d\phi}{dt}dzdsdt$$

$$= -\frac{\hbar^2 2}{(\pi\hbar)^{\frac{1}{2}}} \iint \delta(q-t) \exp\Big(-\frac{1}{2\hbar}(q-s)^2\Big)\Big(g(s) - \frac{\hbar}{4}\Delta_s g\Big)\frac{d}{dt}\exp\left(-\frac{1}{2\hbar}(t-s)^2\right)\frac{d\phi}{dt}dsdt$$

$$= -\frac{\hbar^2 2}{(\pi\hbar)^{\frac{1}{2}}} \int \exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\left(g - \frac{\hbar}{4}\Delta_s g\right)\frac{d}{dq}\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)ds\frac{d\phi}{dq}$$

$$= \frac{\hbar^2 2}{(\pi\hbar)^{\frac{1}{2}}} \int \exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\left(g - \frac{\hbar}{4}\Delta_s g\right)\frac{d}{ds}\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)ds\frac{d\phi}{dq}.$$

Integrate by parts

$$= -\frac{2\hbar^2}{(\pi\hbar)^{\frac{1}{2}}} \int \frac{d}{ds}\left[\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\Big(g - \frac{\hbar}{4}\Delta_s g\Big)\right]\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)ds\frac{d\phi}{dq}$$

$$= -\frac{2\hbar^2}{(\pi\hbar)^{\frac{1}{2}}} \int \left[\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\frac{d}{ds}\left(g - \frac{\hbar}{4}\Delta_s g\right) + \left(g - \frac{\hbar}{4}\Delta_s g\right)\frac{d}{ds}\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\right]$$

$$\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)ds\frac{d\phi}{dq}$$

$$= -\frac{2\hbar^2}{(\pi\hbar)^{\frac{1}{2}}} \int \exp\left(-\frac{1}{\hbar}(q-s)^2\right)\frac{d}{ds}(g - \Delta_s g)ds\frac{d\phi}{dq} -$$

$$\frac{2\hbar^2}{(\pi\hbar)^{\frac{1}{2}}} \int \exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\left(g(s) - \frac{\hbar}{4}\Delta_s g\right)\frac{d}{ds}\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)ds\frac{d\phi}{dq}$$

$$= -\frac{\hbar^2}{2(\pi\hbar)^{\frac{1}{2}}} \int \exp\left(-\frac{1}{\hbar}(q-s)^2\right)\frac{d}{ds}\left(g(s) - \frac{\hbar}{4}\Delta_s g\right)ds\frac{d\phi}{dq}.$$

Now we consider the last term in (1.92)

$$= -\frac{\hbar^2}{2(\pi\hbar)^{\frac{3}{2}}} \iiint \exp(ir(q-t))\exp\Big(-\frac{1}{2\hbar}(q-s)^2\Big)\Big(g(s) - \frac{\hbar}{4}\Delta_s g\Big)\phi(t)\frac{d^2}{dt^2}\exp\Big(-\frac{1}{2\hbar}(t-s)^2\Big)drdtds.$$

Making the substitution (1.93) this becomes

$$= -\frac{\hbar^2\hbar}{2(\pi\hbar)^{\frac{3}{2}}}\iiint \exp(iz(q-t))\exp\Big(-\frac{1}{2\hbar}(q-s)^2\Big)\Big(g(s)-\frac{\hbar}{4}\Delta_s g\Big)\phi(t)\frac{d^2}{dt^2}\exp\Big(-\frac{1}{2\hbar}(t-s)^2\Big)dzdtds$$

$$= -\frac{\hbar^2\hbar(2\pi)}{2(\pi\hbar)^{\frac{3}{2}}}\iint \delta(q-t)\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\Big(g(s)-\frac{\hbar}{4}\Delta_s g\Big)\phi(t)\frac{d^2}{dt^2}\exp\left(-\frac{1}{2\hbar}(t-s)^2\right)dtds$$

$$= -\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\left(g(s)-\frac{\hbar}{4}\Delta_s g\right)\phi(q)\frac{d^2}{dq^2}\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)ds$$

$$= -\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\Big(g(s)-\frac{\hbar}{4}\Delta_s g\Big)\frac{d^2}{ds^2}\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)ds\phi(q).$$

The first term in (1.92) becomes on making the substitution (1.93)

$$-\frac{\hbar^2}{2(\pi\hbar)^{\frac{3}{2}}}\frac{(2\pi)}{(2\pi)}\hbar\iiint \exp(iz(q-t))\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\left(g(s)-\frac{\hbar}{4}\Delta_s g\right)\times$$

$$\exp\left(-\frac{1}{2\hbar}(t-s)^2\right)\frac{d^2}{dt^2}\phi(t)dzdsdt$$

$$= -\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\iint \delta(q-t)\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\Big(g(s)-\frac{\hbar}{4}\Delta_s g\Big)\exp\left(-\frac{1}{2\hbar}(t-s)^2\right)\frac{d^2\phi}{dq^2}dsdt$$

$$= -\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{\hbar}(q-s)^2\right)\Big(g(s)-\frac{\hbar}{4}\Delta_s g\Big)ds\frac{d^2\phi}{dt^2}.$$

In our particular case $g = s$ and this becomes

$$-\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{\hbar}(q-s)^2\right)sds\frac{d^2\phi}{dq^2}.$$

In its entirety the first term in (1.91) i.e. (1.93) becomes

$$-\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{\hbar}(q-s)^2\right)sds\frac{d^2\phi}{dq^2}-\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{\hbar}(q-s)^2\right)ds\frac{d\phi}{dq}-$$

$$\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{2\hbar}(q-s)^2\right)\frac{d^2}{ds^2}\exp\left(-\frac{1}{2\hbar}(q-s)^2\right)sds\phi(q). \tag{1.94}$$

The last term in (1.91) will give

$$-\frac{1}{2(\pi\hbar)^{\frac{3}{2}}}\frac{\hbar}{4}2\frac{(2\pi)\hbar}{(2\pi)}\iiint \exp(iz(q-t))\exp\left(-\frac{1}{2\hbar}((t-s)^2+(q-s)^2)\right)g(s)\phi(t)dzdsdt$$

$$= -\frac{\hbar 2}{4(\pi\hbar)^{\frac{1}{2}}}\iint \delta(q-t)\exp(-\frac{1}{2\hbar}((t-s)^2+(q-s)^2))g(s)\phi(t)dsdt$$

$$= -\frac{\hbar 2}{4(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{\hbar}(q-s)^2\right)sds\phi(q)$$

Substituting this and (1.94) in (1.91) we deduce that

$$qp^2 \to -\left(\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{\hbar}(q-s)^2\right)sds\right)\frac{d^2\phi}{dq^2}-\left(\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{1}{\hbar}(q-s)^2\right)sds\right)\frac{d\phi}{dq}-$$

$$\left(\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\!\left(-\frac{1}{2\hbar}(q-s)^2\right)\frac{d^2}{ds^2}\exp\!\left(-\frac{1}{2\hbar}(q-s)^2\right)sds\right)\phi(q)-\left(\frac{\hbar 2}{4(\pi\hbar)^{\frac{1}{2}}}\int\left(-\frac{1}{\hbar}(q-s)^2\right)sds\right)\phi(q) \tag{1.95}$$

Put

$$u = s - q \tag{1.96}$$

then the coefficient of ϕ'' in the first term in the above becomes

$$\begin{aligned} &-\frac{\hbar^2}{\sqrt{\pi\hbar}}\int \exp\left(-\frac{u^2}{\hbar}\right)(u+q)du \\ &= -\frac{\hbar^2 q}{\sqrt{\pi\hbar}}\int \exp\left(-\frac{u^2}{\hbar}\right)du. \end{aligned} \tag{1.97}$$

Now

$$\int \exp\left(-\frac{\pi}{a}x^2\right)dx = \sqrt{a}. \tag{1.98}$$

Putting $a = \pi\hbar$ in (1.98) we see that (1.97) becomes

$$-\frac{\hbar^2 q}{\sqrt{\pi\hbar}}\sqrt{\pi\hbar} = -\hbar^2 q.$$

Similarly we can evaluate the coefficient of ϕ' in the second term

$$= -\frac{\hbar^2}{(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{u^2}{\hbar}\right)du = -\hbar^2.$$

From (1.96) obviously we have

$$\frac{d}{ds} = \frac{du}{ds}\frac{d}{du} = \frac{d}{du}$$

which enables us to write the coefficient of ϕ' in the third term as

$$\begin{aligned} &-\frac{\hbar^2}{\sqrt{\pi\hbar}}\int \exp\left(-\frac{u^2}{2\hbar}\right)(u+q)\frac{d^2}{du^2}\exp\left(-\frac{u^2}{2\hbar}\right)du \\ &-\frac{\hbar^2}{\sqrt{\pi\hbar}}\int \exp\left(-\frac{u^2}{2\hbar}\right)(u+q)\frac{d}{du}\left[\left(-\frac{2u}{2\hbar}\right)\exp\left(-\frac{u^2}{2\hbar}\right)\right]du \\ &= \frac{\hbar^2}{\sqrt{\pi\hbar}}\int \exp\left(-\frac{u^2}{2\hbar}\right)(u+q)\frac{d}{du}\left[\frac{u}{\hbar}\exp\left(-\frac{u^2}{2\hbar}\right)\right]du \\ &= \frac{\hbar^2}{\sqrt{\pi\hbar}}\int \exp\left(-\frac{u^2}{2\hbar}\right)(u+q)\left[\frac{u}{\hbar}\left(-\frac{2u}{2\hbar}\right)\exp\left(-\frac{u^2}{2\hbar}\right)+\exp\left(-\frac{u^2}{2\hbar}\right)\frac{1}{\hbar}\right]du. \end{aligned}$$

Expanding and discarding odd functions (multiples of u and u^3 to be precise) which will integrate to zero

$$= \frac{\hbar^2}{\sqrt{\pi\hbar}}\int \exp\left(-\frac{u^2}{\hbar}\right)\left[-\frac{u^2}{\hbar^2}+\frac{1}{\hbar}\right]qdu$$

$$= -\frac{q}{\sqrt{\pi\hbar}}\int u^2 \exp\left(-\frac{u^2}{\hbar}\right) du + \frac{\hbar q}{\sqrt{\pi\hbar}}\int \exp\left(-\frac{u^2}{\hbar}\right) du$$

$$= -\frac{q}{\sqrt{\pi\hbar}}\int u^2 \exp\left(-\frac{u^2}{\hbar}\right) du + \frac{\hbar q}{\sqrt{\pi\hbar}}\sqrt{\pi\hbar}$$

$$= -\frac{q}{\sqrt{\pi\hbar}}\int u^2 \exp\left(-\frac{u^2}{\hbar}\right) du + \hbar q. \tag{1.99}$$

Now by differentiating (1.98) under the integral sign with respect to a we obtain

$$\int x^2 \exp\left(-\frac{\pi}{a}x^2\right) dx = \frac{a^{\frac{3}{2}}}{2\pi}.$$

Taking $a = \pi\hbar$ gives the integral that appears in (1.99) which becomes

$$q\left(-\frac{1}{\sqrt{\pi\hbar}}\frac{(\pi\hbar)^{\frac{3}{2}}}{2\pi} + \hbar\right)$$

$$= q\left(-\frac{\pi\hbar}{2\pi} + \hbar\right)$$

$$= \frac{q\hbar}{2}. \tag{1.100}$$

The coefficient of the fourth and final term in (1.95) is

$$-\frac{\hbar 2}{4(\pi\hbar)^{\frac{1}{2}}}\int \exp\left(-\frac{u^2}{\hbar}\right)(u+q)du.$$

Ignoring the odd part of the integrand this becomes

$$= -\frac{\hbar q}{2(\pi\hbar)^{\frac{1}{2}}}\sqrt{\pi\hbar}$$

$$= -\frac{\hbar q}{2}. \tag{1.101}$$

We can see from (1.100) and (1.101) that the third and fourth terms in (1.95) will cancel so that

$$\widehat{qp^2} = -\hbar^2\left(q\frac{d^2}{dq^2} + \frac{d}{dq}\right).$$

Chapter 2

Forms of Relativistic Quantum Mechanics

I am continually trying to find out why people find my procedure obscure... I cannot seriously believe that I ever attain the obscurity that Dirac does.

ARTHUR EDDINGTON

In 1949 Dirac published a paper in which he outlined various ways of combining special relativity with the Hamiltonian formulation of mechanics [75]. The models he proposed were referred to as forms. If we let $\mathcal{M}$ denote Minkowski spacetime then each form was associated with a particular slicing or product decomposition of $\mathcal{M}$ that is in each form we write

$$\mathcal{M} = T \times Q$$

where Q is a three dimensional manifold and T is a 1 dimensional parameter space that defines the notion of time in the form. The hypersurfaces

$$t \times Q$$

where t is an arbitrary fixed point in T are called the moments of the form and represent the spatial universe of the observer at time t. The idea is to develop a description of the physical world in terms of observables defined on the moments. Dirac considered the instant form, the point form and the front form.

In the instant form the moments are constant time spacelike hypersurfaces in some inertial frame. The instant form is associated with the following product decomposition of spacetime

$$T_I \times Q_I$$

where T_I and Q_I are covered by the coordinates t and x^i. With respect to the basis natural to these coordinates the metric has components $\eta_{\mu\nu} = (1, -1, -1, -1)$.

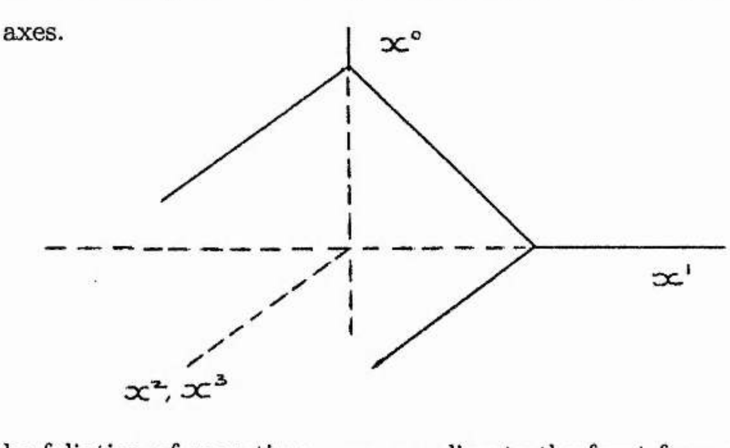
In the point form the moments are the backward light cones of an arbitary observer. For the particular case of an observer at rest at the origin the coordinates (τ, y^i)[1] where

$$\tau = x^0 + \mid \underline{x} \mid \quad \text{and} \quad y^i = x^i$$

are adapted to the point form.

A detailed formulation of classical mechanics in the point form as well as an attempt at quantisation have been given by Derick [58, 59]. More recently the point form has been investigated by Mosley and Farina [73, 74]. We believe that neither of these schemes is entirely satifactory since the quantisation of the classical observables is ad hoc and scant regard is paid to whether or not the operators are self-adjoint. Also no attempt is made to ensure that the spectrum of the operators coincides with the range of the classical observables they purport to represent. It may be that it is impossible to formulate a completely consistent quantum mechanics on a light cone. We shall see that various geometric considerations conspire to thwart any attempt to apply geometric quantisation to the point form.

The front form was considered by Dirac to be mathematically the most interesting. Here the moments are null planes called light fronts which we shall take to be tangent to the x^2 and x^3 axes.



We write the foliation of spacetime corresponding to the front form as

$$T_F \times Q_F.$$

The coordinates (τ, y^i) where

$$\tau = x^0 + x^1 \quad \text{and} \quad y^i = x^i$$

[1] We shall use Greek letters to denote elements of the index set (0,1,2,3). Latin letters from the first half of the alphabet, say i and j, represent the integers (1,2,3) while those from the latter half of the alphabet, say s and t denote an element of the set (2,3).

are adapted to the front form. To see this notice that τ parametrizes the moments (the level sets of τ are the null planes shown above) therefore τ represents front form time. Since y^i fixes a point on a given moment it can represent front form position. With respect to the basis natural to the coordinates (τ, y^i) the components of the metric are given by the matrix $g^{\mu\nu}$ where

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Front form field theories are well known [35, 36, 37, 38, 39, 40, 41, 46, 42, 43] but until now there appears to have been no front form quantum mechanics. We shall see that geometric quantisation furnishes a front form quantum mechanics in a natural way. The constraints on the ranges of the classical observables are respected at the quantum level and the pairing map relates the instant and front forms. We show that Hegerfeldt type causality violations are absent from front form quantum mechanics [76]. This work also appears in [80] and [93].

2.1 The Instant and Front Forms in 2-Dimensional Spacetime

2.1.1 Front Form Classical Mechanics of a Free Scalar Particle in 1+1 Spacetime

We may use the front form time τ to parametrize the path of a particle in spacetime. If we do this the variational principle describing the dynamics of a free particle can be written as [2]

$$\delta\left(-mc\int\left(\left(\frac{dx^0}{d\tau}\right)^2-\left(\frac{dx^1}{d\tau}\right)^2\right)^{\frac{1}{2}}d\tau\right)=0. \tag{2.2}$$

Since

$$x^1=y^1 \quad \text{and} \quad x^0=\tau-y^1$$

we can rewrite (2.2) as

$$\delta\left(-mc\int\left(\left(1-\frac{dy^1}{d\tau}\right)^2-\left(\frac{dy^1}{d\tau}\right)^2\right)^{\frac{1}{2}}d\tau\right)=0$$

or

$$\delta\left(-mc\int\left(1-2\frac{dy^1}{d\tau}\right)^{\frac{1}{2}}d\tau\right)=0.$$

Now τ has the units of length so we introduce a new variable $\tau=c\omega$. The variational principle becomes

$$\delta\left(-mc^2\int\left(1-\frac{2}{c}\frac{dy^1}{d\omega}\right)^{\frac{1}{2}}d\omega\right)=0$$

so

$$L=-mc^2\left(1-\frac{2}{c}\frac{dy^1}{dw}\right)^{\frac{1}{2}}.$$

The canonoical front form momentum is given by

$$\pi_1=-\frac{\partial L}{\partial\frac{dy^1}{dw}}$$

$$=(-1)\left(-\frac{mc^2}{2}\right)\left(1-\frac{2}{c}\frac{dy^1}{dw}\right)^{-\frac{1}{2}}\left(-\frac{2}{c}\right)$$

$$=-mc\left(1-\frac{2}{c}\frac{dy^1}{dw}\right)^{-\frac{1}{2}}$$

([50] page 194). Let τ' be the proper time of the particle. Now

$$\frac{dy^1}{d\omega}=c\frac{dy^1}{d\tau}=c\frac{dy^1}{d\tau'}/\frac{d\tau}{d\tau'}=c\frac{dx^1}{d\tau'}/\left(\frac{dx^0}{d\tau'}+\frac{dx^1}{d\tau'}\right)$$

[2]This is equivalent to the usual variational principle

$$\delta\left(-mc^2\int\left(1-\frac{1}{c^2}\left(\frac{dx^1}{dt}\right)^2\right)dt\right)=0. \tag{2.1}$$

To see this notice that we can write (2.2) as

$$\delta\left(-mc\int\left(1-\left(\frac{dx^1}{d\tau}\right)^2\left(\frac{d\tau}{dx^0}\right)^2\right)^{\frac{1}{2}}\frac{dx^0}{d\tau}d\tau\right)=0 \text{ i.e. } \delta\left(-mc\int\left(1-\left(\frac{dx^1}{dx^0}\right)^2\right)^{\frac{1}{2}}dx^0\right)=0$$

and this is the same as (2.1) if we use $x^0=ct$.

$$= \frac{cp^1}{p^0 + p^1} \tag{2.3}$$

so

$$\pi_1 = -mc\left(1 - \frac{2}{c}\frac{cp^1}{p^0 + p^1}\right)^{-\frac{1}{2}} = -mc\left(\frac{p^0 + p^1 - 2p^1}{p^0 + p^1}\right)^{-\frac{1}{2}}$$

$$= -mc\left(\frac{p^0 - p^1}{p^0 + p^1}\right)^{-\frac{1}{2}}$$

$$= -mc\left(\frac{p^0 + p^1}{p^0 - p^1}\right)^{\frac{1}{2}}.$$

Multiply top and bottom by $(p^0 + p^1)^{\frac{1}{2}}$ and use mass shell condition to obtain

$$\pi_1 = -(p^0 + p^1) \tag{2.4}$$

or

$$\pi_1 = p_1 - p_0. \tag{2.5}$$

Notice that π_1 is negative definite. We shall see that geometric quantisation automatically ensures that this classical constraint is preserved at the quantum level.

Ultimately we shall wish to compare the front and instant form quantum mechanical descriptions of the free scalar particle. As is well known in quantum mechanics different but equivalent representations are related by unitary maps. In classical mechanics on the other hand different representations are related by canonical transformations. To find a canonical transformation that relates the front and instant forms we must think carefully about the geometry of the system.

The classical descriptions of the free scalar particle in the instant and front forms are based on the following exact contact manofolds

$$M_I = T_I \times T^*Q_I \quad \text{and} \quad M_F = T_F \times T^*Q_F$$

([68] page 132) which are covered by the coordinates (t, p_1, q^1) and (τ, π_1, y^1) respectively. If the two descriptions are equivalent they will be linked by a time preserving canonical transformation

$$F :_{t_0} M_I \equiv t_0 \times T^*Q_I \rightarrow_{t_0} M_F \equiv t_0 \times T^*Q_F$$

It is quite easy to find a suitable F. We can show that if q^1 is the position of the free particle in the instant form at $t = 0$ and y^1 is the position of the particle in the front form at $\tau = 0$ then

$$q^1 = y^1 + \frac{p^1}{p^0}y^1$$

$$= \left(\frac{p^0 + p^1}{p^0}\right) y^1 \tag{2.6}$$

so

$$y^1 = \frac{q^1 p^0}{p^0 + p^1}$$

$$= \frac{q^1 p_0}{p_0 - p_1}. \tag{2.7}$$

From (2.4) we obtain

$$\pi_1^2 - 2\pi_1 p_1 + p_1^2 = p_1^2 + (mc)^2$$

and therefore

$$p_1 = \frac{\pi_1^2 - (mc)^2}{2\pi_1}. \tag{2.8}$$

Also from (2.4)

$$p_0 = p_1 - \pi_1 = \frac{\pi_1^2 - (mc)^2}{2\pi_1} - \pi_1$$

$$= -\frac{\pi_1^2 + (mc)^2}{2\pi_1}. \tag{2.9}$$

Using this and (2.4) in (2.6) we obtain

$$q^1 = \left(\frac{-\pi_1}{-\frac{\pi_1^2 + (mc)^2}{2\pi_1}}\right) y^1$$

$$= \frac{2\pi_1^2 y^1}{\pi_1^2 + (mc)^2}. \tag{2.10}$$

Equations (2.5) and (2.7) and the inverse transformations (2.8) and (2.10) define the canonical transformation F between the front and instant form pictures at $t - \tau = 0$. It is easy to see that the transformation is canonical. We have

$$d\pi_1 \wedge dy^1 = \frac{\partial \pi_1}{\partial p_1} dp_1 \wedge \left(\frac{\partial y^1}{\partial q^1} dq^1 + \frac{\partial y^1}{\partial p_1} dp_1\right).$$

Since $dp_1 \wedge dp_1 = 0$ this becomes

$$d\pi_1 \wedge dy^1 = \frac{\partial \pi_1}{\partial p_1} dp_1 \wedge \frac{\partial y^1}{\partial q^1} dq^1 = \left(1 - \frac{p_1}{p_0}\right) dp_1 \wedge \frac{p_0}{p_0 - p_1} dq^1$$

$$= dp_1 \wedge dq^1$$

as required.

The Hamiltonian H of a free relativistic particle in the instant form is given by

$$H = -p_1 \frac{dq^1}{dt} - L$$

$$= -p_1 \frac{dq^1}{dt} + mc^2 \left(1 - \frac{1}{c^2}\left(\frac{dq^1}{dt}\right)^2\right)^{\frac{1}{2}}.$$

Since

$$\frac{dq^1}{dt} = c\frac{p^1}{p^0}$$

we have

$$H = -\frac{cp_1p^1}{p^0} + mc^2\left(1 - \frac{c^2}{c^2}\frac{p^{1^2}}{p^{0^2}}\right)^{\frac{1}{2}} = \frac{cp^{1^2}}{p^0} + mc^2\frac{(p^{0^2} - p^{1^2})^{\frac{1}{2}}}{p^0}$$
$$= \frac{c}{p^0}(p^{1^2} + (mc)^2) = \frac{c}{p^0}p^{0^2}$$
$$= cp^0.$$

Because the front and instant form pictures are linked by a canonical transformation the front form Hamiltonian is also cp^0. Alternatively we can proceed canonically and say that the front form Hamiltonian K is

$$-\pi_1\frac{dy}{d\omega} - L$$
$$= -\pi_1\frac{dy}{d\omega} + mc^2\left(1 - \frac{2}{c}\frac{dy^1}{d\omega}\right)^{\frac{1}{2}} = (p^0+p^1)\frac{cp^1}{p^0+p^1} + mc^2\left(1 - \frac{2}{c}\frac{cp^1}{p^0+p^1}\right)^{\frac{1}{2}}$$
$$= cp^1 + mc^2\left(\frac{p^0+p^1-2p^1}{p^0+p^1}\right)^{\frac{1}{2}} = cp^1 + mc^2\left(\frac{p^0-p^1}{p^0+p^1}\right)^{\frac{1}{2}}$$
$$= cp^1 + mc^2\frac{mc}{p^0+p^1} = -cp_1 - \frac{m^2c^3}{\pi_1}.$$

If we use (2.8) then the above becomes

$$K = -c\frac{(\pi_1^2 - (mc)^2)}{2\pi_1} - \frac{m^2c^3}{\pi_1}$$
$$= -c\left(\frac{\pi_1^2 + (mc)^2}{2\pi_1}\right).$$

From (2.9) we see that $K = cp_0 = H$. This is what we would expect. Under a canonical transformation the new Hamiltonian is derived from the old by taking the push forward.

2.1.2 Instant Form and Front Form Quantum Mechanics in 1+1 Space-time

We shall quantise each of the classical observables π_1, y^1, q^1, p_1 and H separately in the instant and front forms using half density geometric quantisation and then show how the two pictures are related. It is the choice of polarisation that determines which form we are working in; a general polarisation F is associated with a form whose moments are M/F. In this way we see that the instant and front forms are related to the polarisations

$$P = \frac{\partial}{\partial p_1} \quad \text{and} \quad \Pi = \frac{\partial}{\partial \pi_1}$$

respectively. However this is not sufficient for our purposes since not all of π_1, y^1, q^1, p_1 and H are $C^\infty({}_{t_0}M_I, P, 1)$ and $C^\infty({}_{t_0}M_F, \Pi. 1)$. For this reason we also associate the following barred polaisations with the instant and front forms

$$\overline{P} = \frac{\partial}{\partial q} \qquad \overline{\Pi} = \frac{\partial}{\partial y}$$

and we say, for example, that a classical observable can be quantised in the instant form if it can be quantised in P or $\overline{P}$ or both and the result is unique up unitary equivalence. Mutatis mutandis quantisation in the front form. Admitting quantisation with respect to the barred polarisations is quite natural since it corresponds to working in the instant and front form momentum spaces.

Quantisation in the Instant Form

It will be convenient to introduce new coordinates $(\overline{q}^1, \overline{p}_1)$ where

$$p_1 = -\overline{q}^1 \quad \text{and} \quad q^1 = \overline{p}_1.$$

These are canonical coordinates since they are related to the (p_1, q^1) system via the generating function $f = -\overline{q}^1 q$. In fact they are the canonical coordinates adapted to the instant form horizontal polarisation since

$$\frac{\partial}{\partial q^1} = \frac{\partial}{\partial \overline{p}_1}.$$

We have

$$P = \frac{\partial}{\partial p_1} \quad \text{and} \quad \overline{P} = \frac{\partial}{\partial \overline{p}_1}.$$

Since ${}_{t_0}M_I/\overline{P}$ and ${}_{t_0}M/P$ are diffeomorphic to $\mathbb{R}$

$$\mathcal{H}_P = L^2(\mathbb{R}, dq^1) \quad \text{and} \quad \mathcal{H}_{\overline{P}} = L^2(\mathbb{R}, d\overline{q}^1).$$

The pairing construction gives a unitary map between $\mathcal{H}_P$ and $\mathcal{H}_{\overline{P}}$ as follows

$$(U_{P\overline{P}}\phi_P)(\overline{q}^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp[i\overline{q}^1 q^1]\phi_P(q^1) dq^1$$

$$(U_{\overline{P}P}\phi_{\overline{P}})(q^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp[-i\overline{q}^1 q^1]\phi_{\overline{P}}(\overline{q}^1) d\overline{q}^1$$

which we recognise as being the usual Fourier transform. This intertwines quantisations in $\mathcal{H}_{\overline{P}}$ and $\mathcal{H}_P$. For example q^1 can be quantised geometrically in $\mathcal{H}_{\overline{P}}$ and $\mathcal{H}_P$ to give the self-adjoint operators

$$Q^1_P = q^1 \quad \text{and} \quad Q^1_{\overline{P}} = -i\hbar \frac{d}{d\overline{q}^1}$$

and it is easy to show that

$$Q^1_{\overline{P}} = U_{P\overline{P}} Q^1_P U_{\overline{P}P}.$$

Similarly p_1 can be quantised in $\mathcal{H}_{\overline{P}}$ and $\mathcal{H}_P$ to give the self-adjoint operators

$$P_{1P} = -i\hbar \frac{d}{dq^1} \quad \text{and} \quad P_{1\overline{P}} = -\overline{q}^1 = p_1$$

and we find that

$$P_{1\overline{P}} = U_{P\overline{P}} P_{1P} U_{\overline{P}P}.$$

We can also quantise the front form momentum π_1 in $\mathcal{H}_{\overline{P}}$. We have

$$\begin{aligned} \Pi_{1\overline{P}} &= -(\overline{q}^1 + \overline{q}^0) \\ &= -(p^0 + p^1) \end{aligned} \tag{2.11}$$

where $\overline{q}^0 = (\overline{q}^{1^2} + (mc)^2)^{\frac{1}{2}}$. Notice that $\Pi_{1\overline{P}}$ is self-adjoint and negative definite (an immediate consequence of the specral mapping theorem) so it is a good candidate for the representation of π_1 in $\mathcal{H}_{\overline{P}}$.

Clearly $\pi_1 \notin C^\infty({}_{t_0}M_I, P, 1)$. If required the representation of π_1 in $\mathcal{H}_P$ can be taken as

$$\Pi_{1P} = U_{\overline{P}P} \Pi_{1\overline{P}} U_{P\overline{P}}.$$

We now consider the quantisation of front form position y^1 in the instant form. Since

$$y^1 = \frac{\overline{q}^0 \overline{p}_1}{\overline{q}^0 + \overline{q}^1}$$

(see (2.7)) we have $y \in C^\infty({}_{t_0}M_I, \overline{P}, 1)$. However $Y^1_{\overline{P}}$ is not self-adjoint since the vector field

$$\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \frac{\partial}{\partial \overline{q}^1}$$

is not complete. To see this note that the integral curves are solutions of

$$\frac{d\overline{q}^1}{dt} = \frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}.$$

Separating variables we obtain

$$\int \left(1 + \frac{\overline{q}^1}{\overline{q}^0}\right) d\overline{q}^1 = \int 1 dt$$

so

$$\overline{q}^1 + \overline{q}^0 = t + k$$

or

$$\overline{q}^0 = (t+k) - \overline{q}^1.$$

Squaring both sides and using the mass shell condition gives

$$\overline{q}^{1^2} + (mc)^2 = (t+k)^2 - 2\overline{q}^1(t+k) + \overline{q}^{1^2}$$

and therefore

$$\overline{q}^1 = \frac{(t+k)^2 - (mc)^2}{2(t+k)} \tag{2.12}$$

which is not well defined for all t. Geometric quantisation therefore leads us to quantise the front form position observable in the instant form momentum space as the symmetric operator

$$\begin{aligned} Y^1_{\overline{P}} &= -i\hbar\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\frac{\partial}{\partial\overline{q}^1} + \frac{1}{2}\frac{\partial}{\partial\overline{q}^1}\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right) \\ &= -i\hbar\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\frac{\partial}{\partial\overline{q}^1} + \frac{(\overline{q}^0+\overline{q}^1)\frac{\overline{q}^1}{\overline{q}^0} - \overline{q}^0(\frac{\overline{q}^1}{\overline{q}^0}+1)}{(\overline{q}^0+\overline{q}^1)^2}\right) \\ &= -i\hbar\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\frac{\partial}{\partial\overline{q}^1} + \frac{\overline{q}^{1^2} - \overline{q}^{0^2}}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)^2}\right) \\ &= -i\hbar\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\frac{\partial}{\partial\overline{q}^1} + \frac{(\overline{q}^1 - \overline{q}^0)}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}\right). \end{aligned} \tag{2.13}$$

We can investigate whether $Y^1_{\overline{P}}$ has self-adjoint extensions or not using the methods described in Chapter 1. It turns out that $Y^1_{\overline{P}}$ is maximally symmetric.

Notice that $y^1 \notin C^\infty({}_{t_0}M_I, P, 1)$. If required we can take

$$Y^1_P = U_{\overline{P}P} Y^1_{\overline{P}} U_{P\overline{P}}.$$

Clearly H can be quantised in $\mathcal{H}_{\overline{P}}$ to give the self-adjoint operator

$$H_{\overline{P}} = c\overline{q}^0$$

Quantisation in the Front Form

Define a new coordinate system $(\overline{y}^1, \overline{\pi}_1)$ where

$$\pi_1 = -\overline{y}^1 \quad \text{and} \quad y^1 = \overline{\pi}_1.$$

The coordinates (π_1, y^1) and $(\overline{\pi}_1, \overline{y}^1)$ are related by the generating function $f = -\overline{y}^1 y^1$ i.e.

$$\pi_1 = \frac{\partial f}{\partial y^1} \quad \text{and} \quad \overline{\pi}_1 = -\frac{\partial f}{\partial \overline{y}^1}.$$

Consider the polarisations

$$\Pi = \frac{\partial}{\partial \pi_1} \quad \text{and} \quad \overline{\Pi} = \frac{\partial}{\partial \overline{\pi}_1}.$$

Since ${}_{t_0}M_F/\Pi$ is diffeomorphic to $\boldsymbol{R}$ and ${}_{t_0}M_F/\overline{\Pi}$ is diffeomorphic to $\mathbb{R}^+$ we have

$$\mathcal{H}_\Pi = L^2\left(\boldsymbol{R}, \frac{dy^1}{\mid y^1 \mid}\right) \quad \text{and} \quad \mathcal{H}_{\overline{\Pi}} = L^2(\mathbb{R}^+, d\overline{y}^1).$$

$\mathcal{H}_{\overline{\Pi}}$ corresponds to the front form momentum space but $\mathcal{H}_\Pi$ will not serve as the state space of the front form configuration representation in this scheme. We define a subspace $\mathcal{H}_\Pi^+$ of $\mathcal{H}_\Pi$ as follows

$$\mathcal{H}_\Pi^+ = \{\phi \in \mathcal{H}_\Pi : \phi/\sqrt{\mid y^1 \mid} \in \mathcal{H}\}$$

where $\mathcal{H}$ denotes the set of Hardy class functions [82]. This is the Hilbert space we shall associate with Π, an element of $\mathcal{H}_\Pi^+$ will be denoted ϕ_Π. We shall see later that restricting the Hilbert space in this way is nessarsary to ensure that the classical constraint $\pi_1 < 0$ is respected at the quantum level. We find that the model simply does not hang together unless we enforce the constraint; for example notice that the pairing construction does not lead to a unitary map between $\mathcal{H}_\Pi$ and $\mathcal{H}_{\overline{\Pi}}$ but does give the following unitary map between $\mathcal{H}_\Pi^+$ and $\mathcal{H}_{\overline{\Pi}}$

$$(U_{\Pi\overline{\Pi}}\phi_\Pi)(\overline{y}^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp(i\overline{y}^1 y^1)\phi_\Pi(y^1)\frac{1}{\sqrt{\mid y^1 \mid}} dy^1$$

$$(U_{\overline{\Pi}\Pi}\phi_{\overline{\Pi}})(y^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_0^{\infty} \exp(-i\overline{y}^1 y')\phi_{\overline{\Pi}}(\overline{y}')\sqrt{\mid y^1 \mid} d\overline{y}^1.$$

In addition we shall see that it is only because we have respected the constraint $\pi_1 < 0$ that we are able to intertwine quantisations in the instant and front forms. In short, geometric quantisation knows about the constraint and demands that it be taken seriously.

All the observables can be quantised in $\mathcal{H}_{\overline{\Pi}}$. We have $y^1 = \overline{\pi}_1$ therefore $y^1 \in C^\infty({}_{t_0}M_F, \overline{\Pi}, 1)$ and

$$Y_{\overline{\Pi}}^1 = -i\hbar \frac{d}{d\overline{y}^1}.$$

Since $d/d\overline{y}^1$ is not complete on $\mathbb{R}^+$ this operator is not essentially self-adjoint. In fact using the methods of Chapter 1 it is easy to show that the defect indicies of $Y_{\overline{\Pi}}^1$ are (0,1) so that $Y_{\overline{\Pi}}^1$ is maximally symmetric.

Now y^1 can also be quantised in $\mathcal{H}_\Pi^+$. Geometric quantisation tells us that y^1 should be quantised in $\mathcal{H}_\Pi$ as the self-adjoint multiplication operator

$$Y_\Pi^1 = y^1.$$

It is easy to see that Y^1_Π leaves $\mathcal{H}^+_\Pi$ invariant since

$$\frac{y^1\widetilde{\phi(y^1)}}{\sqrt{|\, y^1\,|}} = \frac{1}{i}\frac{d}{d\overline{y}^1}\frac{\widetilde{\phi(y^1)}}{\sqrt{|\, y^1\,|}}$$

(where tilde denotes Fourier transform) which clearly has support in $\boldsymbol{R}^+$ if $\phi(y^1)/\sqrt{|\, y^1\,|}$ is Hardy class. Thus we are led to re-define Y^1_Π as the restriction to $\mathcal{H}^+_\Pi$ of the self-adjoint multiplication operator. The multiplication operator is not reduced by $\mathcal{H}^+_\Pi$ and so Y^1_Π is not self-adjoint in fact Y^1_Π is maximally symmetric.

Internal consistency of the front form is assured because

$$Y^1_\Pi = U_{\overline{\Pi}\Pi} Y^1_{\overline{\Pi}} U_{\Pi\overline{\Pi}} \tag{2.14}$$

Consider now the front form momentum observable π_1. We have $\pi_1 = -\overline{y}^1$ so that $\Pi_{1\overline{\Pi}}$ is the negative of the self-adjoint mutiplication operator in $\mathcal{H}_{\overline{\Pi}}$ i.e.

$$\Pi_{1\overline{\Pi}} = -\overline{y}^1.$$

Since $\mathcal{H}_{\overline{\Pi}} = L^2(\boldsymbol{R}^+, d\overline{y})$ it is easy to see that $\Pi_{1\overline{\Pi}}$ will have a strictly negative spectrum as required.

We have $\pi_1 \in C^\infty({}_{t_0}M_F, \Pi, 1)$ so π_1 can also be quantised in $\mathcal{H}_\Pi$

$$\Pi_{1\Pi} = -i\hbar\left(\frac{d}{dy^1} + \frac{|\, y^1\,|}{2}\frac{d}{dy^1}\left(\frac{1}{|\, y^1\,|}\right)\right)$$

$$-i\hbar\left(\frac{d}{dy^1} + \frac{|\, y^1\,|}{2}\frac{d\,|\, y'\,|}{dy^1}\frac{d\,|\, y^1\,|^{-1}}{d\,|\, y^1\,|}\right)$$

$$= -i\hbar\left(\frac{d}{dy^1} + \frac{|\, y\,|}{2} sgn(y')(-1)\,|\, y^1\,|^{-2}\right)$$

$$= -i\hbar\left(\frac{d}{dy^1} - \frac{sgn(y^1)}{2\,|\, y^1\,|}\right).$$

Since d/dy^1 is complete on $\boldsymbol{R}$ we know that $\Pi_{1\Pi}$ is self-adjoint in $\mathcal{H}_\Pi$. The front form momentum observable is the restriction of this operator to $\mathcal{H}^+_\Pi$. We shall see later that $\mathcal{H}^+_\Pi$ corresponds to the negative spectrum of $\Pi_{1\Pi}$ and so by construction is a reducing subspace. Therefore the restriction of $\Pi_{1\Pi}$ to $\mathcal{H}^+_\Pi$ is self-adjoint and negative definite.

The front form is internally consistent because

$$\Pi_{1\Pi} = U_{\overline{\Pi}\Pi}\Pi_{1\overline{\Pi}}U_{\Pi\overline{\Pi}}. \tag{2.15}$$

We now consider the instant form observables. Since

$$q^1 = \frac{2\overline{y}^{1^2}}{\overline{y}^{1^2} + (mc)^2}\overline{\pi}_1$$

we have $q \in C^\infty({}_{t_0}M_F, \overline{\Pi}, 1)$ and

$$Q^1_{\overline{\Pi}} = -i\hbar \left(\frac{2\overline{y}^{1^2}}{\overline{y}^{1^2} + (mc)^2} \frac{d}{d\overline{y}^1} + \frac{1}{2} \frac{d}{d\overline{y}^1} \left(\frac{2\overline{y}^{1^2}}{\overline{y}^{1^2} + (mc)^2} \right) \right)$$

$$= -i\hbar \left(\frac{2\overline{y}^{1^2}}{(\overline{y}^{1^2} + (mc)^2)} \frac{d}{d\overline{y}^1} + \frac{2(mc)^2\overline{y}^1}{(\overline{y}^{1^2} + (mc)^2)^2} \right).$$

We can also write this as

$$Q^1_{\overline{\Pi}} = \frac{2(-\Pi_{1\overline{\Pi}})^2}{(-\Pi_{1\overline{\Pi}})^2 + (mc)^2} Y^1_{\overline{\Pi}} - \frac{i\hbar 2(mc)^2(-\Pi_{1\overline{\Pi}})}{((-\Pi_{1\overline{\Pi}})^2 + (mc)^2)^2}$$

$$= \frac{2\Pi^2_{1\overline{\Pi}}}{\Pi^2_{1\overline{\Pi}} + (mc)^2} Y^1_{\overline{\Pi}} + \frac{i\hbar 2(mc)^2 \Pi_{1\overline{\Pi}}}{(\Pi^2_{1\overline{\Pi}} + (mc)^2)^2}. \tag{2.16}$$

$Q^1_{\overline{\Pi}}$ is self-adjoint since

$$\frac{2\overline{y}^{1^2}}{\overline{y}^{1^2} + (mc)^2} \frac{d}{d\overline{y}^1}$$

is complete on $\boldsymbol{R}^+$; the integral curves of this vector field are given by

$$\overline{y}^1 = (t+k) + \sqrt{(t+k)^2 + (mc)^2}$$

and these are well defined and contained in $\boldsymbol{R}^+$ for all t.

It is obvious from (2.10) that $q^1 \notin C^\infty({}_{t_0}M_F, \Pi, 1)$ so that q^1 cannot be quantised directly in $\mathcal{H}^+_\Pi$. We can take

$$Q^1_\Pi = U_{\overline{\Pi}\Pi} Q^1_{\overline{\Pi}} U_{\Pi\overline{\Pi}}.$$

Using (2.14) and (2.16) this becomes

$$Q^1_\Pi = \frac{2\Pi^2_{1\Pi}}{\Pi^2_{1\Pi} + (mc)^2} Y^1_\Pi + i\hbar 2(mc)^2 \frac{\Pi_{1\Pi}}{(\Pi^2_{1\Pi} + (mc)^2)^2}. \tag{2.17}$$

Clearly $p_1 \in C^\infty({}_{t_0}M_I, \overline{\Pi}, 1)$ and can be quantised to give the following self-adjoint operator in $\mathcal{H}_{\overline{\Pi}}$

$$P_{1\overline{\Pi}} = -\left(\frac{\overline{y}^{1^2} - (mc)^2}{2\overline{y}^1} \right).$$

p_1 cannot be quantised directly in $\mathcal{H}^+_\Pi$ so we can use the pairing construction to obtain

$$P_{1\Pi} = \frac{\Pi^2_{1\Pi} - (mc)^2}{2\Pi_{1\Pi}}.$$

A similar situation occurs with the Hamiltonian. H can be quantised directly in $\mathcal{H}_{\overline{\Pi}}$ to obtain the following self-adjoint operator

$$H_{\overline{\Pi}} = c \left(\frac{\overline{y}^{1^2} + (mc)^2}{2\overline{y}^1} \right).$$

Since $H \notin C^\infty({}_{t_0}M_F, \Pi, 1)$ we can use the pairing map to give

$$H_\Pi = -c \left(\frac{\Pi^2_{1\Pi} + (mc)^2}{2\Pi_{1\Pi}} \right).$$

Connection Between Instant and Front Forms

The coordinate systems (π_1, y^1) and $(\overline{p}_1, \overline{q}^1)$ are related by the generating function

$$f = -(\overline{q}^1 + (\overline{q}^{1^2} + (mc)^2)^{\frac{1}{2}})y^1$$

i.e.

$$\pi_1 = \frac{\partial f}{\partial y^1} \quad \text{and} \quad \overline{p}_1 = -\frac{\partial f}{\partial \overline{q}^1}.$$

The pairing construction gives rise to a map between $\mathcal{H}_\Pi^+$ and $\mathcal{H}_{\overline{P}}$ as follows

$$(U_{\Pi\overline{P}}\phi_\Pi)(\overline{q}^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \phi_\Pi(y^1)\exp(i(\overline{q}^1 + \overline{q}^0)y^1)\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \frac{1}{\sqrt{|\, y^1 \,|}}dy^1 \qquad (2.18)$$

and

$$(U_{\overline{P}\Pi}\phi_{\overline{P}})(y^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \phi_{\overline{\Pi}}(\overline{q}^1)\exp(-i(\overline{q}^1 + \overline{q}^0)y^1)\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \sqrt{|\, y^1 \,|}d\overline{q}^1.$$

This map is unitary (Appendix 1).

To show that our scheme is completely consistent we must demonstrate that when a classical observable can be quantised in both the instant and front form pictures the resulting operators are unitarily related via the pairing i.e. we require

$$Y^{1}_{\overline{P}} = U_{\Pi\overline{P}}Y^1_\Pi U_{\overline{P}\Pi}, \;\; \Pi_{1\overline{P}} = U_{\Pi\overline{P}}\Pi_{1\Pi}U_{\overline{P}\Pi}, \;\; Q^1_{\overline{P}} = U_{\Pi\overline{P}}Q^1_\Pi U_{\overline{P}\Pi}$$

$$P_{1\overline{P}} = U_{\Pi\overline{P}}P_{1\Pi}U_{\overline{P}\Pi}, \;\; H_{\overline{P}} = U_{\Pi\overline{P}}H_\Pi U_{\overline{P}\Pi}.$$

The validity of these relations is demonstrated in Appendix 2.

Geometric quantisation has led to a front form quantum mechanics that respects the classical constraint $\pi_1 < 0$ and is unitarily related to the instant form via the pairing. The only peculiar feature of the model is the maximally symmetric front form position operator but as described in Chapter 1 maximally symmetric operators are easy to interpret as quantum observables. It should be noticed that the restriction of $\mathcal{H}_\Pi$ to $\mathcal{H}_\Pi^+$ is not at all ad hoc. Suppose we were to quantise π_1 geometrically in $\mathcal{H}_\Pi$ to obtain

$$\Pi_{1\Pi} = -i\hbar\left(\frac{d}{dy^1} - \frac{sgn(y^1)}{2\,|\, y^1 \,|}\right).$$

This operator is clearly not negative definite since its generalised eigenvectors are given by

$$\phi_\lambda(y^1) = A \,|\, y \,|^{\frac{1}{2}} \exp(i\lambda y^1)$$

where A is some normalisation constant and $\lambda \in (-\infty, \infty)$. To obtain a negative definite momentum operator we must restrict $\mathcal{H}_\Pi$ to that subspace corresponding to the negative part of the spectrum of $\Pi_{1\Pi}$. Now an element ϕ of $\mathcal{H}_\Pi$ is in $E(\mathbb{R}^-, \Pi_{1\Pi})\mathcal{H}_\Pi$ (where $E(\Delta, \Pi_{1\Pi})$ is the orthogonal spectral measure corresponding to Π_1) if

$$\int_{-\infty}^{0} A \mid y^1 \mid^{\frac{1}{2}} \exp(i\lambda y^1) < \phi(\hat{y}^1), A \mid \hat{y}^1 \mid^{\frac{1}{2}} \exp(i\lambda \hat{y}^1) > d\lambda =$$

$$\int_{-\infty}^{\infty} A \mid y^1 \mid^{\frac{1}{2}} \exp(i\lambda y^1) < \phi(\hat{y}^1), A \mid \hat{y}^1 \mid^{\frac{1}{2}} \exp(i\lambda \hat{y}^1) > d\lambda$$

i.e. if

$$\text{Supp} < \phi(\hat{y}^1), A \mid \hat{y}^1 \mid^{\frac{1}{2}} \exp(i\lambda \hat{y}^1) > \subset (-\infty, 0)$$

or

$$\text{Supp} < \phi(\hat{y}^1), A \mid \hat{y}^1 \mid^{\frac{1}{2}} \exp(-i\lambda \hat{y}^1) > \subset (0, \infty).$$

More specifically we require

$$\text{Supp} \int_{-\infty}^{\infty} \phi(\hat{y}^1) \mid \hat{y}^1 \mid^{\frac{1}{2}} \frac{\exp(i\lambda \hat{y}^1)}{\mid \hat{y}^1 \mid} d\hat{y}^1 \subset (0, \infty)$$

or

$$\text{Supp} \int_{-\infty}^{\infty} \phi(\hat{y}^1) \frac{\exp(i\lambda \hat{y}^1}{\mid \hat{y}^1 \mid^{\frac{1}{2}}}) d\hat{y}^1 \subset (0, \infty)$$

which is precisely the condition that $\phi(y^1)/ \mid y^1 \mid^{\frac{1}{2}}$ be Hardy class so

$$E(\mathbb{R}^-, \Pi_{1\Pi})\mathcal{H}_\Pi = \mathcal{H}_\Pi^+.$$

2.1.3 A Comment on Hegerfeldt's Theorem in Front Form Quantum Mechanics

It appears that Hegerfeldt's theorem [76] is irrelevant in the context of a quantum mechanical theory developed in the front form since there is no sense in which a particle can be said to be initially localised. This circumstance arises from the nature of the state space $\mathcal{H}_\Pi^+$. As our notion of localisation we take the unique P.O.V measure F associated with the maximally symmetric front form position operator Y_Π^1. Accordingly a state ϕ_Π is localised in a finite region V of the light front if

$$< \phi_\Pi \mid F(V)\phi_\Pi >= 1. \tag{2.19}$$

We can easily determine the explicit form of F since we know that the self adjoint multiplication operator on $\mathcal{H}_\Pi$ is a self-adjoint extension of Y_Π^1. The spectral function of the multiplication operator is simply the characteristic function χ_V so that by Naimarks theorem (see Chapter 1) we have

$$F(V) = P\chi_V$$

where P is the projection operator on $\mathcal{H}_\Pi$ with range $\mathcal{H}_\Pi^+$. If we substitute this expression for $F(V)$ in (2.19) and remember that P is self-adjoint and $\phi_\Pi \in \mathcal{H}_\Pi^+$ we find that an initially localised wave function satisfies

$$\int_{-\infty}^{\infty} \phi_\Pi(y^1)\chi_V(y^1)\phi_\Pi^*(y^1)\frac{dy^1}{|\, y^1 \,|} = 1$$

or

$$\int_V \frac{|\, \phi_\Pi(y^1) \,|^2}{|\, y^1 \,|} dy^1 = 1. \tag{2.20}$$

Of course the localised state vector must also be normalised i.e. we also require that

$$\int_{-\infty}^{\infty} \frac{|\, \phi_\Pi(y^1) \,|^2}{|\, y^1 \,|} dy^1 = 1.$$

However this is incompatible with (2.20) unless the support of ϕ_Π is contained in V. But $\phi_\Pi / \,|y^1|^{\frac{1}{2}}$ is Hardy class and the Paley Wiener theorem tells that there are no Hardy class functions of compact support. Therefore there is no notion of localization on the front and Hegerfeldts results do not apply.

2.2 The Instant and Front Forms in 4-Dimensional Spacetime

We now wish to generalise the results of the previous section to four dimensional spacetime.

2.2.1 Front Form Classical Mechanics of Free Scalar Particle in 3+1 Spacetime

The Lagrangian of a free scalar particle is given by

$$L = -mc^2 \left(1 - \frac{2}{c}\frac{dy'}{d\omega} - \frac{1}{c^2}\left(\frac{dy^2}{d\omega}\right)^2 - \frac{1}{c^2}\left(\frac{dy^3}{d\omega}\right)^2 \right)^{\frac{1}{2}}.$$

Put

$$\mathcal{A} = 1 - \frac{2}{c}\frac{dy'}{d\omega} - \frac{1}{c^2}\left(\frac{dy^2}{d\omega}\right)^2 - \frac{1}{c^2}\left(\frac{dy^3}{d\omega}\right)^2.$$

It is fairly easy to show that in a 3+1 spacetime (2.3) becomes

$$\frac{dy^i}{d\omega} = \frac{cp^i}{p^0 + p^1}$$

so

$$\mathcal{A} = 1 - \frac{2}{c}\frac{cp^1}{p^0+p^1} - \frac{1}{c^2}\frac{c^2 p^{2^2}}{(p^0+p^1)^2} - \frac{1}{c^2}\frac{c^2 p^{3^2}}{(p^0+p^1)^2}$$

$$= \frac{p^{0^2} - p^{1^2} - p^{2^2} - p^{3^2}}{(p^0 + p^1)^2} = \frac{(mc)^2}{(p^0 + p^1)^2}.$$

Therefore

$$\pi_1 = -\frac{\partial L}{\partial(\frac{dy^1}{d\omega})} = (-1)\left(-\frac{mc^2}{2}(\mathcal{A})^{-\frac{1}{2}}\left(-\frac{2}{c}\right)\right) = -mc(\mathcal{A})^{-\frac{1}{2}}$$

$$= -mc\left(\frac{p^0 + p^1}{mc}\right) = -(p^0 + p^1)$$

$$= p_1 - p_0 \tag{2.21}$$

and

$$\pi_s = (-1)\left(-mc^2\frac{1}{2}(\mathcal{A})^{-\frac{1}{2}}\left(-\frac{2}{c^2}\right)\frac{dy^s}{d\omega}\right)$$

$$= -m\frac{dy^s}{d\omega}(\mathcal{A})^{-\frac{1}{2}} = -m\frac{cp^s}{p^0 + p^1}\frac{p^0 + p^1}{mc} = -p^s$$

$$= p_s.$$

By considering the dynamics of the free particle we can show that if $\underline{q}$ is the position of the particle in the instant form at $t = 0$ and $\underline{y}$ the position of the particle in the front form at $\tau = 0$ then

$$q^1 = y^1 + \frac{p^1}{p^0}y^1 \tag{2.22}$$

or

$$q^1 = \left(\frac{p_0 - p_1}{p_0}\right)y^1 \tag{2.23}$$

and

$$q^s = y^s + \frac{p^s}{p^0}y^1 \tag{2.24}$$

or

$$q^s = y^s - \frac{p_s}{p_0}y^1. \tag{2.25}$$

From (2.22) we obtain

$$y^1 = \frac{p^0 q^1}{p^0 + p^1}. \tag{2.26}$$

Substituing this in (2.24) gives

$$q^s = y^s + \frac{p^s}{p^0}\frac{p^0 q^1}{p^0 + p^1}$$

so

$$y^s = q^s - \frac{p^s q^1}{p^0 + p^1}. \tag{2.27}$$

In terms of covariant momenta (2.26) and (2.27) become

$$y^1 = \frac{p_0 q^1}{p_0 - p_1} \tag{2.28}$$

and

$$y^s = q^s + \frac{p_s q^1}{p_0 - p_1}. \tag{2.29}$$

Now from (2.21) we have

$$\pi_1 - p_1 = -p_0.$$

Squaring both sides gives

$$\pi_1^2 - 2\pi_1 p_1 + p_1^2 = (mc)^2 + p_1^2 + p_2^2 + p_3^2$$

$$\pi_1^2 - 2\pi_1 p_1 = (mc)^2 + \pi_2^2 + \pi_3^2.$$

From this we can obtain an expression for p_1 in terms of front form momentum variables so altogether we have

$$p_1 = \frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2}{2\pi_1}$$

and

$$p_s = \pi_s. \tag{2.30}$$

From (2.21) we have

$$p_0 = p_1 - \pi_1 = \frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2}{2\pi_1} - \pi_1$$

$$= -\frac{(\underline{\pi}^2 + (mc)^2)}{2\pi_1}. \tag{2.31}$$

From (2.23) and using (2.21) and (2.31) we obtain

$$q^1 = \frac{(-\pi_1)}{-\frac{(\underline{\pi}^2 + (mc)^2)}{2\pi_1}} y^1 = \frac{2\pi_1^2 y^1}{\underline{\pi}^2 + (mc)^2}$$

and from (2.25), (2.30) and (2.31) we have

$$q^s = y^s - \frac{\pi_s y^1}{-\frac{(\underline{\pi}^2 + (mc)^2)}{2\pi_1}} = y^s + \frac{2\pi_1 \pi_s y^1}{\underline{\pi}^2 + (mc)^2}$$

We can show that these relations define a canonical transformation between the front and instant forms. After some algebra we obtain

$$d\pi_1 \wedge dy^1 = -\frac{1}{p_0(p_0 - p_1)^2} q^1 p_1 p_s dp_1 \wedge dp_s + dp_1 \wedge dq^1 - \frac{p_s}{p_0 - p_1} dp_s \wedge dq^1 - \frac{p_s q^1}{(p_0 - p_1)^2} dp_s \wedge dp_1 \tag{2.32}$$

and

$$d\pi_t \wedge dy^t = -\frac{q^1 p_t p_1}{p_0(p_0 - p_1)^2} dp_t \wedge dp_1 + \frac{q^1 p_t}{(p_0 - p_1)^2} dp_t \wedge dp_1 + dp_t \wedge dq^t + \frac{p_t}{p_0 - p_1} dp_t \wedge dq^1$$

where we have used $dp_s \wedge dp_s = 0$ and $p_s p_t dp_s \wedge dp_t = 0$. Adding this last expression to (2.32) gives

$$d\pi_i \wedge dy^i = dp_i \wedge dq^i$$

as required.

Since the Hamiltonian is cp^0 from (2.31) we have

$$H = -c\frac{(\underline{\pi}^2 + (mc)^2)}{2\pi_1}.$$

2.2.2 Instant and Front Form Quantum Mechanics in 3+1 Spacetime

Quantization in the Instant Form

Define the coordinates $(\overline{p}_i, \overline{q}^i)$ as follows

$$p_i = -\overline{q}^i \quad \text{and} \quad q^i = \overline{p}_i. \tag{2.33}$$

These are canonical coordinates since they are related to the (p_i, q^i) system via the generating function $f = -\overline{q}^i q^i$ where we sum over i. They are the canonical coordinates adapted to the instant form horizontal polarisation or instant form mommentum space since

$$\frac{\partial}{\partial q^i} = \frac{\partial}{\partial \overline{p}_i}.$$

Introduce the following polarisations

$$P = \left\{\frac{\partial}{\partial p_i}\right\} \quad \text{and} \quad \overline{P} = \left\{\frac{\partial}{\partial \overline{p}_i}\right\}$$

then

$$\mathcal{H}_P = L^2(\mathbb{R}^3, d^3\underline{q}) \quad \text{and} \quad \mathcal{H}_{\overline{P}} = L^2(\mathbb{R}^3, d^3\underline{\overline{q}}).$$

The pairing leads to a unitary map between $\mathcal{H}_P$ and $\mathcal{H}_{\overline{P}}$

$$(U_{P\overline{P}}\phi_P)(\underline{\overline{q}}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_{-\infty}^{\infty} \exp[i\underline{\overline{q}}.\underline{q}]\phi_P(\underline{q})d^3\underline{q}$$

$$(U_{\overline{P}P}\phi)(\underline{q}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_{-\infty}^{\infty} \exp[-i\underline{\overline{q}}.\underline{q}]\phi_{\overline{P}}(\underline{\overline{q}})d^3\underline{\overline{q}}.$$

This map intertwines quantisations in $\mathcal{H}_P$ and $\mathcal{H}_{\overline{P}}$. For example the q^i can be quantised in $\mathcal{H}_P$ and $\mathcal{H}_{\overline{P}}$ to give the self-adjoint operators

$$Q^i_P = q^i \quad \text{and} \quad Q^i_{\overline{P}} = -i\hbar\frac{\partial}{\partial \overline{q}^i}$$

and it is easy to show that

$$Q^i_{\overline{P}} = U_{P\overline{P}} Q^i_P U_{\overline{P}P}.$$

Similarly the p_i can be quantised with respect to $\overline{P}$ and P to give the self-adjoint operators

$$P_{iP} = -i\hbar\frac{\partial}{\partial q^i} \quad \text{and} \quad P_{i\overline{P}} = -\overline{q}^i$$

and we have

$$P_{i\overline{P}} = U_{P\overline{P}}P_{iP}U_{\overline{P}P}.$$

We can also quantise the front form momenta π_i in $\mathcal{H}_{\overline{P}}$. They are represented by the self-adjoint operators

$$\Pi_{1\overline{P}} = -(\overline{q}^1 + \overline{q}^0)$$

and

$$\Pi_{s\overline{P}} = -\overline{q}^s \tag{2.34}$$

where $\overline{q}^0 = (|\ \underline{\overline{q}}\ |^2 + m^2)^{\frac{1}{2}}$. Notice that $\Pi_{1\overline{P}}$ is negative definite so the spectrum of the quantum observable coincides with the range of the classical observable as required.

From (2.28) we have

$$y^1 = \frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\overline{p}_1$$

so $y^1 \in C^\infty({}_{t_0}M_I, \overline{P}, 1)$. However y^1 does not generate a complete vector field so can only be quantised as the symmetric operator

$$Y^1_{\overline{P}} = -i\hbar\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\frac{\partial}{\partial\overline{q}^1} + \frac{\overline{q}^1 - \overline{q}^0}{2\overline{q}^0(\overline{q}^0 + \overline{q}^1)}\right). \tag{2.35}$$

We can show that this operator is maximally symmetric. From (2.29) we have

$$y^s = \overline{p}_s - \frac{\overline{q}^s\overline{p}_1}{\overline{q}^0 + \overline{q}^1}$$

so the y^s can be quantised in $\mathcal{H}_{\overline{P}}$ to obtain the self-adjoint operators

$$Y^s_{\overline{P}} = -i\hbar\left(\frac{\partial}{\partial\overline{q}^s} - \left(\frac{\overline{q}^s}{\overline{q}^1 + \overline{q}^0}\right)\frac{\partial}{\partial\overline{q}^1} + \frac{\overline{q}^s}{2\overline{q}^0(\overline{q}^0 + \overline{q}^1)}\right). \tag{2.36}$$

Clearly we have

$$H_{\overline{P}} = c\overline{q}^0.$$

Quantisation in the Front Form

Define new coordinates $(\overline{\pi}_i, \overline{y}^i)$ as follows

$$\pi_i = -\overline{y}^i \quad \text{and} \quad y^i = \overline{\pi}_i.$$

The coordinates (π_i, y^i) and $(\overline{\pi}_i, \overline{y}^i)$ are related by the generating function $f = -\overline{y}^i y^i$ i.e.

$$\pi_i = \frac{\partial f}{\partial y^i} \quad \text{and} \quad \overline{\pi}_i = -\frac{\partial f}{\partial \overline{y}^i}.$$

Consider the polarisations

$$\Pi = \left\{ \frac{\partial}{\partial \pi_i} \right\} \quad \text{and} \quad \overline{\Pi} = \left\{ \frac{\partial}{\partial \overline{\pi}_i} \right\}.$$

We have

$$\mathcal{H}_\Pi = L^2\left(\boldsymbol{R}^3, \frac{d^3\underline{y}}{|y^1|}\right) \quad \text{and} \quad \mathcal{H}_{\overline{\Pi}} = L^2(\boldsymbol{R}^+ \times \boldsymbol{R}^2, d^3\underline{\overline{y}}).$$

Define a subspace $\mathcal{H}_\Pi^+$ of $\mathcal{H}_\Pi$ as follows

$$\mathcal{H}_\Pi^+ = \{\phi \in \mathcal{H}_\Pi : \int_{-\infty}^{\infty} \frac{\phi(\underline{y})}{\sqrt{|y^1|}} \exp[i\underline{\overline{y}}.\underline{y}] d^3\underline{y} \in L^2(\boldsymbol{R}^+ \times \boldsymbol{R}^2, d^3\underline{\overline{y}})\}.$$

Clearly[3] $\mathcal{H}_\Pi^+ = {}_1\mathcal{H}_\Pi^+ \hat{\otimes} L^2(\boldsymbol{R}) \hat{\otimes} L^2(\boldsymbol{R})$. This is the Hilbert space we shall associate with Π from now on.

The pairing construction leads to the following unitary map between $\mathcal{H}_\Pi^+$ and $\mathcal{H}_{\overline{\Pi}}$

$$(U_{\Pi\overline{\Pi}}\phi_\Pi)(\underline{\overline{y}}) = \int_{-\infty}^{\infty} \phi(\underline{y}) \exp(i\underline{\overline{y}}.\underline{y}) \frac{1}{\sqrt{|y^1|}} d^3\underline{y}$$

$$(U_{\overline{\Pi}\Pi}\phi_{\overline{\Pi}})(\underline{y}) = \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\underline{\overline{y}}) \exp(-i\underline{\overline{y}}.\underline{y}) \sqrt{|y^1|} d^3\underline{\overline{y}}.$$

If we quantise the front form position observables in $\mathcal{H}_\Pi^+$ we obtain

$$Y_\Pi^i = y^i.$$

Now Y_Π^1 is maximally symmetric (see Appendix 3) since it can be identified with the closure of ${}_1Y_\Pi^1 \otimes I \otimes I$ on ${}_1\mathcal{H}_\Pi^+ \otimes L^2(\boldsymbol{R}) \otimes L^2(\boldsymbol{R})$ in ${}_1\mathcal{H}_\Pi^+ \hat{\otimes} L^2(\boldsymbol{R}^2)$. Also Y_Π^2 and Y_Π^3 are self-adjoint since they can be identified with the closures of $I \otimes y^2 \otimes I$ and $I \otimes I \otimes y^3$. Recall that the tensor product of self-adjoint operators is essentially self-adjoint so the closure is self-adjoint.

If we quantise the front form position operator in $\mathcal{H}_{\overline{\Pi}}$ we have

$$Y_{\overline{\Pi}}^i = -i\hbar \frac{\partial}{\partial \overline{y}^i}.$$

Again $Y_{\overline{\Pi}}^1$ is maximally symmetric since it can be identified with the closure of ${}_1Y_{\overline{\Pi}}^1 \otimes I \otimes I$. We know that $Y_{\overline{\Pi}}^2$ and $Y_{\overline{\Pi}}^3$ are self-adjoint since y^2 and y^3 generate complete vector fields on $\boldsymbol{R}^+ \times \boldsymbol{R} \times \boldsymbol{R}$.

Now consider the front form momentum observables. It is easy to show that

$$\Pi_{1\Pi} = -i\hbar\left(\frac{\partial}{\partial y^1} - \frac{sgn(y^1)}{2\,|y^1|}\right) \quad \text{and} \quad \Pi_{s\Pi} = -i\hbar \frac{\partial}{\partial y^s}.$$

[3]We use an extra subscript 1 as in ${}_1\mathcal{H}_\Pi^+$ to denote a Hilbert space or operator from the 2-dimensional theory described in the last section.

All these operators are self-adjoint since the vector fields $\partial/\partial y^i$ are complete on $\boldsymbol{R}^3$. Also $\Pi_{1\Pi}$ is negative definite since it is the closure of ${}_1\Pi_{1\Pi} \otimes I \otimes I$ on ${}_1\mathcal{H}^+_\Pi \otimes L^2(\boldsymbol{R}) \otimes L^2(\boldsymbol{R})$.

We can quantise the front form momenta in $\mathcal{H}_{\overline{\Pi}}$ to obtain the self-adjoint operators

$$\Pi_{i\overline{\Pi}} = -\overline{y}^i.$$

Clearly $\Pi_{1\overline{\Pi}}$ is negative definite because it can be identified with the closure of ${}_1\Pi_{1\overline{\Pi}} \otimes I \otimes I$ and since $\sigma(A \otimes B) = \sigma(A)\sigma(B)$ we have $\sigma(\Pi_{1\overline{\Pi}}) = \sigma({}_1\Pi_{1\overline{\Pi}}).1$ and ${}_1\Pi_{1\overline{\Pi}}$ is negative definite[4].

The fact that y^i and π^i can be quantised in both $\mathcal{H}^+_\Pi$ and H_Π does not lead to any inconsistency because

$$Y^i_\Pi = U_{\overline{\Pi}\Pi} Y^i_{\overline{\Pi}} U_{\Pi\overline{\Pi}} \quad \text{and} \quad \Pi_{i\Pi} = U_{\overline{\Pi}\Pi} \Pi_{i\overline{\Pi}} U_{\Pi\overline{\Pi}}.$$

We now turn our attention to the quantisation of cartesian position in the front form. We have

$$q^s = \overline{\pi}_s + \frac{2(-\overline{y}^1)(-\overline{y}^s)\overline{\pi_1}}{\underline{\overline{y}}^2 + (mc)^2}. \tag{2.37}$$

These can be quantised in $\mathcal{H}_{\overline{\Pi}}$ to obtain self-adjoint operators because

$$\frac{\partial}{\partial \overline{y}^s} + \frac{2\overline{y}^1\overline{y}^s}{(\underline{\overline{y}}^2 + (mc)^2)} \frac{\partial}{\partial \overline{y}^1}$$

is complete on $\boldsymbol{R}^+ \times \boldsymbol{R} \times \boldsymbol{R}$ (Appendix 4). From (2.37) we have that

$$Q^s_{\overline{\Pi}} = -i\hbar \left(\frac{\partial}{\partial \overline{y}^s} + \frac{2\overline{y}^1\overline{y}^s}{\underline{\overline{y}}^2 + (mc)^2} \frac{\partial}{\partial \overline{y}^1} + \frac{\overline{y}^s}{(\underline{\overline{y}}^2 + (mc)^2)^2} (\overline{y}^{2^2} + \overline{y}^{3^2} - \overline{y}^{1^2} + (mc)^2) \right).$$

The remaining component of the cartesian position is given by

$$q^1 = \frac{2\pi_1^2 y^1}{\underline{\pi}^2 + (mc)^2}$$

or

$$q^1 = \frac{2\overline{y}^{1^2}}{\underline{\overline{y}}^2 + (mc)^2} \overline{\pi}_1. \tag{2.38}$$

We can show that q^1 can be quantized in $\mathcal{H}_{\overline{\Pi}}$ to give a self-adjoint operator because

$$\frac{2\overline{y}^{1^2}}{\underline{\overline{y}}^2 + (mc)^2} \frac{\partial}{\partial \overline{y}^1}$$

is complete (Appendix 5). We have

$$Q^1_{\overline{\Pi}} = -i\hbar \left(\frac{2\overline{y}^{1^2}}{\underline{\overline{y}}^2 + (mc)^2} \frac{\partial}{\partial \overline{y}^1} + \frac{1}{2} \frac{\partial}{\partial \overline{y}^1} \left(\frac{2\overline{y}^{1^2}}{\underline{\overline{y}}^2 + (mc)^2} \right) \right)$$

[4] σ denotes the spectrum of an operator

$$= -i\hbar\left(\frac{2\overline{y}^{1^2}}{\underline{\overline{y}}^2+(mc)^2}\frac{\partial}{\partial\overline{y}^1}+\frac{(\underline{\overline{y}}^2+(mc)^2)2\overline{y}^1-\overline{y}^1(2\overline{y}^1)^2}{(\underline{\overline{y}}^2+(mc)^2)^2}\right)$$

$$= -i\hbar\left(\frac{2\overline{y}^{1^2}}{(\underline{\overline{y}}^2+(mc)^2)}\frac{\partial}{\partial\overline{y}^1}+\frac{2\overline{y}^1(\overline{y}^{2^2}+\overline{y}^{3^2}+(mc)^2)}{(\underline{\overline{y}}^2+(mc)^2)^2}\right).$$

We cannot quantise the cartesian position operators directly in $\mathcal{H}_\Pi^+$ but using the pairing map between $\mathcal{H}_\Pi^+$ and $\mathcal{H}_{\overline{\Pi}}$ we obtain

$$Q_\Pi^s = Y_\Pi^s + 2\frac{(-\Pi_{1\Pi})(-\Pi_{s\Pi})}{(\underline{\Pi}_\Pi^2+(mc)^2)}Y_\Pi^1 + (-i\hbar)\frac{(-\Pi_{s\Pi})}{(\underline{\Pi}_\Pi^2+(mc)^2)^2}((-\Pi_{2\Pi})^2+(-\Pi_{3\Pi})^2-(-\Pi_{1\Pi})^2+(mc)^2)$$

$$= Y_\Pi^s + 2\frac{\Pi_{1\Pi}\Pi_{s\Pi}}{(\underline{\Pi}_\Pi^2+(mc)^2)}Y_\Pi^1 + i\hbar\frac{\Pi_{s\Pi}}{(\underline{\Pi}_\Pi^2+(mc)^2)^2}(\Pi_{2\Pi}^2+\Pi_{3\Pi}^2-\Pi_{1\Pi}^2+(mc)^2) \qquad (2.39)$$

and

$$Q_\Pi^1 = \frac{2(-\Pi_{1\Pi})^2}{(\underline{\Pi}_\Pi^2+(mc)^2)}Y_\Pi^1 + (-i\hbar)\frac{2(-\Pi_{1\Pi})((-\Pi_{2\Pi})^2+(-\Pi_{3\Pi})^2+(mc)^2)}{(\underline{\Pi}_\Pi^2+(mc)^2)^2}$$

$$= \frac{2\Pi_{1\Pi}^2}{(\underline{\Pi}_\Pi^2+(mc)^2)}Y_\Pi^1 + (i\hbar)2\Pi_{1\Pi}\frac{(\Pi_{2\Pi}^2+\Pi_{3\Pi}^2+(mc)^2)}{(\underline{\Pi}_\Pi^2+(mc)^2)^2}. \qquad (2.40)$$

We can quantise the cartesian momenta p_i in $\mathcal{H}_{\overline{\Pi}}$. We obtain we following self-adjoint operators

$$P_{1\overline{\Pi}} = -\frac{\overline{y}^{1^2}-\overline{y}^{2^2}-\overline{y}^{3^2}-(mc)^2}{2\overline{y}^1} \quad \text{and} \quad P_{s\overline{\Pi}} = -\overline{y}^s$$

Since $p_i \notin C^\infty({}_{t_0}M_F, \Pi, 1)$ we can take

$$P_{i\Pi} = U_{\overline{\Pi}\Pi}P_{i\overline{\Pi}}U_{\Pi\overline{\Pi}}$$

so

$$P_{1\Pi} = \frac{\Pi_{1\Pi}^2-\Pi_{2\Pi}^2-\Pi_{3\Pi}^2-(mc)^2}{2\Pi_{1\Pi}} \quad \text{and} \quad P_{s\Pi} = \Pi_{s\Pi}.$$

Also $H \in C^\infty({}_{t_0}M_F, \overline{\Pi}, 1)$ and can be quantised as the self-adjoint multiplication operator

$$H_{\overline{\Pi}} = c\left(\frac{\underline{\overline{y}}^2+(mc)^2}{2\overline{y}^1}\right).$$

Notice that H cannot be quantised directly in $\mathcal{H}_\Pi$ so we are free to use the pairing map to obtain

$$H_\Pi = -c\left(\frac{\underline{\Pi}_\Pi^2+(mc)^2}{2\Pi_{1\Pi}}\right).$$

Connection between the Instant and Front Form Pictures

The pairing construction gives the following unitary map between $\mathcal{H}_\Pi^+$ and $\mathcal{H}_{\overline{P}}$.

$$(U_{\Pi\overline{P}}\phi_\Pi)(\underline{\overline{q}}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\phi_\Pi(\underline{y})\exp[i((\overline{q}^1+\overline{q}^0)y^1+\overline{q}^2y^2+\overline{q}^3y^3)]\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\frac{1}{\sqrt{|\,y'\,|}}d^3\underline{y}$$

and

$$(U_{\overline{P}\Pi}\phi_{\overline{P}})(\underline{y}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\phi_{\overline{P}}(\underline{\overline{q}})\exp[-i((\overline{q}^1+\overline{q}^0)y^1+\overline{q}^2y^2+\overline{q}^3y^3)]\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\sqrt{|\,y^1\,|}d^3\underline{\overline{q}}.$$

We can show (Appendix 6) that this map completely intertwines quantisations with respect to the front and instant forms i.e.

$$Y^i_{\overline{P}} = U_{\Pi\overline{P}}Y^i_{\Pi}U_{\overline{P}\Pi},\;\; \Pi_{i\overline{P}} = U_{\Pi\overline{P}}\Pi_{i\Pi}U_{\overline{P}\Pi},\;\; Q^i_{\overline{P}} = U_{\Pi\overline{P}}Q^i_{\Pi}U_{\overline{P}\Pi}$$

$$P_{i\overline{P}} = U_{\Pi\overline{P}}P_{i\Pi}U_{\overline{P}\Pi} \text{ and } H_{\overline{P}} = U_{\Pi\overline{P}}H_{\Pi}U_{\overline{P}\Pi}.$$

2.3 Kinematic and Dynamic Subgroups of the Point, Instant and Front Forms

With each form is associated a subgroup of the Lorentz group which leaves the moment invariant. This subgroup is called the kinematic subgroup. Generators of transformations that do not leave the moment invariant are called the Hamiltonians of the form. We shall say that these generate the dynamic subgroup of the form although only in the point form does the cannonical choice of Hamiltonians close to form a subalgebra. It is important to identify the generators of the kinematic and dynamic subgroups of a particular form because many of them will have a physical interpretation. Some of the generators of the kinematic subgroup will correspond to momenta for example. We shall investigate the dynamic and kinematic subgroup structures of the point, instant and front forms.

2.3.1 Kinematic and Dynamic Subgroups of the Point Form

For the case of the point form it is very easy to isolate the dynamic and kinematic subalgebras. From formula 0.17 page 6 [47] we know that a general infinitesimal[5] Lorentz transformation can be written

$$q^\mu \to (g^{\mu\nu} + \epsilon^{\mu\nu})q_\nu = q^\mu + \epsilon^{\mu\nu}q_\nu$$

where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$. Suppose we choose a transformation where the only non zero ϵ's are ϵ^{01} and of course ϵ^{10} then

$$\begin{aligned} q^0 &\to q^0 + \epsilon^{01}q_1 \\ &= q^0 - \epsilon^{01}q^1 \end{aligned} \tag{2.41}$$

[5]See Appendix 7 for finite forms

and

$$q^1 \to q^1 + \epsilon^{1\nu} q_\nu$$
$$= q^1 - \epsilon^{01} q^0. \tag{2.42}$$

We can also express this transformation in terms of light cone coordinates. Since $\tau = t + \mid \underline{q} \mid$ if we put $\alpha = \epsilon^{10}$ we have

$$\tau \to t + \alpha q^1 + ((q^1 + \alpha t)^2 + q^{2^2} + q^{3^2})^{\frac{1}{2}}$$
$$= t + \alpha q + (\mid \underline{q} \mid^2 + 2\alpha q t)^{\frac{1}{2}} = t + \alpha q + \mid \underline{q} \mid \left(1 + \alpha \frac{qt}{\mid \underline{q} \mid^2}\right).$$
$$= t + \alpha q + \mid \underline{q} \mid + \alpha \frac{qt}{\mid \underline{q} \mid} = \tau + \alpha q \left(1 + \frac{t}{\mid \underline{q} \mid}\right).$$

For points on the light cone with apex at the origin we have $t = - \mid \underline{q} \mid$ so that

$$\tau \to \tau \ \ (= 0).$$

We also have that $y^1 = q^1$ so

$$y^1 \to q^1 + \alpha t = y^1 + \alpha t.$$

For a particle on the light cone through the origin this becomes

$$y^1 \to y^1 - \alpha \mid \underline{q} \mid = y^1 - \alpha \mid \underline{y} \mid .$$

Finally

$$y^2 \to y^2 \quad \text{and} \quad y^3 \to y^3$$

[62]. The fact that $\tau \to \tau$ shows that this type of transformation leaves the light cone with apex at the origin invariant. We can also show this just using Minkowski coordinates. Supose (t, q) lies on the light cone i.e. $t = - \mid \underline{q} \mid$. Put $\alpha = -\epsilon^{01}$ then from (2.41) and (2.42)

$$t' = q^0 + \alpha q^1$$

and

$$\mid \underline{q}' \mid = ((q^1 + \alpha q^0)^2 + q^{2^2} + q^{3^2})^{\frac{1}{2}}$$
$$= (\mid \underline{q} \mid^2 + 2\alpha q^0 q^1)^{\frac{1}{2}} = \mid \underline{q} \mid \left(1 + \frac{\alpha 2 q^0 q^1}{\mid \underline{q} \mid^2}\right)^{\frac{1}{2}} = \mid \underline{q} \mid + \alpha \frac{q^0 q^1}{\mid \underline{q} \mid}$$
$$= -(q^0 + \alpha q^1)$$

since $q^0 = - \mid \underline{q} \mid$. Therefore $t' = - \mid \underline{q}' \mid$ and the light cone is preserved. This effectively demonstrates the invariance of the light cone under boosts. We shall denote the generators of the special Lorentz transformations $\overline{J}^{0\nu}$. Therefore we have shown that the $\overline{J}^{0\nu}$ belong to the kinematic subalgebra of the point form.

Suppose now we consider a transformation where the only non zero ϵ's are ϵ^{13} and ϵ^{31}. In this case

$$q^1 \to q^1 + \epsilon^{1\nu} q_\nu = q^1 - \epsilon^{13} q^3$$

and

$$q^3 \to q^3 + \epsilon^{3\nu} q_\nu = q^3 + \epsilon^{13} q^1.$$

If we interpret the transformation passively we might put

$$t' = t, \;\; q^{2'} = q^2, \;\; q^{3'} = q^3 + \alpha q^1, \;\; q^{1'} = q^1 - \alpha q^3 \tag{2.43}$$

where $\alpha = \epsilon^{13}$. It is also possible to express this Lorentz transformation in terms of light cone coordinates. Since

$$\tau = t + \mid \underline{q} \mid \quad \text{and} \quad \underline{y} = \underline{q}$$

we have

$$\tau' = t + [(q^1 - \alpha q^3)^2 + q^{2^2} + (q^3 + \alpha q^1)^2]^{\frac{1}{2}}$$

$$= t + (q^{1^2} - 2\alpha q^1 q^3 + \alpha^2 q^{3^2} + q^{3^2} + 2\alpha q^3 q^1 + \alpha^2 q^{1^2})^{\frac{1}{2}}$$

$$= t + \mid \underline{q} \mid = \tau.$$

Similarly

$$y^{1'} = q^{1'} = q^1 - \alpha q^3 = y^1 - \alpha y^3$$

$$y^{2'} = q^{2'} = q^2 = y^2$$

$$y^{3'} = q^{3'} = q^3 + \alpha q^1 = y^3 + \alpha y^1.$$

Taking the active point of view we would express this as

$$\tau \to \tau, \;\; y^1 \to y^1 - \alpha y^3, \;\; y^2 \to y^2, \;\; y^3 \to y^3 + \alpha y^1$$

[62]. Notice that we have $\tau \to \tau$ showing that this transformation preserves the light cone. We can also see this just working with the Minkowski coordinate description (2.43). Suppose t lies on the light cone with apex at τ then $t = \tau - \mid \underline{q} \mid$. Now

$$\mid \underline{q}' \mid = (q^{2^2} + (q^3 + \alpha q^1)^2 + (q^1 - \alpha q^3)^2)^{\frac{1}{2}}$$

$$= (q^{2^2} + q^{3^2} + 2\alpha q^3 q^1 + \alpha^2 q^{1^2} + q^{1^2} - 2\alpha q^1 q^3 + \alpha^2 q^{3^2})^{\frac{1}{2}} = \mid \underline{q} \mid .$$

Also

$$t' = t$$

so that

$$t' = \tau - \mid \underline{q}' \mid .$$

Therefore (t, q) is moved to a new point on the light cone. This demontrates that the light cone is invariant under spatial rotations. The generators of spatial rotations, denoted $\overline{J}^{ij}$, also belong to the kinematic subalgebra of the point form.

It is easy to show that none of the translations $\overline{P}^{\mu}$ leave the light cone invariant. For example under p_1

$$t \rightarrow t, \;\; q^1 \rightarrow q^1 + \alpha, \;\; q^2 \rightarrow q^2 \;\; q^3 \rightarrow q^3.$$

Clearly $t' = t = - \mid \underline{q} \mid$ whereas

$$\mid \underline{q}' \mid = ((q^1 + \alpha)^2 + q^{2^2} + q^{3^2})^{\frac{1}{2}} = (\mid \underline{q} \mid^2 + 2q^1\alpha)^{\frac{1}{2}} = \mid \underline{q} \mid - \frac{q^1 \alpha}{\mid \underline{q} \mid}.$$

The last term in the above spoils the invariance of the light cone since with it $t' \neq - \mid \underline{q}' \mid$.

Summarising we see that the kinematic and dynamic subgroups of the point form are the homogeneous subgroup and the group of translations respectively.

In a similar way we can show that the generators of kinematic subgroup of the instant form are $\overline{P}^i$ and $\overline{J}^{ij}$ and the Hamiltonians are $\overline{P}^0$ and $\overline{J}^{i0}$.

The Front Form Operator Representation of the Poincare Algebra in Basis Adapted to the Point and Instant Forms

The Poincare Algebra is given by

$$[\overline{J}^{\mu\nu}, \overline{J}^{\rho\sigma}] = \eta^{\nu\rho}\overline{J}^{\mu\sigma} + \eta^{\mu\rho}\overline{J}^{\sigma\nu} + \eta^{\nu\sigma}\overline{J}^{\rho\mu} + \eta^{\mu\sigma}\overline{J}^{\nu\rho}$$

$$[\overline{J}^{\mu\nu}, \overline{P}^{\rho}] = \eta^{\nu\rho}\overline{P}^{\mu} - \eta^{\mu\rho}\overline{P}^{\nu} \tag{2.44}$$

$$[\overline{P}^{\rho}, \overline{P}^{\sigma}] = 0$$

[50] page 150. Suppose we put

$$\overline{P}^{\mu} = p^{\mu}, \;\; \overline{J}^{0i} = p^0 q^i, \;\; \overline{J}^{ij} = p^i q^j - q^i p^j \tag{2.45}$$

where $p^0 = (\underline{p}^2 + m^2)^{\frac{1}{2}}$. These classical observables obey the commutation relations (2.44) under Poisson bracket { , } which is defined in the usual way from the symplectic 2-form as

$$\{f, g\} = \frac{\partial f}{\partial q^s}\frac{\partial g}{\partial p_s} - \frac{\partial g}{\partial q^s}\frac{\partial f}{\partial p_s}.$$

For example

$$[\overline{J}^{0i}, \overline{P}^i] = \eta^{ii}\overline{P}^0 - \eta^{0i}\overline{P}^i = -\overline{P}^0$$

and

$$\{\overline{J}^{0i}, \overline{P}^i\} = \{p^0 q^i, p^i\} = p^0 \frac{\partial q^i}{\partial q^s}\frac{\partial p^i}{\partial p_s} = -p^0 \delta_{is}\delta_{is} = -p^0 = -\overline{P}^0$$

as required. Also

$$[\overline{J}^{0i}, \overline{P}^0] = \eta^{i0}\overline{P}^0 - \eta^{00}\overline{P}^i = -\overline{P}^i$$

whereas

$$\begin{aligned}\{\overline{J}^{0i}, \overline{P}^0\} &= \{p^0 q^i, p^i\} = p^0 \frac{\partial q^i}{\partial q^s}\frac{\partial p^0}{\partial p_s} \\ &= p_0 \delta_{is}\frac{p_s}{p_0} = p_i = -p^i = -\overline{P}^i.\end{aligned}$$

As a final example consider

$$[\overline{J}^{ij}, \overline{P}^i] = \eta^{ji}\overline{P}^i - \eta^{ii}\overline{P}^j = \overline{P}^j$$

where of course we have assumed $i \neq j$. We have

$$\begin{aligned}\{\overline{J}^{ij}, \overline{P}^i\} &= \{-p_i q^j + q^i p_j, -p_i\} \\ &= -p_j \delta_{is}\delta_{is} + p_i \delta_{js}\delta_{is}.\end{aligned}$$

Since $i \neq j$ this becomes

$$-p_j = \overline{P}^j$$

as required.

As is well known these classical observables can be quantised geometrically in $\mathcal{H}_{\overline{P}}$ to obtain an operator representation of the Poincare algebra. We can show that when expressed in terms of front form variables the classical generators are in $C^\infty({}_{t_0}M_F, \overline{\Pi}.1)$ and in fact can be quantised to give self-adjoint operators in $\mathcal{H}_{\overline{\Pi}}$. What is more the pairing maps effect a unitary transformation between the representations.

In terms of front form variables we have

$$\overline{J}^{32} = \overline{y}^3\overline{\pi}_2 - \overline{y}_2\overline{\pi}_3$$

$$\overline{J}^{1s} = \frac{(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2)}{2\overline{y}^1}\overline{\pi}_s - \overline{y}^s\overline{\pi}_1$$

$$\overline{J}^{s0} = -\overline{y}^s\overline{\pi}_1 - \left(\frac{\underline{\overline{y}}^2 + (mc)^2}{2\overline{y}^1}\right)\overline{\pi}_s$$

$$\overline{J}^{10} = -\overline{y}^1\overline{\pi}_1$$

$$\overline{P}^0 = -\left(\frac{\underline{\pi}^2 + (mc)^2}{2\pi_1}\right), \quad \overline{P}^1 = -\left(\frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2}{2\pi_1}\right), \quad \overline{P}^s = -\pi_s.$$

In Appendix 8 we show that these classical observables generate complete vector fields and can be quantised to give the following self adjoint operators

$$\overline{J}^{32}_{\overline{\Pi}} = -i\hbar\left(\overline{y}^3\frac{\partial}{\partial\overline{y}^2} - \overline{y}^2\frac{\partial}{\partial\overline{y}^3}\right)$$

$$\overline{J}^{1s}_{\overline{\Pi}} = -i\hbar\left(\frac{(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2)}{2\overline{y}^1}\frac{\partial}{\partial\overline{y}^s} - \frac{\overline{y}^s}{2\overline{y}^1} - \overline{y}^s\frac{\partial}{\partial\overline{y}^1}\right)$$

$$\overline{J}^{s0}_{\overline{\Pi}} = i\hbar\left(\overline{y}^s\frac{\partial}{\partial\overline{y}^1} + \left(\frac{\underline{\overline{y}}^2 + (mc)^2}{2\overline{y}^1}\right)\frac{\partial}{\partial\overline{y}^s} + \frac{\overline{y}^s}{2\overline{y}^1}\right)$$

$$\overline{J}^{10}_{\overline{\Pi}} = i\hbar\left(\overline{y}^1\frac{\partial}{\partial\overline{y}^1} + \frac{1}{2}\right)$$

$$\overline{P}^0 = \left(\frac{\underline{\overline{y}}^2 + (mc)^2}{2\overline{y}^1}\right) \quad \overline{P}^1 = \left(\frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1}\right), \quad \overline{P}^s = \overline{y}^s.$$

We can show that these operators are unitarily related, via the pairing maps, to the usual representation of the Poincare algebra in $\mathcal{H}_{\overline{P}}$. For example

$$\overline{J}^{30} = -p^0q^3 = -\overline{q}^0\overline{p}_3.$$

Clearly $\overline{J}^{30} \in C^\infty({}_{t_0}M_I, \overline{P}, 1)$ and in fact generates a complete vector field so that it can be quantised in $\mathcal{H}_{\overline{P}}$ to obtain the self-adjoint operator

$$\overline{J}^{30}_{\overline{P}} = i\hbar\left(\overline{q}^0\frac{\partial}{\partial\overline{q}^3} + \frac{\overline{q}^3}{\overline{q}^0}\right).$$

[6] Now we shall show that

$$U_{\Pi\overline{P}}U_{\overline{\Pi}\Pi}\overline{J}^{30}_{\overline{\Pi}}U_{\Pi\overline{\Pi}}U_{\overline{P}\Pi} = \overline{J}^{30}_{\overline{P}}.$$

Well

$$U_{\overline{\Pi}\Pi}\overline{J}^{30}_{\overline{\Pi}}U_{\Pi\overline{\Pi}} = (-\Pi^3_{\Pi})(-1)Y^1_{\Pi} + \frac{\underline{\Pi}^2_{\Pi} + (mc)^2}{2(-\Pi^1_{\Pi})}(-1)Y^3_{\Pi} + (i\hbar)\frac{(-\Pi^3_{\Pi})}{2(-\Pi^1_{\Pi})}$$

[6]This can also be written

$$\overline{J}^{30}_{\overline{P}} = -i\hbar\frac{1}{\sqrt{p_0}}p_0\frac{\partial}{\partial p_3}\sqrt{p_0}.$$

This makes it particularly easy to see that our representation of the Poincare generators is the same as appears in [94] remembering that we use a different metric on the cartesian momentum space.

$$= \Pi^3_\Pi Y^1_\Pi + \frac{(\Pi^2_\Pi + (mc)^2)}{2\Pi^1_\Pi} Y^3_\Pi + i\hbar \frac{\Pi^3_\Pi}{2\Pi^1_\Pi}$$

so

$$U_{\Pi\overline{P}} U_{\overline{\Pi}\Pi} \overline{J}^{30}_{\overline{\Pi}} U_{\Pi\overline{\Pi}} U_{\overline{P}\Pi} = (-\overline{q}^3)(-i\hbar)\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\frac{\partial}{\partial \overline{q}^1} + \frac{\overline{q}^1-\overline{q}^0}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}\right)$$

$$-\overline{q}^0(-i\hbar)\left(\frac{\partial}{\partial \overline{q}^3} - \frac{\overline{q}^3}{\overline{q}^1+\overline{q}^0}\frac{\partial}{\partial \overline{q}^1} + \frac{\overline{q}^3}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}\right) + \frac{i\hbar}{2}\frac{\overline{q}^3}{\overline{q}^1+\overline{q}^0}$$

$$= i\hbar \frac{\overline{q}^3\overline{q}^0}{\overline{q}^0+\overline{q}^1}\frac{\partial}{\partial \overline{q}^1} + i\hbar \frac{\overline{q}^3(\overline{q}^1-\overline{q}^0)}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)} + i\hbar \overline{q}^0 \frac{\partial}{\partial \overline{q}^3} -$$

$$i\hbar \frac{\overline{q}^0\overline{q}^3}{(\overline{q}^1+\overline{q}^0)}\frac{\partial}{\partial \overline{q}^1} + \frac{\overline{q}^3\overline{q}^0}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)} + i\hbar \frac{\overline{q}^3}{2(\overline{q}^1+\overline{q}^0)}$$

$$= i\hbar\left(\overline{q}^0\frac{\partial}{\partial \overline{q}^3} + \frac{\overline{q}^3\overline{q}^1}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)} + \frac{\overline{q}^3}{(\overline{q}^1+\overline{q}^0)}\right)$$

$$= i\hbar\left(\overline{q}^0\frac{\partial}{\partial \overline{q}^3} + \frac{\overline{q}^3}{2\overline{q}^0}\right) = -i\hbar\left(p_0\frac{\partial}{\partial p_3} + \frac{p_3}{2p_0}\right)$$

as required. As a further example we shall show that

$$U_{\Pi\overline{P}} U_{\overline{\Pi}\Pi} \overline{J}^{32}_{\overline{\Pi}} U_{\Pi\overline{\Pi}} U_{\overline{P}\Pi} = \overline{J}^{32}_{\overline{P}}.$$

We have

$$U_{\overline{\Pi}\Pi} \overline{J}^{32}_{\overline{\Pi}} U_{\Pi\overline{\Pi}} = (-\Pi^3_\Pi) Y^2_\Pi - (-\Pi^2_\Pi) Y^3_\Pi$$

so

$$U_{\Pi\overline{P}} U_{\overline{\Pi}\Pi} \overline{J}^{32}_{\overline{\Pi}} U_{\Pi\overline{\Pi}} U_{\overline{P}\Pi} = -\Pi^3_{\overline{P}} Y^2_{\overline{P}} + \Pi^2_{\overline{P}} Y^3_{\overline{P}}$$

$$= \overline{q}^3(-i\hbar)\left(\frac{\partial}{\partial \overline{q}^2} - \left(\frac{\overline{q}^2}{\overline{q}^1+\overline{q}^0}\right)\frac{\partial}{\partial \overline{q}^1} + \frac{\overline{q}^2}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}\right) -$$

$$\overline{q}^2(-i\hbar)\left(\frac{\partial}{\partial \overline{q}^3} - \left(\frac{\overline{q}^3}{\overline{q}^1+\overline{q}^0}\right)\frac{\partial}{\partial \overline{q}^1} + \frac{\overline{q}^3}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}\right)$$

$$= -i\hbar \overline{q}^3 \frac{\partial}{\partial \overline{q}^2} + i\hbar \frac{\overline{q}^3\overline{q}^2}{\overline{q}^1+\overline{q}^0}\frac{\partial}{\partial \overline{q}^1} - i\hbar \frac{\overline{q}^3\overline{q}^2}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}$$

$$+ i\hbar \overline{q}^2 \frac{\partial}{\partial \overline{q}^3} - i\hbar \frac{\overline{q}^2\overline{q}^3}{}\frac{\partial}{\partial \overline{q}^1} + i\hbar \frac{\overline{q}^3\overline{q}^2}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}$$

$$= i\hbar\left(\overline{q}^2\frac{\partial}{\partial \overline{q}^3} - \overline{q}^3\frac{\partial}{\partial \overline{q}^2}\right) = i\hbar\left(p_2\frac{\partial}{\partial p_3} - p_3\frac{\partial}{\partial p_2}\right)$$

whereas according to [94] or using geometric quantisation

$$\overline{J}^{32}_{\overline{P}} = i\hbar \frac{1}{\sqrt{p_0}}\left(p_2\frac{\partial}{\partial p_3} - p_3\frac{\partial}{\partial p_2}\right)\sqrt{p_0} = i\hbar\left(p_2\frac{\partial}{\partial p_3} - p_3\frac{\partial}{\partial p_2}\right)$$

as required. In this way we demonstrate that the classical Poincare generators can be quantised in the instant or front form pictures and lead to operator representations of the Poincare algebra that are unitarily related by the pairing maps.

2.3.2 Kinematic and Dynamic Subgroups of the Front Form

We shall now follow Dirac [75] in calculating the classical generators of the kinematic and dynamic subgroups of the front form. Basically, since we work on the light front $\tau = 0$ we must find generators that do not contain the variable conjugate to τ, i.e. π_0. We shall denote the elements of the new basis $J^{\mu\nu}$ and P^{ν}. We cannot simply take over the expressions given by Dirac since he uses different front form coordinates.

Nievely we might put

$$P^1 = \pi^1$$

but this is not acceptable because

$$\pi^1 = g^{1\mu}\pi_\mu = -\pi_0 - \pi_1.$$

Instead we put

$$P^1 = -\pi_0 - \pi_1 + \lambda^1(-2\pi_0\pi_1 - \underline{\pi}^2 - (mc)^2)$$

and choose λ^1 so as to make P^1 independent of π_0. If we put $\lambda = -1/2\pi_1$ we find that

$$\begin{aligned} P^1 &= -\pi_1 + \frac{1}{2\pi_1}(\underline{\pi}^2 + (mc)^2) \\ &= -\frac{(\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2)}{2\pi_1}. \end{aligned} \tag{2.46}$$

However we can put

$$P^2 = \pi^2 = -\pi_2$$

$$P^3 = \pi^3 = -\pi_3$$

and

$$P^0 = \pi^0 = g^{0\mu}\pi_\mu = -\pi_1.$$

Similarly we can make the assignments

$$J^{0i} = \pi^0 y^i = g^{0\mu}\pi_\mu y^i = -\pi_1 y^i$$

and

$$J^{23} = \pi^2 y^3 - \pi^3 y^2 = \pi_3 y^2 - \pi_2 y^3.$$

However we cannot put

$$J^{31} = \pi^3 y^1 - y^3\pi^1.$$

Again we follow Dirac and seek λ^{31} such that if we put

$$J^{31} = -y^1\pi_3 - y^3(-\pi_0 - \pi_1) + \lambda^{31}(-2\pi_0\pi_1 - \underline{\pi}^2 - (mc)^2)$$

then J^{31} is independent of π_0. Choose $\lambda^{31} = y^3/2\pi_1$ then

$$J^{31} = -y^1\pi_3 + \frac{y^3}{2\pi_1}(\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2).$$

Similarly we cannot put

$$J^{21} = \pi^2 y^1 - y^2\pi^1.$$

We must choose λ^{21} such that

$$J^{21} = -\pi_2 y^1 - y^2(-\pi_0 - \pi_1) + \lambda^{21}(-2\pi_0\pi_1 - \underline{\pi}^2 - (mc)^2)$$

is independent of π_0. Choose $\lambda^{21} = y^2/2\pi_1$ then

$$J^{21} = -y^1\pi_2 + \frac{y^2}{2\pi_1}(\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2).$$

According to Dirac the generators of the kinematic subgroup will consist of those $J^{\mu\nu}$ and P^ν that are simple functions of the front form canonical variables, the others will generate the kinematic subgroup. We are therefore led to posit

$$\{P^2, P^3, P^0, J^{0i}, J^{23}\} \tag{2.47}$$

as the generators of the kinematic subgroup and

$$\{P^1, J^{31}, J^{21}\} \tag{2.48}$$

as the generators of the dynamic subgroup. In fact it is quite easy to show explicitly that the generators (2.47) leave the front invariant and so are indeed the generators of the kinematic subgroup. We shall show that $P^0 = -\pi_1 = p^0 + p^1$ preserves [7] the light front $q^0 + q^1 = 0$. Using 8.67 page 261 [63] we have

$$q^0 \to q^0 + \alpha[q^0, p^0 + p^1] = q^0 + \alpha$$

and

$$q^1 = q^1 + \alpha[q^1, p^0 + p^1] = q^1 - \alpha.$$

[7]It is interesting to note that P^0 generates quantum mechanical translations. We have $P^0 = \overline{q}^1 + \overline{q}^0$. Now

$$\exp(i\lambda(\overline{q}^1 + \overline{q}^0))\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}$$

is the generalised eigenvector of the light front position operator with eigenvalue λ (Appendix 11) clearly multiplying this by $\exp(ia(\overline{q}^0 + \overline{q}^1))$ gives the eigenvector with eigenvalue $\lambda + a$. This much is required of any sensible position and translation operators. See equations (37) and (40) pages 61 and 62 [92].

The effect of the generator on the other coordinates is irrelevant. It is obvious from these expressions that

$$q^{1'} + q^{0'} = 0$$

so that P^0 generates transformations that leave the front invariant. Finally we show that J^{03} also preserves the light front. Since $J^{03} = -\pi_1 y^3 = (p^0 + p^1)q^3$ we have

$$q^0 \to q^0 + \alpha[q^0, (p^0 + p^1)q^3]$$

$$= q^0 + \alpha[q^0, p^0 q^3 + p^1 q^3]$$

$$= q^0 + \alpha[q^0, p^0 q^3] - \alpha[q^0, p^1 q^3]$$

$$= q^0 + \alpha[q^0, p^0]q^3 = q^0 + \alpha q^3$$

and

$$q^1 \to q^1 + \alpha[q^1, p^0 q^3 + p^1 q^3]$$

$$= q^1 + \alpha[q^1, p^1 q^3]$$

$$= q^1 + \alpha[q^1, p^1]q^3 = q^1 - \alpha q^3.$$

Clearly

$$q^{0'} + q^{1'} = q^0 + q^1 = 0$$

as required.

Notice that the generators of the kinematic subgroup generate Lorentz transformations of points on the light front. Consider the transformation generated by J^{03}. The condition for it to be a Lorentz transformation is that the interval

$$(q^0 - \overline{q}^0)^2 - (q^1 - \overline{q}^1)^2$$

be preserved. Under J^{03} this goes to

$$(q^0 + \alpha q^3 - (\overline{q}^0 + \alpha\overline{q}^3))^2 - (q^1 - \alpha q^3 - (\overline{q}^1 - \alpha\overline{q}^3))^2$$

$$= (q^0 + \alpha q^3)^2 - 2(q^0 + \alpha q^3)(\overline{q}^0 + \alpha\overline{q}^3) + (\overline{q}^0 + \alpha\overline{q}^3)^2$$

$$-(q^1 - \alpha q^3)^2 + 2(q^1 - \alpha q^3)(\overline{q}^1 - \alpha\overline{q}^3) - (\overline{q}^1 - \alpha\overline{q}^3)^2$$

$$= q^{0^2} + 2\alpha q^0 q^3 - 2(q^0\overline{q}^0 + \alpha q^0\overline{q}^3 + \alpha q^3\overline{q}^0) + \overline{q}^{0^2} + 2\alpha\overline{q}^0\overline{q}^3$$

$$-q^{1^2} + 2\alpha q^1 q^3 - 2(q^1\overline{q}^1 - \alpha q^1\overline{q}^3 + \alpha q^3\overline{q}^1) - (\overline{q}^2 - 2\alpha\overline{q}^1\overline{q}^3)$$

where we have abandoned terms of order α^2. The coefficient of α is

$$q^0 q^3 - q^0 \overline{q}^3 - q^3 \overline{q}^0 + \overline{q}^0 \overline{q}^3 + q^1 q^3 - q^1 \overline{q}^3 - q^3 \overline{q}^1 + \overline{q}^1 \overline{q}^3.$$

This is identically zero due to $q^0 + q^1 = 0$ and $\overline{q}^0 + \overline{q}^1 = 0$ so we are left with

$$q^{0^2} - 2q^0\overline{q}^0 + \overline{q}^{0^2} - q^{1^2} + 2q^1\overline{q}^1 - \overline{q}^{1^2} = (q^0 - \overline{q}^0)^2 - (q^1 - \overline{q}^1)^2$$

as required.

In the same way it is easy to show that the generators (2.48) do not leave the front invariant and so are the generators of the dynamic subgroup. In fact it easy to see that we have the correct Hamiltonians since these are well known to be the generators of the rotations around the two axis tangent to the front and a generator that moves the plane along its normal [64]. Now the generator $P^1 = p^1$ moves each point on the front an equal distance parallel to the q^1 axis and so shifts the front along its normal. Also we have

$$\begin{aligned} J^{31} &= -y^1\pi_3 + y^3 \frac{\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2}{2\pi_1} \\ &= -y^1 p_3 - y^3 p^1 = q^1 p^3 - q^3 p^1 = \overline{J}^{31} \end{aligned}$$

and

$$J^{21} = \overline{J}^{21}$$

and these are the afore mentioned rotations.

Clearly P^1, J^{21}, J^{31} generate Lorentz transformations.

It is important to notice that while the point and instant forms have four Hamiltonians the front form has just three. Dirac believed that this gave the front form an advantage when it comes to developing relativisticly invariant interacting field theories. Here the generators are expressed in terms of the field variables. Only the Hamiltonians contain extra, complicated terms arrising from the coupling terms in the Lagrangian. If there are fewer complicated generators then it should be easier to verify Lorentz invariance[8].

We wish to express the Poincare algebra with respect the generators $J^{\mu\nu}$ and P^ν since they are more natural to the front form. In this new basis the Poincare algebra becomes

$$\begin{aligned} [J^{\mu\nu}, J^{\rho\sigma}] &= g^{\nu\rho} J^{\mu\sigma} + g^{\mu\rho} J^{\sigma\nu} + g^{\nu\sigma} J^{\rho\mu} + g^{\mu\sigma} J^{\nu\rho} \\ [J^{\mu\nu}, P^\rho] &= g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu \\ [P^\rho, P^\sigma] &= 0. \end{aligned} \tag{2.49}$$

[8]See also Conclusion and Prospect.

We can show that the classical front form generators satisfy these commutation relations (see also Appendix 9). For example

$$[J^{01}, P^1] = -g^{01}P^1 + g^{11}P^0$$

$$= P^1 - P^0$$

and we have

$$\{J^{01}, P^1\} = \{-y^1\pi_1, -\frac{(\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2)}{2\pi_1}\}$$

$$= \frac{\partial(-y^1\pi_1)}{\partial y^1}\frac{\partial}{\partial \pi_1}\left(\frac{-(\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2)}{2\pi_1}\right)$$

$$= \pi_1\left(\frac{2\pi_1(2\pi_1) - (\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2)2}{4\pi_1^2}\right)$$

$$= \frac{\underline{\pi}^2 + (mc)^2}{2\pi_1} = P^1 - P^0$$

as required. Next consider

$$[J^{31}, J^{23}] = -g^{32}J^{13} + g^{12}J^{33} - g^{33}J^{21} + g^{13}J^{23}$$

$$= -J^{33} + J^{21} = J^{21}$$

whereas

$$\{J^{31}, J^{23}\} = \{-y^1\pi_3 + \frac{(\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2)}{2\pi_1}y^3, y^2\pi_3 - \pi_2 y^3\}$$

$$= \frac{(\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2)}{2\pi_1}y^2 - (-\frac{2\pi_2}{2\pi_1})y^3\pi_3 - (-y^1 - \frac{2\pi_3 y^3}{2\pi_1})(-\pi_2)$$

$$= \frac{(\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2)}{2\pi_1}y^2 + \frac{\pi_3\pi_2 y^3}{\pi_1} - y^1\pi_2 - \frac{\pi_3\pi_2 y^3}{\pi_1}$$

$$= J^{21}$$

as required. Also

$$[J^{02}, J^{13}] = -g^{01}J^{23} + g^{21}J^{03} - g^{03}J^{12} + g^{23}J^{10}$$

$$= J^{23}$$

and

$$\{J^{02}, J^{13}\} = \{-y^2\pi_1, -\frac{y^3}{2\pi_1}(\pi_1^2 - \pi_2^2 - \pi_3^2 - (mc)^2) + y^1\pi_3\}$$

$$= (-\pi_1)\left(\frac{y^3}{2\pi_1}2\pi_2\right) - (-y^2)(\pi_3)$$

$$= y^2\pi_3 - y^3\pi_2$$

$$= J^{23}.$$

We have

$$[J^{01}, P^0] = -g^{00}P^1 + g^{10}P^0$$

$$= -P^0$$

whereas

$$\{J^{01}, P^0\} = \{-y^1\pi_1, -\pi_1\}$$

$$= \pi_1$$

$$= -P^0.$$

Finally

$$[J^{01}, J^{02}] = -g^{00}J^{12} + g^{10}J^{02} - g^{02}J^{01} + g^{12}J^{00}$$

$$= g^{10}J^{02}$$

$$= -J^{02}$$

and

$$\{J^{01}, J^{02}\} = \{-y^1\pi_1, -y^2\pi_1\}$$

$$= \pi_1 y^2$$

$$= -J^{02}$$

as required.

Quantisation of Basis Adapted to Front Form

We can obtain an operator representation of the front form generators by quantising the classical expressions geometrically in the front form momentum space. Notice that although $J^{\mu\nu}$ and P^μ are linear combinations of $\overline{J}^{\mu\nu}$ and $\overline{P}^\mu$ we cannot simply use the quantisation of the latter derived in (2.3.1) to obtain operators representing $J^{\mu\nu}$ and P^μ because the linearity axiom does not hold in geometric quantisation i.e. if f and g are in $C^\infty(M, P, 1)$ and generate complete vector fields it does not follow that $f + g$ generates a complete vector field.

In terms of front form variables we have

$$J^{01} = \overline{y}^1\overline{\pi}_1$$

$$J^{0s} = \overline{y}^1\overline{\pi}_s$$

$$J^{1s} = -\overline{\pi}_1\overline{y}^s + \frac{\overline{\pi}_s}{2\overline{y}^1}(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2)$$

$$J^{23} = -\overline{y}^3\overline{\pi}_2 + \overline{y}^2\overline{\pi}^3$$

$$P^0 = \overline{y}^1, \quad P^s = \overline{y}^s, \quad P^1 = \frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1}.$$

In Appendix 10 we show that these classical observables generate complete vector fields and can be quantised to give the following self-adjoint operators

$$J^{01}_{\overline{\Pi}} = -i\hbar\left(\overline{y}^1\frac{\partial}{\partial\overline{y}^1} + \frac{1}{2}\right)$$

$$J^{0s}_{\overline{\Pi}} = -i\hbar\overline{y}^1\frac{\partial}{\partial\overline{y}^s}$$

$$J^{1s}_{\overline{\Pi}} = -i\hbar\left(-\overline{y}^s\frac{\partial}{\partial\overline{y}^1} + \left(\frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1}\right)\frac{\partial}{\partial\overline{y}^s} - \frac{\overline{y}^s}{2\overline{y}^1}\right)$$

$$J^{23}_{\overline{\Pi}} = -i\hbar\left(\overline{y}^2\frac{\partial}{\partial\overline{y}^3} - \overline{y}^3\frac{\partial}{\partial\overline{y}^2}\right)$$

$$P^0_{\overline{\Pi}} = \overline{y}^1$$

$$P^s_{\overline{\Pi}} = \overline{y}^s$$

$$P^1_{\overline{\Pi}} = \frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1}.$$

Because the classical observables $J^{\mu\nu}$, P^μ have been quantised geometriclly we are assured that the operators $J^{\mu\nu}_{\overline{\Pi}}$ and $P^{\mu}_{\overline{\Pi}}$ satisfy the commutation relations (2.49) with Poisson bracket replaced by operator commutator. We shall give just one example. We shall show explicitly that

$$[J^{01}_{\overline{\Pi}}, P^1_{\overline{\Pi}}] = i\hbar[P^1_{\overline{\Pi}} - P^0_{\overline{\Pi}}].$$

The right hand side is

$$\frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1} - \overline{y}^1 = -\left(\frac{\underline{y} + (mc)^2}{2\overline{y}^1}\right).$$

The left hand side is

$$[-i\hbar(\overline{y}^1\frac{\partial}{\partial\overline{y}^1} + \frac{1}{2}), \frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1}]. \tag{2.50}$$

Put

$$z = \frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1}$$

then

$$\frac{\partial z}{\partial\overline{y}^1} = \frac{\underline{y}^2 + (mc)^2}{2\overline{y}^{1^2}}.$$

Using this (2.50) becomes (if we imagine it acts on a function f)

$$-i\hbar\left(\overline{y}^1\frac{\partial}{\partial\overline{y}^1}+\frac{1}{2}\right)(zf)-z(-i\hbar)\left(\overline{y}^1\frac{\partial}{\partial\overline{y}^1}+\frac{1}{2}\right)f$$

$$=(-i\hbar)\left(\overline{y}^1z\frac{\partial}{\partial\overline{y}^1}+\overline{y}^1\frac{\partial z}{\partial\overline{y}^1}+\frac{z}{2}-z\overline{y}^1\frac{\partial}{\partial\overline{y}^1}-\frac{z}{2}\right)$$

$$=-i\hbar\left(\frac{\overline{y}^2+(mc)^2}{2\overline{y}^1}\right)$$

as required.

2.4 More on the Point Form

For the purposes of this critique it is sufficient to restrict ourselves to a discussion of the point form in 1+1 spacetime. Recall that the following expressions define the light cone coordinates (τ, y^1)

$$\tau = x^0 + \mid x \mid, \quad y^1 = x^1. \tag{2.51}$$

The moments are the manifolds corresponding to constant values of τ and are clearly backward lightcones with apex on the time axis. If we use the light cone time τ to parametrize the path of the particle then the variational principle

$$\delta\left(-mc\int\left(\left(\frac{dx^0}{d\tau}\right)^2-\left(\frac{dx^1}{d\tau}\right)^2\right)^{\frac{1}{2}}d\tau\right)=0 \tag{2.52}$$

is equivalent to that usually employed in describing the relativistic motion of a free particle. Since

$$x^0 = \tau - \mid y^1 \mid \quad \text{and} \quad x^1 = y^1$$

we have

$$\frac{dx^0}{d\tau}=1-\frac{d\mid y^1\mid}{d\tau}=1-\frac{dy^1}{d\tau}\frac{d\mid y^1\mid}{dy^1}=1-sgn(y^1)\frac{dy^1}{d\tau}$$

and

$$\frac{dx^1}{d\tau}=\frac{dy^1}{d\tau}$$

so the variational principle (2.52) becomes

$$\delta\left(-mc\int\left(1-2sgn(y^1)\frac{dy^1}{d\tau}\right)^{\frac{1}{2}}d\tau\right)=0$$

or

$$\delta\left(-mc^2\int\left(1-sgn(y^1)\frac{2}{c}\frac{dy^1}{d\omega}\right)^{\frac{1}{2}}d\omega\right)=0$$

so

$$L = -mc^2 \left(1 - sgn(y^1)\frac{2}{c}\frac{dy^1}{d\omega}\right)^{\frac{1}{2}}.$$

Now let τ' denote the proper time of the particle then

$$\frac{dy^1}{d\omega} = c\frac{dy^1}{d\tau} = c\frac{\frac{dx^1}{d\tau'}}{\frac{dx^0}{d\tau'} + \frac{d|x^1|}{d\tau'}} = \frac{cp^1}{p^0 + sgn(y^1)p^1}.$$

We have

$$\begin{aligned}
\pi_1 &= -\frac{\partial L}{\partial \frac{dy^1}{d\omega}} \\
&= mc^2\frac{1}{2}\left(1 - sgn(y^1)\frac{2}{c}\frac{dy^1}{d\omega}\right)^{-\frac{1}{2}} (-sgn(y^1))\frac{2}{c} \\
&= -sgn(y^1)mc\left(1 - sgn(y^1)\frac{2}{c}\frac{dy^1}{d\omega}\right)^{-\frac{1}{2}} \\
&= -sgn(y^1)mc\left(1 - sgn(y^1)\frac{2}{c}\frac{cp^1}{(p^0 + sgn(y^1)p^1)}\right)^{-\frac{1}{2}} \\
&= p_1 - sgn(q^1)p_0. \qquad (2.53)
\end{aligned}$$

Since

$$\pi_1 - p_1 = -sgn(y^1)p_0 \qquad (2.54)$$

we can square both sides and use the mass shell condition to obtain

$$p_1 = \frac{\pi_1^2 - (mc)^2}{2\pi_1}. \qquad (2.55)$$

Also from (2.54) we have

$$\begin{aligned}
p_0 &= -sgn(y)(\pi_1 - p_1) \\
&= -sgn(y^1)\frac{\pi_1^2 + (mc)^2}{2\pi_1}.
\end{aligned}$$

The instant and point form positions q^1 and y^1 of the particle at $t = \tau = 0$ are related by the equation

$$\begin{aligned}
q^1 &= y^1 + \frac{p^1}{p^0}sgn(y^1)y^1 \qquad (2.56) \\
&= y^1\left(1 + \frac{p^1}{p^0}sgn(y^1)\right).
\end{aligned}$$

Now

$$\frac{p^1}{p^0} = -\frac{p_1}{p_0} = sgn(y^1)\left(\frac{\pi_1^2 - (mc)^2}{\pi_1^2 + (mc)^2}\right)$$

so

$$q^1 = y^1\left(\frac{2\pi_1^2}{\pi_1^2 + (mc)^2}\right). \qquad (2.57)$$

We can rewrite (2.56) as

$$q^1 = y^1 - \frac{p_1}{p_0} sgn(q^1) y^1 = y^1 \left(1 - \frac{p_1}{p_0} sgn(q^1)\right)$$

and therefore

$$y^1 = \frac{p_0 q^1}{p_0 - p_1 sgn(q^1)}. \tag{2.58}$$

It is easy to show that (2.53) and (2.58) and their inverses (2.55) and (2.57) constitute a canonical transformation between the point and instant forms.

It will be noticed that many of the dynamical variables given above are discontinuous across the apex of the backward lightcone. This is just one indication that the conventional treatment presented here is rather formal and ignores many subtleties that when addressed force us to modify the scheme dramatically. Notice that if we insist on employing light cone coordinates as a global description of spacetime then extra care will have to be taken in overcoming analytic problems that may result from the fact that they are not even C^1 related to the cartesian coordinates. This is immediately apparent from (2.51) since the transition functions are not continuously differentiable at $x^1 = 0$. To avoid this we define two new coordinate charts as follows. The first has as its patch the open submanifold of spacetime corresponding to $x^1 > 0$; call it M^+. The coordinates (τ^+, y^{1^+}) are related to the cartesian coordinates via the transition functions

$$\tau^+ = x^0 + x^1 \quad \text{and} \quad y^{1^+} = x^1.$$

The second chart has as its patch that region of spacetime, which we shall denote M^-, corresponding to $x^1 < 0$. The coordinates (τ^-, y^-) are related to the cartesian coordinates via

$$\tau^- = x^0 - x^1, \quad \text{and} \quad y^- = x^1.$$

These charts are C^∞ related to the cartesian coordinates however they do not constitute an atlas for spacetime since the world line of the observer, the region of spacetime corresponding to $x = 0$, is not covered by either chart. The new charts are merely the restriction of the light cone coordinates (2.51) to M^+ and M^- so that point form quantum mechanics is really based on spacetime modulo the observers world line. In 1+1 dimensions this is clearly $M^+ \times M^-$. Even physically we can see that this is the correct arena for the discussion of the dynamics of a free particle since if the particle were to pass from M^+ to M^- its world line would inevitably intersect that of the observer and some interaction would result. A purely kinematic description of the dynamics would be inappropriate.

The presence of the observer splits spacetime into two regions that are classically and quantum mechanically disjoint. The quantum description of the entire system will be the direct sum of the representations in M^+ and M^-. Here it will be sufficient to disscuss the difficulty of quantising M^+ where the phase space of the instant form is described by the coordinates $p_1^+ \in \mathbb{R}$ and $q^{1^+} \in \mathbb{R}^+$ and the phase space of the point form is desribed by the coordinates $\pi_1^+ \in \mathbb{R}^+$ and $y^{1^+} \in \mathbb{R}^+$. From now on, to ease notation, we shall omit the superscript + which shall be understood. The point and instant form pictures of M^+ are related by the canonical transformation

$$q^1 = y^1 \left(\frac{2\pi_1^2}{\pi_1^2 + (mc)^2} \right), \quad p_1 = \frac{\pi_1^2 - (mc)^2}{2\pi_1}$$

or

$$y^1 = \frac{p_0 q^1}{p_0 - p_1}, \quad \pi_1 = p_1 - p_0.$$

2.4.1 Quantisation in the Instant Form

Consider the polarisation $P = d/dp_1$ then $\mathcal{H}_P = L^2(\mathbb{R}^+, dq^1)$. We have

$$P_{1P}^+ = -i\hbar \frac{d}{dq^1}$$

which is maximally symmetric and

$$Q_P^{1^+} = q^1$$

which is self-adjoint and positive definite as required by the classical constraint $q^1 > 0$. Introduce new coordinates $(\overline{p}_1, \overline{q}^1)$ where

$$p_1 = -\overline{q}^1 \quad \text{and} \quad q^1 = \overline{p}_1.$$

Since $p_1 \in \mathbb{R}$ and $q^1 \in \mathbb{R}^+$ we have $\overline{q}^1 \in \mathbb{R}$ and $\overline{p}_1 \in \mathbb{R}^+$. If we quantise in the horizontal polarisation i.e. with respect to $\overline{P} = d/d\overline{p}_1$ so $\mathcal{H}_{\overline{P}} = L^2(\mathbb{R}, d\overline{q}^1)$ we obtain

$$P_{1\overline{P}}^+ = -\overline{q}^1$$

which is self-adjoint and

$$Q_{\overline{\Pi}}^{1^+} = -i\hbar \frac{d}{d\overline{q}^1}$$

which is also self-adjoint because $d/d\overline{q}^1$ is complete on $\mathbb{R}$. There are two problems here. Notice that whereas $P_{1\overline{P}}^+$ is self-adjoint P_{1P}^+ is maximally symmetric. This threatens to make the instant form inconsistent. Also $Q_{\overline{P}}^{1^+}$ is not positive definite. However we can solve both these problems simultaneously by identifying the correct physical Hilbert space.

We must restrict $\mathcal{H}_{\overline{P}}$ to $\mathcal{H}^{+}_{\overline{P}} = E(\mathbf{R}^{+}, Q^{1^{+}}_{\overline{P}})\mathcal{H}_{\overline{P}}$ which will comprise those functions in $\mathcal{H}_{\overline{P}}$ that are Hardy class. On $\mathcal{H}^{+}_{\overline{P}}$ the operator $Q^{1^{+}}_{\overline{P}}$ will be self-adjoint and positive whilst $P^{+}_{1\overline{P}}$ will be maximally symmetric. This is most easilly seen by noticing that $\mathcal{H}_{P}$ and $\mathcal{H}^{+}_{\overline{P}}$ are unitarily related by the pairing construction which leads to the map

$$(U_{P\overline{P}}\phi_P)(\overline{q}^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_0^{\infty}\phi_P(q^1)\exp(i\overline{q}^1q^1)$$

with inverse

$$(U_{\overline{P}P}\phi_{\overline{P}})(q^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\phi_{\overline{P}}(\overline{q}^1)\exp(-i\overline{q}^1q^1)d\overline{q}^1.$$

It is easy to show that

$$Q^{1^{+}}_{P} = U_{\overline{P}P}Q^{1^{+}}_{\overline{P}}U_{P\overline{P}} \quad \text{and} \quad P^{+}_{1P} = U_{\overline{P}P}P^{+}_{1\overline{P}}U_{P\overline{P}}.$$

Quantisation of point form observables in the instant form picture leads to complications. Since $y^1 \in C^{\infty}({}_{t_0}M_P, \overline{P}, 1)$ it can be quantised in $\mathcal{H}_{\overline{P}}$ to give the symmetric operator

$$Y^{1}_{\overline{P}} = -i\hbar\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\frac{d}{d\overline{q}^1} + \frac{\overline{q}^1-\overline{q}^0}{2\overline{q}^0+\overline{q}^1}\right).$$

In fact this operator is maximally symmetric. To see this notice that the solutions to the equations

$$(Y^{1}_{\overline{P}} \pm i)\phi_{\overline{P}} = 0$$

are given by

$$\phi_{\overline{P}}(\overline{q}^1) = \left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\exp[\mp\frac{1}{\hbar}(\overline{q}^1+\overline{q}^0)].$$

(Appendix 11). Consider

$$\int_{-\infty}^{\infty}\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)\exp(-2(\overline{q}^1+\overline{q}^0))d\overline{q}^1.$$

Put $u^1 = \overline{q}^0 + \overline{q}^1$ to obtain

$$\int_0^{\infty}\exp(-2u^1)du^1 = \frac{1}{2} < \infty.$$

Also

$$\int_{-\infty}^{\infty}\left(\frac{\overline{q}+\overline{q}^1}{\overline{q}^0}\right)\exp(2(\overline{q}^1+\overline{q}^0))d\overline{q}^1 = \int_0^{\infty}\exp(2u^1)du^1 > \infty$$

Therefore $Y^{1}_{\overline{P}}$ in $\mathcal{H}_{\overline{P}}$ has defect indicies (1,0). Of course $Y^{1}_{\overline{P}}$ ought to be positive definite. we shall ignore this added complication. Similarly we can quantise π_1 in $\mathcal{H}_{\overline{P}}$ to obtain the negative definite self-adjoint operator

$$\Pi_{1\Pi} = -(\overline{q}^0 + \overline{q}^1).$$

Superficially at least it may appear that we have sucessfully quantised the point form observables in the instant form but this is not really the case since for this we require the restrictions of Y^1_Π and $\Pi_{1\Pi}$ to $\mathcal{H}^+_{\bar{P}}$ and it is difficult to see if these are well defined. If we cannot quantise y^1 and π_1 in $\mathcal{H}^+_{\bar{P}}$ then we cannot use the pairing map to find their representations in $\mathcal{H}_P$. The instant form cannot accomodate point form observables.

2.4.2 Quantisation in the Point Form

If $\Pi = d/dy^1$ then

$$\mathcal{H}_\Pi = L^2\left(\mathbf{R}^+, \frac{dy^1}{y^1}\right).$$

We can quantise y^1 in $\mathcal{H}_\Pi$ to give the self-adjoint operator

$$Y^1_\Pi = y^1$$

which is positive definite as required. However recall that $Y^1_{\bar{P}}$ was maximally symmetric. Is it possible that quantising π_1 and enforcing the classical constraint $\pi_1 < 0$ will compell us to modfify $\mathcal{H}_\Pi$ in such a way that Y^1_Π will become maximal symmetric? Unfortunately we are stuck again because the vector field d/dy^1 is not complete on $\mathbf{R}^+$ so we cannot quantise π_1 in $\mathcal{H}_\Pi$ to obtain a self-adjoint operator. The best we can do is find a self-adjoint operator that represents a classical observable that approximates π_1. Let $\eta \in C^\infty(\mathbf{R}^+)$ such that

$$\eta(0) = 0,\ \ \eta(y^1) \in (0,1)\ \text{ for }\ y^1 > 0\ \text{ and }\ y^1 \in \Lambda - \Lambda_0,\ \ \eta(y^1) = 1\ \text{ for }\ y^1 \in \Lambda_0 = [a, \infty)$$

where a is some real constant > 0. Suppose we put

$$\sigma(y) = \int_{y_0}^{y} \frac{dy^1}{\eta(y^1)} + y_0$$

then σ is a bijective map $\mathbf{R}^+ \to \mathbf{R}$, i.e. σ has an inverse σ^{-1}. Consider the classical observable $\eta(y^1)\pi_1$. This is identical to π_1 when $y^1 \in \Lambda_0$. This classical observable can be made to approximate π_1 to any accuracy by choosing a sufficiently close to 0. Now the vector field $\eta(y^1)d/dy^1$ is complete on $\mathbf{R}^+$. This is obvious because the integral curves are given by $y^1 = \sigma^{-1}()$ [9] and σ^{-1} is a bijective map $\mathbf{R} \to \mathbf{R}^+$. We may therefore quantise

[9] We have $\sigma(y^1) = t$. Differentiating with respect to t we obtain

$$\frac{1}{\eta(y^1)}\frac{dy^1}{dt} = 1$$

as required.

$\eta(y^1)\pi_1$ geometrically to obtain the self-adjoint operator

$$\Pi_1 = -i\hbar\left(\eta(y^1)\frac{d}{dy^1} - \frac{\eta(y^1)}{2y^1} + \frac{\eta'(y^1)}{2}\right).$$

This operator is not negative definite. To see this notice that the generalised eigenfunctions of Π_1 are given by

$$G(\lambda, y, \Pi_1) = y^{1^{1/2}}[2\pi\hbar\eta(y^1)]^{-1/2}e^{i\lambda\sigma(y^1)}$$

where $\lambda \in \mathbb{R}$. These satisfy the usual orthogonality and completeness relations (Appendices 12 and 13). We could obtain a negative definite point form momentum operator by restricting $\mathcal{H}_\Pi$ to $E(\mathbb{R}^-, \Pi_1)\mathcal{H}_\Pi$ but it is difficult to give anything more than a very implicit definition of this space. There are other ways of constructing negative definite operators from Π^1 but they are all rather contrived. For example we could take

$$\Pi_{1\Pi} = -\mid\Pi_1\mid. \tag{2.59}$$

The only way of obtaining a more concrete realisation of this operator[10] is via a generalised eigenfunction expansion. Π_1 is self-adjoint and has the associated resolution of the identity

$$(E(\lambda, \Pi_1)\phi_\Pi)(y^1) = \int_{-\infty}^{\lambda} G(\alpha, y^1, \Pi_1) < G(\alpha, y, \Pi_1) \mid \phi_\Pi(y) >_{\mathcal{H}_\Pi} d\alpha$$

[70]. Therefore

$$(-\mid\Pi_1\mid\phi_\Pi)(y^1) = \int_{-\infty}^{\infty} -\mid\lambda\mid d_\lambda(E(\lambda, \Pi_1)\phi_\Pi)(y^1)$$

$$= \int_{-\infty}^{\infty} -\mid\lambda\mid G(\lambda, y^1, \Pi_1) < G(\lambda, y, \Pi_1) \mid \phi_\Pi(y) >_{\mathcal{H}_\Pi} d\lambda.$$

Having obtained a negative definite point form momentum operator we turn our attention to quantising the Hamiltonian. We have

$$H_\Pi = \frac{c}{2}\left(\mid\Pi_1\mid + \frac{(mc)^2}{\mid\Pi_1\mid}\right).$$

This operator is self-adjoint and positive definite[11]. We can now ask wether this Hamiltonian leads to sensible dynamics. An initial state ϕ^0_Π will evolve as

$$\phi^\tau_\Pi(y^1) = \int_0^\infty G(\lambda, y, \Pi_1)e^{-ic\frac{\tau}{2|\lambda|}(\lambda^2+(mc)^2)} < G(\lambda, y, \Pi_1) \mid \phi^0_\Pi(y) >_{\mathcal{H}_\Pi} d\lambda$$

[10] Notice that $\mid\Pi_1\mid$ has a degenerate spectrum. Its generalised eigenvectors are discussed in Appendices 14, 15 and 16

[11] Had we simply ignored the fact that the point form momentum operator should be negative definite we would have ended up with an indefinite Hamiltonian. This would have been difficult to interpret.

or in terms of the generalized eigenvectors $G(\mu,\nu,w,H_\Pi,y^1)$ of H_Π (see Appendices 17,18 and 19)

$$\phi_\Pi^\tau(y^1) = \int_{mc^2}^\infty \sum_{\mu\nu} G(\mu,\nu,w,H_\Pi,y^1)\exp(-i\tau w) < \phi_\Pi^0 \mid G(\mu,\nu,w,H_\Pi,y) > dw \quad (2.60)$$

where

$$G(\mu,\nu,w,H_\Pi,y) = \frac{y^{1/2}[2\pi \text{h}\eta(y)]^{-1/2}\exp\left(\frac{i}{\hbar}(\frac{\mu \text{W}}{c} + \nu\sqrt{(\frac{w}{c})^2-(mc)^2})\sigma(y)\right)}{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(\frac{\mu \text{W}}{c}-\nu\sqrt{(\frac{w}{c})^2-(mc)^2})^2}\right)\right|^{1/2}}.$$

Since Y_Π^1 is simply the self-adjoint multiplication operator there is a notion of localisation on the light cone. We have

$$(E(\Lambda, Y_\Pi^1)\phi)_\Pi(y^1) = \chi_\Lambda(y^1)\phi_\Pi(y^1)$$

so a normalised state ϕ_Π is localised in Λ if $\| E(\Lambda, Y_\Pi^1)\phi_\Pi \| = 1$ i.e. if the function has support in Λ. It is now possible to show that a Hegerfeldt type result holds in this model.

Theorem 22 *A state vector initially localized in Λ_0 undergoes instantaneous spreading in the sense of Hegerfeldt when H_Π generates time evolution.*

Proof. Consider a $C_0^\infty(\boldsymbol{R}^+)$ wave function ϕ_Π^0 such that supp $\phi_\Pi^0 \subset \Lambda_0$ then

$$\hat{\phi}^0(\lambda) = \int_0^\infty [2\pi\hbar\eta(y)]^{-\frac{1}{2}} y^{-1/2}\exp(-i\lambda\sigma(y))\phi_\Pi^0(y)dy.$$

Put $x = \sigma(y)$ so

$$\hat{\phi}^0(\lambda) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^\infty \frac{(\sigma^{-1}(x))^{-1/2}\phi_\Pi^0(\sigma^{-1}(x))\exp(-i\lambda x)}{\eta^{3/2}(\sigma^{-1}(x))}dx.$$

Since supp $\phi_\Pi^0 \subset \Lambda_0$ integrand becomes ϕ_Π^0 when $x \in \Lambda_0$ since then $\sigma^{-1}(x) = x$. Also the integrand is 0 when $x \in \boldsymbol{R} - \Lambda_0$. The integrand is clearly $C_0^\infty(\boldsymbol{R})$ so $\hat{\phi}^0(\lambda)$ is integrable [71] and

$$\exp\left(\frac{ic\tau}{2\mid\lambda\mid}(\lambda^2 + (mc)^2\right)\hat{\phi}_0(\lambda) \quad (2.61)$$

is integrable because absolute integrability $\leftrightarrow$ integrability. Therefore the inverse Fourier transform of (2.61) is given by

$$F(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^\infty \exp\left(\frac{ic\tau}{2\mid\lambda\mid}(\lambda^2 + (mc)^2)\right)\hat{\phi}_0(\lambda)\exp(i\lambda x)d\lambda.$$

Notice that (2.61) is not entire since it is not continuous at 0. Therefore by the Paley Wiener theorem we have that $F(x)$ is not of compact support. Now

$$\phi_\Pi^\tau(y) = y^{1/2}[2\pi \text{h}\eta(y)]^{-1/2}F(\sigma(y))$$

so clearly $\phi^{\tau}_{\Pi}(y)$ is not of compact support in $\boldsymbol{R}^+$.

Notice that the instantaneous spreading of an initially localised wavefunction along a null hypersurface does not imply a violation of causality[12].

Another problem with the point form is that some of the state spaces are not unitarily related by the pairing construction. For example, although formally the pairing maps between $\mathcal{H}_\Pi$ and $\mathcal{H}_{\overline{P}}$ are defined by

$$(U_{\Pi\overline{P}}\phi_\Pi)(\overline{q}^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_0^\infty \phi_\Pi(y^1)\exp(i(\overline{q}^1+\overline{q}^0)y^1)\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\frac{1}{\sqrt{y^1}}dy^1$$

and

$$(U_{\overline{P}\Pi}\phi_{\overline{P}})(y^1) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^\infty \phi_{\overline{\Pi}}(\overline{q}^1)\exp(-i(\overline{q}^1+\overline{q}^0)y^1)\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\sqrt{y^1}d\overline{q}^1 \qquad (2.62)$$

it is not difficult to show that these are meaningless. Put $u^1 = \overline{q}^0 + \overline{q}^1$ so

$$du^1 = \left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)d\overline{q}^1.$$

Using this we can rewrite (2.62) as

$$\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_0^\infty \phi_{\overline{P}}(\overline{q}^1)\exp(-iu^1y^1)\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\sqrt{y^1}du^1. \qquad (2.63)$$

It is easy to show that

$$\overline{q}^1 = \frac{{u^1}^2-(mc)^2}{2u^1}$$

and

$$\overline{q}^0 = ({\overline{q}^1}^2+(mc)^2)^{\frac{1}{2}} = \left(\frac{{u^1}^2+(mc)^2}{2u^1}\right)^{\frac{1}{2}}$$

so that (2.63) becomes

$$\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_0^\infty \phi_{\overline{P}}\left(\frac{{u^1}^2-(mc)^2}{2u^1}\right)\exp(-iu^1y^1)\left(\frac{{u^1}^2+(mc)^2}{2{u^1}^2}\right)^{\frac{1}{2}}\sqrt{y^1}du^1.$$

However

$$\phi_{\overline{P}}\left(\frac{{u^1}^2-(mc)^2}{2u^1}\right)\left(\frac{{u^1}^2+(mc)^2}{2{u^1}^2}\right)^{\frac{1}{2}}$$

is square integrable since

$$\int_0^\infty \left|\phi_{\overline{P}}\left(\frac{{u^1}^2-(mc)^2}{2u^1}\right)\right|^2\left(\frac{{u^1}^2+(mc)^2}{2{u^1}^2}\right)du^1 = \int_0^\infty |\phi(\overline{q}^1)|^2 d\overline{q}^1.$$

[12] We can prove other results about the behaviour at temporal infinity of a wave function that is initially localised on the light cone. We can show, for example, that a state vector initially localized in Λ_0 is a scattering state (Appendix 20).

Therefore the pairing map requires that there exist functions $\in L^2(\mathbb{R}^+)$ whose Fourier transforms are also in $L^2(\mathbb{R}^+)$. Unfortunately there are no such functions. The Paley Wiener condition [95] states that a necessary condition for a square integrable function to have a Fourier transform in $L^2(\mathbb{R}^+)$ is that the magnitude of the function $A(x)$ say should satisfy

$$\int_{-\infty}^{\infty} \frac{| \ln A(x) |}{1+x^2} dx < \infty.$$

Clearly for a function in $L^2(\mathbb{R}^+)$ we have $A(x) = 0 \; \forall x \in \mathbb{R}^-$ so the integrand in the above is infinite on a set that is far from measure zero so the integral does not converge.

We have seen that it is by no means easy to develop a rigorous point form quantum mechanics. The many formal schemes that have been proposed cannot be regarded as a solution to the problem.

Geometric quantisation yields a front form quantum mechanics that is entirely consistent with the familiar instant form. The pairing construction reveals that intra and interform quantisations are unitarily equivalent. Modified or physical Hilbert spaces are a necessary feature of the theory and result in our being able to circumvent Hegerfeldt's famous no go theorem. Constraints on the ranges of certain classical observables are respected at the quantum level. Contrarywise the point form is catalogue of disasters. Coupled with the success of the front form in strong interaction particle physics and string theory (where it appears as the light cone guage) it is easy to see why Dirac, even as late as 1978, was stressing the utility of the front form [83].

2.5 Appendices

Appendix 1

Notice that the integral defining $U_{\Pi\overline{P}}$ exists and is an element of $\mathcal{H}_{\overline{P}}$ for every ϕ_Π in $\mathcal{H}_\Pi^+$ since it is really the Fourier transform of the square integrable function $\phi_\Pi/\sqrt{|y^1|}$ and a simple change of variable shows that if $\psi(\overline{q}^1)$ is square integrable then so is

$$\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\psi(\overline{q}^1+\overline{q}^0).$$

We shall first show that $U_{\Pi\overline{P}}$ is an isometry. We have

$$\int_{-\infty}^{\infty}(U_{\Pi\overline{P}}\phi_\Pi)(\overline{q}^1)(U_{\Pi\overline{P}}\phi_\Pi)^*(\overline{q}^1)d\overline{q}^1 =$$

$$\int_{-\infty}^{\infty}\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\phi_\Pi(y^1)\exp(i(\overline{q}^1+\overline{q}^0)y^1)(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0})^{\frac{1}{2}}\frac{1}{\sqrt{|y^1|}}dy^1$$
$$\times\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\phi_\Pi^*(\hat{y}^1)\exp(-i(\overline{q}^1+\overline{q}^0)\hat{y}^1)(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0})^{\frac{1}{2}}\frac{1}{\sqrt{|\hat{y}^1|}}d\hat{y}^1d\overline{q}^1.$$

Put

$$u^1=\overline{q}^1+\overline{q}^0 \tag{2.64}$$

so

$$du=\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)d\overline{q}^1 \tag{2.65}$$

and the above becomes

$$\frac{1}{2\pi\hbar}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_\Pi(y^1)\phi_\Pi^*(\hat{y}^1)\exp(iu^1(y^1-\hat{y}))\frac{1}{\sqrt{|y^1|}}\frac{1}{\sqrt{|\hat{y}^1|}}dy^1d\hat{y}^1du^1. \tag{2.66}$$

Since $\phi_\Pi/\sqrt{|y^1|}$ is a function of Hardy class

$$\int_{-\infty}^{\infty}\frac{\phi_\Pi(y^1)}{|y^1|^{\frac{1}{2}}}\exp(iu^1y^1)dy^1$$

has support in $(0,\infty)$ so (2.66) becomes

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_\Pi(y^1)\phi_\Pi^*(\hat{y}^1)\exp(iu^1(y^1-\hat{y}))\frac{1}{\sqrt{|y^1|}}\frac{1}{\sqrt{|\hat{y}^1|}}dy^1d\hat{y}^1du^1$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_\Pi(y^1)\phi_\Pi^*(\hat{y}^1)\frac{1}{\sqrt{|y^1|}}\frac{1}{\sqrt{|\hat{y}^1|}}\delta(y^1-\hat{y}^1)dy^1d\hat{y}^1$$
$$=\int_{-\infty}^{\infty}\frac{|\phi_\Pi(y^1)|^2}{|y^1|}dy^1$$

as required. We now show that the range of $U_{\Pi\overline{P}}$ is all of $\mathcal{H}_{\overline{P}}$. First note that the range of any isometry U with a closed domain is closed. Proof. Consider a cauchy sequence of vectors $\in R_U$ i.e. vectors $\phi_m \in R_U$ such that

$$\| \phi_m - \phi_n \| < \epsilon \quad \forall\ m, n > N.$$

Clearly there exist vectors $\varphi_m \in D_U$ such that $U\varphi_n = \phi_n$ so

$$\| U\varphi_m - U\varphi_n \| < \epsilon \quad n, m > N$$

i.e.

$$\| U(\varphi_m - \varphi_n) \| < \epsilon \quad n, m > N.$$

Since U is an isometry this gives

$$\| \varphi_m - \varphi_n \| < \epsilon \quad n, m > N.$$

Now D_U is closed so there exist $\varphi \in D_U$ such that $\varphi_m \to \varphi$, i.e.

$$\| \varphi_m - \varphi \| \to 0.$$

Also U is an isometry so it is clearly bounded, its norm is ≤ 1, i.e. U is continuous. From [69] page 496 we see that this implies

$$\| U\varphi_n - U\varphi \| \to 0$$

i.e.

$$\| \phi_n - \phi \| \to 0$$

where $\phi = U\varphi$. Therefore every cauchy sequence in R_U tends to a limit in R_U so that R_U is closed as required.

Notice that from (2.18) the range of $U_{\Pi\overline{P}}$ certainly contains all elements of $\mathcal{H}_{\overline{P}}$ that can be written as

$$\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \tilde{\phi}(\overline{q}^1 + \overline{q}^0)$$

where $\phi \in \mathcal{H}$. Now take any function $\phi_{\overline{P}}(\overline{q}^1) \in \mathcal{H}_{\overline{P}}$ that is $C_0^\infty(\boldsymbol{R})$ then

$$\eta(\overline{q}^1) \equiv \sqrt{\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}}\phi_{\overline{P}}(\overline{q}^1) \tag{2.67}$$

is also $C_0^\infty(\boldsymbol{R})$. Notice that since $\phi_{\overline{P}}(\overline{q}^1) \in C_0^\infty(\boldsymbol{R})$ we have $\eta(\overline{q}^1) \in L^2(\boldsymbol{R})$ even though

$$\sqrt{\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}}$$

is unbounded. Now there exists a function ψ in $L^2(\boldsymbol{R}^+)$ such that

$$\eta(\overline{q}^1) = \psi(\overline{q}^1 + \overline{q}^0),$$

simply take

$$\psi(u^1) = \eta\left(\frac{{u^1}^2 - (mc)^2}{2u^1}\right)$$

where u^1 is defined in (2.64) then

$$\psi(\overline{q}^1 + \overline{q}^0) = \eta\left(\frac{(\overline{q}^1 + \overline{q}^0)^2 - (mc)^2}{2(\overline{q}^1 + \overline{q}^0)}\right)$$

$$= \eta\left(\frac{{\overline{q}^1}^2 + 2\overline{q}^1\overline{q}^0 + {\overline{q}^0}^2 - (mc)^2}{2(\overline{q}^1 + \overline{q}^0)}\right)$$

$$= \eta\left(\frac{{\overline{q}^1}^2 + 2\overline{q}^1\overline{q}^0 + {\overline{q}^1}^2}{2(\overline{q}^1 + \overline{q}^0)}\right) = \eta\left(\frac{2\overline{q}^1(\overline{q}^1 + \overline{q}^0)}{2(\overline{q}^1 + \overline{q}^0)}\right)$$

$$= \eta(\overline{q}^1)$$

as required. Also since ψ is $C_0^\infty(\boldsymbol{R}^+)$, ψ is the Fourier transform of a function ϕ of Hardy class, i.e. $\psi = \tilde{\phi}(\overline{q}^1 + \overline{q}^0)$. From (2.67) it follows that

$$\phi_{\overline{P}}(\overline{q}^1) = \sqrt{\frac{\overline{q}^1 + \overline{q}^0}{\overline{q}^0}}\tilde{\phi}(\overline{q}^1 + \overline{q}^0).$$

Therefore $U_{\Pi\overline{P}}$ is onto $C_0^\infty(\boldsymbol{R})$ which is dense in $\mathcal{H}_{\overline{P}}$ and so, since the range of $U_{\Pi\overline{P}}$ is closed (its domain is closed since it is all of $\mathcal{H}_\Pi^+$), it is onto $\mathcal{H}_{\overline{P}}$.

We shall sketch a proof that we have the correct expression for the inverse. First we shall show that

$$(U_{\overline{P}\Pi}U_{\Pi\overline{P}}\phi_\Pi)(y^1) = \phi_\Pi(y^1).$$

We have

$$(U_{\overline{P}\Pi}U_{\Pi\overline{P}}\phi_\Pi)(y^1) =$$

$$\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\left(\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\phi_\Pi(\hat{y}^1)\exp(i(\overline{q}^1 + \overline{q}^0)\overline{y}^1)\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)\frac{1}{\sqrt{|\hat{y}^1|}}d\hat{y}^1\right)$$

$$\exp(-i(\overline{q}^1 + \overline{q}^0)y^1)\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\sqrt{|y^1|}d\overline{q}^1.$$

Let $u^1 = \overline{q}^1 + \overline{q}^0$ and the above becomes

$$\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_0^{\infty}\left(\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{\phi_\Pi(\hat{y}^1)}{\sqrt{|\hat{y}^1|}}\exp(iu^1\hat{y}^1)d\hat{y}^1\right)\exp(-iu^1y^1)\sqrt{|y^1|}du^1$$

where we have used (2.64) and (2.65). Since the term in brackets has support in $\mathbb{R}^+$ this becomes

$$\frac{1}{(2\pi\hbar)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_\Pi(\hat{y}^1)\frac{\sqrt{|\,y^1\,|}}{\sqrt{|\,\hat{y}^1\,|}}\exp(iu^1(\hat{y}^1-y^1))d\hat{y}^1du^1.$$

If we assume that we can invert the orders of integration we obtain

$$\int_{-\infty}^{\infty}\phi_\Pi(\hat{y}^1)\frac{\sqrt{|\,y^1\,|}}{\sqrt{|\,\hat{y}^1\,|}}\delta(\hat{y}^1-y^1)d\hat{y}^1$$

$$=\phi_\Pi(y^1)$$

as required. Now we shall show that

$$(U_{\Pi\overline{P}}U_{\overline{P}\Pi}\phi_{\overline{P}})(\overline{q}^1)=\phi_{\overline{P}}(\overline{q}^1).$$

Well

$$(U_{\Pi\overline{P}}U_{\overline{P}\Pi}\phi_{\overline{P}})(\overline{q}^1)=$$

$$\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\left(\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\phi(\hat{\overline{q}}^1)\exp(-i(\hat{\overline{q}}^1+\hat{\overline{q}}^0)y^1)\sqrt{|\,y^1\,|}\left(\frac{\hat{\overline{q}}^1+\hat{\overline{q}}^0}{\hat{\overline{q}}^0}\right)^{\frac{1}{2}}d\hat{\overline{q}}^1\right)$$

$$\exp(i(\overline{q}^1+\overline{q}^0)y^1)\left(\frac{\overline{q}^1+\overline{q}^0}{\overline{q}^0}\right)^{\frac{1}{2}}\frac{1}{\sqrt{|\,y^1\,|}}dy^1$$

$$=\frac{1}{(2\pi\hbar)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\hat{\overline{q}}^1)\exp(iy^1((\overline{q}^1+\overline{q}^0)-(\hat{\overline{q}}^1+\hat{\overline{q}}^0)))\left(\frac{\overline{q}^1+\overline{q}^0}{\hat{\overline{q}}^0}\right)^{\frac{1}{2}}\left(\frac{\hat{\overline{q}}^1+\hat{\overline{q}}^0}{\overline{q}^0}\right)^{\frac{1}{2}}d\hat{\overline{q}}^1dy^1.$$

Assuming that we can invert the order of integration we obtain

$$\int_{-\infty}^{\infty}\phi(\hat{\overline{q}}^1)\left(\frac{\overline{q}^1+\overline{q}^0}{\hat{\overline{q}}^0}\right)^{\frac{1}{2}}\left(\frac{\hat{\overline{q}}^1+\hat{\overline{q}}^0}{\overline{q}^0}\right)^{\frac{1}{2}}\delta(\overline{q}^0+\overline{q}^1-(\hat{\overline{q}}^0+\hat{\overline{q}}^1))$$

$$=\int_{-\infty}^{\infty}\phi(\hat{\overline{q}}^1)\left(\frac{\overline{q}^1+\overline{q}^0}{\hat{\overline{q}}^0}\right)^{\frac{1}{2}}\left(\frac{\hat{\overline{q}}^1+\hat{\overline{q}}^0}{\overline{q}^0}\right)^{\frac{1}{2}}\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)\delta(\overline{q}^1-\overline{q}^1)d\overline{q}^1$$

$$=\phi(\overline{q}^1)$$

showing that we have the correct form for the inverse.

Appendix 2

First we shall show that

$$Y^1_{\overline{P}}=U_{\Pi\overline{P}}Y^1_\Pi U_{\overline{P}\Pi}. \tag{2.68}$$

We have

$$(U_{\Pi\overline{P}}Y^1_\Pi U_{\overline{P}\Pi}\phi_{\overline{P}})=U_{\Pi\overline{P}}\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty}y^1\phi_{\overline{P}}(\overline{q}^1)\exp(-i(\overline{q}^1+\overline{q}^0)y^1)\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\sqrt{|\,y^1\,|}d\overline{q}^1.$$

As before let

$$u^1 = \overline{q}^1 + \overline{q}^0 \quad \text{so that} \quad d\overline{q}^1 = \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right) du^1.$$

Therefore

$$(U_{\Pi\overline{P}} Y^1_\Pi U_{\overline{P}\Pi} \phi_{\overline{P}}) = U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_0^\infty \phi_{\overline{P}}(\overline{q}^1) \left(-\frac{1}{i}\right) \frac{d}{du^1} \exp(-iu^1 y^1) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}} \sqrt{|\, y^1 \,|} du^1.$$

Integrate by parts

$$(U_{\Pi\overline{P}} Y^1_\Pi U_{\overline{P}\Pi} \phi_{\overline{P}}) = U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_0^\infty \left[\frac{d}{du^1}\left(\frac{1}{i}\right) \phi_{\overline{P}}(u^1) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right] \exp[-iu^1 y^1] \sqrt{|\, y^1 \,|} du^1$$

$$= U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_0^\infty \left[\frac{d}{du^1}\left(\frac{1}{i}\right) \phi_{\overline{P}}(u^1) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right] \exp[-iu^1 y^1] \sqrt{|\, y^1 \,|} \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right) d\overline{q}^1$$

$$= U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_{-\infty}^\infty \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \left[\frac{d}{du^1}\left(\frac{1}{i}\right) \phi_{\overline{P}}(u^1) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right] \times$$

$$\exp[-i(\overline{q}^1 + \overline{q}^0) y^1] \sqrt{|\, y^1 \,|} \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} d\overline{q}^1$$

$$= \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \left[\frac{d}{du^1}\left(\frac{1}{i}\right) \phi_{\overline{P}}(u^1) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right].$$

Since

$$\frac{d}{du^1} = \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right) \frac{d}{d\overline{q}^1}$$

we have

$$(U_{\Pi\overline{P}} Y^1_\Pi U_{\overline{P}\Pi} \phi_{\overline{P}}) = \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \left[\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right) \frac{d}{d\overline{q}^1}\left(\frac{1}{i}\right) \phi_{\overline{P}}(\overline{q}^1) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right]$$

$$= -i\hbar \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}} \left[\frac{d}{d\overline{q}^1}\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}} \phi_{\overline{P}}(\overline{q}^1)\right]$$

$$= -i\hbar \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}} \left[\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}} \frac{d}{d\overline{q}^1} + \frac{d}{d\overline{q}^1}\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right] \phi_{\overline{P}}(\overline{q}^1)$$

so

$$U_{\Pi\overline{P}} Y^1_\Pi U_{\overline{P}\Pi} = -i\hbar \left[\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right) \frac{d}{d\overline{q}^1} + \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}} \frac{1}{2} \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{-\frac{1}{2}} \left(\frac{(\overline{q}^0 + \overline{q}^1)\frac{\overline{q}^1}{\overline{q}^0} - \overline{q}^0(\frac{\overline{q}^1}{\overline{q}^0} + 1)}{(\overline{q}^0 + \overline{q}^1)^2}\right)\right]$$

$$= -i\hbar \left[\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right) \frac{d}{d\overline{q}^1} + \frac{1}{2} \frac{(\overline{q}^0\overline{q}^1 + \overline{q}^{1^2} - \overline{q}^0\overline{q}^1 - \overline{q}^{0^2})}{\overline{q}^0(\overline{q}^0 + \overline{q}^1)^2}\right]$$

$$= -i\hbar \left[\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right) \frac{d}{d\overline{q}^1} + \frac{(\overline{q}^1 - \overline{q}^0)}{2\overline{q}^0(\overline{q}^0 + \overline{q}^1)}\right]$$

$$= Y^1_{\overline{P}}$$

as required. Similarly it is easy to show that

$$\Pi_{1\overline{P}} = U_{\Pi\overline{P}} \Pi_{1\Pi} U_{\overline{P}\Pi} \tag{2.69}$$

or

$$\Pi_{1\Pi} = U_{\overline{P}\Pi} \Pi_{1\overline{P}} U_{\Pi\overline{P}}.$$

We have

$$(U_{\overline{P}\Pi}\Pi_{1\overline{P}}U_{\Pi\overline{P}}\phi_\Pi) = U_{\overline{P}\Pi} - (\overline{q}^1 + \overline{q}^0) \int_{-\infty}^{\infty} \phi_\Pi(y^1) \exp(i(\overline{q}^1 + \overline{q}^0)y^1) \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \frac{1}{\sqrt{| y^1 |}} dy^1$$

$$= U_{\overline{P}\Pi} \int_{\infty}^{\infty} \phi_\Pi(y^1) \left[i\hbar \frac{d}{dy^1} \exp(i(\overline{q}^1 + \overline{q}^0)y^1) \right] \left(\frac{\overline{q}^1 + \overline{q}^0}{\overline{q}^0}\right)^{\frac{1}{2}} \frac{1}{\sqrt{| y^1 |}} dy^1$$

$$= U_{\overline{P}\Pi} \int_{-\infty}^{\infty} \left(-i\hbar \sqrt{| y^1 |} \frac{d}{dy^1} \frac{\phi_\Pi}{\sqrt{| y^1 |}} \right) \exp(i(\overline{q}^1 + \overline{q}^0)y^1) \left(\frac{\overline{q}^1 + \overline{q}^0}{\overline{q}^0}\right)^{\frac{1}{2}} \frac{1}{\sqrt{| y^1 |}} dy^1$$

so

$$U_{\overline{P}\Pi}\Pi_{1\overline{P}}U_{\Pi\overline{P}} = -i\hbar \sqrt{| y^1 |} \frac{d}{dy^1} \frac{1}{\sqrt{| y^1 |}}$$
$$= \Pi_{1\Pi}$$

as required. Finally, using these results we shall show that

$$Q^1_{\overline{P}} = U_{\Pi\overline{P}} Q^1_\Pi U_{\overline{P}\Pi}.$$

From (2.17), (2.69) and (2.68) we have

$$U_{\Pi\overline{P}} Q^1_\Pi U_{\overline{P}\Pi} = \frac{2\Pi^{1^2}_{\overline{P}}}{(\Pi^{1^2}_{\overline{P}} + (mc)^2)} Y^1_{\overline{P}} + i\hbar 2(mc)^2 \frac{\Pi_{1\overline{P}}}{(\Pi^{1^2}_{\overline{P}} + (mc)^2)^2}.$$

Using (2.11) and (2.13) this becomes

$$U_{\Pi\overline{P}} Q^1_\Pi U_{\overline{P}\Pi} = \frac{2(p^0 + p^1)^2}{(p^0 + p^1)^2 + (mc)^2} (-i\hbar) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \frac{\partial}{\partial \overline{q}^1} + \frac{\overline{q}^1 - \overline{q}^0}{2\overline{q}^0(\overline{q}^0 + \overline{q}^1)} \right) +$$

$$i\hbar 2(mc)^2 \frac{(-1)(p^0 + p^1)}{((p^0 + p^1)^2 + (mc)^2)^2}$$

$$= \frac{2(p_0 - p_1)^2(-i\hbar)}{2p_0(p_0 - p_1)} \left(-\frac{p_0}{p_0 - p_1} \frac{\partial}{\partial p_1} - \frac{(p_1 + p_0)}{2p_0(p_0 - p_1)} \right) - \frac{i\hbar 2(mc)^2(p_0 - p_1)}{[2p_0(p_0 - p_1)]^2}$$

$$= \frac{(p_0 - p_1)(-i\hbar)}{p_0} \left(-\frac{p_0}{p_0 - p_1} \frac{\partial}{\partial p_1} - \frac{p_1 + p_0}{2p_0(p_0 - p_1)} \right) - \frac{i\hbar(mc)^2}{2p_2(p_0 - p_1)}.$$

If we put $(mc)^2 = p_0^2 - p_1^2$ then we obtain

$$U_{\Pi\overline{P}} Q^1_\Pi U_{\overline{P}\Pi} = i\hbar \frac{d}{dp_1} + \frac{i\hbar(p_0^2 - p_1^2)}{2p_0^2(p_0 - p_1)} - \frac{i\hbar(p_0^2 - p_1^2)}{2p_0^2(p_0 - p_1)} = i\hbar \frac{\partial}{\partial p_1}$$
$$= Q^1_{\overline{P}}$$

as required. This demonstrates the consistency of our front form quantum mechanics.

Appendix 3

We wish to prove that $Y^1_{\overline{\Pi}}$ is maximally symmetric. We were able to prove that ${}_1Y^1_{\overline{\Pi}}$ was maximally symmetric using the methods described in chapter 1. Since these were formulated to deal with ordinary and not partial differential operators they are of no use per se in our analysis of $Y^1_{\overline{\Pi}}$. Although we shall give the proof for $Y^1_{\overline{\Pi}}$ the same method can be used to prove that Y^1_{Π} is maximally symmetric directly in $\mathcal{H}_{\Pi}$ given that ${}_1Y^1_{\Pi}$ is maximally symmetric.

Consider the maximally symmetric operator

$$ {}_1Y^1_{\overline{\Pi}} = -i\hbar\frac{d}{d\overline{y}^1} $$

on $D({}_1Y^1_{\overline{\Pi}}) \subset_1 \mathcal{H}_{\overline{\Pi}}$ where

$$ \begin{aligned} \mathcal{H}_{\overline{\Pi}} &= L^2(\mathbb{R}^+ \times \mathbb{R}^2, d^3\underline{y}) \\ &\equiv L^2(\mathbb{R}^+)\hat{\otimes}L^2(\mathbb{R}^2) \\ &={}_1\mathcal{H}_{\overline{\Pi}}\hat{\otimes}L^2(\mathbb{R}^2). \end{aligned} $$

Let $\{e_i\}_{i=1}^{\infty}$ denote an orthonormal basis in $L^2(\mathbb{R}^2)$ and consider the Hilbert spaces $\mathcal{H}_i ={}_1 \mathcal{H}_{\overline{\Pi}} \otimes \{e_i\}$ where $\{e_i\}$ denotes the linear span of e_i. Notice that $\mathcal{H}_{\overline{\Pi}} = \oplus_{i=1}^{\infty}\mathcal{H}_i$. Let $A_i ={}_1Y^1_{\overline{\Pi}} \otimes I$ act on $\mathcal{H}_i$. Let $A = \oplus_{i=1}^{\infty}A_i$ where A acts on that subspace of $\mathcal{H}_{\overline{\Pi}}$ consisting of vectors ϕ of the form $\phi = \oplus_{i=1}^{\infty}\phi_i$ where $\phi_i \in D({}_1Y^1_{\overline{\Pi}}) \otimes \{e_i\}$ and only a finite number of the ϕ_i are non zero. This operator is symmetric and has defect indicies

$$ \left(\sum_{i=1}^{\infty} n_+(A_i), \sum_{i=1}^{\infty} n_-(A_i)\right) $$

([65] page 338). Now define a unitary map $U : \mathcal{H}_i \to_1\mathcal{H}_{\overline{\Pi}}$ by $Uf\otimes e_i = f$ ([67] page 67). We have $UA_iU^{-1} ={}_1Y^1_{\overline{\Pi}}$ so that $n_+(A_i) = 1$ and $n_-(A_i) = 0$. Therefore the defect indicies of A are $(0, \infty)$. Now recall that A is symmetric so it has a closure that will also have defect indicies $(0, \infty)$; an operator and its closure have the same defect indicies since they have the same adjoint. This operator is maximally symmetric and is equal to $Y^1_{\overline{\Pi}}$.

Appendix 4

The vector field

$$ \frac{\partial}{\partial\overline{y}^s} + \frac{2\overline{y}^1\overline{y}^s}{(\underline{\overline{y}}^2 + (mc)^2)}\frac{\partial}{\partial\overline{y}^1} $$

is complete. To see this we must solve

$$ \frac{d\overline{y}^s}{dt} = 1, \quad \frac{d\overline{y}^1}{dt} = \frac{2\overline{y}^1\overline{y}^s}{(\underline{\overline{y}}^2 + (mc)^2)}, \quad \frac{d\overline{y}^{5-s}}{dt} = 0. $$

The first and last equations in this system are easy to solve and we find that

$$\overline{y}^s = t + \overline{y}_0^s \quad \text{and} \quad \overline{y}^{5-s} = \overline{y}_0^{5-s}$$

where $\overline{y}_0^s$ and $\overline{y}_0^{5-s}$ are constants in $\mathbb{R}$. Now we must solve

$$\frac{d\overline{y}^1}{dt} = \frac{2\overline{y}^1(t+\overline{y}_0^s)}{\overline{y}^{1^2} + (t+\overline{y}_0^s)^2 + (\overline{y}_0^{5-s})^2 + (mc)^2}$$

or

$$\frac{dt}{d\overline{y}^1} = \frac{\overline{y}^{1^2} + (t+\overline{y}_0^s)^2 + (\overline{y}_0^{5-s})^2 + (mc)^2}{2\overline{y}^1(t+\overline{y}_0^s)}.$$

Put

$$u^s = (t+\overline{y}_0^s)^2$$

then the differential equation becomes

$$\frac{du^s}{d\overline{y}^1} - \frac{1}{\overline{y}^1}u = \overline{y}^1 + \frac{(\overline{y}_0^{5-s})^2 + (mc)^2}{\overline{y}^1}.$$

We can solve this using an integrating factor $1/\overline{y}^1$. This gives the equation

$$\frac{du^s\overline{y}^{1^{-1}}}{d\overline{y}^1} = 1 + ((\overline{y}^{5-s})^2 + (mc)^2)\overline{y}^{1^{-2}}$$

which is easily integrated to give

$$u^s\overline{y}^{1^{-1}} = \overline{y}^1 - ((y_0^{5-s})^2 + (mc)^2)\overline{y}^{1^{-1}} + z$$

or

$$(t+\overline{y}_0^s)^2 = \overline{y}^{1^2} - ((\overline{y}^{5-s})^2 + (mc)^2) + z\overline{y}^1$$

so

$$\overline{y}^1 = \frac{-z + \sqrt{z^2 + 4[(t+\overline{y}_0^s)^2 + (\overline{y}_0^{5-s})^2 + (mc)^2]}}{2}.$$

Taking

$$z = \frac{\overline{y}_0^{i^2} + (\overline{y_0}^{5-s})^2 + (mc)^2 - \overline{y}_0^{1^2}}{\overline{y}_0^1}$$
$$= \frac{\overline{y}_0^{2^2} + \overline{y_0}^{3^2} + (mc)^2 - \overline{y}_0^{1^2}}{\overline{y}_0^1}$$

ensures that the integral curve starts at $\overline{y}_0^1 \in \mathbb{R}^+$. The integral curve is therefore well defined for all t.

Appendix 5

Proof that

$$\frac{2\overline{y}^{1^2}}{\underline{\overline{y}}^2 + (mc)^2} \frac{\partial}{\partial \overline{y}^1}$$

is complete. To show this we need to solve

$$\frac{d\overline{y}^1}{dt} = \frac{2\overline{y}^{1^2}}{\underline{\overline{y}}^2 + (mc)^2}, \quad \frac{d\overline{y}^2}{dt} = 0, \quad \frac{d\overline{y}^3}{dt} = 0.$$

The first of these is the only one that presents any difficulty. We have

$$\int 1 + (\overline{y}_0^2 + \overline{y}_0^{3^2} + (mc)^2)\overline{y}^{1^{-2}} d\overline{y}^1 = 2(t+k)$$

or

$$\overline{y}^1 - (\overline{y}_0^{2^2} + \overline{y}_0^{3^2} + (mc)^2)\overline{y}^{1^{-1}} = 2(t+k).$$

Rearranging this gives

$$\overline{y}^1 = (t+k) + \sqrt{(t+k)^2 + \overline{y}_0^{2^2} + \overline{y}_0^{3^2} + (mc)^2}.$$

If we take

$$k = \frac{\overline{y}_0^{1^2} - \overline{y}_0^{2^2} - \overline{y}_0^{3^2}}{2\overline{y}_0^1}$$

then at $t = 0$ $\overline{y}^1 = \overline{y}_0^1$. The integral curve is well defined for all t and $\overline{y}^1$ remains in $\mathbb{R}^+$ therefore vector field is complete.

Appendix 6

We begin by showing that

$$Y^1_{\overline{P}} = U_{\Pi\overline{P}} Y^1_{\Pi} U_{\overline{P}\Pi}.$$

We have

$$(U_{\Pi\overline{P}} Y^1_{\Pi} U_{\overline{P}\Pi} \phi_{\overline{P}}) = U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int_{-\infty}^{\infty} y^1 \phi_{\overline{P}}(\underline{\overline{q}}) \exp[-\tfrac{i}{\hbar}((\overline{q}^1 + \overline{q}^0)y^1 + \overline{q}^2 y^2 + \overline{q}^3 y^3)]$$

$$\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \sqrt{|\, y^1 \,|} d^3\underline{\overline{q}}. \tag{2.70}$$

Now let

$$u^1 = \overline{q}^1 + \overline{q}^0, \;\; u^2 = \overline{q}^2, \;\; \text{and} \;\; u^3 = \overline{q}^3. \tag{2.71}$$

It is easy to show that

$$\frac{\partial \underline{u}}{\partial \underline{\overline{q}}} = \begin{vmatrix} \frac{\partial u^1}{\partial \overline{q}^1} & \frac{\partial u^1}{\partial \overline{q}^2} & \frac{\partial u^1}{\partial \overline{q}^3} \\ \frac{\partial u^2}{\partial \overline{q}^1} & \frac{\partial u^2}{\partial \overline{q}^2} & \frac{\partial u^2}{\partial \overline{q}^3} \\ \frac{\partial u^3}{\partial \overline{q}^1} & \frac{\partial u^3}{\partial \overline{q}^2} & \frac{\partial u^3}{\partial \overline{q}^3} \end{vmatrix} = \begin{vmatrix} 1 + \frac{\overline{q}^1}{\overline{q}^0} & \frac{\overline{q}^2}{\overline{q}^0} & \frac{\overline{q}^3}{\overline{q}^0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}.$$

We also require the inverse transformations of (2.71). From the first equation in that set we obtain

$$u^{1^2} - 2u^1\overline{q}^1 + \overline{q}^{1^2} = (mc)^2 + \overline{q}^{1^2} + \overline{q}^{2^2} + \overline{q}^{3^2}$$

so

$$u^{1^2} - 2u^1\overline{q}^1 = (mc)^2 + u^{2^2} + u^{3^2}.$$

From this we obtain the expression for $\overline{q}^1$ in terms of the u's so that

$$\overline{q}^1 = \frac{u^{1^2} - u^{2^2} - u^{3^2} - (mc)^2}{2u^1}, \;\; \overline{q}^2 = u^2, \;\text{ and }\; \overline{q}^3 = u^3. \tag{2.72}$$

Since

$$d^3\underline{\overline{q}} = \frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} d^3\underline{u}$$

we can write (2.70) as

$$(U_{\Pi\overline{P}} Y^1_\Pi U_{\overline{P}\Pi} \phi_{\overline{P}}) =$$

$$U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty y^1 \phi_{\overline{P}}(\underline{u}) \exp[-i\underline{u}.\underline{y}] \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \sqrt{|\, y' \,|} \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right) d^3\underline{u}$$

$$= U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int \phi_{\overline{P}}(\underline{u}) \left(-\frac{1}{i}\right) \cdot \left(\frac{\partial}{\partial u^1} \exp[-i\underline{u}.\underline{y}]\right) \sqrt{|\, y^1 \,|} \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}} d^3\underline{u}.$$

Now integrate by parts

$$(U_{\Pi\overline{P}} Y^1_\Pi U_{\overline{P}\Pi} \phi_{\overline{P}}) = U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int \left[\frac{\partial}{\partial u^1} \left(\frac{1}{i}\right) \phi_{\overline{P}}(\underline{u}) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right] \exp[-i\underline{u}.\underline{y}] \sqrt{|\, y^1 \,|} d^3\underline{u}$$

$$= U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int \left[\frac{\partial}{\partial u^1} \left(\frac{1}{i}\right) \phi_{\overline{P}}(\underline{u}) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right] \exp[-i\underline{u}.\underline{y}] \sqrt{|\, y^1 \,|} \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right) d^3\underline{\overline{q}}$$

$$= U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \left[\frac{\partial}{\partial u^1} \left(\frac{1}{i}\right) \phi_{\overline{P}}(\underline{u}) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right] \exp[-i\underline{u}.\underline{y}] \sqrt{|\, y^1 \,|} \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} d^3\underline{\overline{q}}$$

$$= \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} \left[\frac{\partial}{\partial u^1} \left(\frac{1}{i}\right) \phi_{\overline{P}}(\underline{u}) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)^{\frac{1}{2}}\right].$$

Now

$$\frac{\partial}{\partial u^1} = \frac{\partial \overline{q}^1}{\partial u^1}\frac{\partial}{\partial \overline{q}^1} + \frac{\partial \overline{q}^2}{\partial u^1}\frac{\partial}{\partial \overline{q}^2} + \frac{\partial \overline{q}^3}{\partial u^1}\frac{\partial}{\partial \overline{q}^3} = \left(\frac{2u^1(2u^1) - (u^{1^2} - u^{2^2} - u^{3^2} - (mc)^2)2}{4u^{1^2}}\right)\frac{\partial}{\partial \overline{q}^1}$$

$$= \left(\frac{\underline{u}^2 + (mc)^2}{2u^{1^2}}\right)\frac{\partial}{\partial \overline{q}^1} = \left(\frac{(\overline{q}^1 + \overline{q}^0)^2 + \overline{q}^{2^2} + \overline{q}^{3^2} + (mc)^2}{2(\overline{q}^0 + \overline{q}^1)^2}\right)\frac{\partial}{\partial \overline{q}^1}$$

$$= \left(\frac{\overline{q}^{1^2} + 2\overline{q}^1\overline{q}^0 + \overline{q}^{0^2} + \overline{q}^{2^2} + \overline{q}^{3^2} + (mc)^2}{2(\overline{q}^0 + \overline{q}^1)^2}\right)\frac{\partial}{\partial \overline{q}^1}.$$

Using the mass shell condition this becomes

$$\left(\frac{2\overline{q}^1\overline{q}^0+\overline{q}^{0^2}+\overline{q}^{0^2}}{2(\overline{q}^0+\overline{q}^1)^2}\right)\frac{\partial}{\partial\overline{q}^1}=\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)\frac{\partial}{\partial\overline{q}^1}$$

so that

$$(U_{\Pi\overline{P}}Y^1_{\Pi}U_{\overline{P}\Pi}\phi_{\overline{P}})=\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\left[\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)\frac{\partial}{\partial\overline{q}^1}\left(\frac{1}{i}\right)\phi_{\overline{P}}(\underline{\overline{q}})\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\right]$$

$$=-i\hbar\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\left[\frac{\partial}{\partial\overline{q}^1}\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\phi_{\overline{P}}(\underline{\overline{q}})\right]$$

$$=-i\hbar\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\left[\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\frac{\partial}{\partial\overline{q}^1}+\frac{\partial}{\partial\overline{q}^1}\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\right]\phi_{\overline{P}}(\underline{\overline{q}}).$$

Therefore

$$U_{\Pi\overline{P}}Y^1_{\Pi}U_{\overline{P}\Pi}=-i\hbar\left[\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)\frac{\partial}{\partial\overline{q}^1}+\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\frac{1}{2}\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{-\frac{1}{2}}\left(\frac{(\overline{q}^0+\overline{q}^1)\frac{\overline{q}^1}{\overline{q}^0}-\overline{q}^0(\frac{\overline{q}^1}{\overline{q}^0}+1)}{(\overline{q}^0+\overline{q}^1)^2}\right)\right]$$

$$=-i\hbar\left[\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)\frac{\partial}{\partial\overline{q}^1}+\frac{1}{2}\frac{(\overline{q}^0\overline{q}^1+\overline{q}^{1^2}-\overline{q}^0\overline{q}^1-\overline{q}^{0^2})}{\overline{q}^0(\overline{q}^0+\overline{q}^1)^2}\right]$$

$$=-i\hbar\left[\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)\frac{\partial}{\partial\overline{q}^1}+\frac{(\overline{q}^1-\overline{q}^0)}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}\right] \tag{2.73}$$

$$=Y^1_{\overline{P}}$$

as required.

We now wish to show that

$$Y^s_{\overline{P}}=U_{\Pi\overline{P}}Y^s_{\Pi}U_{\overline{P}\Pi}. \tag{2.74}$$

Well

$$U_{\Pi\overline{P}}Y^s_{\Pi}U_{\overline{P}\Pi}\phi_{\overline{P}}=$$

$$U_{\Pi\overline{P}}\frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_{-\infty}^{\infty}y^s\phi_{\overline{P}}(\underline{\overline{q}})\exp[-i((\overline{q}^1+\overline{q}^0)y^1+\overline{q}^2y^2+\overline{q}^3y^3)]\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\sqrt{|y^1|}d^3\underline{\overline{q}}$$

$$=U_{\Pi\overline{P}}\frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y^s\phi_{\overline{P}}(\underline{u})\exp[-i\underline{u}.\underline{y}]\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\sqrt{|y^1|}\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)d^3\underline{u}$$

$$=U_{\Pi\overline{P}}\frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int\phi_{\overline{P}}(\underline{u})\left[\left(-\frac{1}{i}\right)\frac{\partial}{\partial u^s}\exp[-i\underline{u}.\underline{y}]\right]\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\sqrt{|y^1|}d^3\underline{u}.$$

Integrating by parts

$$U_{\Pi\overline{P}}Y^s_{\Pi}U_{\overline{P}\Pi}\phi_{\overline{P}}=U_{\Pi\overline{P}}\frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int\left[\frac{\partial}{\partial u^s}\left(\frac{1}{i}\right)\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{1}{2}}\phi_{\overline{P}}(\underline{u})\right]\exp[-i\underline{u}.\underline{y}]\sqrt{|y^1|}d^3\underline{u}$$

$$= U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial u^s} \left(\frac{1}{i} \right) \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \right)^{\frac{1}{2}} \phi_{\overline{P}}(\underline{u}) \right] \exp[-i\underline{u}.\underline{y}] \sqrt{| y^1 |} \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0} \right) d^3\underline{q}$$

$$= U_{\Pi\overline{P}} \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int \left[\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0} \right)^{\frac{1}{2}} \left(\frac{1}{i} \right) \frac{\partial}{\partial u^s} \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \right)^{\frac{1}{2}} \phi_{\overline{P}}(\underline{u}) \right] \exp[-i\underline{u}.\underline{y}] \sqrt{| y^1 |} \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0} \right)^{\frac{1}{2}} d^3\underline{\overline{q}}$$

$$= -i\hbar \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0} \right)^{\frac{1}{2}} \frac{\partial}{\partial u^i} \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \right)^{\frac{1}{2}} \phi_{\overline{P}}(\underline{u}).$$

Now

$$\frac{\partial}{\partial u^s} = \frac{\partial \overline{q}^1}{\partial u^s} \frac{\partial}{\partial \overline{q}^1} + \frac{\partial \overline{q}^2}{\partial u^s} \frac{\partial}{\partial \overline{q}^2} + \frac{\partial \overline{q}^3}{\partial u^s} \frac{\partial}{\partial \overline{q}^3} = -\frac{2u^s}{2u^1} \frac{\partial}{\partial \overline{q}^1} + \frac{\partial}{\partial \overline{q}^s} = \frac{\partial}{\partial \overline{q}^s} - \left(\frac{\overline{q}^s}{\overline{q}^1 + \overline{q}^0} \right) \frac{\partial}{\partial \overline{q}^1}$$

so

$$U_{\Pi\overline{P}} Y^s_{\Pi} U_{\overline{P}\Pi} \phi_{\overline{P}} = -i\hbar \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0} \right)^{\frac{1}{2}} \left[\frac{\partial}{\partial \overline{q}^s} - \frac{\overline{q}^s}{(\overline{q}^1 + \overline{q}^0)} \frac{\partial}{\partial \overline{q}^1} \right] \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \right)^{\frac{1}{2}} \phi_{\overline{P}}(\underline{q})$$

i.e.

$$U_{\Pi\overline{P}} Y^s_{\Pi} U_{\overline{P}\Pi} = -i\hbar \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0} \right)^{\frac{1}{2}} \left[\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \right)^{\frac{1}{2}} \frac{\partial}{\partial \overline{q}^s} + \frac{1}{2} \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \right)^{-\frac{1}{2}} \frac{((\overline{q}^0 + \overline{q}^1)\frac{\overline{q}^s}{\overline{q}^0} - \overline{q}^0 \frac{\overline{q}^s}{\overline{q}^0})}{(\overline{q}^0 + \overline{q}^1)^2} - \right.$$

$$\left. \frac{\overline{q}^s}{(\overline{q}^1 + \overline{q}^0)} \left\{ \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \right)^{\frac{1}{2}} \frac{\partial}{\partial \overline{q}^1} + \frac{1}{2} \left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1} \right)^{-\frac{1}{2}} \frac{((\overline{q}^0 + \overline{q}^1)\frac{\overline{q}^1}{\overline{q}^0} - \overline{q}_0(\frac{\overline{q}^1}{\overline{q}^0} + 1))}{(\overline{q}^0 + \overline{q}^1)^2} \right\} \right]$$

$$= -i\hbar \left[\frac{\partial}{\partial \overline{q}^s} + \frac{1}{2} \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0} \right) \frac{\overline{q}^1 \overline{q}^s}{\overline{q}^0 (\overline{q}^0 + \overline{q}^1)^2} - \left(\frac{\overline{q}^s}{\overline{q}^1 + \overline{q}^0} \right) \frac{\partial}{\partial \overline{q}^1} - \right.$$

$$\left. \frac{\overline{q}^s}{\overline{q}^1 + \overline{q}^0} \frac{1}{2} \left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0} \right) \frac{1}{\overline{q}^0 (\overline{q}^1 + \overline{q}^0)^2} (\overline{q}^{1^2} - \overline{q}^{0^2}) \right]$$

$$= -i\hbar \left[\frac{\partial}{\partial \overline{q}^s} - \frac{\overline{q}^s}{\overline{q}^1 + \overline{q}^0} \frac{\partial}{\partial \overline{q}^1} + \frac{1}{2} \frac{\overline{q}^1 \overline{q}^s}{\overline{q}^{0^2} (\overline{q}^0 + \overline{q}^1)} - \frac{\overline{q}^s (\overline{q}^1 - \overline{q}^0)}{2\overline{q}^{0^2} (\overline{q}^0 + \overline{q}^1)} \right]$$

$$= -i\hbar \left[\frac{\partial}{\partial \overline{q}^s} - \left(\frac{\overline{q}^s}{\overline{q}^1 + \overline{q}^0} \right) \frac{\partial}{\partial \overline{q}^1} + \frac{\overline{q}^s}{2\overline{q}^0 (\overline{q}^0 + \overline{q}^1)} \right] \tag{2.75}$$

$$= Y^s_{\overline{P}}$$

as required.

We shall omit the proof that

$$\Pi^i_{\overline{P}} = U_{\Pi\overline{P}} \Pi^i_{\Pi} U_{\overline{P}\Pi}.$$

It is a straightforward generalisation of the proof given in the case of 1+1 spacetime in Appendix 2.

We shall now consider the instant form position operator and show that

$$Q^s_{\overline{P}} = U_{\Pi\overline{P}} Q^s_{\Pi} U_{\overline{P}\Pi}. \tag{2.76}$$

Well from (2.39) and using (2.74) and (2.76) we have

$$U_{\Pi\overline{P}}Q^s_{\Pi}U_{\overline{P}\Pi} = Y^s_{\overline{P}} + 2\frac{\Pi_{1\overline{P}}\Pi_{s\overline{P}}}{(\underline{\Pi}^2_{\overline{P}} + (mc)^2)}Y^1_{\overline{P}} + i\hbar\frac{\Pi_{s\overline{P}}}{(\underline{\Pi}^2_{\overline{P}} + (mc)^2)^2}(\Pi^2_{2\overline{P}} + \Pi^2_{3\overline{P}} - \Pi^2_{1\overline{P}} + (mc)^2).$$

Using (2.35) and (2.36) we obtain

$$U_{\Pi\overline{P}}Q^s_{\Pi}U_{\overline{P}\Pi} = -i\hbar\left(\frac{\partial}{\partial\overline{q}^s} - \left(\frac{\overline{q}^s}{\overline{q}^1 + \overline{q}^0}\right)\frac{\partial}{\partial\overline{q}^1} + \frac{\overline{q}^s}{2\overline{q}^0(\overline{q}^0 + \overline{q}^s)}\right) +$$

$$\frac{2(p_1 - p_0)p_s}{((p_1 - p_0)^2 + p_2^2 + p_3^2 + m^2)}(-i\hbar)\left(\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\right)\frac{\partial}{\partial\overline{q}^1} + \frac{(\overline{q}^1 - \overline{q}^0)}{2\overline{q}^0(\overline{q}^0 + \overline{q}^1)}\right) +$$

$$(i\hbar)\frac{p_s}{((p_1 - p_0)^2 + p_2^2 + p_3^2 + m^2)^2}(p_1^2 + p_3^2 - (p_1 - p_0)^2 + (mc)^2).$$

Since

$$p_1^2 + p_3^2 - p_1^2 + 2p_1p_0 - p_0^2 + (mc)^2 = 2p_1(p_0 - p_1)$$

and

$$(p_1 - p_0)^2 + p_2^2 + p_3^2 + (mc)^2 = 2p_0(p_0 - p_1)$$

we have

$$U_{\Pi\overline{P}}Q^s_{\Pi}U_{\overline{P}\Pi} = -i\hbar\left(-\frac{\partial}{\partial p_s} - \frac{(-p_s)}{(p_0 - p_1)}(-1)\frac{\partial}{\partial p_1} - \frac{p_s}{2(p_0 - p_1)p_0}\right) +$$

$$\frac{2(p_1 - p_0)p_s}{2p_0(p_0 - p_1)}(-i\hbar)\left(\left(\frac{p_0}{p_0 - p_1}\right)(-1)\frac{\partial}{\partial p_1} + \frac{(-p_1 - p_0)}{2p_0(p_0 - p_1)}\right) + \frac{(i\hbar)p_s}{(2p_0(p_0 - p_1))^2}(2p_1(p_0 - p_1))$$

$$= i\hbar\frac{\partial}{\partial p_s} + \frac{i\hbar 2}{(2p_0(p_0 - p_1))^2}(p_sp_0^2 - p_sp_1p_0 + p_1^2p_s - p_0^2p_s + p_sp_1p_0 - p_1^2p_s)$$

$$= i\hbar\frac{\partial}{\partial p_s} = Q^s_{\overline{P}}$$

as required. Lastly we shall show that

$$Q^1_{\overline{P}} = U_{\Pi\overline{P}}Q^1_{\Pi}U_{\overline{P}\Pi}.$$

From (2.40) and using (2.74) and (2.76) and then (2.35), (2.36) and (2.34) we have

$$U_{\Pi\overline{P}}Q^1_{\Pi}U_{\overline{P}\Pi} = \frac{2(p^0 + p^1)^2}{(p^0 + p^1)^2 + p^{2^2} + p^{3^2} + (mc)^2}(-i\hbar)\left(\frac{\overline{q}^0}{\overline{q}^0 + \overline{q}^1}\frac{\partial}{\partial\overline{q}^1} + \frac{\overline{q}^1 - \overline{q}^0}{2\overline{q}^0(\overline{q}^0 + \overline{q}^1)}\right) -$$

$$i\hbar 2(p^0 + p^1)\frac{(p^{2^2} + p^{2^2} + (mc)^2)}{((p^0 + p^1)^2 + p^{2^2} + p^{3^2} + m^2)^2}.$$

Using (2.33) this becomes

$$-\frac{i\hbar 2(p^0 + p^1)^2}{2p_0(p_0 - p_1)}\left(-\frac{p_0}{(p_0 - p_1)}\frac{\partial}{\partial p_1} + \frac{(-1)(p_1 + p_0)}{2p_0(p_0 - p_1)}\right) - (i\hbar)2(p_0 - p_1)\frac{(p_0^2 - p_1^2)}{(2p_0(p_0 - p_1))^2}$$

$$= \frac{i\hbar 2(p_0 - p_1)^2p_0}{2p_0(p_0 - p_1)^2}\frac{\partial}{\partial p_1} + \frac{i\hbar 2(p_0 - p_1)^2(p_1 + p_0)}{2^2p_0^2(p_0 - p_1)^2} - \frac{i\hbar 2(p_0 - p_1)^2(p_1 + p_0)}{2^2p_0^2(p_0 - p_1)^2}$$

$$= i\hbar\frac{\partial}{\partial p_1}$$

as required.

Appendix 7

In (2.3.1) we demonstrated that certain generators left the the light cone with apex at the origin invariant. We worked with the infinitesimal form of the transformations. Here we show how to obtain the finite form of the transformations. Consider the infinitesimal transformation

$$x^{0'} = x^0, \quad x^{1'} = x^1 - \alpha x^3, \quad x^{2'} = x^2, \quad x^{3'} = x^3 + \alpha x^1. \tag{2.77}$$

The infinitesimal generator of this transformation has components

$$(0, -x^3, 0, x^1)$$

i.e. the generator is

$$-x^3 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^3}.$$

To get the finite form of (2.77) we must solve

$$\frac{d\overline{x}^0}{dt} = 0$$

$$\frac{d\overline{x}^1}{dt} = -\overline{x}^3 \tag{2.78}$$

$$\frac{d\overline{x}^2}{dt} = 0$$

$$\frac{d\overline{x}^3}{dt} = \overline{x}^1. \tag{2.79}$$

Clearly we have

$$\overline{x}^0 = x^0 \tag{2.80}$$

and

$$\overline{x}^2 = x^2. \tag{2.81}$$

If we differentiate (2.77) with respect t we obtain

$$\frac{d^2\overline{x}^1}{dt^2} = -\frac{d\overline{x}^3}{dt} \quad \text{so} \quad \frac{d^2\overline{x}^1}{dt^2} = -\overline{x}^1$$

where we have used (2.79). The general solution of this equation is

$$\overline{x}^1 = A\cos t + B\sin t.$$

Since

$$\overline{x}^1 \mid_0 = x^1$$

we have $A = x^1$. Also because

$$\frac{d\overline{x}^1}{dt} \mid_0 = -\overline{x}^3 \mid_0 = -x^3$$

we have $B = -x^3$ and therefore

$$\overline{x}^1 = x^1 \cos t - x^2 \sin t. \tag{2.82}$$

Similarly differentiating (2.79) with respect to t and using (2.78) we obtain

$$\frac{d^2\overline{x}^3}{dt^2} = -\overline{x}^3.$$

Again the general solution is of the form

$$\overline{x}^3 = A \cos t + B \sin t.$$

Since

$$\overline{x}^3 \mid_0 = x^3$$

we have $A = x^3$. Also

$$\frac{d\overline{x}^3}{dt} \mid_0 = \overline{x}^1 \mid_0 = x^1$$

so $B = x^1$. Therefore

$$\overline{x}^3 = x^3 \cos t + x^1 \sin t. \tag{2.83}$$

Of course (2.80), (2.81), (2.82) and (2.83) are the familiar integrated forms of the Lorentz transformation (2.77) (c.f 6.63 page 151 [50]). The finite forms of the other infinitesimal transformations can be found in the same way. All this is well known. What is slightly more interesting is to find the finite form of the Lorentz transformations with respect to light cone coordinates. For example the Lorentz transformation (2.77) becomes in light cone coordinates

$$\overline{\tau} = \tau, \quad \overline{y}^1 = y^1 - \alpha \mid \underline{y} \mid, \quad \overline{y}^2 = y^2, \quad \overline{y}^3 = y^3.$$

The infinitesimal generator is

$$(0, - \mid \underline{y} \mid, 0, 0)$$

so to get the finite form we must solve

$$\frac{d\overline{\tau}}{dt} = 0, \quad \frac{d\overline{y}^1}{dt} = - \mid \underline{\overline{y}} \mid, \quad \frac{d\overline{y}^2}{dt} = 0, \quad \frac{d\overline{y}^3}{dt} = 0.$$

Clearly

$$\overline{\tau} = \tau, \quad \overline{y}^2 = y^2, \quad \overline{y}^3 = y^3. \tag{2.84}$$

It only remains to solve

$$\frac{d\overline{y}^1}{dt} = -(\overline{y}^{1^2} + k^2)^{\frac{1}{2}}$$

where $k = (y^{2^2} + y^{3^2})^{\frac{1}{2}}$. From page 75 [8] we see that this implies

$$\sinh^{-1}\left(\frac{\overline{y}^1}{k}\right) = t + R.$$

Since

$$\overline{y}^1 \mid_0 = y^1$$

we have

$$\frac{\overline{y}^1}{k} = \sinh\left(t + \sinh^{-1}\left(\frac{y^1}{k}\right)\right).$$

Now

$$\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$$

page 178 [66] so

$$\frac{\overline{y}^1}{k} = \sinh t \cosh\left(\sinh^{-1}\left(\frac{y^1}{k}\right)\right) + \sinh\left(\sinh^{-1}\left(\frac{y^1}{k}\right)\right)\cosh t$$

$$= \sinh t \cosh\left(\sinh^{-1}\left(\frac{y^1}{k}\right)\right) + \frac{y^1}{k}\cosh t.$$

Since

$$\cosh z = \sqrt{1 + \sinh^2 z}$$

we have

$$\cosh\left(\sinh^{-1}\left(\frac{y^1}{k}\right)\right) = \sqrt{1 + \frac{y^{1^2}}{k^2}} = \frac{\mid \underline{y} \mid}{k}.$$

Therefore

$$\frac{\overline{y}^1}{k} = \frac{\mid \underline{y} \mid}{k} + \frac{y^1}{k}\cosh t$$

so

$$\overline{y}^1 = \mid \underline{y} \mid \sinh t + y^1 \cosh t. \tag{2.85}$$

Equations (2.84) and (2.85) give the finite form of the Lorentz transformation (2.77) in light cone coordinates. Using this method the integrated forms of the remaining infinitesimal Lorentz transformations can be expressed in terms of light cone coordinates [62].

Appendix 8

In this Appendix we shall make repeated use of the relations (2.37) and (2.38) as well as

$$p_1 = -\frac{(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2)}{2\overline{y}^1} \quad \text{and} \quad p_0 = \frac{\underline{y} + (mc)^2}{2\overline{y}^1}.$$

Integral curves start at $\underline{y}_0 = (\overline{y}_0^1, \overline{y}_0^2, \overline{y}_0^3)$ where $\overline{y}_0^1 \in \mathbb{R}^+$ and $\overline{y}^2$ and $\overline{y}^3 \in \mathbb{R}$. If we express $\overline{J}^{32}$ with respect to light front momentum variables we obtain

$$\overline{J}^{32} = \overline{y}^3 \left(\overline{\pi}_2 + \frac{2\overline{y}^1\overline{y}^2}{\underline{y}^2 + (mc)^2}\overline{\pi}_1 \right) - \overline{y}_2 \left(\overline{\pi}_3 + \frac{2\overline{y}^1\overline{y}^3}{\underline{y}^2 + (mc)^2}\overline{\pi}_1 \right)$$
$$= \overline{y}^3\overline{\pi}_2 - \overline{y}_2\overline{\pi}_3.$$

This can be quantised to give a self-adjoint operator since

$$\overline{y}^3\frac{\partial}{\partial\overline{y}^2} - \overline{y}^2\frac{\partial}{\partial\overline{y}^3}$$

is complete on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. To see this we must solve

$$\frac{d\overline{y}^2}{dt} = \overline{y}^3, \quad \frac{d\overline{y}^3}{dt} = -\overline{y}^2, \quad \frac{d\overline{y}^1}{dt} = 0.$$

The solutions are

$$\overline{y}^1 = \overline{y}_0^1, \;\; \overline{y}^2 = (\overline{y}_0^{3^2} + \overline{y}_0^{2^2})^{\frac{1}{2}} \cos(t+f), \;\; \overline{y}^3 = -(\overline{y}_0^{3^2} + \overline{y}_0^{2^2})^{\frac{1}{2}} \sin(t+f)$$

where

$$\cos f = \frac{\overline{y}_0^2}{(\overline{y}_0^{3^2} + \overline{y}_0^{2^2})^{\frac{1}{2}}}, \quad \sin f = -\frac{\overline{y}_0^3}{(\overline{y}_0^{3^2} + \overline{y}_0^{2^2})^{\frac{1}{2}}}$$

and these are well defined for all t. The self-adjoint operator representing $\overline{J}^{32}$ is

$$\overline{J}^{32}_{\Pi} = -i\hbar \left(\overline{y}^3\frac{\partial}{\partial\overline{y}^2} - \overline{y}^2\frac{\partial}{\partial\overline{y}^3} \right).$$

Next we have

$$\overline{J}^{13} = q^1p_3 - p_1q^3 = \frac{(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2)}{2\overline{y}^1}\overline{\pi}_3 - \overline{y}^3\overline{\pi}_1.$$

This observable generates a complete vector field. To see this we must solve

$$\frac{d\overline{y}^3}{dt} = \frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1}, \quad \frac{d\overline{y}^1}{dt} = -\overline{y}^3, \quad \frac{d\overline{y}^2}{dt} = 0.$$

The solutions are

$$\overline{y}^1 = (p^{1^2}(\underline{y}_0) + \overline{y}_0^{3^2})^{\frac{1}{2}} \sin(t+f) + p^0(\underline{y}_0), \quad \overline{y}^2 = \overline{y}_0^2, \quad \overline{y}^3 = -(p^{1^2}(\underline{y}_0) + \overline{y}_0^{3^2})^{\frac{1}{2}} \cos(t+f)$$

where

$$\cos f = -\frac{\overline{y}_0^3}{(p^{1^2}(\underline{\overline{y}}_0) + \overline{y}_0^3)^{\frac{1}{2}}}$$

and

$$\sin f = \frac{\overline{y}_0^1 - p^0(\underline{\overline{y}}_0)}{(p^{1^2}(\underline{\overline{y}}_0) + \overline{y}_0^3)^{\frac{1}{2}}} = \frac{p^1(\underline{\overline{y}}_0)}{(p^{1^2}(\underline{\overline{y}}_0) + \overline{y}_0^3)^{\frac{1}{2}}}.$$

Here we have used $p^1(\underline{\overline{y}}_0) + p^0(\underline{\overline{y}}_0) = \overline{y}_0^1$. The first expression for $\sin f$ is the easiest to use for showing that the solutions satisfy the initial conditions whilst the second makes it obvious that $\cos^2 f + \sin^2 f = 1$ so that a suitable f exists. Notice that $\overline{y}^1 > 0$ for all t because $p^0(\underline{\overline{y}}_0)$ is positive and $> (p^{1^2}(\underline{\overline{y}})^2 + \overline{y}_0^{3^2})^{\frac{1}{2}}$. The self-adjoint operator representing $\overline{J}^{13}$ is

$$\overline{J}^{13}_{\Pi} = -i\hbar\left(\frac{(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2)}{2\overline{y}^1}\frac{\partial}{\partial\overline{y}^3} - \frac{\overline{y}^3}{2\overline{y}^1} - \overline{y}^3\frac{\partial}{\partial\overline{y}^1}\right).$$

Next consider

$$\overline{J}^{20} = -\overline{y}^2\overline{\pi}_1 - \frac{\underline{\overline{y}}^2 + (mc)^2}{2\overline{y}^1}\overline{\pi}_2.$$

This observable also generates a complete vector field. To see this we must solve

$$\frac{d\overline{y}^1}{dt} = -\overline{y}^2, \quad \frac{d\overline{y}^2}{dt} = -\frac{(\underline{\overline{y}}^2 + (mc)^2)}{2\overline{y}^1}, \quad \frac{d\overline{y}^3}{dt} = 0.$$

The solutions are

$$\overline{y}^1 = (p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{2^2})^{\frac{1}{2}}\cosh(t+f) + p^1(\underline{\overline{y}}_0), \quad \overline{y}^2 = -(p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{2^2})^{\frac{1}{2}}\sinh(t+f), \quad \overline{y}^3 = \overline{y}_0^3$$

where

$$\cosh f = \frac{\overline{y}_0^1 - p^1(\underline{\overline{y}}_0)}{(p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{2^2})^{\frac{1}{2}}} = \frac{p^0(\underline{\overline{y}}_0)}{(p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{2^2})^{\frac{1}{2}}}$$

and

$$\sinh f = -\frac{\overline{y}_0^2}{(p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{2^2})^{\frac{1}{2}}}.$$

The second expression for $\cosh f$ makes it easy to see that $\cosh^2 f - \sinh^2 f = 1$. Notice that we have $\overline{y}^1 > 0$ for all t since, dropping the argument $\underline{\overline{y}}_0$, we have $\cosh(t+f) > 0$ (in fact > 1) so $(p^{1^2} + p^{2^2} + (mc)^2)^{\frac{1}{2}}\cosh(t+f) + p^1 > 0$. The self-adjoint operator representing $\overline{J}^{20}$ is

$$\overline{J}^{20}_{\Pi} = i\hbar\left(\overline{y}^2\frac{\partial}{\partial\overline{y}^1} + \frac{(\underline{\overline{y}}^2 + (mc)^2)}{2\overline{y}^1}\frac{\partial}{\partial\overline{y}^2} + \frac{\overline{y}^2}{2\overline{y}^1}\right).$$

Next Consider

$$\overline{J}^{10} = -\overline{y}^1\overline{\pi}_1.$$

This generates a complete vector field. To see this we must solve

$$\frac{d\overline{y}^1}{dt} = -\overline{y}^1, \quad \frac{d\overline{y}^2}{dt} = 0, \quad \frac{d\overline{y}^3}{dt} = 0.$$

The solutions are

$$\overline{y}^1 = \overline{y}_0^1 \exp(-t), \quad \overline{y}^2 = \overline{y}_0^2, \quad \overline{y}^3 = \overline{y}_0^3.$$

These are well defined for all t and clearly $\overline{y}^1$ is positive for all t. The self-adjoint operator representing $\overline{J}^{10}$ is

$$\overline{J}^{10}_{\overline{\Pi}} = i\hbar \left(\overline{y}^1 \frac{\partial}{\partial \overline{y}^1} + \frac{1}{2} \right).$$

Next consider

$$\overline{J}^{30} = -\frac{(\overline{y}^2 + (mc)^2)}{2\overline{y}^1} \overline{\pi}_3 - \overline{y}^3 \overline{\pi}_1.$$

This classical observable also generates a complete vector field. The equations for the integral curve are

$$\frac{d\overline{y}^3}{dt} = -\frac{(\overline{y}^2 + (mc)^2)}{2\overline{y}^1}, \quad \frac{d\overline{y}^1}{dt} = -\overline{y}^3, \quad \frac{d\overline{y}^2}{dt} = 0.$$

The solutions are given by

$$\overline{y}^1 = (p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f) + p^1(\underline{\overline{y}}_0), \quad \overline{y}^2 = \overline{y}_0^2$$

$$\overline{y}^3 = -(p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{3^2})^{\frac{1}{2}} \sinh(t+f) \tag{2.86}$$

where

$$\cosh f = \frac{\overline{y}_0^1 - p^1(\underline{\overline{y}}_0)}{(p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{3^2})^{\frac{1}{2}}} = \frac{p^0(\underline{\overline{y}}_0)}{(p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{3^2})^{\frac{1}{2}}}$$

and

$$\sinh f = -\frac{\overline{y}_0^3}{(p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{3^2})^{\frac{1}{2}}}.$$

As an example we shall show that these satisfy the differential equations above. The only non trivial case is the first equation in the set. From (2.86) we have

$$\frac{d\overline{y}^3}{dt} = -(p^{0^2}(\underline{\overline{y}}_0) - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f)$$

whereas

$$-\frac{\underline{\overline{y}}^2 + (mc)^2}{2\overline{y}^1} =$$

$$-\frac{(p^{0^2} - \overline{y}_0^{3^2}) \cosh^2(t+f) + 2(p^{0^2} - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f) p^1 + p^{1^2} + y_0^{2^2} + (p^{0^2} - \overline{y}_0^{3^2}) \sinh^2(t+f) + (mc)^2}{2[(p^{0^2} - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f) + p^1]}.$$

Using $\cosh^2 - \sinh^2 = 1$ this becomes

$$-\frac{2(p^{0^2} - \overline{y}_0^{3^2}) \cosh^2(t+f) + 2(p^{0^2} - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f) + p^{1^2} + y_0^{2^2} - (p^{0^2} - \overline{y}_0^{3^2}) + (mc)^2}{2[(p^{0^2} - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f) + p^1]}$$

$$= -\frac{2(p^{0^2} - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f)[(p^{0^2} - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f) + p^1] + (mc)^2 - (mc)^2}{2[(p^{0^2} - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f) + p^1]}$$

$$= -(p^{0^2} - \overline{y}_0^{3^2})^{\frac{1}{2}} \cosh(t+f)$$

as required. The self-adjoint operator representing $\overline{J}^{30}$ is

$$\overline{J}^{30}_{\overline{\Pi}} = i\hbar \left(\overline{y}^3 \frac{\partial}{\partial \overline{y}^1} + \frac{(\underline{\overline{y}}^2 + (mc)^2)}{2\overline{y}^1} \frac{\partial}{\partial \overline{y}^3} + \frac{\overline{y}^3}{2\overline{y}^1} \right).$$

Finally we have

$$\overline{J}^{21} = \overline{y}^2 \overline{\pi}_1 - \frac{(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2)}{2\overline{y}^1} \overline{\pi}_2.$$

This observable generates a complete vector field. The equations for the integral curves are

$$\overline{y}^1 = (p^{1^2}(\underline{\overline{y}}_0) + \overline{y}_0^{2^2})^{\frac{1}{2}} \sin(t+f) + p^0(\underline{\overline{y}}_0), \quad \overline{y}^2 = (p^{1^2}(\underline{\overline{y}}_0) + \overline{y}_0^{2^2})^{\frac{1}{2}} \cos(t+f), \quad \overline{y}_0^3 = \overline{y}_0^3$$

where

$$\cos f = \frac{\overline{y}_0^2}{(p^{1^2}(\underline{\overline{y}}_0) + \overline{y}_0^{2^2})^{\frac{1}{2}}}$$

and

$$\sin f = \frac{\overline{y}_0^1 - p^0(\underline{\overline{y}}_0)}{(p^{1^2}(\underline{\overline{y}}_0) + \overline{y}_0^{2^2})^{\frac{1}{2}}} = \frac{p^1(\underline{y}_0)}{(p^{1^2}(\underline{\overline{y}}_0) + \overline{y}_0^{2^2})^{\frac{1}{2}}}.$$

The self-adjoint operator representing $\overline{J}^{21}$ is

$$-i\hbar \left(\overline{y}^2 \frac{\partial}{\partial \overline{y}^1} - \frac{(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2)}{2\overline{y}^1} \frac{\partial}{\partial \overline{y}^2} + \frac{\overline{y}^2}{2\overline{y}^1} \right)$$

Appendix 9

In this Appendix we explain why it is that the front form generators obey the commutation relations (2.49). This is simply a consequence of the fact that we have

$$J^{\sigma\epsilon} = \frac{\partial y^\sigma}{\partial q^\mu} \frac{\partial y^\epsilon}{\partial q^\nu} \overline{J}^{\mu\nu} \quad \text{and} \quad P^\mu = \frac{\partial y^\mu}{\partial q^\nu} \overline{P}^\nu.$$

For example

$$\frac{\partial y^0}{\partial q^\mu} \frac{\partial y^3}{\partial q^\nu} \overline{J}^{\mu\nu} = \overline{J}^{03} + \overline{J}^{13} = p^0 q^3 - q^0 p^3 + p^1 q^3 - q^1 p^3$$

or restricting this to the front

$$\frac{\partial y^0}{\partial q^\mu} \frac{\partial y^3}{\partial q^\nu} \overline{J}^{\mu\nu} = p^0 q^3 + p^1 q^3 = q^3(p^0 + p^1) = J^{03}.$$

Appendix 10

Firstly we have

$$J^{01} = \overline{y}^1 \overline{\pi}_1.$$

J^{01} generates a complete vector field. To see this we must solve

$$\frac{d\overline{y}^1}{dt} = \overline{y}^1, \quad \frac{d\overline{y}^i}{dt} = 0.$$

The solutions are

$$\overline{y}^1 = \overline{y}_0^1 \exp(t) \quad \text{and} \quad \overline{y}^i = \overline{y}_0^i.$$

Therefore the self-adjoint operator representing J^{01} is

$$J_{\Pi}^{01} = -i\hbar \left(\overline{y}^1 \frac{\partial}{\partial \overline{y}^1} + \frac{1}{2} \right).$$

Next consider

$$J^{0s} = \overline{y}^1 \overline{\pi}_s.$$

Again we find that the J^{0s} generate complete vector fields. To see this we must solve the equations

$$\frac{d\overline{y}^1}{dt} = 0, \quad \frac{d\overline{y}^s}{dt} = 0, \quad \frac{d\overline{y}^{5-s}}{dt} = 0.$$

The solutions are

$$\overline{y}^1 = \overline{y}_0^1, \quad \overline{y}^{5-s} = \overline{y}_0^{5-s}, \quad \overline{y}^s = \overline{y}_0^1 t + \overline{y}_0^s$$

and these are well defined for all t. The self-adjoint operators representing the J^{0s} are

$$J_{\Pi}^{0s} = -i\hbar \overline{y}^1 \frac{\partial}{\partial \overline{y}^s}.$$

Consider

$$J^{1s} = -\overline{\pi}_1 \overline{y}^s + \frac{\overline{\pi}_s}{2\overline{y}^1}(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2).$$

This generates a complete vector field. To see this we must solve

$$\frac{d\overline{y}^1}{dt} = -\overline{y}^s, \quad \frac{d\overline{y}^s}{dt} = \frac{1}{2\overline{y}^1}(\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2), \quad \frac{d\overline{y}^{5-s}}{dt} = 0.$$

The solutions are

$$\overline{y}^1 = (p^{1^2}(\underline{y}_0) + \overline{y}_0^{2^2})^{\frac{1}{2}} \sin(t+f) + p^0(\underline{y}_0) \tag{2.87}$$

$$\overline{y}^s = -(p^{1^2}(\underline{y}_0) + \overline{y}_0^{2^2})^{\frac{1}{2}} \cos(t+f), \quad \overline{y}^{5-s} = \overline{y}_0^3$$

where f is such that

$$\cos f = -\frac{\overline{y}_0^s}{(p^{1^2}(\underline{y}_0) + \overline{y}_0^{s^2})^{\frac{1}{2}}} \quad \text{and} \quad \sin f = \frac{p^1(\underline{y}_0)}{(p^{1^2}(\underline{y}_0) + \overline{y}_0^{s^2})^{\frac{1}{2}}}.$$

It is easy to see by differentiating these expressions that they are solutions to the differential equations. They also satisfy the required initial conditions. This is obvious except in the case of (2.87). Put $t = 0$ in (2.87) then

$$\overline{y}^1 = p^1(\underline{\overline{y}}_0) + p^0(\underline{\overline{y}}_0) = \overline{y}_0^1$$

since

$$p^1 + p^0 = -\pi_1.$$

It is also important to notice that (2.87) ensures that $\overline{y}^1 > 0$ for all t. The self-adjoint operator representing J^{1s} is

$$J^{1s}_{\overline{\Pi}} = -i\hbar\left(-\overline{y}^s\frac{\partial}{\partial\overline{y}^1} + \frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1}\frac{\partial}{\partial\overline{y}^s} - \frac{1}{2}\frac{\overline{y}^s}{\overline{y}^1}\right).$$

Consider

$$J^{23} = -\overline{y}^3\overline{\pi}_2 + \overline{y}^2\overline{\pi}_3.$$

This observable also generates a complete vector field. The equations we must solve are

$$\frac{d\overline{y}^2}{dt} = -\overline{y}^3, \quad \frac{d\overline{y}^3}{dt} = \overline{y}^2, \quad \frac{d\overline{y}^1}{dt} = 0.$$

The solutions are given by

$$\overline{y}^2 = -(\overline{y}_0^{3^2} + \overline{y}_0^{3^2})^{\frac{1}{2}}\sin(t+f), \quad \overline{y}^3 = (\overline{y}_0^{3^2} + \overline{y}_0^{3^2})^{\frac{1}{2}}\cos(t+f), \quad \overline{y}^1 = \overline{y}_0^1$$

where

$$\cos f = \frac{\overline{y}_0^3}{(\overline{y}_0^{3^2} + \overline{y}_0^{2^2})^{\frac{1}{2}}} \quad \text{and} \quad \sin f = -\frac{\overline{y}_0^2}{(\overline{y}_0^{3^2} + \overline{y}_0^{2^2})^{\frac{1}{2}}}.$$

The self-adjoint operator representing J^{23} is

$$J^{23}_{\overline{\Pi}} = -i\hbar\left(\overline{y}^2\frac{\partial}{\partial\overline{y}^3} - \overline{y}^3\frac{\partial}{\partial\overline{y}^2}\right).$$

Finally it is obvious that we have

$$P^0_{\overline{\Pi}} = \overline{y}^1, \quad P^2_{\overline{\Pi}} = \overline{y}^2, \quad P^3_{\overline{\Pi}} = \overline{y}^3, \quad P^1_{\overline{\Pi}} = \frac{\overline{y}^{1^2} - \overline{y}^{2^2} - \overline{y}^{3^2} - (mc)^2}{2\overline{y}^1}.$$

Appendix 11

We have

$$Y^1_{\overline{P}}\exp(\pm(\overline{q}^1 + \overline{q}^0))\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}} = \mp i\exp(\pm(\overline{q}^1 + \overline{q}^0))\left(\frac{\overline{q}^0 + \overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}.$$

For example

$$Y_{\frac{1}{P}} \exp(-(\overline{q}^1+\overline{q}^0))\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}$$

$$= -i\hbar\exp(-(\overline{q}^1+\overline{q}^0))\left[\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)\left(\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\frac{(-1)}{\hbar}\left(1+\frac{\overline{q}^1}{\overline{q}^0}\right)+\right.\right.$$

$$\left.\left.\frac{1}{2}\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{-\frac{1}{2}}\frac{(\overline{q}^0(\frac{\overline{q}^1}{\overline{q}^0}+1)-(\overline{q}^0+\overline{q}^1)\frac{\overline{q}^1}{\overline{q}^0})}{\overline{q}^{0^2}}\right)+\frac{\overline{q}^1-\overline{q}^0}{2(\overline{q}^0+\overline{q}^1)\overline{q}^0}\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\right]$$

$$= -i\hbar\exp(-(\overline{q}^1+\overline{q}^0))\left[-\frac{1}{\hbar}\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}+\frac{1}{2}\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{3}{2}}\frac{1}{\overline{q}^{0^3}}(\overline{q}^0\overline{q}^1+\overline{q}^{0^2}-\overline{q}^0\overline{q}^1-\overline{q}^{1^2})+\right.$$

$$\left.\frac{\overline{q}^1-\overline{q}^0}{2(\overline{q}^0+\overline{q}^1)\overline{q}^0}\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\right]$$

$$= -i\hbar\exp(-(\overline{q}^1+\overline{q}^0))\left[-\frac{1}{\hbar}\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}+\frac{1}{2}\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{\frac{3}{2}}\frac{\overline{q}^{0^2}-\overline{q}^{1^2}}{\overline{q}^{0^3}}+\frac{\overline{q}^1-\overline{q}^0}{2(\overline{q}^0+\overline{q}^1)\overline{q}^0}\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\right]$$

$$= -i\hbar\exp(-(\overline{q}^1+\overline{q}^0))\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\left[-\frac{1}{\hbar}+\frac{1}{2}\left(\frac{\overline{q}^0}{\overline{q}^0+\overline{q}^1}\right)^{2}\frac{\overline{q}^{0^2}-\overline{q}^{1^2}}{\overline{q}^{0^3}}+\frac{\overline{q}^1-\overline{q}^0}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}\right]$$

$$= -i\hbar\exp(-(\overline{q}^1+\overline{q}^0))\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}\left[-\frac{1}{\hbar}+\frac{\overline{q}^0-\overline{q}^1}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}+\frac{\overline{q}^1-\overline{q}^0}{2\overline{q}^0(\overline{q}^0+\overline{q}^1)}\right]$$

$$= i\exp(-(\overline{q}^1+\overline{q}^0))\left(\frac{\overline{q}^0+\overline{q}^1}{\overline{q}^0}\right)^{\frac{1}{2}}$$

as required.

Appendix 12: Proof of Orthogonality of Generalised Eigenvectors of Π_1 in $\mathcal{H}_\Pi$.

Consider

$$< G(\lambda, y, \Pi_1) \mid \mathrm{G}(\lambda', y, \Pi_1) >_{\mathcal{H}_\Pi} = \int_0^\infty y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}$$

$$\exp\left(\frac{i}{\hbar}\lambda\sigma(y)\right) y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp\left(-\frac{i}{\hbar}\lambda'\sigma(y)\right)\frac{1}{y}dy$$

$$= \int_0^\infty [2\pi\hbar\eta(y)]^{-1}\exp\left(\frac{i}{\hbar}\sigma(y)(\lambda-\lambda')\right)dy.$$

Make the substitution $x = \sigma(y)$ then the above becomes

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\exp\left(\frac{i}{\hbar}x(\lambda-\lambda')\right)dx = \frac{1}{\hbar}\delta\left(\frac{\lambda-\lambda'}{\hbar}\right) = \delta(\lambda-\lambda')$$

as required.

Appendix 13: Proof of Completeness of Generalised Eigenvectors of Π_1.

We have

$$\int_{-\infty}^{\infty} < G(\lambda, y, \Pi_1) \mid G(\lambda, y', \Pi_1) > d\lambda = \int_{-\infty}^{\infty} y^{1/2}[2\pi\hbar\eta(y)]^{-1/2} \exp\left(\frac{i}{\hbar}\lambda\sigma(y)\right) y'^{1/2}[2\pi\hbar\eta(y')]^{-1/2}$$

$$\exp\left(-\frac{i}{\hbar}\lambda\sigma(y')\right) d\lambda$$

$$= y^{1/2}y'^{1/2}[\hbar\eta(y)]^{-1/2}[\hbar\eta(y')]^{-1/2}\delta\left(\frac{\sigma(y)-\sigma(y')}{\hbar}\right)$$

$$y^{1/2}y'^{1/2}[\hbar\eta(y)]^{-1/2}[\hbar\eta(y')]^{-1/2}\frac{1}{\mid \frac{1}{\hbar}\frac{d\sigma(y)}{dy} \mid}\delta(y-y')$$

$$= y^{1/2}y'^{1/2}[\hbar\eta(y)]^{-1/2}[\hbar\eta(y')]^{-1/2}\eta(y)\hbar\delta(y-y')$$

$$= y\delta(y-y')$$

as required.

Appendix 14: Derivation of Generalised Eigenfunctions of $\mid \Pi_1 \mid$.

We have

$$(\mid \Pi_1 \mid \phi)(y) = \int_{-\infty}^{\infty} \mid \lambda \mid \mathrm{y}^{1/2}[2\pi\hbar\eta(y)]^{-1/2} \exp(i\lambda\sigma(y))\tilde{\phi}(\lambda)d\lambda \qquad (2.88)$$

where

$$\phi(\lambda) = \int_0^{\infty} \frac{1}{y'} y'^{1/2}[2\pi\hbar\eta(y')]^{-1/2} \exp(i\lambda\sigma(y'))\phi(y')dy'. \qquad (2.89)$$

Substituting for $\tilde{\phi}(\lambda)$ from (2.89) and dividing the range of integration in (2.88) into $(-\infty, 0)$ and $(0, \infty)$ we obtain

$$(|\Pi_1|\phi)(y) = \int_{-\infty}^{0}(-\lambda)y^{1/2}[2\pi\hbar\eta(y)]^{-\frac{1}{2}}\exp(i\lambda\sigma(y))\int_0^{\infty}\frac{1}{y'}y'^{1/2}[2\pi\hbar\eta(y')]^{-\frac{1}{2}}\exp(i\lambda\sigma(y'))\phi(y')dy'd\lambda$$

$$+\int_0^{\infty}\lambda y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\lambda\sigma(y))\int_0^{\infty}\frac{1}{y'}y'^{1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(i\lambda\sigma(y'))\phi(y')dy'd\lambda.$$

Making the change of variable $\lambda \to -\lambda$ in the first integral gives

$$(|\Pi_1|\,\phi)(y) = -\int_{\infty}^{0}\lambda y^{1/2}[2\pi\hbar\eta(y)]^{-\frac{1}{2}}\exp(-i\lambda\sigma(y))\int_0^{\infty}\frac{1}{y'}y'^{1/2}[2\pi\hbar\eta(y')]^{-\frac{1}{2}}\exp(-i\lambda\sigma(y'))\phi(y')dy'd\lambda$$

$$+\int_0^{\infty}\lambda y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\lambda\sigma(y))\int_0^{\infty}\frac{1}{y'}y'^{1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(i\lambda\sigma(y'))\phi(y')dy'd\lambda$$

$$= \int_0^{\infty}\sum_{\nu=\pm1}\lambda \mathrm{y}^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\nu\lambda\sigma(y))\int_0^{\infty}\frac{1}{y'}y'^{1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(i\nu\lambda\sigma(y'))\phi(y')dy'd\lambda$$

and therefore

$$G(\lambda, \nu, y, \mid \Pi_1 \mid) = y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\nu\lambda\sigma(y))$$

as required.

Appendix 15: Proof of Completeness of Generalised Eigenfunctions of $|\Pi_1|$.

We have

$$\int_0^\infty \sum_\nu G(w,\nu,y,|\Pi_1|)G(w,\nu,y',|\Pi_1|)dw$$

$$= \int_0^\infty \sum_\nu y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\nu w\sigma(y))y'^{1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(-i\nu w\sigma(y'))dw$$

$$= \int_0^\infty \sum_\nu y^{1/2}y'^{1/2}[2\pi\hbar\eta(y)]^{-1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(i\nu w(\sigma(y)-\sigma(y'))dw$$

$$= \int_0^\infty y^{1/2}y'^{1/2}[2\pi\hbar\eta(y)]^{-1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(-iw(\sigma(y)-\sigma(y'))dw$$

$$+ \int_0^\infty y^{1/2}y'^{1/2}[2\pi\hbar\eta(y)]^{-1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(iw(\sigma(y)-\sigma(y'))dw.$$

Make the substitution $w \to -w$ in the first integral and the above becomes

$$\int_{-\infty}^0 y^{1/2}y'^{1/2}[2\pi\hbar\eta(y)]^{-1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(iw(\sigma(y)-\sigma(y'))dw$$

$$+ \int_0^\infty y^{1/2}y'^{1/2}[2\pi\hbar\eta(y)]^{-1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(iw(\sigma(y)-\sigma(y'))dw$$

$$= \int_{-\infty}^\infty y^{1/2}y'^{1/2}[2\pi\hbar\eta(y)]^{-1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(iw(\sigma(y)-\sigma(y'))dw$$

$$= y\delta(y-y')$$

as required.

Appendix 16: Proof of Orthogonality of Generalised Eigenfunctions of $|\Pi_1|$.

$$< G(w,\nu,y,|\Pi_1|)\mid G(w',\nu',y,|\Pi_1|) >_{\mathcal{H}_\Pi} =$$

$$\int_0^\infty \frac{y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\nu w\sigma(y)y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(-i\nu' w'\sigma(y)}{y}dy$$

$$= \int_0^\infty \frac{1}{[2\pi\hbar\eta(y)]}\exp(i(\nu' w' - \nu w)\sigma(y))dy.$$

Let $x = \sigma(\mathrm{y})$ and the above becomes

$$\frac{1}{2\pi\hbar}\int_{-\infty}^\infty \exp(i(\nu' w' - \nu w)x)dx = \delta(\nu' w' - \nu w).$$

A little thought shows that since w and $w' > 0$ and ν and ν' take only the values ± 1 we have $\delta(\nu'\mu' - \nu\mu) = \delta_{\nu\nu'}\delta(w - w')$ as required.

Appendix 17: Derivation of Generalised Eigenfunctions of H_{Π}.

We have

$$\phi_\tau(y) = \int_{-\infty}^{\infty} y^{1/2}[2\pi\hbar\eta(y)]^{-1/2} \exp\left(\frac{-ic\tau}{2\mid\lambda\mid}(\mid\lambda\mid^2 + (mc)^2\right) \tilde{\phi}_0(\lambda)\exp(-i\lambda\sigma(y))d\lambda$$

where

$$\tilde{\phi}_0(\lambda) = \int_0^{\infty} y'^{-1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(-i\lambda\sigma(y'))\phi_0(y')dy'.$$

We split up the first integral as

$$\int_{-\infty}^{-mc} ...d\lambda + \int_{-mc}^{0} ...d\lambda + \int_{0}^{mc} ...d\lambda + \int_{mc}^{\infty} ...d\lambda.$$

In the first integral make the substitution

$$\lambda = -\frac{w}{c} - \sqrt{\left(\frac{w}{c}\right)^2 - (mc)^2}$$

with λ restricted to $(-\infty, -mc)$. This map is bijective (on the range of integration) and has inverse

$$w = \frac{c}{2}\left(-\lambda - \frac{(mc)^2}{\lambda}\right).$$

The Jacobian is

$$d\lambda = -\frac{dw}{\left|\frac{c}{2}\left(1 - \frac{(mc)^2}{(-\frac{w}{c} - \sqrt{(\frac{w}{c})^2 - (mc)^2})^2}\right)\right|^{1/2}}.$$

Since $\lambda = -\infty \Rightarrow w = \infty$ and $\lambda = -mc \Rightarrow w = mc^2$ we obtain

$$\int_{-\infty}^{-mc} ...d\lambda = \int_{mc^2}^{\infty} y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\tau w)\exp\left(i\sigma(y)\left[-\frac{w}{c} - \sqrt{\left(\frac{w}{c}\right)^2 - (mc)^2}\right]\right)$$

$$\times\left(\int_0^{\infty} y'^{-1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp\left(-i\sigma(y')\left[-\frac{w}{c} - \sqrt{\left(\frac{w}{c}\right)^2 - (mc)^2}\right]\right)\phi_0(y')dy'\right)$$

$$\frac{dw}{\left|\frac{c}{2}\left(1 - \frac{(mc)^2}{(-\frac{w}{c} - \sqrt{(\frac{w}{c})^2 - (mc)^2})^2}\right)\right|^{1/2}}$$

$$= \int_{mc^2}^{\infty} \frac{y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\sigma(y)[-\frac{w}{c} - \sqrt{(\frac{w}{c})^2 - (mc)^2}])}{\left|\frac{c}{2}\left(1 - \frac{(mc)^2}{(-\frac{w}{c} - \sqrt{(\frac{w}{c})^2 - (mc)^2})^2}\right)\right|^{1/2}}$$

$$\times \int_0^{\infty} \frac{y'^{1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(i\sigma(y')[-\frac{w}{c} - \sqrt{(\frac{w}{c})^2 - (mc)^2}])}{\left|\frac{c}{2}\left(1 - \frac{(mc)^2}{(-\frac{w}{c} - \sqrt{(\frac{w}{c})^2 - (mc)^2})^2}\right)\right|^{1/2}} \frac{1}{y'}\phi_0(y')dy'\exp(-i\tau w)dw.$$

Similarly in the second integral we make the substitution

$$\lambda = -\frac{w}{c} + \sqrt{\left(\frac{w}{c}\right)^2 - (mc)^2}$$

then

$$d\lambda = \frac{dw}{\left|\frac{c}{2}\left(1 - \frac{(mc)^2}{(-\frac{w}{c}+\sqrt{(\frac{w}{c})^2-(mc)^2})^2}\right)\right|^{1/2}}.$$

Now $\lambda = -mc \Rightarrow w = mc^2$, $\lambda = 0 \Rightarrow w = \infty$ so

$$\int_{-mc}^{0} ...d\lambda = \int_{mc^2}^{\infty} y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\tau w)\exp\left(i\sigma(y)\left[-\frac{w}{c}+\sqrt{\left(\frac{w}{c}\right)^2+(mc)^2}\right]\right)$$

$$\times\left(\int_0^\infty y'^{-1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp\left(-i\sigma(y')\left[-\frac{w}{c}+\sqrt{(\frac{w}{c})^2-(mc)^2}\right]\right)\phi_0(y')dy'\right)$$

$$\frac{dw}{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(-\frac{w}{c}+\sqrt{(\frac{w}{c})^2+(mc)^2})^2}\right)\right|}$$

$$= \int_{mc^2}^{\infty} \frac{y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\sigma(y)[-\frac{w}{c}+\sqrt{(\frac{w}{c})^2-(mc)^2}])}{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(-\frac{w}{c}+\sqrt{(\frac{w}{c})^2-(mc)^2})^2}\right)\right|^{1/2}}$$

$$\times\int_0^\infty \frac{y'^{1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp(i\sigma(y')[-\frac{w}{c}+\sqrt{(\frac{w}{c})^2-(mc)^2}])}{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(-\frac{w}{c}+\sqrt{(\frac{w}{c})^2-(mc)^2})^2}\right)\right|^{1/2}}\frac{1}{y'}\phi_0(y')dy'\exp(-i\tau \mathrm{w})dw$$

continuing in this way we obtain (2.60).

Appendix 18: Proof of the Completeness of the Generalised Eigenvectors of H_Π.

$$\int_{mc^2}^{\infty}\sum_{\nu\mu} G(\lambda,\nu,\mu,y',H_\Pi^1)G(\lambda,\nu,\mu,y,H_\Pi^1) =$$

$$\int_{mc^2}^{\infty}\sum_{\nu\mu} y'^{1/2}[2\pi\hbar\eta(y')]^{-1/2}\exp\left(\frac{i}{\hbar}\left(\frac{\mu w}{c}+\nu\sqrt{\left(\frac{w}{c}\right)^2-(mc)^2}\right)\sigma(y')\right)y^{1/2}$$

$$\times[2\pi\hbar\eta(y)]^{-1/2}\exp\left(-\frac{i}{\hbar}\left(\frac{\mu w}{c}+\nu\sqrt{(\frac{w}{c})^2-(mc)^2}\right)\sigma(y)\right)\frac{dw}{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(\frac{\mu w}{c}-\nu\sqrt{(\frac{w}{c})^2-(mc)^2})^2}\right)\right|}$$

$$= \int_{mc^2}^{\infty}\sum_{\nu\mu}\frac{\exp(\frac{i}{\hbar}(\frac{\mu w}{c}+\nu\sqrt{(\frac{w}{c})^2-(mc)^2})(\sigma(y')-\sigma(y)))y^{1/2}y'^{1/2}[2\pi\hbar\eta(y)]^{-1/2}[2\pi\hbar\eta(y')]^{-1/2}}{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(\frac{\mu w}{c}-\nu\sqrt{(\frac{w}{c})^2-(mc)^2})^2}\right)\right|}dw.$$

Make the substitution $Q = \frac{1}{\hbar}(\frac{\mu w}{c}+\nu\sqrt{(\frac{w}{c})^2-(mc)^2})$

$$= y^{\frac{1}{2}}y'^{\frac{1}{2}}[2\pi\eta(y)]^{-\frac{1}{2}}[2\pi\eta(y')]^{-\frac{1}{2}}(-\int_{-mc}^{-\infty}\exp[iQ(\sigma(y')-\sigma(y))]dQ+\int_{-mc}^{0}\exp[iQ(\sigma(y')-\sigma(y))]dQ$$

$$-\int_{mc}^{0}\exp[iQ(\sigma(y')-\sigma(y))]dQ+\int_{mc}^{\infty}\exp[iQ(\sigma(y')-\sigma(y))]dQ)$$
$$=y^{1/2}y'^{1/2}[\eta(y)]^{-1/2}[\eta(y')]^{-1/2}\delta(\sigma(y')-\sigma(y))=y\delta(y-y')$$

as required.

Appendix 19: Proof of the Orthogonality of Generalised Eigenvectors of H_Π.

Consider

$$\int_0^\infty \frac{[2\pi\hbar\eta(y)]\exp\left(i\sigma(y)(\frac{\mu w}{c}+\nu\sqrt{(\frac{w}{c})^2-(mc)^2}-(\frac{\bar\mu\bar w}{c}+\bar\nu\sqrt{(\frac{\bar w}{c})^2-(mc)^2}))\right)}{\sqrt{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(\frac{\mu w}{c}-\nu\sqrt{(\frac{w}{c})^2-(mc)^2})^2}\right)\right|}\sqrt{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(\frac{\bar\mu\bar w}{c}-\bar\nu\sqrt{(\frac{\bar w}{c})^2-(mc)^2})^2}\right)\right|}}dy.$$

Let $x=\sigma(\mathrm{y})$ then

$$=\frac{\delta(\frac{\mu w}{c}+\nu\sqrt{(\frac{w}{c})^2-(mc)^2}-(\frac{\bar\mu\bar w}{c}+\bar\nu\sqrt{(\frac{\bar w}{c})^2-(mc)^2}))}{\sqrt{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(\frac{\mu w}{c}-\nu\sqrt{(\frac{w}{c})^2-(mc)^2})^2}\right)\right|}\sqrt{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(\frac{\bar\mu\bar w}{c}-\bar\nu\sqrt{(\frac{\bar w}{c})^2-(mc)^2})^2}\right)\right|}}.$$

A case by case analysis shows that the only non zero contributions come from

$$=\frac{\delta(\frac{\mu w}{c}+\nu\sqrt{(\frac{w}{c})^2-(mc)^2}-(\frac{\bar\mu\bar w}{c}+\bar\nu\sqrt{(\frac{\bar w}{c})^2-(mc)^2}))\delta_{\nu\bar\nu}\delta_{\mu\bar\mu}}{\sqrt{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(\frac{\mu w}{c}-\nu\sqrt{(\frac{w}{c})^2-(mc)^2})^2}\right)\right|}\sqrt{\left|\frac{c}{2}\left(1-\frac{(mc)^2}{(\frac{\bar\mu\bar w}{c}-\bar\nu\sqrt{(\frac{\bar w}{c})^2-(mc)^2})^2}\right)\right|}}$$

and the result follows by Landau and Lifschitz (vol 3 page15).

Appendix 20: A State Vector Localised in Λ_0 is a Scattering State

Consider

$$\lim_{\tau\to\infty}\int_0^\infty\frac{|\chi_{[a,b]}(y)|^2|\phi_\Pi^\tau(y)|^2}{y}dy.$$

Put

$$f_\tau(y)=\frac{|\chi_{[a,b]}(y)|^2|\phi_\Pi^\tau(y)|^2}{y}$$

$$=\frac{|\chi_{[a,b]}(y)|^2|\int_{-\infty}^{\infty}y^{1/2}[2\pi\hbar\eta(y)]^{-1/2}\exp(i\lambda\sigma(y))\exp(-\frac{ic\tau}{|\lambda|}(\lambda^2+(mc)^2)\tilde\phi_\Pi^0(\lambda)d\lambda|^2}{y}$$

$$\le|\chi_{[a,b]}(y)|^2\,2\pi\mathrm{h}\eta^{-1}(y)(\int_{-\infty}^{\infty}|\tilde\phi_\Pi^0(\lambda)|\,\mathrm{d}\lambda)^2. \qquad (2.90)$$

Now $\tilde\phi_\Pi^0(\lambda)$ is integrable (see proof of spreading) so (2.90) becomes

$$\frac{|\chi_{[a,b]}(y)|^2 M}{\eta(y)}=L_\tau(y)\quad\text{say.} \qquad (2.91)$$

Then

$$\int_0^\infty L_\tau(y)dy\le\int_a^b\frac{1}{\eta(y)}dy<\infty\quad\text{if}\quad a>0. \qquad (2.92)$$

From (2.91) and (2.92) we see that f_τ is bounded above by an integrable function v.i.z $L_\tau(y)$. Now

$$\lim_{\tau\rightarrow} \phi_\Pi^\tau(y) = 0.$$

To see this consider (2.60) and note that the Riemann Lebesque lemma applies since

$$\int_{mc^2}^{\infty} \sum_{\mu\nu} G(\mu,\nu,w,H_\Pi,y) < \phi_\Pi^0(y) \mid \mathrm{G}(\mu,\nu,w,H_\Pi,y') >_{\mathcal{H}_\Pi} dw = \phi_\Pi^o(y).$$

Hence we have

$$\lim_{\tau\rightarrow\infty} f_\tau(y) = 0.$$

Therfore by Lebesgue dominated convergence

$$\lim_{\tau\rightarrow\infty} \int_0^\infty f_\tau(y)dy = \int_0^\infty \lim_{\tau\rightarrow\infty} f_\tau(y)dy = 0.$$

Chapter 3

Field Theories

The idleness I love is not that of an indolent fellow who stands with folded arms...and thinks as little as he acts. It is the idleness of a child who is incessantly on the move without ever doing anything...I love to busy myself about trifles, to begin a hundred things and not finish one of them...eagerly to begin on a ten-years task and to give it up after ten minutes.

JEAN-JACQUES ROUSSEAU; *The Confessions.*

In this chapter we shall explore the possibility of constructing light cone and light front field theories although for the light cone the canonical approach to a second quantised theory will prove impossible to carry out. We shall concentrate for the most part on developing a field theory for the free neutral scalar boson and merely indicate how we might construct a field theory describing free fermions. We show that the field theory is relativistically invariant and naturally related to the light front quantun mechanics described in chapter 2.

3.1 Spin Zero Particles

3.1.1 The Scalar Particle in Light Cone Coordinates in 3+1 Spacetime

It may be thought that the formulation of a quantum field theory in light cone coordinates should be relatively straightforward. In this section we shall see this is not the case.

A free scalar particle is described by the Klein-Gordon equation which in an arbitrary coordinate system is

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial \mathrm{x}^i}\left(\sqrt{g}g^{ij}\frac{\partial \phi}{\partial \mathrm{x}^j}\right)+\mu^2\phi=0$$

where $\mu = mc$ ([52] page 304). Derrick has given the expression for the Klein-Gordon equation in light cone coordinates [58]. The corresponding Lagrangian, which is absent from Derrick's work, is

$$\mathcal{L} = -\frac{1}{2}\left[\sum_k \left(\frac{2y^k}{|\underline{y}|}\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y^k} - \frac{\nu_\lambda y^\lambda}{|\underline{y}|}\left(\frac{\partial\phi}{\partial y^k}\right)^2\right) - \sum_k \frac{2y^k}{|\underline{y}|}\frac{\partial\phi}{\partial y^k}\left(\nu^i\frac{\partial\phi}{\partial y^i}\right) - \frac{\nu_\lambda y^\lambda}{|\underline{y}|}\mu^2\phi^2\right] \tag{3.1}$$

(Appendix 1) where ν^μ are the components of the light cone observer's four velocity. The difficulty in formulating a field theory based on the light cone is due to the explicit dependence of the Lagrangian (3.1) on the spacetime coordinates.

3.1.2 The Scalar Particle in Light Front Coordinates in 1+1 Spacetime

We can deduce the form of the Lagrangian for the Klein-Gordon equation in light front coordinates from (3.1). We need only observe that in a 2-dimensional spacetime the light front coordinates are identical to light cone coordinates adapted to an observer at rest at the origin and restricted to that region of spacetime corresponding to $y > 0$. Therefore the light front Klein-Gordon Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\left[2\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y} + \left(\frac{\partial\phi}{\partial y}\right)^2 + \mu^2\phi^2\right]. \tag{3.2}$$

Since the Lagrangian does not have an explicit spacetime dependence it should be possible to derive a field theory from it. The Euler-Lagrange equation is

$$\frac{\partial\frac{\partial L}{\partial\frac{\partial\phi}{\partial y}}}{\partial y} + \frac{\partial\frac{\partial L}{\partial\frac{\partial\phi}{\partial\tau}}}{\partial\tau} - \frac{\partial L}{\partial\phi} = 0$$

i.e.

$$-2\frac{\partial^2\phi}{\partial y\partial\tau} - \frac{\partial^2\phi}{\partial y^2} + \mu^2\phi = 0. \tag{3.3}$$

Using the results of Appendix 2 we can show that the light front momenta are given in terms[1] of the field by the following expressions

$$P^\tau = \frac{1}{2}\int T^{\tau\tau}dy = \frac{1}{2}\int\left(\frac{\partial\phi}{\partial y}\right)^2 dy \tag{3.4}$$

$$P^y = \frac{1}{2}\int T^{\tau y}dy = -\frac{1}{2}\frac{1}{2}\int\left(\mu^2\phi^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right)dy \tag{3.5}$$

([47] 2.45 page 69). The variable π conjugate to ϕ is given by

$$\pi = n_\mu \pi^1_\mu$$

[1] In this chapter we write P^τ and $g^{y^1y^2}$ to denote P^0 and g^{12} etc. This enables us to reinforce the distinction between Minkowski and light front coordinates.

([46] 2.61) where

$$\pi^1_\mu = \frac{\partial L}{\partial \frac{\partial \phi}{\partial y^\mu}}$$

(see Appendix 2) and n_μ is normal to the light front. In light front coordinates this becomes

$$\pi = \pi^1_\tau = -\frac{\partial \phi}{\partial y}. \tag{3.6}$$

We could pass to the quantum theory by imposing the canonical equal τ commutation relations

$$[\phi(y,\tau), \pi(y',\tau)] = [\frac{\partial \phi}{\partial y'}(y',\tau), \phi(y,\tau)] = i\delta(y'-y) \tag{3.7}$$

with all other commutators vanishing[2]. There is no guarantee that imposing these commutation relations will lead to a relativistically invariant field theory. We must demonstrate that the conserved quantities as given by (3.4) and (3.5), with the canonical commutation relation (3.7), satisfy

$$[P^\mu, \phi] = -i\partial^\mu \phi \tag{3.8}$$

and

$$[J^{\mu\nu}, \phi] = -i(y^\mu \partial^\nu - y^\nu \partial^\mu)\phi \tag{3.9}$$

([57] page 367)[3]. Unfortunately it appears that canonical quantisation fails us in this instance. We have

$$[P^\tau, \phi] = [\frac{1}{2}\int \left(\frac{\partial \phi}{\partial y}\right)^2 (\tau, y)dy, \phi(\tau, y')]$$

$$= -\frac{1}{2}\int dy[\phi(\tau,y'), \left(\frac{\partial \phi}{\partial y}\right)^2 (\tau, y)].$$

If $[B, A]$ is a c number then

$$[B, A^2] = 2[B, A]A. \tag{3.10}$$

[2]In this chapter we take $\hbar = 1$

[3]For example here it is shown that a necessary condition for a field theory to be relativistically invariant is that the generators for the t=constant hypersurfaces satisfy

$$[\overline{P}^\mu, \phi] = -i\eta^{\mu\nu}\frac{\partial \phi}{\partial x^\nu} = -i\eta^{\mu\nu}\frac{\partial y^\theta}{\partial x^\nu}\frac{\partial \phi}{\partial y^\theta}.$$

This is equivalent to

$$[\frac{\partial y^\omega}{\partial x^\mu}\overline{P}^\mu, \phi] = -i\frac{\partial y^\omega}{\partial x^\mu}\frac{\partial y^\theta}{\partial x^\nu}\eta^{\mu\nu}\frac{\partial \phi}{\partial y^\theta}$$

i.e.

$$[P^\omega . o] = -ig^{\omega\theta}\frac{\partial \phi}{\partial y^\theta} = -i\partial^\omega \phi$$

as given above.

so the above becomes

$$-\frac{1}{2}\int dy\left([\phi(\tau,y'),\frac{\partial\phi}{\partial y}(\tau,y)]\frac{\partial\phi}{\partial y}(\tau,y)+\frac{\partial\phi}{\partial y}(\tau,y)[\phi(\tau,y'),\frac{\partial\phi}{\partial y}(\tau,y)]\right).$$

Using (3.7) this reduces to

$$\begin{aligned}\frac{1}{2}\int dy2i\delta(y-y')\frac{\partial\phi}{\partial y}(y,\tau)&=\frac{1}{2}2i\frac{\partial\phi}{\partial y'}(y',\tau)\\&=\frac{1}{2}\left(-2ig^{\tau\mu}\frac{\partial\phi}{\partial y'^{\mu}}\right)=\frac{1}{2}(-2i\partial^{\tau}\phi)\\&=-i\partial^{\tau}\phi\end{aligned}$$

which is the same as the result given by (3.8) . However

$$[P^y,\phi]=[-\frac{1}{2}\frac{1}{2}\int\left(\mu^2\phi^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right)dy,\phi(\tau,y')]$$

$$=-\frac{1}{2}\frac{1}{2}\int dy\left(\mu^2[\phi^2(\tau,y),\phi(\tau,y')]-[\left(\frac{\partial\phi}{\partial y}\right)^2(\tau,y),\phi(\tau,y')]\right)$$

$$=-\frac{1}{2}\frac{1}{2}\int dy[\phi(\tau,y'),\left(\frac{\partial\phi}{\partial y}\right)^2(\tau,y)]$$

$$=-\frac{1}{2}\frac{1}{2}\int dy\left([\phi(\tau,y'),\frac{\partial\phi}{\partial y}(\tau,y)]\frac{\partial\phi}{\partial y}(\tau,y)+\frac{\partial\phi}{\partial y}(\tau,y)[\phi(\tau,y'),\frac{\partial\phi}{\partial y}(\tau,y)]\right)$$

$$=\frac{1}{2}\int dyi\delta(y'-y)\frac{\partial\phi}{\partial y}$$

$$=\frac{i}{2}\frac{\partial\phi}{\partial y'}\tag{3.11}$$

whereas (3.8) gives

$$\begin{aligned}[P^y,\phi]&=-i\partial^y\phi=-ig^{\mu y}\partial_\mu\phi=-i(g^{\tau y}\partial_\tau+g^{yy}\partial_y\phi)\\&=i\left(\frac{\partial\phi}{\partial\tau}+\frac{\partial\phi}{\partial y}\right)\end{aligned}\tag{3.12}$$

which differs from (3.11). It is not at all surprising that the canonical formalism fails. For a field theory whose instants are spacelike hypersurfaces the cannonical commutation relation

$$[\phi(x,t),\phi(x',t)]=0$$

makes intuitive sense since it derives from a need to preserve causality. However, in a theory based on null hypersurfaces things are altogether different. A non vanishing commutator between a field evaluated at two points on a given light front does not imply acausal behaviour.

In the light front theory of the scalar boson developed in [35] (3.17) and [38] (2.15) a consistent field theory was obtained by taking

$$[\phi(y',\tau),\phi(y,\tau)] = \frac{i}{2}\epsilon(y'-y) \tag{3.13}$$

where $\epsilon(x)$ denotes the Heaveside step function which is equal to -1 for $x<0$ and 1 for $x>0$. Can we rescue our scheme in a similar fashion? Suppose we take the equal τ commutation relation as (3.13)[4]. Notice that on differentiating (3.13) with respect to y' we recover (3.7) so the commutators are not contradictory. The commutator $[P^\tau,\phi]$ as calculated previously is unaltered. Consider now

$$[P^y,\phi] = -[\frac{1}{2}\int\frac{1}{2}\left(\mu^2\phi^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right)dy,\phi']$$

$$= -\frac{1}{2}\left(-\int\frac{\mu^2}{2}[\phi',\phi^2]dy+\int\frac{1}{2}[\phi',\left(\frac{\partial\phi}{\partial y}\right)^2]dy\right)$$

$$= -\frac{1}{2}\left(-\int\mu^2[\phi',\phi]\phi dy+\int[\phi',\frac{\partial\phi}{\partial y}]\frac{\partial\phi}{\partial y}dy\right)$$

$$= \frac{1}{2}\left(-\int\mu^2\frac{i}{2}\epsilon(y-y')\phi dy+\int i\delta(y-y')\frac{\partial\phi}{\partial y}dy\right)$$

$$= \frac{1}{2}\left(-\int\mu^2\frac{i}{2}\epsilon(y-y')\phi dy+i\frac{\partial\phi}{\partial y'}\right).$$

Using the field equation this becomes

$$\frac{1}{2}\left(-\int\frac{i}{2}\epsilon(y-y')\left(2\frac{\partial\frac{\partial\phi}{\partial\tau}}{\partial y}+\frac{\partial\frac{\partial\phi}{\partial y}}{\partial y}\right)dy+i\frac{\partial\phi}{\partial y'}\right).$$

Perform an integration by parts with respect to y in the first term

$$= \frac{1}{2}\left(\int i\delta(y-y')\left(2\frac{\partial\phi}{\partial\tau}+\frac{\partial\phi}{\partial y}\right)dy+i\frac{\partial\phi}{\partial y'}\right)$$

$$= \frac{1}{2}\left(2i\frac{\partial\phi'}{\partial\tau}+i\frac{\partial\phi}{\partial y'}+i\frac{\partial\phi}{\partial y'}\right) = \frac{1}{2}\left(2i\left(\frac{\partial\phi'}{\partial\tau}+\frac{\partial\phi}{\partial y'}\right)\right)$$

$$= i\left(\frac{\partial\phi'}{\partial\tau}+\frac{\partial\phi}{\partial y'}\right)$$

which agrees with (3.12). We can also show that (3.9) is satisfied. For example in Appendix 3 we show that

$$[J^{\tau y}.\, o] = i(\tau(\partial_\tau+\partial_y)-y\partial_y)\phi.$$

[4]We shall give a much more compelling reason for imposing this rather peculiar commutation relation in section (3.1.3).

This shows that the field theory is relativistically invariant if we impose the commutation relation (3.13). In Appendix 3 we also show that the generators satisfy the correct commutation relations.

We wish to show that the Heisenberg equation of this quantum field theory reproduces the light front Klein-Gordon equation. We have

$$\begin{aligned} P_\tau &= P^\tau - P^y \\ &= \frac{1}{2}\left(\int \left(\frac{\partial\phi}{\partial y}\right)^2 dy + \frac{1}{2}\int \mu^2\phi^2 - \left(\frac{\partial\phi}{\partial y}\right)^2 dy\right) \\ &= \frac{1}{2}\left(\left(1-\frac{1}{2}\right)\int \left(\frac{\partial\phi}{\partial y}\right)^2 dy + \frac{1}{2}\int \mu^2\phi^2 dy\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\int \left(\frac{\partial\phi}{\partial y}\right)^2 + \mu^2\phi^2 dy\right). \end{aligned} \tag{3.14}$$

The Heisenberg equation is (45.23 page 346 [60])

$$\begin{aligned} \frac{\partial\phi}{\partial\hat{\tau}} &= \frac{1}{i}[\phi, P_\tau] \\ &= -\frac{1}{i}[P_\tau, \phi] \\ &= -\frac{1}{2}\frac{1}{i}[\frac{1}{2}\int \mu^2\phi^2(\hat{\tau},y) + \left(\frac{\partial\phi}{\partial y}\right)^2(\hat{\tau},y)dy, \phi(\hat{\tau},\hat{y})] \end{aligned} \tag{3.15}$$

so

$$\begin{aligned} \frac{\partial^2\phi}{\partial\hat{\tau}\partial\hat{y}} &= -\frac{1}{2}\frac{1}{2i}\int dy[\mu^2\phi^2(\hat{\tau},y) + \left(\frac{\partial\phi}{\partial y}\right)^2(\hat{\tau},y), \frac{\partial\phi}{\partial\hat{y}}(\hat{\tau},\hat{y})] \\ &= -\frac{1}{2i}\frac{1}{2}\int dy\left(\mu^2[\phi^2(\hat{\tau},y), \frac{\partial\phi}{\partial\hat{y}}(\hat{\tau},\hat{y})] + [\left(\frac{\partial\phi}{\partial y}\right)^2(\hat{\tau},y), \frac{\partial\phi}{\partial\hat{y}}(\hat{\tau},\hat{y})]\right). \end{aligned}$$

Now if $[A, C]$ is a c number then $[A^2, C] = [A, C]A + A[A, C] = 2[A, C]A$. Therefore

$$\begin{aligned} \frac{\partial^2\phi}{\partial\hat{\tau}\partial\hat{y}} &= -\frac{1}{2i}\frac{1}{2}\int dy\left(2\mu^2[\phi(\hat{\tau},y), \frac{\partial\phi}{\partial\hat{y}}(\hat{\tau},\hat{y})]\phi(\hat{\tau},y) + 2[\frac{\partial\phi}{\partial y}(\hat{\tau},y), \frac{\partial\phi}{\partial\hat{y}}(\hat{\tau},\hat{y})]\frac{\partial\phi}{\partial y}(\hat{\tau},y)\right) \\ &= \frac{1}{2i}\frac{1}{2}\int dy\left(2\mu^2 i\delta(y-\hat{y})\phi(\hat{\tau},y) - 2\frac{\partial}{\partial y}[\phi(\hat{\tau},y), \frac{\partial\phi}{\partial\hat{y}}(\hat{\tau},\hat{y})]\frac{\partial\phi}{\partial y}(\hat{\tau},y)\right) \\ &= \frac{1}{2i}\frac{1}{2}\left(2i\mu^2\phi(\hat{\tau},\hat{y}) + \int dy 2i\frac{\partial\delta(y-\hat{y})}{\partial y}\frac{\partial\phi}{\partial y}(\hat{\tau},y)\right) \\ &= \frac{1}{2}\left(\mu^2\phi(\hat{\tau},\hat{y}) + \int \frac{\partial\delta(y-\hat{y})}{\partial y}\frac{\partial\phi}{\partial y}(\hat{\tau},y)dy\right) \\ &= \frac{1}{2}\left(\mu^2\phi(\hat{\tau},\hat{y}) - \frac{\partial^2\phi}{\partial\hat{y}^2}(\hat{\tau},\hat{y})\right) \\ &= \frac{1}{2}\left(\mu^2\phi(\hat{\tau},\hat{y}) - \frac{\partial^2\phi}{\partial\hat{y}^2}(\hat{\tau},\hat{y})\right) \end{aligned}$$

so that

$$2\frac{\partial^2\phi}{\partial\hat{y}\partial\hat{\tau}} = \mu^2\phi(\hat{\tau},\hat{y}) - \frac{\partial^2\phi}{\partial\hat{y}^2}(\hat{\tau},\hat{y})$$

which is the light front Klein-Gordon Equation.

3.1.3 Dirac's Theory of Constraints

In this section we look more closely at the light front field theory of the neutral scalar boson. Clearly (3.2) is singular and (3.6) is a constraint. A careful application of the method for handling constraints developed by Dirac [34] gives an elegant justification for imposing the peculiar commutation relation (3.13).

The constraint

$$\pi = -\frac{\partial \phi}{\partial y}$$

is primary but not first class as we shall now demonstrate. Put

$$\Psi = \pi(\overline{y}) + \frac{\partial \phi}{\partial \overline{y}} \tag{3.16}$$

$$= \int \delta(\overline{y} - y) \left(\pi(y) + \frac{\partial \phi}{\partial y} \right) dy. \tag{3.17}$$

Now

$$[\Psi(\overline{y}), \Psi(y')] = \int dy \left(\frac{\delta \Psi(\overline{y})}{\delta \phi(y)} \frac{\delta \Psi(y')}{\delta \pi(y)} - \frac{\delta \Psi(\overline{y})}{\delta \pi(y)} \frac{\delta \Psi(y')}{\delta \phi(y)} \right). \tag{3.18}$$

It is easy to see from (3.17) that

$$\frac{\delta \Psi(\overline{y})}{\delta \phi(y)} = \frac{\partial}{\partial \phi(y)} \left(\delta(\overline{y} - y) \left(\pi(y) + \frac{\partial \phi}{\partial y}(y) \right) \right) - \frac{\partial}{\partial y} \left(\frac{\partial \delta(\overline{y} - y)(\pi(y) + \frac{\partial \phi}{\partial y}(y))}{\partial \frac{\partial \phi}{\partial y}} \right)$$

$$= -\frac{\partial \delta(\overline{y} - y)}{\partial y} \tag{3.19}$$

and

$$\frac{\delta \Psi(\overline{y})}{\delta \pi(y)} = \frac{\partial}{\partial \pi(y)} \left(\delta(\overline{y} - y) \left(\pi(y) + \frac{\partial \phi}{\partial y}(y) \right) \right) - \frac{\partial}{\partial y} \left(\frac{\partial \delta(\overline{y} - y)(\pi(y) + \frac{\partial \phi}{\partial y}(y))}{\partial \frac{\partial \pi}{\partial y}} \right)$$

$$= \delta(\overline{y} - y). \tag{3.20}$$

Using (3.19) and (3.20) in (3.18) we have

$$[\Psi(\overline{y}), \Psi(y')] = \int dy \left(-\delta(y' - y) \frac{\partial}{\partial y} \delta(\overline{y} - y) + \delta(\overline{y} - y) \frac{\partial}{\partial y} \delta(y' - y) \right)$$

$$= \int dy \left(\delta(\overline{y} - y) \frac{\partial}{\partial y} \delta(y' - y) + \delta(\overline{y} - y) \frac{\partial}{\partial y} \delta(y' - y) \right)$$

$$= 2 \int dy \delta(\overline{y} - y) \frac{\partial}{\partial y} \delta(y' - y) = -2 \int dy \delta(\overline{y} - y) \frac{\partial}{\partial y'} \delta(y' - y)$$

$$= -2 \frac{\partial}{\partial y'} \int dy \delta(\overline{y} - y) \delta(y' - y) = -2 \frac{\partial}{\partial y'} \delta(\overline{y} - y')$$

$$\neq 0.$$

hence Ψ is not first class. Let

$$\begin{aligned}\mathcal{H} &= \pi\frac{\partial\phi}{\partial\tau} - \mathcal{L} \\ &= -\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial\tau} + \frac{1}{2}\left(2\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y} + \left(\frac{\partial\phi}{\partial y}\right)^2 + \mu^2\phi^2\right) \\ &= -\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial\tau} + \frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial\tau} + \frac{1}{2}\left(\left(\frac{\partial\phi}{\partial y}\right)^2 + \mu^2\phi^2\right) \\ &= \frac{1}{2}(\pi^2 + \mu^2\phi^2). \end{aligned} \tag{3.21}$$

We must now investigate what kind of consistency equations arise when we examine how the constraints evolve under the Hamiltonian

$$H = \int dy(\mathcal{H} + U\Psi)$$

where U is a Lagrange multiplier. Now

$$\dot{\Psi}\,(y') = [\Psi, H] = [\Psi, \int dy\mathcal{H}] + [\Psi, \int dyU\Psi].$$

Remembering the rule that describes how the Poisson bracket is to be extended to quantities involving Lagrange multipliers (see bottom of page 24 and top of page 25 of [34]) this becomes

$$\begin{aligned} &[\Psi(y'), \int dy\mathcal{H}] + \int dyU(y)[\Psi, \Psi(y)] \\ &= [\Psi(\prime), \int dy\mathcal{H}] - \int dyU(y)2\frac{\partial}{\partial y}\delta(y'-y) \\ &= [\Psi(y'), \int dy\mathcal{H}] + 2U'(y'). \end{aligned} \tag{3.22}$$

If we put

$$P = \int dy\mathcal{H}$$

then it is easy to see that

$$\frac{\delta P}{\delta\pi(y)} = \pi(y) \quad \text{and} \quad \frac{\delta P}{\delta\phi(y)} = \mu^2\phi(y).$$

Using these and (3.19) and (3.20) in (3.18) we obtain

$$[\Psi(y'), \int dy\mathcal{H}] = \int dy\left(-\frac{\partial}{\partial y}\delta(y'-y)\pi(y) - \delta(y'-y)\mu^2\phi(y)\right) \tag{3.23}$$

$$= \frac{\partial\pi}{\partial y'} - \mu^2\phi(y')$$

and subsituting this in (3.22) gives

$$\dot{\Psi} = \frac{\partial\pi}{\partial y'}(y') - \mu^2\phi(y') + 2\frac{\partial U}{\partial y'}(y').$$

Consistency requires

$$\dot{\Psi} \approx 0$$

i.e.

$$\frac{\partial \pi}{\partial y'}(y') - \mu^2 \phi(y') + 2\frac{\partial U}{\partial y'}(y') \approx 0. \tag{3.24}$$

According to Dirac this is a consistency condition of type 3 (see page 14 of [34]) since it imposes conditions on the U's. At least formally we can see that

$$2U = \pi(y') - \mu^2 \int^{y'} \phi(y') dy' + k$$

is a solution of (3.24) (k an arbitrary constant) but Dirac points out that we require the most general solution. To find this we must obtain the solutions of the equations

$$\int V(x)[\Psi(y), \Psi(x)] dx \approx 0 \tag{3.25}$$

(c.f 1.30 page 16 [34]) which become

$$-\int V(x) 2\frac{\partial}{\partial y}\delta(y - x) dx \approx 0$$

i.e.

$$\frac{\partial V}{\partial y} \approx 0. \tag{3.26}$$

Only quantities that are strongly equal to a linear combination of constraints are weakly zero so

$$\frac{\partial V}{\partial y} = \int dx' f(x') \Psi(x')$$

and therefore

$$V(y) = y \int dx' f(x') \Psi(x').$$

This gives as the total Hamiltonian

$$H_T = \int dy [\mathcal{H} + U\Psi + \nu(\tau) V(y) \Psi]$$

(c.f 1.33 page 16 [34]). We have no secondary constraints (first class or otherwise) so this is the final form of our Hamiltonian, we do not require Dirac's notion of the extended Hamiltonian (2.2 page 25 [34]). Dirac has given a prescription for quantising classical systems with second class constraints. Suppose we have a solution $C(x, y)$ of the equation

$$\int dy C(x, y) [\Psi(y), \Psi(z)] = \delta(x - z). \tag{3.27}$$

We define a new Poisson bracket

$$[\zeta(x), \eta(y)]_D = [\zeta(x), \eta(y)] - \int dz \int dp [\zeta(x), \Psi(z)] C(z, p) [\Psi(p), \eta(y)]$$

(2.30. page 41 [34]). This Dirac bracket satisfies the crucial identity

$$[\zeta(x),\Psi(y)]_D = [\zeta(x),\Psi(y)] - \int dz \int dp[\zeta(y),\Psi(z)]C(z,p)[\Psi(p),\Psi(y)]$$

$$= [\zeta(x),\Psi(y)] - \int dz[\zeta(x),\Psi(z)]\delta(z-y)dz$$

$$= 0$$

(page 42 [34]).

We should demonstrate that the classical field equation is equivalent to Hamilton's equation with Poisson or Dirac brackets.

With Poisson bracket we have

$$\dot{\phi} \approx [\phi, H_T]$$

$$\approx [\phi, \int dy\{\mathcal{H}(y) + U(y)\Psi(y) + \nu(\tau)V(y)\Psi(y)\}]$$

$$\approx [\phi, \int dy\mathcal{H}(y)] + [\phi, \int dyU(y)\Psi(y)dy] + [\phi, \int \nu(\tau)V(y)\Psi(y)\Psi(y)]$$

$$\approx [\phi, \int dy\mathcal{H}] + \int U(y)[\phi,\Psi(y)dy] + \nu(\tau)\int dyV(y)[\phi,\Psi(y)].$$

Now

$$[\phi(y'),\Psi(y)] = \int d\hat{y}\left[\left(\frac{\partial}{\partial\phi(\hat{y})} - \frac{\partial}{\partial\hat{y}}\left(\frac{\partial}{\partial\frac{\partial\phi}{\partial\hat{y}}}\right)\right)\delta(\hat{y}-y')\phi(\hat{y})\left(\frac{\partial}{\partial\pi(\hat{y})} - \frac{\partial}{\partial\hat{y}}\left(\frac{\partial}{\partial\frac{\partial\pi}{\partial\hat{y}}}\right)\right)\delta(\hat{y}-y)\left(\pi(\hat{y}) + \frac{\partial\phi}{\partial\hat{y}}\right) - 0\right]$$

$$= \int d\hat{y}\delta(\hat{y}-y')\delta(\hat{y}-y)$$

$$= \delta(y'-y)$$

so

$$\dot{\phi} \approx [\phi, \int dy\mathcal{H}(y)] + \int U(y)\delta(y'-y)dy + \nu(\tau)\int V(y)\delta(y'-y)dy$$

$$\approx [\phi(y'), \int dy\mathcal{H}(y)] + U(y') + \nu(\tau)V(y').$$

Also

$$[\phi(y'),\mathcal{H}(y)] = \int d\hat{y}\left[\left(\frac{\partial}{\partial\phi(\hat{y})} - \frac{\partial}{\partial\hat{y}}\left(\frac{\partial}{\partial\frac{\partial\phi}{\partial\hat{y}}}\right)\right)\delta(y'-\hat{y})\phi(\hat{y})\left(\frac{\partial}{\partial\pi(\hat{y})} - \frac{\partial}{\partial\hat{y}}\left(\frac{\partial}{\partial\frac{\partial\pi}{\partial\hat{y}}}\right)\right)\right.$$

$$\left.\delta(y'-\hat{y})\left(\frac{\delta(y-y')}{2}(\pi^2(\hat{y}) + \mu^2\phi^2)\right)\right]$$

$$= \int d\hat{y}\frac{\delta(y'-\hat{y})}{2}2\pi(\hat{y})(y-\hat{y})$$

$$= \pi(y)\delta(y'-y)$$

and therefore

$$[\phi(y'), \int \mathcal{H} dy] = \int [\phi(y'), \mathcal{H}] dy = \pi(y').$$

Clearly

$$\dot{\phi} \approx \pi(y') + U(y') + \nu(\tau) V(y')$$

so

$$\frac{\partial^2 \phi}{\partial y' \partial \tau} \approx \frac{\partial \pi}{\partial y'}(y') + \frac{\partial U}{\partial y'}(y') + \nu(\tau) \frac{\partial V}{\partial y'}(y').$$

The last term is weakly zero by (3.26) and using (3.24) we obtain

$$\frac{\partial^2 \phi}{\partial y' \partial \tau} \approx \frac{\partial \pi}{\partial y'} + \frac{1}{2} \left(-\frac{\partial \pi}{\partial y'} + \mu^2 \phi(y') \right)$$

i.e.

$$2 \frac{\partial^2 \phi}{\partial y' \partial \tau} \approx -\frac{\partial^2 \phi}{\partial y'^2} + \mu^2 \phi(y')$$

which is the correct equation of motion.

It is easy to see that we shall still have the correct field equation if we work with the Dirac bracket since in that case

$$[\phi, H_T]_D = [\phi, H_T] - \int dz \int dp [\phi(y), \Psi(z)] C(z,p) [\Psi(p), H_T].$$

The second term is clearly weakly zero. To see this note that

$$[\Psi(p), H_T] = [\Psi(p), \mathcal{H}(y) + U(y)\Psi(y) + \nu(\tau) V(y) \Psi(y)]$$

$$= \int [\Psi(p), \mathcal{H}(y) + U(y)] dy + \nu(\tau) \int [\Psi(p), V(y)\Psi(y)] dy.$$

Now

$$[\Psi(p), V(y)\Psi(y)] = [\Psi(p), V(y)]\Psi(y) + V(y)[\Psi(p), \Psi(y)]$$

and first term in this expansion is weakly zero so that

$$[\Psi(p), H_T] \approx \int [\Psi(p), \mathcal{H}(y) + U(y)\Psi(y)] dy + \nu(\tau) \int V(y) [\Psi(p), \Psi(y)] dy$$

$$\approx \frac{\partial \pi}{\partial y} - \mu^2 \phi + 2 \frac{\partial u}{\partial y}$$

$$\approx 0$$

by (3.24) and (3.25).

We now wish to show that the Dirac bracket reproduces the peculiar commutator (3.13). First notice that it is easy to show that a solution to (3.27) is given by

$$C(x,y) = -\frac{1}{4} \epsilon(y - x).$$

To see this remember that

$$[\Psi(y), \Psi(z)] = -2\frac{\partial}{\partial z}\delta(y-z)$$

so

$$\int dy C(x,y)[\Psi(y), \Psi(z)] = \int dy \frac{1}{4}\epsilon(y-x) 2\frac{\partial}{\partial z}\delta(y-z)$$
$$= \frac{1}{2}\frac{\partial}{\partial z}\int dy \epsilon(y-x)\delta(y-z) = \frac{1}{2}\frac{\partial}{\partial z}\epsilon(z-x)$$
$$= \delta(z-x)$$

as required. Now

$$[\phi(y,\tau), \phi(y',\tau)]_D = [\phi(y,\tau), \phi(y',\tau)] +$$
$$\int ds \int dp [\phi(y,\tau), \pi(s) - \frac{\partial \phi}{\partial s}(s,\tau)](-1)\frac{1}{4}\epsilon(p-s)[\pi(p) - \frac{\partial \phi}{\partial p}(p,\tau), \phi(y',\tau)]$$
$$= -\delta(y-s)\epsilon(p-s)\frac{1}{4}\delta(p-y')$$
$$= \frac{1}{4}\epsilon(y-y') \tag{3.28}$$

so the nonstandard commutator (3.13) is the Dirac bracket. We see that (3.28) and (3.13) differ by a factor of 1/2. Previously this factor was absorbed in the definition of the generators (see (3.4) and (3.5)) [5].

3.1.4 Discrete Light Front Field Theory in 1+1 Spacetime

We take as the inner product on the space of solutions of the light front Klein-Gordon Equation

$$(\phi_1, \phi_2) = -i\int_{y=-\infty}^{\infty} \left(\phi_1 \frac{\partial}{\partial \tau}\phi_2^* - \frac{\partial}{\partial \tau}\phi_1 \phi_2^*\right) dy$$

([56] equation 3.28 page 44). It is easy to see that the normal modes are

$$\frac{1}{\sqrt{4\pi}} \frac{((k^2+\mu^2)^{\frac{1}{2}} + k)^{\frac{1}{2}}}{(k^2+\mu^2)^{\frac{1}{2}}} \exp[i(k + (k^2+\mu^2)^{\frac{1}{2}})y - i(k^2+\mu^2)^{\frac{1}{2}}\tau]$$

where $k \in \boldsymbol{R}$. We can check that these satisfy (3.3) and the usual orthonormality and completeness relations (Appendix 4). We may therefore write

$$\phi(y,\tau) = \int \left\{ a(u) \frac{1}{\sqrt{4\pi}} \frac{((k^2+\mu^2)^{\frac{1}{2}} + k)^{\frac{1}{2}}}{(k^2+\mu^2)^{\frac{1}{2}}} \exp[i(k + (k^2+\mu^2)^{\frac{1}{2}})y - i(k^2+\mu^2)^{\frac{1}{2}}\tau] + \right.$$

[5] We remark on a slight inconsistancy in the notation. Properly we should denote the field ϕ of this section by a new symbol say $\overline{\phi}$ which is related to the previous ϕ by the equation $\overline{\phi} = 1/\sqrt{2}\phi$. Then it is obvious that we have $[\overline{\phi}', \overline{\phi}] = 1/2[\phi', \phi] = 1/4\epsilon(y'-y)$ and for example $P = \int (\frac{\partial \overline{\phi}}{\partial y})^2 dy$. Also the equation ('3.21) should be written $\mathcal{H} = 1/2((\frac{\partial \overline{\phi}}{\partial y})^2 + \mu^2 \overline{\phi}^2) = 1/4((\frac{\partial \phi}{\partial y})^2 + \mu^2\phi^2)$ which agrees with the Hamiltonian density in (3.14).

$$a^+(u)\frac{1}{\sqrt{4\pi}}\frac{((k^2+\mu^2)^{\frac{1}{2}}+k)^{\frac{1}{2}}}{(k^2+\mu^2)^{\frac{1}{2}}}\exp[-i(k+(k^2+\mu^2)^{\frac{1}{2}})y+i(k^2+\mu^2)^{\frac{1}{2}}\tau]\Bigg\}\,dk \qquad (3.29)$$

or

$$\phi(y,\tau)=\int_0^\infty\left\{a(u)\frac{1}{\sqrt{u}\sqrt{4\pi}}\exp\left[iuy-i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]+a^+(u)\frac{1}{\sqrt{u}\sqrt{4\pi}}\exp\left[-iuy+i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]\right\}du$$

where $u=(k^2+\mu^2)^{\frac{1}{2}}+k$. If we have

$$[a(u),a^+(\hat{u})]=\delta(u-\hat{u}) \qquad (3.30)$$

then to ensure that

$$[\phi(y,\tau),\phi(\hat{y},\tau)]=\frac{i}{2}\epsilon(y-\hat{y}) \qquad (3.31)$$

we must write

$$\phi(y,\tau)=\int_0^\infty\left\{a(u)\frac{\sqrt{2}}{\sqrt{u}\sqrt{4\pi}}\exp\left[iuy-i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]+a^+(u)\frac{\sqrt{2}}{\sqrt{u}\sqrt{4\pi}}\exp\left[-iuy+i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]\right\}du. \qquad (3.32)$$

We show in Appendix 5 that (3.30) ensures that (3.32) satisfies (3.31).

Expressing Generators in terms of Creation and Annihilation Operators

We wish to obtain expressions for the front form generators in terms of creation and annihilation operators. We shall use box normalisation in a cell of length $2L$. We require $\phi(-L,\tau)=\phi(L,\tau)$ so that $u=n\pi/L$, $n>0$ and

$$\phi(y,\tau)=\sum_{u=1}^{\infty}\left\{\frac{a_u}{\sqrt{u}\sqrt{2L}}\exp\left[iuy-i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]+\frac{a_u^+}{\sqrt{u}\sqrt{2L}}\exp\left[-iuy+i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]\right\}$$

where $[a_u,a_{u'}]=\delta_{uu'}$[6].

We have

$$P^\tau=\frac{1}{2}\int_{-L}^{L}\left(\frac{\partial\phi}{\partial y}\right)^2dy$$

$$=\frac{1}{2}\int_{-L}^{L}dy\left(\sum_u\frac{iua_u}{\sqrt{u}\sqrt{2L}}\exp\left[iuy-i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]+ a_u^+\frac{(-iu)}{\sqrt{u}\sqrt{2L}}\exp\left[-iuy+i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]\right)^2.$$

Since we are dealing with conserved quantities it is sufficient to calculate P^τ at $\tau=0$ (see [47] page 51) so

$$P^\tau=\frac{1}{2}\int_{-L}^{L}dy\left(\sum_u\frac{iua_u}{\sqrt{u}\sqrt{2L}}\exp[iuy]-\frac{iua_u^+}{\sqrt{u}\sqrt{2L}}\exp[-iuy]\right)^2$$

[6]Clearly by summation over u we mean summation over the associated n

$$=\frac{1}{2}\int_{-L}^{L}dy\sum_u\sum_{\hat{u}}\left(\frac{iua_u}{\sqrt{u}\sqrt{2L}}\exp[iuy]-\frac{iua_u^+}{\sqrt{u}\sqrt{2L}}\exp[-iuy]\right)\left(\frac{i\hat{u}a_{\hat{u}}}{\sqrt{\hat{u}}\sqrt{2L}}\exp[i\hat{u}y]-\frac{i\hat{u}a_{\hat{u}}^+}{\sqrt{\hat{u}}\sqrt{2L}}\exp[-i\hat{u}y]\right)$$

$$=\frac{1}{2}\int_{-L}^{L}dy\sum_u\sum_{\hat{u}}\left(-\frac{u\hat{u}a_ua_{\hat{u}}}{\sqrt{u}\sqrt{\hat{u}}2L}\exp[iy(u+\hat{u})]+\frac{\hat{u}ua_ua_{\hat{u}}^+}{\sqrt{u}\sqrt{\hat{u}}2L}\exp[iy(u-\hat{u})]+\right.$$
$$\left.+\frac{\hat{u}ua_u^+a_{\hat{u}}}{\sqrt{u}\sqrt{\hat{u}}2L}\exp[iy(\hat{u}-u)]-\frac{\hat{u}ua_u^+a_{\hat{u}}^+}{\sqrt{u}\sqrt{\hat{u}}2L}\exp[-iy(u+\hat{u})]\right).$$

Put

$$w=\frac{\pi y}{L} \tag{3.33}$$

so

$$dy=\frac{L}{\pi}dw$$

then

$$P^\tau=\frac{1}{2}\int_{-\pi}^{\pi}\frac{L}{\pi}dw\sum_u\sum_{\hat{u}}\left(-\frac{u\hat{u}a_ua_{\hat{u}}}{\sqrt{u}\sqrt{\hat{u}}2L}\exp[i\frac{L}{\pi}w(u+\hat{u})]+\frac{\hat{u}ua_ua_{\hat{u}}^+}{\sqrt{u}\sqrt{\hat{u}}2L}\exp[i\frac{L}{\pi}w(u-\hat{u})]+\right.$$
$$\left.\frac{\hat{u}ua_u^+a_{\hat{u}}}{\sqrt{u}\sqrt{\hat{u}}2L}\exp[i\frac{L}{\pi}w(\hat{u}-u)]-\frac{u\hat{u}a_u^+a_{\hat{u}}^+}{\sqrt{u}\sqrt{\hat{u}}2L}\exp[-i\frac{L}{\pi}w(u+\hat{u})]\right)$$
$$=-\frac{1}{2}\sum_{\hat{u}}\sum_u\left\{-\frac{u\hat{u}}{\sqrt{u}\sqrt{\hat{u}}}a_ua_{\hat{u}}\delta_{u+\hat{u},0}+\frac{\hat{u}u}{\sqrt{u}\sqrt{\hat{u}}}a_ua_{\hat{u}}^+\delta_{u-\hat{u},0}+\right.$$
$$\left.\frac{\hat{u}u}{\sqrt{\hat{u}}\sqrt{u}}a_u^+a_{\hat{u}}\delta_{\hat{u}-u,0}-\frac{u\hat{u}}{\sqrt{u}\sqrt{\hat{u}}}a_u^+a_{\hat{u}}^+\delta_{-u-\hat{u},0}\right\}$$

since Lu/π is an integer. Now $\delta_{-u-\hat{u},0}$ and $\delta_{u+\hat{u},0}$ are never other than zero (remember that u and $\hat{u}>0$). $\delta_{\hat{u}-u,0}$ and $\delta_{u-\hat{u},0}$ are non zero only when $\hat{u}=u$ so that

$$P^\tau=\frac{1}{2}\sum_u(ua_ua_u^++ua_u^+a_u).$$

After normal ordering

$$P^\tau=\sum_u ua_u^+a_u. \tag{3.34}$$

We now consider the operator

$$P^y=-\frac{1}{2}\frac{1}{2}\int\left\{\mu^2\phi^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right\}dy$$
$$=-\frac{1}{2}\left(\frac{\mu^2}{2}\int\phi^2dy-\frac{1}{2}\int\left(\frac{\partial\phi}{\partial y}\right)^2dy\right)=-\frac{1}{2}\left(\frac{\mu^2}{2}\int\phi^2dy-P^\tau\right)$$
$$=-\frac{1}{2}\left(\frac{\mu^2}{2}\int\phi^2dy-\sum_u ua_u^+a_u\right). \tag{3.35}$$

Now

$$\int\phi^2dy=\int_{-L}^{L}dy\left(\frac{1}{\sqrt{2L}}\sum_u\left(\frac{a_u}{\sqrt{u}}\exp\left[iuy-i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]+\frac{a_u^+}{\sqrt{u}}\exp\left[-iuy+i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]\right)\right)\times$$

$$\left(\frac{1}{\sqrt{2L}}\sum_{\hat{u}}\left(\frac{a_{\hat{u}}}{\sqrt{\hat{u}}}\exp\left[i\hat{u}y - i\left(\frac{\hat{u}^2+\mu^2}{2\hat{u}}\right)\tau\right] + \frac{a_{\hat{u}}^+}{\sqrt{\hat{u}}}\exp\left[-i\hat{u}y + i\left(\frac{\hat{u}^2+\mu^2}{2\hat{u}}\right)\tau\right]\right)\right).$$

As before we can assume $\tau = 0$ so

$$\int \phi^2 dy = \int_{-L}^{L} dy \frac{1}{2L}\sum_u\sum_{\hat{u}}\left(\frac{a_u}{\sqrt{u}}\exp[iuy] + \frac{a_u^+}{\sqrt{u}}\exp[-iuy]\right)$$

$$\left(\frac{a_{\hat{u}}}{\sqrt{\hat{u}}}\exp[i\hat{u}y] + \frac{a_{\hat{u}}^+}{\sqrt{\hat{u}}}\exp[-i\hat{u}y]\right)$$

$$= \int_{-L}^{L} dy \frac{1}{2L}\sum_u\sum_{\hat{u}}\left(\frac{a_u a_{\hat{u}}}{\sqrt{u}\sqrt{\hat{u}}}\exp[iy(u+\hat{u})] + \frac{a_u a_{\hat{u}}^+}{\sqrt{u}\sqrt{\hat{u}}}\exp[iy(u-\hat{u})]\right.$$

$$\left. + \frac{a_u^+ a_{\hat{u}}}{\sqrt{u}\sqrt{\hat{u}}}\exp[iy(\hat{u}-u)] + \frac{a_u^+ a_{\hat{u}}^+}{\sqrt{u}\sqrt{\hat{u}}}\exp[iy(-u-\hat{u})]\right).$$

We can see that the first and last terms in the above can be dropped at this stage since ultimately they will generate Kronecker delta functions that are always zero. If we introduce the change of variable (3.33) we obtain

$$\int \phi^2 dy = \int_{-\pi}^{\pi} \frac{1}{2\pi} dw \left(\sum_u\sum_{\hat{u}}\left(\frac{a_u a_{\hat{u}}^+}{\sqrt{u}\sqrt{\hat{u}}}\exp[i\frac{Lw}{\pi}(u-\hat{u})] + \frac{a_u^+ a_{\hat{u}}}{\sqrt{u}\sqrt{\hat{u}}}\exp[i\frac{Lw}{\pi}(\hat{u}-u)]\right)\right)$$

$$= \sum_u\sum_{\hat{u}}\left(\frac{a_u a_{\hat{u}}^+}{\sqrt{u}\sqrt{\hat{u}}}\delta_{u-\hat{u},0} + \frac{a_u^+ a_{\hat{u}}}{\sqrt{u}\sqrt{\hat{u}}}\delta_{\hat{u}-u,0}\right).$$

Normal order to give

$$\int \phi^2 dy = \sum_u \frac{2}{u} a_u^+ a_{\hat{u}}. \tag{3.36}$$

Subsituting this in (3.35) we find that

$$P^y = -\frac{1}{2}\left(\mu^2 \sum_u \frac{1}{u} a_u^+ a_{\hat{u}} - \sum_u u a_u^+ a_{\hat{u}}\right)$$

$$= -\frac{1}{2}\left(-\sum_u \left(\frac{u^2-\mu^2}{u}\right) a_u^+ a_u\right)$$

$$= \sum_u \left(\frac{u^2-\mu^2}{2u}\right) a_u^+ a_u. \tag{3.37}$$

Now

$$P_\tau = g_{\tau\mu}P^\mu = P^\tau - P^y$$

(formula 2.47 page 69 [47]). Using (3.34) and (3.37) it is easy to show that

$$P_\tau = \sum_u \left(\frac{u^2+\mu^2}{2u}\right) a_u^+ a_u.$$

The creation operator a_u^+ creates from the vacuum[7] an eigenvector with eigenvalue $-u$ $(u > 0)$ of the operator π_1. Written in terms of creation operators (still denoted a_u [8]) of eigenstates with eigenvalue u $(u < 0)$ of π_1 the above becomes

$$P_\tau = -\sum_{-1}^{-\infty} \left(\frac{u^2 + \mu^2}{2u} \right) a_u^+ a_u. \tag{3.38}$$

It is now quite easy to see that the field theory we have described here is simply the many body generalisation of the 1 particle quauntum mechanics described in chapter 2. We form the Fock space

$$\oplus_{i=0} \mathcal{H}^i$$

where $\mathcal{H}^i$ denotes the symmetric subspace of

$$\otimes_{j=1}^{i} \mathcal{H}_\Pi^+.$$

Suppose as a basis for the light front Hilbert space $\mathcal{H}_\Pi^+$ we choose the generalised eigenfunctions $V_u(y)$ of the light front momentum operator π_1 then we lift Π_1 (suppressing the subscript Π) to an operator on Fock space. With some abuse of notation we denote this field operator by π_1. We have

$$\pi_1 = \int dy \hat{F}^+(y) \Pi_1 \hat{F}(y)$$

where

$$F(y) = \sum_u V_u(y) a_u \quad \text{and} \quad \hat{F}^+(y) = \sum_u V_u^*(y) a_u^+$$

so that

$$\pi_1 = \int dy \sum_u V_u^*(y) a_u^+ \Pi_1 \sum_{u'} V_{u'}(y) a_{u'}.$$

From the orthogonality of generalised eigenvectors

$$\pi_1 = \sum_{uu'} \delta_{uu'} u' a_u^+ a_{u'}$$

$$= \sum_u u a_u^+ a_u$$

and therefore

$$P^y = \int dy \sum V_u^*(y) a_u^+ (-1) \left(\frac{\pi_1^2 - (mc)^2}{2\pi_1} \right) \sum_{u'} V_{u'}(y) a_{u'}$$

[7]The vacumn $| 0 >$ is defined by $a_u | 0 >= 0$

[8]Really we should write a_{-u}^+ since $\pi_1 a_u^+ | 0 >= (-u) a_u^+ | 0 >$ when $u > 0$ so $\pi_1 a_{-u}^+ | 0 >= u a_{-u}^+ | 0 >$ when $u < 0$.

$$= -\sum_u \left(\frac{u^2 - (mc)^2}{u}\right) a_u^+ a_u$$

which is identical to (3.37) when the latter is expressed in terms of creation and annhilation operators of π_1 (recall $\mu = mc$). Also notice that the lift of the light front Hamiltonian K is given by

$$P_\tau = \int dy \sum_u V_u^*(y) a_u^+ K \sum_{u'} V_{u'}(y) a_{u'}.$$
$$= -\sum_u \left(\frac{u^2 + \mu^2}{2u}\right) a_u^+ a_u \tag{3.39}$$

which is the same as (3.38).

3.1.5 The Scalar Particle in Light Front Coordinates in 3+1 Spacetime

It is a straightforward matter to generalise the results of the previous sections to the case of a full 4-dimensional spacetime. Here we shall confine ourselves to presenting formulae for 3+1 light front Klein-Gordon equation and its solution in terms of normal modes as well as indicating how we expand the generators in terms of creation and annhilation operators.

Light Front Klein-Gordon Equation in 3+1 Spacetime

The Lagrangian in Minkowski coordinates is

$$\frac{1}{2}\left[\eta^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} - \mu^2\phi^2\right]$$
$$= \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \sum_i \left(\frac{\partial\phi}{\partial x^i}\right)^2 - \mu^2\phi^2\right]. \tag{3.40}$$

Now

$$\frac{\partial}{\partial x^0} = \frac{\partial\tau}{\partial x^0}\frac{\partial}{\partial\tau} + \frac{\partial y^i}{\partial x^0}\frac{\partial}{\partial y^i} = \frac{\partial}{\partial\tau}$$

and

$$\frac{\partial}{\partial x^j} = \frac{\partial\tau}{\partial x^j}\frac{\partial}{\partial\tau} + \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i}$$
$$= \delta_{1j}\frac{\partial}{\partial\tau} + \frac{\partial}{\partial y^j}$$

so that (3.40) becomes

$$\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \sum_j \left(\delta_{1j}\frac{\partial\phi}{\partial\tau} + \frac{\partial\phi}{\partial y^j}\right)^2 - \mu^2\phi^2\right]$$
$$= \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \sum_j \left(\delta_{1j}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + 2\delta_{1j}\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y^j} + \left(\frac{\partial\phi}{\partial y^j}\right)^2\right) - \mu^2\phi^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \left(\frac{\partial\phi}{\partial\tau}\right)^2 - 2\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y^1} - \sum_j\left(\frac{\partial\phi}{\partial y^j}\right)^2 - \mu\phi^2\right]$$

$$= -\frac{1}{2}\left[2\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y^1} + \sum_j\left(\frac{\partial\phi}{\partial y^j}\right)^2 + \mu^2\phi^2\right]. \tag{3.41}$$

The Klein-Gordon equation is the Euler-Lagrange equation, i.e.

$$\frac{\partial}{\partial y^1}\left(-\left(\frac{1}{2}\right)\left[2\frac{\partial\phi}{\partial\tau} + 2\frac{\partial\phi}{\partial y^1}\right]\right) +$$

$$\frac{\partial}{\partial y^2}\left(\left(-\frac{1}{2}\right)\left[2\frac{\partial\phi}{\partial y^2}\right]\right) + \frac{\partial}{\partial y^3}\left(\left(-\frac{1}{2}\right)\left[2\left(\frac{\partial\phi}{\partial y^3}\right)\right]\right) + \frac{\partial}{\partial\tau}\left(\left(-\frac{1}{2}\right)\left[2\frac{\partial\phi}{\partial y^1}\right]\right) - \left[\left(-\frac{1}{2}\right)(2\mu^2\phi)\right] = 0$$

$$-\frac{\partial^2\phi}{\partial\tau\partial y^1} - \frac{\partial^2\phi}{\partial {y^1}^2} - \frac{\partial^2\phi}{\partial {y^2}^2} - \frac{\partial^2\phi}{\partial {y^3}^2} - \frac{\partial^2\phi}{\partial y^1\partial\tau} + \mu^2\phi = 0$$

$$-2\frac{\partial^2\phi}{\partial y^1\partial\tau} - \sum_i\frac{\partial^2\phi}{\partial {y^i}^2} + \mu^2\phi = 0. \tag{3.42}$$

Expansion of Fields in terms of Normal Modes

The 3 dimensional analogue of (3.29), i.e. the expansion of ϕ in terms of normal modes and their creation and annhilation operators, is

$$\phi(\underline{y},\tau) = \int d^3\underline{k}_L\Bigg\{a(\underline{u})\frac{((\underline{k}_L^2+\mu^2)^{\frac{1}{2}}+k^{y^1})^{\frac{1}{2}}}{\sqrt{2}(2\pi)^{\frac{3}{2}}(\underline{k}_L^2+\mu^2)^{\frac{1}{2}}}\exp[i(k^{y^1}+(\underline{k}_L^2+\mu^2)^{\frac{1}{2}})y^1 - i(\underline{k}_L^2+\mu^2)^{\frac{1}{2}}\tau + ik^{y^2}y^2 + ik^{y^3}y^3]$$

$$+a^+(\underline{u})\frac{((\underline{k}_L^2+\mu^2)^{\frac{1}{2}}+k^{y^1})^{\frac{1}{2}}}{\sqrt{2}(2\pi)^{\frac{3}{2}}(\underline{k}_L^2+\mu^2)^{\frac{1}{2}}}\exp[-i(k^{y^1}+(\underline{k}_L^2+\mu^2)^{\frac{1}{2}})y^1 + i(\underline{k}_L^2+\mu^2)^{\frac{1}{2}}\tau - ik^{y^2}y^2 - ik^{y^3}y^3]\Bigg\}$$

(Appendix 6) where $u^1 = (\underline{k}_L^2+\mu^2)^{\frac{1}{2}} + k^{y^1}$, $u^i = k^{y^i}$ $i = 1,2$ and k_L is the four momentum with respect to the basis natural to the front form. Alternatively we can express the fields in terms of $\underline{u}$. We have

$$(u^1 - k^{y^1})^2 = \underline{k}_L^2 + \mu^2$$

so

$$ {u^1}^2 - 2u^1k^{y^1} + {k^{y^1}}^2 = \underline{k}_L^2 + \mu^2 = {k^{y^1}}^2 + {k^{y^2}}^2 + {k^{y^3}}^2 + \mu^2.$$

Therefore

$$ {u^1}^2 - 2u^1k^{y^1} = {u^2}^2 + {u^3}^2 + \mu^2$$

and

$$k^{y^1} = \frac{{u^1}^2 - {u^2}^2 - {u^3}^2 - \mu^2}{2u^1}.$$

Clearly

$$(\underline{k}_L^2+\mu^2)^{\frac{1}{2}} = u - \frac{({u^1}^2 - {u^2}^2 - {u^3}^2 - \mu^2)}{2u^1} = \frac{({u^1}^2 + {u^2}^2 + {u^3}^2 + \mu^2)}{2u^1}$$

$$= \frac{\underline{u}^2 + \mu^2}{2u^1}$$

so

$$\phi(\underline{y}, \tau) = \int \frac{d^3\underline{u}}{u^1} \left(\frac{\underline{u}^2 + \mu^2}{2u^1} \right) \left(\frac{a(\underline{u})}{\sqrt{2}(2\pi)^{\frac{3}{2}}} u^{1\frac{1}{2}} \frac{2u^1}{(\underline{u}^2 + \mu^2)} \exp\left[-i \frac{(\underline{u}^2 + \mu^2)\tau}{2u^1} + i\underline{u}.\underline{y} \right] + \frac{a^+(\underline{u})}{\sqrt{2}(2\pi)^{\frac{3}{2}}} u^{1\frac{1}{2}} \frac{2u^1}{(\underline{u}^2 + \mu^2)} \exp\left[i \frac{(\underline{u}^2 + \mu^2)}{2u^1} \tau - i\underline{u}.\underline{y} \right] \right).$$

If

$$[a(\underline{u}), a^+(\hat{\underline{u}})] = \delta(\underline{u} - \hat{\underline{u}})$$

then we must write

$$\phi(\underline{y}, \tau) = \int d^3\underline{u} \left(\frac{a(\underline{u})\sqrt{2}}{\sqrt{2}(2\pi)^{\frac{3}{2}} u^{1\frac{1}{2}}} \exp\left[i\underline{u}.y - i \frac{(\underline{u}^2 + \mu^2)\tau}{2u^1} \right] + \frac{a^+(\underline{u})\sqrt{2}}{\sqrt{2}(2\pi)^{\frac{3}{2}} u^{1\frac{1}{2}}} \exp\left[i \frac{(\underline{u}^2 + \mu^2)\tau}{2u^1} - i\underline{u}.y \right] \right)$$

so that

$$[\phi(\underline{y}, \tau), \phi(\hat{\underline{y}}, \tau)] = \frac{i}{2} \epsilon(y^1 - \hat{y}^1) \delta(\hat{y}^2 - y^2) \delta(\hat{y}^3 - y^3).$$

Expressing Generators in Terms of Creation and Annihilation Operators

In this section we shall have recourse to the results of Appendix 7 where we derive the components of the stress energy tensor. We shall only derive the expressions for P^τ and P^{y^1} since the method is rather routine. We start with P^τ. First we require an expression for the field in terms of creation and annihilation operators. This is given by

$$\phi(\underline{y}, \tau) = \left(\frac{\pi}{L} \right)^{\frac{3}{2}} \sum_{\underline{u}} \left(\frac{a(\underline{u})}{(2\pi)^{\frac{3}{2}} u^{1\frac{1}{2}}} \exp\left[i\underline{u}.y - i \frac{(\underline{u}^2 + \mu^2)\tau}{2u^1} \right] + \frac{a^+(\underline{u})}{(2\pi)^{\frac{3}{2}} u^{1\frac{1}{2}}} \exp\left[i \frac{(\underline{u}^2 + \mu^2)\tau}{2u^1} - i\underline{u}.y \right] \right)$$

where $\underline{u} = \underline{n}\pi/L$, $n^1 > 0$. Now

$$P^\tau = \frac{1}{2} \int_{\underline{y}} T^{\tau\tau} d^3\underline{y}$$

$$= \frac{1}{2} \int_{\underline{y}} \left(\frac{\partial \phi}{\partial y^1} \right)^2 d^3\underline{y}. \qquad (3.43)$$

We can take $\tau = 0$ so

$$P^\tau = \frac{1}{2} \int_{\underline{y}} d^3\underline{y} \left(\sum_{\underline{u}} \left(\frac{a(\underline{u}) i u^1}{(2L)^{\frac{3}{2}} u^{1\frac{1}{2}}} \exp[i\underline{u}.\underline{y}] + \frac{a^+(\underline{u})(-iu^1)}{(2L)^{\frac{3}{2}} u^{1\frac{1}{2}}} \exp[-i\underline{u}.\underline{y}] \right) \right)^2$$

$$= \frac{1}{2} \int_{\underline{y}} d^3\underline{y} \sum_{\underline{u}} \left(\frac{i a(\underline{u}) u^{1\frac{1}{2}}}{(2L)^{\frac{3}{2}}} \exp[i\underline{u}.\underline{y}] - \frac{i a^+(\underline{u}) u^{1\frac{1}{2}}}{(2L)^{\frac{3}{2}}} \exp[-i\underline{u}.\underline{y}] \right)$$

$$\sum_{\hat{\underline{u}}} \left(\frac{i a(\hat{\underline{u}}) \hat{u}^{1\frac{1}{2}}}{(2L)^{\frac{3}{2}}} \exp[i\hat{\underline{u}}.\underline{y}] - \frac{i a^+(\hat{\underline{u}}) \hat{u}^{1\frac{1}{2}}}{(2L)^{\frac{3}{2}}} \exp[-i\hat{\underline{u}}.\underline{y}] \right)$$

$$= \frac{1}{2}\int_{\underline{y}} d^3\underline{y}\sum_{\underline{u}}\sum_{\underline{\hat{u}}}\left(-\frac{1}{(2L)^3}a(\underline{u})a(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\exp[i\underline{y}.(\underline{u}+\underline{\hat{u}})]+\right.$$

$$\frac{1}{(2L)^3}a(\underline{u})a^+(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\exp[i\underline{y}.(\underline{u}-\underline{\hat{u}})]+$$

$$\frac{1}{(2L)^3}a^+(\underline{u})a(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\exp[i\underline{y}.(\underline{\hat{u}}-\underline{u})]-$$

$$\left.\frac{1}{(2L)^3}a^+(\underline{u})a^+(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\exp[-i\underline{y}.(\underline{\hat{u}}+\underline{u})]\right). \tag{3.44}$$

Put

$$w_i = \frac{\pi}{L}y^1$$

so that

$$d^3\underline{y} = \left(\frac{L}{\pi}\right)^3 d^3\underline{w}$$

then (3.44) becomes

$$\frac{1}{2}\int_{-\underline{\pi}}^{\underline{\pi}} d^3\underline{w}\sum_{\underline{u}}\sum_{\underline{\hat{u}}}\left(-\frac{1}{(2\pi)^3}a(\underline{u})a(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\exp\left[i\frac{L}{\pi}\underline{w}.(\underline{u}+\underline{\hat{u}})\right]+\right.$$

$$\frac{1}{(2\pi)^3}a(\underline{u})a^+(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\exp\left[i\frac{L}{\pi}\underline{w}.(\underline{u}-\underline{\hat{u}})\right]+$$

$$\frac{1}{(2\pi)^3}a^+(\underline{u})a(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\exp\left[i\frac{L}{\pi}\underline{w}.(\underline{\hat{u}}-\underline{u})\right]-$$

$$\left.\frac{1}{(2\pi)^3}a^+(\underline{u})a^+(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\exp\left[-i\frac{L}{\pi}\underline{w}.(\underline{\hat{u}}+\underline{u})\right]\right).$$

Since

$$\frac{1}{(2\pi)^3}\int_{-\underline{\pi}}^{\underline{\pi}} d^3\underline{\epsilon}\exp[i\underline{m}.\underline{\epsilon}] = \delta_{\underline{m},\underline{0}}$$

the above becomes

$$\frac{1}{2}\sum_{\underline{u}\underline{\hat{u}}}\left(-a(\underline{u})a(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\delta_{\underline{u}+\underline{\hat{u}},\underline{0}} + a(\underline{u})a^+(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\delta_{\underline{u}-\underline{\hat{u}},\underline{0}}+\right.$$

$$\left.a^+(\underline{u})a(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\delta_{\underline{\hat{u}}-\underline{u},\underline{0}} - a^+(\underline{u})a^+(\underline{\hat{u}})u^{1\frac{1}{2}}\hat{u}^{1\frac{1}{2}}\delta_{\underline{\hat{u}}+\underline{\hat{u}},\underline{0}}\right)$$

where, for example, $\delta_{\underline{u}+\underline{\hat{u}},\underline{0}}$ is non zero only when $u^i + \hat{u}^i = 0\ \forall i$ where $i = 1, 2, 3$. Clearly $\delta_{\underline{u}+\underline{\hat{u}},\underline{0}}$ is never non-zero so

$$P^\tau = \frac{1}{2}\sum_{\underline{u}}\left(a(\underline{u})a^+(\underline{u})u^1 + a^+(\underline{u})a(\underline{u})u^1\right).$$

Normal ordering gives

$$P^\tau = \sum_{\underline{u}} u^1 a^+(\underline{u})a(\underline{u})$$

c.f (3.34).

We now wish to obtain the expression for P^{y^1} in terms of creation and annihilation operators. We have

$$P^{y^1} = \frac{1}{2}\int_{\underline{y}} T^{\tau y^1} d^3\underline{y}$$

$$= \frac{1}{2}\int \left(\frac{1}{2}\left(\frac{\partial\phi}{\partial y^1}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial y^2}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial y^3}\right)^2 - \frac{\mu^2}{2}\phi^2\right) d^3\underline{y}$$

$$= \frac{1}{2}\left(\frac{1}{2}\int\left(\frac{\partial\phi}{\partial y^1}\right)^2 d^3\underline{y} - \frac{1}{2}\int\left(\frac{\partial\phi}{\partial y^2}\right)^2 d^3\underline{y} - \frac{1}{2}\int\left(\frac{\partial\phi}{\partial y^3}\right)^2 d^3\underline{y} - \int\frac{\mu^2}{2}\phi^2 d^3\underline{y}\right).$$

By (3.43) this becomes

$$= \frac{1}{2}\left(P^\tau - \frac{1}{2}\int\left(\frac{\partial\phi}{\partial y^2}\right)^2 d^3\underline{y} - \frac{1}{2}\int\left(\frac{\partial\phi}{\partial y^3}\right)^2 d^3\underline{y} - \int\frac{\mu^2}{2}\phi^2 d^3\underline{y}\right). \qquad (3.45)$$

Now it is quite easy to see that for $i = 2, 3$

$$\int_{-\underline{L}}^{\underline{L}}\left(\frac{\partial\phi}{\partial y^i}\right)^2 d^3\underline{y} = \int_{-\underline{L}}^{\underline{L}} d^3\underline{y}\left(\sum_{\underline{u}}\left(\frac{a(\underline{u})iu^i}{(2L)^{\frac{3}{2}}u^{1\frac{1}{2}}}\exp[i\underline{u}.\underline{y}] + \frac{a^+(\underline{u})(-iu^i)}{(2L)^{\frac{3}{2}}u^{1\frac{1}{2}}}\exp[-i\underline{u}.\underline{y}]\right)\right)^2$$

$$= 2\sum_{\underline{u}}\frac{u^{i2}}{u^1}a^+(\underline{u})a(\underline{u}).$$

We can also show that

$$\frac{\mu^2}{2}\int\phi^2 d^3\underline{y} = \sum_{\underline{u}}\frac{\mu^2}{2}\frac{2}{u^1}a^+(\underline{u})a(\underline{u})$$

so that (3.45) becomes

$$P^{y^1} = \frac{1}{2}\left(\sum_{\underline{u}} u^1 a^+(\underline{u})a(\underline{u}) - \sum_{\underline{u}}\left(\frac{(u^{22}+u^{32})}{u^1} + \frac{\mu^2}{u^1}\right)a^+(\underline{u})a(\underline{u})\right)$$

$$= \sum_{\underline{u}}\left(\frac{u^{12} - u^{22} - u^{32} - \mu^2}{2u^1}\right)a^+(\underline{u})a(\underline{u})$$

c.f (2.46). Now

$$P_\tau = P^\tau - P^{y^1}$$

$$= \sum_{\underline{u}} u^1 a^+(\underline{u})a(\underline{u}) - \sum_{\underline{u}}\left(\frac{u^{12} - u^{22} - u^{32} - \mu^2}{2u^1}\right)a^+(\underline{u})a(\underline{u})$$

$$= \sum_{\underline{u}}\left(\frac{2u^{12} + u^{22} + u^{32} - u^{12} + \mu^2}{2u^1}\right)a^+(\underline{u})a(\underline{u})$$

$$= \sum_{\underline{u}}\left(\frac{u^{12} + u^{22} + u^{32} + \mu^2}{2u^1}\right)a^+(\underline{u})a(\underline{u})$$

$$= \sum_{\underline{u}}\left(\frac{\underline{u}^2 + \mu^2}{2u^1}\right)a^+(\underline{u})a(\underline{u}).$$

These expressions for P^{y^1} and P^τ clearly agree with those we obtain by lifting the operators of Chapter 2.

3.2 Particles with Spin

In this section we discuss the possibility of developing front and point form field theories of free fermions. The work is inchoate and is merely intended to illustrate the problems that must be overcome.

3.2.1 Light Cone Dirac Equation

If we choose a tetrad [51] with components h_a^μ satisfying

$$h_a^\mu h_b^\nu g_{\mu\nu} = \eta_{ab} \tag{3.46}$$

then the Dirac equation in an arbitrary coordinate system is

$$i\hat{\gamma}^\mu \Psi;\mu + m\Psi = o \tag{3.47}$$

where

$$\hat{\gamma}^\mu = h_a^\mu \gamma^a \tag{3.48}$$

(the γ^a are the usual Dirac matrices) and

$$\Psi;_\mu = (\partial_\mu - \Gamma_\mu)\Psi$$

with

$$\Gamma_\mu = -\frac{1}{4}(\partial_\mu h_a^\rho + \{{}^{\rho}_{\sigma\mu}\} h_a^\sigma) g_{\nu\rho} h_b^\nu \gamma^b \gamma^a \tag{3.49}$$

[53, 54, 55, 56]. Finding the explicit form of the Dirac equation in a given coordinate system is very complicated in general as it requires a knowledge of the Christoffel symbols and the calculation of some lenghty contractions. A judicious choice of tetrad can simplify things considerably. For this reason one often works in the diagonal tetrad gauge i.e. one chooses three of the tetrad vectors to be parallel to coordinate vectors so that the greatest possible number of tetrad components vanish. In the particular case of light cone coordinates in 3+1 spacetime, even working in the diagonal tetrad gauge, the algebra involved in calculating the Lagrangian by the bare hands method is still very involved. We shall therefore obtain the light cone Dirac equation in a different way, by transforming the Dirac Lagrangian from Cartesian to light cone coordinates and deriving the field equations by the usual variational method. We shall need the Lagrangian anyway to discuss the fermion field theory. The Lagrangian in Minkowski coordinates is

$$i\overline{\Psi}\gamma^\nu \partial_\nu \Psi - m\overline{\Psi}\Psi$$

where $\partial_\nu = \frac{\partial}{\partial x^\nu}$. Using (3.67) and (3.68) in Appendix 1 this becomes

$$i\overline{\Psi}\gamma^0\left(-\frac{\mid \underline{y} \mid}{\nu_\lambda y^\lambda}\frac{\partial}{\partial\tau}+\nu^i\frac{\partial}{\partial y^i}\right)\Psi+i\overline{\Psi}\gamma^k\left(-\frac{y^k}{\nu_\lambda y^\lambda}\frac{\partial}{\partial\tau}+\left(\delta_{ik}+\frac{y^k\nu^i}{\nu_\lambda y^\lambda}\right)\frac{\partial}{\partial y^i}\right)\Psi-m\overline{\Psi}\Psi.$$

Rearranging gives

$$-\frac{i}{\nu_\lambda y^\lambda}\overline{\Psi}(\mid \underline{y}\mid\gamma^0+y^k\gamma^k)\frac{\partial\Psi}{\partial\tau}+i\overline{\Psi}\left(\nu^i\gamma^0+\gamma^i+\frac{y^k\nu^i}{\nu_\lambda y^\lambda}\gamma^k\right)\frac{\partial\Psi}{\partial y^i}-m\overline{\Psi}\Psi.$$

The Lagrangian in light cone coordinates is obtained from the above by multiplying by the Jacobian so that

$$L=-\frac{i}{\mid \underline{y}\mid}\overline{\Psi}(\mid \underline{y}\mid\gamma^0+y^k\gamma^k)\frac{\partial\Psi}{\partial\tau}+i\frac{\nu_\lambda y^\lambda}{\mid \underline{y}\mid}\overline{\Psi}\left(\nu^i\gamma^0+\gamma^i+\frac{y^k\nu^i}{\nu_\lambda y^\lambda}\gamma^k\right)\frac{\partial\Psi}{\partial y^i}-m\overline{\Psi}\Psi\frac{\nu_\lambda y^\lambda}{\mid \underline{y}\mid}. \quad (3.50)$$

Clearly

$$\frac{\partial L}{\partial\overline{\Psi}}=\frac{-i}{\mid \underline{y}\mid}(\mid \underline{y}\mid\gamma^0+y^k\gamma^k)\frac{\partial\Psi}{\partial\tau}+i\frac{\nu_\lambda y^\lambda}{\mid \underline{y}\mid}\left(\nu^i\gamma^0+\gamma^i+\frac{y^k\nu^i}{\nu_\lambda y^\lambda}\gamma^k\right)\frac{\partial\Psi}{\partial y^i}-m\frac{\nu_\lambda y^\lambda}{\mid \underline{y}\mid}\Psi$$

and

$$\frac{\partial L}{\partial\frac{\partial\overline{\Psi}}{\partial y^\mu}}=0$$

which gives the Dirac equations as

$$-\frac{i}{\mid \underline{y}\mid}(\mid \underline{y}\mid+y^k\gamma^k)\frac{\partial\Psi}{\partial\tau}+i\frac{\nu_\lambda y^\lambda}{\mid \underline{y}\mid}\left(\nu^i\gamma^0+\gamma^i+\frac{y^k\nu^i}{\nu_\lambda y^\lambda}\gamma^k\right)\frac{\partial\Psi}{\partial y^i}-m\frac{\nu_\lambda y^\lambda}{\mid \underline{y}\mid}\Psi=0. \quad (3.51)$$

Notice that it is difficult to derive a field theory from (3.50) because it has an explicit space time dependence.

3.2.2 2-Dimensional Light Front Field Theory of Spin 1/2 Particle

Suppose we consider the special case of a 1+1 dimensional spacetime and lightcone coordinates adapted to an observer at rest at the origin then, when $y > 0$, (3.50) becomes

$$-i\overline{\Psi}(\gamma^0+\gamma^1)\frac{\partial\Psi}{\partial\tau}+m\overline{\Psi}\Psi-i\overline{\Psi}\gamma^1\frac{\partial\Psi}{\partial y}.$$

This expression represents the Lagrangian for the light front Dirac equation. The Lagrangian has lost its explicit spacetime dependance so it should be possible to develop a field theory. The light front Dirac equation is

$$i(\gamma^0+\gamma^1)\frac{\partial\Psi}{\partial\tau}+i\gamma^1\frac{\partial\Psi}{\partial y}+m\Psi=0. \quad (3.52)$$

In fact it is easy to derive the Dirac equation in light front coordinates directly from (3.47). Recall that the coordinate transformation from Cartesian to light front coordinates is

$$\tau=x^0+x^1, \quad y=x^1.$$

From the usual transformation law of the metric tensor we can see that a solution of (3.46) is given by

$$h_0^\tau = \frac{\partial \tau}{\partial x^0} = 1, \quad h_1^\tau = \frac{\partial \tau}{\partial x^1} = 1$$
$$h_0^y = \frac{\partial y}{\partial x^0} = 0, \quad h_1^y = \frac{\partial y}{\partial x^1} = 1.$$

Since all the components of the tetrad and metric are constants (the later ensuring that the Christoffel symbols vanish) we have from 3.49 that

$$\Gamma^\mu = 0.$$

From (3.48) we obtain

$$\begin{aligned}\hat{\gamma}^y &= h_0^y \gamma^0 + h_1^y \gamma^1 \\ &= \gamma^1\end{aligned}$$

and

$$\begin{aligned}\hat{\gamma}^\tau &= h_0^\tau \gamma^0 + h_1^\tau \gamma^1 \\ &= \gamma^0 + \gamma^1.\end{aligned}$$

Substituting these results in (3.47) gives (3.52). Suppose we assume a particular 2-dimensional representation of the Dirac matrices [38]

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.53}$$

Plane wave solutions of (3.52) can be found in the usual way ([47] section 1.3 page 48 also [48] section 7.2 page 216). Suppose we express the solution in the form

$$\Psi = \int U(k) \exp[i(k^\tau \tau - k^\tau y - k^y \tau)] dk^\tau dk^y. \tag{3.54}$$

Substituting this in (3.52) we see that we must have

$$\left(i \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} i(k^\tau - k^y) + i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} i(-k^\tau) + m \right) U = 0$$

i.e.

$$\begin{pmatrix} -k^\tau + k^y + m & k^y \\ -k^y & k^\tau - k^y + m \end{pmatrix} \begin{pmatrix} U_A \\ U_B \end{pmatrix} = 0.$$

From these matrix equations we obtain

$$U_A = -\frac{k^y}{k^y - k^\tau + m} U_B \quad \text{and} \quad U_A = -\frac{k^y - k^\tau - m}{k^y} U_B$$

so we can only have a solution when

$$k^{y^2} = (k^y - k^\tau)^2 - \mu^2$$

i.e.

$$k^{\tau^2} - 2k^y k^\tau = \mu^2$$

which is simply the mass shell condition in light front coordinates. Therefore

$$U = \delta(k^{\tau^2} - 2k^y k^\tau - \mu^2) \begin{pmatrix} \frac{(-k^y)}{k^y - k^\tau + m} \\ 1 \end{pmatrix}.$$

Substituting this in (3.54) gives

$$\Psi = \int \delta(k^{\tau^2} - 2k^y k^\tau - \mu^2) \begin{pmatrix} \frac{(-k^y)}{k^y - k^\tau + m} \\ 1 \end{pmatrix} \exp[i(k^\tau \tau - k^\tau y - k^y \tau)] dk^\tau dk^y. \qquad (3.55)$$

Let k_1^τ and k_2^τ denote the zeros of $k^{\tau^2} - 2k^y k^\tau - \mu^2$ then

$$k_1^\tau = k^y + (k^{y^2} + \mu^2)^{\frac{1}{2}} \quad \text{and} \quad k_2^\tau = k^y - (k^{y^2} + \mu^2)^{\frac{1}{2}}.$$

Also we have

$$\frac{d}{dk^\tau}(k^{\tau^2} - 2k^y k^\tau - \mu^2) = 2(k^\tau - k^y).$$

Using [50] formula A.6 page 470 (3.55) becomes

$$\Psi = \left(\int \frac{\delta(k^\tau - (k^y + (k^{y^2} + \mu^2)^{\frac{1}{2}}))}{\mid 2[k^y + (k^{y^2} + \mu^2)^{\frac{1}{2}} - k^y] \mid} \begin{pmatrix} \frac{(-k^y)}{k^y - k^\tau + m} \\ 1 \end{pmatrix} \exp[i(k^\tau \tau - k^\tau y - k^y \tau)]\, dk^\tau dk^y + \right.$$

$$\left. \int \frac{\delta(k^\tau - (k^y - (k^{y^2} + \mu^2)^{\frac{1}{2}}))}{\mid 2[k^y - (k^{y^2} + \mu^2)^{\frac{1}{2}} - k^y] \mid} \begin{pmatrix} \frac{(-k^y)}{k^y - k^\tau + m} \\ 1 \end{pmatrix} \exp[i(k^\tau \tau - k^\tau y - k^y \tau)] dk^\tau dk^y \right)$$

$$= \left(\int \frac{\delta(k^\tau - (k^y + (k^{y^2} + \mu^2)^{\frac{1}{2}}))}{2(k^{y^2} + \mu^2)^{\frac{1}{2}}} \begin{pmatrix} \frac{(-k^y)}{k^y - k^\tau + m} \\ 1 \end{pmatrix} \exp[i(k^\tau \tau - k^\tau y - k^y \tau)]\, dk^\tau dk^y + \right.$$

$$\left. \int \frac{\delta(k^\tau - (k^y - (k^{y^2} + \mu^2)^{\frac{1}{2}}))}{2(k^{y^2} + \mu^2)^{\frac{1}{2}}} \begin{pmatrix} \frac{(-k^y)}{k^y - k^\tau + m} \\ 1 \end{pmatrix} \exp[i(k^\tau \tau - k^\tau y - k^y \tau)] dk^\tau dk^y \right).$$

Now perform the k^τ integral

$$= - \left(\int \begin{pmatrix} \frac{k^y}{(m - (k^{y^2} + \mu^2)^{\frac{1}{2}})} \\ -1 \end{pmatrix} \frac{1}{2(k^{y^2} + \mu^2)^{\frac{1}{2}}} \exp[i(\tau (k^{y^2} + \mu^2)^{\frac{1}{2}} - y(k^y + (k^{y^2} + \mu^2)^{\frac{1}{2}}))] \right.$$

$$+\int\begin{pmatrix}\frac{k^y}{(m+(k^{y^2}+\mu^2)^{\frac{1}{2}})}\\ -1\end{pmatrix}\frac{1}{2(k^{y^2}+\mu^2)^{\frac{1}{2}}}\exp[i(-\tau(k^{y^2}+\mu^2)^{\frac{1}{2}}-y(k^y-(k^{y^2}+\mu^2)^{\frac{1}{2}}))]dk^y\Bigg).$$

In the second integral let $k^y \to -k^y$ then the above becomes

$$-\Bigg(\int\begin{pmatrix}\frac{k^y}{(m-(k^{y^2}+\mu^2)^{\frac{1}{2}})}\\ -1\end{pmatrix}\frac{1}{2(k^{y^2}+\mu^2)^{\frac{1}{2}}}\exp[i(\tau(k^{y^2}+\mu^2)^{\frac{1}{2}}-y(k^y+(k^{y^2}+\mu^2)^{\frac{1}{2}}))]$$

$$+\int\begin{pmatrix}\frac{-k^y}{(m+(k^{y^2}+\mu^2)^{\frac{1}{2}})}\\ -1\end{pmatrix}\frac{1}{2(k^{y^2}+\mu^2)^{\frac{1}{2}}}\exp[-i(\tau(k^{y^2}+\mu^2)^{\frac{1}{2}}-y(k^y+(k^{y^2}+\mu^2)^{\frac{1}{2}}))]dk^y\Bigg)$$

$$=\int a(k^y)u(k^y)\exp[i(\tau,y)(\omega(k^y),k^y)]+a^\dagger(k^y)v(k^y)\exp[-i(\tau,y)(\omega(k^y),k^y)]dk^y$$

where

$$\omega(k^y)=(k^{y^2}+\mu^2)^{\frac{1}{2}}$$

$$(\tau,y)(\omega(k^y),k^y)=\tau\omega(k^y)-y(k^y+\omega(k^y))$$

$$u(k^y)=\begin{pmatrix}\frac{-k^y}{m-(k^{y^2}+\mu^2)^{\frac{1}{2}}}\\ 1\end{pmatrix}\quad\text{and}\quad v(k^y)=\begin{pmatrix}\frac{k^y}{m+(k^{y^2}+\mu^2)^{\frac{1}{2}}}\\ 1\end{pmatrix}. \tag{3.56}$$

It is easy to see that $u(k^y)$ and $v(k^y)$ are orthogonal ([45] page 67) i.e.

$$u(k^y)v(-k^y)=\frac{(-k^y)^2}{\mu^2-(k^{y^2}+\mu^2)}+1=0.$$

It is a far from trivial matter to quantise this system. To begin with we have to be careful to isolate the proper degrees of freedom. If we expand (3.52) we obtain

$$i\frac{\partial\psi_1}{\partial\tau}+i\frac{\partial\psi_2}{\partial\tau}+i\frac{\partial\psi_2}{\partial y}+m\psi_1=0$$

and

$$-i\frac{\partial\psi_1}{\partial\tau}-i\frac{\partial\psi_2}{\partial\tau}-i\frac{\partial\psi_1}{\partial y}+m\psi_2=0.$$

If we add these we find that

$$i\frac{\partial\psi_2}{\partial y}-i\frac{\partial\psi_1}{\partial y}+m\psi_1+m\psi_2=0.$$

This is a constraint and it tells us that only one of the fields ψ_1 or ψ_2 should be quantised [36]. We find that if we ignore this constraint and quantise canonically then the commutators are contrdictory. We have not had time to develop a light front fermion field theory. Undoubtedly this would be worth looking into. It may even be possible to construct a light front field theory that accurately describes an interacting fermion boson system. This may be of more than purely theoretical interest since it is likely that the Feynmann rules would be much simpler than those obtained in field theories based on spacelike instants. This is a hall mark of field theories with constraints. The constraints rule out many diagrams which would otherwise contribute to a given process.

3.3 Appendices

Appendix 1

We wish to find the expression for the Lagrangian corresponding to the light cone Klein-Gordon equation. The Lagrangian for the Klein-Gordon equation in Minkowski coordinates is

$$\frac{1}{2}[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \mu^2\phi^2]. \tag{3.57}$$

The transformation to light cone coordinates is given by equations 2 and 3 of [59] [9]. We have that

$$z^0(\tau) - x^0 = \mid \underline{z} - \underline{x} \mid . \tag{3.58}$$

It is easy to show that

$$\frac{\partial \mathrm{z}^0}{\partial \mathrm{x}^i} = \mid \underline{z} - \underline{x} \mid^{-1} \left(\sum_k (z^k - x^k) \frac{\partial \mathrm{z}^k}{\partial \mathrm{x}^i} - (z^i - x^i) \right)$$

which can be rewritten as

$$\frac{\partial \mathrm{z}^0}{\partial \tau}\frac{\partial \tau}{\partial \mathrm{x}^i} = \mid \underline{z} - \underline{x} \mid^{-1} \left(\sum_k (z^k - x^k) \frac{\partial \mathrm{z}^k}{d\tau}\frac{\partial \tau}{\partial \mathrm{x}^i} + (z^i - x^i) \right).$$

This follows from a simple application of the chain rule remembering that z is a function of τ only. Of course this also implies

$$\frac{\partial \mathrm{z}^\mu}{\partial \tau} = \frac{dz^\mu}{d\tau} = \nu^\mu. \tag{3.59}$$

Using this and

$$y^i = x^i - z^i \tag{3.60}$$

we obtain

$$\nu^0 \frac{\partial \tau}{\partial \mathrm{x}^i} = \mid \underline{y} \mid^{-1} (-\underline{y}.\underline{\nu}\frac{\partial \tau}{\partial \mathrm{x}^i} + y^i)$$

so

$$\frac{\partial \tau}{\partial \mathrm{x}^i} = -\frac{y^i}{\nu_\lambda y^\lambda} \tag{3.61}$$

where, following Derrick, we define

$$\nu_\lambda y^\lambda = \nu^0 \mid \underline{y} \mid - \underline{\nu}.\underline{y}. \tag{3.62}$$

Because

$$\frac{\partial z^j}{\partial x^i} = \frac{\partial \tau}{\partial x^i}\frac{\partial z^j}{\partial \tau} + \frac{\partial y^k}{\partial x^i}\frac{\partial z^j}{\partial y^k} = \frac{\partial \tau}{\partial x^i}\nu^j$$

[9] In this Appendix the Einstein summation convention and explicit summations may appear in the same expression. Latin indices denote an element of the index set 1,2,3 and greek lettres an element of 0,1,2,3.

we have using (3.61)

$$\frac{\partial z^j}{\partial x^i} = -\frac{y^i \nu^j}{\nu_\lambda y^\lambda}. \tag{3.63}$$

From (3.60) and (3.63) we obtain

$$\frac{\partial y^i}{\partial x^j} = \delta_{ij} + \frac{y^j \nu^i}{\nu_\lambda y^\lambda}. \tag{3.64}$$

From (3.58) we have

$$\frac{\partial z^0}{\partial x^0} - 1 = \mid \underline{z} - \underline{x} \mid^{-1} \sum_k (z^k - x^k) \frac{\partial z^k}{\partial x^o}.$$

Using (3.60) and (3.59) and rearranging we find that

$$\frac{\partial \tau}{\partial x^0} = -\frac{\mid y \mid}{\nu_\lambda y^\lambda}. \tag{3.65}$$

Now

$$\frac{\partial z^i}{\partial x^0} = \frac{\partial z^i}{\partial \tau}\frac{\partial \tau}{\partial x^0} + \frac{\partial z^i}{\partial y^j}\frac{\partial y^j}{\partial x^0} = \nu^i \frac{\partial \tau}{\partial x^0} = -\frac{\nu^i \mid \underline{y} \mid}{\nu_\lambda y^\lambda}.$$

where we have used (3.65). Also from (3.60)

$$\frac{\partial y^i}{\partial x^0} = -\frac{\partial z^i}{\partial x^o} = -\frac{\partial z^i}{\partial \tau}\frac{\partial \tau}{\partial x^0}.$$

Using (3.59) and (3.65)

$$= \frac{\nu^i \mid \underline{y} \mid}{\nu_\lambda y^\lambda}. \tag{3.66}$$

We are now in a position to transform the partial derivatives. We have

$$\frac{\partial}{\partial x^0} = \frac{\partial \tau}{\partial x^0}\frac{\partial}{\partial \tau} + \frac{\partial y^i}{\partial x^0}\frac{\partial}{\partial y^i}.$$

Using (3.66) and (3.65) gives

$$\frac{\partial}{\partial x^0} = -\frac{\mid \underline{y} \mid}{\nu_\lambda y^\lambda}\left(\frac{\partial}{\partial \tau} - \nu^i \frac{\partial}{\partial y^i}\right). \tag{3.67}$$

Also

$$\frac{\partial}{\partial x^k} = \frac{\partial \tau}{\partial x^k}\frac{\partial}{\partial \tau} + \frac{\partial y^i}{\partial x^k}\frac{\partial}{\partial y^i}$$

so from (3.61) and (3.64) we have

$$\frac{\partial}{\partial x^k} = -\frac{y^k}{\nu_\lambda y^\lambda}\frac{\partial}{\partial \tau} + \frac{\partial}{\partial y^k} + \frac{y^k \nu^i}{\nu_\lambda y^\lambda}\frac{\partial}{\partial y^i}. \tag{3.68}$$

Substituting these in (3.57) we find that the right hand side of that equation becomes

$$\frac{1}{2}\left[\sum_k \left(\frac{2y^k}{\nu_\lambda y^\lambda}\frac{\partial \phi}{\partial \tau}\frac{\partial \phi}{\partial y^k} - \left(\frac{\partial \phi}{\partial y^k}\right)^2\right) - \sum_k \frac{2y^k}{\nu_\lambda y^\lambda}\frac{\partial \phi}{\partial y^k}\left(\nu^i \frac{\partial \phi}{\partial y^i}\right) - \mu^2 \phi\right].$$

The light cone Lagrangian is obtained from the above by multiplying by the Jacobian i.e. $\frac{\nu_\lambda y^\lambda}{|\underline{y}|}$ so that

$$L = \frac{1}{2}\left[\sum_k \left(\frac{2y^k}{|\underline{y}|}\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y^k} - \frac{\nu_\lambda y^\lambda}{|\underline{y}|}\left(\frac{\partial\phi}{\partial y^k}\right)^2\right) - \sum_k \frac{2y^k}{|\underline{y}|}\frac{\partial\phi}{\partial y^k}\left(\nu^i\frac{\partial\phi}{\partial y^i}\right) - \frac{\nu_\lambda y^\lambda}{|\underline{y}|}\mu^2\phi^2\right] \quad (3.69)$$

(c.f 3.1). We can check that this is the correct expression for the Lagrangian by showing that the Euler Lagrange equation coincides with the light cone Klein-Gordon equation derived by Derrick. To begin with

$$\frac{\partial L}{\partial\frac{\partial\phi}{\partial y^r}} = \frac{1}{2}\left[\frac{2y^r}{|\underline{y}|}\frac{\partial\phi}{\partial\tau} - 2\frac{\partial\phi}{\partial y^r}\frac{\nu_\lambda y^\lambda}{|\underline{y}|} - \sum_k\frac{2y^k}{|\underline{y}|}\frac{\partial\phi}{\partial y^k}\nu^r - \left(\nu^i\frac{\partial\phi}{\partial y^i}\right)\frac{2y^r}{|\underline{y}|}\right]$$

so

$$\frac{\partial\frac{\partial L}{\partial\frac{\partial\phi}{\partial y^r}}}{\partial y^r} = \frac{1}{2}\left[\frac{2y^r}{|\underline{y}|}\frac{\partial^2\phi}{\partial y^r\partial\tau} + \frac{2}{|\underline{y}|^2}\left(|\underline{y}| - \frac{y^{r^2}}{|\underline{y}|}\right)\frac{\partial\phi}{\partial\tau} - 2\frac{\partial\phi}{\partial y^r}\frac{1}{|\underline{y}|^2}\left(|\underline{y}|\frac{\partial\nu_\lambda y^\lambda}{\partial y^r} - \nu_\lambda y^\lambda\frac{y^r}{|\underline{y}|}\right) -\right.$$

$$2\frac{\partial^2\phi}{\partial y^{r^2}}\frac{\nu_\lambda y^\lambda}{|\underline{y}|} - \sum_k\frac{2y^k}{|\underline{y}|}\nu^r\frac{\partial^2\phi}{\partial y^r\partial y^k} - \sum_k\frac{2}{|\underline{y}|^2}\left(|\underline{y}|\,\delta_{kr} - \frac{y^k y^r}{|\underline{y}|}\right)\frac{\partial\phi}{\partial y^k}\nu^r - \left(\nu^i\frac{\partial\phi}{\partial y^i}\right)\frac{2}{|\underline{y}|^2}\left(|\underline{y}| - \frac{y^{r^2}}{|\underline{y}|}\right) -$$

$$\left.\left(\nu^i\frac{\partial^2\phi}{\partial y^r\partial y^i}\right)\frac{2y^r}{|\underline{y}|}\right].$$

Now from (3.62) we have

$$\frac{\partial\nu_\lambda y^\lambda}{\partial y^r} = -\left(\frac{\nu^0 y^r}{|\underline{y}|} + \nu^r\right)$$

so that the above becomes

$$\frac{y^r}{|\underline{y}|}\frac{\partial^2\phi}{\partial y^r\partial\tau} + \frac{2}{|\underline{y}|^3}(|\underline{y}|^2 - y^{r^2})\frac{\partial\phi}{\partial\tau} - \frac{\partial\phi}{\partial y^r}\frac{1}{|\underline{y}|^2}\left(-|\underline{y}|\left(\frac{\nu^0 y^r}{|\underline{y}|} + \nu^r\right) - \nu_\lambda y^\lambda\frac{y^r}{|\underline{y}|}\right) - \frac{\partial^2\phi}{\partial y^{r^2}}\frac{\nu_\lambda y^\lambda}{|\underline{y}|}$$

$$-\sum_k\frac{y^k\nu^r}{|\underline{y}|}\frac{\partial^2\phi}{\partial y^r\partial y^k} - \frac{\nu^r}{|\underline{y}|}\frac{\partial\phi}{\partial y^r} + \sum_k\frac{1}{|\underline{y}|^3}y^k y^r\nu^r\frac{\partial\phi}{\partial y^k} - \left(\underline{\nu}.\frac{\partial\phi}{\partial\underline{y}}\right)\frac{1}{|\underline{y}|^3}(|\underline{y}|^2 - y^{r^2}) - \left(\nu^i\frac{\partial^2\phi}{\partial y^r\partial y^i}\right)\frac{y^r}{|\underline{y}|}$$

and therefore

$$\sum_r\frac{\partial\frac{\partial L}{\partial\frac{\partial\phi}{\partial y^r}}}{\partial y^r} = \frac{1}{|\underline{y}|}\underline{y}.\frac{\partial\frac{\partial\phi}{\partial\tau}}{\partial\underline{y}} + \frac{2}{|\underline{y}|}\frac{\partial\phi}{\partial\tau} + \frac{\nu^0}{|\underline{y}|^2}\underline{y}.\frac{\partial\phi}{\partial\underline{y}} + \frac{1}{|\underline{y}|}\underline{\nu}.\frac{\partial\phi}{\partial\underline{y}} + \frac{\nu_\lambda y^\lambda}{|\underline{y}|^3}\underline{y}.\frac{\partial\phi}{\partial\underline{y}} - \frac{\nu_\lambda y^\lambda}{|\underline{y}|}\frac{\partial^2\phi}{\partial\underline{y}^2} -$$

$$\frac{1}{|\underline{y}|}\underline{y}.\frac{\partial}{\partial\underline{y}}\left(\underline{\nu}.\frac{\partial\phi}{\partial\underline{y}}\right) - \frac{1}{|\underline{y}|}\underline{\nu}.\frac{\partial\phi}{\partial\underline{y}} + \frac{1}{|\underline{y}|^3}(\underline{\nu}.\underline{y})\underline{y}.\frac{\partial\phi}{\partial\underline{y}} - \frac{2}{|\underline{y}|^3}\left(\underline{\nu}.\frac{\partial\phi}{\partial\underline{y}}\right) - \frac{1}{|\underline{y}|}\underline{y}.\frac{\partial}{\partial\underline{y}}\left(\underline{\nu}.\frac{\partial\phi}{\partial\underline{y}}\right).$$

In the above the 3rd, 5th and 9th terms cancel. The 7th and 11th combine to give

$$\frac{1}{|\underline{y}|}\underline{y}.\frac{\partial\frac{\partial\phi}{\partial\tau}}{\partial\underline{y}} + \frac{2}{|\underline{y}|}\frac{\partial\phi}{\partial\tau} - \frac{\nu_\lambda y^\lambda}{|\underline{y}|}\frac{\partial^2\phi}{\partial\underline{y}^2} - \frac{2}{|\underline{y}|^3}\left(\underline{\nu}.\frac{\partial\phi}{\partial\underline{y}}\right) - \frac{2}{|\underline{y}|}\underline{y}.\frac{\partial}{\partial\underline{y}}\left(\underline{\nu}.\frac{\partial\phi}{\partial\underline{y}}\right). \quad (3.70)$$

It is not difficult to see that

$$\underline{\nu}.\frac{\partial}{\partial\underline{y}}\left(\underline{y}.\frac{\partial\phi}{\partial\underline{y}}\right) = \underline{y}.\frac{\partial}{\partial\underline{y}}\left(\underline{\nu}.\frac{\partial\phi}{\partial\underline{y}}\right) + \underline{\nu}.\frac{\partial\phi}{\partial\underline{y}}$$

so (3.70) becomes

$$\frac{1}{|\underline{y}|}\underline{y}.\frac{\partial\frac{\partial\phi}{\partial\tau}}{\partial\underline{y}}+\frac{2}{|\underline{y}|}\frac{\partial\phi}{\partial\tau}-\frac{\nu_\lambda y^\lambda}{|\underline{y}|}\frac{\partial^2\phi}{\partial\underline{y}^2}-\frac{2}{|\underline{y}|}\underline{\nu}.\frac{\partial}{\partial\underline{y}}\left(\underline{y}.\frac{\partial\phi}{\partial\underline{y}}\right). \tag{3.71}$$

Also it is easy to see that

$$\frac{\partial L}{\partial\frac{\partial\phi}{\partial\tau}}=\frac{1}{|\underline{y}|}\underline{y}.\frac{\partial\phi}{\partial\underline{y}}$$

so

$$\frac{\partial\frac{\partial L}{\partial\frac{\partial\phi}{\partial\tau}}}{\partial\tau}=\frac{1}{|\underline{y}|}\underline{y}.\frac{\partial\frac{\partial\phi}{\partial\tau}}{\partial\underline{y}}. \tag{3.72}$$

From (3.71) and (3.72) we have

$$\sum_r\frac{\partial\frac{\partial L}{\partial\frac{\partial\phi}{\partial y^r}}}{\partial y^r}+\frac{\partial\frac{\partial L}{\partial\frac{\partial\phi}{\partial\tau}}}{\partial\tau}=\frac{2}{|\underline{y}|}\underline{y}.\frac{\partial\frac{\partial\phi}{\partial\tau}}{\partial\underline{y}}+\frac{2}{|\underline{y}|}\frac{\partial\phi}{\partial\tau}-\frac{\nu_\lambda y^\lambda}{|\underline{y}|}\frac{\partial^2\phi}{\partial\underline{y}^2}-\frac{2}{|\underline{y}|}\underline{\nu}.\frac{\partial}{\partial\underline{y}}\left(\underline{y}.\frac{\partial\phi}{\partial\underline{y}}\right).$$

Finally, since

$$-\frac{\partial L}{\partial\phi}=\frac{\nu_\lambda y^\lambda}{|\underline{y}|}\mu^2\phi$$

the Euler equation is now easily found to be

$$2\left(\underline{y}.\frac{\partial}{\partial\underline{y}}+1\right)\frac{\partial\phi}{\partial\tau}+\nu^0|\underline{y}|\left(\frac{\partial^2\phi}{\partial\underline{y}^2}-\mu^2\phi\right)+\nu.\left(\underline{y}\left(\frac{\partial^2\phi}{\partial\underline{y}^2}-\mu^2\right)-2\frac{\partial}{\partial\underline{y}}\left(\underline{y}.\frac{\partial\phi}{\partial\underline{y}}\right)\right)=0$$

which agrees with Derricks expression.

Appendix 2

The energy momentum tensor is given by

$$T^{\mu\nu}=\pi^1_\mu g^{\theta\nu}\partial_\theta\phi-g^{\mu\nu}L$$

where

$$\pi^1_\mu=\frac{\partial L}{\partial\frac{\partial\phi}{\partial y^\mu}}$$

([47] 2.41 page 68 and 2.15 page 63). For for the time being $y^0=\tau$. It follows that

$$\pi^1_\tau=-\frac{\partial\phi}{\partial y} \tag{3.73}$$

and

$$\pi^1_y=-\left(\frac{\partial\phi}{\partial\tau}+\frac{\partial\phi}{\partial y}\right) \tag{3.74}$$

so

$$\begin{aligned}T^{\tau\tau}&=\pi^1_\tau g^{s\tau}\partial_s\phi-g^{\tau\tau}L=\pi^1_\tau(g^{\tau\tau}\partial_\tau\phi+g^{y\tau}\partial_y\phi)\\&=\left(\frac{\partial\phi}{\partial y}\right)^2\end{aligned} \tag{3.75}$$

and

$$T^{\tau y} = \pi_\tau^1 g^{\theta y}\partial_\theta\phi - g^{\tau y}L = \pi_\tau^1(g^{\tau y}\partial_\tau\phi + g^{yy}\partial_y\phi) - g^{\tau y}L$$
$$= -\left(\frac{\partial\phi}{\partial y}\left(-\frac{\partial\phi}{\partial\tau} - \frac{\partial\phi}{\partial y}\right) + \frac{1}{2}\left[2\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y} + (\frac{\partial\phi}{\partial y})^2 + \mu^2\phi^2\right]\right)$$
$$= -\frac{1}{2}\left[\mu^2\phi^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right].$$

We note in passing that the stress tensor is symmetric. For example

$$T^{y\tau} = \pi_y^1 g^{\theta\tau}\partial_\theta\phi - g^{\tau y}L = -\frac{\partial L}{\partial\frac{\partial\phi}{\partial y}}\frac{\partial\phi}{\partial y} + L$$

$$= \frac{1}{2}\left[2\frac{\partial\phi}{\partial\tau} + 2\frac{\partial\phi}{\partial y}\right]\frac{\partial\phi}{\partial y} - \frac{1}{2}\left[2\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y} + \left(\frac{\partial\phi}{\partial y}\right)^2 + \mu^2\phi^2\right]$$
$$= -\left(\frac{\mu^2\phi^2}{2} - \left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2\right) = -\frac{1}{2}\left(\mu^2\phi^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right)$$
$$= T^{\tau y}.$$

There is no reason why the stress tensor should be symmetric in general ([47] page 69).

Appendix 3

We have

$$[J^{\tau y}, \phi] = \frac{1}{2}\left(\int dy[\tau T^{\tau y} - yT^{\tau\tau}, \phi']\right)$$
$$= \frac{1}{2}\int dy(\tau[T^{\tau y}, \phi'] - y[T^{\tau\tau}, \phi'])$$
$$= -\frac{1}{2}\frac{1}{2}\int dy\left(\tau[\mu^2\phi^2 - \left(\frac{\partial\phi}{\partial y}\right)^2, \phi'] + 2[y\left(\frac{\partial\phi}{\partial y}\right)^2, \phi']\right)$$
$$= -\frac{1}{2}\frac{1}{2}\int dy\left(\tau\left(\mu^2[\phi^2, \phi'] - [\left(\frac{\partial\phi}{\partial y}\right)^2, \phi']\right) + 2y[\left(\frac{\partial\phi}{\partial y}\right)^2, \phi']\right)$$
$$= -\frac{1}{2}\frac{1}{2}\int dy\left(\tau\mu^2[\phi^2, \phi'] + (2y - \tau)[\left(\frac{\partial\phi}{\partial y}\right)^2, \phi']\right)$$
$$= \frac{1}{2}\frac{1}{2}\int dy\left(\tau\mu^2[\phi', \phi^2] + (2y - \tau)[\phi', \left(\frac{\partial\phi}{\partial y}\right)^2]\right)$$
$$= \frac{1}{2}\frac{1}{2}\int dy\left(\tau\mu^2 2[\phi', \phi]\phi + (2y - \tau)2[\phi', \frac{\partial\phi}{\partial y}]\frac{\partial\phi}{\partial y}\right)$$
$$= -\frac{1}{2}\frac{1}{2}\int dy\left(2\tau\mu^2\frac{i}{2}\epsilon(y - y')\phi + 2(2y - \tau)i\delta(y' - y)\frac{\partial\phi}{\partial y}\right).$$

Using the field equation

$$= -\frac{1}{2}\frac{i}{2}\left(\int 2\tau\frac{1}{2}\epsilon(y - y')\left(2\frac{\partial\frac{\partial\phi}{\partial\tau}}{\partial y} + \frac{\partial\frac{\partial\phi}{\partial y}}{\partial y}\right)dy + 2(2y' - \tau)\frac{\partial\phi}{\partial y'}\right).$$

Integrating by parts

$$= -\frac{1}{2}\frac{i}{2}\left(-\int 2\tau\delta(y-y')\left(2\frac{\partial\phi}{\partial\tau}+\frac{\partial\phi}{\partial y}\right)dy + 2(2y'-\tau)\frac{\partial\phi}{\partial y'}\right)$$

$$= -\frac{1}{2}\frac{i}{2}\left(-4\tau\frac{\partial\phi'}{\partial\tau} - 2\tau\frac{\partial\phi}{\partial y'} + 4y'\frac{\partial\phi}{\partial y'} - 2\tau\frac{\partial\phi}{\partial y'}\right)$$

$$= -\frac{1}{2}\frac{i}{2}\left(-4\tau\frac{\partial\phi'}{\partial\tau} - 4\tau\frac{\partial\phi}{\partial y'} + 4y'\frac{\partial\phi}{\partial y'}\right)$$

$$= -\frac{1}{2}\frac{i}{2}\left(-4\tau\left(\frac{\partial\phi'}{\partial\tau}+\frac{\partial\phi}{\partial y'}\right) + 4y\frac{\partial\phi}{\partial y'}\right)$$

$$= -i\left(-\tau\left(\frac{\partial\phi'}{\partial\tau}+\frac{\partial\phi}{\partial y'}\right) + y\frac{\partial\phi}{\partial y'}\right).$$

On the other hand (3.9) gives

$$[J^{\tau y},\phi] = -i(\tau\partial^y - y\partial^\tau)\phi$$

$$= -i(\tau g^{y\theta}\partial_\theta - y g^{\tau\mu}\partial_\mu)\phi = -i(\tau(g^{y\tau}\partial_\tau + g^{yy}\partial_y) - y g^{\tau y}\partial_y)\phi$$

$$= -i(-\tau(\partial_\tau+\partial_y) + y\partial_y)\phi$$

which agrees with the above. In this way we show that, with the new commutator, the field theory is relativistically invariant. We can also show explicitly that the generators satisfy the correct commutation relations. For example consider

$$[P^y,P^\tau] = [-\frac{1}{2}\frac{1}{2}\int\left(\mu^2\phi^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right)dy, \frac{1}{2}\int\left(\frac{\partial\phi}{\partial y'}\right)^2 dy']$$

$$= -\frac{1}{2}\frac{1}{4}\int dy\int dy'[\mu^2\phi^2 - \left(\frac{\partial\phi}{\partial y}\right)^2, \left(\frac{\partial\phi}{\partial y'}\right)^2]$$

$$= -\frac{1}{2}\frac{1}{4}\int dy\int dy'\left(\mu^2[\phi^2,\left(\frac{\partial\phi}{\partial y'}\right)^2] - [\left(\frac{\partial\phi}{\partial y}\right)^2,\left(\frac{\partial\phi}{\partial y'}\right)^2]\right).$$

Now we use the result that if [A,B] is a c number then $[A^2,B^2] = 2[A,B](BA+AB)$ so that the above becomes

$$-\frac{1}{2}\frac{1}{4}\int dy\int dy'\left(\mu^2 2[\phi,\frac{\partial\phi}{\partial y'}]\left(\frac{\partial\phi}{\partial y'}\phi + \phi\frac{\partial\phi}{\partial y'}\right) - 2[\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial y'}]\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right)\right).$$

The first term is

$$i\mu^2\frac{1}{4}\int dy\int dy'\left(\delta(y-y')\frac{\partial\phi}{\partial y'}\phi + \delta(y-y')\phi\frac{\partial\phi}{\partial y'}\right)$$

where we have used (3.7). In the second summand integrate by parts with respect to y'

$$i\mu^2\frac{1}{4}\int dy\int dy'\left(\delta(y-y')\frac{\partial\phi}{\partial y'}\phi - \frac{\partial}{\partial y'}\delta(y-y')\phi\phi'\right).$$

From [61] page 4 this can be written as

$$i\mu^2\frac{1}{4}\int dy\int dy'\left(\delta(y-y')\frac{\partial\phi}{\partial y'}\phi+\frac{\partial}{\partial y}\delta(y-y')\phi\phi'\right).$$

Integrate second term by parts with respect to y

$$i\mu^2\frac{1}{4}\int dy\int dy'\left(\delta(y-y')\frac{\partial\phi}{\partial y'}\phi-\delta(y-y')\frac{\partial\phi}{\partial y}\phi'\right)=0.$$

The second term is

$$-\frac{1}{4}i\int dy\int dy'\frac{\partial}{\partial y}\delta(y-y')\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right)$$

$$=-\frac{1}{4}\left(i\int dy\int dy'\frac{\partial}{\partial y}\delta(y-y')\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}-i\int dy\int dy'\frac{\partial}{\partial y'}\delta(y-y')\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right)$$

$$=0$$

so that P^y and P^τ commute as required (2.49). Next consider

$$[J^{\tau y},P^\tau]=[\frac{1}{2}\int dy(\tau T^{\tau y}-yT^{\tau\tau}),\frac{1}{2}\int\left(\frac{\partial\phi}{\partial y'}\right)^2dy']$$

$$=-\frac{1}{4}\int dy\int dy'\left(\tau[\frac{1}{2}\left(\mu^2\phi^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right),\left(\frac{\partial\phi}{\partial y'}\right)^2]-y[-\left(\frac{\partial\phi}{\partial y}\right)^2,\left(\frac{\partial\phi}{\partial y'}\right)^2]\right)$$

$$=\frac{1}{4}\left(-\int dy\int dy'\left(\tau[\frac{1}{2}\left(\mu^2\phi^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right),\left(\frac{\partial\phi}{\partial y'}\right)^2]-\int dy\int dy' y[-\left(\frac{\partial\phi}{\partial y}\right)^2,\left(\frac{\partial\phi}{\partial y'}\right)^2]\right)\right).$$

The first term vanishes since it is essentially $[P^y,P^\tau]$ and we have just shown that P^y and P^τ commute. The second term is

$$-\frac{1}{4}\int dy\int dy' 2y[\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial y'}]\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right)$$

$$=\frac{1}{4}\int dy\int dy' 2yi\frac{\partial}{\partial y}\delta(y-y')\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right).$$

From [61] page 4

$$=-\frac{1}{4}\int dy\int dy' 2yi\frac{\partial}{\partial y'}\delta(y-y')\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right).$$

Integrate by parts with respect to y'

$$=\frac{1}{4}\int dy\int dy' 2yi\delta(y-y')\left(\frac{\partial^2\phi}{\partial y'^2}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y'^2}\right)$$

$$=\frac{1}{4}\int dy 2yi\left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2}\right)$$

$$=\frac{1}{4}\int dy 2yi\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y}\right)dy$$

$$= \frac{1}{4} \int dy 2yi \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y}\right)^2.$$

Integrate by parts with respect to y

$$= -2i \frac{1}{4} \int dy \left(\frac{\partial \phi}{\partial y}\right)^2$$

$$= -iP^\tau. \tag{3.76}$$

From 5.21 page 277 [57] or (2.49) we see that Lorentz invariance requires

$$[J^{\tau y}, P^\tau] = i(g^{y\tau} P^\tau - g^{\tau\tau} P^y) = i g^{y\tau} P^\tau$$

$$= -iP^\tau$$

which agrees with (3.76). Next we have

$$[J^{\tau y}, J^{\tau y}] = [\frac{1}{2} \int dy (\tau T^{\tau y} - y T^{\tau\tau}), \frac{1}{2} \int dy' (\tau \hat{T}^{\tau y} - y' \hat{T}^{\tau\tau})]$$

where we use the hat to show that the stress energy component is a function of y'. This becomes

$$\frac{1}{4} \int dy \int dy' (\tau\tau [T^{\tau y}, \hat{T}^{\tau y}] - \tau y' [T^{\tau y}, \hat{T}^{\tau\tau}] - y\tau [T^{\tau\tau}, \hat{T}^{\tau y}] + yy' [T^{\tau\tau}, \hat{T}^{\tau\tau}]).$$

Since

$$-y\tau [T^{\tau y}, \hat{T}^{\tau y}] = \tau y [\hat{T}^{\tau y}, T^{\tau y}]$$

it is easy to see that the 2nd and 3rd terms in the above will cancel on performing integrations. The last term is

$$\frac{1}{4} \int dy \int dy' yy' [\left(\frac{\partial \phi}{\partial y}\right)^2, \left(\frac{\partial \phi}{\partial y'}\right)^2]$$

$$= \frac{1}{4} \int dy \int dy' yy' 2 [\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial y'}] \left(\frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial y'} + \frac{\partial \phi}{\partial y'}\frac{\partial \phi}{\partial y}\right)$$

$$= -\frac{1}{4} \int dy \int dy' yy' i 2 \frac{\partial}{\partial y} \delta(y - y') \left(\frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial y'} + \frac{\partial \phi}{\partial y'}\frac{\partial \phi}{\partial y}\right)$$

$$= -\frac{1}{4} \left(2 \int dy \int dy' yy' i \frac{\partial}{\partial y} \delta(y - y') \frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial y'} - 2 \int dy \int dy' yy' i \frac{\partial}{\partial y'} \delta(y - y') \frac{\partial \phi}{\partial y'}\frac{\partial \phi}{\partial y}\right)$$

$$= 0$$

by symmetry. This leaves only the first term

$$= \frac{1}{4} \tau^2 \int dy \int dy' [\frac{1}{2} \left(\mu^2 \phi^2 - \left(\frac{\partial \phi}{\partial y}\right)^2\right), \frac{1}{2} \left(\mu^2 \phi'^2 - \left(\frac{\partial \phi}{\partial y'}\right)^2\right)]$$

$$= \frac{1}{4}\frac{\tau^2}{4} \int dy \int dy' \left(\mu^4 [\phi^2, \hat{\phi}^2] - \mu^2 [\phi^2, \left(\frac{\partial \phi}{\partial y'}\right)^2] - \mu^2 [\left(\frac{\partial \phi}{\partial y}\right)^2, \hat{\phi}^2] + [\left(\frac{\partial \phi}{\partial y}\right)^2, \left(\frac{\partial \phi}{\partial y'}\right)^2]\right)$$

$$= \frac{1}{4}\frac{\tau^2}{4}\int dy\int dy'\left(\mu^4 2[\phi,\hat{\phi}](\hat{\phi}\phi+\phi\hat{\phi}) - \mu^2 2[\phi,\frac{\partial\phi}{\partial y'}]\left(\frac{\partial\phi}{\partial y'}\phi+\phi\frac{\partial\phi}{\partial y'}\right) - \mu^2 2[\frac{\partial\phi}{\partial y},\hat{\phi}]\left(\hat{\phi}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\hat{\phi}\right) - 2[\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial y'}]\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right)\right)$$

$$= -\frac{1}{4}\frac{\tau^2}{4}\int dy\int dy'\left(\mu^4 2\frac{i}{2}\epsilon(y'-y)(\hat{\phi}\phi+\phi\hat{\phi}) - \mu^2 2i\delta(y-y')\left(\frac{\partial\phi}{\partial y'}\phi+\phi\frac{\partial\phi}{\partial y'}\right) + \mu^2 2i\delta(y-y')\left(\hat{\phi}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\hat{\phi}\right) + 2i\frac{\partial}{\partial y}\delta(y-y')\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right)\right).$$

The second and third terms cancel on carrying out integrations so we have

$$-\frac{1}{4}\frac{\tau^2}{2}\int dy\int dy'\left(\mu^4\frac{i}{2}\epsilon(y-y')(\hat{\phi}\phi+\phi\hat{\phi}) + i\frac{\partial}{\partial y}\delta(y-y')\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right)\right).$$

The second term is

$$-\frac{1}{4}\frac{i\tau^2}{2}\int dy\int dy'\left(\frac{\partial}{\partial y}\delta(y-y')\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'} - \frac{\partial}{\partial y'}\delta(y-y')\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}\right)$$

[61] which is obviously 0. This leaves only the first term which can be written as

$$-\frac{1}{4}\left(i\frac{\mu^4\tau^2}{2}\int dy\int dy'\frac{1}{2}\epsilon(y'-y)(\hat{\phi}\phi+\phi\hat{\phi}) - i\frac{\mu^4\tau^2}{2}\int dy\int dy'\frac{1}{2}\epsilon(y-y')(\hat{\phi}\phi+\phi\hat{\phi})\right)$$

since $\epsilon(-x) = -\epsilon(x)$. Clearly this is zero. Lorentz invariance requires that

$$[J^{\tau y}, J^{\tau y}] = i(g^{\tau\tau}J^{y\tau} - g^{\tau y}J^{\tau y} + g^{\tau y}J^{\tau y} - g^{yy}J^{\tau\tau}) = 0$$

since $J^{\mu\mu} = 0$ and $g^{\tau\tau} = 0$. This agrees with the above. Finally

$$[J^{\tau y}, P^y] = [\frac{1}{2}\int(\tau T^{\tau y} - yT^{\tau\tau})dy, \frac{1}{2}\int\hat{T}^{\tau y}dy']$$

$$= \frac{1}{4}\int dy\int dy'[-yT^{\tau\tau},\hat{T}^{\tau y}] = \frac{1}{4}\int dy\int dy' y[\hat{T}^{\tau y},T^{\tau\tau}]$$

$$= -\frac{1}{4}\int dy\int dy' y[\frac{1}{2}\left(\mu^2\hat{\phi}^2 - \left(\frac{\partial\phi}{\partial y'}\right)^2\right), \left(\frac{\partial\phi}{\partial y}\right)^2]$$

$$= \frac{1}{4}\int dy\int dy'\frac{y}{2}[\left(\frac{\partial\phi}{\partial y}\right)^2, \mu^2\hat{\phi}^2 - \left(\frac{\partial\phi}{\partial y'}\right)^2]$$

$$= \frac{1}{4}\int dy\int dy'\frac{y}{2}\left(\mu^2[\left(\frac{\partial\phi}{\partial y}\right),\hat{\phi}^2] + [\left(\frac{\partial\phi}{\partial y'}\right)^2, \left(\frac{\partial\phi}{\partial y}\right)^2]\right)$$

$$= \frac{1}{4}\int dy\int dy'\frac{y}{2}\left(\mu^2 2[\frac{\partial\phi}{\partial y},\hat{\phi}]\left(\frac{\partial\phi}{\partial y}\hat{\phi}+\hat{\phi}\frac{\partial\phi}{\partial y}\right) + 2[\frac{\partial\phi}{\partial y'},\frac{\partial\phi}{\partial y}]\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right)\right)$$

$$= \frac{1}{4}\int dy\int dy' y\left(\mu^2\delta(y-y')i\left(\frac{\partial\phi}{\partial y}\hat{\phi}+\hat{\phi}\frac{\partial\phi}{\partial y}\right) - i\frac{\partial}{\partial y'}\delta(y-y')\left(\frac{\partial\phi}{\partial y'}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y'}\right)\right).$$

Integrate second term by parts with respect to y' and then perform y' integral throughout

$$= \frac{1}{4}i\int dyy\left(\mu^2\left(\frac{\partial\phi}{\partial y}\phi+\phi\frac{\partial\phi}{\partial y}\right) + \left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial\phi}{\partial y}+\frac{\partial o}{\partial y}\frac{\partial^2\phi}{\partial y^2}\right)\right)$$

$$= \frac{1}{4} i \int dy y \frac{\partial}{\partial y} \left(\mu^2 \phi^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right)$$

$$= -\frac{i}{4} \int dy \left(\phi^2 \mu^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right). \qquad (3.77)$$

From 5.21 page 277 [57] we see that Lorentz invarience requires

$$[J^{\tau y}, P^y] = i(g^{yy} P^\tau - g^{\tau y} P^y) = i(-P^\tau + P^y)$$

$$= -i \frac{1}{2} \int \left(\frac{\partial \phi}{\partial y} \right)^2 dy + \frac{1}{2} \frac{1}{2} \int \left(\mu^2 \phi^2 - \left(\frac{\partial \phi}{\partial y} \right)^2 \right) dy$$

$$= -\frac{i}{4} \int \left(\mu^2 \phi^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right) dy$$

which agrees with (3.77).

Appendix 4: Orthogonality and Completeness of Normal Modes

We have

$$(u_k, u_{k'}) = (-i) \int_y \frac{1}{\sqrt{4\pi}} \frac{((k^2 + \mu^2)^{\frac{1}{2}} + k)^{\frac{1}{2}}}{(k^2 + \mu^2)^{\frac{1}{2}}} \exp[i(k + (k^2 + \mu^2)^{\frac{1}{2}})y - i(k^2 + \mu^2)^{\frac{1}{2}} \tau] \times$$

$$\frac{\partial}{\partial \tau} \left(\frac{1}{\sqrt{4\pi}} \frac{((\hat{k}^2 + \mu^2)^{\frac{1}{2}} + \hat{k})^{\frac{1}{2}}}{(\hat{k}^2 + \mu^2)^{\frac{1}{2}}} \exp[-i(\hat{k} + (\hat{k}^2 + \mu^2)^{\frac{1}{2}})y + i(\hat{k}^2 + \mu^2)^{\frac{1}{2}} \tau] \right) -$$

$$\frac{\partial}{\partial \tau} \left(\frac{1}{\sqrt{4\pi}} \frac{((k^2 + \mu^2)^{\frac{1}{2}} + k)^{\frac{1}{2}}}{(k^2 + \mu^2)^{\frac{1}{2}}} \exp[i(k + (k^2 + \mu^2)^{\frac{1}{2}})y - i(k^2 + \mu^2)^{\frac{1}{2}} \tau] \right) \times$$

$$\frac{1}{\sqrt{4\pi}} \frac{((\hat{k}^2 + \mu^2)^{\frac{1}{2}} + \hat{k})^{\frac{1}{2}}}{(\hat{k}^2 + \mu^2)^{\frac{1}{2}}} \exp[-i(\hat{k} + (\hat{k}^2 + \mu^2)^{\frac{1}{2}})y + i(\hat{k}^2 + \mu^2)^{\frac{1}{2}} \tau] dy$$

$$= (-i) \int_y \frac{i(\hat{k}^2 + \mu^2)^{\frac{1}{2}}}{4\pi} \frac{((k^2 + \mu^2)^{\frac{1}{2}} + k)^{\frac{1}{2}} ((\hat{k}^2 + \mu^2)^{\frac{1}{2}} + \hat{k})^{\frac{1}{2}}}{(k^2 + \mu^2)^{\frac{1}{2}} (\hat{k}^2 + \mu^2)^{\frac{1}{2}}} \exp[i(k + (k^2 + \mu^2)^{\frac{1}{2}})y - i(k^2 + \mu^2)^{\frac{1}{2}} \tau] \times$$

$$\exp[-i(\hat{k} + (\hat{k}^2 + \mu^2)^{\frac{1}{2}})y + i(\hat{k}^2 + \mu^2)^{\frac{1}{2}} \tau] -$$

$$\frac{(-i)(k^2 + \mu^2)^{\frac{1}{2}}}{4\pi} \frac{((k^2 + \mu^2)^{\frac{1}{2}} + k)^{\frac{1}{2}} ((\hat{k}^2 + \mu^2)^{\frac{1}{2}} + \hat{k})^{\frac{1}{2}}}{(k^2 + \mu^2)^{\frac{1}{2}} (k^2 + \mu^2)^{\frac{1}{2}}} \exp[i(k + (k^2 + \mu^2)^{\frac{1}{2}})y - i(k^2 + \mu^2)^{\frac{1}{2}} \tau] \times$$

$$\exp[-i(\hat{k} + (\hat{k}^2 + \mu^2)^{\frac{1}{2}})y + i(\hat{k}^2 + \mu^2)^{\frac{1}{2}} \tau] dy$$

$$= (-i) \int_y dy \frac{i}{4\pi} \frac{((k^2 + \mu^2)^{\frac{1}{2}} + k)^{\frac{1}{2}} ((\hat{k}^2 + \mu^2)^{\frac{1}{2}} + \hat{k})^{\frac{1}{2}}}{(k^2 + \mu^2)^{\frac{1}{2}}} \exp[i(k + (k^2 + \mu^2)^{\frac{1}{2}})y - i(k^2 + \mu^2)^{\frac{1}{2}} \tau] \times$$

$$\exp[-i(\hat{k} + (\hat{k}^2 + \mu^2)^{\frac{1}{2}})y + i(\hat{k}^2 + \mu^2)^{\frac{1}{2}} \tau] +$$

$$\frac{i}{4\pi} \frac{((k^2 + \mu^2)^{\frac{1}{2}} + k)^{\frac{1}{2}} (\hat{k}^2 + \mu^2)^{\frac{1}{2}} + \hat{k})^{\frac{1}{2}}}{(\hat{k}^2 + \mu^2)^{\frac{1}{2}}} \exp[i(k + (k^2 + \mu^2)^{\frac{1}{2}})y - i(k^2 + \mu^2)^{\frac{1}{2}} \tau] \times$$

$$\exp[-i(\hat{k} + (\hat{k}^2 + \mu^2)^{\frac{1}{2}})y + i(\hat{k}^2 + \mu^2)^{\frac{1}{2}} \tau]$$

$$= (-i)\frac{i}{4\pi}((k^2+\mu^2)^{\frac{1}{2}}+k)^{\frac{1}{2}}((\hat{k}^2+\mu^2)^{\frac{1}{2}}+\hat{k})^{\frac{1}{2}}\left(\frac{1}{(k^2+\mu^2)^{\frac{1}{2}}}+\frac{1}{(\hat{k}^2+\mu^2)^{\frac{1}{2}}}\right)\times$$

$$\int dy \exp[iy\{(k+(k^2+\mu^2)^{\frac{1}{2}})-(\hat{k}+(\hat{k}^2+\mu^2)^{\frac{1}{2}})\}]\exp[i\tau\{(\hat{k}^2+\mu^2)^{\frac{1}{2}}-(k^2+\mu^2)^{\frac{1}{2}}\}]$$

$$= \frac{1}{2}((k^2+\mu^2)^{\frac{1}{2}}+k)^{\frac{1}{2}}((\hat{k}^2+\mu^2)^{\frac{1}{2}}+\hat{k})^{\frac{1}{2}}\left(\frac{1}{(k^2+\mu^2)^{\frac{1}{2}}}+\frac{1}{(\hat{k}^2+\mu^2)^{\frac{1}{2}}}\right)\exp[i\tau\{(\hat{k}^2+\mu^2)^{\frac{1}{2}}-(k^2+\mu^2)^{\frac{1}{2}}\}]\times$$

$$\delta(k+(k^2+\mu^2)^{\frac{1}{2}}-(\hat{k}+(\hat{k}^2+\mu^2)^{\frac{1}{2}})). \tag{3.78}$$

Put

$$f(k) = k+(k^2+\mu^2)^{\frac{1}{2}}-\hat{k}+(\hat{k}^2+\mu^2)^{\frac{1}{2}}$$

then

$$f(k) = 0 \to k = \hat{k}.$$

This is easy to see. Let

$$a = \hat{k}+(\hat{k}^2+\mu^2)^{\frac{1}{2}}$$

then $f(k) = 0$ becomes

$$k+(k^2+\mu^2)^{\frac{1}{2}}-a = 0$$

so

$$(k-a)^2 = k^2+\mu^2 \quad \text{or} \quad k^2-2ka+a^2 = k^2+\mu^2$$

and therefore

$$k = \frac{a^2-\mu^2}{2a}$$

$$= \frac{\hat{k}^2+2\hat{k}(\hat{k}^2+\mu^2)^{\frac{1}{2}}+\hat{k}^2+\mu^2-\mu^2}{2(\hat{k}+(\hat{k}^2+\mu^2)^{\frac{1}{2}})} = \hat{k}$$

as required. Also

$$\frac{\partial f}{\partial k}(k) = \frac{(k^2+\mu^2)^{\frac{1}{2}}+k}{(k^2+\mu^2)^{\frac{1}{2}}}$$

so that (3.78) becomes

$$\frac{1}{2}((k^2+\mu^2)^{\frac{1}{2}}+k)^{\frac{1}{2}}((\hat{k}^2+\mu^2)^{\frac{1}{2}}+\hat{k})^{\frac{1}{2}}\left(\frac{1}{(k^2+\mu^2)^{\frac{1}{2}}}+\frac{1}{(\hat{k}^2+\mu^2)^{\frac{1}{2}}}\right)\exp[i\tau\{(\hat{k}^2+\mu^2)^{\frac{1}{2}}-(k^2+\mu^2)^{\frac{1}{2}}\}]\times$$

$$\frac{(k^2+\mu^2)^{\frac{1}{2}}}{(k^2+\mu^2)^{\frac{1}{2}}+k}\delta(k-\hat{k})$$

$$= \frac{1}{2}((k^2+\mu^2)^{\frac{1}{2}}+k)\frac{2}{(k^2+\mu^2)^{\frac{1}{2}}}\frac{(k^2+\mu^2)^{\frac{1}{2}}}{(k^2+\mu^2)^{\frac{1}{2}}+k}\delta(k-\hat{k})$$

$$= \delta(k-\hat{k}).$$

This demonstrates orthonormality and completeness follows similarly.

Appendix 5

We show that (3.30) ensures that (3.32) satisfies (3.31).

$$[\phi(y,\tau),\phi(\hat{y},\tau)]=\int_0^\infty du\int_0^\infty d\hat{u}[a_u\frac{\sqrt{2}}{\sqrt{u}\sqrt{4\pi}}\exp\left[iuy-i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]+$$

$$a_u^+\frac{\sqrt{2}}{\sqrt{u}\sqrt{4\pi}}\exp\left[-iuy+i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right],a_{\hat{u}}\frac{\sqrt{2}}{\sqrt{\hat{u}}\sqrt{4\pi}}\exp\left[i\hat{u}\hat{y}-i\left(\frac{\hat{u}^2+\mu^2}{2\hat{u}}\right)\tau\right]+$$

$$a_{\hat{u}}^+\frac{\sqrt{2}}{\sqrt{\hat{u}}\sqrt{4\pi}}\exp\left[-i\hat{u}\hat{y}+i\left(\frac{\hat{u}^2+\mu^2}{2\hat{u}}\right)\tau\right]].$$

Non vanishing terms in this commutator are

$$\int_0^\infty d\hat{u}\int_0^\infty du\left([a_u,a_{\hat{u}}^+]\frac{2}{4\pi\sqrt{u}\sqrt{\hat{u}}}\exp\left[iuy-i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]\exp\left[-i\hat{u}\hat{y}+i\left(\frac{\hat{u}^2+\mu^2}{2\hat{u}}\right)\tau\right]+\right.$$

$$\left.[a_u^+,a_{\hat{u}}]\frac{2}{4\pi\sqrt{u}\sqrt{\hat{u}}}\exp\left[i\hat{u}\hat{y}-i\left(\frac{\hat{u}^2+\mu^2}{2\hat{u}}\right)\tau\right]\exp\left[-iuy+i\left(\frac{u^2+\mu^2}{2u}\right)\tau\right]\right)$$

$$=-\int_0^\infty du\left(-\frac{1}{2\pi u}\exp[iu(y-\hat{y})]+\frac{1}{2\pi u}\exp[iu(\hat{y}-y)]\right).$$

Put $u=-u$ in the first integral and this becomes

$$-\left(\int_0^{-\infty}-\frac{(-du)}{(-u)(2\pi)}\exp[-iu(y-\hat{y})]+\int_0^\infty\frac{1}{2\pi u}\exp[i\hat{u}(\hat{y}-y)]du\right)$$

$$=-\left(-\int_0^{-\infty}\frac{du}{2\pi u}\exp[iu(\hat{y}-y)]+\int_0^\infty\frac{du}{2\pi u}\exp[i\hat{u}(\hat{y}-y)]\right)$$

$$=-\frac{i}{2(2\pi)}\int_{-\infty}^\infty\frac{2}{iu}\exp[iu(\hat{y}-y)]du=-\frac{i}{2}\epsilon(\hat{y}-y)$$

$$=\frac{i}{2}\epsilon(y-\hat{y}).$$

Appendix 6

We have

$$g_{\mu\nu}k^\mu y^\nu=g_{\tau\tau}k^\tau y^\tau+g_{\tau y^1}k^\tau y^1+g_{\tau y^1}k^{y^1}\tau+g_{y^2y^2}k^{y^2}y^2+g_{y^3y^3}k^{y^3}y^3$$

$$=k^\tau\tau-k^\tau y^1-k^{y^1}\tau-k^{y^2}y^2-k^{y^3}y^3$$

so we can express the mass shell condition as

$$k^{\tau 2}-2k^\tau k^{y^1}-k^{y^2}-k^{y^3}=\mu^2$$

and a solution ϕ of (3.42) as

$$\phi(\underline{y},\tau)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\delta(k^{\tau 2}-2k^\tau k^{y^1}-k^{y^2}-k^{y^3}-\mu^2)\exp[i(k^\tau\tau-k^\tau y^1-k^{y^1}\tau-k^{y^2}y^2-k^{y^3}y^3)]dk^\tau dk^{\underline{y}}.$$

We shall now perform the k^τ integral. Notice that if we differentiate the argument of the delta function with respect to k^τ we obtain

$$2(k^\tau - k^{y^1}).$$

The zeros of the delta function occur when

$$k^\tau = k^{y^1} \pm \sqrt{k^{y^1 2} + k^{y^2 2} + k^{y^3 2} + \mu^2}$$

$$= k^{y^1} \pm \sqrt{\underline{k}_L^2 + \mu^2}$$

so that

$$\phi(\underline{y}, \tau) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int b^+(\underline{u}) \frac{\delta(k^\tau - (k^{y^1} + \sqrt{\underline{k}_L^2 + \mu^2}))}{\mid 2(k^{y^1} + \sqrt{\underline{k}_L^2 + \mu^2} - k^{y^1}) \mid} \exp[i g_{\mu\nu} k^\mu y^\nu]$$

$$+ b^+(\underline{u}) \frac{\delta(k^\tau - (k^{y^1} - \sqrt{\underline{k}_L^2 + \mu^2}))}{\mid 2(k^{y^1} - \sqrt{\underline{k}_L^2 + \mu^2} - k^{y^1}) \mid} \exp[i g_{\mu\nu} k^\mu y^\nu] dk^\tau d^3 \underline{k}_L$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int b^+(\underline{u}) \frac{1}{2(\underline{k}_L^2 + \mu^2)^{\frac{1}{2}}} \exp(i[\sqrt{\underline{k}_L^2 + \mu^2}\tau - (k^{y^1} + \sqrt{\underline{k}_L^2 + \mu^2}) y^1 - k^{y^2} y^2 - k^{y^3} y^3]) +$$

$$b(\underline{u}) \frac{1}{2(\underline{k}_L^2 + \mu^2)^{\frac{1}{2}}} \exp(i[-\sqrt{\underline{k}_L^2 + \mu^2}\tau - (k^{y^1} - \sqrt{\underline{k}_L^2 + \mu^2}) y^1 - k^{y^2} y^2 - k^{y^3} y^3]) d^3 \underline{k}_L.$$

In the second integral let $k^{y^i} \to -k^{y^i}$ then

$$\phi(\underline{y}, \tau) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{b^+(\underline{u})}{2(\underline{k}_L^2 + \mu^2)^{\frac{1}{2}}} \exp i[\sqrt{\underline{k}_L^2 + \mu^2}\tau - (k^{y^1} + \sqrt{\underline{k}_L^2 + \mu^2}) y^1 - k^{y^2} y^2 - k^{y^3} y^3] +$$

$$\frac{b(\underline{u})}{2(\underline{k}_L^2 + \mu^2)^{\frac{1}{2}}} \exp -i[\sqrt{\underline{k}_L^2 + \mu^2}\tau - (k^{y^1} + \sqrt{\underline{k}_L^2 + \mu^2}) y^1 - k^{y^2} y^2 - k^{y^3} y^3] d^3 \underline{k}_L.$$

Appendix 7: Components of Stress Energy Tensor in 3+1 Spacetime

The stress energy tensor is given by

$$T^{\mu\nu} = \pi^1_\mu g^{\theta,\nu} \frac{\partial \phi}{\partial y^\theta} - g^{\mu\nu} L$$

where

$$\pi^1_\mu = \frac{\partial L}{\partial \frac{\partial \phi}{\partial y^\mu}}$$

so

$$\pi^1_\tau = \frac{\partial L}{\partial \frac{\partial \phi}{\partial \tau}} = -\frac{\partial \phi}{\partial y^1}$$

and

$$\pi^1_{y^1} = \frac{\partial L}{\partial \frac{\partial \phi}{\partial y^1}} = -\left(\frac{\partial \phi}{\partial \tau} + \frac{\partial \phi}{\partial y^1}\right)$$

c.f (3.73) and (3.74). Similarly we can show that

$$\pi^1_{y^2} = -\frac{\partial\phi}{\partial y^2}$$

and

$$\pi^1_{y^3} = -\frac{\partial\phi}{\partial y^3}.$$

Recall that

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We can now show that

$$T^{\tau\tau} = \pi^1_\tau g^{\theta\tau}\frac{\partial\phi}{\partial y^\theta} - g^{\tau\tau}L$$

$$= \pi^1_\tau\left(g^{1\tau}\frac{\partial\phi}{\partial y^1} + g^{\tau\tau}\frac{\partial\phi}{\partial y^2} + g^{3\tau}\frac{\partial\phi}{\partial y^3}\right) = \pi^1_\tau\left(-\frac{\partial\phi}{\partial y^1}\right) = \left(\frac{\partial\phi}{\partial y^1}\right)^2.$$

Similarly

$$T^{\tau y^1} = \pi^1_\tau g^{\theta y^1}\frac{\partial\phi}{\partial y^\theta} - g^{\tau y^1}L$$

$$= \pi^1_\tau\left(g^{\tau y^1}\frac{\partial\phi}{\partial\tau} + g^{y^1y^1}\frac{\partial\phi}{\partial y^1}\right) - g^{\tau y^1}L$$

$$= \frac{\partial\phi}{\partial y^1}\left(\frac{\partial\phi}{\partial\tau} + \frac{\partial\phi}{\partial y^1}\right) - \frac{1}{2}\left(2\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y^1} + \sum_j\left(\frac{\partial\phi}{\partial y^j}\right)^2 + \mu^2\phi^2\right)$$

$$= \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial y^1}\right)^2 - \left(\frac{\partial\phi}{\partial y^2}\right)^2 - \left(\frac{\partial\phi}{\partial y^3}\right)^2 - \mu^2\phi^2\right] \tag{3.79}$$

c.f (3.75). Also

$$T^{\tau y^2} = \pi^1_\tau g^{\theta y^2}\frac{\partial\phi}{\partial y^\theta} - g^{\tau y^2}L = \pi^1_\tau g^{y^2y^2}\frac{\partial\phi}{\partial y^2}$$

$$= \frac{\partial\phi}{\partial y^1}\frac{\partial\phi}{\partial y^2} \tag{3.80}$$

In this way we can calculate all the components of the stress tensor. The stress tensor is symmetric. For example we have that

$$T^{y^1\tau} = \pi^1_{y^1}g^{\theta\tau}\frac{\partial\phi}{\partial y^\theta} - g^{\tau y^1}L$$

$$= -\left(\frac{\partial\phi}{\partial\tau} + \frac{\partial\phi}{\partial y^1}\right)g^{y^1\tau}\frac{\partial\phi}{\partial y^1} - g^{\tau y^1}L$$

$$= \left(\frac{\partial\phi}{\partial\tau} + \frac{\partial\phi}{\partial y^1}\right)\frac{\partial\phi}{\partial y^1} - \frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial y^1} - \frac{1}{2}\sum_j\left(\frac{\partial\phi}{\partial y^j}\right)^2 - \frac{\mu^2}{2}\phi^2$$

which is the same as (3.79). Also

$$T^{y^2\tau} = \pi^1_{y^2}g^{\theta\tau}\frac{\partial\phi}{\partial y^\theta} - g^{\tau y^2}L = -\pi^1_{y^2}\frac{\partial\phi}{\partial y^1} = \frac{\partial\phi}{\partial y^2}\frac{\partial\phi}{\partial y^1}$$

c.f (3.80).

Chapter 4

Conclusion and Prospect

It is evident that both the front form quantum mechanics of Chapter 2 and the field theory described in Chapter 3, although entirely satisfactory in themselves, are limited in that they are concerned only with free particles. Although logically the next step it is not easy to generalise to a theory incorporating interactions. Many attempts have been made to circumvent the well known no go theorms that appear to exclude relativistically invariant theories of interacting particles. To see how intricate the problem is we shall give a brief review of work of Coester [84, 85].

At the very least we require that the state space $\mathcal{H}$ be a module for a representation U of the Poincare group with generators corresponding to total momentum $\overline{P}^i$, time translation $\overline{P}^0 = \overline{H} = (\overline{P}^2 + \overline{M}^2)^{\frac{1}{2}}$, spatial rotation $\underline{\overline{J}} = (\overline{J}^{23}, \overline{J}^{13}, \overline{J}^{12})$ and boosts $\underline{\overline{K}} = (\overline{J}^{01}, \overline{J}^{02}, \overline{J}^{03})$. These generators represent an interacting system. Generators associated with the system with interactions switched off are denoted by the same symbol with the suffix 0. The question arises as to what sort of terms can be added to the non interacting multiparticle generators to give $\underline{\overline{P}}, \overline{H}, \underline{\overline{J}}$ and $\underline{\overline{K}}$ satisfying the the usual commutation relations. Coester has shown that the answer depends on the Dirac form under consideration. Following Bakamjian and Thomas [87, 88, 89] we express the free generators in terms of the mass operator and another set of operators that depends on the Dirac form and which are functions of the Hamiltonians and mass operator. The generators of the kinematic subgroup are unaltered by this change of variables. We add an interaction ν to the mass operator. Preserving the commutation relations generally requires that additional terms be added to the new form-dependent[1] operators and the Hamiltonians but if we require that ν commute with the new operators and the generators of the kinematic subgroup then the

[1] In the sense of Dirac.

form-dependent operators need not be altered and the perturbed generators automatically satisfy the Poincare algebra. We restrict ourselves to the instant form for the time being so we put

$$\overline{M} = \overline{M}_0 + \overline{v}$$

and require that $\overline{v}$ commute with $\underline{\overline{P}} = \underline{\overline{P}}_0$, $\underline{\overline{J}} = \underline{\overline{J}}_0$ (these being the generators of the kinematic subgroup of the instant form) and $\underline{\overline{X}} = \underline{\overline{X}}_0$ (the Newton Wigner position operator). The dynamical generators $\overline{H}$ and $\underline{\overline{K}}$ are peturbed by $\overline{v}$ i.e. $\underline{\overline{K}} \neq \underline{\overline{K}}_0$ and $\overline{H} \neq \overline{H}_0$. The reqirement that $\overline{v}$ commute with the generators of the kinematic subgroup puts some restrictions on $\overline{v}$; however we also require that the model satisfy the condition of macrolocality, essentially that when components of the composite are separated by large spacelike intervals or when interactions between particles in different clusters are actually switched off, they should behave as autonomous systems. This places more stringent conditions on $\overline{v}$.

We shall try to make the notion of macrolocality more precise. There are a number of ways of defining macrolocality not all of which are equivalent. Suppose we have a collection of N particles which, quantum mechanically, is described by some Hilbert space $\mathcal{H}$. Let a denote a partition of N into n_a clusters where a_i is the ith cluster of a. Let $\mathcal{H}_{a_i}$ be the Hilbert space associated with the cluster a_i. Clearly

$$\mathcal{H} = \otimes_{i=1}^{n_a} \mathcal{H}_{a_i}.$$

Let U_{a_i} denote a representation of the Poincare group on $\mathcal{H}_{a_i}$. The representation corresponding to non interacting clusters is therefore

$$U_a = \otimes_{i=1}^{n_a} U_{a_i}.$$

Let $G(a_i)$ be a generator of U_{a_i} then the generators G for U on $\mathcal{H}$ are said to satisfy the cluster decomposition condition if

$$G_a = G(a_1) \otimes_{i=2}^{n_a} I_i + .. + \otimes_{i=1}^{j-1} I_i \otimes G(a_j) \otimes_{i=j+1}^{n_a} I_i + .. + \otimes_{i=1}^{n_a-1} I_i \otimes G(a_{n_a})$$

for every partition a. In the above G_a represents the operator obtained from G by switching off interactions between clusters and I_i is the identity operator on $\mathcal{H}_{a_i}$. Intuitively we can see that this corresponds to one conception of macrolocality. Alternatively we might expect that as clusters are separated by larger spacelike distances[2] the representations U

[2]Without the intercluster interactions being swithched off which was the case in cluster seperability

and U_a should become identical. This is realised by requiring that

$$\underset{\min|\underline{d}_i-\underline{d}_j|\to\infty}{\text{s-lim}}[U(r,\Lambda)-U_a(r,\Lambda)]T_a(\underline{d}_1,..,\underline{d}_{n_a})=0$$

where

$$T_a(\underline{d}_1,..,\underline{d}_{n_a})=\otimes_{i=1}^{n_a}U_{a_i}(\underline{d}_i,I).$$

A third definition of macrolocality requires that widely separated subsytems should evolve as free components. This is the definition with which we shall be concerned. To make it more precise we need to develop the dynamics of the interacting system.

Denote by $\overline{M}_{a_i}$ the mass operator of U_{a_i} in $\mathcal{H}_{a_i}$. Suppose it has eigenvalues $m_{\alpha,i}$ with associated eigenvectors $\phi_{\alpha,i}$. The $\phi_{\alpha,i}$ are simultaneously eigenvectors of $\overline{\underline{j}}^2_{a_i}$ (where $\overline{\underline{j}}_{a_i}$ is the spin operator in $\mathcal{H}_{a_i}$) with eigenvalues $S_{\alpha,i}$. In general these eigenvalues will be highly degenerate. We continue to denote by $\phi_{\alpha,i}$ the eigenvectors in $\mathcal{H}_{a_i}$ corresponding to a *particular* $(m_{\alpha,i},S_{\alpha,i})$. The subspace of $\mathcal{H}_{a_i}$ spanned by the $\phi_{\alpha,i}$, which we shall denote by $\mathcal{H}_{f,\alpha,i}$[3], is a module for the irreducible representation $(m_{\alpha,i},S_{\alpha,i})$ of the Poincare group. Denote the generators of this representation $g(\alpha,i)$. We have

$$\phi_{\alpha,i}=\int d^3\underline{p}_{\alpha,i}\sum_{\overline{\mu}_{\alpha,i}}|\,\overline{\underline{p}}_{\alpha,i},\overline{\mu}_{\alpha,i}>\chi_{\alpha,i}(\overline{\underline{p}}_{\alpha,i},\overline{\mu}_{\alpha,i})$$

$(-S_{\alpha,i}\leq\overline{\mu}_{\alpha,i}\leq S_{\alpha,i})$ where $|\,\overline{\underline{p}}_{\alpha,i},\overline{\mu}_{\alpha,i}>$ are simultaneous generalised eigenvectors of $\overline{\underline{P}}_{a_i}$ and $\overline{j}_{a_i,3}$ with eigenvalues $\overline{\underline{p}}_{\alpha,i}$ and $\overline{\mu}_{\alpha,i}$. Therefore $\mathcal{H}_{f,\alpha,i}$ is the subspace of $\mathcal{H}_{a_i}$ spanned by the $|\,\overline{\underline{p}}_{\alpha,i},\overline{\mu}_{\alpha,i}>$. We can interpret this in another way. We identify $\mathcal{H}_{f,\alpha,i}$ with square integrable functions $\chi_{\alpha,i}$ of $\overline{\underline{p}}_{\alpha,i}$ and $\overline{\mu}_{\alpha,i}$ and construct an injection operator $\overline{\Phi}_{\alpha,i}:\mathcal{H}_{f,\alpha,i}\to\mathcal{H}_{a_i}$ by writing

$$\overline{\Phi}_{\alpha,i}\chi=\int d^3\overline{\underline{p}}_{\alpha,i}\sum_{\alpha\overline{\mu}_{\alpha,i}}|\,\overline{\underline{p}}_{\alpha,i},\overline{\mu}_{\alpha,i}>\chi_{\alpha,i}(\overline{\underline{p}}_{\alpha,i},\overline{\mu}_{\alpha,i}).$$

We now form the *channel subspace* $\mathcal{H}_{f,\alpha}$ of $\mathcal{H}$

$$\mathcal{H}_{f,\alpha}=\otimes_{i=1}^{n_a}\mathcal{H}_{f,\alpha,i}$$

and define the following operators therein

$$G_{f,\alpha}=g(\alpha,1)\otimes_{i=2}^{n_a-1}I_i+....+\otimes_{i=1}^{j-1}I_i\otimes g(\alpha,j)\otimes_{k=j+1}^{n_a}I_k+....+\otimes_{i=1}^{n_a-1}I_i\otimes g(\alpha,n_a)$$

[3]The suffix f stands for free. A state in this space belongs to an irreducible representation and so can represent an elementary particle emerging from the fire ball of the interaction. These are the asymptotic spaces in the sense of scattering theory.

$$\overline{\Phi}_\alpha = \otimes_{i=1}^{n_a} \overline{\Phi}_{\alpha,i}.$$

We then form

$$\mathcal{H}_f = \oplus_\alpha \mathcal{H}_{f,\alpha}$$

and define the operators on this space by taking direct sums of the expressions above i.e.

$$G_f = \oplus_\alpha G_{f\alpha} \tag{4.1}$$

and

$$\overline{\Phi} = \oplus_\alpha \overline{\Phi}_\alpha.$$

$\mathcal{H}$ and $\mathcal{H}_f$ can be written as direct integrals over total momentum i.e.

$$\mathcal{H}_f = \int_\oplus d^3\underline{\overline{p}}\, \hat{\mathcal{H}}_f(\overline{p}) \quad \text{and} \quad \mathcal{H} = \int_\oplus d^3\underline{\overline{p}}\, \hat{\mathcal{H}}(p).$$

Coester argues that we must formulate relativistic theories of interacting particles as scattering experiments where observables can only be measured at temporal infinity. If not then position can be measured at any moment and relativistic invariance requires invariant world lines. These are forbidden by the no interaction theorem [90]. We define a scattering state ψ as one that becomes equal to a state describing non-interacting particles in the remote past and future; i.e. a state ψ in $\mathcal{H}$ is a scatering state if there exists $\chi \in \mathcal{H}_f$ such that

$$\lim_{t \to -\infty} \| \psi(t) - \overline{\Phi} \exp(-it\overline{H}_f)\chi \| = 0$$

where $\overline{H}_f$ is defined via (4.1). We define the wave operators, when they exist, as

$$\Omega_\pm(\overline{H}, \overline{\Phi}, \overline{H}_f) = \underset{t \to \pm\infty}{\text{s-lim}} \exp(i\overline{H}t)\overline{\Phi} \exp(-i\overline{H}_f t)$$

then the scattering operator is given by

$$S = \Omega_+^* \Omega_-.$$

We know that wave operators generally exist only when the interacting Hamiltonian is a small perturbation of the free Hamiltonian. In the present case we require that

$$\overline{H\Phi} = \overline{\Phi H}_f + \overline{V}$$

where V is short range [84]. Naturally we require that S is Lorentz invariant. The existence of the wave operator is sufficient to ensure that S commutes with $\overline{H}_f$ since the wave operators then satisfy the so called intertwining relations i.e.

$$\overline{H}\Omega_\pm = \Omega_\pm \overline{H}_f.$$

It is then easy to show from the defintion of S that

$$[\overline{H}_f, S] = 0.$$

Because of the way $\overline{\Phi}_{\alpha,i}$ was constructed

$$\overline{P^i}\,\overline{\Phi} = \overline{\Phi P^i_f} \quad \text{and} \quad \overline{J^i}\,\overline{\Phi} = \overline{\Phi J^i_f}.$$

Using these relations and the defintion of S we obtain

$$[\overline{P}_f, S] = [\overline{J}_f, S] = 0.$$

To show that S is Lorentz invariant it only remains to prove that

$$[\overline{K}^i_f, S] = 0 \tag{4.2}$$

but this is more difficult. For example we can show that (4.2) holds iff

$$\lim_{t\to\infty} \| \, (\overline{K^i}\,\overline{\Phi} - \overline{\Phi K^i_f}) \exp(\pm i \overline{H}_f t)\chi \, \| = 0 \tag{4.3}$$

for all χ in some dense set in $\mathcal{H}_f$. Recall that $\overline{K^i}$ depends on $\overline{\nu}$ so this places more restrictions on $\overline{\nu}$.

If we consider the special case of two particles, where all observed particles are elementary, the above formalism becomes a little less abstract. The operator algebras in $\mathcal{H}$ and $\mathcal{H}_f$ are isomorphic and the same symbol may be used to denote a state in either space that is we can take $\overline{\Phi} = I$. Initially one might describe the particles in their momentum space representations, the state space of the combined system is then

$$\mathcal{H} = L^2(\underline{\overline{p}}^{(1)}, d^3\underline{\overline{p}}^{(1)}) \otimes L^2(\underline{\overline{p}}^{(2)}, d^3\underline{\overline{p}}^{(2)}).$$

We may pass to centre of mass coordinates $\underline{\overline{p}}$ and $\underline{\overline{k}}$ and it is well known that we can define an isomorphism

$$L^2(\underline{\overline{p}}^{(1)}, d^3\underline{\overline{p}}^{(1)}) \otimes L^2(\underline{\overline{p}}^{(2)}, d^3\underline{\overline{p}}^{(2)}) \to L^2(\underline{\overline{p}}, d^3\underline{\overline{p}}) \otimes L^2(\underline{\overline{k}}, d^3\underline{\overline{k}}).$$

This gives a new representation of $\mathcal{H}$ i.e.

$$\mathcal{H} = \mathcal{H}_{\text{cm}} \otimes \mathcal{H}_{\text{int}}$$

with the obvious definitions of $\mathcal{H}_{\text{cm}}$ and $\mathcal{H}_{\text{int}}$. The operators representing total mommentum are of the form $\overline{P} \otimes I$. We now see how to describe $\mathcal{H} = \mathcal{H}_f$ as a direct integral. Suppose the $\underline{\overline{P}}$ had pure point spectra with corresponding simultaneous eigenvectors $\phi_{\underline{\overline{p}}}$

then since $\mathcal{H}_{CM} = \oplus\mathcal{H}_{\underline{\overline{p}}}$ (where $\mathcal{H}_{\underline{\overline{p}}}$ is the 1-dimensional space spanned by the eigenvector with eigenvalues $\underline{\overline{p}}$) we have

$$\mathcal{H} = (\oplus\mathcal{H}_{\underline{\overline{p}}}) \otimes \mathcal{H}_{\text{int}} = \oplus(\mathcal{H}_{\underline{\overline{p}}} \otimes \mathcal{H}_{\text{int}}).$$

but since the spectra of the $\underline{\overline{P}}$ are purely continuous we should write

$$\mathcal{H} = \int_{\oplus} \mathcal{H}_{\underline{\overline{p}}} \otimes \mathcal{H}_{\text{int}} d^3\underline{\overline{p}}.$$

We now define

$$\overline{H} = (|\,\underline{\overline{p}}\,|^2 + (\overline{M}_0 + \overline{v})^2)^{\frac{1}{2}} \quad \text{and} \quad \overline{H}_0 = (|\,\underline{\overline{p}}\,|^2 + \overline{M}_0)^{\frac{1}{2}}.$$

These can be written as direct integral operators

$$\overline{H} = \int_{\oplus} \overline{H}'(|\,\underline{\overline{p}}\,|) d^3\underline{\overline{p}} \quad \text{and} \quad H_0 = \int_{\oplus} \overline{H}'_0(|\,\underline{\overline{p}}\,|) d^3\underline{\overline{p}}$$

where

$$\overline{H}'_0(|\,\underline{\overline{p}}\,|) = I \otimes (|\,\underline{\overline{p}}\,|^2 + \overline{M}_0^2)^{\frac{1}{2}} \quad \text{and} \quad \overline{H}'(|\,\underline{\overline{p}}\,|) = I \otimes (|\,\underline{\overline{p}}\,|^2 + (\overline{M}_0 + \overline{v})^2)^{\frac{1}{2}}$$

which are well defined fibre operators because $\overline{M}$ and $\overline{M}_0$ are operators on $\mathcal{H}_{\text{int}}$ [86]. Similarly we can represent the wave operators as direct integrals. We have

$$\Omega(\overline{H}, \overline{H}_0) = \int_{\oplus} \Omega(\overline{H}'(\underline{\overline{p}}), \overline{H}'_0(\underline{\overline{p}})) d^3\underline{\overline{p}} = \int_{\oplus} I \otimes \Omega(\overline{M}_0 + \overline{v}, \overline{M}_0) d^3\underline{\overline{p}}$$

where the last step follows from the application, fibrewise, of a well known result from scattering theory. If we impose further restrictions on $\overline{v}$ then we arrive at a macrocausal theory describing 2 interacting particles i.e we have

$$\operatorname*{s-lim}_{|\underline{d}_1 - \underline{d}| \to \infty} (S - I) T(\underline{d}_1, \underline{d}_2) = 0.$$

S tends to I because we considered *elementary* particles [4]. This result allows Coester to induce macrocausal theories for larger systems. Unfortunately it appears that no one has managed to exhibit a $\overline{v}$ satisfying the necessary restrictions. His work remains a rather abstract existence theorem.

[4]In general macrolacality requires

$$\operatorname*{s-lim}_{\min|\underline{d}_1 - \underline{d}_2| \to \infty} (S - S_a) T(\underline{d}_1, \underline{d}_2) = 0.$$

Coester shows that his results can equally well be developed in the front form. In this case the analogue of the Newton Wigner operator is the front form spin. The interaction ν is added to $\overline{M}_0$ and is required to commute with the generators of the kinematic subgroup (2.47) and the front form spin which are therefore left invariant by the addition of the interaction. The Hamiltonians (2.48) are now ν dependent. By construction Φ and hence S will be invariant under the kinematic generators while invariance of S under P^1 will follow from the intertwining relations. Invariance under the perturbed J^{31} and J^{21} will not be automatic and will require that ν satisfy other criteria. Notice however that now we only require two further relations from ν as opposed to the three conditions (4.3) needed in the instant form. It is because of this that of all the forms the front form is considered as offering the best means of constructing relativistically invariant theories of directly interacting particles. Having said that, no specific operators ν have been forthcoming. The difficulty of obtaining interaction terms that lead to macrolocal theories has led some to abandon this restriction. In [91] Mosley develops an interacting, relativistic, point form classical mechanics that is not macrolocal. It does not even reduce to the free particle theory advanced by the same author when interactions are switched off. Since it has long been hoped that relativity would place just this kind of severe constraint on subatomic particle interactions it may be premature to acknowledge defeat.

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