# GLOBAL TIME ESTIMATES FOR SOLUTIONS TO HIGHER ORDER STRICTLY HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

by

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**Abstract**

In this thesis we consider the Cauchy problem for general higher order constant coefficient strictly hyperbolic PDEs with lower order terms and show how the behaviour of the characteristic roots determine the rate of decay in the associated $L^p - L^q$ estimates.

In particular, we show under what conditions the solution behaves like that of the standard wave equation, the wave equation with dissipation or the Klein–Gordon equation. We explain the various factors involved, such as the presence of multiple roots, the size of the sets of multiplicity and the order with which characteristics meet the real axis, yield different rates of decay. As an example, we show how the results obtained can be applied to the Fokker–Planck equation.

In the second part, we derive $L^p - L^q$ estimates for wave equations with a bounded time dependent coefficient. A classification of the oscillating behaviour of the coefficient is given and related to the estimate which can be obtained.

# Contents

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# Chapter 0: Preliminaries

## 0.1 Notation

Throughout this thesis we will use the following notation:

**Arbitrary Constant:** We will use $C$, sometimes with suffices, to denote an arbitrary constant (depending on its suffices); it may differ at each occurrence, unless explicitly stated otherwise.

**Derivatives:** Use $D_{x_j} = -i\partial_{x_j} = -i\frac{\partial}{\partial x_j}$, where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $i = \sqrt{-1}$; and $D_x = (D_{x_1}, \dots, D_{x_n})$. Similarly, $D_t = -i\partial_t = \frac{1}{i}\frac{\partial}{\partial t}$. We use multi-index notation: if $\alpha = (\alpha_1, \dots, \alpha_n)$ and each $\alpha_l \geq 0$ is an integer then $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$. Also, write $|\alpha| = \sum_{l=1}^n \alpha_l$.

**Function Spaces:** We use the standard notation for $L^p(U)$ spaces, that is the space of functions that are $p$-times integrable on the set $U \subset \mathbb{R}^n$. We denote Sobolev spaces by

$$W_p^s := W_p^s(\mathbb{R}^n) = \left\{ f \in L^p : \|f\|_{W_p^s(\mathbb{R}^n)} < \infty \right\} ,$$

where $s \in \mathbb{R}$, $p \geq 1$, and $\|\cdot\|_{W_p^s(\mathbb{R}^n)}$ is the standard Sobolev norm.

**Fourier Transform** For $f \in \mathcal{S}$ (the space of Schwartz functions) and $f \in L^1(\mathbb{R}^n)$ define the Fourier transform of a function

$$\widehat{f}(\xi) \equiv (\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$

and

$$(\mathcal{F}^{-1}f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} f(\xi)\, d\xi\,;$$

for more general $f$, such as $f \in L^2(\mathbb{R}^n)$ or even $f \in \mathcal{D}'(\mathbb{R}^n)$ the corresponding natural definitions are employed.

**Symbol Classes** For $m \in \mathbb{R}$, we write $S^m \equiv S^m_{(1,0)}$ for symbols of order $m$; that is, functions $f \in \mathcal{S}$ such that $|D^\alpha f(x)| \leq C_\alpha (1 + |x|)^{m-|\alpha|}$.

**Balls** We write $B_r(c)$ to denote the ball in $\mathbb{R}^n$ of radius $r$ with centre $c$; also, use $B'_r(c) = B_r(c) \setminus \{c\}$ to denote the punctured ball.

## 0.2 Hyperbolic Differential Operators

We give some standard definitions from the theory of hyperbolic partial differential equations, as used in, for example, [ES92], [Hör83b], [Nis00] and [Trè80].

**Definition 0.1.** *Let $L = L(D_x, D_t)$ be a linear constant coefficient $m^{th}$ order partial differential operator. We define the* principal part *of the operator $L$ to be the homogeneous $m^{th}$ order part and denote this by $L_m = L_m(D_x, D_t)$.*

**Definition 0.2.** *A linear constant coefficient $m^{th}$ order partial differential operator $L = L(D_x, D_t)$ is called* hyperbolic *if, for each $\xi \in \mathbb{R}^n$, the auxiliary polynomial of the principal part, $L_m(\xi, \tau)$, only has* $(m)$ real roots *with respect to $\tau$. $L$ is said to be* strictly hyperbolic *if, at each $\xi \in \mathbb{R}^n \backslash \{0\}$, these roots are pairwise distinct. We denote the roots of $L_m(\xi, \tau)$ with respect to $\tau$ by $\varphi_1(\xi) \leq \cdots \leq \varphi_m(\xi)$, and if $L$ is strictly hyperbolic the above inequalities are strict for $\xi \neq 0$.*

Examples of such operators include the well-known wave equation; other examples such as those arising from the Fokker–Planck equation are discussed in Chapter 5 and explored more fully in, for example, [VR04].

**Remarks:**

**1.** The condition for hyperbolicity arises naturally in the study of the Cauchy problem for linear partial differential operators and it can be shown that it is a necessary condition for $C^\infty$ well-posedness of the problem (that is existence and uniqueness of a solution which depends continuously on the initial data); this is discussed in [ES92] and [Hör83b], for example. Strict hyperbolicity is sufficient for $C^\infty$ well-posedness of the Cauchy problem for such an operator with any lower order terms; if the operator is only hyperbolic (sometimes called *weakly hyperbolic*) the lower order terms must satisfy additional conditions for $C^\infty$ well-posedness, so-called *Levi conditions.* For this reason, we only consider strictly hyperbolic operators (with lower order terms), and thus we know that there exists a unique solution to

$$\left.\begin{aligned} D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0\,, \\ D_t^l u(x,0) = f_l(x),\ l = 0,\dots,m-1. \end{aligned}\right\}$$

**2.** Sometimes, for example in [Trè80], in the definition of a hyperbolic operator the polynomial $L(\xi, i\tau)$ is used as it is better suited to taking the partial Fourier transform in $x$, corresponding as it does to $L(D_x, \partial_t)$; in this case, we require the roots with respect to $\tau$ to be purely imaginary. However, the definition given above is more standard, and thus adopted here throughout.

**Definition 0.3.** *Given a linear constant coefficient $m^{th}$ order partial differential operator $L = L(D_x, D_t)$, we denote the roots of the associated auxiliary polynomial $L(\xi,\tau)$ with respect to $\tau$ by $\tau_1(\xi), \dots, \tau_m(\xi)$. $L(\xi,\tau)$ is called the* characteristic polynomial *of $L$ and the roots are called the* characteristic roots *of the full operator.*

Clearly, if $L$ is a homogeneous operator then the characteristic roots $\tau_k(\xi)$, $k = 1, \dots, m$, coincide, possibly after reordering, with the roots $\varphi_k(\xi)$, $k = 1, \dots, m$, of the operator $L_m$ from Definition 0.2. However, in general there is no natural ordering on the roots $\tau_k(\xi)$ as they may be complex-valued.

# Part I

# Constant Coefficient Operators

# Chapter 1: Introduction

## 1.1 Background

The study of $L^p - L^q$ decay estimates, or *Strichartz estimates*, for linear evolution equations began in 1970 when Robert Strichartz published two papers, [Str70a] and [Str70b]. He proved that if $u = u(x,t)$ satisfies the Cauchy problem (that is, the initial value problem) for the homogeneous linear wave equation

$$\left.\begin{aligned} &\partial_t^2 u(x,t) - \Delta_x u(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty)\,, \\ &u(x,0) = \phi(x),\ \partial_t u(x,0) = \psi(x), \quad x \in \mathbb{R}^n\,, \end{aligned}\right\} \tag{1.1}$$

where the initial data $\phi$ and $\psi$ lie in suitable function spaces such as $C_0^\infty(\mathbb{R}^n)$, then the *a priori* estimate

$$\|(u(\cdot,t), u_t(\cdot,t), \nabla_x u(\cdot,t))\|_{L^q} \le C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\nabla_x \phi, \psi)\|_{W_p^{N_p}} \tag{1.2}$$

holds when $n \ge 2$, $p^{-1} + q^{-1} = 1$, $1 < p \le 2$ and $N_p \ge n(p^{-1} - q^{-1})$. Using this estimate, Strichartz proved global existence and uniqueness of solutions to the Cauchy problem for nonlinear wave equations with suitable ("small") initial data. This procedure of proving an *a priori* estimate for a linear equation and using it, together with local existence of a nonlinear equation, to prove global existence and uniqueness for a variety of nonlinear evolution equations is now standard; a systematic overview, with examples including the equations of elasticity, Schrödinger equations and heat equations, can be found in [Rac92].

There are two main approaches used in order to prove (1.2); firstly, one may write the solution to (1.1) using the d'Alembert ($n = 1$), Poisson

($n=2$) or Kirchhoff ($n=3$) formulae, and their generalisation to large $n$,

$$u(x,t)=\begin{cases}\dfrac{1}{\prod_{j=1}^{\frac{n-1}{2}}(2j-1)}\Big[\partial_t(t^{-1}\partial_t)^{\frac{n-3}{2}}\Big(t^{n-1}⨍_{\partial B_t(x)}\phi\,dS\Big)\\ \qquad\qquad +(t^{-1}\partial_t)^{\frac{n-3}{2}}\Big(t^{n-1}⨍_{\partial B_t(x)}\psi\,dS\Big)\Big] \quad (\text{odd } n\geq 3)\\ \dfrac{1}{\prod_{j=1}^{n/2}2j}\Big[\partial_t(t^{-1}\partial_t)^{\frac{n-2}{2}}\Big(t^{n}⨍_{B_t(x)}\dfrac{\phi(y)}{\sqrt{t^2-|y-x|^2}}\,dy\Big)\\ \qquad\qquad +(t^{-1}\partial_t)^{\frac{n-2}{2}}\Big(t^{n}⨍_{B_t(x)}\dfrac{\psi(y)}{\sqrt{t^2-|y-x|^2}}\,dy\Big)\Big] \quad (\text{even } n)\,,\end{cases}$$

(for the derivation of these formulae see, for example, [Eva98]), as is done in [vW71] and [Rac92]. Alternatively, one may write the solution as a sum of Fourier integral operators:

$$u(x,t)=\mathcal{F}^{-1}\Big(\frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\,\hat{\phi}(\xi)+\frac{e^{it|\xi|}-e^{-it|\xi|}}{2|\xi|}\,\hat{\psi}(\xi)\Big)\,. \tag{1.3}$$

This is done in [Str70a], [Bre75] and [Pec76], for example. Using one of these representations for the solution and techniques from either the theory of Fourier integral operators ([Pec76]), Bessel functions ([Str70a]) or standard analysis ([vW71]), the estimate (1.2) may be obtained. The representation (1.3) and the method of Pecher is the one that is perhaps the most useful since it can be generalised to other hyperbolic equations (see Section 0.2 for the definition of such operators and other important related concepts that we shall use throughout) for which the solution may also be written as the sum of Fourier integral operators.

Another problem of interest where an $L^p-L^q$ decay estimate for the linear equation is used to prove existence and uniqueness for the related nonlinear problem is the Cauchy problem for the Klein–Gordon equation. Precisely, if $u=u(x,t)$ satisfies the initial value problem

$$\left.\begin{aligned}&u_{tt}(x,t)-\Delta_x u(x,t)+m^2u(x,t)=0, \quad (x,t)\in\mathbb{R}^n\times(0,\infty)\,,\\ &u(x,0)=\phi(x),\ u_t(x,0)=\psi(x), \quad x\in\mathbb{R}^n\,,\end{aligned}\right\} \tag{1.4}$$

where $\phi,\psi\in C_0^\infty(\mathbb{R}^n)$, say, and $m$ is a constant (representing a *mass term*), then

$$\|(u(\cdot,t),u_t(\cdot,t),\nabla_x u(\cdot,t))\|_{L^q} \le C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|(\nabla_x\phi,\psi)\|_{W_p^{N_p}}, \quad (1.5)$$

where $p, q, N_p$ are as before. Comparing (1.2) to (1.5), we see that the estimate for the solution to the Klein–Gordon equation decays more rapidly—there is an improvement in the exponent of the decay function of $-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})$. The estimate is proved in [vW71], [Pec76] and [Hör97] in different ways, each suggesting reasons for this improvement: in [vW71], the function

$$v = v(x, x_{n+1}, t) := e^{-imx_{n+1}}u(x,t)\,, \quad x_{n+1}\in\mathbb{R}\,,$$

is defined; using (1.4), it is simple to show that $v$ satisfies the wave equation in $\mathbb{R}^{n+1}$, and thus the Strichartz estimate (1.2) holds for $v$, yielding the desired estimate for $u$. This is elegant, but cannot easily be adapted to other situations due to the importance of the structures of the Klein–Gordon and wave equations for this proof. In [Pec76] and [Hör97], a representation of the solution via Fourier integral operators is used and the stationary phase method then applied in order to obtain estimate (1.5).

A third problem of interest is the Cauchy problem for the dissipative wave equation,

$$\left.\begin{array}{l} u_{tt}(x,t)-\Delta_x u(x,t)+u_t(x,t)=0\,, \quad (x,t)\in\mathbb{R}^n\times(0,\infty) \\ u(x,0)=\phi(x),\ u_t(x,0)=\psi(x), \quad x\in\mathbb{R}^n\,, \end{array}\right\}$$

where $\psi,\phi\in C_0^\infty(\mathbb{R}^n)$, say. In this case,

$$\|\partial_t^r\partial_x^\alpha u(\cdot,t)\|_{L^q} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-r-\frac{|\alpha|}{2}}\|(\phi,\nabla\psi)\|_{W_p^{N_p}}\,.$$

This is proved in [Mat76] with a view to showing well-posedness of related semilinear equations. Once again, this estimate (for the solution $u(x,t)$ itself) is better than that for the solution to the wave equation by $-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})$; there is an even greater improvement for higher derivatives of the solution. As before, the proof of this may be done via a representation of the solution using the Fourier transform:

$$u(x,t) = \begin{cases} \mathcal{F}^{-1}\Big(\Big[\dfrac{e^{-t/2}\sinh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)}{\sqrt{1-4|\xi|^2}} + e^{-t/2}\cosh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)\Big]\hat{\phi}(\xi) \\ \qquad\qquad + \dfrac{2e^{-t/2}\sinh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)}{\sqrt{1-4|\xi|^2}}\hat{\psi}(\xi)\Big), \quad |\xi| \le 1/2, \\ \mathcal{F}^{-1}\Big(\Big[\dfrac{e^{-t/2}\sin\left(\frac{t}{2}\sqrt{4|\xi|^2-1}\right)}{\sqrt{4|\xi|^2-1}} + e^{-t/2}\cos\left(\frac{t}{2}\sqrt{4|\xi|^2-1}\right)\Big]\hat{\phi}(\xi) \\ \qquad\qquad + \dfrac{2e^{-t/2}\sin\left(\frac{t}{2}\sqrt{4|\xi|^2-1}\right)}{\sqrt{4|\xi|^2-1}}\hat{\psi}(\xi)\Big), \quad |\xi| > 1/2. \end{cases}$$

Matsumura divides the phase space into the regions where the solution has different properties and then uses standard techniques from analysis.

It is, therefore, interesting to ask why the addition of lower order terms improves the rate of decay of the solution to the equation; furthermore, we would like to understand why the improvement in the decay is the same for both the addition of a mass term and for the addition of a dissipative term. In the proof of each of the estimates (see the papers cited above), the critical role is played by the characteristic roots (see Definition 0.3) of the equations. Let us list the significant properties in each of the cases:

- Wave equation. The characteristic roots are $\varphi_\pm(\xi) = \pm|\xi|$; they are real and homogeneous of order 1.
- Klein–Gordon equation. Here, the roots, $\tau_\pm(\xi) = \pm\sqrt{|\xi|^2 + m^2}$, are real, but not homogeneous. Furthermore, the Hessian of each of the roots is everywhere non-singular, whereas the Hessian of each of the characteristic roots of the wave equation is zero at the point of stationary phase, $\xi = 0$.
- Dissipative wave equation. Here $\tau_{1,2}(\xi) = \frac{i}{2} \pm \frac{1}{2}\sqrt{4|\xi|^2 - 1}$. In this case, the improvement in the decay rate occurs because the roots lie away from the imaginary axis when $\xi \neq 0$. The rate of decay is determined by the behaviour of $\tau_1(\xi)$ near $\xi = 0$; this can be seen from the proof of Lemma 1 in [Mat76] and will be studied in more detail later in the thesis.

In conclusion, it is the difference in the behaviour of the characteristic roots of the Klein–Gordon equation and the dissipative wave equation which yield improvement over the Strichartz decay rate for the wave equation.

The aim of this thesis is to investigate this phenomenon for higher order hyperbolic equations and see how lower order terms affect the rate of decay compared to that for the homogeneous $m^{\text{th}}$ order equation and the examples above.

## 1.2 Statement of Main Problem

Consider the Cauchy problem for the model $m^{\text{th}}$ order constant coefficient linear strictly hyperbolic equation with solution $u = u(x,t)$:

$$\left.\begin{aligned} &D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, \quad t > 0, \\ &D_t^l u(x,0) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1,\ x \in \mathbb{R}^n\,, \end{aligned}\right\} \tag{1.6}$$

where $P_j(\xi)$, the polynomial obtained from the operator $P_j(D_x)$ by replacing each $D_{x_i}$ by $\xi_i$, is a constant coefficient homogeneous polynomial of order $j$, and the $c_{\alpha,r}$ are constants.

**Remark 1.2.1:** For a hyperbolic equation with real coefficients we note that the constants $c_{\alpha,r}$ satisfy $i^{m-|\alpha|-l} c_{\alpha,r} \in \mathbb{R}$; the equation is written in the form above since our results may be used to study hyperbolic systems, which can be reduced to an $m^{\text{th}}$ order equation with complex coefficients.

We seek *a priori* estimates for the solution to this problem of the type

$$\|D_x^\alpha D_t^r u(\cdot,t)\|_{L^q} \le K(t) \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p-l}}\,, \tag{1.7}$$

where $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $N_p = N_p(\alpha, r)$ is a constant depending on $p, \alpha$ and $r$, and $K(t)$ is a function to be determined.

## 1.3 Homogeneous Operators

The case where the operator in (1.6) is homogeneous has been studied extensively:

$$\left.\begin{array}{l} L(D_x, D_t)u = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ D_t^l u(x,0) = f_l(x), \quad l = 0,\dots,m-1,\ x \in \mathbb{R}^n\,, \end{array}\right\} \tag{1.8}$$

where $L$ is a homogeneous $m^{\text{th}}$ order constant coefficient strictly hyperbolic differential operator; the symbol of $L$ may be written in the form

$$L(\tau,\xi) = (\tau - \varphi_1(\xi)) \dots (\tau - \varphi_m(\xi)), \text{ with } \varphi_1(\xi) > \dots > \varphi_m(\xi) \quad (\xi \neq 0).$$

In a series of papers, [Sug94], [Sug96] and [Sug98], Mitsuru Sugimoto showed how the geometric properties of the characteristic roots $\varphi_1(\xi), \dots, \varphi_m(\xi)$ affect the $L^p - L^q$ estimate. To understand this, let us summarise the method of approach.

Firstly, the solution can be written as the sum of Fourier multipliers:

$$u(x,t) = \sum_{l=0}^{m-1} [E_l(t) f_l](x), \quad \text{where } E_l(t) = \sum_{k=1}^{m} \mathcal{F}^{-1} e^{it\varphi_k(\xi)} a_{k,l}(\xi) \mathcal{F}$$

and $a_{k,l}(\xi)$ is homogeneous of order $-l$. Now, the problem of finding an $L^p - L^q$ decay estimate for the solution is reduced to showing that operators of the form

$$M_r(D) := \mathcal{F}^{-1} e^{i\varphi(\xi)} |\xi|^{-r} \chi(\xi) \mathcal{F}\,,$$

where $\varphi(\xi) \in C^\omega(\mathbb{R}^n \setminus \{0\})$ is homogeneous of order 1 and $\chi \in C^\infty(\mathbb{R}^n)$ is equal to 1 for large $\xi$ and zero near the origin, are $L^p - L^q$ bounded for suitably large $r \geq l$. In particular, this means that, for such $r$,

$$\|E_l(1) f\|_{L^q} \leq C \|f\|_{W_p^{r-l}}\,.$$

Indeed, it may be assumed, without loss of generality, that $t = 1$ since:

**Lemma 1.3.1.** *For $t > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$,*

$$[E_l(t)f](x) = t^l[E_l(1)f(t\cdot)](t^{-1}x)\,.$$

*Proof.* By the homogeneity of $\varphi_k(\xi)$ and $a_{k,l}(\xi)$,

$$\begin{aligned}
[E_l(t)f](x) &= \sum_{k=1}^m \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\varphi_k(\xi)t)}a_{k,l}(\xi)\hat{f}(\xi)\,d\xi \\
&= \sum_{k=1}^m \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\varphi_k(t\xi))}a_{k,l}(\xi)\hat{f}(\xi)\,d\xi \\
&= \sum_{k=1}^m \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\eta t^{-1}+\varphi_k(\eta))}a_{k,l}(t^{-1}\eta)\hat{f}(t^{-1}\eta)t^{-n}\,d\eta\,,
\end{aligned}$$

where in the last line the change of coordinates $\xi \mapsto t^{-1}\eta$ has been used. Now, with $y \mapsto ty'$,

$$\hat{f}(t^{-1}\eta) = \int_{\mathbb{R}^n} e^{-iy\cdot t^{-1}\eta}f(y)\,dy = \int_{\mathbb{R}^n} e^{-iy'\cdot\eta}f(ty')t^n\,dy' = t^n\hat{f}_t(\eta)\,,$$

where $f_t(x) := f(tx)$. Thus,

$$\begin{aligned}
[E_l(t)f](x) &= \sum_{k=1}^m \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\eta t^{-1}+\varphi_k(\eta))}t^l a_{k,l}(\eta)\hat{f}_t(\eta)\,d\eta \\
&= t^l[E_l(1)f(t\cdot)](t^{-1}x)\,.
\end{aligned}$$

□

Using this identity gives

$$\begin{aligned}
\|E_l(t)f\|_{L^q}^q &= t^{lq}\|[E_l(1)f_t](t^{-1}\cdot)\|_{L^q}^q = t^{lq}\int_{\mathbb{R}^n}|[E_l(1)f_t](t^{-1}x)|^q\,dx \\
&\overset{(x=tx')}{=} t^{lq}\int_{\mathbb{R}^n} t^n|[E_l(1)f_t](x')|^q\,dx' = t^{lq+n}\|E_l(1)f_t\|_{L^q}^q\,.
\end{aligned}$$

Then, noting that a simple change of variables yields

$$\|f_t\|_{W_p^k}^p \leq Ct^{kp-n}\|f\|_{W_p^k}^p\,,$$

we have,

$$\|E_l(t)f\|_{L^q} \leq Ct^{l+\frac{n}{q}}\|f_t\|_{W_p^{r-l}} \leq Ct^{r-n(\frac{1}{p}-\frac{1}{q})}\|f\|_{W_p^{r-l}}\,;$$

hence,

$$\|u(\cdot,t)\|_{L^q} \leq Ct^{r-n(\frac{1}{p}-\frac{1}{q})}\sum_{l=0}^{m-1}\|f_l\|_{W_p^{r-l}}\,.$$

It has long been known that the values of $r$ for which $M_r(D)$ is $L^p - L^q$ bounded depends on the geometry of the level set

$$\Sigma_\varphi = \{\xi \in \mathbb{R}^n \setminus \{0\} : \varphi(\xi) = 1\}\ .$$

In [Lit73], [Bre75] it is shown that if the Gaussian curvature of $\Sigma_\varphi$ is never zero then $M_r(D)$ is $L^p - L^q$ bounded when $r \geq \frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$. This is extended in [Bre77], in which it is proven that $M_r(D)$ is $L^p - L^q$ bounded provided $r \geq \frac{2n-\rho}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$, where $\rho = \min_{\xi\neq 0}\operatorname{rank}\operatorname{Hess}\varphi(\xi)$.

Sugimoto extended this further in [Sug94], where he showed that if $\Sigma_\varphi$ is convex then $M_r(D)$ is $L^p - L^q$ bounded when $r \geq \left(n - \frac{n-1}{\gamma(\Sigma)}\right)\left(\frac{1}{p}-\frac{1}{q}\right)$; here,

$$\gamma(\Sigma) := \sup_{\sigma\in\Sigma}\sup_{P}\gamma(\Sigma;\sigma,P)\,, \quad \Sigma \subset \mathbb{R}^n \text{ a hypersurface}\,,$$

where $P$ is a plane containing the normal to $\Sigma$ at $\sigma$ and $\gamma(\Sigma;\sigma,P)$ denotes the order of the contact between the line $T_\sigma \cap P$, $T_\sigma$ is the tangent plane at $\sigma$, and the curve $\Sigma \cap P$. See Section 3.2.3 for more on this maximal order of contact.

In order to apply this result to the solution of (1.8), it is necessary to find a condition under which the level sets of the characteristic roots are convex. The following notion is the one that is sufficient:

**Definition 1.1.** *Let $L = L(D_x, D_t)$ be a homogeneous $m^{th}$ order constant coefficient partial differential operator. It is said to satisfy the* convexity condition *if the Hessian,* $\operatorname{Hess}\varphi_k(\xi)$, *corresponding to each of its characteristic roots $\varphi_1(\xi),\ldots,\varphi_m(\xi)$ is semi-definite for $\xi \neq 0$.*

It can be shown that if an operator $L$ does satisfy this convexity condition, then the above results can be applied to the solution and thus an estimate of the form (1.7) holds with

$$K(t) = (1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}, \quad \text{where } \gamma \leq m\,.$$

Finally, the case when this convexity condition does not hold is discussed; in [Sug96] and [Sug98] it is shown that, in general, $M_r(D)$ is $L^p - L^q$ bounded when $r \geq \big(n - \frac{1}{\gamma_0(\Sigma)}\big)\big(\frac{1}{p} - \frac{1}{q}\big)$, where

$$\gamma_0(\Sigma) := \sup_{\sigma \in \Sigma} \inf_{P} \gamma(\Sigma; \sigma, P) \leq \gamma(\Sigma).$$

For $n = 2$, $\gamma_0(\Sigma) = \gamma(\Sigma)$, so, the convexity condition may be lifted in that case. However, in [Sug96], examples are given when $n \geq 3$, $p = 1, 2$ where this lower bound for $r$ is the best possible and, thus, the convexity condition is necessary for the above estimate. It turns out that the case $n \geq 3$, $1 < p < 2$ is more interesting and is studied in greater depth in [Sug98], where microlocal geometric properties must be looked at in order to obtain an optimal result.

Two remarks are worth making; firstly, the convexity condition result recovers the Strichartz decay estimate for the wave equation, since that clearly satisfies such a condition, Secondly, the convexity condition is an important restriction on the geometry of the characteristic roots that affects the $L^p - L^q$ decay rate; hence, in the case of an $m^{\text{th}}$ order operator with lower order terms we must expect some geometrical conditions on the characteristic roots to obtain decay.

The discussion here, and that on the second order equations, suggests that the properties of the characteristics play the key role in determining the rate of decay; we concentrate on this in Chapters 2–4: in Chapter 2 we state the main results in this thesis for the case where the operator has lower order terms and outline the approach used to prove them; in Chapter 3, some key results needed for the main proof are proved, and Chapter 4 contains the proof of the main theorem. In Chapter 5, there are some results which give characterisations in terms of the structure of the lower order, as well as an application to systems and some examples.

# Chapter 2: Main Results and Outline of Approach

## 2.1 Main Results

In this thesis, we shall analyse the conditions under which we can obtain $L^p - L^q$ decay estimates for the general $m^{\text{th}}$ order linear, constant coefficient, strictly hyperbolic Cauchy problem

$$\left.\begin{aligned} &D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, \quad t > 0, \\ &D_t^l u(x,0) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1,\ x \in \mathbb{R}^n\,. \end{aligned}\right\} \tag{2.1}$$

The main theorem below states how different behaviours of the characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$ affect the rate of decay that can be obtained. Ideally, of course, we would like to have conditions on the lower order terms for different rates of decay; in Chapter 5 we shall give some results in this direction. For now, though, we concentrate on conditions on the characteristic roots.

It is natural to impose the condition:

$$\operatorname{Im} \tau_k(\xi) \geq 0 \quad \text{for } k = 1, \dots, m\,; \tag{2.2}$$

this is equivalent to requiring the characteristic polynomial of the operator to be stable at all points $\xi \in \mathbb{R}^n$, and thus cannot be expected to be lifted.

Also, we shall show that it is sensible to divide the considerations of how characteristic roots behave into two parts: their behaviour for large values of $|\xi|$ and for bounded values of $|\xi|$. These two cases are then subdivided further; in particular the following are the key properties to consider:

- multiplicities of roots (this only occurs in the case of bounded $|\xi|$—see Lemma 3.1.4);
- whether roots lie on the real axis or are separated from it;
- behaviour as $|\xi| \to \infty$ (only in the case of large $|\xi|$);
- how roots meet the real axis (if they do);
- properties of the Hessian of the root, $\operatorname{Hess} \tau_k(\xi)$;
- a convexity-type condition, as in the case of homogeneous roots (Section 1.3).

Here is the main theorem, which in Chapter 4 we shall prove; included in it are notions such as "convexity condition $\gamma$", "no convexity condition, $\gamma_0$", "codimension $l$"; these will be defined and discussed in the relevant places in Chapters 3 and 4.

**Theorem 2.1.1.** *Suppose $u = u(x,t)$ satisfies the $m^{th}$ order linear, constant coefficient, strictly hyperbolic Cauchy problem* (2.1)*. Denote the characteristic roots of the operator by $\tau_1(\xi), \dots, \tau_m(\xi)$ and assume that* (2.2) *holds.*

*We introduce two functions, $K^{(l)}(t)$ and $K^{(b)}(t)$, which take values as follows*:

I. *Consider the behaviour of each characteristic root, $\tau_k(\xi)$, in the region $|\xi| \geq N$, where $N$ is a large real number to be chosen later. The following table gives values for the function $K_k^{(l)}(t)$ corresponding to possible properties of $\tau_k(\xi)$; if $\tau_k(\xi)$ satisfies more than one, then take $K_k^{(l)}(t)$ to be function that decays the slowest as $t \to \infty$.*

| Location of $\tau_k(\xi)$ | Additional Property | $K_k^{(l)}(t)$ |
|---|---|---|
| away from real axis | | $e^{-\delta t}$, some $\delta > 0$ |
| on real axis | $\det \operatorname{Hess} \tau_k(\xi) \neq 0$ | $(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$ |
| | $\operatorname{rank} \operatorname{Hess} \tau_k(\xi) = n-1$ | $(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$ |
| | convexity condition $\gamma$ | $(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$ |
| | no convexity condition, $\gamma_0$ | $(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$ |
| asymptotic to real axis | $\det \operatorname{Hess} \tau_k(\xi) \neq 0$ | $(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$ |
| | $\operatorname{rank} \operatorname{Hess} \tau_k(\xi) = n-1$ | $(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$ |
| | no convexity condition, $\gamma_0$ | $(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$ |

*Then take* $K^{(l)}(t) = \max_{k=1\dots,n} K_k^{(l)}(t)$.

II. *Consider the behaviour of the characteristic roots in the bounded region* $|\xi| \leq N$; *again, take* $K^{(b)}(t)$ *to be the maximum (slowest decaying) function for which there are roots satisfying the conditions in the following table:*

| Location of Root(s) | Properties | $K^{(\mathrm{b})}(t)$ |
|---|---|---|
| away from axis | no multiplicities | $e^{-\delta t}$, some $\delta > 0$ |
| | $L$ roots coinciding | $(1+t)^{L-1}e^{-\delta t}$ |
| on axis, no multiplicities* | $\det \operatorname{Hess} \tau_k(\xi) \neq 0$ | $(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$ |
| | convexity condition $\gamma$ | $(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$ |
| | no convexity condition, $\gamma_0$ | $(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$ |
| on axis, multiplicities* | $L$ roots coincide on set of codimension $\ell$ | $(1+t)^{L-1-\ell(\frac{1}{p}-\frac{1}{q})}$ |
| meeting axis with finite order $s$ | $L$ roots coincide on set of codimension $\ell$ | $(1+t)^{L-1-\frac{\ell}{s}(\frac{1}{p}-\frac{1}{q})}$ |

* *These two cases of roots lying on the real axis require some additional regularity assumptions; see proof for details.*

*Then, with* $K(t) = \max\big(K^{(b)}(t), K^{(l)}(t)\big)$, *the following estimate holds:*

$$\|D_x^\alpha D_t^r u(\cdot,t)\|_{L^q} \leq K(t) \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p - l}},$$

*where* $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $N_p = N_p(\alpha, r)$ *is a constant depending on* $p, \alpha$ *and* $r$.

## 2.2 Schematic of Method

**Step 1:** Representation of the solution.

Using the Fourier transform, this reduces the problem to studying oscillatory integrals.

**Step 2:** Division of the integral.

Reduce the problem to several model cases using suitable cut-off functions; in view of interpolation techniques, it then suffices to prove $L^2 - L^2$ and $L^1 - L^\infty$ estimates. The problem is divided into studying the behaviour of the characteristic roots in three regions of the phase space—large $|\xi|$, bounded $|\xi|$ away from multiplicities of roots and bounded $|\xi|$ in a neighbourhood of multiplicities.

**Step 3:** Interpolation reduces problem to finding $L^1 - L^\infty$ and $L^2 - L^2$ estimates.

**Step 4:** Large $|\xi|$:

- root separated from the real axis;
- root asymptotic to the real axis;
- root lying on the real axis.

**Step 5:** Bounded $|\xi|$, away from multiplicities:

- root away from the real axis;
- root meeting the real axis with finite order;
- root lying on the real axis.

**Step 6:** Bounded $|\xi|$, around multiplicities of roots:

- all intersecting roots away from the real axis;
- all intersecting roots lie on the real axis around the multiplicity;
- all intersecting roots meet the real axis with finite order;
- one or more of the roots meets the real axis with infinite order.

# Chapter 3: Auxiliary Results

In this chapter, we collect together various tools that enable us to prove the main theorem. First, we prove some important properties for the characteristics of strictly hyperbolic operators, which motivate the division into large and bounded $|\xi|$. Then we give several theorems about oscillatory integrals divided into the cases where a convexity condition holds on the phase function and where no such condition holds.

## 3.1 Strictly Hyperbolic Operators and Polynomials

In order to study the solution $u(x,t)$ to (1.6), we must first know some properties of the characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$. Naturally, we do not have explicit formulae for the roots, unlike in the cases of the dissipative wave equation and the Klein–Gordon equation, but we do know properties for the roots of the principal symbol. For general hyperbolic operators, the roots $\varphi_1(\xi), \dots, \varphi_m(\xi)$ of the characteristic polynomial of the *principal part* are homogeneous functions of order 1 since the principal part is homogeneous. Furthermore, for strictly hyperbolic polynomials these roots are distinct when $\xi \neq 0$. Since these two properties are very useful when studying homogeneous (strictly) hyperbolic equations, it is useful to know whether the characteristic roots of the full equation, $\tau_1(\xi), \dots, \tau_m(\xi)$, have similar properties. Indeed, if we regard the full equation as a perturbation of the principal part, we can show that similar properties hold for large $|\xi|$; these results are the focus of this section. In particular, we shall show that they are continuous everywhere, analytic away from multiplicities, are symbols and have no multiplicities for sufficiently large $|\xi|$. In the schematic of the proof (Section 2.2), we subdivided the phase space into large $|\xi|$ and bounded $|\xi|$, and it is these properties that motivate this step.

First, we give some properties of general polynomials which are useful to us. For constant coefficient polynomials, the following result holds:

**Lemma 3.1.1.** *Consider the polynomial over* $\mathbb{C}$ *with complex coefficients*

$$z^m + c_1 z^{m-1} + \cdots + c_{m-1} z + c_m = \prod_{k=1}^{m} (z - z_k).$$

*If there exists* $M > 0$ *such that* $|c_j| \leq M^j$ *for each* $j = 1, \ldots, m$, *then* $|z_k| \leq 2M$ *for all* $k = 1, \ldots, m$.

*Proof.* Assume that $|z| > 2M$. Then

$$\begin{aligned} |z^m + c_1 z^{m-1} + \cdots + c_{m-1} z + c_m| &\geq |z|^m \left(1 - \frac{|c_1|}{|z|} - \cdots - \frac{|c_{m-1}|}{|z|^{m-1}} - \frac{|c_m|}{|z|^m}\right) \\ &\geq (2r)^m (1 - 2^{-1} - \cdots - 2^{-(m-1)} - 2^{-m}) > 0. \end{aligned}$$

That is, no zero of the polynomial lies outside of the ball about the origin of radius $2M$; hence $|z_k| \leq 2M$ for each $k = 1, \ldots, m$. □

**Remark 3.1.1:** If we replace the hypothesis $|c_j| \leq M^j$ by $|c_j| \leq M$ for each $j = 1, \ldots, m$, then by a similar argument we obtain that $|z_k| \leq \max\{2, 2M\}$. The $\max\{2, 2M\}$ term appears because we need $M \geq 1$ for the sum on the right hand side to be positive.

For general polynomials with variable coefficients, we have:

**Lemma 3.1.2.** *Consider the* $m^{th}$ *order polynomial with coefficients depending on* $\xi \in \mathbb{R}^n$

$$p(\xi, \tau) = \tau^m + a_1(\xi) \tau^{m-1} + \cdots + a_m(\xi).$$

*If each of the coefficient functions* $a_j(\xi)$, $j = 1, \ldots, m$, *is continuous in* $\mathbb{R}^n$ *then each of the roots* $\tau_1(\xi), \ldots, \tau_m(\xi)$ *with respect to* $\tau$ *of* $p(\xi, \tau)$ *is also continuous in* $\mathbb{R}^n$.

*Proof.* Define $\rho : \mathbb{C}^m \to \mathbb{C}^m$ by $\rho(z_1, \ldots, z_m) = (c_1, \ldots, c_m)$ where the $c_j$ satisfy

$$z^m + c_1 z^{m-1} + \cdots + c_m = \prod_{j=1}^{m} (z - z_j).$$

Then $\rho$ is:

(a) surjective by the Fundamental Theorem of Algebra;

(b) continuous since each of the $c_j$ may be written as polynomials of the $z_j$ (by the Vièta formulae);

(c) proper (that is, the preimage of each compact set is compact) by Remark 3.1.1;

properties (b) and (c) imply that $\rho$ is a closed mapping.

Now, fix $\xi_0 \in \mathbb{R}^n$. For any given $\varepsilon > 0$, consider the set

$$U = \bigcup_{\alpha \in S_m} \bigcap_{k=1}^{m} \{\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m : |\zeta_{\alpha_k} - \tau_k(\xi_0)| < \varepsilon\} ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in S_m$ denotes the set of permutations of $\{1, \ldots, m\}$ (see Fig. 3.1 for a diagram of this). Note that $U$ is, by construction, symmet-

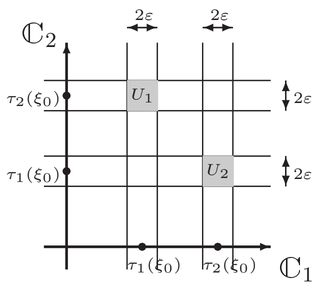


Figure 3.1: $U = U_1 \cup U_2$

ric, i.e. if $(z_1, \ldots, z_m) \in U$ then $(z_{\alpha_1}, \ldots, z_{\alpha_m}) \in U$ for all $(\alpha_1, \ldots, \alpha_m) \in S_m$. Let $F$ denote the complement to $U$:

$$F = \bigcap_{\alpha \in S_m} \{\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m : |\zeta_{\alpha_k} - \tau_k(\xi_0)| \geq \varepsilon \;\exists\, k = 1, \ldots, m\} .$$

We need to show that there exists $\delta > 0$ such that $(\tau_1(\xi), \ldots, \tau_m(\xi)) \in U$ whenever $|\xi - \xi_0| < \delta$; note:

- $\rho^{-1}(\rho(F)) = F$ by construction—if $\rho(w) = \rho(w')$ then both $w$ and $w'$ give rise to the same polynomial, and hence their entries are permutations of each other, and so either both or neither lie in $F$;

- by the surjectivity of $\rho$,

$$\rho(U) = \rho(F^c) = \rho([\rho^{-1}(\rho(F))]^c) = \rho(\rho^{-1}(\rho(F)^c)) = \rho(F)^c\,;$$

- $\rho(F)$ is closed since $F$ a closed set and $\rho$ is a closed mapping;

therefore, $\rho(U)$ is open. Thus, there exists an open ball in $\rho(U)$ of radius $\delta'$ (for some $\delta' > 0$) about $a(\xi_0) \equiv (a_1(\xi_0), \ldots, a_m(\xi_0)) = \rho(\tau_1(\xi_0), \ldots, \tau_m(\xi_0))$:

$$B_{\delta'}(a(\xi_0)) = \big\{(c_1, \ldots, c_m) \in \mathbb{C}^m : |c_j - a_j(\xi_0)| < \delta' \;\forall\, j = 1, \ldots, m\big\} \subset \rho(U).$$

By the continuity of the $a_j(\xi)$, there exists $\delta > 0$ such that

$$|\xi - \xi_0| < \delta \implies |a_j(\xi) - a_j(\xi_0)| < \delta' \text{ for all } j = 1, \ldots, m\,;$$

hence,

$$|\xi - \xi_0| < \delta \implies (a_1(\xi), \ldots, a_m(\xi)) \in B_{\delta'}(a(\xi_0)) \subset \rho(U)\,.$$

Finally, since $\rho(\tau_1(\xi), \ldots, \tau_m(\xi)) = (a_1(\xi), \ldots, a_m(\xi))$ and $U$ is symmetric (this is needed as different root orderings give the same coefficients), we find $(\tau_1(\xi), \ldots, \tau_m(\xi)) \in U$ when $|\xi - \xi_0| < \delta$ as required; this completes the proof of the lemma. □

Now, let us turn to proving properties of the characteristic roots.

**Proposition 3.1.3** (Continuity of Roots)**.** *Let $L = L(D_x, D_t)$ be a linear $m^{th}$ order constant coefficient partial differential operator. Then each of the characteristic roots of $L$, denoted $\tau_1(\xi), \ldots, \tau_m(\xi)$, is continuous in $\mathbb{R}^n$; furthermore, for each $k = 1, \ldots, m$, the characteristic root $\tau_k(\xi)$ is analytic in*

$$\{\xi \in \mathbb{R}^n : \tau_k(\xi) \neq \tau_l(\xi)\, \forall\, l \neq k\}\;.$$

*Proof.* The characteristic polynomial of $L$ is of the form

$$L(\xi, \tau) = \tau^m + a_1(\xi)\tau^{m-1} + \cdots + a_m(\xi)\,,$$

where the $a_j(\xi)$ are polynomials in $\xi$ of order $j$. Thus, by Lemma 3.1.2

each of the roots is continuous. The second statement follows by the Real Analytic Implicit Function Theorem[1], using that the $a_j(\xi)$ are polynomials and hence analytic in $\xi$. $\square$

**Remark 3.1.2:** In the case where $L(D_x, D_t)$ is a differential operator in $t$ with coefficients that are *pseudodifferential* operators in $x$, the $a_j(\xi)$ are classical symbols of order $j$; so, in this case, the characteristic roots are continuous in $\mathbb{R}^n$ and *smooth* away from multiplicities.

**Lemma 3.1.4** (Location of Root Multiplicities)**.** *Let $L = L(D_x, D_t)$ be a linear $m^{th}$ order constant coefficient strictly hyperbolic partial differential operator. Then there exists a constant $N$ such that, if $|\xi| > N$ then the characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$ of $L$ are pairwise distinct.*

*Proof.* We use the notation and results from Chapter 12 of [GKZ94] concerning the discriminant $\Delta_p$ of the polynomial $p(x) = p_m x^m + \cdots + p_1 x + p_0$,

$$\Delta_p \equiv \Delta(p_0, \dots, p_m) := (-1)^{\frac{m(m-1)}{2}} p_m^{2m-2} \prod_{i<j} (x_i - x_j)^2 ,$$

where the $x_j$ $(j = 1, \dots, m)$ are the roots of $p(x)$; that is, the irreducible polynomial in the coefficients of the polynomial which vanishes when the polynomial has multiple roots. We note that $\Delta_p$ is a continuous function of the coefficients $p_0, \dots, p_m$ of $p(x)$ and it is a homogeneous function of degree $2m - 2$ in them; in addition, it satisfies the quasi-homogeneity property:

$$\Delta(p_0, \lambda p_1, \lambda^2 p_2, \dots, \lambda^m p_m) = \lambda^{m(m-1)} \Delta(p_0, \dots, p_m).$$

Furthermore, $\Delta_p = 0$ if and only if $p(x)$ has a double root.

We write $L(\xi, \tau)$ in the form

$$L(\xi, \tau) = L_m(\xi, \tau) + a_1(\xi)\tau^{m-1} + a_2(\xi)\tau^{m-2} + \cdots + a_{m-1}(\xi)\tau + a_m(\xi),$$

where

$$L_m(\xi, \tau) = \tau^m + \sum_{k=1}^{m} P_j(\xi)\tau^{m-j}$$

[1] see, for example, [KP02]

is the principal part of $L(\xi,\tau)$; note that the $P_j(\xi)$ are homogeneous polynomials of degree $j$ and the $a_j(\xi)$ are polynomials of degree $\leq j$. By the homogeneity and quasi-homogeneity properties of $\Delta_L$, we have, for $\lambda \neq 0$,

$$\begin{aligned}
\Delta_L(\lambda\xi) &= \Delta(P_m(\lambda\xi) + a_m(\lambda\xi), \ldots, P_1(\lambda\xi) + a_1(\lambda\xi), 1) \\
&= \Delta(\lambda^m[P_m(\xi) + \tfrac{a_m(\lambda\xi)}{\lambda^m}], \ldots, \lambda[P_1(\xi) + \tfrac{a_1(\lambda\xi)}{\lambda}], 1) \\
&= \lambda^{m(2m-2)}\Delta(P_m(\xi) + \tfrac{a_m(\lambda\xi)}{\lambda^m}, \ldots, \lambda^{-(m-1)}[P_1(\xi) + \tfrac{a_1(\lambda\xi)}{\lambda}], \lambda^{-m}) \\
&\qquad\qquad \text{(using that } \Delta \text{ is homogenous of degree } 2m-2\text{)} \\
&= \lambda^{m(m-1)}\Delta(P_m(\xi) + \tfrac{a_m(\lambda\xi)}{\lambda^m}, \ldots, P_1(\xi) + \tfrac{a_1(\lambda\xi)}{\lambda}, 1) \\
&\qquad\qquad \text{(by quasi-homogeneity).}
\end{aligned}$$

Now, since $L$ is strictly hyperbolic, the characteristic roots $\varphi_1(\xi), \ldots, \varphi_m(\xi)$ of $L_m$ are pairwise distinct for $\xi \neq 0$, so

$$\Delta_{L_m}(\xi) = \Delta(P_m(\xi), \ldots, P_1(\xi), 1) \neq 0 \text{ for } \xi \neq 0.$$

Since the discriminant is continuous in each argument, there exists $\delta > 0$ such that if $\left|\frac{a_j(\lambda\xi)}{\lambda^j}\right| < \delta$ for all $j = 1, \ldots, m$ then

$$\left|\Delta(P_m(\xi) + \tfrac{a_0(\lambda\xi)}{\lambda^m}, \ldots, P_1(\xi) + \tfrac{a_{m-1}(\lambda\xi)}{\lambda}, 1)\right| > 0,$$

and hence the roots of the associated polynomial are pairwise distinct. So, fix $\xi \in \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ and let $\lambda \to \infty$. Since the $a_j(\xi)$ are polynomials of degree $\leq j$ it follows that $\frac{a_j(\lambda\xi)}{\lambda^j} \to 0$ for each $j = 1, \ldots, m$. So, there exists $N > 0$ such that if $\lambda > N$ then $\left|\frac{a_j(\lambda\xi)}{\lambda^j}\right| < \delta$ for all $j = 1, \ldots, m$. Hence when $|\xi| > N$ the characteristic roots of $L$ are pairwise distinct. □

**Remark 3.1.3:** As in the case above, this statement is true when $L(D_x, D_t)$ is a (strictly hyperbolic) differential operator in $t$ with coefficients that are pseudodifferential operators in $x$. The only modification to the proof needed is we must use the simple fact that $P_j(\xi) - \frac{P_j(\lambda\xi)}{\lambda^j} \to 0$ as $\lambda \to \infty$ when $\xi \in S^{n-1}$ (since the $P_j(\xi)$ are symbols of order $j$) to ensure

$$\Delta_L(\lambda\xi) = \lambda^{m(m-1)}\Delta\big(\tfrac{P_m(\lambda\xi)+a_m(\lambda\xi)}{\lambda^m}, \ldots, \tfrac{P_1(\lambda\xi)+a_1(\lambda\xi)}{\lambda}, 1\big)$$

is bounded away from zero for large $\lambda$.

**Proposition 3.1.5** (Symbolic Properties of Roots). *Let $L = L(D_x, D_t)$ be a linear $m^{th}$ order constant coefficient hyperbolic partial differential operator with characteristic roots $\tau_1(\xi), \ldots, \tau_m(\xi)$; then*

I. *for each $k = 1, \ldots, m$, there exists a constant $C > 0$ such that*

$$|\tau_k(\xi)| \leq C(1 + |\xi|) \quad \textit{for all } \xi \in \mathbb{R}^n \,. \tag{3.1}$$

*Furthermore, if we insist that $L$ is* strictly *hyperbolic, and denote the roots of the principal part $L_m(\xi, \tau)$ by $\varphi_1(\xi), \ldots, \varphi_m(\xi)$, then we have the following:*

II. *Suppose that the maximum order of the lower order terms is $0 \leq K \leq m-1$. Then, for each $\tau_k(\xi)$, $k = 1, \ldots, m$, there exists a corresponding root of the principal symbol $\varphi_k(\xi)$ (possibly after reordering) such that*

$$|\tau_k(\xi) - \varphi_k(\xi)| \leq C(1 + |\xi|)^{K+1-m} \quad \textit{for all } \xi \in \mathbb{R}^n \,. \tag{3.2}$$

*In particular, for arbitrary lower terms, we have*

$$|\tau_k(\xi) - \varphi_k(\xi)| \leq C \quad \textit{for all } \xi \in \mathbb{R}^n \,. \tag{3.3}$$

III. *There exists $N > 0$ such that, for each characteristic root of $L$ and for each multi-index $\alpha$, we can find constants $C = C_{k,\alpha} > 0$ such that*

$$\left|\partial_\xi^\alpha \tau_k(\xi)\right| \leq C|\xi|^{1-|\alpha|} \quad \textit{for all } |\xi| \geq N \,, \tag{3.4}$$

*In particular, there exists a constant $C > 0$ such that*

$$|\nabla \tau_k(\xi)| \leq C \quad \textit{for all } |\xi| \geq N \,. \tag{3.5}$$

IV. *There exists $N > 0$ such that, for each $\tau_k(\xi)$ a corresponding root of the principal symbol $\varphi_k(\xi)$ can be found (possibly after reordering) which satisfies, for each multi-index $\alpha$ and $k = 1, \ldots, m$,*

$$\left|\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi)\right| \leq C|\xi|^{-|\alpha|} \quad \textit{for all } |\xi| \geq N \,, \tag{3.6}$$

*for some constants* $C = C_{k,\alpha} > 0$.

*If we further assume that the maximum order of the lower order terms is* $0 \le K \le m-1$, *then we get the estimate*

$$\left|\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi)\right| \le C|\xi|^{K+1-m-|\alpha|} \quad \textit{for all } |\xi| \ge N \tag{3.7}$$

*for each multi-index* $\alpha$ *and* $k = 1, \dots, m$.

First, we need the following lemma about perturbation properties of general smooth functions.

**Lemma 3.1.6.** *Let* $p : \mathbb{C} \to \mathbb{C}$ *and* $q : \mathbb{C} \to \mathbb{C}$ *be smooth functions and suppose* $z_0$ *is a simple zero of* $p(z)$ *(i.e.* $p(z_0) = 0$, $p'(z_0) \neq 0$*). Consider, for each* $\varepsilon > 0$, *the following "perturbation" of* $p(z)$:

$$p_\varepsilon(z) := p(z) + \varepsilon q(z)\,,$$

*and suppose* $z_\varepsilon$ *is a root of* $p_\varepsilon(z)$; *then, for all sufficiently small* $\varepsilon > 0$,

$$|z_\varepsilon - z_0| \le C\varepsilon\left|\frac{q(z_0)}{p'(z_0)}\right|. \tag{3.8}$$

*Proof.* By Taylor's Theorem, we have, near $z_0$,

$$\begin{aligned} p_\varepsilon(z) &= p_\varepsilon(z_0) + p_\varepsilon'(z_0)(z - z_0) + O(|z - z_0|^2) \\ &= \varepsilon q(z_0) + (p'(z_0) + \varepsilon q'(x_0))(z - z_0) + O(|z - z_0|^2)\,. \end{aligned}$$

Thus, setting $z = z_\varepsilon$,

$$0 = \varepsilon q(z_0) + (p'(z_0) + \varepsilon q'(z_0))(z_\varepsilon - z_0) + O(|z_\varepsilon - z_0|^2)\,. \tag{3.9}$$

Now, consider the function of $\varepsilon$, $z(\varepsilon) := z_\varepsilon$; this is clearly smooth since $p$ and $q$ are smooth and $z_0$ is a simple zero of $p(z)$, and thus, near the origin,

$$z(\varepsilon) = z(0) + \varepsilon z'(0) + O(\varepsilon^2)\,. \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$0 = \varepsilon q(z_0) + (p'(z_0) + \varepsilon q'(z_0))(\varepsilon z'(0) + O(\varepsilon^2)) + O(\varepsilon^2)\,,$$

or,

$$0 = q(z_0) + p'(z_0)z'(0) + O(\varepsilon)\,,$$

which is equivalent to

$$|q(z_0) + p'(z_0)z'(0)| \leq C\varepsilon \quad \text{as } \varepsilon \to 0\,.$$

Therefore, by the triangle inequality, for each $\varepsilon > 0$ small enough,

$$|z'(0)| \leq \frac{C\varepsilon}{|p'(z_0)|} + \Big|\frac{q(z_0)}{p'(z_0)}\Big|\,,$$

and, thus,

$$|z'(0)| \leq C\Big|\frac{q(z_0)}{p'(z_0)}\Big|\,. \tag{3.11}$$

Finally, combining (3.11) with (3.10), we obtain (3.8) as required. □

*Proof of Proposition 3.1.5.*

**Part I:** We may write $L(\xi, \tau)$ in the form

$$L(\xi, \tau) = \tau^m + a_1(\xi)\tau^{m-1} + \cdots + a_{m-1}(\xi)\tau + a_m(\xi),$$

where the $a_j(\xi)$ are polynomials in $\xi$ of order $\leq j$. Now, for each $j = 1, \ldots, m$, there exists a constant $M_j$ such that, for some constant $C_j > 0$, we have $|a_j(\xi)| \leq C_j|\xi|^j$ when $|\xi| > M_j$; this is because the $a_j(\xi)$ are polynomials. Then, taking $M = \max_j M_j$, we have by Lemma 3.1.1 that there exists $C > 0$ such that $|\tau_k(\xi)| \leq C|\xi|$ when $|\xi| > M$; since $\tau(\xi)$ is continuous in $\mathbb{R}^n$ (Proposition 3.1.3) and thus bounded on compact sets, we have (3.1) as desired.

**Part II:** In the proof of this part, let us write $L(\xi,\tau)$ in the form

$$L(\xi,\tau) = \sum_{i=0}^{R} L_{m-r_i}(\xi,\tau)\,,$$

where $r_0 = 0$, $m - r_1 = K$ (the maximum order of the lower order terms), $1 \le r_1 < \cdots < r_R \le m$,

$$L_m(\xi,\tau) = \tau^m + \sum_{k=1}^{m} P_j(\xi)\tau^{m-j}$$
$$\text{and } L_{m-r_i}(\xi,\tau) = \sum_{|\alpha|+j=m-r_i} c_{\alpha,j}\xi^\alpha\tau^j \quad \text{for } 1 \le i \le R;$$

here, as usual, the $P_j(\xi)$ are homogeneous polynomials in $\xi$ of order $j$.

Denote the roots of

$$\mathfrak{L}_l(\xi,\tau) := \sum_{i=0}^{l} L_{m-r_i}(\xi,\tau)\,, \quad 0 \le l \le R\,,$$

with respect to $\tau$ by $\tau_1^l(\xi),\ldots,\tau_m^l(\xi)$. Note that $\mathfrak{L}_0(\xi,\tau) = L_m(\xi,\tau)$, i.e. $\mathfrak{L}_0(\xi,\tau)$ is the principal symbol with no lower order terms.

Here, let us choose constants $N_l \ge 1$ so that, for each fixed $l = 0,\ldots,R$, $\tau_1^l(\xi),\ldots,\tau_m^l(\xi)$ are distinct when $|\xi| > N_l$—this can be done by Lemma 3.1.4 since each $\mathfrak{L}_l(\xi,\tau)$ is a strictly hyperbolic polynomial. Next, set $N \equiv \max_l N_l$, and from now on, we assume that $|\xi| \ge N_{\max}$ throughout.

We shall show that there exists $M \ge N_{\max}$ so that, possibly after re-ordering the roots, for all $k = 1,\ldots,m$,

$$|\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| \le C|\xi|^{-r_{l+1}+1} \text{ for all } l = 0,\ldots,R-1 \text{ and } |\xi| > M\,. \quad (3.12)$$

Assuming this, and noting that $\tau_k^0(\xi) = \varphi_k(\xi)$ and $\tau_k^R(\xi) = \tau_k(\xi)$ for each $k = 1,\ldots,m$ (possibly after reordering), we obtain

$$|\tau_k(\xi) - \varphi_k(\xi)| \le \sum_{l=0}^{r} |\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| \le C|\xi|^{-r_1+1} \text{ when } |\xi| > M;$$

this, together with the continuity of the $\tau_k(\xi)$ and $\varphi_k(\xi)$—and thus the boundedness of $|\tau_k(\xi)-\varphi_k(\xi)|$ in $B_M(0)$, gives (3.2). Then, (3.3) follows by setting $K=m-1$.

So, with the aim of proving (3.12), we first introduce some notation: set

$$\begin{aligned}\tilde{L}_{m-r_i}: S^n\times\mathbb{C}\to\mathbb{C}: \quad & \tilde{L}_{m-r_i}(\omega,\tau)=L_{m-r_i}(\omega,\tau)\,, \quad i=0,\dots,R,\\ \tilde{\mathfrak{L}}_l:(N_l,\infty)\times S^n\times\mathbb{C}\to\mathbb{C}: \quad & \tilde{\mathfrak{L}}_l(\rho,\omega,\tau)=\rho^{-m}\mathfrak{L}_l(\rho\omega,\rho\tau), \quad l=0,\dots,R;\end{aligned}$$

observe that $\tilde{L}_{m-r_i}$ is just the restriction of $L_{m-r_i}(\xi,\tau)$ to $S^n\times\mathbb{C}\to\mathbb{C}$. Denote by $\tilde{\varphi}_1(\omega),\tilde{\varphi}_2(\omega),\dots,\tilde{\varphi}_m(\omega)$ the roots of $\tilde{L}_m(\omega,\tau)=\tilde{\mathfrak{L}}_0(\rho,\omega,\tau)$ with respect to $\tau$, and by $\tilde{\tau}^k_1(\rho,\omega),\tilde{\tau}^k_2(\rho,\omega),\dots,\tilde{\tau}^k_m(\rho,\omega)$ those of $\tilde{\mathfrak{L}}_k(\rho,\omega,\tau)$. Now, for $1\le l\le R$,

$$\begin{aligned}|\xi|^{-m}\mathfrak{L}_l(\xi,\tau)=&|\xi|^{-m}\Big(\tau^m+\sum_{j=1}^{m}P_j(\xi)\tau^{m-j}+\sum_{i=1}^{l}\sum_{|\alpha|+j=m-r_i}c_{\alpha,j}\xi^\alpha\tau^j\Big)\\ =&\tilde{\tau}^m+\sum_{j=1}^{m}P_j\big(\tfrac{\xi}{|\xi|}\big)\tilde{\tau}^{m-j}+\sum_{i=1}^{l}|\xi|^{-r_i}\sum_{|\alpha|+j=m-r_i}c_{\alpha,l}\big(\tfrac{\xi}{|\xi|}\big)^\alpha\tilde{\tau}^j\\ =&\tilde{L}_m\big(\tfrac{\xi}{|\xi|},\tilde{\tau}\big)+\sum_{i=1}^{l}|\xi|^{-r_i}\tilde{L}_{m-r_i}\big(\tfrac{\xi}{|\xi|},\tilde{\tau}\big)\,,\end{aligned}\tag{3.13}$$

where $\tilde{\tau}=\frac{\tau}{|\xi|}$. Since,

$$\tilde{L}_m\big(\tfrac{\xi}{|\xi|},\tilde{\tau}\big)=L_m\big(\tfrac{\xi}{|\xi|},\tilde{\tau}\big)=|\xi|^{-m}L_m(\xi,\tau)=|\xi|^{-m}\mathfrak{L}_0(\xi,\tau)=\tilde{\mathfrak{L}}_0\big(|\xi|,\tfrac{\xi}{|\xi|},\tilde{\tau}\big)$$

for $\xi\in\mathbb{R}^n$, $\tau\in\mathbb{C}$, and

$$\begin{aligned}\tilde{\mathfrak{L}}_{l+1}(\rho,\omega,\tau)=\rho^{-m}\mathfrak{L}_{l+1}(\rho\omega,\rho\tau)=&\rho^{-m}\sum_{i=0}^{l+1}L_{m-r_i}(\rho\omega,\rho\tau)\\ =&\rho^{-m}\sum_{i=0}^{l}L_{m-r_i}(\rho\omega,\rho\tau)+\rho^{-m}\sum_{|\alpha|+j=m-r_{l+1}}c_{\alpha,j}(\rho\omega)^\alpha(\rho\tau)^j\\ =&\tilde{\mathfrak{L}}_l(\rho,\omega,\tau)+\rho^{-r_{l+1}}\sum_{|\alpha|+j=m-r_i}c_{\alpha,j}\omega^\alpha\tau^j\\ =&\tilde{\mathfrak{L}}_l(\rho,\omega,\tau)+\rho^{-r_{l+1}}\tilde{L}_{m-r_{l+1}}(\omega,\tau)\end{aligned}\tag{3.14}$$

for $\omega\in S^{n-1}$, $\rho>N_{\max}$, $\tau\in\mathbb{C}$, $l=0,\dots,R-1$, we have, by repeated application of (3.14) in (3.13),

$$|\xi|^{-m}\mathfrak{L}_l(\xi,\tau) = \tilde{\mathfrak{L}}_l\big(|\xi|, \tfrac{\xi}{|\xi|}, \tilde{\tau}\big)\,. \tag{3.15}$$

As the left-hand side of this is zero when $\tau = \tau_k^l(\xi)$, $k = 1,\ldots,m$, and the right-hand side is zero when $\tilde{\tau} = \tilde{\tau}_k^l(|\xi|, \frac{\xi}{|\xi|})$, $k = 1,\ldots,m$, we see that $|\xi|\tilde{\tau}_k^l(|\xi|, \frac{\xi}{|\xi|}) = \tau_k^l(\xi)$ for each $k = 1,\ldots,m$ (possibly after reordering). Hence, for all $|\xi| > N_{\max}$, $k = 1,\ldots,m$ and $l = 0,\ldots,R-1$, we have

$$|\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| = |\tilde{\tau}_k^{l+1}\big(|\xi|, \tfrac{\xi}{|\xi|}\big) - \tilde{\tau}_k^l\big(|\xi|, \tfrac{\xi}{|\xi|}\big)||\xi|\,.$$

Next, observe that applying Lemma 3.1.6 with $\varepsilon = \rho^{r_{l+1}}$ to

$$\tilde{\mathfrak{L}}_l(\rho,\omega,\tau) + \rho^{-r_{l+1}}\tilde{L}_{m-r_{l+1}}(\omega,\tau)$$

yields, for all $\omega \in \mathcal{S}^{n-1}$ and $k = 1,\ldots,m$,

$$|\tilde{\tau}_k^{l+1}(\rho,\omega) - \tilde{\tau}_k^l(\rho,\omega)| \le C\rho^{-r_{l+1}}\left|\frac{\tilde{L}_{m-r_{l+1}}(\omega, \tilde{\tau}_k^l(\rho,\omega))}{\partial_\tau\tilde{\mathfrak{L}}_l(\rho,\omega,\tilde{\tau}_k^l(\rho,\omega))}\right|.$$

provided we take $\rho > M'$ for a sufficiently large constant $M' \ge N_{\max}$. Therefore, for all $|\xi| > M'$, $k = 1,\ldots,m$ and $l = 0,\ldots,R-1$, we have

$$|\tau_k^{l+1}(\xi) - \tau_k^l(\xi)| \le C|\xi|^{-r_{l+1}+1}\left|\frac{L_{m-r_{l+1}}\big(\frac{\xi}{|\xi|}, \frac{\tau_k^l(\xi)}{|\xi|}\big)}{\partial_\tau\tilde{\mathfrak{L}}_l\big(|\xi|, \frac{\xi}{|\xi|}, \frac{\tau_k^l(\xi)}{|\xi|}\big)}\right|. \tag{3.16}$$

Thus, it suffices to show the following two inequalities when $|\xi| > M$ for some $M \ge M'$:

- there exists a constant $C_1$ so that, for all $1 \le i \le R$,

$$\Big|L_{m-r_i}\big(\tfrac{\xi}{|\xi|}, \tfrac{\tau_k^l(\xi)}{|\xi|}\big)\Big| = \Big|\sum_{|\alpha|+j=m-r_i} c_{\alpha,l}\big(\tfrac{\xi}{|\xi|}\big)^\alpha\big(\tfrac{\tau_k^l(\xi)}{|\xi|}\big)^j\Big| \le C_1; \tag{3.17}$$

and

- there exists a constant $C_2 > 0$ so that, for all $0 \le l \le R-1$,

$$\left|\partial_\tau \tilde{\mathfrak{L}}_l\big(|\xi|, \tfrac{\xi}{|\xi|}, \tfrac{\tau_k^l(\xi)}{|\xi|}\big)\right| = |\xi|^{-m+1} |\partial_\tau \mathfrak{L}_l(\xi, \tau_k^l(\xi))| \ge C_2. \tag{3.18}$$

Then, combining (3.16), (3.17) and (3.18) gives (3.12).

Let us now show (3.17) and (3.18):

The first, (3.17), follows immediately from Part I since the $\tau_k^l(\xi)$ are roots of strictly hyperbolic equations.

The second, (3.18), in the case $l = 0$ is clear: the homogeneity of $L_m(\xi, \tau)$ and its roots give

$$|\xi|^{-m+1} |\partial_\tau \mathfrak{L}_0(\xi, \tau_k^0(\xi))| = \left|\partial_\tau L_m\big(\tfrac{\xi}{|\xi|}, \varphi_k\big(\tfrac{\xi}{|\xi|}\big)\big)\right|,$$

which is never zero due to the strict hyperbolicity of $L_m$ and hence (using that the sphere $S^{n-1}$ is compact and $L_m(\xi, \tau)$ is continuous and thus achieves its minimum) is bounded below by some positive constant as required.

For $1 \le l \le R-1$, we know that $\tau_k^l(\xi)$, $k = 1, \dots, m$, are simple zeros of $\mathfrak{L}_l(\xi, \tau)$ for $|\xi| > N_{\max}$ by the earlier choice of $N_{\max}$. Observe,

$$\frac{(\partial_\tau \mathfrak{L}_l)(\xi, \tau_k^l(\xi))}{|\xi|^{m-1}} = \frac{(\partial_\tau L_m)(\xi, \tau_k^l(\xi))}{|\xi|^{m-1}} + \sum_{i=1}^{l} \frac{(\partial_\tau L_{m-r_i})(\xi, \tau_k^l(\xi))}{|\xi|^{m-1}}.$$

Now,

$$\frac{(\partial_\tau L_{m-r_i})(\xi, \tau_k^l(\xi))}{|\xi|^{m-1}} = |\xi|^{-r_i} (\partial_\tau L_{m-r_i})\big(\tfrac{\xi}{|\xi|}, \tfrac{\tau_k^l(\xi)}{|\xi|}\big) \to 0 \text{ as } |\xi| \to \infty$$

for $i = 1, \dots, l$, because $\partial_\tau L_{m-r_i}(\xi, \tau)$ is homogeneous of order $m - r_i - 1$ and $\left|(\partial_\tau L_{m-r_i})\big(\tfrac{\xi}{|\xi|}, \tfrac{\tau_k^l(\xi)}{|\xi|}\big)\right| \le C$ for all $\xi \in \mathbb{R}^n$ for some $C \ge 0$ (here we use Part I once more). Also, using the Mean Value Theorem,

$$\begin{aligned}(\partial_\tau L_m)(\xi, \tau_k^l(\xi)) &= (\partial_\tau L_m)(\xi, \varphi_k(\xi)) + [(\partial_\tau L_m)(\xi, \tau_k^l(\xi)) - (\partial_\tau L_m)(\xi, \varphi_k(\xi))] \\ &= (\partial_\tau L_m)(\xi, \varphi_k(\xi)) + (\partial_\tau^2 L_m)(\xi, \bar{\tau}_k^l(\xi)),\end{aligned}$$

where $\bar{\tau}_k^l(\xi)$ lies on the path $\gamma$ connecting $\varphi_k(\xi)$ and $\tau_k^l(\xi)$, $\gamma(s) = \varphi_k(\xi) + s[\tau_k^l(\xi) - \varphi_k(\xi)]$, $0 \le s \le 1$ for each $\xi \in \mathbb{R}^n$, $k = 1, \dots, m$ and $l = 1, \dots, R-1$,

and

$$\frac{\left|(\partial_\tau^2 L_m)(\xi,\bar{\tau}_k^l(\xi))\right|}{|\xi|^{m-1}} = |\xi|^{-1}\left|\partial_\tau^2 L_m(\tfrac{\xi}{|\xi|},\tfrac{\bar{\tau}_k(\xi)}{|\xi|})\right| \le C|\xi|^{-1} \to 0 \text{ as } |\xi| \to \infty\,.$$

Therefore, for a sufficiently large constant $M \ge M'$, there exists a constant $C_2 > 0$ such that

$$\frac{\left|\partial_\tau L_m(\xi,\tau_k^l(\xi))\right|}{|\xi|^{m-1}} \ge C\frac{|\partial_\tau L_m(\xi,\varphi_k(\xi))|}{|\xi|^{m-1}} \ge C_2\,,\ \text{when } |\xi| > M.$$

This completes the proof of (3.17) and thus of Part II.

**Part III:** We take $N > 0$ as given by Lemma 3.1.4, that is, for $|\xi| > N$, the roots $\tau_1(\xi),\ldots,\tau_m(\xi)$ are distinct.

To prove the statement, we do induction on $|\alpha|$.

First, assume $|\alpha| = 1$. Since $L(\xi,\tau_k(\xi)) = 0$ for each $k = 1,\ldots,m$, we have, for each $i = 1,\ldots,n$,

$$\frac{\partial L}{\partial \xi_i}(\xi,\tau_k(\xi)) + \frac{\partial L}{\partial \tau}(\xi,\tau_k(\xi))\frac{\partial \tau_k}{\partial \xi_i}(\xi) = 0\,.$$

The first term is a polynomial of order $m-1$ in $(\xi,\tau_k(\xi))$, hence, by Part I, there exists a constant $C$ such that, when $|\xi| \ge M_1$ for some suitably large constant $M_1 \ge N$,

$$\left|\frac{\partial L}{\partial \xi_i}(\xi,\tau_k(\xi))\right| \le C|\xi|^{m-1}\,.$$

The inequality (3.4) for $|\alpha| = 1$ (i.e. (3.5)) then follows immediately from:

**Lemma 3.1.7.** *There exists constants $C > 0$, $M_2 \ge N$ such that, for each $k = 1,\ldots,m$,*

$$\left|\frac{\partial L}{\partial \tau}(\xi,\tau_k(\xi))\right| \ge C|\xi|^{m-1} \quad \textit{when } |\xi| > M_2\,.$$

*Proof.* Note that

$$\left|\frac{\partial L}{\partial \tau}(\xi,\tau_k(\xi))\right| \ge \left|\frac{\partial L_m}{\partial \tau}(\xi,\varphi_k(\xi))\right| - \left|\frac{\partial L}{\partial \tau}(\xi,\tau_k(\xi)) - \frac{\partial L_m}{\partial \tau}(\xi,\varphi_k(\xi))\right|, \quad (3.19)$$

where $L_m(\xi,\tau)$ is the principal symbol of $L$ and $\varphi_1(\xi),\ldots,\varphi_m(\xi)$ are the corresponding characteristic roots, ordered in the same way as in Part II. We look at each of the terms on the right-hand side in turn:

- By strict hyperbolicity, $\frac{\partial L_m}{\partial \tau}(\xi, \varphi_k(\xi))$ is non-zero for $\xi \neq 0$, so there is some constant $C > 0$ such that $|\frac{\partial L_m}{\partial \tau}(\omega, \varphi_k(\omega))| \geq C$ for all $\omega \in S^{n-1}$ (here we have used the compactness of the sphere). It is also clearly homogeneous of order $m-1$; thus, for all $\xi \neq 0$,

$$\Big|\frac{\partial L_m}{\partial \tau}(\xi, \varphi_k(\xi))\Big| = |\xi|^{m-1}\Big|\frac{\partial L_m}{\partial \tau}\Big(\tfrac{\xi}{|\xi|}, \varphi\big(\tfrac{\xi}{|\xi|}\big)\Big)\Big| \geq C|\xi|^{m-1}\,. \tag{3.20}$$

- Observe,

$$\frac{\partial L}{\partial \tau}(\xi, \tau_k(\xi)) - \frac{\partial L_m}{\partial \tau}(\xi, \varphi_k(\xi))$$
$$= \frac{\partial L_m}{\partial \tau}(\xi, \tau_k(\xi)) - \frac{\partial L_m}{\partial \tau}(\xi, \varphi_k(\xi)) + \sum_{r=0}^{m-1} \sum_{|\alpha|+l=r} c_{\alpha,l}\, l\, \xi^\alpha \tau_k(\xi)^{l-1}\,.$$

  Now,

$$\frac{\partial L_m}{\partial \tau}(\xi, \tau_k(\xi)) - \frac{\partial L_m}{\partial \tau}(\xi, \varphi_k(\xi))$$
$$= m(\tau_k(\xi)^{m-1} - \varphi_k(\xi)^{m-1}) + \sum_{j=1}^{m}(m-j)P_j(\xi)(\tau_k(\xi)^{m-j-1} - \varphi_k(\xi)^{m-j-1}),$$

  and

$$|\tau_k(\xi)^r - \varphi_k(\xi)^r| =$$
$$|\tau_k(\xi) - \varphi_k(\xi)||\tau_k(\xi)^{r-1} + \tau_k(\xi)^{r-2}\varphi_k(\xi) + \cdots + \varphi_k(\xi)^{r-1}|\,.$$

  So, by Part I and Part II (specifically inequality (3.3)) and the fact that the $P_j(\xi)$ are homogeneous polynomials in $\xi$ of order $j$, we have, for some suitably large $M_2 \geq N$,

$$\Big|\frac{\partial L_m}{\partial \tau}(\xi, \tau_k(\xi)) - \frac{\partial L_m}{\partial \tau}(\xi, \varphi_k(\xi))\Big| \leq C|\xi|^{m-2} \quad \text{when } |\xi| > M_2\,.$$

This, together with

$$\Big|\sum_{|\alpha|+l=r} c_{\alpha,r} l \xi^\alpha \tau_k(\xi)^{l-1}\Big| \le C|\xi|^{m-2} \quad \text{when } |\xi| > M_2,\ r = 0,\dots,m-1\,,$$

which again follows straight from Part I, yields

$$\Big|\frac{\partial L}{\partial \tau}(\xi,\tau_k(\xi)) - \frac{\partial L_m}{\partial \tau}(\xi,\varphi_k(\xi))\Big| \le C|\xi|^{m-2} \quad \text{for } |\xi| > M_2\,. \tag{3.21}$$

The result now follows by combining (3.19), (3.21) and (3.20). □

For $|\alpha| = J > 1$, assume inductively that,

$$\big|\partial_\xi^\alpha \tau_k(\xi)\big| \le C|\xi|^{1-|\alpha|} \quad \text{when } |\xi| > M,\ |\alpha| \le J-1\,,$$

for some fixed $M \ge \max(M_1, M_2)$.

Then, for $|\alpha| = J$, we use $\partial_\xi^\alpha[L(\xi,\tau_k(\xi))] = 0$, i.e.

$$\partial_\xi^\alpha \tau_k(\xi)\partial_\tau L(\xi,\tau_k(\xi)) + \sum_{\substack{\beta^1+\dots+\beta^r\le\alpha,\\ \beta^j\ne 0,\beta^j\ne\alpha}} c_{\alpha,\beta^1,\dots,\beta^r}\Big(\prod_{j=1}^r \partial_\xi^{\beta^j}\tau_k(\xi)\Big)\partial_\xi^{\alpha-\beta^1-\dots-\beta^r}\partial_\tau^r L(\xi,\tau_k(\xi)) = 0\,.$$

By the inductive hypothesis and the fact that $\partial_\xi^\beta \partial_\tau^j L(\xi,\tau_k(\xi))$ is a polynomial of degree $m - j - |\beta|$, we have, for all multi-indices $\beta^1,\dots,\beta^r \ne 0$ or $\alpha$ satisfying $\beta^1 + \dots + \beta^r \le \alpha$,

$$\left|\Big(\prod_{j=1}^r \partial_\xi^{\beta^j}\tau_k(\xi)\Big)\partial_\xi^{\alpha-\beta^1-\dots-\beta^r}\partial_\tau^r L(\xi,\tau_k(\xi))\right| \le C_{k,\alpha}|\xi|^{m-|\alpha|} \text{ when } |\xi| \ge M.$$

Thus, using Lemma 3.1.7 again, we have

$$|\partial_\xi^\alpha \tau_k(\xi)| \le \frac{C_\alpha|\xi|^{m-|\alpha|}}{|\partial_\tau L(\xi,\tau_k(\xi))|} \le C_{k,\alpha}|\xi|^{1-|\alpha|} \text{ when } |\xi| \ge M,$$

which completes the proof of the induction step.

**Part IV:** Once again, assume that the roots $\tau_k(\xi)$, $k = 1\,dots, m$ corresponds to $\varphi_k(\xi)$ $k = 1, \dots, m$ in the manner of Part II.

The proof of this part for general multi-index $\alpha$ is quite complicated, so we first give the proof in the case $|\alpha| = 1$ to demonstrate the main ideas required, and then show how it can be extended when $|\alpha| > 1$.

From $L(\xi, \tau_k(\xi)) = 0 = L_m(\xi, \varphi_k(\xi))$, we have for each $i = 1, \dots, n$,

$$\frac{\partial L}{\partial \xi_i}(\xi, \tau_k(\xi)) + \frac{\partial L}{\partial \tau}(\xi, \tau_k(\xi))\frac{\partial \tau_k}{\partial \xi_i}(\xi) = 0\,,$$
$$\frac{\partial L_m}{\partial \xi_i}(\xi, \varphi_k(\xi)) + \frac{\partial L_m}{\partial \tau}(\xi, \varphi_k(\xi))\frac{\partial \varphi_k}{\partial \xi_i}(\xi) = 0\,.$$

Therefore,

$$\frac{\partial L}{\partial \tau}(\xi, \tau_k(\xi))\Big(\frac{\partial \tau_k}{\partial \xi_i}(\xi) - \frac{\partial \varphi_k}{\partial \xi_i}(\xi)\Big) = \frac{\partial L_m}{\partial \xi_i}(\xi, \varphi_k(\xi)) - \frac{\partial L_m}{\partial \xi_i}(\xi, \tau_k(\xi))$$
$$+ \frac{\partial \varphi_k}{\partial \xi_i}\Big[\frac{\partial L_m}{\partial \tau}(\xi, \varphi_k(\xi)) - \frac{\partial L}{\partial \tau}(\xi, \tau_k(\xi))\Big] - \frac{\partial (L - L_m)}{\partial \xi_i}(\xi, \tau_k(\xi))\,. \quad (3.22)$$

It suffices to show that the right-hand side is bounded absolutely by $C|\xi|^{m-2}$ when $|\xi| > M$ for some suitably large $M_1 \geq N$; this is because an application of Lemma 3.1.7 then yields

$$\Big|\frac{\partial \tau_k}{\partial \xi_i}(\xi) - \frac{\partial \varphi_k}{\partial \xi_i}(\xi)\Big| \leq \frac{C|\xi|^{m-2}}{\big|\frac{\partial L}{\partial \tau}(\xi, \tau_k(\xi))\big|} \leq C|\xi|^{-1} \quad \text{for } |\xi| > M\,,$$

where $M = \max(M_1, M_2)$.

Since $\partial_{\xi_i}(L - L_m)(\xi, \tau)$ is a polynomial of degree $\leq m - 2$ in $(\xi, \tau)$ (it is the derivative of a polynomial of order $\leq m-1$), it is immediately clear that the final term of (3.22) is absolutely bounded by $C|\xi|^{m-2}$; here we have also used Part I. Also, noting that $|\partial_{\xi_i}\varphi_k(\xi)| \leq C$ by the homogeneity of $\varphi_k(\xi)$, we have, by (3.21),

$$\Big|\frac{\partial \varphi_k}{\partial \xi_i}(\xi)\Big|\Big|\frac{\partial L_m}{\partial \tau}(\xi, \varphi_k(\xi)) - \frac{\partial L_m}{\partial \tau}(\xi, \tau_k(\xi))\Big| \leq C|\xi|^{m-2}\,.$$

Finally, by the Mean Value Theorem,

$$\Big|\frac{\partial L_m}{\partial \xi_i}(\xi, \varphi_k(\xi)) - \frac{\partial L_m}{\partial \xi_i}(\xi, \tau_k(\xi))\Big| \leq C\Big|\frac{\partial^2 L_m}{\partial \tau \partial \xi_i}(\xi, \bar{\tau})\Big||\varphi_k(\xi) - \tau_k(\xi)|\,,$$

where $\bar\tau$ lies on the linear path between $\varphi_k(\xi)$ and $\tau_k(\xi)$—which means that (using Part I once more) $|\bar\tau| \le C|\xi|$ for $|\xi| \ge M$. Since $\partial_\tau \partial_{\xi_i} L_m(\xi,\tau)$ is a polynomial of degree $m-2$ in $(\xi,\tau)$, and $|\varphi_k(\xi) - \tau_k(\xi)| \le C$ by Part II, this term is bounded by $C|\xi|^{m-2}$, completing the proof in the case $|\alpha| = 1$.

For $|\alpha| = J > 1$, we assume inductively that

$$\left|\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi)\right| \le C|\xi|^{-|\alpha|} \quad \text{for } |\xi| > M\,, |\alpha| \le J-1\,.$$

As in the proof of Part III, we have

$$\partial_\xi^\alpha \tau_k(\xi) \partial_\tau L(\xi, \tau_k(\xi)) \\ + \sum_{\substack{\beta^1+\cdots+\beta^r \le \alpha, \\ \beta^j \ne 0, \beta^j \ne \alpha}} c_{\alpha,\beta^1,\ldots,\beta^r} \Big( \prod_{j=1}^r \partial_\xi^{\beta^j} \tau_k(\xi) \Big) \partial_\xi^{\alpha-\beta^1-\cdots-\beta^r} \partial_\tau^r L(\xi, \tau_k(\xi)) = 0\,;$$

similarly,

$$\partial_\xi^\alpha \varphi_k(\xi) \partial_\tau L_m(\xi, \varphi_k(\xi)) \\ + \sum_{\substack{\beta^1+\cdots+\beta^r \le \alpha, \\ \beta^j \ne 0, \beta^j \ne \alpha}} c_{\alpha,\beta^1,\ldots,\beta^r} \Big( \prod_{j=1}^r \partial_\xi^{\beta^j} \varphi_k(\xi) \Big) \partial_\xi^{\alpha-\beta^1-\cdots-\beta^r} \partial_\tau^r L_m(\xi, \varphi_k(\xi)) = 0\,.$$

Thus,

$$(\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi))\partial_\tau L(\xi, \tau_k(\xi)) =$$
$$\partial_\xi^\alpha \varphi_k(\xi)\big(\partial_\tau L_m(\xi, \varphi_k(\xi)) - \partial_\tau L(\xi, \tau_k(\xi))\big)$$
$$+ \sum_{\substack{\beta^1+\cdots+\beta^r\le\alpha,\\ \beta^j\neq 0, \beta^j\neq\alpha}} c_{\alpha,\beta^1,\ldots,\beta^r} \Big(\prod_{j=1}^r \partial_\xi^{\beta^j}\varphi_k(\xi)\Big)[\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\xi,\varphi_k(\xi)) -$$
$$\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\xi,\tau_k(\xi))]$$
$$+ \sum_{\substack{\beta^1+\cdots+\beta^r\le\alpha,\\ \beta^j\neq 0, \beta^j\neq\alpha}} c_{\alpha,\beta^1,\ldots,\beta^r} \Big(\prod_{j=1}^r [\partial_\xi^{\beta^j}\varphi_k(\xi) - \partial_\xi^{\beta^j}\tau_k(\xi)]\Big)\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\xi,\tau_k(\xi))$$
$$- \sum_{\substack{\beta^1+\cdots+\beta^r\le\alpha,\\ \beta^j\neq 0, \beta^j\neq\alpha}} c_{\alpha,\beta^1,\ldots,\beta^r} \Big(\prod_{j=1}^r \partial_\xi^{\beta^j}\tau_k(\xi)\Big)\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r (L-L_m)(\xi,\tau_k(\xi)) \,.$$

We claim the right-hand side is then bounded absolutely by $C_\alpha|\xi|^{m-1-|\alpha|}$, which, together with Lemma 3.1.7, yields the desired estimate.

To see this, let us look at each of the terms in turn:

- $|\partial_\xi^\alpha \varphi_k(\xi)| \le C_\alpha |\xi|^{1-|\alpha|}$ by the homogeneity of $\varphi_k(\xi)$; using this with (3.21) gives the desired bound.
- Using the Mean Value Theorem as in the case $|\alpha| = 1$, we get

$$\big|[\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\xi,\varphi_k(\xi)) - \partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\xi,\tau_k(\xi))]\big|$$
$$\le C_\alpha |\xi|^{m-|\alpha|+|\beta^1|+\cdots+|\beta^r|-r-1}\,;$$

  coupled with $|\partial_\xi^\beta \varphi_k(\xi)| \le C_\alpha |\xi|^{1-|\beta|}$, this gives the correct bound.
- By the inductive hypothesis,

$$|\partial_\xi^{\beta^j}\varphi_k(\xi) - \partial_\xi^{\beta^j}\tau_k(\xi)| \le C_\beta |\xi|^{1-|\beta^j|}\,;$$

  together with

$$|\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r L_m(\xi,\tau_k(\xi))| \le C_\alpha |\xi|^{m-|\alpha|+|\beta^1|+\cdots+|\beta^r|-r}\,,$$

which follows from Part I and the homogeneity of $L_m(\xi, \tau)$, this gives the correct estimate.

- To show the final term is bounded absolutely by $|\xi|^{m-1-|\alpha|}$, first note that $\partial_\xi^{\alpha-\beta^1-\cdots-\beta^r}\partial_\tau^r(L-L_m)(\xi,\tau_k(\xi))$ is a polynomial of degree $\leq m-|\alpha|+|\beta^1|+\cdots+|\beta^r|-r-1$; applying Part III to estimate the $\partial_\xi^{\beta^j}\tau_k(\xi)$ terms, we have the required result.

This completes the proof of (3.6); (3.7) is proved in a similar way in the proof using the set-up of the proof of Part II. □

**Remark 3.1.4:** As with Proposition 3.1.3 and Lemma 3.1.4, this Proposition holds when $L(D_t, D_x)$ is a (strictly hyperbolic) differential operator with respect to $t$ with coefficients that are pseudodifferential operators in $x$ (see Remarks 3.1.2 and 3.1.3).

Indeed, for the proof of Part I, it suffices for the $a_j(\xi)$ to be (classical) symbols of order $j$; for the proof of Part II, it suffices to have inequality in place of the equality in (3.15), and the proofs of (3.17) and (3.18) rely on the symbolic nature of the coefficients and the strict hyperbolicity of the operator. The proofs of Part III and Part IV similarly only use symbolic estimates and Lemma 3.1.7 (which is a consequence of strict hyperbolicity and the fact that the coefficients are smooth, and hence bounded from below on the unit sphere).

To complete this section, we give one final property which tells us about the image of the set of multiplicities of the characteristic roots.

**Corollary 3.1.8.** *Let $L = L(D_x, D_t)$ be a linear $m^{th}$ order constant coefficient strictly hyperbolic partial differential operator. Let $\tau_1(\xi), \ldots, \tau_m(\xi)$ be the characteristic roots of $L$. Then there exists a constant $R \geq 0$ such that, if $\tau_k(\xi_0) = \tau_l(\xi_0)$ for some $\xi_0 \in \mathbb{R}^n$, $l \neq k$, it follows that $|\tau_k(\xi_0)| < R$. That is, there exists a disc in $\mathbb{C}$ centred at $0$ of radius $R$ such that no multiple roots lie outside it.*

*Proof.* By Part I of Proposition 3.1.5 there exists $C > 0$ and $M > 0$ such that if $|\xi| > M$ then $|\tau_k(\xi)| \leq C|\xi|$ for all $k = 1, \ldots, m$; by Lemma 3.1.4, if $\tau_{k_1}(\xi_0) = \tau_{k_2}(\xi_0)$ for $k_1 \neq k_2$ then $|\xi_0| \leq N$. Also, by the continuity of the $\tau_k(\xi)$ (Lemma 3.1.3), there exists $B \geq 0$ such that if $|\xi| \leq M$ then $|\tau_k(\xi)| \leq B$ for all $k = 1, \ldots, m$. The claim then holds with $R = \max(NC, B)$. □

## 3.2 Convexity Results

As discussed in Section 1.3, in the case of homogeneous $m^{\text{th}}$ order strictly hyperbolic operators, geometric properties of the characteristic roots play the fundamental role in determining the $L^p - L^q$ decay; in particular, if the characteristic roots satisfy the convexity condition of Definition 1.1, then the decay is, in general, more rapid than when they do not. We will show that a similar improvement can be obtained for operators with lower order terms when a suitable 'convexity condition' holds. In section 3.2.3, we shall extend this notion of the convexity condition to functions $\tau : \mathbb{R}^n \to \mathbb{R}$ and prove a decay estimate for an oscillatory integral (related to the solution representation for a strictly hyperbolic operator) with phase function $\tau$.

First, we give a general result for oscillatory integrals and show how the concept of functions of "convex type" make this a useful result.

### 3.2.1 A General Theorem for Oscillatory Integrals

The following theorem is central in proving results involving convexity conditions. In some sense, it bridges the gap between the Van der Corput Lemma and the Method of Stationary Phase, in that the former is used when there is no convexity but gives a weaker result, while the former can be used when a stronger condition than simply convexity holds and gives a better result. Here, we state and prove a result that has no reference to convexity; however, in the following section, we show how convexity (in some sense) enables this result to be used in applications.

**Theorem 3.2.1.** *Consider the oscillatory integral*

$$I(\lambda, \nu) = \int_{\mathbb{R}^N} e^{i\lambda\Phi(y,\nu)} A(y, \nu) g(y)\, dy\,, \tag{3.23}$$

*where* $N \in \mathbb{N}$, $I : [0, \infty) \times \mathcal{N} \to \mathbb{C}$, $\mathcal{N}$ *is any set of parameters* $\nu$ *and*

(I1) *there exists a bounded open set* $U \subset \mathbb{R}^N$ *such that* $g \in C_0^\infty(U)$;

(I2) $\Phi(y,\nu)$ *is a complex-valued function such that* $\operatorname{Im}\Phi(y,\nu) \geq 0$ *for all* $y \in U$, $\nu \in \mathcal{N}$;

(I3) *for some fixed* $z \in \mathbb{R}^N$ *and some* $\gamma \in \mathbb{N}$, $\gamma \geq 2$, *the function*

$$F(\rho,\omega,\nu) := \Phi(\rho\omega + z,\nu)$$

*satisfies*

$$|\partial_\rho F(\rho,\omega,\nu)| \geq C\rho^{\gamma-1} \text{ and } |\partial_\rho^m F(\rho,\omega,\nu)| \leq C_m \rho^{1-m} |\partial_\rho F(\rho,\omega,\nu)|$$

*for all* $(\omega,\nu) \in S^{N-1} \times \mathcal{N}$, $m \in \mathbb{N}$ *and* $\delta > \rho \geq \lambda^{-1/\gamma} > 0$;

(I4) *for each multi-index* $\alpha$ *such that* $|\alpha| \leq \left[\frac{N}{\gamma}\right] + 1$, *there exists a constant* $C_\alpha > 0$ *such that* $|\partial_y^\alpha A(y,\nu)| \leq C_\alpha$ *for all* $y \in U$, $\nu \in \mathcal{N}$.

*Then there exists a constant* $C = C_{N,\gamma} > 0$ *such that*

$$|I(\lambda,\nu)| \leq C(1+\lambda)^{-\frac{N}{\gamma}} \quad \text{for all } \lambda \in [0,\infty),\, \nu \in \mathcal{N}. \tag{3.24}$$

**Remark 3.2.1:** This Theorem extends to the case where $A(y,\nu)$ is replaced by $A(y,\nu')$, where $\nu'$ is independent of the variable $\nu$ appearing in the phase function $\Phi(y,\nu)$; these parameters do not have to be related in any way, provided the estimates in hypotheses (I2) and (I4) hold uniformly in the appropriate parameters.

*Proof.* It is clear that (3.24) holds for $0 \leq \lambda < 1$ since $|I(\lambda,\nu)|$ is bounded for such $\lambda$.

Now, consider the case where $\lambda > 1$. Set $y = \rho\omega + z$, where $\omega \in S^{N-1}$ (using the convention that $S^0 = \{-1,1\}$), $\rho > 0$ and $z \in \mathbb{R}^N$ is some fixed point; then

$$I(\lambda,\nu) = \int_{S^{N-1}} \int_0^\infty e^{i\lambda\Phi(\rho\omega+z,\nu)} A(\rho\omega+z,\nu) g(\rho\omega+z) \rho^{N-1}\, d\rho\, d\omega\,.$$

By the compactness of $S^{N-1}$, it suffices to prove (3.24) for the inner integral.

Choose a function $\chi \in C_0^\infty(\mathbb{R}_+)$, $0 \le \chi(s) \le 1$ for all $s$, which is identically 1 on $0 \le s \le \frac{1}{2}$ and is zero when $s \ge 1$; then, writing $F(\rho,\omega,\nu) = \Phi(\rho\omega + z, \nu)$, we split the inner integral into the sum of the two integrals

$$I_1(\lambda,\nu,\omega,z) = \int_0^\infty e^{i\lambda F(\rho,\omega,\nu)} A(\rho\omega+z,\nu) g(\rho\omega+z)\chi(\lambda^{\frac{1}{\gamma}}\rho)\rho^{N-1}\,d\rho\,,$$
$$I_2(\lambda,\nu,\omega,z) = \int_0^\infty e^{i\lambda F(\rho,\omega,\nu)} A(\rho\omega+z,\nu) g(\rho\omega+z)(1-\chi)(\lambda^{\frac{1}{\gamma}}\rho)\rho^{N-1}\,d\rho\,.$$

Let us first look at $I_1 = I_1(\lambda,\nu,\omega,z)$; since $\chi(\lambda^{\frac{1}{\gamma}}\rho)$ is zero for $\lambda^{\frac{1}{\gamma}}\rho \ge 1$, we have, by the change of variables $\tilde\rho = \lambda^{\frac{1}{\gamma}}\rho$,

$$\begin{aligned} |I_1| &\le C\int_0^\infty \chi(\lambda^{\frac{1}{\gamma}}\rho)\rho^{N-1}\,d\rho = C\int_0^\infty (\tilde\rho)^{N-1}\lambda^{-\frac{N-1}{\gamma}}\chi(\tilde\rho)\lambda^{-\frac{1}{\gamma}}\,d\tilde\rho \\ &\le C\lambda^{-\frac{N}{\gamma}}\int_0^1 (\tilde\rho)^{N-1}\,d\tilde\rho = C\lambda^{-\frac{N}{\gamma}}\,, \end{aligned}$$

where we have used $|e^{i\lambda F(\rho,\omega,\nu)}| \le 1$ since $\operatorname{Im} F(\rho,\omega,\nu) \ge 0$ for all $\rho,\omega,\nu$ by hypothesis (I2); this is the desired estimate for $|I_1|$.

In order to estimate $I_2 = I_2(\lambda,\nu,\omega,z)$, let us first define the operator $L := (i\lambda\partial_\rho F(\rho,\omega,\nu))^{-1}\frac{\partial}{\partial\rho}$ and observe that

$$L(e^{i\lambda F(\rho,\omega,\nu)}) = e^{i\lambda F(\rho,\omega,\nu)}\,.$$

Denoting the adjoint of $L$ by $L^*$, we have, for each $l \in \mathbb{N}\cup\{0\}$,

$$I_2 = \int_0^\infty e^{i\lambda F(\rho,\omega,\nu)}(L^*)^l[A(\rho\omega+z,\nu)g(\rho\omega+z)(1-\chi)(\lambda^{\frac{1}{\gamma}}\rho)\rho^{N-1}]\,d\rho\,.$$

Now,

$$(L^*)^l = \Big(\frac{i}{\lambda}\Big)^l \sum C_{s_1,\dots,s_p,p,r,l}\frac{\partial_\rho^{s_1}F\dots\partial_\rho^{s_p}F}{(\partial_\rho F)^{l+p}}(\rho,\omega,\nu)\frac{\partial^r}{\partial\rho^r}\,,$$

where the sum is over all integers $s_1,\dots,s_p,p,r \ge 0$ such that $s_1+\dots+s_p+r-p = l$. By Hypothesis (I3),

$$\Big|\frac{\partial_\rho^{s_1}F\dots\partial_\rho^{s_p}F}{(\partial_\rho F)^{l+p}}(\rho,\omega,\nu)\Big| \le C\rho^{p-s_1-\dots-s_p-l\gamma+l} = C\rho^{r-l\gamma}\,.$$

Also, we claim that, for $r \leq [\frac{N}{\gamma}] + 1$,

$$\Big|\frac{\partial^r}{\partial\rho^r}[A(\rho\omega+z,\nu)g(\rho\omega+z)(1-\chi)(\lambda^{\frac{1}{\gamma}}\rho)\rho^{N-1}]\Big| \leq C_N\rho^{N-1-r}\tilde{\chi}(\lambda,\rho)\,, \quad (3.25)$$

where $\tilde{\chi}(\lambda,\rho)$ is a smooth function in $\rho$ which is zero for $\lambda^{\frac{1}{\gamma}}\rho < \frac{1}{2}$. Assuming this is true, we see that, for large enough $l$—it suffices to take $l = [\frac{N}{\gamma}] + 1$, i.e. $N - l\gamma < 0$—we have,

$$\begin{aligned}|I_2| \leq& C_N\lambda^{-l}\int_0^\infty \sum C_{s_1,\ldots,s_p,p,r,l}\rho^{r-l\gamma}[\rho^{N-1-r}]\tilde{\chi}(\lambda,\rho)\,d\rho \\ \leq& C_N\lambda^{-l}\int_{\frac{1}{2}\lambda^{-\frac{1}{\gamma}}}^\infty \rho^{N-1-l\gamma}\,d\rho = C_N\lambda^{-l}\Big[\frac{\rho^{N-l\gamma}}{N-l\gamma}\Big]_{\frac{1}{2}\lambda^{-\frac{1}{\gamma}}}^\infty = C_{N,\gamma}\lambda^{-\frac{N}{\gamma}};\end{aligned}$$

together with the estimate for $|I_1|$, this yields the desired estimate (3.24).

Finally, let us check (3.25). It holds because:

(i) $|\partial_\rho^r(\rho^{N-1})| \leq C_{r,N}\rho^{N-1-r}$ for all $r \in \mathbb{N}$.

(ii) For each $r \in \mathbb{N}$, $\partial_\rho^r[(1-\chi)(\lambda^{\frac{1}{\gamma}}\rho)] = -\lambda^{\frac{r}{\gamma}}(\partial_s^r\chi)(\lambda^{\frac{1}{\gamma}}\rho)$; now, $(\partial_s\chi)(\lambda^{\frac{1}{\gamma}}\rho)$ is supported on the set $\Big\{(\lambda,\rho) \in (0,\infty)\times(0,\infty) : \frac{1}{2} < \lambda^{\frac{1}{\gamma}}\rho < 1\Big\}$, so, in particular, on its support $\lambda^{\frac{1}{\gamma}} < \rho^{-1}$; therefore,

$$|\partial_\rho^r[(1-\chi)(\lambda^{\frac{1}{\gamma}}\rho)]| \leq C\rho^{-r}(\partial_s^r\chi)(\lambda^{\frac{1}{\gamma}}\rho) \quad \text{for all } r \in \mathbb{N}\,,$$

and $(\partial_s^r\chi)(\lambda^{\frac{1}{\gamma}}\rho)$ is smooth in $\rho$ and zero for $\lambda^{\frac{1}{\gamma}}\rho \leq \frac{1}{2}$.

(iii) By hypothesis (I4), $|\partial_\rho^r A(\rho\omega+z,\nu)| \leq C_r$ for each $r \leq [\frac{N}{\gamma}]+1$ (this can be seen for $r = 1$ by noting that $\partial_\rho A(\rho\omega+z,\nu) = \omega\cdot\nabla_y A(y,\nu)\big|_{y=\rho\omega+z}$, and then for $r \geq 2$ by calculating the higher derivatives). Also, $g$ is smooth in $U$, so, $|\partial_\rho^r[A(\rho\omega+z,\nu)g(\rho\omega+z)]| \leq C_r$ for $r \leq [\frac{N}{\gamma}] + 1$. Furthermore, by hypothesis (I1), there exists a constant $\rho_0 > 0$ so that $g(\rho\omega+z) = 0$ for $\rho > \rho_0$; thus, $\partial_\rho^r[A(\rho\omega+z,\nu)g(\rho\omega+z)]$ is zero for $\rho > \rho_0$; hence, for $r \leq [\frac{N}{\gamma}] + 1$,

$$|\partial_\rho^r[A(\rho\omega+z,\nu)g(\rho\omega+z)]| \leq C_r\rho_0^r\rho^{-r}\,.$$

This completes the proof of the claim, and thus the theorem. □

### 3.2.2 Functions of Convex Type

Hypothesis (I3) of Theorem 3.2.1 is sufficient for the result of the Theorem to hold; however, it is often difficult to check. For this reason, we now introduce the concept of a function of convex type—a condition that is far simpler to verify—and show that for such functions, (I3) automatically holds.

**Definition 3.1.** *Let $F = F(\rho, \upsilon) : [0, \infty) \times \Upsilon \to \mathbb{C}$ be a function that is smooth in $\rho$ for each fixed $\upsilon \in \Upsilon$, where $\Upsilon$ is some parameter space. Write its $N^{th}$ order Taylor expansion in $\rho$ about 0 in the form*

$$F(\rho, \upsilon) = \sum_{j=0}^{N} a_j(\upsilon)\rho^j + R_N(\rho, \upsilon)\,, \tag{3.26}$$

*where $R_N(\rho, \upsilon) = \int_0^\rho \partial_\rho^{N+1} F(s, \upsilon) \frac{(\rho-s)^N}{N!}\, ds$ is the $N^{th}$ remainder term.*

*We say $F$ is a* function of convex type $\gamma$ *if, for some $\gamma \in \mathbb{N}$, $\gamma \geq 2$, and for some $\delta > 0$,*

(CT1) $a_0(\upsilon) = a_1(\upsilon) = 0$ *for all $\upsilon \in \Upsilon$;*

(CT2) *there exists a constant $C > 0$ such that $\sum_{j=2}^{\gamma} |a_j(\upsilon)| \geq C$ for all $\upsilon \in \Upsilon$;*

(CT3) *for each $\upsilon \in \Upsilon$, $|\partial_\rho F(\rho, \upsilon)|$ is increasing in $\rho$ for $0 < \rho < \delta$;*

(CT4) *for each $k \in \mathbb{N}$, $\partial_\rho^k F(\rho, \upsilon)$ is bounded uniformly in $0 < \rho < \delta$, $\upsilon \in \Upsilon$.*

**Remark 3.2.2:** Note that, if $F$ is *real-valued*, then (CT3) is equivalent to requiring either $\partial_\rho^2 F(\rho, \upsilon) \geq 0$ for all $0 < \rho < \delta$, or $\partial_\rho^2 F(\rho, \upsilon) \leq 0$ for all $0 < \rho < \delta$—this is because $\partial_\rho F(0, \nu) = 0$. This is the connection with convexity, hence the name of such functions.

Such functions have the following useful property:

**Lemma 3.2.2.** *Let $F(\rho, \upsilon)$ be a function of convex type $\gamma$. Then, for each sufficiently small $0 < \delta \leq 1$ there exist constants $C, C_m > 0$ such that*

$$|\partial_\rho F(\rho, \upsilon)| \geq C\rho^{\gamma-1} \tag{3.27}$$

$$\text{and } |\partial_\rho^m F(\rho, \upsilon)| \leq C_m \rho^{1-m} |\partial_\rho F(\rho, \upsilon)| \tag{3.28}$$

*for all* $0 < \rho < \delta$, $\upsilon \in \Upsilon$ *and* $m \in \mathbb{N}$.

**Remark 3.2.3:** A version of this Lemma appeared in [Sug94] for *analytic* functions without dependence on $\upsilon$ and is based on Lemmas 3, 4 and 5 of Randol [Ran69] (which also appeared in Beals [Bea82], Lemmas 3.2, 3.3). This result extends it to functions that are only smooth and which depend on an additional parameter.

*Proof.* First, let us note that, for $0 < \rho \le 1$ we have, by (CT2),

$$\pi(\rho, \upsilon) := \sum_{j=2}^{\gamma} j|a_j(\upsilon)|\rho^{j-1} \ge C\rho^{\gamma-1} \,. \tag{3.29}$$

Thus, in order to prove (3.27), it suffices to show

$$|\partial_\rho F(\rho, \upsilon)| \ge C\pi(\rho, \upsilon) \quad \text{for all } 0 < \rho < \delta,\ \upsilon \in \Upsilon \,; \tag{3.30}$$

For $1 \le m \le \gamma$, we have, using (3.26),

$$\partial_\rho^m F(\rho, \upsilon) = \sum_{k=0}^{\gamma-m} \frac{(k+m)!}{k!} a_{k+m}(\upsilon)\rho^k + R_{m,\gamma-m}(\rho, \upsilon) \,, \tag{3.31}$$

where $R_{m,\gamma-m}(\rho, \upsilon) = \int_0^\rho \partial_\rho^{\gamma+1} F(s, \upsilon) \frac{(\rho-s)^{\gamma-m}}{(\gamma-m)!}\, ds$ is the remainder term of the $(\gamma - m)^{\text{th}}$ Taylor expansion of $\partial_\rho^m F(\rho, \upsilon)$. By (CT4) and (3.29), we see

$$|R_{m,\gamma-m}(\rho, \upsilon)| \le C_{\gamma,m}\rho^{\gamma+1-m} \le C_{\gamma,m}\pi(\rho, \upsilon)\rho^{2-m} \quad \text{for } 0 < \rho < \delta \,. \tag{3.32}$$

Hence, for $0 < \rho < \delta$,

$$\begin{aligned} |\partial_\rho F(\rho, \upsilon)| &= \Big|\sum_{k=0}^{\gamma-1}(k+1)a_{k+1}(\upsilon)\rho^k + R_{1,\gamma-1}(\rho, \upsilon)\Big| \\ &\ge \Big|\sum_{j=2}^{\gamma} j a_j(\upsilon)\rho^{j-1}\Big| - \Big|R_{1,\gamma-1}(\rho, \upsilon)\Big| \ge \Big|\sum_{j=2}^{\gamma} j a_j(\upsilon)\rho^{j-1}\Big| - C_\gamma \pi(\rho, \upsilon)\rho \,. \end{aligned}$$

Now, by (CT3), $|\partial_\rho F(\rho, \upsilon)|$ is increasing in $\rho$ for each $\upsilon \in \Upsilon$ and, by

(CT1), $\partial_\rho F(0,\upsilon)=0$; therefore,

$$\begin{aligned}
|\partial_\rho F(\rho,\upsilon)| &= \max_{0\le\sigma\le\rho}|\partial_\rho F(\sigma,\upsilon)| \\
&\ge \max_{0\le\sigma\le\rho}\Big|\sum_{j=2}^{\gamma} j a_j(\upsilon)\sigma^{j-1}\Big| - \max_{0\le\sigma\le\rho} C_\gamma \pi(\sigma,\upsilon)\sigma \\
&= \max_{0\le\bar\sigma\le 1}\Big|\sum_{j=2}^{\gamma} j a_j(\upsilon)\rho^{j-1}\bar\sigma^{j-1}\Big| - C_\gamma\pi(\rho,\upsilon)\rho\,,
\end{aligned}$$

since $\pi(\sigma,\upsilon)\sigma=\sum_{j=2}^{\gamma} j|a_j(\upsilon)|\sigma^j$ clearly achieves its maximum on $0\le\sigma\le\rho$ at $\sigma=\rho$. Noting that

$$\max_{0\le\bar\sigma\le 1}\Big|\sum_{j=1}^{M} z_j\bar\sigma^{j-1}\Big| \quad\text{and}\quad \sum_{j=1}^{M}|z_j|$$

are norms on $\mathbb{C}^M$ and, hence, are equivalent, we immediately get

$$\begin{aligned}
|\partial_\rho F(\rho,\upsilon)| \ge& C\sum_{j=2}^{\gamma} j|a_j(\upsilon)|\rho^{j-1} - C_\gamma\pi(\rho,\upsilon)\rho \\
\ge& (C-C_\gamma\delta)\pi(\rho,\upsilon) = C_{\gamma,\delta}\pi(\rho,\upsilon)\,,
\end{aligned}$$

which completes the proof of (3.30).

To prove (3.28), we consider the cases $1\le m\le\gamma$ and $m>\gamma$ separately.

For $m>\gamma$, we have, by (CT4),

$$|\partial_\rho^m F(\rho,\upsilon)| \le C_m \le C_{m,\delta}\rho^{\gamma+1-m} \quad \text{for } 0<\rho<\delta\,,$$

since $\gamma+1-m\le 0$, and, thus, $\rho^{\gamma+1-m}\ge\delta^{\gamma+1-m}>0$; so, by (3.27), we have

$$|\partial_\rho^m F(\rho,\upsilon)| \le C_{m,\delta}\rho^{2-m}|\partial_\rho F(\rho,\upsilon)| \quad \text{for } 0<\rho<\delta,\ m>\gamma\,. \tag{3.33}$$

For $1\le m\le\gamma$, we have the representation (3.31). It is clear that

$$\Big|\sum_{k=0}^{\gamma-m}\frac{(k+m)!}{k!}\,a_{k+m}(\upsilon)\rho^k\Big| \le C_m\pi(\rho,\upsilon)\rho^{1-m}\,,$$

which, together with (3.32) and (3.30), yields

$$|\partial_\rho^m F(\rho,\upsilon)| \le C_{m,\delta}\rho^{1-m}|\partial_\rho F(\rho,\upsilon)| \quad \text{for } 0<\rho<\delta,\ 1\le m\le\gamma\,.$$

This, together with (3.33), completes the proof of (3.28) and, thus, the Lemma. □

This Lemma means we have the following alternative version of Theorem 3.2.1.

**Corollary 3.2.3.** *Hypothesis* (I3) *of Theorem 3.2.1 may be replaced by:*

(I3′) *for some fixed* $z\in\mathbb{R}^N$*, the function* $F(\rho,\omega,\nu) := \Phi(\rho\omega+z,\nu)$ *is a function of convex type* $\gamma$*, for some* $\gamma\in\mathbb{N}$*, in the sense of Definition 3.1 with* $(\omega,\nu)\in S^{N-1}\times\mathcal{N}\equiv\Upsilon$.

### 3.2.3 Convexity Condition for Real-Valued Phase Functions

Using the results of the previous two sections, we can now prove a series of results for which a so-called convexity condition holds; here we give important definitions and prove the basic result for real-valued functions.

Given a smooth function $\tau:\mathbb{R}^n\to\mathbb{R}$ and $\lambda\in\mathbb{R}$, set

$$\Sigma_\lambda \equiv \Sigma_\lambda(\tau) := \{\xi\in\mathbb{R}^n : \tau(\xi)=\lambda\}\ .$$

In the case where $\tau(\xi)$ is homogeneous of order 1, write $\Sigma_\tau := \Sigma_1(\tau)$—for such $\tau$, we then have $\Sigma_\lambda(\tau)=\lambda\Sigma_\tau$.

**Definition 3.2.** *A smooth function* $\tau:\mathbb{R}^n\to\mathbb{R}$ *is said to satisfy the* convexity condition *if* $\Sigma_\lambda$ *is convex for each* $\lambda\in\mathbb{R}$*. Note that the empty set is considered to be convex.*

Another important notion is that of the *maximal order of contact* of a hypersurface:

**Definition 3.3.** *Let* $\Sigma$ *be a hypersurface in* $\mathbb{R}^n$ (*i.e. a manifold of codimension* $n-1$); *let* $\sigma\in\Sigma$*, and denote the tangent plane at* $\sigma$ *by* $T_\sigma$*. Now let* $P$ *be* *a plane containing the normal to $\Sigma$ at $\sigma$ and denote the order of the contact between the line $T_\sigma \cap P$ and the curve $\Sigma \cap P$ by $\gamma(\Sigma; \sigma, P)$. Then set*

$$\gamma(\Sigma) := \sup_{\sigma\in\Sigma} \sup_{P} \gamma(\Sigma; \sigma, P)\,.$$

**Examples 3.2.1:**

(a) $\gamma(S^n) = 2$, as $\gamma(S^n; \sigma, P) = 2$ for all $\sigma \in S^n$ and all planes $P$ containing $\sigma$ and the origin.

(b) If $\varphi_l(\xi)$ is a characteristic root of an $m^{\text{th}}$ order homogeneous strictly hyperbolic constant coefficient operator, then $\gamma(\Sigma_{\varphi_l}) \leq m$—see [Sug96] for a proof of this.

Also, let us introduce some useful notation for a family of cut-off functions $g_R \in C_0^\infty(\mathbb{R}^n)$, $R \in [0, \infty)$: suppose $g \in C_0^\infty(\mathbb{R}^n)$ such that, for some constants $c_0, c_1 \geq 0$, it is supported in the set

$$\{\xi : c_0 < |\xi| < c_1\}\ ,$$

and let $g_0 \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ be another (arbitrary) compactly supported function. Then, for $R \geq 0$, set

$$g_R(\xi) := \begin{cases} g(\xi/R) & \text{if } R \geq 1, \\ g_0(\xi) & \text{if } 0 \leq R < 1. \end{cases} \tag{3.34}$$

Now we can prove the main convexity theorem:

**Theorem 3.2.4.** *Suppose $\tau : \mathbb{R}^n \to \mathbb{R}$ satisfies the convexity condition; furthermore, assume:*

(i) *for all multi-indices $\alpha$ there exists a constant $C_\alpha > 0$ such that*

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|} \quad \textit{for all } \ \xi \in \mathbb{R}^n\,;$$

(ii) *there exist constants $M, C > 0$ such that for all $|\xi| \geq M$ we have $|\tau(\xi)| \geq C|\xi|$;*

(iii) *there exists a constant* $C_0 > 0$ *such that* $|\partial_\omega \tau(\lambda\omega)| \geq C_0$ *for all* $\omega \in S^{n-1}$, $\lambda > 0$; *in particular,* $|\nabla\tau(\xi)| \geq C_0$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$;

(iv) *there exists a constant* $R_1 > 0$ *such that, for all* $\lambda > 0$,

$$\Sigma'_\lambda := \frac{1}{\lambda}\Sigma_\lambda(\tau) \subset B_{R_1}(0)\,.$$

*Also, set* $\gamma := \sup_{\lambda>0}\gamma(\Sigma_\lambda(\tau))$ *and assume this is finite and let* $a(\xi)$ *be a symbol of order* $\frac{n-1}{\gamma} - n$ *of type* $(1,0)$ *on* $\mathbb{R}^n$. *Then, the following estimate holds for all* $R \geq 0$, $x \in \mathbb{R}^n$, $t > 1$:

$$\Big|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a(\xi)g_R(\xi)\,d\xi\Big| \leq Ct^{-\frac{n-1}{\gamma}}\,, \tag{3.35}$$

*where* $g_R(\xi)$ *is as given in* (3.34) *and* $C > 0$ *is independent of* $R$.

**Remark 3.2.4:** For an integral of this type with some specific compactly supported function, $\chi \in C_0^\infty(\mathbb{R}^n)$ say, in place of $g_R$, then use the results for $R = 0$.

*Proof.* We may assume throughout, without loss of generality, that either $\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ or $\tau(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$. Indeed, hypothesis (ii) and the continuity of $\tau$ ensure that either $\tau(\xi)$ is positive for all $|\xi| \geq M$ or negative for all $|\xi| \geq M$. In the case where $\tau(\xi)$ is positive for all $|\xi| \geq M$, set

$$\tau_+(\xi) := \tau(\xi) + \min(0, \inf_{|\xi|<M}\tau(\xi)) \geq 0 \;\text{ for all } \xi \in \mathbb{R}^n.$$

Now, $\tau(\xi) - \tau_+(\xi)$ is a constant (in particular, it is independent of $\xi$) and $|e^{i[\tau(\xi)-\tau_+(\xi)]t}| = 1$, so it suffices to show

$$\Big|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_+(\xi)t)}a(\xi)g_R(\xi)\,d\xi\Big| \leq Ct^{-\frac{n-1}{\gamma}}\,.$$

In the case where $\tau(\xi)$ is negative for $|\xi| \geq M$, set $\tilde{\tau}(\xi) := -\tau(\xi)$ and by similar reasoning to above, it is sufficient to show

$$\Big|\int_{\mathbb{R}^n} e^{i(x\cdot\xi-\tilde{\tau}_+(\xi)t)}a(\xi)g_R(\xi)\,d\xi\Big| \leq Ct^{-\frac{n-1}{\gamma}}\,,$$

where $-\tilde{\tau}_+(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$.

We begin by dividing the integral into two parts: near to the wave-front set, i.e. points where $\nabla_\xi[i(x\cdot\xi+\tau(\xi)t)]=0$, and away from such points. To this end, we introduce a cut-off function $\kappa\in C_0^\infty(\mathbb{R}^n)$, $0\le\kappa(y)\le 1$, which is identically 1 in the ball of radius $r>0$ (which will be fixed below) centred at the origin, $B_r(0)$, and identically 0 outside the ball of radius $2r$, $B_{2r}(0)$. Then we estimate the following two integrals separately:

$$\begin{aligned} I_1(x,t) &:= \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) \kappa\big(t^{-1}x+\nabla\tau(\xi)\big)\,d\xi\,, \\ I_2(x,t) &:= \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi)(1-\kappa)\big(t^{-1}x+\nabla\tau(\xi)\big)\,d\xi\,. \end{aligned}$$

For $I_2(x,t)$ we have the following result:

**Lemma 3.2.5.** *Suppose $a(\xi)$ is a symbol of order $j\in\mathbb{R}$. Then, for each $l\in\mathbb{N}$ with $l>n+j$, we have, for all $t>1$,*

$$|I_2(x,t)|\le C_{r,l}t^{-l}\,, \tag{3.36}$$

*where the constants $C_{r,l}>0$ are independent of $R$.*

*Proof.* In the support of $(1-\kappa)(t^{-1}x+\nabla\tau(\xi))$, $|x+t\nabla\tau(\xi)|\ge rt>0$, so we can write

$$\frac{(x+t\nabla\tau(\xi))}{i|x+t\nabla\tau(\xi)|^2}\cdot\nabla_\xi\big(e^{i(x\cdot\xi+\tau(\xi)t)}\big)=e^{i(x\cdot\xi+\tau(\xi)t)}\,;$$

therefore, denoting the adjoint to $P\equiv\frac{(x+t\nabla\tau(\xi))}{i|x+t\nabla\tau(\xi)|^2}\cdot\nabla_\xi$ by $P^*$,

$$I_2(x,t)=\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}(P^*)^l\big[a(\xi)g_R(\xi)(1-\kappa)\big(t^{-1}x+\nabla\tau(\xi)\big)\big]\,d\xi$$

for each $l\in\mathbb{N}$. We claim that for each $l$ there exists some constant $C_{r,l}>0$ independent of $R$ so that, when $t>1$,

$$(P^*)^l\big[a(\xi)g_R(\xi)(1-\kappa)\big(t^{-1}x+\nabla\tau(\xi)\big)\big]\le C_{r,l}t^{-l}(1+|\xi|)^{j-l}\,; \tag{3.37}$$

assuming this, we obtain,

$$|I_2(x,t)| \le C_{r,l} t^{-l} \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^{l-j}}\, d\xi\,.$$

Noting that $\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^{l-j}}\, d\xi$ converges for $l-j>n$ yields the desired estimate (3.36).

It remains to prove (3.37). Let $f \equiv f(\xi; x,t)$ be a function that is zero for $|x+t\nabla\tau(\xi)| \le rt$ and is continuously differentiable with respect to $\xi$; then,

$$\begin{aligned} P^* f = \nabla_\xi \cdot \Big[\frac{(x+t\nabla\tau(\xi))}{i|x+t\nabla\tau(\xi)|^2} f\Big] &= \frac{t\Delta\tau(\xi)}{i|x+t\nabla\tau(\xi)|^2} f + \frac{(x+t\nabla\tau(\xi))}{i|x+t\nabla\tau(\xi)|^2}\cdot\nabla_\xi f \\ &\quad - \frac{2t(x+t\nabla\tau(\xi))\cdot[\nabla^2\tau(\xi)\cdot(x+t\nabla\tau(\xi))]}{i|x+t\nabla\tau(\xi)|^4} f\,. \quad (3.38)\end{aligned}$$

Hence, using $|x+t\nabla\tau(\xi)| \ge rt$ (hypothesis on $f$) and $|\partial^\alpha\tau(\xi)| \le C(1+|\xi|)^{1-|\alpha|}$ (hypothesis (i)),

$$|P^* f| \le C_r t^{-1}[(1+|\xi|)^{-1}|f| + |\nabla_\xi f|]\,. \quad (3.39)$$

Now, for all multi-indices $\alpha$ and for all $\xi\in\mathbb{R}^n$,

- $|\partial^\alpha a(\xi)| \le C_\alpha(1+|\xi|)^{j-|\alpha|}$ for all $\xi\in\mathbb{R}^n$ as $a\in S^j_{1,0}(\mathbb{R}^n)$;
- $|\partial^\alpha_\xi\big[(1-\kappa)\big(t^{-1}x+\nabla\tau(\xi)\big)\big]| \le C_\alpha(1+|\xi|)^{-|\alpha|}$, for all $\xi\in\mathbb{R}^n$—here we have used hypothesis (i) once more. Also, it is zero for each $\alpha$ when $|x+t\nabla\tau(\xi)|\le rt$ by the definition of $\kappa$.

Furthermore, $|\partial^\alpha g_R(\xi)| = |\partial^\alpha g_0(\xi)| \le C_\alpha(1+|\xi|)^{-|\alpha|}$ for $0\le R<1$, since $C_0^\infty(\mathbb{R}^n\setminus\{0\}) \subset S^0_{1,0}(\mathbb{R}^n)$. For $R\ge 1$, we have:

$$\begin{aligned} &\partial^\alpha g_R(\xi) = \partial^\alpha[g(\xi/R)] = R^{-|\alpha|}(\partial^\alpha g)(\xi/R) \text{ and } g\in S^0_{1,0}(\mathbb{R}^n) \\ &\implies |\partial^\alpha g_R(\xi)| \le C_\alpha R^{-|\alpha|}(1+|\xi/R|)^{-|\alpha|} \le C_\alpha(1+|\xi|)^{-|\alpha|}\,. \end{aligned}$$

Therefore,

$$|\partial^\alpha g_R(\xi)| \le C_\alpha(1+|\xi|)^{-|\alpha|} \text{ for all } \xi\in\mathbb{R}^n \text{ and multi-indices } \alpha\,, \quad (3.40)$$

where the $C_\alpha > 0$ are independent of $R$.

Hence, by (3.39),

$$\big|P^*[a(\xi)g_R(\xi)(1-\kappa)\big(t^{-1}x+\nabla\tau(\xi)\big)]\big| \leq C_r t^{-1}(1+|\xi|)^{j-1}\,.$$

To prove (3.37) for $l \geq 2$ we do induction on $l$. Note that

$$|(P^*)^l f| \leq C_r t^{-1}[(1+|\xi|)^{-1}|(P^*)^{l-1}f| + |\nabla_\xi\{(P^*)^{l-1}f\}|]\,.$$

The first term satisfies the desired estimate by the inductive hypothesis. For the second term, repeated application of the properties of $a(\xi)$, $g(\xi)$ and $(1-\kappa)(t^{-1}x+\nabla\tau(\xi))$ noted above to inductively estimate derivatives of $(P^*)^{l'}f$, $1 \leq l' \leq l-2$ yields the desired estimate. This completes the proof of the lemma. □

This lemma, with $j = \frac{n-1}{\gamma} - n$, means that it suffices to prove (3.35) for $I_1(x,t)$, where $|t^{-1}x+\nabla\tau(\xi)| < 2r$.

Let $\{\Psi_\ell(\xi)\}_{\ell=1}^L$ be a partition of unity in $\mathbb{R}^n$ where $\Psi_\ell(\xi) \in C^\infty(\mathbb{R}^n)$ is supported in a narrow (the breadth is fixed below) open cone $K_\ell$, $\ell = 1,\ldots,L$—we may take the partition to be finite due to the compactness of $S^{n-1}$; let us assume that $K_1$ contains the point $e_n = (0,\ldots,0,1)$ (if necessary, relabel the cones to ensure this) and also that each $K_\ell$, $\ell = 1,\ldots,L$, can be mapped onto $K_1$ by rotation. Then, it suffices to estimate

$$I_1'(x,t) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a(\xi)g_R(\xi)\Psi_1(\xi)\kappa\big(t^{-1}x+\nabla\tau(\xi)\big)\,d\xi\,, \tag{3.41}$$

since the properties of $\tau(\xi)$, $a(\xi)$, $g_R(\xi)$ and $\kappa(t^{-1}x+\nabla\tau(\xi))$ used throughout are invariant under rotation.

By hypothesis (iii), the level sets $\Sigma_\lambda = \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\}$ are all non-degenerate (or empty). Furthermore, the Implicit Function Theorem allows us to parameterise the intersection of the surface $\Sigma_\lambda' \equiv \frac{1}{\lambda}\Sigma_\lambda$ and the cone $K_1$:

$$K_1 \cap \Sigma_\lambda' = \{(y, h_\lambda(y)) : y \in U\}\ ;$$

here $U \subset \mathbb{R}^{n-1}$ is the bounded open set for which $p(U) = S^{n-1} \cap K_1$ where $p(y) = (y, \sqrt{1-|y|^2})$, and $h_\lambda : U \to \mathbb{R}$ is a smooth function for each

$\lambda > 0$; in particular, each $h_\lambda$ is concave due to $\tau(\xi)$ satisfying the convexity condition, i.e. $\Sigma'_\lambda$ is convex for each $\lambda \in \mathbb{R}$. Then, in the case that $\tau(\xi) \geq 0$ all $\xi \in \mathbb{R}^n$, the cone $K_1$ is parameterised by

$$K_1 = \{(\lambda y, \lambda h_\lambda(y)) : \lambda > 0,\, y \in U\}\ ,$$

and when $\tau(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$,

$$K_1 = \{(\lambda y, \lambda h_\lambda(y)) : \lambda < 0,\, y \in U\}\ .$$

Now, let $\underline{\mathbf{n}} : K_1 \cap \Sigma'_\lambda \to S^{n-1}$ be the Gauss map,

$$\underline{\mathbf{n}}(\zeta) = \frac{\nabla\tau(\zeta)}{|\nabla\tau(\zeta)|}\,.$$

By the definition of $\kappa(t^{-1}x + \nabla\tau(\xi))$, we have

$$|t^{-1}x - (-\nabla\tau(\xi_\lambda))| < 2r$$

for each $\xi_\lambda \in K_1 \cap \Sigma'_\lambda$ that is also in the support of the integrand of (3.41). Hence, provided $r > 0$ is taken sufficiently small, the convexity of $\Sigma'_\lambda$ ensures that the points $t^{-1}x/|t^{-1}x|$ and $-\underline{\mathbf{n}}(\xi_\lambda)$ are close enough so that there exists $z(\lambda) \in U$ (for each $\xi_\lambda \in K_1 \cap \Sigma'_\lambda$) satisfying

$$\underline{\mathbf{n}}\big(z(\lambda), h_\lambda(z(\lambda))\big) = -t^{-1}x/|t^{-1}x| = -x/|x| \in S^{n-1}\,.$$

Also, $(-\nabla_y h_\lambda(y), 1)$ is normal to $\Sigma'_\lambda$ at $(y, h_\lambda(y))$, so, writing $x = (x', x_n)$,

$$-\frac{x}{|x|} = \frac{(-\nabla_y h_\lambda(z(\lambda)), 1)}{|(-\nabla_y h_\lambda(z(\lambda)), 1)|} \implies -\frac{x_n}{|x|} = \frac{1}{|(-\nabla_y h_\lambda(z(\lambda)), 1)|}$$
$$\text{and } -\frac{x'}{|x|} = \frac{-\nabla_y h_\lambda(z(\lambda))}{|(-\nabla_y h_\lambda(z(\lambda)), 1)|} = \frac{x_n \nabla_y h_\lambda(z(\lambda))}{|x|}\,;$$

therefore, $-x' = x_n \nabla_y h_\lambda(z(\lambda))$. We claim that $x_n$ is away from 0 provided the breadth of the cone $K_1$ is chosen to be sufficiently narrow, so

$$\frac{x'}{x_n} = -\nabla_y h_\lambda(z(\lambda))\,. \tag{3.42}$$

To prove this claim, first recall that $\Sigma'_\lambda \subset B_{R_1}(0)$ for all $\lambda > 0$ (hypothesis (iv)) and note that $\partial_{\xi_n}\tau(\xi)$ is absolutely continuous on $\overline{B_{R_1}(0)}$ (it is continuous in $\mathbb{R}^n$): taking $C_0 > 0$ as in hypothesis (iii),

$$\begin{gathered} \text{there exists } \delta > 0 \text{ so that } |\eta^1 - \eta^2| < \delta, \quad \text{where } \eta^1, \eta^2 \in \overline{B_{R_1}(0)}, \\ \text{implies } |\partial_{\xi_n}\tau(\eta^1) - \partial_{\xi_n}\tau(\eta^2)| < C_0/4\,. \end{gathered} \tag{3.43}$$

Then, fix the breadth of $K_1$ so that the maximal shortest distance from a point $\xi \in K_1 \cap (\bigcup_{\lambda>0}\Sigma'_\lambda)$ to the ray $\{\mu e_n : \mu > 0\}$ is less than this $\delta$, i.e.

$$\sup\left\{\inf_{\mu>0}|\xi - \mu e_n| : \xi \in K_1 \cap \big(\bigcup_{\lambda>0}\Sigma'_\lambda\big)\right\} < \delta\,.$$

Now, observe that for any $\xi_0 \in \mathbb{R}^n$, $\mu > 0$,

$$\left|\tfrac{x_n}{t}\right| \geq |\partial_{\xi_n}\tau(\mu e_n)| - |\partial_{\xi_n}\tau(\xi_0) - \partial_{\xi_n}\tau(\mu e_n)| - |\tfrac{x_n}{t} + \partial_{\xi_n}\tau(\xi_0)|\,.$$

Choose $\xi_0 \in K_1 \cap \Sigma'_\lambda \cap \operatorname{supp}[\kappa(t^{-1}x + \nabla\tau(\xi))]$ and $\mu > 0$ so that $|\xi_0 - \mu e_n| < \delta$ and, hence,

$$|\partial_{\xi_n}\tau(\xi_0) - \partial_{\xi_n}\tau(\mu e_n)| < C_0/4;$$

also, by hypothesis (iii), $|\partial_{\xi_n}\tau(\mu e_n)| \geq C_0$, so

$$|t^{-1}x_n| \geq 3C_0/4 - 2r.$$

Taking $r$ sufficiently small, less than $C_0/8$ say, (ensuring $r > 0$ satisfies the earlier condition also) we get

$$|x_n| \geq ct > 0 \tag{3.44}$$

proving the claim.

Before estimating (3.41), we introduce some useful notation: by the definition of $g_R(\xi)$, (3.34), when $R \geq 1$

$$\xi \in \operatorname{supp} g_R \implies Rc_0 < |\xi| < Rc_1;$$

also, if $0 \leq R < 1$, then there exist constants $\tilde{c}_0, \tilde{c}_1 > 0$ so that $\tilde{c}_0 < |\xi| < \tilde{c}_1$

for $\xi \in \operatorname{supp} g_R$. Thus, by hypotheses (i) and (ii), there exist constants $c_0', c_1' > 0$ such that

$$\begin{cases} Rc_0' < |\tau(\xi)| < Rc_1' & \text{if } R \geq 1 \text{ and } \xi \in \operatorname{supp} g_R, \\ c_0' < |\tau(\xi)| < c_1' & \text{if } 0 \leq R < 1 \text{ and } \xi \in \operatorname{supp} g_R. \end{cases}$$

Let $G \in C_0^\infty(\mathbb{R})$ which is identically one on the set $\{s \in \mathbb{R} : c_0' < s < c_1'\}$ and identically zero in a neighbourhood of the origin; writing $\mathcal{R} = \max(R, 1)$, this then satisfies

$$g_R(\xi) = g_R(\xi) G(\tau(\xi)/\mathcal{R}) \,.$$

Also, for simplicity, write

$$\tilde{a}(\xi) \equiv \tilde{a}_R(\xi) := a(\xi) g_R(\xi) \Psi_1(\xi) \,; \tag{3.45}$$

this is a type (1,0) symbol of order $\frac{n-1}{\gamma} - n$ supported in the cone $K_1$, and the constants in the symbolic estimates are all independent of $R$ as each $g_R(\xi)$, $R \geq 0$, is a symbol of order 0 with constants independent of $R$ (see (3.40)).

We now turn to estimating (3.41). Using the change of variables $\xi \mapsto (\lambda y, \lambda h_\lambda(y))$ and equality (3.42), it becomes

$$\begin{aligned} I_1'(x,t) = & \int_0^\infty \int_U e^{i[\lambda x' \cdot y + \lambda x_n h_\lambda(y) + \tau(\lambda y, \lambda h_\lambda(y))t]} a(\lambda y, \lambda h_\lambda(y)) \\ & g_R(\lambda y, \lambda h_\lambda(y)) \Psi_1(\lambda y, \lambda h_\lambda(y)) \kappa\big(t^{-1}x + \nabla\tau(\lambda y, \lambda h_\lambda(y))\big) \frac{d\xi}{d(\lambda, y)} \, dy \, d\lambda \\ = & \int_0^\infty \int_U e^{i\lambda x_n[-\nabla_y h_\lambda(z(\lambda)) \cdot y + h_\lambda(y) + t x_n^{-1}]} \tilde{a}(\lambda y, \lambda h_\lambda(y)) \\ & G(\lambda/\mathcal{R}) \kappa\big(t^{-1}x + \nabla\tau(\lambda y, \lambda h_\lambda(y))\big) \frac{d\xi}{d(\lambda, y)} \, dy \, d\lambda, \end{aligned} \tag{3.46}$$

where we have used $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ (definition of $\Sigma_\lambda$) in the last line. Here, note that

$$\frac{d\xi}{d(\lambda, y)} = \begin{vmatrix} \lambda I & y \\ \lambda \nabla_y h_\lambda(y) & \partial_\lambda[\lambda h_\lambda(y)] \end{vmatrix} = \lambda^{n-1} (\partial_\lambda[\lambda h_\lambda(y)] - y \cdot \nabla_y h_\lambda(y)) \,,$$

where $I$ is the identity matrix. Differentiating $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ with respect to $\lambda$ in the first case and with respect to $y$ in the second, gives

$$y \cdot \nabla_{\xi'} \tau(\lambda y, \lambda h_\lambda(y)) + \partial_\lambda[\lambda h_\lambda(y)] \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y)) = 1\,,$$
$$\lambda \nabla_{\xi'} \tau(\lambda y, \lambda h_\lambda(y)) + \lambda \nabla_y h_\lambda(y) \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y)) = 0\,;$$

Substituting the second of these equalities into the first yields

$$\big(\partial_\lambda[\lambda h_\lambda(y)] - y \cdot \nabla_y h_\lambda(y)\big) \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y)) = 1\,.$$

We claim that

$$|\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \geq C > 0\,. \tag{3.47}$$

To see this, first note that

$$|\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \geq |\partial_{\xi_n} \tau(\lambda \mu e_n)| - \big|\partial_{\xi_n} \tau(\lambda \mu e_n) - \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))\big|$$

where $\mu > 0$ is chosen as above so that $|\mu e_n - (y, h_\lambda(y))| \leq \delta$; now, $|\partial_{\xi_n} \tau(\lambda \mu e_n)| \geq C_0$ by hypothesis (iii). Also, by the Mean Value Theorem, there exists $\bar{\xi}$ lying on the segment between $(\lambda y, \lambda h_\lambda(y))$ and $\lambda \mu e_n$ such that

$$|\partial_{\xi_n} \tau(\lambda \mu e_n) - \partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))| \leq C |\nabla_\xi \partial_{\xi_n} \tau(\bar{\xi})| \lambda \delta \leq C |\bar{\xi}|^{-1} \lambda \delta \leq C \delta\,;$$

choosing $\delta > 0$ small enough (also ensuring it satisfies condition (3.43) above) completes the proof of the claim. Hence,

$$\Big|\frac{d\xi}{d(\lambda, y)}\Big| = \Big|\frac{\lambda^{n-1}}{\partial_{\xi_n} \tau(\lambda y, \lambda h_\lambda(y))}\Big| \leq C \lambda^{n-1}\,. \tag{3.48}$$

Also, note that this Jacobian is bounded below away from zero because $|\partial_{\xi_n} \tau(\xi)| \leq C$ for all $\xi \in \mathbb{R}^n$ (hypothesis (i)), which this means that the transformation above is valid in $K_1$.

Next, using the change of variables $\tilde{\lambda} = \lambda x_n = \lambda \tilde{x}_n t$ in (3.46), writing

$h(\lambda, y) \equiv h_\lambda(y)$ and setting $\tilde{x} := t^{-1}x$ (so $\tilde{x}_n = t^{-1}x_n$), we obtain

$$\int_0^\infty \int_U e^{i\tilde{\lambda}(-\nabla_y h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, z\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}\right)\right)\cdot y + h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right) + \tilde{x}_n^{-1})} \tilde{a}\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\Big)\Big)$$
$$G\Big(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\Big) \kappa\Big(\tilde{x} + \nabla\tau\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\Big)\Big)\Big) \frac{d\xi}{d(\lambda, y)} t^{-1} \tilde{x}_n^{-1} \, dy \, d\tilde{\lambda}\,.$$

Therefore, using $\left|\frac{d\xi}{d(\lambda,y)}\right| \leq C\tilde{\lambda}^{n-1}|\tilde{x}_n|^{-(n-1)}t^{-(n-1)}$ (by (3.48)) and recalling that $|\kappa(\eta)| \leq 1$, we have,

$$|I_1'(x,t)| \leq Ct^{-\frac{n-1}{\gamma}} |\tilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty \left| I\Big(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; z\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}\Big)\Big) G\Big(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\Big) \tilde{\lambda}^{\frac{n-1}{\gamma}-1} \right| d\tilde{\lambda}\,, \tag{3.49}$$

where,

$$I\Big(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; z\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}\Big)\Big) = \int_U e^{i\tilde{\lambda}\left[h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right) - h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, z\right) - (y-z)\cdot\nabla_y h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, z\right)\right]}$$
$$\tilde{a}\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t} y, \frac{\tilde{\lambda}}{\tilde{x}_n t} h\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\Big)\Big) \Big(\frac{\tilde{\lambda}}{t|\tilde{x}_n|}\Big)^{n-\frac{n-1}{\gamma}} dy\,.$$

With Theorem 3.2.1 in mind, let us rewrite this in the form of (3.23):

$$I(\lambda, \mu; z) = \int_{\mathbb{R}^{n-1}} e^{i\lambda\Phi(y,\mu;z)} a_0(\mu y, \mu h_\mu(y)) b(y) \, dy\,,$$

with arbitrary $\lambda > 0$, $\mu > 0$ and $z \in \mathbb{R}^{n-1}$, where

- $\Phi(y, \mu; z) = h_\mu(y) - h_\mu(z) - (y - z) \cdot \nabla_y h_\mu(z)$;
- $a_0(\xi) := \tilde{a}(\xi)|\xi|^{n-\frac{n-1}{\gamma}}$;
- $b \in C_0^\infty(\mathbb{R}^{n-1})$ with support contained in $U$.

We shall show that the following conditions (numbered as in Theorem 3.2.1 and Corollary 3.2.3) are satisfied by $I(\lambda, \mu; z)$:

(I1) there exists a bounded set $U \subset \mathbb{R}^{n-1}$ such that $b \in C_0^\infty(U)$;

(I2) $\operatorname{Im}\Phi(y, \mu; z) \geq 0$ for all $y \in U$, $\mu > 0$;

(I3′) $F(\rho, \omega, \mu; z) = \Phi(\rho\omega + z, \mu; z)$, $\omega \in S^{n-2}$, $\rho > 0$, is a function of convex type $\gamma$ (see Definition 3.1);

(I4) there exist constants $C_\alpha$ such that $|\partial_y^\alpha[a_0(\mu y, \mu h_\mu(y))]| \leq C_\alpha$ for all $y \in U$, $\mu > 0$ and $|\alpha| \leq [\frac{n-1}{\gamma}] + 1$.

Assuming for now that these hold, Theorem 3.2.1 (or, more precisely, Corollary 3.2.3) states that, for all $\lambda > 0$, $\mu > 0$,

$$|I(\lambda, \mu; z)| \leq C(1+\lambda)^{-\frac{n-1}{\gamma}} \leq C\lambda^{-\frac{n-1}{\gamma}}.$$

This, together with (3.49), gives

$$|I_1'(x,t)| \leq Ct^{-\frac{n-1}{\gamma}} |\tilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty \tilde{\lambda}^{-\frac{n-1}{\gamma}} G\Big(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\Big) \tilde{\lambda}^{\frac{n-1}{\gamma}-1} \, d\tilde{\lambda};$$

then, setting $\nu = \frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}$, we have

$$\begin{aligned} |I_1'(x,t)| &\leq Ct^{-\frac{n-1}{\gamma}} |\tilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty (\mathcal{R}\tilde{x}_n t\nu)^{-1} G(\nu) \mathcal{R}\tilde{x}_n t \, d\nu \\ &= Ct^{-\frac{n-1}{\gamma}} |\tilde{x}_n|^{-\frac{n-1}{\gamma}} \int_0^\infty \nu^{-1} G(\nu) \, d\nu \leq Ct^{-\frac{n-1}{\gamma}} \quad \text{for all } t > 1. \end{aligned}$$

Here we have used that $G$ is identically zero in a neighbourhood of the origin and that it is compactly supported and also (3.44) ($|\tilde{x}_n| \geq C > 0$); also, note the constant here is independent of $R$. Since this inequality holds for $I_1'(x,t)$, it also holds for $I_1(x,t)$; thus, together with Lemma 3.2.5, this proves the desired estimate (3.35), provided we show that the four properties (I1)–(I4) above hold.

Now, clearly (I1) holds automatically and (I2) is true since $h_\mu(y)$ is real-valued, so $\operatorname{Im} \Phi(y, \mu; z) = 0$ for all $y \in U$, $\mu > 0$.

For (I3′) and (I4), we need an auxiliary result about the boundedness of the derivatives of $h_\lambda(y)$:

**Lemma 3.2.6.** *All derivatives of $h_\lambda(y)$ with respect to $y$ are bounded uniformly in $y$. That is, for each multi-index $\alpha$ there exists a constant $C_\alpha > 0$ such that*

$$|\partial_y^\alpha h_\lambda(y)| \leq C_\alpha \quad \textit{for all } y \in U,\ \lambda > 0.$$

*Proof.* By definition, $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$. So,

$$(\nabla_{\xi'}\tau)(\lambda y, \lambda h_\lambda(y)) + (\partial_{\xi_n}\tau)(\lambda y, \lambda h_\lambda(y))\nabla_y h_\lambda(y) \\ = \lambda^{-1}\nabla_y[\tau(\lambda y, \lambda h_\lambda(y))] = 0\,,$$

or, equivalently,

$$\nabla_y h_\lambda(y) = -\frac{(\nabla_{\xi'}\tau)(\lambda y, \lambda h_\lambda(y))}{(\partial_{\xi_n}\tau)(\lambda y, \lambda h_\lambda(y))}\,. \tag{3.50}$$

Hypothesis (i) ($|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha(1+|\xi|)^{1-|\alpha|}$ for all $\xi \in \mathbb{R}^n$) and (3.47) ($|\partial_{\xi_n}\tau(\lambda y, \lambda h_\lambda(y))| \geq C > 0$) then ensure that $|\nabla_y h_\lambda(y)| \leq C$ for all $y \in U$, $\lambda > 0$.

For higher derivatives, note that $|(y, h_\lambda(y))| \leq R_1$ by hypothesis (iv); so, using hypothesis (i) once more, for all multi-indices $\alpha$, there exists a constant $C_\alpha > 0$ such that

$$|(\partial_\xi^\alpha \tau)(\lambda y, \lambda h_\lambda(y))| \leq C_\alpha \lambda^{1-|\alpha|}\,.$$

Then, differentiating (3.50), this ensures, by an inductive argument, that the desired result for higher derivatives of $h_\lambda(y)$ holds, proving the Lemma. □

Returning to the proof of (I4), note that,

$$|\partial_\xi^\alpha a_0(\xi)| \leq C_\alpha(1+|\xi|)^{-|\alpha|} \text{ for all } \xi \in \mathbb{R}^n\,,$$

since, $\tilde{a}(\xi)$ is a symbol of order $\frac{n-1}{\gamma} - 1$ (see (3.45) for its definition). Together with Lemma 3.2.6, this ensures that $\partial_y^\alpha[a_0(\mu y, \mu h_\mu(y))$ is absolutely bounded for all $y \in U$, $\mu > 0$ and $|\alpha| \leq [\frac{n-1}{\gamma}] + 1$ as required.

Finally, we show (I3′): observe that for $|\rho| < \delta'$, some suitably small $\delta' > 0$,

$$F(\rho, \omega, \mu; z) = h_\mu(\rho\omega + z) - h_\mu(z) - \rho\omega \cdot \nabla_y h_\mu(z) \\ = \sum_{k=2}^{\gamma+1}\Big[\sum_{|\alpha|=k}\frac{1}{\alpha!}(\partial_y^\alpha h_\mu)(z)\omega^\alpha\Big]\rho^k + R_{\gamma+1}(\bar{\rho}, \omega, \mu; z)\rho^{\gamma+2}\,.$$

So, $F(\rho, \omega, \mu; z)$ is a function of convex type $\gamma$ if (using the numbering of

Definition 3.1)

(CT2) $\sum_{k=2}^{\gamma+1}\left|\sum_{|\alpha|=k}\frac{1}{\alpha!}(\partial_y^\alpha h_\mu)(z)\omega^\alpha\right| \geq C > 0$ for all $\omega \in S^{n-2}$, $\mu > 0$, $z \in \mathbb{R}^{n-1}$.

(CT3) $|\partial_\rho F(\rho, \omega, \mu; z)|$ is increasing in $\rho$ for $0 < \rho < \delta$, for each $\omega \in S^{n-2}$, $\mu > 0$;

(CT4) for each $k \in \mathbb{N}$, $\partial_\rho^k F(\rho, \omega, \mu; z)$ is bounded uniformly in $0 < \rho < \delta'$, $\omega \in S^{n-2}$, $\mu > 0$.

Condition (CT4), follows straight from Lemma 3.2.6.

The concavity of $h_\mu(y)$ means that

$$\partial_\rho^2 F(\rho, \omega, \mu; z) = \partial_\rho^2[h_\mu(\rho\omega + z)] = \omega^t \operatorname{Hess} h_\mu(\rho\omega + z)\omega \leq 0$$

for all $0 < \rho < \delta'$ and for each $\omega \in S^{n-1}$, $\mu > 0$, $z \in \mathbb{R}^{n-1}$; coupled with the fact that $\partial_\rho F(0, \omega, \mu; z) = 0$, this ensures Condition (CT3) holds.

Lastly, recall that, by definition, $\gamma \geq \gamma(\Sigma_\lambda)$ for all $\lambda > 0$, which is the maximal order of contact between $\Sigma_\lambda$ and its tangent plane; furthermore, $\gamma$ is assumed to be finite; thus, for some $k \leq \gamma + 1 < \infty$,

$$\partial_\rho^k[h_\mu(z + \rho\omega)]\big|_{\rho=0} \neq 0\,.$$

Now, $\partial_\rho^k[h_\mu(z + \rho\omega)]\big|_{\rho=0} = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\partial_y^\alpha h_\mu(z)\omega^\alpha$, so for some $k \leq \gamma + 1$, $|\sum_{|\alpha|=k}\frac{k!}{\alpha!}\partial_y^\alpha h_\mu(z)\omega^\alpha| \geq C > 0$ for all $\omega \in S^{n-1}$ (the function achieves its minimum, which is non-zero, by compactness of the sphere). Thus, condition (CT2) holds.

This completes the proof of conditions (I1)–(I4), and, hence, the Theorem. □

## 3.3 Results without Convexity

Theorem 3.2.4 requires the phase function to satisfy the convexity condition of Definition 3.2; however, we will also investigate solutions to hyperbolic equations for which the characteristic roots do not necessarily satisfy such a condition. In this section we state and prove a theorem for this case. First, we give the key results that replaces Theorem 3.2.1 in the proof, the well-known Van der Corput Lemma; in addition, we study the case where the phase function is complex-valued with the imaginary part non-negative.

### 3.3.1 Van der Corput Lemma

The standard Van der Corput Lemma is given in, for example, [Sog93, Lemma 1.1.2]:

**Lemma 3.3.1.** *Let $\Phi \in C^\infty(\mathbb{R})$, $a \in C_0^\infty(\mathbb{R})$ and $m \geq 2$ be an integer such that $\Phi^{(j)}(0) = 0$ for $0 \leq j \leq m-1$ and $\Phi^{(m)}(0) \neq 0$; then*

$$\Big| \int_0^\infty e^{i\lambda\Phi(x)} a(x)\, dx \Big| \leq C\lambda^{-1/m}\,,$$

*provided the support of $a$ is sufficiently small. The constant on the left-hand side is independent of $\lambda$ and $\Phi$.*

**Remark 3.3.1:** If $m = 1$, then the same result holds provided $\Phi'(x)$ is monotonic on the support of $a$.

We extend this to the case where the phase function is complex-valued:

**Lemma 3.3.2.** *Let $\phi \in C^\infty(\mathbb{R})$, $\psi(x,\nu) \in C^\infty(\mathbb{R}\times\mathbb{R})$ be real-valued smooth functions and $a \in C_0^\infty(\mathbb{R}_+)$ that satisfy the following conditions for some integer $m \geq 2$:*

(i) $\phi^{(j)}(0) = 0$ *for* $0 \leq j \leq m-1$;

(ii) $|\phi^{(m)}(x)| \geq 1$ *for all* $x \in \operatorname{supp} a$;

(iii) $\psi(x,\nu) \geq 0$ *for all* $x \in \operatorname{supp} a$, $\nu \in \mathbb{R}$;

(iv) $|\partial_x^k \psi(x,\nu)| \leq C_k$ *for all* $k \geq 2$, $x \in \operatorname{supp} a$ *and all* $\nu \in \mathbb{R}$.

*Set $\Phi(x,\nu) = \phi(x) + i\psi(x,\nu)$; then,*

$$\Big| \int_0^\infty e^{i\lambda\Phi(x,\nu)} a(x)\, dx \Big| \leq C\lambda^{-1/m} \quad \text{for all } \nu \in \mathbb{R}\,; \tag{3.51}$$

*the constant $C$ is independent of $\phi, \psi$ and $\lambda$.*

*Proof.* The proof given here is an extension of the method used in [Ste93, VIII, 1.2, Proposition 2] where the phase function $\Phi(x)$ is real-valued.

First, assume $m = 2$. Let $\rho(y) \in C^\infty(\mathbb{R})$ that is identically 1 on $y \leq 1$ and identically 0 on $y \geq 2$; then, for some $\delta > 0$ (to be chosen later), split the integral into two parts:

$$I_1 + I_2 := \int_0^\infty e^{i\lambda\Phi(x,\nu)} a(x)\rho(x/\delta)\, dx + \int_0^\infty e^{i\lambda\Phi(x,\nu)} a(x)[1-\rho(x/\delta)]\, dx\,.$$

The integral $I_1$ is straightforward to estimate since $\psi(x,\nu) \geq 0$ for all $x, \nu$:

$$|I_1| \leq \int_0^{2\delta} |e^{-\lambda\psi(x,\nu)}||a(y)\rho(y/\delta)|\, dy \leq 2\delta \sup|a|\,.$$

For $I_2$, integrating by parts, after noting that

$$(i\lambda\partial_x\Phi(x,\nu))^{-1}\partial_x e^{i\lambda\Phi(x,\nu)} = e^{i\lambda\Phi(x,\nu)}\,,$$

yields

$$\begin{aligned} I_2 =& \frac{1}{i\lambda}\int_\delta^\infty e^{i\lambda\Phi(x,\nu)}\partial_x\Big[\frac{a(x)(1-\rho(x/\delta))}{\partial_x\Phi(x,\nu)}\Big]\, dx \\ =& \frac{1}{i\lambda}\int_\delta^\infty e^{i\lambda\Phi(x,\nu)}(\partial_x\Phi(x,\nu))^{-2}\big[\partial_x\Phi(x,\nu)\big(a'(x)(1-\rho(x/\delta))- \\ & \qquad a(x)\delta^{-1}\rho'(x/\delta)\big) - a(x)(1-\rho(x/\delta))\partial_x^2\Phi(x,\nu)\big]\, dx\,; \end{aligned}$$

there are no boundary terms since (i) $a \in C_0^\infty(\mathbb{R})$ and (ii) $\rho(1) = 1$. Now, by hypothesis $\phi'(x) = \phi''(0)x + O(x^2)$ as $x \to 0$, and $\phi''(0) \neq 0$; so, on $[\delta,\infty) \cap \operatorname{supp} a$,

$$|\partial_x\Phi(x,\nu)| \geq |\mathrm{Re}\,\partial_x\Phi(x,\nu)| = |\phi'(x)| \geq Cx > 0$$

and

$$|\partial_x^2\Phi(x,\nu)| \leq |\phi''(x)| + |\partial_x^2\psi(x,\nu)| \leq C\,.$$

Therefore, because the integrand is compactly supported,

$$|I_2| \leq \frac{C}{\lambda} \int_{\delta}^{c} [x^{-1} + \delta^{-1} x^{-1} + x^{-2}]\, dy \leq Cc\lambda^{-1}(\delta^{-1} + \delta^{-2}) \leq C\lambda^{-1}\delta^{-1}\,,$$

(we are implicitly assuming that $\delta \leq 1$ in the last inequality). Thus,

$$\Big|\int_0^{\infty} e^{i\lambda\Phi(x,\nu)} a(x)\, dx\Big| \leq C(\lambda^{-1}\delta^{-1} + \delta)\,,$$

which achieves its minimum when the two expressions on the right-hand side are equal, i.e. when $\lambda^{-1}\delta^{-1} = \delta$, or $\delta = \lambda^{-1/2}$. Hence, for $m = 2$, we have shown (3.51) when $\lambda \geq 1$ (we needed $\delta \leq 1$); furthermore, the estimate clearly holds when $\lambda < 1$ since the integral is bounded above by $C' = |\operatorname{supp} a||\sup a|$, so also by $C'\lambda^{-1/2}$.

For $m \geq 3$ this integration by parts cannot be carried out since we have no control over $\partial_x \Phi'(x, \nu)$ away from the origin; instead, we do induction on $m$: assume the theorem holds for $m = K$ and that the hypothesis holds for $m = K + 1$. As in the case for $m = 2$, split the integral into $I_1$ and $I_2$ and estimate $I_1$ as before. For $I_2$, let $z \in [\delta, \infty) \cap \operatorname{supp} a$ be the point such that $|\phi^{(K)}(x)|$ achieves its minimum—this point is unique since $|\phi^{(K+1)}(x)| \geq 1$, so $\phi^K(x)$ is either increasing or decreasing everywhere.

Now, introduce another cut-off function around $z$, $\rho_2(y) \in C_0^{\infty}(\mathbb{R})$, which is identically 1 in $(z - 1, z + 1)$ and identically zero outside $(z - 2, z + 2)$, and split $I_2$ into two parts: (set $\tilde{a}(x) := [1 - \rho(x/\delta)]a(x)$)

$$\int_{\delta}^{\infty} e^{i\lambda\Phi(x,\nu)} \tilde{a}(x)\rho_2(x/\delta)\, dx + \int_{\delta}^{\infty} e^{i\lambda\Phi(x,\nu)} \tilde{a}(x)[1 - \rho_2(x/\delta)]\, dx =: I_2^1 + I_2^2\,.$$

Integral $I_2^1$ can be estimated in the same way as $I_1$, namely $|I_2^1| \leq C\delta$. For $I_2^2$ , we use the inductive hypothesis. On the support of its integrand, $|\phi^{(K)}(x)| \geq \delta$. Indeed,

(i) if $\phi^{(K)}(z) = 0$, then $\phi^{(K)}(x) = (x - z)\phi^{(K+1)}(z) + O(|x - z|^2)$, as $x \to z$, $|x - z| \geq \delta$ and $|\phi^{(K+1)}(z)| \geq 1$, so $|\phi^{(K)}(x)| \geq \delta$;

(ii) on the other hand, if $\phi^{(K)}(z) \neq 0$ (in which case, either $z = \delta$ or $z$ is a boundary point of $\operatorname{supp} a$) then $\phi^{(K)}(x) = x\phi^{(K+1)}(0) + O(x^2)$ as $x \to 0$, $|x| \geq \delta$ and $|\phi^{(K+1)}(0)| \geq 1$, so, again, $|\phi^{(K)}(x)| \geq \delta$.

Therefore,

$$|I_2^2| = \Big|\int_\delta^\infty e^{i(\lambda\delta)[\Phi(x,\nu)/\delta]}\tilde{a}(x)[1-\rho_2(x/\delta)]\,dx\Big| \le C(\lambda\delta)^{-1/K}\,.$$

Hence,

$$\Big|\int_0^\infty e^{i\lambda\Phi(x,\nu)}a(x)\,dx\Big| \le C(\lambda^{-1/K}\delta^{-1/K}+\delta)\,.$$

This achieves its minimum when the two expressions on the right-hand side are equal, i.e. when $\lambda^{-1/K}\delta^{-1/K}=\delta$, or $\delta=\lambda^{-1/(K+1)}$, thus proving (3.51) for $m=K+1$, completing the induction step. □

### 3.3.2 Real-Valued Phase Function

In the case when the convexity condition holds the estimate of Theorem 3.2.4 is given in terms of the constant $\gamma$; as in the case of the homogeneous operators (see the Introduction, Section 1.3) we introduce an analog to this in the case where the convexity condition does not hold.

**Definition 3.4.** *Let $\Sigma$ be a hypersurface in $\mathbb{R}^n$; set*

$$\gamma_0(\Sigma) := \sup_{\sigma\in\Sigma}\inf_P \gamma(\Sigma;\sigma,P) \le \gamma(\Sigma)$$

*where $\gamma(\Sigma;\sigma,P)$ is as in Definition 3.3.*

**Remark 3.3.2:**

(a) When $n=2$, $\gamma_0(\Sigma)=\gamma(\Sigma)$;

(b) If $p(\xi)$ is a polynomial of order $m$ and $\Sigma=\{\xi\in\mathbb{R}^n : p(\xi)=0\}$ is compact then $\gamma_0(\Sigma)\le\gamma(\Sigma)\le m$; this is useful when applying the result below to hyperbolic differential equations and is proved in [Sug96].

An important result for calculating this value is the following:

**Lemma 3.3.3** ([Sug96])**.** *Suppose $\Sigma=\{(y,h(y)) : y\in U\}$, $h\in C^\infty(U)$, $U\subset\mathbb{R}^{n-1}$ is an open set, and let*

$$F(\rho)=h(\eta+\rho\omega)-h(\eta)-\rho\nabla h(\eta)\cdot\omega$$

*where* $\eta \in U$, $\omega \in S^{n-2}$. *Taking* $\sigma = (\eta, h(\eta)) \in \Sigma$, $\omega \in S^{n-2}$ *and*

$$P = \{\sigma + s(\omega, \nabla h(\eta) \cdot \omega) + t(-\nabla h(\eta), 1) \in \mathbb{R}^n : s, t \in \mathbb{R}\} \ ,$$

*then*

$$\gamma(\Sigma; \sigma, P) = \min \left\{ k \in \mathbb{N} : F^{(k)}(0) \neq 0 \right\} =: \gamma(h; \eta, \omega) \, .$$

*Therefore,*

$$\gamma(\Sigma) = \sup_{\eta} \sup_{\omega} \gamma(h; \eta, \omega),$$
$$\gamma_0(\Sigma) = \sup_{\eta} \inf_{\omega} \gamma(h; \eta, \omega) \, .$$

Now we are in a position to state and prove the result for oscillatory integrals with a real-valued phase function that does not satisfy the earlier convexity condition:

**Theorem 3.3.4.** *Suppose* $\tau : \mathbb{R}^n \to \mathbb{R}$ *satisfies the following conditions*:

(i) *for all multi-indices* $\alpha$ *there exists a constant* $C_\alpha > 0$ *such that*

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|} \quad \textit{for all } \ \xi \in \mathbb{R}^n \, ;$$

(ii) *there exist constants* $M, C > 0$ *such that for all* $|\xi| \geq M$ *we have* $|\tau(\xi)| \geq C|\xi|$;

(iii) *there exists a constant* $C_0 > 0$ *such that* $|\partial_\omega \tau(\lambda\omega)| \geq C_0$ *for all* $\omega \in S^{n-1}$, $\lambda > 0$;

(iv) *there exists a constant* $R_1 > 0$ *such that, for all* $\lambda > 0$,

$$\Sigma'_\lambda := \frac{1}{\lambda} \Sigma_\lambda(\tau) \subset B_{R_1}(0) \, .$$

*Set* $\gamma_0 := \sup_{\lambda > 0} \gamma_0(\Sigma_\lambda(\tau))$ *and assume it is finite*; *also, let* $a(\xi)$ *be a symbol of order* $\frac{1}{\gamma_0} - n$ *of type* $(1, 0)$ *on* $\mathbb{R}^n$. *Then, the following estimate holds for all* $R \geq 0$, $x \in \mathbb{R}^n$, $t > 1$:

$$\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \tau(\xi) t)} a(\xi) g_R(\xi) \, d\xi \right| \leq C t^{-\frac{1}{\gamma_0}} \, ,$$

*where* $g_R(\xi)$ *is as given in* (3.34) *and* $C > 0$ *is independent of* $R$.

*Proof.* We follow the proof of Theorem 3.2.4 as far as possible, and shall show how the absence of the convexity condition affects the estimate. Thus, as in that proof, we may first assume, without loss of generality, that either $\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ or $\tau(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$.

Divide the integral into two parts:

$$I_1(x,t) := \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) \kappa\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,,$$
$$I_2(x,t) := \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) (1-\kappa)\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,,$$

where $\kappa \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \kappa(y) \leq 1$, which is identically 1 in the ball of radius $r > 0$ centred at the origin, $B_r(0)$, and identically 0 outside the ball of radius $2r$, $B_{2r}(0)$. By Lemma 3.2.5 (which does not require the phase function to satisfy the convexity condition),

$$|I_2(x,t)| \leq C_r t^{-1/\gamma_0} \quad \text{for all } t > 1.$$

To estimate $|I_1(z,t)|$ we introduce, as before, a partition of unity $\{\Psi_\ell(\xi)\}_{\ell=1}^L$ and restrict attention to

$$I_1'(x,t) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) \Psi_1(\xi) \kappa\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,,$$

where $\Psi_1(\xi)$ is supported in a cone, $K_1$, that contains $e_n = (0,\dots,0,1)$. Parameterise this cone in the same way as above: with $U \subset \mathbb{R}^{n-1}$,

$$K_1 = \begin{cases} \{(\lambda y, \lambda h_\lambda(y)) : \lambda > 0,\ y \in U\} & \text{if } \tau(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}^n \\ \{(\lambda y, \lambda h_\lambda(y)) : \lambda < 0,\ y \in U\} & \text{if } \tau(\xi) \leq 0 \text{ for all } \xi \in \mathbb{R}^n\,. \end{cases}$$

Here the Implicit Function Theorem ensures the existence of a smooth function $h_\lambda : U \to \mathbb{R}$ for each $\lambda > 0$, but there is one major difference: the functions $h_\lambda$ are not necessarily concave, in contrast to the earlier proof. Using the change of variables $\xi \mapsto (\lambda y, \lambda h_\lambda(y))$—note that

$$0 < C \leq \left|\frac{d\xi}{d(\lambda,y)}\right| \leq C\lambda^{n-1}$$

by the same argument as in the proof of Theorem 3.2.4, providing the width of $K_1$ is taken to be sufficiently small—gives

$$I_1'(x,t) = \int_0^\infty \int_U e^{i[\lambda x'\cdot y + \lambda x_n h_\lambda(y) + \tau(\lambda y, \lambda h_\lambda(y))t]} a(\lambda y, \lambda h_\lambda(y))$$
$$g_R(\lambda y, \lambda h_\lambda(y))\Psi_1(\lambda y, \lambda h_\lambda(y))\kappa\big(t^{-1}x + \nabla\tau(\lambda y, \lambda h_\lambda(y))\big)\frac{d\xi}{d(\lambda,y)}\,dy\,d\lambda\,.$$

Once again, let $G \in C_0^\infty(\mathbb{R})$ so that $g_R(\xi) = g_R(\xi)G(\tau(\xi)/\mathcal{R})$ (where $\mathcal{R} = \max(R,1)$) and $\tilde{a}(\xi) = a(\xi)g_R(\xi)\Psi_1(\xi)$, which is a symbol of order $\frac{1}{\gamma_0} - n$ supported in $K_1$ and with all the constants in the symbolic estimates independent of $R$. So, recalling that $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ and writing $h(\lambda, y) \equiv h_\lambda(y)$,

$$I_1'(x,t) = \int_0^\infty \int_U e^{i\lambda[x'\cdot y + x_n h_\lambda(y) + t]}\tilde{a}(\lambda y, \lambda h_\lambda(y))$$
$$G(\lambda/\mathcal{R})\kappa\big(t^{-1}x + \nabla\tau(\lambda y, \lambda h_\lambda(y))\big)\frac{d\xi}{d(\lambda,y)}\,dy\,d\lambda$$
$$= \int_0^\infty \int_U e^{i\tilde{\lambda}[\frac{\tilde{x}'}{\tilde{x}_n}\cdot y + h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right) + \tilde{x}_n^{-1}]}\tilde{a}\Big(\tfrac{\tilde{\lambda}}{\tilde{x}_n t}y, \tfrac{\tilde{\lambda}}{\tilde{x}_n t}h\Big(\tfrac{\tilde{\lambda}}{\tilde{x}_n t}, y\Big)\Big)$$
$$G\Big(\tfrac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\Big)\kappa\Big(\tilde{x} + \nabla\tau\Big(\tfrac{\tilde{\lambda}}{\tilde{x}_n t}y, \tfrac{\tilde{\lambda}}{\tilde{x}_n t}h\Big(\tfrac{\tilde{\lambda}}{\tilde{x}_n t}, y\Big)\Big)\Big)\frac{d\xi}{d(\lambda,y)}\tilde{x}_n^{-1}t^{-1}\,dy\,d\tilde{\lambda}\,,$$

where $\tilde{\lambda} = \lambda x_n = \lambda\tilde{x}_n t$. Thus, using $|\kappa(\eta)| \le 1$,

$$|I_1'(x,t)| \le C|\tilde{x}_n|^{-1/\gamma_0}t^{-1/\gamma_0}\int_0^\infty \Big|I\Big(\tilde{\lambda}, \tfrac{\tilde{\lambda}}{\tilde{x}_n t}; \tilde{x}_n^{-1}\tilde{x}\Big)G\Big(\tfrac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\Big)\tilde{\lambda}^{-1+(1/\gamma_0)}\Big|\,d\tilde{\lambda} \tag{3.52}$$

where

$$I\Big(\tilde{\lambda}, \tfrac{\tilde{\lambda}}{\tilde{x}_n t}; \tilde{x}_n^{-1}\tilde{x}'\Big)$$
$$= \int_U e^{i\tilde{\lambda}\left[\tilde{x}_n^{-1}\tilde{x}'\cdot y + h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right]}\tilde{a}\Big(\tfrac{\tilde{\lambda}}{\tilde{x}_n t}y, \tfrac{\tilde{\lambda}}{\tilde{x}_n t}h\Big(\tfrac{\tilde{\lambda}}{\tilde{x}_n t}, y\Big)\Big)\Big(\tfrac{\tilde{\lambda}}{|\tilde{x}_n|t}\Big)^{n-\frac{1}{\gamma_0}}\,dy\,.$$

At this point, we diverge from the proof of the earlier theorem since we cannot apply Theorem 3.2.1; instead, note that, for some $b \in C_0^\infty(\mathbb{R}^{n-1})$

with support contained in $U$,

$$\left|I\Big(\tilde{\lambda},\tfrac{\tilde{\lambda}}{\tilde{x}_n t};\tilde{x}_n^{-1}\tilde{x}'\Big)\right| \leq \int_{\mathbb{R}^{n-2}}\left|\int_{\mathbb{R}} e^{i\tilde{\lambda}\left[\tilde{x}_n^{-1}\tilde{x}'\cdot y+h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t},y\right)\right]} \tilde{a}\Big(\tfrac{\tilde{\lambda}}{\tilde{x}_n t}y,\tfrac{\tilde{\lambda}}{\tilde{x}_n t}h\Big(\tfrac{\tilde{\lambda}}{\tilde{x}_n t},y\Big)\Big)\Big(\tfrac{\tilde{\lambda}}{|\tilde{x}_n|t}\Big)^{n-\frac{1}{\gamma_0}} b(y)\,dy_1\right| dy'\,.$$

We wish to apply the Van der Corput Lemma, Lemma 3.3.2, to the inner integral. Set $\Phi(y,\mu;z) := z\cdot y + h_\mu(y)$, which is real-valued, and consider the integral

$$\int_{\mathbb{R}} e^{i\lambda\Phi(y,\mu;z)}a_0(y,\mu)b(y)\,dy_1$$

where $a_0(y,\mu) := \mu^{n-(1/\gamma_0)}\tilde{a}(\mu y,\mu h_\mu(y))$. Recall that

$$\Sigma_\mu = \{(y,h_\mu(y)) : y\in U\}\ ,$$

so by Lemma 3.3.3,

$$\min\left\{k\in\mathbb{N} : \partial_{y_1}^k\Phi(y,\mu;z)\big|_{y_1=0}\neq 0\right\} = \gamma(h_\mu;0,(1,0,\ldots,0)) =: m\,.$$

Fixing the size of $U$ so that $|\partial_{y_1}^{(m)}\Phi(y,\mu;z)|\geq\varepsilon>0$ for all $y\in U$ ensures that the hypotheses of Lemma 3.3.2 are satisfied. Thus, since the support of $b$ is compact in $\mathbb{R}^{n-1}$ and $a_0$ is smooth,

$$\left|\int_{\mathbb{R}} e^{i\lambda\Phi(y,\mu;z)}a_0(y,\mu)b(y)\,dy_1\right| \leq C\lambda^{-1/m}\,.$$

Carry out a suitable change of coordinates so that $m=\inf_\omega\gamma(h_\mu;0,\omega)$ (this is possible due to the rotational invariance of all properties used); then, since $m\leq\gamma_0$ by definition,

$$\left|I\Big(\tilde{\lambda},\tfrac{\tilde{\lambda}}{\tilde{x}_n t};\tilde{x}_n^{-1}\tilde{x}'\Big)\right| \leq C\tilde{\lambda}^{-1/\gamma_0}\,,$$

for all $\tilde{\lambda}$ such that $\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\in\operatorname{supp}G$ (this is to ensure $\tilde{\lambda}$ is away from the origin). Combining this with (3.52) then gives the required estimate:

$$\begin{aligned}|I_1'(x,t)| \leq & C|\tilde{x}_n|^{-\frac{1}{\gamma_0}} t^{-\frac{1}{\gamma_0}} \int_0^\infty \left|\tilde{\lambda}^{-1} G\Big(\tfrac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\Big)\right| d\tilde{\lambda} \\ =& C|\tilde{x}_n|^{-\frac{1}{\gamma_0}} t^{-\frac{1}{\gamma_0}} \int_0^\infty (\nu\mathcal{R}^{-1}\tilde{x}_n t)^{-1} G(\nu)\mathcal{R}\tilde{x}_n t\, d\nu \leq Ct^{-\frac{1}{\gamma_0}}. \qquad \square\end{aligned}$$

### 3.3.3 Complex-Valued Phase Function

Having studied the case where the phase function $\tau(\xi)$ is real-valued (which will correspond to the case where the characteristic roots of the strictly hyperbolic operator are real), we now turn to the situation where it is complex-valued—of particular interest in the situation where characteristic roots tend asymptotically to the real axis (see Chapter 4), and for this reason, we only need consider the case where the imaginary part is non-negative.

We shall show that under suitable conditions on the imaginary part, a similar result to Theorem 3.3.4 holds:

**Theorem 3.3.5.** *Suppose $\tau : \mathbb{R}^n \to \mathbb{C}$ is a smooth function with $\operatorname{Im}\tau(\xi) \geq 0$ that satisfies:*

(i) *for all multi-indices $\alpha$ there exist constants $C_\alpha, C'_\alpha > 0$ such that*

$$|\partial_\xi^\alpha \operatorname{Re}\tau(\xi)| \leq C_\alpha(1+|\xi|)^{1-|\alpha|}$$

*and*

$$|\partial_\xi^\alpha \operatorname{Im}\tau(\xi)| \leq C'_\alpha(1+|\xi|)^{-|\alpha|}$$

*for all $\xi \in \mathbb{R}^n$;*

(ii) *there exist constants $M, C > 0$ such that for all $|\xi| \geq M$ we have $|\operatorname{Re}\tau(\xi)| \geq C|\xi|$;*

(iii) *there exists a constant $C_0 > 0$ such that $|\partial_\omega \operatorname{Re}\tau(\lambda\omega)| \geq C_0$ for all $\omega \in S^{n-1}$ and $\lambda > 0$;*

(iv) *there exists a constant $R_1 > 0$ such that, for all $\lambda > 0$,*

$$\frac{1}{\lambda}\{\xi \in \mathbb{R}^n : \operatorname{Re}\tau(\xi) = \lambda\} \subset B_{R_1}(0)\,.$$

*Set* $\gamma_0 := \sup_{\lambda>0} \gamma_0(\Sigma_\lambda(\operatorname{Re}\tau))$ *and assume it is finite and let* $a(\xi)$ *be a symbol of order* $n - \frac{1}{\gamma_0}$ *of type* $(1,0)$ *on* $\mathbb{R}^n$*. Then, the following estimate holds for all* $R \geq 0$, $x \in \mathbb{R}^n$, $t > 1$:

$$\Big|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi)\, d\xi\Big| \leq Ct^{-1/\gamma_0}\,,$$

*where* $g_R(\xi)$ *is as given in* (3.34) *and* $C > 0$ *is independent of* $R$.

**Remark 3.3.3:** Note that here the level sets under consideration are those of the *real* part; we use the same notation as before:

$$\Sigma_\lambda \equiv \Sigma_\lambda(\operatorname{Re}\tau) = \{\xi \in \mathbb{R}^n : \operatorname{Re}\tau(\xi) = \lambda\}\ .$$

*Proof.* For the most part, we once again follow the proof of Theorem 3.2.4; however, there are a few significant differences—some of which we deal with similarly to those resolved in the proof of Theorem 3.3.4, and others that must be considered separately since they arise due to the imaginary part of the phase function being non-zero.

First, we may assume without loss of generality that either $\operatorname{Re}\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ or $\operatorname{Re}\tau(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$ in the same way as in the earlier proofs. Then, split the integral into the usual two parts:

$$I_1(x,t) := \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi) \kappa\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,,$$
$$I_2(x,t) := \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) g_R(\xi)(1-\kappa)\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,,$$

where $\kappa \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \kappa(y) \leq 1$, which is identically 1 in the ball of radius $r > 0$ centred at the origin, $B_r(0)$, and identically 0 outside the ball of radius $2r$, $B_{2r}(0)$.

Now, the result of Lemma 3.2.5 continues to hold, so

$$|I_2(x,t)| \leq C_r t^{-1/\gamma_0}\,;$$

indeed, hypothesis (i) ensures that

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha (1+|\xi|)^{1-|\alpha|} \quad \text{for all } \xi \in \mathbb{R}^n\,,$$

and the assumption $\operatorname{Im}\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ implies that

$$|e^{i\tau(\xi)t}| = |e^{-\operatorname{Im}\tau(\xi)t}| \leq 1\,,$$

which is sufficient to prove Lemma 3.2.5 when $\tau(\xi)$ is complex-valued.

For $I_1(\xi, t)$, we again introduce the conic partition of unity $\{\Psi_\ell\}_{\ell=1}^L$ and focus on the integral supported in cone $K_1$ that contains $e_n = (0, \ldots, 0, 1)$:

$$I_1'(x,t) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau(\xi)t)} a(\xi) g_R(\xi) \Psi_1(\xi) \kappa\big(t^{-1}x + \nabla\tau(\xi)\big)\, d\xi\,.$$

In order to parameterise $K_1$, first note that hypothesis (iii) ensures that the level sets of the real part of the phase function,

$$\Sigma_\lambda = \{\xi \in \mathbb{R}^n : \operatorname{Re}\tau(\xi) = \lambda\}\,, \quad \lambda > 0\,,$$

are all non-degenerate or empty; then, $K_1$ may be written as

$$K_1 = \begin{cases} \{(\lambda y, \lambda h_\lambda(y)) : \lambda > 0,\, y \in U\} & \text{if } \operatorname{Re}\tau(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R}^n \\ \{(\lambda y, \lambda h_\lambda(y)) : \lambda < 0,\, y \in U\} & \text{if } \operatorname{Re}\tau(\xi) \leq 0 \text{ for all } \xi \in \mathbb{R}^n\,; \end{cases}$$

here the functions $h_\lambda : U \to \mathbb{R}$, $\lambda > 0$, are smoothly parameterise the surfaces $\Sigma_\lambda$.

Again, take $G \in C_0^\infty(\mathbb{R})$ and $\mathcal{R} = \max(R, 1)$ so that

$$g_R(\xi) = g_R(\xi) G(\operatorname{Re}\tau(\xi)/\mathcal{R})\,,$$

and write $\tilde{a}(\xi) = a(\xi) g_R(\xi) \Psi_1(\xi)$; this is valid by the earlier reasoning as hypotheses (ii) and (i) ensure that, with the earlier notation,

$$\begin{cases} Rc_0' < |\operatorname{Re}\tau(\xi)| < Rc_1' & \text{when } R \geq 1 \text{ and } \xi \in \operatorname{supp} g_R, \\ c_0' < |\operatorname{Re}\tau(\xi)| < c_1' & \text{when } 0 \leq R < 1 \text{ and } \xi \in \operatorname{supp} g_R. \end{cases}$$

Then, changing variables $\xi \mapsto (\lambda y, \lambda h_\lambda(y))$ gives

$$
\begin{aligned}
I_1'(x,t) &= \int_0^\infty \int_U e^{i[\lambda x'\cdot y + \lambda x_n h_\lambda(y) + \tau(\lambda y, \lambda h_\lambda(y))t]} a(\lambda y, \lambda h_\lambda(y)) g_R(\lambda y, \lambda h_\lambda(y)) \\
&\qquad \Psi_1(\lambda y, \lambda h_\lambda(y)) \kappa\big(t^{-1}x + \nabla\tau(\lambda y, \lambda h_\lambda(y))\big) \frac{d\xi}{d(\lambda, y)}\, dy\, d\lambda \\
&= \int_0^\infty \int_U e^{i\lambda[x'\cdot y + x_n h_\lambda(y) + t + it\lambda^{-1}\operatorname{Im}\tau(\lambda y, \lambda h_\lambda(y))]} \tilde{a}(\lambda y, \lambda h_\lambda(y)) \\
&\qquad G(\lambda/\mathcal{R}) \kappa\big(t^{-1}x + \nabla\tau(\lambda y, \lambda h_\lambda(y))\big) \frac{d\xi}{d(\lambda, y)}\, dy\, d\lambda\,,
\end{aligned}
$$

where we have used $\operatorname{Re}\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ by the definition of $\Sigma_\lambda$; note the difference in the phase function. Once again, it is straightforward to show that the Jacobian is absolutely bounded above by $C\lambda^{n-1}$ and is bounded from below away from zero using hypotheses (i) and (iii) (replace occurrences of $\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ by $\operatorname{Re}\tau(\lambda y, \lambda h_\lambda(y)) = \lambda$ and the argument in Theorem 3.2.4 carries straight through).

Next, setting $\tilde{x} = t^{-1}x$ and $h(\lambda, y) = h_\lambda(y)$ and changing variables with $\lambda \mapsto x_n^{-1}\tilde{\lambda} = (\tilde{x}_n t)^{-1}\tilde{\lambda}$, yields

$$
\begin{aligned}
I_1'(x,t) = \int_0^\infty \int_U & e^{i\tilde{\lambda}\left[\frac{\tilde{x}'}{\tilde{x}_n}\cdot y + h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right) + \tilde{x}_n^{-1} + it\tilde{\lambda}^{-1}\operatorname{Im}\tau\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}y, \frac{\tilde{\lambda}}{\tilde{x}_n t}h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right)\right]} G\Big(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\Big) \\
& \tilde{a}\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}y, \frac{\tilde{\lambda}}{\tilde{x}_n t}h\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\Big)\Big) \kappa\Big(\tilde{x} + \nabla\tau\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}y, \frac{\tilde{\lambda}}{\tilde{x}_n t}h\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\Big)\Big)\Big) \\
& \frac{d\xi}{d(\lambda, y)} t^{-1}\tilde{x}_n^{-1}\, dy\, d\tilde{\lambda}\,.
\end{aligned}
$$

Hence,

$$
|I_1'(x,t)| \le C|\tilde{x}_n|^{-1/\gamma_0} t^{-1/\gamma_0} \int_0^\infty \Big| I\Big(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; \tilde{x}_n^{-1}\tilde{x}\Big) G\Big(\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\Big) \tilde{\lambda}^{-1+(1/\gamma_0)} \Big|\, d\tilde{\lambda} \tag{3.53}
$$

(recall $|\kappa(\eta)| \le 1$), where

$$
\begin{aligned}
I\Big(\tilde{\lambda}, \frac{\tilde{\lambda}}{\tilde{x}_n t}; \tilde{x}_n, \tilde{x}_n^{-1}\tilde{x}'\Big) = \int_U & e^{i\tilde{\lambda}\left[\tilde{x}_n^{-1}\tilde{x}'\cdot y + h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right) + it\tilde{\lambda}^{-1}\operatorname{Im}\tau\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}y, \frac{\tilde{\lambda}}{\tilde{x}_n t}h\left(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\right)\right)\right]} \\
& \tilde{a}\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}y, \frac{\tilde{\lambda}}{\tilde{x}_n t}h\Big(\frac{\tilde{\lambda}}{\tilde{x}_n t}, y\Big)\Big) \Big(\frac{\tilde{\lambda}}{|\tilde{x}_n| t}\Big)^{n-\frac{1}{\gamma_0}} dy\,.
\end{aligned}
$$

As in the proof of Theorem 3.3.4, we will use Lemma 3.3.2 to estimate this final integral, so we observe that there exists $b \in C_0^\infty(\mathbb{R}^{n-1})$ with support contained in $U$ such that

$$\begin{aligned}&\Big|I\Big(\tilde\lambda, \tfrac{\tilde\lambda}{\tilde x_n t}; \tilde x_n \tilde x_n^{-1}\tilde x'\Big)\Big|\\ &\quad\le \int_{\mathbb{R}^{n-2}}\Big|\int_{\mathbb{R}} e^{i\tilde\lambda\left[\tilde x_n^{-1}\tilde x'\cdot y + h\left(\frac{\tilde\lambda}{\tilde x_n t},y\right)+it\tilde\lambda^{-1}\operatorname{Im}\tau\left(\frac{\tilde\lambda}{\tilde x_n t}y, \frac{\tilde\lambda}{\tilde x_n t}h\left(\frac{\tilde\lambda}{\tilde x_n t},y\right)\right)\right]}\\ &\qquad\qquad \tilde a\Big(\tfrac{\tilde\lambda}{\tilde x_n t}y, \tfrac{\tilde\lambda}{\tilde x_n t}h\Big(\tfrac{\tilde\lambda}{\tilde x_n t}, y\Big)\Big)\Big(\tfrac{\tilde\lambda}{|\tilde x_n|t}\Big)^{n-\frac{1}{\gamma_0}} b(y)\,dy_1\Big|\,dy'\,.\end{aligned}$$

Set $\phi(y,\mu;z) := z\cdot y + h_\mu(y)$ and $\psi(y,\mu,\tilde x_n) := \mu^{-1}\tilde x_n^{-1}\operatorname{Im}\tau(\mu y, \mu h_\mu(y))$; we shall show that these, as functions of $(y_1,\nu)$, satisfy the hypotheses of Lemma 3.3.2:

- By Lemma 3.3.3 and the definition of $\Sigma_\mu$,

  $$\min\Big\{k\in\mathbb{N} : \partial_{y_1}^k \phi(y,\mu;z)\big|_{y_1=0}\ne 0\Big\} = \gamma(h_\mu; 0, (1,0,\dots,0)) =: m\,,$$

  so the first two hypotheses are satisfied providing $U$ is chosen sufficiently small;

- $\psi(y,\mu,\tilde x_n)\ge 0$ for all $y$, $\mu$ and $\tilde x_n$ by our initial assumption on the positivity of $\operatorname{Im}\tau(\xi)$;

- to show $|\partial_{y_1}^k\psi(y,\mu,\tilde x_n)|\le C_k$ for all $k\ge 2$, $\mu\in\mathbb{R}$ and $y\in\operatorname{supp} a$, we note that

  $$\begin{aligned}\partial_{y_1}^k\psi(y,\mu,\tilde x_n) = \tilde x_n\mu^{k-1}[(\partial_{\xi_1}\operatorname{Im}\tau)(\mu y,\mu h_\mu(y))\\ + \partial_{y_1}h_\mu(y)(\partial_{\xi_n}\operatorname{Im}\tau)(\mu y, \mu h_\mu(y))]\,;\end{aligned}$$

  then, an analog of Lemma 3.2.6 for this $h_\lambda(y)$ (possible due to hypotheses (i) and (iv)) and the assumption that $\operatorname{Im}\tau(\xi)$ is a symbol of order 0 (hypothesis (i)) ensure that

  $$|\partial_{y_1}^k\psi(y,\mu,\tilde x_n)|\le C\frac{\mu^{k-1}}{(1+\mu)^k}\le C_k \quad \text{for all } y\in U,\ \mu>0\,;$$

  we have $|\tilde x_n|\ge C>0$ by a similar argument to that in the proof of Theorem 3.2.4.

So, with $a_0(y,\mu) := \mu^{n-(1/\gamma_0)}\tilde{a}(\mu y, \mu h_\mu(y))$, Lemma 3.3.2 implies that

$$\Big|\int_{\mathbb{R}} e^{i\lambda[\phi(y,\mu;z)+i\psi(y,\mu,\nu)]} a_0(y,\mu) b(y)\, dy_1\Big| \le C\lambda^{-1/m}\,.$$

Thus,

$$\Big|I\Big(\tilde{\lambda}, \tfrac{\tilde{\lambda}}{\tilde{x}_n t}, t\tilde{\lambda}^{-1}; \tilde{x}_n^{-1}\tilde{x}'\Big)\Big| \le C\tilde{\lambda}^{-1/\gamma_0}\,,$$

for all $\tilde{\lambda}$ such that $\frac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t} \in \operatorname{supp} G$ (this is to ensure $\tilde{\lambda}$ is away from the origin). Combining this with (3.53) completes the proof:

$$\begin{aligned}|I_1'(x,t)| \le{}& C|\tilde{x}_n|^{-\frac{1}{\gamma_0}} t^{-\frac{1}{\gamma_0}} \int_0^\infty \Big|\tilde{\lambda}^{-1} G\Big(\tfrac{\tilde{\lambda}}{\mathcal{R}\tilde{x}_n t}\Big)\Big|\, d\tilde{\lambda}\\ ={}& C|\tilde{x}_n|^{-\frac{1}{\gamma_0}} t^{-\frac{1}{\gamma_0}} \int_0^\infty (\nu\mathcal{R}^{-1}\tilde{x}_n t)^{-1} G(\nu)\mathcal{R}\tilde{x}_n t\, d\nu \le Ct^{-\frac{1}{\gamma_0}}\,. \qquad \square\end{aligned}$$

# Chapter 4: Proof of Main Theorem

## 4.1 Step 1: Representation of the Solution

Recall that we begin with the Cauchy problem with solution $u = u(x,t)$ as stated in Section 1.2:

$$\left.\begin{aligned} &D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, \quad t > 0, \\ &D_t^l u(x,0) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1, \ x \in \mathbb{R}^n\,, \end{aligned}\right\} \tag{4.1}$$

where $P_j(\xi)$, the polynomial obtained from the operator $P_j(D_x)$ by replacing each $D_{x_i}$ by $\xi_i$, is a constant coefficient homogeneous polynomial of order $j$, and the $c_{\alpha,r}$ are constants.

Applying the partial Fourier transform with respect to $x$ yields an ordinary differential equation for $\hat{u} = \hat{u}(\xi,t) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x,t)\,dx$:

$$D_t^m \hat{u} + \sum_{j=1}^{m} P_j(\xi) D_t^{m-j} \hat{u} + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} \xi^\alpha D_t^r \hat{u} = 0\,, \tag{4.2a}$$

$$D_t^l \hat{u}(\xi,0) = \hat{f}_l(\xi), \quad l = 0, \dots, m-1, \tag{4.2b}$$

where $(\xi,t) \in \mathbb{R}^n \times [0,\infty)$. Let $E_j = E_j(\xi,t)$, $j = 0, \dots, m-1$, be the solutions to (4.2a) with initial data

$$D_t^l E_j(\xi,0) = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{if } l \neq j. \end{cases} \tag{4.2c}$$

Then the solution $u$ of (4.1) can be written in the form

$$u(x,t) = \sum_{j=0}^{m-1} (\mathcal{F}^{-1} E_j \mathcal{F} f_j)(x,t), \tag{4.3}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the partial Fourier transform with respect to $x$ and its inverse respectively.

Now, as (4.2a), (4.2c) is the Cauchy problem for a linear ordinary differential equation, we can write, denoting the characteristic roots of (4.1) by $\tau_1(\xi), \dots, \tau_m(\xi)$ (see Definition 0.3),

$$E_j(\xi, t) = \sum_{k=1}^{m} A_j^k(\xi, t) e^{i\tau_k(\xi)t}$$

where $A_j^k(\xi, t)$ are polynomials in $t$ whose coefficients depend on $\xi$. Moreover, for each $k = 1, \dots, m$ and $j = 0 \dots, m-1$, the $A_j^k(\xi, t)$ are independent of $t$ at points of the (open) set $\{\xi \in \mathbb{R}^n : \tau_k(\xi) \neq \tau_l(\xi) \, \forall \, l \neq k\}$; when this is the case, we write $A_j^k(\xi, t) \equiv A_j^k(\xi)$. In particular, by Lemma 3.1.4, there exists $N > 0$ such that if $|\xi| > N$, the roots are pairwise distinct. For $A_j^k(\xi)$, we have the following properties:

**Lemma 4.1.1.** *Suppose $\xi \in S_k := \{\xi \in \mathbb{R}^n : \tau_k(\xi) \neq \tau_l(\xi) \, \forall \, l \neq k\}$; then we have the following formula*:

$$A_j^k(\xi) = \frac{(-1)^j \displaystyle\sum^{k}_{1 \leq s_1 < \dots < s_{m-j-1} \leq m} \prod_{q=1}^{m-j-1} \tau_{s_q}(\xi)}{\displaystyle\prod_{l=1, l \neq k}^{m} (\tau_l(\xi) - \tau_k(\xi))}, \tag{4.4}$$

*where $\sum^k$ means sum over the range indicated excluding $k$. Furthermore, we have, for each $j = 0, \dots, m-1$ and $k = 1, \dots, m$,*

(*i*) *$A_j^k(\xi)$ is smooth in $S_k$;*

(*ii*) *$A_j^k(\xi) = O(|\xi|^{-j})$ as $|\xi| \to \infty$.*

*Proof.* The representation (4.4) follows from Cramer's rule (and is done explicitly in [Kli67]): $A_j^k(\xi) = \frac{\det V_j^k}{\det V}$, where $V := \left(\tau_i^{l-1}(\xi)\right)_{i,l=1}^m$ is the Vandermonde matrix and $V_j^k$ is the matrix obtained by taking $V$ and replacing the $k^{\text{th}}$ column by $(\underbrace{0 \; \dots \; 0 \; 1}_{j} \; 0 \; \dots \; 0)^{\mathrm{T}}$.

Smoothness of $A_j^k(\xi)$ then follows by Proposition 3.1.3 and the asymptotic behaviour is a consequence of Part I of Proposition 3.1.5 since (4.4) holds for all $|\xi| > N$. $\square$

## 4.2 Step 2: Division of the Integral

In view of Lemmas 3.1.4 and 4.1.1, choose $N_1 > 0$ so that the $\tau_k(\xi)$, $k = 1, \ldots, n$, are distinct for $|\xi| > N_1$. Also, choose $N_2 > 0$ so that all points at which any of the roots, $\tau_k(\xi)$, meet the real axis—i.e. points $\xi \in \mathbb{R}^n$ such that, for all $\varepsilon > 0$, there exist $\xi_1, \xi_2 \in B_\varepsilon(\xi)$ with $\operatorname{Im} \tau_k(\xi_1) = 0$ and $\operatorname{Im} \tau_k(\xi_2) \neq 0$—lie in $B_{N_2}(0)$. Set $N = \max(N_1, N_2)$.

Let $\chi(\xi) = \chi_N(\xi) \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \chi(\xi) \leq 1$, be a cut-off function that is identically 1 for $|\xi| < N$ and identically zero for $|\xi| > 2N$. Then, using the linearity of the (inverse) Fourier transform, (4.3) can be rewritten as:

$$u(x,t) = \sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j \chi \mathcal{F} f_j)(x,t) + \sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j(1-\chi)\mathcal{F} f_j)(x,t)\,. \qquad (4.5)$$

**Large $|\xi|$:** The second term of (4.5) is the most straightforward to study: by the choice of $N$,

$$E_j(\xi,t)(1-\chi)(\xi) = \sum_{k=1}^{m} A_j^k(\xi)(1-\chi)(\xi) e^{i\tau_k(\xi)t}\,;$$

therefore, since each summand is smooth in $\mathbb{R}^n$,

$$\sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j(1-\chi)\mathcal{F} f_j)(x,t)$$
$$= \frac{1}{(2\pi)^n} \sum_{j=0}^{m-1} \sum_{k=1}^{m} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau_k(\xi)t)} A_j^k(\xi)(1-\chi)(\xi) \hat{f}_j(\xi)\, d\xi\,.$$

Each of these integrals may be studied separately. Note that, unlike in the cases of the wave equation, Brenner [Bre75], and the general $m^{\text{th}}$ order homogeneous strictly hyperbolic equations, Sugimoto [Sug94], we may not assume that $t = 1$ (see Lemma 1.3.1 for the homogeneous case). The $L^p - L^q$ estimates obtained under different conditions on the phase function for operators of this type are calculated in Section 4.4 below.

**Bounded $|\xi|$:** We turn our attention to the terms of the first sum in (4.5), the case of low frequencies,

$$\mathcal{F}^{-1}(E_j\chi\mathcal{F}f)(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\Big(\sum_{k=1}^{m} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi(\xi)\hat{f}(\xi)\,d\xi\,. \quad (4.6)$$

Unlike in the case above, here the characteristic roots $\tau_1(\xi),\dots,\tau_m(\xi)$ are not necessarily distinct at all points in the support of the integrand (which is contained in the ball of radius $2N$ about the origin); in particular, this means that the $A_j^k(\xi,t)$ genuinely depend on $t$ and we have no simple formula valid for them in the whole region.

For this reason, we begin by systematically separating neighbourhoods of points where roots meet—referred to henceforth as multiplicities—from the rest of the region, and then considering the two cases separately. In Section 4.5 we find $L^p - L^q$ estimates in the region away from multiplicities under various conditions; in Section 4.6 we show how these differ in the neighbourhoods of singularities, to give us the final results in Theorem 2.1.1.

First, we need to understand in what type of sets the roots $\tau_k(\xi)$ can intersect:

**Lemma 4.2.1.** *The complement of the set of multiplicities of a linear strictly hyperbolic constant coefficient partial differential operator $L(D_x, D_t)$,*

$$S := \{\xi \in \mathbb{R}^n : \tau_j(\xi) \neq \tau_k(\xi) \text{ for all } j \neq k\}\ ,$$

*is dense in $\mathbb{R}^n$.*

*Proof.* First note

$$S = \{\xi \in \mathbb{R}^n : \Delta_L(\xi) \neq 0\}\ ,$$

where $\Delta_L$ is the discriminant of $L(\xi,\tau)$ (see the proof of Lemma 3.1.4 for definition and some properties). Now, by Sylvester's Formula (see, for example, [GKZ94]), $\Delta_L$ is a polynomial in the coefficients of $L(\xi,\tau)$, which are themselves polynomials in $\xi$. Hence, $\Delta_L$ is a polynomial in $\xi$; as it is not identically zero (for large $|\xi|$, the characteristic roots are distinct, and hence it is non-zero at such points), it cannot be zero on an open set, and hence its complement is dense in $\mathbb{R}^n$. □

**Corollary 4.2.2.** *Let $L(\xi, \tau)$ be a linear strictly hyperbolic constant coefficient partial differential operator with characteristic roots $\tau_1(\xi), \ldots, \tau_m(\xi)$. Suppose $\mathcal{M}_{kl} \subset \mathbb{R}^n$ is a set such that $\tau_k(\xi) = \tau_l(\xi)$, for some $k \neq l$, for all $\xi \in \mathcal{M}_{kl}$. For $\varepsilon > 0$, define*

$$\mathcal{M}_{kl}^{\varepsilon} := \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}_{kl}) \leq \varepsilon\} ;$$

*denote the minimal $\nu \in \mathbb{N}$ such that* $\operatorname{meas}(\mathcal{M}_{kl}^{\varepsilon}) \leq C\varepsilon^{\nu}$ *for all sufficiently small $\varepsilon > 0$ by* $\operatorname{codim} \mathcal{M}_{kl}$. *Then* $\operatorname{codim} \mathcal{M}_{kl} \geq 1$.

*Proof.* Follows straight from Lemma 4.2.1: the fact that $\mathcal{M}_{kl}$ has non-empty interior ensures that its $\varepsilon$-neighbourhood is bounded by $C\varepsilon$ in at least one dimension for all small $\varepsilon > 0$. □

With this in mind, we shall subdivide the integral (4.6): suppose $L$ roots meet on a set $\mathcal{M}$ with $\operatorname{codim} \mathcal{M} = \ell$; without loss of generality, assume the coinciding roots are $\tau_1(\xi), \ldots, \tau_L(\xi)$. By continuity, there exists an $\varepsilon > 0$ such that only characteristic roots coinciding with $\tau_k(\xi)$, $k \in \{1, \ldots, L\}$, in $\mathcal{M}^{\varepsilon}$ are $\tau_1(\xi), \ldots, \tau_L(\xi)$. Furthermore, we may assume that $\partial\mathcal{M}^{\varepsilon} \in C^1$: for each $\varepsilon > 0$ there exists a set $S_{\varepsilon}$ with $C^1$ boundary such that $\mathcal{M}^{\varepsilon} \subset S_{\varepsilon}$ and $\operatorname{meas}(\mathcal{M}^{\varepsilon}) \to \operatorname{meas}(S_{\varepsilon})$ as $\varepsilon \to 0$. Then:

1. Let $\chi_{\mathcal{M},\varepsilon} \in C^{\infty}(\mathbb{R}^n)$ be a smooth function identically 1 on $\mathcal{M}^{\varepsilon}$ and identically zero outside $\mathcal{M}^{2\varepsilon}$; now consider the subdivision of (4.6):

$$\int_{B_{2N}(0)} e^{ix\cdot\xi} E_j(\xi, t)\hat{f}(\xi)\, d\xi = \int_{B_{2N}(0)} e^{ix\cdot\xi} E_j(\xi, t)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\, d\xi + \int_{B_{2N}(0)} e^{ix\cdot\xi} E_j(\xi, t)(1 - \chi_{\mathcal{M},\varepsilon})(\xi)\hat{f}(\xi)\, d\xi ;$$

   for the second integral, simply repeat the above procedure around any root multiplicities in $B_{2N}(0) \setminus \mathcal{M}^{\varepsilon}$.

2. For the first integral, the case where the integrand is supported on $\mathcal{M}^{\varepsilon}$,

split off the coinciding roots from the others:

$$\begin{aligned}\int_{B_{2N}(0)} & e^{ix\cdot\xi} E_j(\xi,t)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi \\ &= \int_{B_{2N}(0)} e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi \\ &\quad + \int_{B_{2N}(0)} e^{ix\cdot\xi}\Big(\sum_{k=L+1}^{m} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi. \quad (4.7)\end{aligned}$$

3. For the first integral, we use techniques discussed in Section 4.6 below to estimate it.

4. For the second there are two possibilities: firstly, two or more of the roots $\tau_{L+1}(\xi),\dots,\tau_m(\xi)$ coincide in $\mathcal{M}^{2\varepsilon}$—in this case, repeat the procedure above for this integral. Alternatively, these roots are all distinct in $\mathcal{M}^{2\varepsilon}$—in this case, it suffices to study each integral separately as the $A_k^j(\xi,t)$ are independent of $t$, and thus the expression (4.4) is valid and we can write

$$\begin{aligned}\int_{B_{2N}(0)} & e^{ix\cdot\xi}\Big(\sum_{k=L+1}^{m} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi \\ &= \sum_{k=L+1}^{m}\int_{B_{2N}(0)} e^{i[x\cdot\xi+\tau_k(\xi)t]}A_j^k(\xi)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi\ ;\end{aligned}$$

estimates for integrals of the type on the right-hand side are found in Section 4.5—note that in this case we may use that the region is bounded to ensure the continuous functions are also bounded.

Continue this procedure until all multiplicities are accounted for in this way.

## 4.3 Step 3: Interpolation

The following result can be found in [BL76, Theorem 6.4.5]:

**Theorem 4.3.1.** *Suppose $T$ is a linear map such that it maps*

$$T : L^{q_0} \to W^{s_0}_{p_0}\,, \quad T : L^{q_1} \to W^{s_1}_{p_1}\,,$$

*where* $s_0 \neq s_1$*,* $1 \leq p_0, p_1 < \infty$*; then* $T$ *also maps:*

$$T : L^{q_\theta} \to W^{s_\theta}_{p_\theta}\,,$$

*where*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,, \quad s_\theta = (1-\theta)s_0 + \theta s_1\,.$$

*That is,* $\|Tf\|_{L^{q_\theta}} \leq C\|f\|_{W^{s_\theta}_{p_\theta}}$ *and* $C$ *is independent of* $f \in W^{s_\theta}_{p_\theta}$*.*

In particular, this means that if we have estimates

$$\|Tf\|_{L^\infty} \leq Ct^{d_0}\|f\|_{W_1^{N_0}}\,, \quad \|Tf\|_{L^2} \leq Ct^{d_1}\|f\|_{W_2^{N_1}}\,,$$

then

$$\|Tf\|_{L^q} \leq C(1+t)^{d_p}\|f\|_{W_p^{N_p}}$$

where $p^{-1} + q^{-1} = 1$, $N_p = N_0\big(\frac{1}{p} - \frac{1}{q}\big) + \frac{2}{q}N_1$ and $d_p = d_0\big(\frac{1}{p} - \frac{1}{q}\big) + \frac{2}{q}d_1$.

This reduces our task to finding $L^1 - L^\infty$ and $L^2 - L^2$ estimates in each case.

## 4.4 Step 4: Estimates for Large $|\xi|$

Via the division of the integral above, it suffices to find $L^p - L^q$ estimates for integrals of the form

$$\int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau(\xi)t)} a_j(\xi)\hat{f}(\xi)\,d\xi\,,$$

where $a_j(\xi) = O(|\xi|^{-j})$ as $\xi \to \infty$ is smooth and is zero in a neighbourhood of 0, and $\tau(\xi)$ is a complex-valued, inhomogeneous smooth function which is $O(|\xi|)$ as $\xi \to \infty$ and $\operatorname{Im}\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$.

By further judicious use of cut-off functions, it is clear that we can split the considerations into three main cases:

1. $\tau(\xi)$ is separated from the real axis, i.e. there exists $\delta > 0$ such that $\operatorname{Im}\tau(\xi) \geq \delta$ for all $|\xi| > N$;

2. $\tau(\xi)$ lies on the real axis;

3. $\tau(\xi)$ tends asymptotically to the real axis as $|\xi| \to \infty$.

Let us look at each of these in turn.

### 4.4.1 Phase function separated from the real axis

In this section, we consider the case where characteristic root $\tau(\xi)$ is separated from the real axis for large $|\xi|$; let us define $\delta > 0$ to be a constant such that $\operatorname{Im}\tau(\xi) \geq \delta$ for all $|\xi| \geq N$.

We claim that, for all $t > 0$,

$$\left\| D_t^r D_x^\alpha \Big( \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi) \hat{f}(\xi)\, dx \Big) \right\|_{L^\infty} \leq C e^{-\delta t} \|f\|_{W_1^{N_1+|\alpha|+r-j}} ,$$
$$\left\| D_t^r D_x^\alpha \Big( \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi) \hat{f}(\xi)\, dx \Big) \right\|_{L^2} \leq C e^{-\delta t} \|f\|_{W_2^{|\alpha|+r-j}} ,$$

where $N_1 > n$, $r \geq 0$, $\alpha$ multi-index. Indeed, these follow immediately from:

**Proposition 4.4.1.** *Let $\tau : U \to \mathbb{C}$ be a smooth function, $U \subset \mathbb{R}^n$ open, and $a_j(\xi) \in \mathcal{S}^{-j}_{(1,0)}(U)$. Assume:*

(i) *there exists $\delta > 0$ such that $\operatorname{Im}\tau(\xi) \geq \delta$ for all $\xi \in U$;*

(ii) *$|\tau(\xi)| \leq C(1+|\xi|)$ for all $\xi \in U$.*

*Then,*

$$\left\| \int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \hat{f}(\xi)\, d\xi \right\|_{L^\infty(\mathbb{R}^n_x)} \leq C e^{-\delta t} \|f\|_{W_1^{N_0+|\alpha|+r-j}}$$

*and*

$$\left\| \int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \hat{f}(\xi)\, d\xi \right\|_{L^2(\mathbb{R}^n_x)} \leq C e^{-\delta t} \|f\|_{W_2^{|\alpha|+r-j}}$$

*for all $t > 0$, $N_0 > n$, multi-indices $\alpha$, $r \in \mathbb{R}$ and $f \in C_0^\infty(U)$.*

*Proof.* By the hypotheses on $\tau(\xi)$ and $a_j(\xi)$,

$$\begin{aligned}\Big|\int_U e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\xi^\alpha\tau(\xi)^r\hat f(\xi)\,d\xi\Big| &\le \int_U |e^{i\tau(\xi)t}a_j(\xi)||\xi|^{|\alpha|}|\tau(\xi)|^r|\hat f(\xi)|d\xi\\ = \int_U e^{-\operatorname{Im}\tau(\xi)t}|a_j(\xi)||\xi|^{|\alpha|}|\tau(\xi)|^r|\hat f(\xi)|d\xi &\le Ce^{-\delta t}\int_U |\xi|^{|\alpha|+r-j}|\hat f(\xi)|\,d\xi\\ \le Ce^{-\delta t}\int_U|\xi|^{-N_0}d\xi\,\big\||\xi|^{N_0+|\alpha|+r-j}|\hat f(\xi)|\big\|_{L^\infty} &\le Ce^{-\delta t}\|f\|_{W_1^{N_0+|\alpha|+r-j}}\,.\end{aligned}$$

This proves the first inequality. For the second, note the Plancherel Theorem implies

$$\Big\|\int_U e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\xi^\alpha\tau(\xi)^r\hat f(\xi)\,d\xi\Big\|_{L^2(\mathbb{R}^n_x)} = \big\|e^{i\tau(\xi)t}a_j(\xi)\xi^\alpha\tau(\xi)^r\hat f(\xi)\big\|_{L^2(U)};$$

then,

$$\begin{aligned}&\int_U\big|e^{i\tau(\xi)t}a_j(\xi)\xi^\alpha\tau(\xi)^r\hat f(\xi)\big|^2\,d\xi\\ &\qquad\le \int_U e^{-2\operatorname{Im}\tau(\xi)t}|a_j(\xi)|^2|\xi|^{2|\alpha|}|\tau(\xi)|^{2r}|\hat f(\xi)|^2d\xi\\ &\qquad\le Ce^{-2\delta t}\int_U|\xi|^{2(|\alpha|+r-j)}|\hat f(\xi)|^2\,d\xi \le Ce^{-2\delta t}\|f\|^2_{W_2^{|\alpha|+r-j}}\,.\end{aligned}$$

Taking square roots on both sides completes the proof. ☐

So, by the interpolation Theorem 4.3.1,

$$\Big\|D_t^rD_x^\alpha\Big(\int_{\mathbb{R}^n}e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\hat f(\xi)\,dx\Big)\Big\|_{L^q}\le Ce^{-\delta t}\|f\|_{W_p^{N_p+|\alpha|+r-j}}\,,$$

where $p^{-1}+q^{-1}=1$, $1\le p\le 2$, $N_p\ge n\big(\frac1p-\frac1q\big)$, $r\ge 0$, $\alpha$ a multi-index and $f\in C_0^\infty(\mathbb{R}^n)$. Thus, in this case we have exponential decay of the solution.

This proves Part I of the main Theorem 2.1.1 for roots away from the real axis.

### 4.4.2 Phase function lies on the real axis

This case subdivides into the following subcases, each of which yields a different decay rate:

(i) $\det\operatorname{Hess}\tau(\xi)\neq 0$; in this case we use the method of stationary phase;

(ii) $\det \operatorname{Hess} \tau(\xi) = 0$ and $\tau(\xi)$ satisfies the convexity condition of Definition 3.2; in this case we use Theorem 3.2.4;

(iii) the general case when $\det \operatorname{Hess} \tau(\xi) = 0$ (i. e. $\tau(\xi)$ does not satisfy the convexity condition); in this case, we use Theorem 3.3.4.

We assume throughout that $\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ or $\tau(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$. This is valid because for the characteristic roots lying on the real axis, there exists $\tilde{\tau}(\xi)$ such that $\tilde{\tau}_k(\xi) := \tau_k(\xi) - \tilde{\tau}(\xi)$ is either everywhere non-negative or everywhere non-positive, and, if $\tau_k(\xi)$ satisfies the convexity condition, so does $\tilde{\tau}_k(\xi)$. A proof for this in the homogeneous case is given in [Sug94]; the generalisation to the nonhomogeneous case follows using the perturbation results in Chapter 3.

4.4.2.1 $\det \operatorname{Hess} \tau(\xi) \neq 0$

In this section, we consider the case where we have

$$\int_{\mathbb{R}^n} e^{it(\tilde{x}\cdot\xi+\tau(\xi))} a_j(\xi) \hat{f}(\xi)\, d\xi\,,$$

and $\operatorname{Hess} \tau(\xi) \neq 0$ for all $\xi \in \operatorname{supp} a_j$, To estimate this, we first consider the oscillatory integral

$$\int_{\mathbb{R}^n} e^{i(\tilde{x}\cdot\xi+\tau(\xi))t} a(\xi)\, d\xi\,,$$

where $a(\xi) \in S^{-\mu}_{(1,0)}$, some $\mu \in \mathbb{R}$, $\operatorname{Im} \tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$, and, for some $\xi_0 \in \mathbb{R}^n$, $\tilde{x} + \nabla_\xi \tau(\xi_0) = 0$ and $\operatorname{Hess} \tau(\xi_0) \neq 0$; we refer to $\xi_0$ as a (non-degenerate) critical point. Let us assume that $\xi_0$ is the only such critical point—if there are more than one, we use suitable cut-off functions to localise around each separately (we assume the set of critical point has no accumulation points). Indeed, let $\chi \in C_0^\infty(\mathbb{R}^n)$ which is supported in a neighbourhood $U$ of $\xi_0$ so that there are no other critical points in $U$. Then consider separately

$$\int_{\mathbb{R}^n} e^{i(\tilde{x}\cdot\xi+\tau(\xi))t} a(\xi)\chi(\xi)\, d\xi \text{ and } \int_{\mathbb{R}^n} e^{i(\tilde{x}\cdot\xi+\tau(\xi))t} a(\xi)(1-\chi)(\xi)\, d\xi\,.$$

The second integral, which we may assume contains no critical points in its support (otherwise introduce further cut-off functions around those), can be shown to decay exponentially: note that away from the critical points,

$$e^{i(\tilde{x}\cdot\xi+\tau(\xi))t} = \frac{x+\nabla\tau(\xi)}{it|x+\nabla\tau(\xi)|^2}\cdot\nabla_\xi[e^{i(\tilde{x}\cdot\xi+\tau(\xi))t}]\,;$$

so, integrating by parts repeatedly shows that for any $N \in \mathbb{N}$ sufficiently large,

$$\Big|\int_{\mathbb{R}^n} e^{i(\tilde{x}\cdot\xi+\tau(\xi))t}a(\xi)(1-\chi)(\xi)\,d\xi\Big| \le Ct^{-N}\,.$$

Let us return to the case when there is a critical point.

We claim

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} e^{i(\tilde{x}\cdot\xi+\tau(\xi))t}a(\xi)\chi(\xi)\,d\xi\Big| &\le Ct^{-n/2}|\det\operatorname{Hess}(\xi_0)|^{-1/2}|a(\xi_0)\chi(\xi_0)| \\ &\le Ct^{-n/2}|\det\operatorname{Hess}(\xi_0)|^{-1/2}(1+|\xi_0|)^{-\mu}\,. \qquad (4.8)\end{aligned}$$

This is a consequence of the following theorem, found in [Hör83a, Theorem 7.7.12, p. 228]:

**Theorem 4.4.2.** *Suppose* $\Phi(x,y) \in \mathbb{R}^n \times \mathbb{R}^p$ *is a complex-valued smooth function in a neighbourhood of the origin* $(0,0) \in \mathbb{R}^n \times \mathbb{R}^p$ *such that*:
*(i)* $\operatorname{Im}\Phi \ge 0$; *(ii)* $\operatorname{Im}\Phi(0,0) = 0$; *(iii)* $\Phi'_x(0,0) = 0$; *(iv)* $\det\Phi''_{xx}(0,0) \ne 0$. *Also, suppose* $u \in C_0^\infty(K)$ *where* $K$ *is a small neighbourhood of* $(0,0)$. *Then*

$$\Big|\int_{\mathbb{R}^n} e^{i\omega\Phi(x,y)}u(x,y)\,dx - \big((\det(\omega\Phi''_{xx}/2\pi i))^0\big)^{-1/2}e^{i\omega\Phi^0}\sum_{j=0}^{N-1}(L_{\Phi,j}u)^0\omega^{-j}\Big| \le C_N\omega^{-N-n/2}\,,$$

*where the notation* $G^0(y)$ *(where* $G(x,y)$ *is the function) means the function of* $y$ *only which is in the same residue class modulo the ideal generated by* $\partial\Phi/\partial x_j$, $j=1,\ldots,n$.

The proof of this result uses the method of stationary phase; similar results (with slightly differing conditions and conclusions) can be found in [Sog93, (1.1.20), p. 49], [Ste93, Ch. VIII, 2.3, Proposition 6, p. 344], [Dui96, Proposition 1.2.4, p. 14] and [Trè80, p. 432, Ch. VIII, (2.15)–(2.16)], for example.

So, we have (4.8) as a simple consequence of this theorem; now, in order to show that

$$\Big|\int_{\mathbb{R}^n} e^{i(\tilde{x}\cdot\xi+\tau(\xi))t} a(\xi)\chi(\xi)\, d\xi\Big| \le Ct^{-n/2}\,,$$

we must choose $\mu \in \mathbb{R}$ suitably. Assume that $|\det \operatorname{Hess} \tau(\xi)| \ge C(1+|\xi|)^{-M}$ for some $M \in \mathbb{R}$; then taking $\mu = M/2$, we have this estimate. Compare this to the case of Klein–Gordon equation (which is done in [Hör97] pp.146–155) where $\det \operatorname{Hess} \tau(\xi) = (1+|\xi|)^{-n-2}$, so $M = n+2$.

Let us now apply this result to our situation. We have

$$\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\hat{f}(\xi)\, d\xi\,,$$

where $a_j(\xi) = O(|\xi|^{-j})$ as $\xi \to \infty$; we assume $|\det \operatorname{Hess} \tau(\xi)| \ge C(1+|\xi|)^{-M}$. Now, for each $\nu \in \mathbb{R}$, we have

$$\begin{aligned} a_j(\xi) &= (1+|\xi|)^{-2\nu}(1+|\xi|)^{2\nu} a_j(\xi) \\ &= \sum_{|\alpha|\le\nu} (1+|\xi|)^{-2\nu}\xi^\alpha a_j(\xi)\xi^\alpha = \sum_{|\alpha|\le\nu} a_{j,\alpha}(\xi)\xi^\alpha\,, \end{aligned}$$

where $a_{j,\alpha} \in S^{-j-2\nu+|\alpha|}$. Taking $\nu = M/2 - j$, ensure that the worst order of any of these symbols is $-M/2$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\hat{f}(\xi)\, d\xi &= \sum_{|\alpha|\le\nu} \int e^{it((\tilde{x}-\tilde{y})\cdot\xi+\tau(\xi))} a_{j,\alpha}(\xi)\widehat{D^\alpha f}(\xi)\, d\xi \\ &= \sum_{|\alpha|\le\nu} \int e^{it((\tilde{x}-\tilde{y})\cdot\xi+\tau(\xi))} a_{j,\alpha}(\xi)\, d\xi * D^\alpha f(x)\,, \end{aligned}$$

since $\xi^\alpha \hat{f}(\xi) = \widehat{D^\alpha f}(\xi)$. Then

$$\begin{aligned} &\Big\|\sum_{|\alpha|\le\nu} \int e^{it((\tilde{x}-\tilde{y})\cdot\xi+\tau(\xi))} a_{j,\alpha}(\xi)\, d\xi * D^\alpha f(x)\Big\|_{L^\infty} \\ &\quad\le \sum_{|\alpha|\le\nu} \Big\|\int e^{it((\tilde{x}-\tilde{y})\cdot\xi+\tau(\xi))} a_{j,\alpha}(\xi)\, d\xi\Big\|_{L^\infty} \|D^\alpha f\|_{L^1} \le Ct^{-n/2}\|f\|_{W_1^{M/2-j}} \end{aligned}$$

Thus, we have an $L^1 - L^\infty$ estimate in this case. To find an $L^2 - L^2$ estimate is simpler: by the Plancherel Theorem,

$$\Big\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\hat{f}(\xi)\,d\xi\Big\|_{L^2(\mathbb{R}^n_x)} = C\big\|e^{i\tau(\xi)t}a_j(\xi)\hat{f}(\xi)\big\|_{L^2(\mathbb{R}^n_\xi)}$$
$$\leq C\big\||\xi|^{-j}\hat{f}(\xi)\big\|_{L^2} \leq C\|f\|_{W_2^{-j}}\,.$$

Using the interpolation Theorem 4.3.1, this gives us Theorem 2.1.1, Part I, for roots on the real axis with $\det \mathrm{Hess}\,\tau_k(\xi) \neq 0$.

**Behaviour of Critical Points:** Above, we assumed that $\xi_0$ was the only critical point of the phase function; this is not such an unreasonable assumption as the following result shows:

**Lemma 4.4.3.** *If* $\mathrm{Hess}\,\tau(\xi)$ *is positive definite for all* $\xi$, *then the integral*

$$\int_{\mathbb{R}^n} e^{i(\tilde{x}\cdot\xi+\tau(\xi))t}a(\xi)\,d\xi\,,$$

*has only one critical point.*

*Proof.* Suppose $\xi^1, \xi^2 \in \mathbb{R}^n$ are two such critical points. So $\tilde{x} + \nabla_\xi\tau(\xi^1) = \tilde{x} + \nabla_\xi\tau(\xi^2)$, or $\nabla_{\xi_j}\tau(\xi^1) = \nabla_{\xi_j}\tau(\xi^2)$ for each $j = 1, \ldots, n$. Thus, by the fundamental theorem of calculus, for all $j = 1, \ldots, n$,

$$0 = \partial_{\xi_j}\tau(\xi^1) - \partial_{\xi_j}\tau(\xi^2) = \int_0^1 (\xi^2 - \xi^1)\cdot\nabla_\xi(\partial_{\xi_j})\tau(\xi^1 + s(\xi^2 - \xi^1)\,ds\,.$$

But this means that $(\xi^2 - \xi^1)\,\mathrm{Hess}\,\tau(\xi^1 + s(\xi^2 - \xi^1))(\xi^2 - \xi^1) = 0$ for all $s$ since the Hessian is positive definite; and since it is never zero, we have that $\xi^2 - \xi^1 = 0$, which shows that there is at most one critical point. □

An example of such an operator is the Klein–Gordon equation.

**Remark 4.4.1:** Another consequence of $\mathrm{Hess}\,\tau(\xi)$ being positive definite is that the level sets $\{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\}$, $\lambda \in \mathbb{R}$ are all strictly convex; indeed, if we parameterise the set by $\xi(s)$, $s \in [0, \infty)$, where $\xi(0) = \xi_0$ and, by assumption, $\dot{\xi}(s) \neq 0$, then $\nabla\tau(\xi(s))\cdot\dot{\xi}(s) = 0$ (differentiate $\tau(\xi(s)) = \lambda$), and (differentiating again)

$$\dot{\xi}(s)^T\cdot\mathrm{Hess}\,\tau(\xi(s))\cdot\xi(s) + \nabla\tau(\xi(s))\cdot\ddot{\xi}(s) = 0.$$

Then, since $\operatorname{Hess}\tau(\xi)$ is positive definite, the first term in this sum is positive, hence the second is negative—which means that the angle between $\nabla\tau(\xi(s))$, that is, the normal to the level set, and $\ddot{\xi}(s)$ is strictly greater than $\pi/2$, so the level set is strictly convex. In particular, this shows that imposing the condition $\operatorname{Hess}\tau(\xi)$ positive definite is stronger than imposing the convexity condition of Definition 3.2, and making it clear why we get a faster rate of decay in this case (see the next section for that case).

Finally, we remark on the differences in the general case, $\det\operatorname{Hess}\tau(\xi) \neq 0$. In this case, there may be many critical points; let us assume that they are all isolated, so there is a small neighbourhood around each critical point $\xi_0$, $U_{\xi_0}$, such that $\xi_0$ is the only critical point in it and chosen so that $|\tilde{x} + \nabla\tau(\xi)| \leq r$ for some $r > 0$ (outside of this set we use the integration by parts argument above). We estimate the volume of $U_{\xi_0}$: suppose $\xi^1, \xi^2 \in U_{\xi_0}$; then, by the mean value theorem,

$$\partial_{\xi_j}\tau(\xi^1) - \partial_{\xi_j}\tau(\xi^2) = \partial^2_{\xi_j\xi_k}\tau(\xi')(\xi^1_k - \xi^2_k)\,,$$

for some $\xi'$ . Now, by assumption,

$$\left|\partial^2_{\xi_j\xi_k}\tau(\xi')\right| \geq C(1+|\xi'|)^{-M} \text{ and } \left|\partial_{\xi_j}\tau(\xi^1) - \partial_{\xi_j}\tau(\xi^2)\right| \leq 2r\,.$$

Therefore, $|\xi^1_k - \xi^2_k| \leq Cr(1+|\xi'|)^M$, and $|\xi'| \leq \max(|\xi^1|, |\xi^2|)$. Thus, choosing $r$ sufficiently small, we see that the volume of $U_{\xi_0}$ is bounded by $C(1+|\xi^1|)^{Mn}$. So, using this in the above argument, we see that the regularity of $f$ must be made greater—take $[Mn]+1$ more derivatives—to compensate that the amplitude is a symbol of order $-M/2 - Mn$.

**Remark 4.4.2:** If $\operatorname{rank}\operatorname{Hess}\tau(\xi) = n-1$, then a similar argument can be used to prove the corresponding part of Theorem 2.1.1, i.e. that there is decay of order $-\frac{n-1}{2}$. This is a consequence of an extension to Theorem 4.4.2—see

4.4.2.2 $\det\operatorname{Hess}\tau(\xi) = 0$ *and* $\tau(\xi)$ *satisfies the convexity condition*

Assume that $\tau(\xi)$ satisfies the convexity condition of Definition 3.2. Set

$\gamma \equiv \gamma(\tau) := \sup_{\lambda>0} \gamma(\Sigma_\lambda(\tau))$, where, as before,

$$\Sigma_\lambda(\tau) = \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \,.$$

and

$$\gamma(\Sigma_\lambda(\tau)) := \sup_{\sigma \in \Sigma_\lambda(\tau)} \sup_P \gamma(\Sigma_\lambda(\tau); \sigma, P)$$

where the second supremum is over planes $P$ containing the normal to $\Sigma_\lambda(\tau)$ at $\sigma$ and $\gamma(\Sigma_\lambda(\tau); \sigma, P)$ denotes the order of the contact between the line $T_\sigma \cap P$—$T_\sigma$ is the tangent plane at $\sigma$—and the curve $\Sigma_\lambda(\tau) \cap P$.

We have the following results which ensures that this is finite:

**Lemma 4.4.4.** *Suppose $\tau : \mathbb{R}^n \to \mathbb{R}$ is a characteristic root of a linear $m^{th}$ order constant coefficient strictly hyperbolic partial differential operator; then, there exists a homogeneous function of order 1, $\varphi(\xi)$, a characteristic root of the principal symbol, such that*

$$\gamma(\Sigma_\lambda(\tau)) \to \ell \leq \gamma(\Sigma_1(\varphi)) \text{ as } \lambda \to \infty \,.$$

*In particular, $\gamma(\tau) < \infty$.*

*Proof.* This is true because:

(a) by Proposition 3.1.5, Part II, $\Sigma_\lambda(\tau)$ is near to $\Sigma_\lambda(\varphi)$ for large $\lambda$ in a suitable metric;

(b) by the homogeneity of $\varphi$, if $|\lambda - \lambda'|$ is sufficiently small, then $\Sigma_\lambda(\varphi)$ is near to $\Sigma_{\lambda'}(\varphi)$ for large $\lambda$ in the same metric;

(c) Proposition 3.1.5, Part IV, ensures that $T_\sigma(\tau)$ is near to $T_\sigma(\varphi)$ (because derivatives of $\tau$ tend to those of $\varphi$) for large $\lambda$;

(d) so, with $\Sigma_\lambda(\tau)$ and $T_\sigma(\tau)$ near to (in a suitable sense) the corresponding data of $\varphi$ for large $\lambda$, it is clear that the $\gamma(\Sigma_\lambda(\tau); \sigma, P)$ is near to $\gamma(\Sigma_\lambda(\varphi); \sigma, P)$, and hence $\gamma(\Sigma_\lambda(\tau))$ is near to $\gamma(\Sigma_\lambda(\varphi))$;

(e) finally, $\gamma(\Sigma_1(\varphi)) = \gamma(\Sigma_\lambda(\varphi))$ by homogeneity.

□

We shall show

$$\Big\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\hat{f}(\xi)\,d\xi\Big\|_{L_q} \le C(1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{W_p^{N_{p,j}}}\,, \qquad (4.9)$$

for all $t \ge 0$, where $p^{-1}+q^{-1}=1$, $1<p\le 2$, $N_{p,j}\ge n(p^{-1}-q^{-1})-j$ and $f\in C_0^\infty(\mathbb{R}^n)$.

**Besov Space Reduction:** We begin by following Brenner [Bre75] and Sugimoto [Sug94] in using the theory of Besov spaces to reduce this to showing, for all $t\ge 0$,

$$\Big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi))(x,t)\Big\|_{L^q} \le C(1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{W_p^{N_{p,j}}}\,; \quad (4.10)$$

here $\{\Phi_l(\xi)\}_{l=0}^{\infty}$ is a Hardy–Littlewood partition: let $\Phi\in C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\Phi = \left\{\xi\in\mathbb{R}^n : \frac{1}{2}\le|\xi|\le 2\right\}, \quad \Phi(\xi)>0 \text{ for } \frac{1}{2}<|\xi|<2\,,$$
$$\text{and } \sum_{k=-\infty}^{\infty}\Phi(2^{-k}\xi)=1 \text{ for } \xi\ne 0\,,$$

and set

$$\Phi_0(\xi)=1-\sum_{l=1}^{\infty}\Phi(2^{-l}\xi)\,, \quad \Phi_l(\xi):=\Phi(2^{-l}\xi)\,,\ l\in\mathbb{N}\,.$$

Now, recall the definition of a Besov space, as given in, for example, Bergh and Löfström [BL76]:

**Definition 4.1.** *For suitable $p,q,s\in\mathbb{R}$ define the Besov norm by*

$$\|f\|_{B^s_{p,q}} := \|\mathcal{F}^{-1}(\Phi_0(\xi)\hat{f}(\xi))\|_{L^p} + \Big(\sum_{l=1}^{\infty}(2^{sl}\|\mathcal{F}^{-1}(\Phi_l(\xi)\hat{f}(\xi))\|_{L^p})^p\Big)^{1/q}\,;$$

*the* Besov space $B^s_{p,q}$ *is the space of functions in $\mathcal{S}(\mathbb{R}^n)$ for which this norm is finite.*

This result is the main one we shall need:

**Theorem 4.4.5** ([BL76], Theorem 6.4.4)**.** *The following inclusions hold:*

$$B^s_{pp} \subset W^s_p \subset B^s_{p2} \text{ and } B^s_{q2} \subset W^s_q \subset B^s_{qq}$$

*for all* $s \in \mathbb{R}$, $1 < p \le 2$, $2 \le q < \infty$.

Using this theorem, we have

$$\begin{aligned}
\Big\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\hat{f}(\xi)\,d\xi \Big\|_{L^q(\mathbb{R}^n)} &= (2\pi)^n \big\| \mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\hat{f}(\xi))(x,t)\big\|_{L^q} \\
&\le C\big\| \mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\hat{f}(\xi))(x,t)\big\|_{B^0_{q,2}} \\
&= C\Big(\sum_{l=0}^{\infty} \big\| \mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi))(x,t)\big\|^2_{L^q}\Big)^{1/2} \\
&= C\Big(\sum_{l=0}^{\infty} \Big\| \mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\sum_{r=l-1}^{l+1}\Phi_r(\xi)\hat{f}(\xi))(x,t)\Big\|^2_{L^q}\Big)^{1/2};
\end{aligned}$$

in the final line we have used that $\sum_{r=l-1}^{l+1}\Phi_r(\xi) = 1$ on $\operatorname{supp}\Phi_l(\xi)$ by the structure of the partition of unity. Now, assuming that (4.10) holds, this can be further estimated:

$$\begin{aligned}
&\Big(\sum_{l=0}^{\infty} \Big\| \mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\sum_{r=l-1}^{l+1}\Phi_r(\xi)\hat{f}(\xi))(x,t)\Big\|^2_{L^q}\Big)^{1/2} \\
&\qquad\le C(1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\Big(\sum_{l=0}^{\infty}\Big(\sum_{r=l-1}^{l+1}\|\mathcal{F}^{-1}(\Phi_r(\xi)\hat{f}(\xi))\|_{W_p^{N_{p,j}}}\Big)^2\Big)^{1/2} \\
&\qquad\le C(1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\Big(\sum_{l=0}^{\infty}\sum_{r=l-1}^{l+1}\|\mathcal{F}^{-1}(\Phi_r(\xi)\hat{f}(\xi))\|^2_{W_p^{N_{p,j}}}\Big)^{1/2} \\
&\qquad\le C(1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\Big(\sum_{l=0}^{\infty}\|\mathcal{F}^{-1}(\Phi_l(\xi)\hat{f}(\xi))\|^2_{W_p^{N_{p,j}}}\Big)^{1/2}.
\end{aligned}$$

Finally, using Theorem 4.4.5 once again,

$$\Big(\sum_{l=0}^{\infty}\|\mathcal{F}^{-1}(\Phi_l(\xi)\hat{f}(\xi))\|^2_{W_p^{N_{p,j}}}\Big)^{\frac{1}{2}} \leq C\Big(\sum_{l=0}^{\infty}\sum_{|\alpha|\leq N_{p,j}}\|D_x^{\alpha}[\mathcal{F}^{-1}(\Phi_l(\xi)\hat{f}(\xi))]\|^2_{L_p}\Big)^{\frac{1}{2}}$$
$$= C\sum_{|\alpha|\leq N_{p,j}}\Big(\sum_{l=0}^{\infty}\|\mathcal{F}^{-1}(\Phi_l(\xi)\widehat{D^{\alpha}f}(\xi))]\|^2_{L_p}\Big)^{1/2}$$
$$= C\sum_{|\alpha|\leq N_{p,j}}\|D^{\alpha}f\|_{B^0_{p,2}} \leq C\|f\|_{W_p^{N_{p,j}}}\,.$$

Combining these estimates produces (4.9) as desired. So, it suffices to prove (4.10); indeed, as shown above, this requires us to show two estimates and then interpolate—Theorem 4.3.1 yields:

$$\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi))(x,t)\big\|_{L^{\infty}} \leq C(1+t)^{-\frac{n-1}{\gamma}}\|f\|_{W_1^{N_1-j}}\,, \tag{4.11}$$
$$\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi))(x,t)\big\|_{L^2} \leq C\|f\|_{W_2^{-j}}\,, \tag{4.12}$$

where $N_1 > n$.

$L^2 - L^2$ **estimate:** By Plancherel's Theorem,

$$\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi))(x,t)\big\|_{L^2} = C\big\|e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi)\big\|_{L^2}\,.$$

Then, since $\tau(\xi)$ is real-valued and $a_j(\xi) = O(|\xi|^{-j})$ as $|\xi| \to \infty$,

$$\int_{\mathbb{R}^n}|e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi)|^2\,d\xi \leq C\int_{|\xi|\geq N}|\xi|^{-2j}|\hat{f}(\xi)|^2\,d\xi \leq C\|f\|_{W_2^{-j}}\,.$$

Note that $C$ is independent of $l$ because $a_j(\xi)|\xi|^j$ is uniformly bounded in $\mathbb{R}^n$. This proves the required estimate (4.12).

$L^1 - L^\infty$ **estimate:** First, suppose $0 \le t < 1$; then

$$\Big\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\hat f(\xi)\,d\xi\Big\|_{L^\infty} \le C\int_{|\xi|\ge N} |\xi|^{-j}|\hat f(\xi)|\,d\xi$$
$$\le C\int_{|\xi|\ge N}|\xi|^{-N_1}\,d\xi\,\big\||\xi|^{N_1-j}\hat f(\xi)\big\|_{L^\infty}$$
$$\le C\|D^{N_1-j}f\|_{L^1} = C\|f\|_{W_1^{N_1-j}}\,, \quad (4.13)$$

where $N_1 > n$.

For $t \ge 1$, we show

$$\Big\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\hat f(\xi)\,d\xi\Big\|_{L^\infty} \le Ct^{-(n-1)/\gamma}\|f\|_{W_1^{n-\frac{n-1}{\gamma}-j}}\,; \quad (4.14)$$

we claim it suffices to prove that there exists a constant $C > 0$ which is independent of $l$ such that, for all $t \ge 1$,

$$\Big\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)|\xi|^{\frac{n-1}{\gamma}-n+j}\Phi_l(\xi)\,d\xi\Big\|_{L^\infty} \le Ct^{-(n-1)/\gamma}\,. \quad (4.15)$$

Indeed,

$$\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\hat f(\xi)\,d\xi = (2\pi)^n\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat f(\xi))$$
$$= (2\pi)^n\mathcal{F}^{-1}_{\xi\to x}[e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)](x,t) * f(x)$$
$$= \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\Phi_l(\xi)\,d\xi\Big) * f(x)\,,$$

where we have used $\mathcal{F}^{-1}[\hat g\hat h] = g * h$, and

$$\Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\Phi_l(\xi)\,d\xi\Big) * f(x)$$
$$= \Big(\int_{\mathbb{R}^n} |D_x|^{n-\frac{n-1}{\gamma}-j} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\Phi_l(\xi)|\xi|^{\frac{n-1}{\gamma}-n+j}\,d\xi\Big) * f(x)$$
$$= |D_x|^{n-\frac{n-1}{\gamma}-j}\Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\Phi_l(\xi)|\xi|^{\frac{n-1}{\gamma}-n+j}\,d\xi\Big) * f(x)$$
$$= \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\Phi_l(\xi)|\xi|^{\frac{n-1}{\gamma}-n+j}\,d\xi\Big) * |D_x|^{n-\frac{n-1}{\gamma}-j}f(x)\,;$$

also,

$$\|g*h\|_{L^\infty} \le \|g\|_{L^\infty}\|h\|_{L^1}\,,$$

for all $g \in L^\infty(\mathbb{R}^n)$, $h \in L^1(\mathbb{R}^n)$. Combining all these proves the claim.

In order to show (4.15), we can Theorem 3.2.4 as $\tau : \mathbb{R}^n \to \mathbb{R}$ is assumed to satisfy the convexity condition; let us check each hypothesis holds:

- Hypothesis (i): by Proposition 3.1.5, Part III,

$$|\partial_\xi^\alpha \tau(\xi)| \leq C_\alpha |\xi|^{1-|\alpha|} \quad \text{for all } |\xi| \geq N\,,$$

  for all multi-indices $\alpha$; this suffices for the hypothesis to hold since $a_j(\xi)$ is supported away from the origin.

- Hypothesis (ii) and hypothesis (iii): these follow by using perturbation methods. By Proposition 3.1.5, Part IV, there exists a homogeneous function $\varphi(\xi)$ of order 1 such that, for all $|\xi| \geq N$ and $k = 1, \ldots, n$,

$$|\tau(\xi) - \varphi(\xi)| \leq C_0 \text{ and } |\partial_{\xi_k} \tau(\xi) - \partial_{\xi_k} \varphi(\xi)| \leq C_k |\xi|^{-1}\,,$$

  for some constants $C_0, C_k > 0$. Now, the homogeneity of $\varphi(\xi)$ implies that $\varphi(\xi) = |\xi| \varphi(\frac{\xi}{|\xi|})$ and $e_k \cdot \nabla\varphi(e_k) = \varphi(e_k)$, where $e_k = (\underbrace{0, \ldots, 0, 1}_{k}, 0, \ldots, 1)$, so

$$|\varphi(\xi)| \geq C' |\xi| \text{ for all } \xi \in \mathbb{R}^n \text{ and } |\partial_\omega \varphi(\lambda\omega)| \geq C' \text{ for all } \omega \in S^{n-1},\ \lambda > 0\,,$$

  for some constant $C' > 0$. Thus,

$$|\tau(\xi)| \geq |\varphi(\xi)| - |\tau(\xi) - \varphi(\xi)| \geq C'|\xi| - C_0 \geq C|\xi| \text{ for } |\xi| \geq M\,, \quad (4.16)$$

  some constants $M, C > 0$, and

$$|\partial_\omega \tau(\lambda\omega)| \geq |\partial_\omega \varphi(\lambda\omega)| - |\partial_\omega \varphi(\lambda\omega) - \partial_\omega \tau(\lambda\omega)| \geq C' - C_k \lambda^{-1} \geq C > 0$$

  for all $\omega \in S^{n-1}$ and suitably large $\lambda$; for small $\lambda > 0$, $\partial_\omega \tau(\lambda\omega)$ is separated from 0 by the convexity condition, so $|\partial_\omega \tau(\lambda\omega)| \geq C > 0$ for all $\omega \in S^{n-1}$, $\lambda > 0$, as required.

- Hypothesis (iv)—there exists a constant $R_1 > 0$ such that, for all $\lambda > 0$, $\frac{1}{\lambda}\Sigma_\lambda(\tau) \subset B_{R_1}(0)$—holds by Proposition 3.1.5, Part II, and the fact that

$\frac{1}{\lambda}\Sigma_\lambda(\varphi) = \Sigma_1(\varphi)$ for the characteristic root of the principal symbol $\varphi$ corresponding to $\tau$. Also, $\gamma < \infty$ by Lemma 4.4.4 above.

- $a_j(\xi)|\xi|^{\frac{n-1}{\gamma}-n+j}$ is a symbol of order $\frac{n-1}{\gamma} - n$ since $a(\xi) = O(|\xi|^{-j})$ as $|\xi| \to \infty$ and because it is zero in a neighbourhood of the origin.

- the partition of unity $\{\Phi_l(\xi)\}_{l=1}^{\infty}$ is in the form of $g_R(\xi)$ as required by Theorem 3.2.4.

Therefore, for $t \geq 1$,

$$\Big|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)|\xi|^{\frac{n-1}{\gamma}-n+j}\Phi_l(\xi)\, d\xi\Big| \leq Ct^{-(n-1)/\gamma} .$$

Hence, we have (4.14), which, together with (4.13), proves (4.11); this completes the proof of Theorem 2.1.1, Part I, for roots on the real axis with convexity condition $\gamma$.

4.4.2.3 *General case when* $\det \operatorname{Hess} \tau(\xi) = 0$

The general case depends upon Theorem 3.3.4, just as the case where the convexity condition holds depends upon Theorem 3.2.4; for this reason we introduce $ga_0 \equiv \gamma_0(\tau) := \sup_{\lambda>0} \gamma_0(\Sigma_\lambda(\tau))$, where,

$$\gamma_0(\Sigma_\lambda(\tau)) := \sup_{\sigma\in\Sigma_\lambda(\tau)} \inf_P \gamma(\Sigma_\lambda(\tau); \sigma, P)$$

(all notation as before). For this quantity we have the analogous result to Lemma 4.4.4, which can be proved in the same way:

**Lemma 4.4.6.** *If $\tau : \mathbb{R}^n \to \mathbb{R}$ is a characteristic root of a linear $m^{th}$ order constant coefficient strictly hyperbolic partial differential operator, then, there exists a homogeneous function of order* 1, $\varphi(\xi)$, *a characteristic root of the principal symbol, such that*

$$\gamma_0(\Sigma_\lambda(\tau)) \to \ell \leq \gamma_0(\Sigma_1(\varphi)) \text{ as } \lambda \to \infty .$$

*In particular,* $\gamma_0(\tau) < \infty$.

We shall show

$$\Big\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\hat{f}(\xi)\,d\xi \Big\|_{L_q} \le C(1+t)^{-\frac{1}{\gamma_0}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{W_p^{N_{p,j}}} ,$$

for all $t \ge 0$, where $p^{-1}+q^{-1}=1$, $1<p\le 2$, $N_{p,j} \ge n(p^{-1}-q^{-1})-j$ and $f \in C_0^\infty(\mathbb{R}^n)$.

As in the case of above, this can be reduced, via a Besov space reduction the interpolation Theorem 4.3.1, to showing

$$\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi))(x,t)\big\|_{L^\infty} \le Ct^{-\frac{1}{\gamma_0}}\|f\|_{W_1^{N_1-j}} ,$$
$$\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi))(x,t)\big\|_{L^2} \le C\|f\|_{W_2^{-j}} ,$$

where the partition of unity $\{\Phi_l(\xi)\}_{l=1}^\infty$ is as above and $N_1>1$.

The $L^2$ estimate follows by the Plancherel Theorem in the same way as before.

For the $L^1-L^\infty$ estimate, the case $0\le t<1$ is as in (4.13); for $t\ge 1$ it suffices to show (see the earlier argument),

$$\Big\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)|\xi|^{\frac{1}{\gamma_0}-n+j}\Phi_l(\xi)\,d\xi \Big\|_{L^\infty} \le Ct^{-1/\gamma_0} .$$

This follows by Theorem 3.3.4: the hypotheses of this hold by the same arguments as above—the convexity condition is not required for the perturbation methods employed—and Lemma 4.4.6.

This completes the proof of Theorem 2.1.1 for roots on the real axis.

### 4.4.3 Phase function asymptotic to real axis

Here we consider the case where the phase function $\tau(\xi)$ satisfies

$$\operatorname{Im}\tau(\xi) \to 0 \text{ as } |\xi|\to\infty ;$$

we remark that the results here are consistent with those when $\operatorname{Im}\tau(\xi)\equiv 0$.

As in the case of the phase function lying on the real axis, we split into subcases; we consider the following two:

(i) $\det \operatorname{Hess}\tau(\xi) \ne 0$; in this case we use the method of stationary phase;

(ii) $\det \operatorname{Hess} \tau(\xi) = 0$; in this case, we use Theorem 3.3.5.

**Remark 4.4.3:** Unlike in the case of the phase function $\tau(\xi)$ lying on the real axis, we do not consider a case where the phase function satisfies a "convexity condition". The reason for this is twofold: firstly, there is no straightforward analog of the convexity condition for real-valued phase functions as the presence of the non-zero imaginary part causes problems; secondly, there are no common examples of this situation, and hence it does not seem worthwhile developing a complicated theory for a situation which may not arise.

4.4.3.1 $\det \operatorname{Hess} \tau(\xi) \neq 0$

This can be done in exactly the same way as that in Section 4.4.2.1, since Theorem 4.4.2 holds for integrals with complex phase functions.

4.4.3.2 $\det \operatorname{Hess} \tau(\xi) = 0$

The result for this case is similar to that for the general case when the imaginary part is zero; instead, though, we shall use Theorem 3.3.5; this time, set $\gamma_0 = \gamma_0(\operatorname{Re} \tau) = \sup_{\lambda > 0} \gamma_0(\Sigma_\lambda(\operatorname{Re} \tau))$, and note that

**Lemma 4.4.7.** *If $\tau : \mathbb{R}^n \to \mathbb{C}$ is a characteristic root of a linear $m^{th}$ order constant coefficient strictly hyperbolic partial differential operator such that $\operatorname{Im} \tau(\xi) \to 0$ as $|\xi| \to \infty$, then, there exists a homogeneous function of order 1, $\varphi(\xi)$, a characteristic root of the principal symbol, such that*

$$\gamma_0(\Sigma_\lambda(\operatorname{Re} \tau)) \to \ell \leq \gamma_0(\Sigma_1(\varphi)) \text{ as } \lambda \to \infty\,.$$

*In particular, $\gamma_0(\operatorname{Re} \tau) < \infty$.*

*Proof.* The hypothesis that the imaginary part goes to zero as $|\xi| \to \infty$ implies that $|\tau(\xi) - \operatorname{Re} \tau(\xi)| \to 0$ as $|\xi| \to \infty$. With this additional observation, the proof of Lemma 4.4.4 can then be used once more. □

We claim that, as in the general case when $\tau$ is real-valued,

$$\Big\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau(\xi)t)} a_j(\xi) \hat{f}(\xi)\, d\xi \Big\|_{L^q} \leq C(1+t)^{-\frac{1}{\gamma_0}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{W_p^{N_{p,j}}}\,,$$

for all $t \geq 0$, where $p^{-1} + q^{-1} = 1$, $1 < p \leq 2$, $N_{p,j} \geq n(p^{-1} - q^{-1}) - j$ and $f \in C_0^\infty(\mathbb{R}^n)$.

Again, a Besov space reduction and Theorem 4.3.1 mean that it suffices to show

$$\left\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi))(x,t)\right\|_{L^\infty} \leq Ct^{-\frac{1}{\gamma_0}}\|f\|_{W_1^{N_1-j}},$$
$$\left\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\hat{f}(\xi))(x,t)\right\|_{L^2} \leq C\|f\|_{W_2^{-j}}.$$

As $\operatorname{Im}\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$, the $L^2$ estimate, and the $L^1 - L^\infty$ estimate in the case $0 \leq t < 1$, hold just as in the case for a real-valued phase function.

To prove the $L^1 - L^\infty$ estimate for $t \geq 1$, for which it suffices to show

$$\left\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)|\xi|^{\frac{1}{\gamma_0}-n+j}\Phi_l(\xi)\,d\xi\right\|_{L^\infty} \leq Ct^{-1/\gamma_0}, \tag{4.17}$$

we use Theorem 3.3.5; we must check that the phase function satisfies the hypotheses of this theorem:

- Hypothesis (i) follows by Proposition 3.1.5: Part III implies that for all $|\xi| \geq N$ and multi-indices $\alpha$,

  $$|\partial_\xi^\alpha \operatorname{Re}\tau(\xi)| \leq |\partial_\xi^\alpha \tau(\xi)| \leq C|\xi|^{1-|\alpha|},$$

  which suffices for the first part of the hypothesis to hold. Furthermore, Part IV tells us that for all $|\xi| \geq N$ and multi-indices $\alpha$,

  $$|\partial_\xi^\alpha[\operatorname{Re}\tau(\xi) - \varphi(\xi)] + i\partial_\xi^\alpha \operatorname{Im}\tau(\xi)| = |\partial_\xi^\alpha \tau(\xi) - \partial_\xi^\alpha \varphi(\xi)| \leq C|\xi|^{-|\alpha|},$$

  where $\varphi(\xi)$ is a characteristic root of the principal part (and is thus real-valued by definition of hyperbolicity); this implies that, for all $|\xi| \geq N$ and multi-indices $\alpha$,

  $$|\partial_\xi^\alpha[\operatorname{Re}\tau(\xi) - \varphi(\xi)]| \leq C|\xi|^{-|\alpha|} \text{ and } |\partial_\xi^\alpha \operatorname{Im}\tau(\xi)| \leq C|\xi|^{-|\alpha|}. \tag{4.18}$$

  The second of these gives us the second part of the hypothesis.

- For hypothesis (ii), note that there exist constants $C, C', C'', M > 0$ such that, for all $|\xi| \geq M$,

$$|\mathrm{Re}\,\tau(\xi)| \geq |\tau(\xi)| - |\mathrm{Im}\,\tau(\xi)| \geq C'|\xi| - C'' \geq C|\xi|\,.$$

Here we have used (4.16), which did not require $\tau$ to be real-valued (nor to satisfy the convexity condition), simply to be a characteristic root of a linear constant coefficient strictly hyperbolic partial differential equation, and the second part of (4.18).

- Hypothesis (iii) is shown to hold in a similar way: using the corresponding hypothesis for the real-valued case above (which does not use that $\tau$ is real-valued) and (4.18), we have, for $\lambda \geq M$, some $M > 0$,

  $$|\partial_\omega \,\mathrm{Re}\,\tau(\lambda\omega)| \geq |\partial_\omega \tau(\lambda\omega)| - |\partial_\omega \,\mathrm{Im}\,\tau(\lambda\omega)| \geq C' - C''\lambda^{-1} \geq C\,.$$

  For small $\lambda > 0$, we simply extend $\mathrm{Re}\,\tau(\xi)$ so that this holds—this is possible as the integrand is supported for large $|\xi|$ only.

- Hypothesis (iv) follows from $|\mathrm{Re}\,\tau(\xi) - \varphi(\xi)| \leq C$ for all $\xi \in \mathbb{R}^n$ (see earlier—this holds in all $\mathbb{R}^n$ by Part II of Proposition 3.1.5. We have $\gamma_0 < \infty$ by Lemma 4.4.7. Finally, $a_j(\xi)$ and $\Phi_l(\xi)$ satisfy the required conditions in the same way as for the real-valued case.

So, all the hypotheses of Theorem 3.3.5 hold, and hence we have (4.17). This completes the proof of Theorem 2.1.1, Part I.

## 4.5 Step 5: Estimates for Bounded $|\xi|$ Away from Multiplicities

In this section we find $L^p - L^q$ estimates for integrals of the kind

$$\int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi)\,d\xi\,,$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded, $f \in C_0^\infty(\mathbb{R}^n)$, $a \in C_0^\infty(\Omega)$, $\tau \in C^\omega(\Omega)$ (by Proposition 3.1.3) and $\mathrm{Im}\,\tau(\xi) \geq 0$ for all $\xi \in \Omega$.

As in the case of large $|\xi|$, we can further split this into three main cases by using suitable cut-off functions:

1. $\tau(\xi)$ is separated from the real axis for all $\xi \in \Omega$;

2. $\tau(\xi)$ meets the real axis with order $s < \infty$ at a point $\xi_0 \in \Omega$;

3. $\tau(\xi)$ lies on the real axis for all $\xi \in \Omega$.

We look at each in turn.

### 4.5.1 Phase function separated from the real axis

Similarly to the case for large $|\xi|$, we show that when the phase function $\tau(\xi)$ is separated from the real axis (here, for $\xi \in \Omega$),

$$\left\| D_t^r D_x^\alpha \Big( \int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \hat{f}(\xi)\, dx \Big) \right\|_{L^q} \leq C e^{-\delta t} \|f\|_{L^p}\,, \tag{4.19}$$

where $p^{-1} + q^{-1} = 1$, $1 \leq p \leq 2$, $N_p \geq n\big(\frac{1}{p} - \frac{1}{q}\big)$, $r \geq 0$, $\alpha$ a multi-index, $f \in C_0^\infty(\mathbb{R}^n)$, $\delta > 0$ is a constant such that $\operatorname{Im}\tau(\xi) \geq \delta$ for all $\xi \in \Omega$ and $C \equiv C_{\Omega,r,\alpha,p} > 0$. So, in this case we have also have exponential decay of the solution.

By interpolating (Theorem 4.3.1), it suffices to show for such $\tau(\xi)$

$$\left\| D_t^r D_x^\alpha \Big( \int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \hat{f}(\xi)\, dx \Big) \right\|_{L^\infty} \leq C e^{-\delta t} \|f\|_{L^1}\,,$$
$$\left\| D_t^r D_x^\alpha \Big( \int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \hat{f}(\xi)\, dx \Big) \right\|_{L^2} \leq C e^{-\delta t} \|f\|_{L^2}\,,$$

for $t > 0$, where $N_1 > n$, $r \geq 0$ and $\alpha$ is a multi-index.

These are proved in a similar way to Proposition 4.4.1, but noting that the boundedness of $\Omega$ and the continuity in $\Omega$ of $\tau(\xi)^r a(\xi)$ ensure there exists a constant $C_{\Omega,r,\alpha} \equiv C > 0$ such that $|\tau(\xi)|^r |a(\xi)||\xi|^{|\alpha|} \leq C$ for all $\xi \in \Omega$. Then, for all $t > 0$ and $r, \alpha$ as above,

$$\begin{aligned}
\left| D_t^r D_x^\alpha \Big( \int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \hat{f}(\xi)\, dx \Big) \right| &= \left| \int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \xi^\alpha \tau(\xi)^r \hat{f}(\xi)\, dx \right| \\
&\leq C \int_\Omega e^{-\operatorname{Im}\tau(\xi)t} |a(\xi)||\xi|^{|\alpha|} |\tau(\xi)|^r |\hat{f}(\xi)|\, dx \\
&\leq C \int_\Omega e^{-\operatorname{Im}\tau(\xi)t} |\hat{f}(\xi)|\, dx \leq C e^{-\delta t} \|\hat{f}\|_{L^\infty(\Omega)} \leq C e^{-\delta t} \|f\|_{L^1}\,,
\end{aligned}$$

and

$$\Big\| D_t^r D_x^\alpha \Big( \int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi)\,dx \Big) \Big\|_{L^2(\mathbb{R}^n_x)} = \big\| e^{i\tau(\xi)t} a(\xi)\xi^\alpha \tau(\xi)^r \hat{f}(\xi) \big\|_{L^2(\Omega)}$$
$$= \Big( \int_\Omega e^{-2\operatorname{Im}\tau(\xi)t} |a(\xi)|^2 |\xi^\alpha|^2 |\tau(\xi)|^{2r} |\hat{f}(\xi)|^2\,dx \Big)^{1/2}$$
$$\le Ce^{-\delta t}\|\hat{f}\|_{L^2(\Omega)} \le Ce^{-\delta t}\|f\|_{L^2}\,.$$

We have proved Theorem 2.1.1, Part II for roots away from the axis with no multiplicities.

### 4.5.2 Phase function meeting the real axis with finite order

In the case of bounded $|\xi|$, we must also consider the situation where the phase function $\tau(\xi)$ meets the real axis. Suppose $\xi_0 \in \Omega$ is such a point, i.e. $\operatorname{Im}\tau(\xi_0) = 0$, while in each punctured ball around $\xi_0$, $B'_\varepsilon(\xi_0) \subset \Omega$, $\varepsilon > 0$, there exists $\xi \in B'_\varepsilon(\xi_0)$ so that $\operatorname{Im}\tau(\xi) > 0$. Then, we claim that $\xi_0$ is a root of $\operatorname{Im}\tau(\xi)$ of finite order $s$: indeed, if $\xi_0$ were a zero of $\operatorname{Im}\tau(\xi)$ of infinite order, then, by the analyticity of $\operatorname{Im}\tau(\xi)$ at $\xi_0$ (which follows straight from the analyticity of $\tau(\xi)$ at $\xi_0$) it would be identically zero in a neighbourhood of $\xi_0$, contradicting the assumption.

Furthermore, we claim $s \ge 2$, $s$ is even, and that there exist constants $c_0, c_1 > 0$ such that, for all $\xi$ sufficiently close to $\xi_0$,

$$c_0|\xi-\xi_0|^s \le |\operatorname{Im}\tau(\xi)| \le c_1|\xi-\xi_0|^s\,.$$

The Taylor expansion of $\operatorname{Im}\tau(\xi)$ around $\xi_0$,

$$\operatorname{Im}\tau(\xi) = \sum_{i=1}^n \partial_{\xi_i}\operatorname{Im}\tau(\xi_0)(\xi_i-(\xi_0)_i) + O(|\xi-\xi_0|^2)\,,$$

is valid for $\xi \in B_\varepsilon(\xi_0) \subset \Omega$ for some small $\varepsilon > 0$. Now, if $\xi \in B_\varepsilon(\xi_0)$, then $-\xi + 2\xi_0 \in B_\varepsilon(\xi_0)$ also. However,

$$\operatorname{Im}\tau(-\xi+2\xi_0) = -\sum_{i=1}^n \partial_{\xi_i}\operatorname{Im}\tau(\xi_0)(\xi_i-(\xi_0)_i) + O(|\xi-\xi_0|^2)\,;$$

thus, for $\varepsilon > 0$ chosen small enough, this means that either $\operatorname{Im}\tau(\xi) \le 0$

or $\operatorname{Im}\tau(-\xi+2\xi_0) \le 0$—contradicting the hypothesis that $\operatorname{Im}\tau(\xi) \ge 0$ for all $\xi \in \Omega$; hence, $\partial_{\xi_i} \operatorname{Im}\tau(\xi_0) = 0$ for each $i = 1, \ldots, n$. In conclusion, $\operatorname{Im}\tau(\xi) = O(|\xi - \xi_0|^2)$ for all $\xi \in B_\varepsilon(\xi_0)$, which means that the zero is of order $s \ge 2$, and a similar argument shows that $s$ must be even; also, this means that there exist $c_0, c_1 > 0$ so that the above inequality holds for $\xi \in B_\varepsilon(\xi_0)$, proving the claim.

Now, we need the following result, which is based in the calculation of the $L^p - L^q$ decay estimate for the dissipative wave equation in [Mat76], but is here extended to a more general situation so that it can be used on a wider class of equations:

**Proposition 4.5.1.** *Let $\phi : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, be a continuous function and suppose $\xi_0 \in U$ such that $\phi(\xi_0) = 0$ and that $\phi(\xi) > 0$ in a punctured open neighbourhood of $\xi_0$, denoted by $V \setminus \{\xi_0\}$. Furthermore, assume that, for some $s > 0$, there exists a constant $c_0 > 0$ such that, for all $\xi \in V$,*

$$\phi(\xi) \ge c_0 |\xi - \xi_0|^s \, .$$

*Then, for any function $a(\xi)$ that is bounded and compactly supported in $U$, and for all $t \ge 0$, $f \in C_0^\infty(\mathbb{R}^n)$, and $r \in \mathbb{R}$,*

$$\int_V e^{-\phi(\xi)t} |\xi - \xi_0|^r |a(\xi)| |\hat{f}(\xi)| \, d\xi \le C(1+t)^{-(n+r)/s} \|f\|_{L^1} \, , \tag{4.20}$$

*and*

$$\left\| e^{-\phi(\xi)t} |\xi - \xi_0|^r a(\xi) \hat{f}(\xi) \right\|_{L^2(V)} \le C(1+t)^{-r/s} \|f\|_{L^2} \, . \tag{4.21}$$

*Proof.* First, we give a straightforward result that is useful in proving each of the estimates:

**Lemma 4.5.2.** *For each $\rho, M \ge 0$ and $\varsigma, c > 0$ there exists $C \equiv C_{\rho,\varsigma,M,c} \ge 0$ such that, for all $t \ge 0$,*

$$\int_0^M x^\rho e^{-cx^\varsigma t} \, dx \le C(1+t)^{-(\rho+1)/\varsigma} \text{ and } \sup_{0 \le x \le M} x^\rho e^{-cx^\varsigma t} \le C(1+t)^{-\rho/\varsigma} \, .$$

*Proof.* For $0 \le t \le 1$, each is clearly bounded: the first by $\frac{M^{\rho+1}}{\rho+1}$ and the second by $M^\rho$. For $t > 1$, set $y = xt^{1/\varsigma}$; with this substitution, the first becomes

$$\int_0^{Mt^{1/\varsigma}} y^\rho t^{-\rho/\varsigma} e^{-cy^\varsigma} t^{-1/\varsigma}\, dy \le t^{-(\rho+1)/\varsigma} \int_0^\infty y^\rho e^{-cy^\varsigma}\, dy\,,$$

while the second becomes

$$\sup_{0\le y\le Mt^{1/\varsigma}} y^\rho t^{-\rho/\varsigma} e^{-cy^\varsigma} \le t^{-\rho/\varsigma} \sup_{y\ge 0} y^\rho e^{-cy^\varsigma}\,;$$

that the right-hand side of each is then bounded follows from standard results. □

Returning to the proof of (4.20), as $a(\xi)$ is bounded in $U$ by assumption, we have

$$\int_V e^{-\phi(\xi)t}|\xi-\xi_0|^r|a(\xi)||\hat f(\xi)|\,d\xi \le C\int_{V'} e^{-\phi(\xi)t}|\xi-\xi_0|^r|\hat f(\xi)|\,d\xi\,,$$

where $V' = V\cap \operatorname{supp} a$; this, in turn, can be estimated in the following manner using the hypothesis on $\phi(\xi)$ and Hölder's inequality:

$$\begin{aligned}\int_{V'} e^{-\phi(\xi)t}|\xi-\xi_0|^r|\hat f(\xi)|\,d\xi &\le C\int_{V'} e^{-c_0|\xi-\xi_0|^s t}|\xi-\xi_0|^r|\hat f(\xi)|\,d\xi\\ &\le C\int_{V'} e^{-c_0|\xi-\xi_0|^s t}|\xi-\xi_0|^r\,d\xi\|\hat f\|_{L^\infty(V')}\,.\end{aligned}$$

Then, transforming to polar coordinates and using the Hausdorff–Young inequality, we find that, for some $\varepsilon > 0$ (chosen so that $V'\subset B_\varepsilon(\xi_0)$, possible since $a(\xi)$ is compactly supported),

$$\begin{aligned}&\int_{V'} e^{-c_0|\xi-\xi_0|^s t}|\xi-\xi_0|^r\,d\xi\|\hat f\|_{L^\infty(V')}\\ &\qquad\le C\int_{S^{n-1}}\int_0^\varepsilon |\eta|^{r+n-1}e^{-c_0|\eta|^s t}\,d|\eta| d\omega\|f\|_{L^1(\mathbb{R}^n)}\,,\end{aligned}$$

Finally, by the first part of Lemma 4.5.2, we find

$$\begin{aligned}\int_V e^{-\phi(\xi)t}|\xi-\xi_0|^r|a(\xi)||\hat f(\xi)|\,d\xi &\le C\int_0^\varepsilon y^{r+n-1}e^{-c_0y^s t}\,dy\|f\|_{L^1(\mathbb{R}^n)}\\ &\le C(1+t)^{-(n+r)/s}\|f\|_{L^1}\,.\end{aligned}$$

This completes the proof of the first part.

Now let us look at the second part. By the second part of Lemma 4.5.2,

$$\begin{aligned}\left\|e^{-\phi(\xi)t}|\xi-\xi_0|^r a(\xi)\hat{f}(\xi)\right\|^2_{L^2(V)} &\le \int_{V'} e^{-2c_0|\xi-\xi_0|^s t}|\xi-\xi_0|^{2r}|\hat{f}(\xi)|^2\,d\xi \\ &\le C(1+t)^{-2r/s}\int_{V'} e^{-c_0|\xi-\xi_0|^s t}|\hat{f}(\xi)|^2\,d\xi\,.\end{aligned}$$

The Hölder inequality implies that

$$\int_{V'} e^{-c_0|\xi-\xi_0|^s t}|\hat{f}(\xi)|^2\,d\xi \le \sup_{V'}\left|e^{-c_0|\xi-\xi_0|^s t}\right|\|\hat{f}\|^2_{L^2(V')} \le C\|f\|^2_{L^2}\,.$$

Together these give the required estimate (4.21). □

So, using this proposition, we have, for all $t>0$, and sufficiently small $\varepsilon>0$,

$$\begin{aligned}&\left\|D_t^r D_x^\alpha \int_{B_\varepsilon(\xi_0)} e^{i(x\cdot\xi+\tau(\xi)t)}a(\xi)\hat{f}(\xi)\,d\xi\right\|_{L^\infty(\mathbb{R}^n_x)} \\ &\qquad\le \int_{B_\varepsilon(\xi_0)} e^{-\operatorname{Im}\tau(\xi)t}|a(\xi)||\tau(\xi)|^r|\xi|^\alpha|\hat{f}(\xi)|\,d\xi \le C(1+t)^{-n/s}\|f\|_{L^1}\,,\end{aligned}$$

and, using the Plancherel Theorem,

$$\begin{aligned}&\left\|D_t^r D_x^\alpha \int_{B_\varepsilon(\xi_0)} e^{i(x\cdot\xi+\tau(\xi)t)}a(\xi)\hat{f}(\xi)\,d\xi\right\|_{L^2(\mathbb{R}^n_x)} \\ &\qquad= C\left\|e^{i\tau(\xi)t}\tau(\xi)^r\xi^\alpha a(\xi)\hat{f}(\xi)\right\|_{L^2(B_\varepsilon(\xi_0))} \le C\|f\|_{L^2}\,;\end{aligned}$$

here we have used that $|\xi|^{|\alpha|}|\tau(\xi)|^r \le C$ for $\xi\in V'$ for $r\in\mathbb{N}$, $\alpha$ a multi-index.

Thus, by Theorem 4.3.1, for all $t>0$,

$$\left\|D_t^r D_x^\alpha \int_{B_\varepsilon(\xi_0)} e^{i(x\cdot\xi+\tau(\xi)t)}a(\xi)\hat{f}(\xi)\,d\xi\right\|_{L^q(\mathbb{R}^n_x)} \le C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p}\,, \tag{4.22}$$

where $1\le p\le 2$, $p^{-1}+q^{-1}=1$. This completes the proof of Theorem 2.1.1, Part II for roots meeting the axis with finite order and no multiplicities.

**Remark 4.5.1:** If $\xi_0 = 0$, then Proposition 4.5.1 further tells us that

$$\left\| D_t^r D_x^\alpha \int_{B_\varepsilon(0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi)\, d\xi \right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n+|\alpha|}{s}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p}\,,$$

while, if $\operatorname{Re}\tau(\xi_0) = 0$, and so $|\tau(\xi)| \leq |\operatorname{Im}\tau(\xi)| \leq c_1|\xi-\xi_0|^s$ for $\xi$ near $\xi_0$, then we get

$$\left\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi_0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi)\, d\xi \right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\left(\frac{n}{s}+r\right)\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p}\,.$$

### 4.5.3 Phase function lies on the real axis

As in the case of large $|\xi|$, we can subdivide into several subcases:

(i) $\det \operatorname{Hess}\tau(\xi) \neq 0$;

(ii) $\det \operatorname{Hess}\tau(\xi) = 0$ and $\tau(\xi)$ satisfies the convexity condition;

(iii) the general case when $\det \operatorname{Hess}\tau(\xi) = 0$.

For the first case, the approach used in Section 4.4.2.1 can be used here also, since there we do not use that $|\xi|$ is large other than to ensure that $\tau(\xi)$ was smooth; here, we are away from multiplicities, so that still holds. Therefore, the conclusion is the same, giving Theorem 2.1.1, Part II for roots on the axis, with no multiplicities satisfying $\det \operatorname{Hess}\tau_k(\xi) \neq 0$

The other two cases are considered in the next section alongside the case where there are multiplicities since it is important precisely how the integral is split up for such cases.

## 4.6 Step 6: Estimates for Bounded $|\xi|$ Around Multiplicities

Finally, let us turn to finding estimates for the first term of (4.7), which we may write in the form

$$\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi,t)\Big)\chi(\xi)\hat{f}(\xi)\, d\xi\,,$$

where the characteristic roots $\tau_1(\xi), \dots, \tau_L(\xi)$ coincide on a set $\mathcal{M} \subset \Omega$ of codimension $\ell$ (in the sense of Corollary 4.2.2), $\Omega \subset \mathbb{R}^n$ is a bounded open set and $\chi \in C_0^\infty(\Omega)$.

As in Section 4.6, we must consider the cases where the image of the phase function(s) either lie on the real axis, are separated from the real axis or meet the real axis. One additional thing to note in this case is that the order of contact at points of multiplicity may be infinite as the roots are not necessarily analytic at such points; we have no examples of such a situation occurring, so it is not worth studying too deeply unless such an example can be found—for now, we can use the same technique as if the point(s) were points where the roots lie entirely on the real axis, and the results in these two situations are given together in Theorem 2.1.1.

Unlike in the case away from multiplicities of characteristic roots, we have no explicit representation for the coefficients $A_j^k(\xi, t)$, which in turn means we cannot split this into $L$ separate integrals. To overcome this, we first show, in Section 4.6.1, that a useful representation for the above integral exists that allows us to use techniques from earlier. Using this alternative representation, it is a simple matter to find estimates in the case where the image of the set $\mathcal{M}$ is separated from the real axis and when it arises on the real axis as a result of all the roots meeting the axis with finite order, and these are done in Sections 4.6.2 and 4.6.3 respectively.

The situations where the roots meet on the real axis and at least one has a zero of infinite order there (either because it fully lies on the axis, or because it meets the axis with infinite order) is slightly more complicated; this is discussed in Section 4.6.4.

### 4.6.1 Resolution of multiple roots

In this section, we find estimates for

$$\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\,,$$

where $\tau_1(\xi), \dots, \tau_L(\xi)$ coincide on a set $\mathcal{M}$ of codimension $\ell$. For simplicity, first consider the simplest case, $L = 2$ and $\mathcal{M} = \{\xi_0\}$; the general case works in a similar way, and we shall show how it differs below. So, assume

$$\tau_1(\xi_0) = \tau_2(\xi_0) \text{ and } \tau_k(\xi_0) \neq \tau_1(\xi_0) \text{ for } k = 3, \ldots, m\,;$$

by continuity, there exists a ball of radius $\varepsilon > 0$ about $\xi_0$, $B_\varepsilon(\xi_0)$, in which the only root which coincides with $\tau_1(\xi)$ is $\tau_2(\xi)$. Then:

**Lemma 4.6.1.** *For all $t \geq 0$ and $\xi \in B_\varepsilon(\xi_0)$,*

$$\Big|\sum_{k=1}^{2} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\Big| \leq C(1+t)e^{-\min(\operatorname{Im}\tau_1(\xi), \operatorname{Im}\tau_2(\xi))t}\,. \tag{4.23}$$

*Proof.* First, note that in the set

$$S := \{\xi \in \mathbb{R}^n : \tau_1(\xi) \neq \tau_k(\xi)\ \forall k = 2, \ldots, m \text{ and}\, \tau_2(\xi) \neq \tau_l(\xi)\, \forall l = 3, \ldots, m\}$$

the formula (4.4) is valid for $A_j^1(\xi)$ and $A_j^2(\xi)$. Now, recall that $E_j(\xi, t) = \sum_{k=1}^m e^{i\tau_k(\xi)t} A_j^k(\xi, t)$ is the solution to the Cauchy problem (4.2a), (4.2c), and thus is continuous; therefore, for all $\eta \in \mathbb{R}^n$ such that $\tau_1(\eta) \neq \tau_k(\eta)$ and $\tau_2(\eta) \neq \tau_k(\eta)$ for $k = 3, \ldots, m$ (but allow $\tau_1(\eta) = \tau_2(\eta)$),

$$\sum_{k=1}^{2} e^{i\tau_k(\eta)t} A_j^k(t, \eta) = \lim_{\xi \to \eta} \left(e^{i\tau_1(\xi)t} A_j^1(\xi) + e^{i\tau_2(\xi)t} A_j^2(\xi)\right),$$

provided $\xi$ varies in the set $S$ (thus, ensuring $e^{i\tau_1(\xi)t} A_j^1(\xi) + e^{i\tau_2(\xi)t} A_j^2(\xi)$ is well-defined). Hence, to obtain (4.23) for all $\xi \in B_\varepsilon(\xi_0)$, it suffices to show

$$\left|e^{i\tau_1(\xi)t} A_j^1(\xi) + e^{i\tau_2(\xi)t} A_j^2(\xi)\right| \leq Cte^{-\min(\operatorname{Im}\tau_1(\xi), \operatorname{Im}\tau_2(\xi))t}$$

for all $t \geq 0$, $\xi \in B'_\varepsilon(\xi_0) = B_\varepsilon(\xi_0) \setminus \{\xi_0\}$.

Now, note the following trivial equality:

$$
\begin{aligned}
K_1e^{iy_1} + K_2e^{iy_2} &= K_1e^{iy_2}e^{i(y_1-y_2)} + K_2e^{iy_1}e^{-i(y_1-y_2)} \\
&= \frac{e^{i(y_1-y_2)} - e^{-i(y_1-y_2)}}{2}K_1e^{iy_2} + \frac{e^{i(y_1-y_2)} + e^{-i(y_1-y_2)}}{2}K_1e^{iy_2} \\
&\quad + \frac{e^{-i(y_1-y_2)} - e^{i(y_1-y_2)}}{2}K_2e^{iy_1} + \frac{e^{-i(y_1-y_2)} + e^{i(y_1-y_2)}}{2}K_2e^{iy_1} \\
&= \sinh(y_1-y_2)[K_1e^{iy_2} - K_2e^{iy_1}] + \cosh(y_1-y_2)[K_1e^{iy_2} + K_2e^{iy_1}]\,.
\end{aligned}
$$

Using this, we have, for all $\xi \in B'_\varepsilon(\xi_0)$, $t \geq 0$,

$$
\begin{aligned}
&e^{i\tau_1(\xi)t}A_j^1(\xi) + e^{i\tau_2(\xi)t}A_j^2(\xi) \\
&\qquad = \sinh[(\tau_1(\xi)-\tau_2(\xi))t](e^{i\tau_2(\xi)t}A_j^1(\xi) - e^{i\tau_1(\xi)t}A_j^2(\xi)) \\
&\qquad\quad + \cosh[(\tau_1(\xi)-\tau_2(\xi))t](e^{i\tau_2(\xi)t}A_j^1(\xi) + e^{i\tau_1(\xi)t}A_j^2(\xi))\,. \qquad (4.24)
\end{aligned}
$$

We estimate each of these terms:

(a) <u>"sinh" term</u>: The first term is simple to estimate: since

$$
\frac{\sinh[(\tau_1(\xi)-\tau_2(\xi))t]}{(\tau_1(\xi)-\tau_2(\xi))} \to t \quad \text{as} \quad (\tau_1(\xi)-\tau_2(\xi)) \to 0\,,
$$

or, equivalently, as $\xi \to \xi_0$ through $S$, and $A_j^k(\xi)(\tau_1(\xi)-\tau_2(\xi))$ is continuous in $B_\varepsilon(\xi_0)$ for $k = 1, 2$, it follows that, for all $\xi \in B'_\varepsilon(\xi_0)$, $t \geq 0$,

$$
\begin{aligned}
&\left|\sinh[(\tau_1(\xi)-\tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t} - A_j^2(\xi)e^{i\tau_1(\xi)t})\right| \\
&\qquad \leq Ct[|e^{i\tau_2(\xi)t}| + |e^{i\tau_1(\xi)t}|] \leq Cte^{-\min(\operatorname{Im}\tau_1(\xi),\operatorname{Im}\tau_2(\xi))t}\,. \qquad (4.25)
\end{aligned}
$$

(b) <u>"cosh" term</u>: Estimating the second term is slightly more complicated. First, recall the explicit representation (4.4) for the $A_j^k(\xi)$ at points away from multiplicities of $\tau_k(\xi)$:

$$
A_j^k(\xi) = \frac{(-1)^j \displaystyle\sum_{1\leq s_1<\cdots<s_{m-j-1}\leq m}^{k} \prod_{q=1}^{m-j-1} \tau_{s_q}(\xi)}{\displaystyle\prod_{l=1,l\neq k}^{m} (\tau_l(\xi)-\tau_k(\xi))}\,.
$$

So, we can write

$$\cosh[(\tau_1(\xi) - \tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t} + A_j^2(\xi)e^{i\tau_1(\xi)t})$$
$$= \frac{\cosh[(\tau_1(\xi) - \tau_2(\xi))t]}{\prod_{k=3}^m(\tau_k(\xi) - \tau_1(\xi))(\tau_k(\xi) - \tau_2(\xi))} \frac{e^{i\tau_2(\xi)t}F_{j+1}^{1,2}(\xi) - e^{i\tau_1(\xi)t}F_{j+1}^{2,1}(\xi)}{\tau_1(\xi) - \tau_2(\xi)},$$

where

$$F_i^{\rho,\sigma}(\xi) := \left( \sum_{1\le s_1<\cdots<s_{m-i}\le m}^{\rho} \prod_{q=1}^{m-i} \tau_{s_q}(\xi) \right) \prod_{k=1,k\ne\rho,\sigma}^{m} (\tau_k(\xi) - \tau_\sigma(\xi)).$$

Now, $\big(\cosh[(\tau_1(\xi) - \tau_2(\xi))t]\big)/\big(\prod_{k=3}^m(\tau_k(\xi) - \tau_1(\xi))(\tau_k(\xi) - \tau_2(\xi))\big)$ is continuous in $S$, hence it is bounded there, and, thus, absolutely converges to a constant, $C \ge 0$ say, as $\xi \to \xi_0$ through $S$. This leaves the $[e^{i\tau_2(\xi)t}F_{j+1}^{1,2}(\xi) - e^{i\tau_1(\xi)t}F_{j+1}^{2,1}(\xi)]/(\tau_1(\xi) - \tau_2(\xi))$ term.

For this, write

$$F_i^{\rho,\sigma}(\xi) = \sum_{\kappa=0}^{m-1} Q_{\kappa,i}^{\rho,\sigma}(\xi)\tau_\sigma(\xi)^\kappa,$$

where the $Q_{\kappa,i}^{\rho,\sigma}(\xi)$ are polynomials in the $\tau_k(\xi)$ for $k \ne \rho, \sigma$ (which depend on $i$); also, note $Q_{\kappa,i}^{\rho,\sigma}(\xi) = Q_{\kappa,i}^{\sigma,\rho}(\xi)$. Then,

$$\frac{e^{i\tau_2(\xi)t}F_{j+1}^{1,2}(\xi) - e^{i\tau_1(\xi)t}F_{j+1}^{2,1}(\xi)}{\tau_1(\xi) - \tau_2(\xi)} = \frac{\sum_{\kappa=0}^{m-1}\left[Q_{\kappa,j+1}^{1,2}(\xi)(\tau_2(\xi)^\kappa e^{i\tau_2(\xi)t} - \tau_1(\xi)^\kappa e^{i\tau_1(\xi)t})\right]}{\tau_1(\xi) - \tau_2(\xi)}. \quad (4.26)$$

Let us show that this is continuous in $B_\varepsilon(\xi_0)$ and is bounded absolutely by $Cte^{-\min\{\lambda_1,\lambda_2\}}$: for $y_1 \ne y_2$, and for all $r, s \in \mathbb{N}$, $t \ge 0$,

$$\frac{y_2^s y_1^r e^{iy_2t} - y_1^s y_2^r e^{iy_1t}}{y_1 - y_2} = \frac{y_2^s y_1^r(e^{iy_2t} - e^{iy_1t})}{y_1 - y_2} + \frac{y_2^s e^{iy_1t}(y_1^r - y_2^r)}{y_1 - y_2} + \frac{e^{iy_1t}y_2^r(y_2^s - y_1^s)}{y_1 - y_2}.$$

Furthermore, for all $y_1, y_2 \in \mathbb{C}$, $t \in [0, \infty)$, $s \in \mathbb{N}$,

$$\Big|\frac{e^{iy_2 t} - e^{iy_1 t}}{y_1 - y_2}\Big| \leq C_0 t e^{-\min(\operatorname{Im} y_1, \operatorname{Im} y_2)t} \quad \text{and} \quad \Big|\frac{y_1^s - y_2^s}{y_1 - y_2}\Big| \leq C_s\,,$$

for some constants $C_0, C_s$. Using these with $y_1 = \tau_1(\xi)$, $y_2 = \tau_2(\xi)$, $r = \kappa$, and $s$ chosen appropriately for $Q^{1,2}_{\kappa,j+1}(\xi)$, the continuity and upper bound follow immediately. Thus, for all $\xi \in B'_\varepsilon(\xi_0)$, $t \geq 0$,

$$\begin{aligned} &|\cosh[(\tau_1(\xi) - \tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t} + A_j^2(\xi)e^{i\tau_1(\xi)t})| \\ &\qquad \leq C t e^{-\min(\operatorname{Im}\tau_1(\xi), \operatorname{Im}\tau_2(\xi))t}\,. \end{aligned} \tag{4.27}$$

Combining (4.24), (4.25) and (4.27) we have (4.23), which completes the proof of the lemma. □

Now we show that a similar result holds in the general case: suppose the characteristic roots $\tau_1(\xi), \ldots, \tau_L(\xi)$, $2 \leq L \leq m$, coincide on a set $\mathcal{M}$ of codimension $\ell$, and that $\tau_1(\xi) \neq \tau_k(\xi)$ for all $\xi \in \mathcal{M}$ when $k = L+1, \ldots, m$. By continuity, we may take $\varepsilon > 0$ so that the set $\mathcal{M}^\varepsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) < \varepsilon\}$ contains no points $\eta$ at which $\tau_1(\eta), \ldots, \tau_L(\eta) = \tau_k(\eta)$ for $k = L+1, \ldots, m$. With this notation, we have:

**Lemma 4.6.2.** *For all $t \geq 0$ and $\xi \in \mathcal{M}^\varepsilon$,*

$$\Big|\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\Big| \leq C(1+t)^{L-1} e^{-t\min_{k=1,\ldots,L}\operatorname{Im}\tau_k(\xi)}\,. \tag{4.28}$$

**Remark 4.6.1:** Note that this estimate does not depend on the codimension of $\mathcal{M}$.

*Proof.* First note that, just as in the previous proof, for all $\eta \in \mathbb{R}^n$ such that $\tau_1(\eta)\ldots, \tau_L(\eta) \neq \tau_k(\eta)$ when $k = L+1, \ldots, m$ (but allowing any or all of $\tau_1(\eta), \ldots, \tau_L(\eta)$ to be equal),

$$\sum_{k=1}^{L} e^{i\tau_k(\eta)t} A_j^k(t, \eta) = \lim_{\xi \to \eta} \left(e^{i\tau_1(\xi)t} A_j^1(\xi) + \cdots + e^{i\tau_L(\xi)t} A_j^L(\xi)\right),$$

provided $\xi$ to varies the set $S := \bigcup_{l=1}^{L} S_l$, where

$$S_l := \{\xi \in \mathbb{R}^n : \tau_l(\xi) \neq \tau_k(\xi) \ \forall k \neq l\},$$

to ensure that each term of the sum on the right-hand side is well-defined. Note that Lemma 4.2.1 ensures every point in $\mathcal{M}$ is the limit of a sequence of points in $S$. Thus, we must simply show, for all $t \geq 0$, $\xi \in (\mathcal{M}^\varepsilon)' = \mathcal{M}^\varepsilon \backslash \mathcal{M}$,

$$\left|e^{i\tau_1(\xi)t} A_j^1(\xi) + \cdots + e^{i\tau_L(\xi)t} A_j^L(\xi)\right| \leq C t^{L-1} e^{-t \min_{k=1,\ldots,L} \operatorname{Im} \tau_k(\xi)} .$$

Now, we claim that we can write $\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)$, for $\xi \in (\mathcal{M}^\varepsilon)'$ and $t \geq 0$, as a sum of terms involving products of $\frac{(L-1)L}{2}$ sinh and cosh terms of differences of coinciding roots; to clarify, (4.24) is this kind of representation for $L = 2$, while for $L = 3$, we want sums of terms such as

$$\sinh[\alpha_1(\tau_1(\xi) - \tau_2(\xi))t] \cosh[\alpha_2(\tau_1(\xi) - \tau_3(\xi))t] \sinh[\alpha_3(\tau_2(\xi) - \tau_3(\xi))t] ,$$

where the $\alpha_i$ are appropriately chosen constants; incidentally, a comparison to the $L = 2$ case suggests that the term above is multiplied by

$$\left(A_j^1(\xi) e^{i\tau_2(\xi)t} - A_j^2(\xi) e^{i\tau_1(\xi)t}\right)$$

in the full representation.

To show this, we do induction on $L$; the previous Lemma gives us the case $L = 2$ (note that the proof holds with $\xi_0$ and $B_\varepsilon(\xi_0)$ replaced throughout by $\mathcal{M}$ and $\mathcal{M}^\varepsilon$ respectively). Assume there is such a representation for $L = K \leq m - 1$. Observe,

$$\begin{aligned}\sum_{k=1}^{K+1} e^{i\tau_k(\xi)t} A_j^k(\xi) = \frac{1}{K} \sum_{k=1}^{K} e^{i\tau_k(\xi)t} A_j^k(\xi) + \frac{1}{K} \sum_{k=1, k\neq K}^{K+1} e^{i\tau_k(\xi)t} A_j^k(\xi) \\ + \cdots + \frac{1}{K} \sum_{k=2}^{K+1} e^{i\tau_k(\xi)t} A_j^k(\xi) \,;\end{aligned}$$

by the induction hypothesis, there is a representation for each of these terms by means of products of $\frac{(K-1)K}{2}$

$$\sinh[\alpha_{k,l}(\tau_k(\xi) - \tau_l(\xi))t] \text{ and } \cosh[\beta_{k,l}(\tau_k(\xi) - \tau_l(\xi))t] \text{ terms,}$$

where $1 \leq k, l \leq K+1$ and the $\alpha_{k,l}, \beta_{k,l}$ are some non-zero constants. Next, note that we can write $(\tau_1(\xi) - \tau_2(\xi))$ (or, indeed, the difference of any pair of roots from $\tau_1(\xi), \ldots, \tau_{K+1}(\xi)$) as a linear combination of the $\frac{K(K+1)}{2}$ differences $\tau_k(\xi) - \tau_l(\xi)$ such that $1 \leq k < l \leq K+1$; that is

$$\sinh[\alpha_{1,2}(\tau_1(\xi) - \tau_2(\xi))t] = \sinh\Big[\sum_{1\leq k<l\leq K+1} \alpha'_{k,l}(\tau_k(\xi) - \tau_l(\xi))t\Big],$$

for some non-zero constants $\alpha'_{k,l}$; similarly, there is such a representation for $\cosh[\beta_{1,2}(\tau_1(\xi) - \tau_2(\xi))t]$. Lastly, repeated application of the double angle formulae

$$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b\,,$$
$$\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b\,,$$

yields products of $\frac{K(K+1)}{2}$ terms, which completes the induction step.

Now, as in the previous proof, each of these terms must be estimated. The key fact to observe is that

$$A_j^k(\xi) \prod_{l=1, l\neq k}^{L} (\tau_l(\xi) - \tau_k(\xi))$$

is continuous in $\mathcal{M}^\varepsilon$ for all $k = 1, \ldots, L$. Then, using the same arguments as for each of the terms in the earlier proof, and observing that the exponent of $t$ is determined by the products involving either

(a) $(\sinh[\alpha_{k,l}(\tau_k(\xi) - \tau_l(\xi)t)])/(\tau_k(\xi) - \tau_l(\xi))$ terms, or

(b) $(e^{i\tau_k(\xi)t} - e^{i\tau_l(\xi)t})/(\tau_k(\xi) - \tau_l(\xi))$ terms (see (4.26)),

the estimate (4.28) is immediately obtained. □

### 4.6.2 Phase function separated from the real axis

We now turn back to finding $L^p - L^q$ estimates for

$$\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi(\xi)\hat{f}(\xi)\,d\xi\,,$$

when $\tau_1(\xi),\ldots,\tau_L(\xi)$ coincide on a set $\mathcal{M}$ of codimension $\ell$; choose $\varepsilon > 0$ so that these roots do not intersect with any of the roots $\tau_{L+1}(\xi),\ldots,\tau_m(\xi)$ in $\mathcal{M}^\varepsilon$.

In this section, we assume that there exists $\delta > 0$ such that $\operatorname{Im}\tau_k(\xi) \geq \delta$ for all $\xi \in \mathcal{M}^\varepsilon$—so, $\min_k \operatorname{Im}\tau_k(\xi) \geq \delta$. For this, we use the same approach as in Section 4.5.1, but using Lemma 4.6.2 to estimate the sum. Firstly, the $L^1 - L^\infty$ estimate:

$$\begin{aligned}
&\Big\|D_t^r D_x^\alpha\Big(\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi(\xi)\hat{f}(\xi)\,dx\Big)\Big\|_{L^\infty(\mathbb{R}^n_x)}\\
&\qquad= \Big\|\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\tau_k(\xi)^r\Big)\xi^\alpha\chi(\xi)\hat{f}(\xi)\,dx\Big\|_{L^\infty(\mathbb{R}^n_x)}\\
&\qquad\leq \max_k \sup_\Omega|\tau_k(\xi)|^r\int_\Omega\Big|\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big||\xi|^{|\alpha|}|\hat{f}(\xi)|\,dx\\
&\qquad\leq C(1+t)^{L-1}e^{-\delta t}\|\hat{f}\|_{L^\infty(\Omega)} \leq C(1+t)^{L-1}e^{-\delta t}\|f\|_{L^1}\,.
\end{aligned}$$

Similarly, the $L^2 - L^2$ estimate:

$$\begin{aligned}
&\Big\|D_t^r D_x^\alpha\Big(\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi(\xi)\hat{f}(\xi)\,dx\Big)\Big\|_{L^2(\mathbb{R}^n_x)}\\
&\qquad= \Big\|\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\tau_k(\xi)^r\Big)\xi^\alpha\chi(\xi)\hat{f}(\xi)\Big\|_{L^2(\Omega)}\\
&\qquad\leq C(1+t)^{L-1}e^{-\delta t}\|\hat{f}\|_{L^2(\Omega)} \leq C(1+t)^{L-1}e^{-\delta t}\|f\|_{L^2}\,.
\end{aligned}$$

Then, Theorem 4.3.1 yields

$$\begin{aligned}
&\Big\|D_t^r D_x^\alpha\Big(\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi(\xi)\hat{f}(\xi)\,dx\Big)\Big\|_{L^q(\mathbb{R}^n_x)}\\
&\qquad\qquad\leq C(1+t)^{L-1}e^{-\delta t}\|f\|_{L^p}\,,
\end{aligned}$$

where $p^{-1} + q^{-1} = 1$, $1 \le p \le 2$. Once again, we have exponential decay. This, together with (4.19) gives the statement when there are multiplicities away from the axis in Theorem 2.1.1, Part II.

### 4.6.3 Phase function meeting the real axis with finite order

We next look at the case where the characteristic roots $\tau_1(\xi), \ldots, \tau_L(\xi)$ that coincide on the set $\mathcal{M}$ of codimension $\ell$ meet the real axis in $\mathcal{M}$ with finite orders. Suppose $\xi_0 \in \mathcal{M}$ satisfies $\operatorname{Im} \tau_1(\xi_0) = 0$; then by the assumption that it is a finite zero for each root, there exists $\varepsilon > 0$ such that $\operatorname{Im} \tau_k(\xi) > 0$ for all $\xi \in B_\varepsilon(\xi_0)$, $k = 1, \ldots, L$. If there are more points in $\mathcal{M}$ at which the above roots meet the axis with finite order (or even with infinite order/lying on the axis), they may be considered separately in the same way (or using the method below where necessary), while away from such points, the roots are separated from the axis, and the previous argument may be used.

Since the characteristic roots are not necessarily analytic (or even differentiable) on $\mathcal{M}$, we must look at each branch of the roots as they approach the real axis; set $s_k$ to be the maximal order of the contact with the real axis for $\tau_k(\xi)$, that is, the maximal value for which there exist constants $c_{0,k}, c_{1,k} > 0$ such that

$$c_{0,k}|\xi - \xi_0|^{s_k} \le |\operatorname{Im} \tau_k(\xi)| \le c_{1,k}|\xi - \xi_0|^{s_k},$$

for all $\xi$ sufficiently near $\xi_0$. Set $s = \max(s_1, \ldots, s_L)$. Then, by Proposition 4.5.1, using Lemma 4.6.2 to estimate the sum in the amplitude, for all $t > 0$,

$$\begin{aligned}
&\Big\| D_t^r D_x^\alpha \Big( \int_{B_\varepsilon(\xi_0)} e^{ix\cdot\xi} \Big( \sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t) \Big) \chi(\xi) \hat{f}(\xi)\, dx \Big) \Big\|_{L^\infty(\mathbb{R}_x^n)} \\
&\qquad = \Big\| \int_{B_\varepsilon(\xi_0)} e^{ix\cdot\xi} \Big( \sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t) \tau_k(\xi)^r \Big) \xi^\alpha \chi(\xi) \hat{f}(\xi)\, dx \Big\|_{L^\infty(\mathbb{R}_x^n)} \\
&\qquad \le \int_{B_\varepsilon(\xi_0)} t^{L-1} e^{-t \min_{k=1,\ldots,L} \operatorname{Im} \tau_k(\xi)} |\chi(\xi)| |\hat{f}(\xi)|\, d\xi \\
&\qquad \le C(1+t)^{L-1-(n/s)} \|f\|_{L^1},
\end{aligned}$$

and, also using the Plancherel Theorem,

$$\begin{aligned}
&\Big\| D_t^r D_x^\alpha \Big( \int_{B_\varepsilon(\xi_0)} e^{ix\cdot\xi} \Big( \sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi,t) \Big) \chi(\xi) \hat{f}(\xi)\, dx \Big) \Big\|_{L^2(\mathbb{R}_x^n)} \\
&\qquad = \Big\| \int_{B_\varepsilon(\xi_0)} e^{ix\cdot\xi} \Big( \sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi,t) \tau_k(\xi)^r \Big) \xi^\alpha \chi(\xi) \hat{f}(\xi)\, dx \Big\|_{L^2(\mathbb{R}_x^n)} \\
&\qquad = \Big\| \Big( \sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi,t) \tau_k(\xi)^r \Big) \xi^\alpha \chi(\xi) \hat{f}(\xi) \Big\|_{L^2(B_\varepsilon(\xi_0))} \\
&\qquad \le C t^{L-1} \big\| e^{-t \min_{k=1,\dots,L} \operatorname{Im} \tau_k(\xi)} |\chi(\xi)| |\hat{f}(\xi)| \big\|_{L^2(B_\varepsilon(\xi_0))} \\
&\qquad \le C(1+t)^{L-1} \|f\|_{L^2}\,.
\end{aligned}$$

Therefore, interpolation Theorem 4.3.1 says, for all $t > 0$,

$$\begin{aligned}
&\Big\| D_t^r D_x^\alpha \Big( \int_{B_\varepsilon(\xi_0)} e^{ix\cdot\xi} \Big( \sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi,t) \Big) \chi(\xi) \hat{f}(\xi)\, dx \Big) \Big\|_{L^q(\mathbb{R}_x^n)} \\
&\qquad\qquad\qquad \le C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)+L-1} \|f\|_{L^p}\,,
\end{aligned}$$

where $p^{-1} + q^{-1} = 1$, $1 \le p \le 2$; this, together with (4.22) proves Theorem 2.1.1, Part II for roots meeting the axis with finite order.

#### 4.6.4 Phase function lies on the real axis for bounded $|\xi|$

Recall that in the division of the integral in Section 4.2, we have

$$\int_{B_{2N}(0)} e^{ix\cdot\xi} \Big( \sum_{k=1}^{m} e^{i\tau_k(\xi)t} A_j^k(\xi,t) \Big) \hat{f}(\xi)\, d\xi\,,$$

which we then subdivide around and away from multiplicities. The cases where the root or roots are either separated from the real axis or meet it with finite order have already been discussed; here we shall complete the analysis by proving estimates for the situation where a root or roots lie on the real axis, or, in the case of multiple roots, meet it with a zero of infinite order.

To have any possibility of obtaining estimates, we must impose additional conditions on the characteristic roots at low frequencies—for large $|\xi|$, these properties were obtained by using perturbation results, but naturally such results are no longer valid for $|\xi| < N$. Also, we can impose the convexity condition on the roots to obtain a better result than the general case.

Again, throughout we assume that either $\tau(\xi) \geq 0$ for all $\xi \in \Omega$ or $\tau(\xi) \leq 0$ for all $\xi \in \Omega$. The key point is to use a carefully chosen cut off function to isolate the multiplicities and then use Theorem 3.2.4 or Theorem 3.3.4 to estimate the integrals where there are no multiplicities (and hence the coefficients $A_j^k(\xi,t)$ are independent of $t$) and use suitable adjustments around the singularities. For these purposes, let us assume that the only multiplicity is at a point $\xi_0 \in B_{2N}(0)$ and $\tau_1(\xi_0) = \tau_2(\xi_0)$ are the only coinciding roots. Then, we must consider the sum of the first two roots, where we have a multiplicity at $\xi_0$,

$$\int_{B_{2N}(0)} e^{ix\cdot\xi}\Big(\sum_{k=1}^{2} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi(\xi)\hat{f}(\xi)\,d\xi\,,$$

and terms involving the remaining roots, which are all distinct,

$$\sum_{k=3}^{m}\int_{B_{2N}(0)} e^{ix\cdot\xi+\tau_k(\xi)t}A_j^k(\xi,t)\chi(\xi)\hat{f}(\xi)\,d\xi\,.$$

**Case of no multiplicities:** For the second of these, we apply wish to apply Theorem 3.2.4 if $\tau_k(\xi)$ satisfies the convexity condition, and 3.3.4 otherwise.

In order to ensure the hypotheses of these theorems are satisfied, however, we need to impose an additional regularity condition on the behaviour of the characteristic roots for small frequencies (i.e. $\xi \in B_N(0)$) to avoid pathological situations:

$$\text{Assume } |\partial_\omega \tau_k(\lambda\omega)| \geq C_0 \text{ for all } \omega \in S^{n-1},\ \lambda > 0. \tag{4.29}$$

Since this is satisfied for large $|\xi|$ and for roots of homogeneous operators, it is quite a natural extra assumption.

The other hypotheses of these theorems hold: hypothesis (i) is satisfied because $|\partial_\xi^\alpha \tau_k(\xi)| \leq C\alpha$ for all $\xi$ since the characteristic roots are smooth in $\mathbb{R}^n$; hypothesis (ii) only requires information about high frequencies; and hypotheses (iv) holds by the same argument as for large $|\xi|$, where only Part II of Proposition 3.1.5 is needed, and that holds for all $\xi \in \mathbb{R}^n$. Also, the coefficients $A_k^j(\xi)$ are smooth away from multiplicities, so the symbolic behaviour (i.e. bounded for small frequencies) holds.

Now $L^1 - L^\infty$ and $L^2 - L^2$ estimates can be found as in the case for large $|\xi|$, and the interpolation theorem used to give the desired results.

Thus, with condition 4.29, we have proved the on axis, no multiplicities case of Theorem 2.1.1, Part II.

**Case of multiplicities:** Now we can turn to the other integral. First, introduce a cut off function $\psi \in C_0^\infty([0,\infty))$, $0 \le \psi(s) \le 1$, which is identically 0 for $s > 1$ and 1 for $s < \frac{3}{4}$; then it can be rewritten as the sum of two integrals:

$$I_1 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(t|\xi - \xi_0|)\chi(\xi) \sum_{k=1}^{2} A_j^k(\xi,t) e^{i\tau_k(\xi)t} \hat{f}(\xi)\, d\xi\,,$$

$$I_2 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (1-\psi)(t|\xi - \xi_0|)\chi(\xi) \sum_{k=1}^{2} A_j^k(\xi,t) e^{i\tau_k(\xi)t} \hat{f}(\xi)\, d\xi\,.$$

To study $I_1$, we use the resolution of multiplicities technique above: by Lemma 4.6.1,

$$\Big|\sum_{k=1}^{2} A_j^k(\xi,t) e^{i\tau_k(\xi)t}\Big| \le Ct$$

in $|\xi - \xi_0| < t^{-1}$. Now, we may estimate the integral using the compactness of the support of $\psi(s)$: for $0 \le t \le 1$, $I_1$ is clearly bounded; for $t > 1$,

$$\begin{aligned} |I_1| &\le Ct \int_{\mathbb{R}^n} |\psi(t|\xi - \xi_0|)||\hat{f}(\xi)|\, d\xi \\ &= Ct^{1-n} \|\hat{f}\|_{L^\infty} \int_{\mathbb{R}^n} \psi(|\eta|)\, d\eta \le C(1+t)^{1-n}\|f\|_{L^1}. \end{aligned}$$

Similarly, $\|I_2\|_{L^2} \le C(1+t)\|f\|_{L^2}$ (using the Plancherel Theorem as usual).

We remark that such a calculation generalises to the case when $L$ roots meet on a set of codimension $\ell$—clearly the cut off function must be adjusted, but using Lemma 4.6.2, we can show that the corresponding integral is bounded above by $Ct^{L-\ell}$.

For $I_2$ we are away from the singularity, so

$$\sum_{k=1}^{2} A_j^k(\xi,t)e^{i\tau_k(\xi)t} = A_j^1(\xi)e^{i\tau_1(\xi)t} + A_j^2(\xi)e^{i\tau_2(\xi)t}\,;$$

Now, we would like to apply Theorem 3.2.4 (for the case where the root satisfies the convexity condition) and 3.3.4 (for the general case), as in the case of simple roots; however, the proximity of the multiplicity brings the additional cut-off function, $(1-\psi)(t|\xi-\xi_0|)$, into play, and this depends on $t$. Therefore, the aforementioned results cannot be used directly. However, a similar result does hold, provided we impose some additional conditions:

**Proposition 4.6.3.** *Suppose $\tau_k(\xi)$, $k=1,2$, satisfy the following assumptions:*

(i) *for each multi-index $\alpha$ there exists a constant $C_\alpha > 0$ such that, for some $\delta > 0$,*

$$|\partial_\eta^\alpha[(\nabla_\xi\tau_k)(\xi_0+s\eta)]| \le C_\alpha(1+|\eta|)^{-|\alpha|}\,,\ \textit{for small } s,\ |\eta|>\delta\,;$$

(ii) *there exists a constant $C_0 > 0$ such that $|\partial_\omega\tau_k(\xi_0+t^{-1}\lambda\omega)| \ge C > 0$ for all $\omega\in S^{n-1}$; in particular, each of the level sets*

$$\lambda\Sigma'_\lambda \equiv \Sigma_\lambda = \left\{\eta\in\mathbb{R}^n : \tau_k(\xi_0+t^{-1}\eta)=\lambda\right\}$$

*is non-degenerate;*

(iii) *there exists a constant $R_1 > 0$ such that, for all $\lambda > 0$,*

$$\Sigma'_\lambda := \frac{1}{\lambda}\Sigma_\lambda(\tau) \subset B_{R_1}(0)\,.$$

*Furthermore, assume that $A_j^k(\xi)$ satisfies the following condition: for each multi-index $\alpha$ there exists a constant $C_\alpha > 0$ such that*

(iv)

$$|\partial_\eta^\alpha[A_j^k(\xi_0+s\eta)]| \le C_\alpha s^{-j}(1+|\eta|)^{-j-|\alpha|}\,,\ \textit{for small } s,\ |\eta|>\delta\,.$$

*Then, the following estimate holds for all $R \geq 0$, $x \in \mathbb{R}^n$, $t > 1$:*

$$\Big|\sum_{k=1}^{2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)}A_j^k(\xi)(1-\psi)(t|\xi-\xi_0|)\chi(\xi)\kappa\big(t^{-1}x+\nabla\tau_k(\xi)\big)\,d\xi\Big| \leq Ct^{j-n}\,, \tag{4.30}$$

*for $j \geq n - \frac{n-1}{\gamma}$, where $\gamma := \sup_{\lambda>0}\gamma(\Sigma_\lambda(\tau_k))$, if $\tau_k^j(\xi)$ satisfies the convexity condition, and for $j \geq n - \frac{1}{\gamma_0}$, where $\gamma_0 := \sup_{\lambda>0}\gamma_0(\Sigma_\lambda(\tau_k))$, if it does not.*

**Remark 4.6.2:** Conditions (i), (ii) and (iv) arise naturally when $\tau_k(\xi)$ is a homogeneous function of order 1—for example, the wave equation.

**Remark 4.6.3:** Assumption (iv) is needed because, $A_j^k(\xi)$ has a singularity at $\xi_0$, so we must ensure we are away from that—this is the role of the cut-off function $(1-\psi)(|\eta|)$ in this proposition; similarly,

*Proof.* As before, cut-off near the wave front: let $\kappa \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function supported in $B(0,r)$. Then, consider

$$I_1(x,t) := \sum_{k=1}^{2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)}A_j^k(\xi)(1-\psi)(t|\xi-\xi_0|)\chi(\xi) \\ \kappa\big(t^{-1}x+\nabla\tau_k(\xi)\big)\,d\xi,$$

and

$$I_2(x,t) := \sum_{k=1}^{2}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)}A_j^k(\xi)(1-\psi)(t|\xi-\xi_0|)\chi(\xi) \\ (1-\kappa)\big(t^{-1}x+\nabla\tau_k(\xi)\big)\,d\xi.$$

**Away from the wave front set:** First, we estimate $I_2(x,t)$; we claim that

$$|I_2(x,t)| \leq C_r t^{j-n} \text{ for all } t > 0\,, x \in \mathbb{R}^n\,. \tag{4.31}$$

In order to show this, we consider each term of the sum separately,

$$I_2^k(x,t) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau_k(\xi)t)}A_j^k(\xi)(1-\psi)(t|\xi-\xi_0|)\chi(\xi)(1-\kappa)\big(\tfrac{x}{t}+\nabla\tau_k(\xi)\big)\,d\xi\,,$$

and imitate the proof of Lemma 3.2.5 (in which the corresponding term was estimated in Theorem 3.2.4), but noting that in place of $g_R(\xi) \in C_0^\infty(\mathbb{R}^n)$

we have $(1-\psi)(t(\xi-\xi_0))$, which depends also on $t$; in particular, this means that care must be taken when carrying out the integration by parts when derivatives fall on $(1-\psi)(t|\xi-\xi_0|)$. To take this into account, use the change of variables $\xi = \xi_0 + t^{-1}\eta$:

$$I_2^k(x,t) = e^{ix\cdot\xi_0} \int_{\mathbb{R}^n} e^{i(t^{-1}x\cdot\eta+\tau_k(\xi_0+t^{-1}\eta)t)} A_j^k(\xi_0+t^{-1}\eta)(1-\psi)(|\eta|) \\ \chi(\xi_0+t^{-1}\eta)(1-\kappa)\big(t^{-1}x+(\nabla_\xi\tau_k)(\xi_0+t^{-1}\eta)\big)t^{-n}\,d\eta.$$

Integrating by parts, with respect to $\eta$ gives

$$I_2^k(x,t) = e^{ix\cdot\xi_0}t^{-n} \int_{\mathbb{R}^n} e^{i(t^{-1}x\cdot\eta+\tau_k(\xi_0+t^{-1}\eta)t)} P^*\big[A_j^k(\xi_0+t^{-1}\eta)(1-\psi)(|\eta|) \\ \chi(\xi_0+t^{-1}\eta)(1-\kappa)\big(t^{-1}x+(\nabla_\xi\tau_k)(\xi_0+t^{-1}\eta)\big)\big]\,d\eta\,,$$

where $P^*$ is the adjoint operator to $P = \frac{t^{-1}x+(\nabla_\xi\tau_k)(\xi_0+t^{-1}\eta)}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi_0+t^{-1}\eta)|^2}\cdot\nabla_\eta$; this integration by parts is valid as $|t^{-1}x+(\nabla_\xi\tau_k)(\xi_0+t^{-1}\eta)| \geq r > 0$, in the support of $(1-\kappa)\big(t^{-1}x+\nabla\tau_k(\xi_0+t^{-1}\eta)\big)$. For suitable functions $f \equiv f(\xi;x,t)$,

$$\begin{aligned} P^*f =& \nabla_\eta\cdot\Big[\frac{t^{-1}x+(\nabla_\xi\tau_k)(\xi)}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi)|^2}f\Big] \\ =& \frac{\nabla_\eta\cdot(\nabla_\xi\tau_k)(\xi)}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi)|^2}f + \frac{t^{-1}x+(\nabla_\xi\tau_k)(\xi)}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi)|^2}\cdot\nabla_\eta f \\ &- \frac{2(t^{-1}x+(\nabla_\xi\tau_k)(\xi))\cdot[\nabla_\eta[(\nabla_\xi\tau_k)(\xi)]\cdot(t^{-1}x+(\nabla_\xi\tau_k)(\xi))]}{i|t^{-1}x+(\nabla_\xi\tau_k)(\xi)|^4}f. \end{aligned}$$

Comparing this to (3.38), observe that the first and third terms have one power of $t$ fewer in the denominator due to the transformation; this is critical in this case where we are approaching a singularity in $A_j^k(\xi_0+t^{-1}\eta)$ when $t\to\infty$. By hypothesis (i), for $\eta$ in the support of the integrand of $I_2^k(x,t)$,

$$\frac{\nabla_\eta\cdot[(\nabla_\xi\tau_k)(\xi_0+t^{-1}\eta)]}{|t^{-1}x+(\nabla_\xi\tau_k)(\xi_0+t^{-1}\eta)|^2} \leq C_r(1+|\eta|)^{-1}\,;$$

thus,

$$|P^*f| \leq C_r[(1+|\eta|)^{-1}|f| + |\nabla_\eta f|]\,.$$

In Lemma 3.2.5, we carried out this integration by parts repeatedly in order to estimate the integral. Here, however, note that differentiating $(1-\psi)(|\eta|)$ once is sufficient: by definition of $\psi(s)$,

$$\partial_{\eta_j}[(1-\psi)(|\eta|)] = -\frac{\eta_j}{|\eta|}(\partial_s\psi)(|\eta|)$$

is supported in $\frac{3}{4} \le |\eta| \le 1$, so

$$|\partial_{\eta_j}[(1-\psi)(|\eta|)]| \le C\mathbb{1}_{|\eta|\ge 3/4}(\eta)\,,$$

where $\mathbb{1}_{|\eta|\ge 3/4}(\eta)$ denotes the characteristic function of $\{\eta \in \mathbb{R}^n : |\eta| > 3/4\}$; hence, by hypothesis (iv),

$$\begin{aligned}
\int_{\mathbb{R}^n} \Big| \frac{t^{-1}x + (\nabla_\xi \tau_k)(\xi_0 + t^{-1}\eta)}{i|t^{-1}x + (\nabla_\xi \tau_k)(\xi_0 + t^{-1}\eta)|^2} \Big| |A_j^k(\xi_0 + t^{-1}\eta)||\partial_{\eta_j}[(1-\psi)(|\eta|)]| \\
|\chi(\xi_0 + t^{-1}\eta)||(1-\kappa)\big(t^{-1}x + \nabla\tau_k(\xi_0 + t^{-1}\eta)\big)|t^{-n}\,d\eta \\
\le C_r \int_{\frac{3}{4}\le|\eta|\le 1} |A_j^k(\xi_0 + t^{-1}\eta)|t^{-n}\,d\eta \\
\le C_r t^j \int_{\frac{3}{4}\le|\eta|\le 1} \frac{1}{(1+|\eta|)^j} t^{-n}\,d\eta \le C_r t^{j-n}\,,
\end{aligned} \tag{4.32}$$

which is the desired estimate (4.31).

On the other hand, if, when integrating by parts, the derivative does not fall on $\psi(|\eta|)$, we use a similar argument to that in the earlier proof; let us look at the effect of differentiating each of the other terms: in the support of $\psi(|\eta|)$, for each multi-index $\alpha$ and $t > 0$,

- $|\partial_\eta^\alpha[A_j^k(\xi_0 + t^{-1}\eta)]| \le C_\alpha t^j (1+|\eta|)^{-j-|\alpha|}$ by hypothesis (iv);
- $|\partial_\eta^\alpha[\chi(\xi_0 + t^{-1}\eta)]| \le C_\alpha(1+|\eta|)^{-|\alpha|}$: for $\alpha = 0$, take $C_\alpha = 1$; for $|\alpha| \ge 1$, note that
$$\partial_\eta^\alpha[\chi(\xi_0 + t^{-1}\eta)] = t^{-|\alpha|}(\partial_\xi^\alpha\chi)(\xi_0 + t^{-1}\eta)\,,$$
and that $(\partial_\xi^\alpha\chi)(\xi_0 + t^{-1}\eta)$ is supported in $N \le |\xi_0 + t^{-1}\eta| \le 2N$, so $t^{-1} \le C_{N,\xi_0}|\eta|^{-1}$;
- $|\partial_\eta^\alpha[(1-\kappa)\big(t^{-1}x + (\nabla_\xi\tau_k)(\xi_0 + t^{-1}\eta)\big)]| \le C_\alpha(1+|\eta|)^{-|\alpha|}$: obvious for

$\alpha = 0$; for $|\alpha| \geq 1$, note

$$\partial_\eta^\alpha[(1-\kappa)(t^{-1}x + (\nabla_\xi \tau_k)(\xi_0 + t^{-1}\eta))] \\ = -(\partial_\xi^\alpha \kappa)(t^{-1}x + \nabla_\xi \tau_k(\xi))\partial_\eta^\alpha[(\nabla_\xi \tau_k)(\xi_0 + t^{-1}\eta)]\,,$$

which yields the desired estimate by hypothesis (i).

Summarising, this means

$$\big|(1-\psi)(|\eta|)\partial_\eta^\alpha\big[A_j^k(\xi_0+t^{-1}\eta)\chi(\xi_0+t^{-1}\eta)(1-\kappa)\big(t^{-1}x+(\nabla_\xi \tau_k)(\xi_0+t^{-1}\eta)\big)\big]\big| \\ \leq C_r(1+|\eta|)^{-|\alpha|-j}t^j \mathbb{1}_{|\eta|>\frac{3}{4}}(\eta)\,.$$

So, repeatedly integrating by parts we find that either a derivative falls on $(1-\psi)(|\eta|)$ (in which case a similar argument to that in (4.32) above works) or we eventually get the integrable function $C(1+|\eta|)^{-n-1}\mathbb{1}_{|\eta|>3/4}(\eta)$ as an upper bound; in either case, we have (4.31).

**On the wave front set:** Next, we look at the term supported in the wave front set, $I_1(x,t)$. As in the case away from the wave front, set $\xi = \xi_0 + t^{-1}\eta$: consider, for $k = 1, 2$,

$$I_1^k(x,t) := e^{ix\cdot\xi_0}\int_{\mathbb{R}^n} e^{i(t^{-1}x\cdot\eta+\tau_k(\xi_0+t^{-1}\eta)t)}A_j^k(\xi_0+t^{-1}\eta)(1-\psi)(|\eta|) \\ \chi(\xi_0+t^{-1}\eta)\kappa\big(t^{-1}x+(\nabla_\xi \tau_k)(\xi_0+t^{-1}\eta)\big)t^{-n}\,d\eta\,.$$

As in the proof of Theorems 3.2.4 and 3.3.4, let $\{\Psi_\ell(\eta)\}_{\ell=1}^L$ be a conic partition of unity, where the support of $\Psi_\ell(\eta)$ is a cone $K_\ell$, and each cone can be mapped by rotation onto $K_1$, which contains $e_n = (0,\ldots,0,1)$. Then, it suffices to estimate

$$t^{-n}\int_{\mathbb{R}^n} e^{i(t^{-1}x\cdot\eta+\tau_k(\xi_0+t^{-1}\eta)t)}A_j^k(\xi_0+t^{-1}\eta)(1-\psi)(|\eta|) \\ \Psi_1(\eta)\chi(\xi_0+t^{-1}\eta)\kappa\big(t^{-1}x+(\nabla_\xi \tau_k)(\xi_0+t^{-1}\eta)\big)\,d\eta\,,$$

for $k = 1, 2$.

Let us parameterise the cone $K_1$: by hypothesis (ii), each of the level sets

$$\lambda\Sigma'_\lambda \equiv \Sigma_\lambda = \left\{\eta \in \mathbb{R}^n : \tau_k(\xi_0 + t^{-1}\eta) = \lambda\right\}$$

is non-degenerate; so, for some $U \subset \mathbb{R}^{n-1}$, and smooth function $h_k(\lambda, \cdot) : U \to \mathbb{R}$,

$$K_1 = \{(\lambda y, \lambda h_k(\lambda, y)) : \lambda > 0, y \in U\} \ .$$

If $\tau_k(\xi)$ satisfies the convexity condition, then $h_k$ is also a concave function in $y$. Now, change variables $\eta \mapsto (\lambda y, \lambda h_k(\lambda, y))$, to obtain:

$$t^{-n} \int_0^\infty \int_U e^{i\lambda(t^{-1}x'\cdot y + t^{-1}x_n h_k(\lambda,y) + t)} A_j^k(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))$$
$$(1-\psi)(\lambda|(y, h_k(\lambda, y))|)\Psi_1(\lambda(y, h_k(\lambda, y)))\chi(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))$$
$$\kappa\big(t^{-1}x + (\nabla_\xi \tau_k)(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))\big)\frac{d\eta}{d(\lambda, y)}\, d\lambda dy, \quad (4.33)$$

where we have used $\tau_k(\xi_0 + t^{-1}(\lambda y, \lambda h_k(\lambda, y))) = \lambda$. As in the earlier proofs, we ensure $x_n$ is away from zero in the cone—this requires hypotheses (i) and (iii)). So, in the general case, we can write this as, with $\tilde{x} = t^{-1}x$, $\tilde{\lambda} = \lambda\tilde{x}_n = \lambda t^{-1}x_n$,

$$t^{-n} \int_0^\infty \int_U e^{i\lambda x_n(t^{-1}x_n^{-1}x'\cdot y + t^{-1}h_k(\lambda,y) + \tilde{x}_n^{-1})} A_j^k(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))$$
$$(1-\psi)(\lambda|(y, h_k(\lambda, y))|)\Psi_1(\lambda(y, h_k(\lambda, y)))\chi(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))$$
$$\kappa\big(t^{-1}x + (\nabla_\xi \tau_k)(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))\big)\frac{d\eta}{d(\lambda, y)}\, d\lambda dy\,.$$

If the convexity condition holds, then, as in the proof of Theorem 3.2.4, we have the Gauss map

$$\underline{\mathbf{n}}_k : K_1 \cap \Sigma'_\lambda \to S^{n-1},\ \underline{\mathbf{n}}_k(\zeta) = \frac{\nabla_\zeta[\tau_k(\xi_0 + t^{-1}\zeta)]}{|\nabla_\zeta[\tau_k(\xi_0 + t^{-1}\zeta)]|} = \frac{(\nabla_\xi \tau_k)(\xi_0 + t^{-1}\zeta)}{|(\nabla_\xi \tau_k)(\xi_0 + t^{-1}\zeta)|}\,,$$

and, as before, can define $z_k(\lambda) \in U$ so that

$$\underline{\mathbf{n}}_k(z_k(\lambda), h_k(\lambda, z(\lambda))) = -x/|x|\,.$$

Then,

$$\frac{x'}{x_n} = -\nabla_y h_k(\lambda, z(\lambda))\,.$$

So, in this case, (4.33) becomes:

$$(I_1^k)' := t^{-n} \int_0^\infty \int_U e^{i\lambda x_n[-t^{-1}\nabla_y h_k(\lambda,z(\lambda))\cdot y + t^{-1}h_k(\lambda,y)+\tilde{x}_n^{-1}]}$$
$$A_j^k(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))(1-\psi)(\lambda|(y, h_k(\lambda, y))|)\Psi_1(\lambda(y, h_k(\lambda, y)))$$
$$\chi(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))\kappa\big(\tilde{x} + (\nabla_\xi \tau_k)(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))\big)\frac{d\eta}{d(\lambda, y)}\, d\lambda dy,$$

Let us estimate the second (i.e. the case where the convexity condition holds):

- The same argument as in the earlier proof (which uses hypothesis (ii)), shows
$$\Big|\frac{d\eta}{d(\lambda, y)}\Big| \le C\lambda^{n-1}\,;$$
- Now, with $\tilde{A}_k^j(\nu) = A_k^j(\nu)\chi(\nu)\kappa\big(\tilde{x} + (\nabla_\xi\tau_k)(\nu)\big)\Psi_1(\lambda(y, h_k(\lambda, y)))$, where $\nu = \xi_0 + t^{-1}\lambda(y, h_k(\lambda, y))$,
$$|(I_1^k)'| \le t^{j-n} \int_0^\infty \Big|\int_U e^{i\lambda\tilde{x}_n[-(y-z(\lambda))\nabla_y h_k(\lambda,z(\lambda))\cdot y + h_k(\lambda,y)+h_k(\lambda,z(\lambda))]}$$
$$t^{-j}\lambda^j \tilde{A}_j^k(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))(1-\psi)(\lambda|(y, h_k(\lambda, y))|)\, dy\Big|\lambda^{n-1-j}\, d\lambda\,.$$
- Now, applying Theorem 3.2.1—this may be used due to the properties of $A_j^k(\xi)$ and $\tau_k(\xi)$ stated in hypotheses (iv) and (i)—we find that
$$\Big|\int_U e^{i\lambda\tilde{x}_n[-(y-z(\lambda))\nabla_y h_k(\lambda,z(\lambda))\cdot y + h_k(\lambda,y)+h_k(\lambda,z(\lambda))]}$$
$$t^{-j}\lambda^j \tilde{A}_j^k(\xi_0 + t^{-1}\lambda(y, h_k(\lambda, y)))(1-\psi)(\lambda|(y, h_k(\lambda, y))|)\, dy\Big| \le C\lambda^{j-n}\tilde{\chi}(\lambda)\,,$$
where $\tilde{\chi}(\lambda)$ is a compactly smooth function that is zero in a neighbourhood of the origin.
- Hence,
$$|(I_1^k)'| \le t^{j-n} \int_0^\infty \tilde{\chi}(\lambda)\lambda^{-1}\, d\lambda \le Ct^{j-n}\,.$$

Finally, the general case can be estimated in exactly the same way, with the necessary changes used in the proof of Theorem 3.3.4 to account for the change in the phase function—in particular, the use of the Van der Corput Lemma, Lemma 3.3.2, in place of Theorem 3.2.1. This completes the proof of (4.30). □

Using this, it is clear that

$$\Big\|\int_{\mathbb{R}^n} e^{ix\cdot\xi}(1-\psi)(t|\xi-\xi_0|)\chi(\xi)\sum_{k=1}^{2} A_j^k(\xi,t)e^{i\tau_k(\xi)t}\hat{f}(\xi)\,d\xi\Big\|_{L^\infty}$$
$$\leq C(1+t)^{-\frac{n-1}{\gamma}}\|f\|_{W_1^{\frac{n-1}{\gamma}-n-j}}$$

if the roots satisfy the convexity condition, and

$$\Big\|\int_{\mathbb{R}^n} e^{ix\cdot\xi}(1-\psi)(t|\xi-\xi_0|)\chi(\xi)\sum_{k=1}^{2} A_j^k(\xi,t)e^{i\tau_k(\xi)t}\hat{f}(\xi)\,d\xi\Big\|_{L^\infty}$$
$$\leq C(1+t)^{-\frac{1}{\gamma_0}}\|f\|_{W_1^{\frac{1}{\gamma_0}-n-j}}$$

otherwise. The $L^2-L^2$ estimates are simple via the Plancherel Theorem as in other cases.

Finally, we must consider the case where $L$ roots meet on a set of codimension $\ell$; the above proof can easily be adapted for such a case, leaving the same results. Due to the earlier bound near the multiplicity, we can combine the results and the interpolation Theorem 4.3.1 to complete the proof of the final part of Theorem 2.1.1. □

# Chapter 5:
# Examples and Extensions

## 5.1 Introduction

Theorem 2.1.1 give estimates for operators provided the characteristic roots satisfy certain hypotheses. However, in order to test the validity of such an estimate for an arbitrary linear, constant coefficient $m^{\text{th}}$ order strictly hyperbolic operator with lower order terms, it is desirable to find conditions on the structure of the lower order terms under which certain conditions for the characteristic roots hold. For the case $m = 2$ a characterisation for wave-type equations with (possibly negative) mass and dissipation terms can be given, and this is done in Section 5.2. However, for large $m$, it is difficult to do such an analysis; nevertheless, certain conditions can be found that do make the task of checking the conditions of the characteristic roots, and these are discussed in Section 5.3, where a method is also given that can be used to find many examples. Finally, in Section 5.4, we give a few applications of our results.

## 5.2 Complete Analysis of Wave Equation with Mass and Dissipation

Consider the general Cauchy problem for the wave equation with lower order terms

$$\begin{cases} \partial_t^2 u - c^2 \Delta u + \delta \partial_t u + \mu u = 0\,, \\ u(0,x) = 0,\ u_t(0,x) = g(x)\,. \end{cases}$$

Then the associated characteristic polynomial is

$$\tau^2 - c^2|\xi|^2 - i\delta\tau - \mu = 0\,,$$

which has roots

$$\tau_{\pm}(\xi) = \frac{i\delta}{2} \pm \sqrt{c^2|\xi|^2 + \mu - \delta^2/4}\,.$$

Now, we have the following cases and apply Theorem 2.1.1 in each:

- $\delta = \mu = 0$. This is the wave equation.

- $\delta = 0$, $\mu > 0$. This is the Klein–Gordon equation.

- $\mu = 0$, $\delta > 0$. This is the dissipative wave equation.

- $\delta < 0$. In this case, $\operatorname{Im} \tau_-(\xi) \leq \frac{\delta}{2} < 0$ for all $\xi$, hence we cannot expect any decay in general.

- $\delta, \mu > 0$. In this case the discriminant is always strictly greater than $-\delta^2/4$, and thus the roots always lie in the half plane $\operatorname{Im} z > 0$ and are separated from the real axis. So we have exponential decay.

- $\delta \geq 0$, $\mu < 0$. In this case we note that $\operatorname{Im} \tau_-(\xi) \geq 0$ if and only if $c^2|\xi|^2 + \mu \geq 0$, i.e. the critical value is $|\xi| = \frac{\sqrt{|\mu|}}{c}$. We must, therefore, look at the space in which the initial condition $g$ (or, more precisely, its Fourier transform) lies. If no function under consideration has Fourier transform with support in $c^2|\xi|^2 + \mu < 0$, then we may get decay of some type. That is:

  – if we have initial data $g$ such that $\operatorname{supp} \hat{g} \cap B(0, \frac{\sqrt{|\mu|}}{c}) \neq \varnothing$, then we cannot expect decay;

  – if, for some $\varepsilon > 0$, $\operatorname{supp} \hat{g} \subset \mathbb{R}^n \setminus B(0, \frac{\sqrt{|\mu|}}{c} + \varepsilon)$, then the roots are either separated from the real axis (if $\delta > 0$), and we get exponential decay, or lie on the real axis (if $\delta = 0$), and we get Klein-Gordon type behaviour since the Hessian of $\tau$ is

$$\operatorname{Hess} \tau_{\pm}(\xi) = \left( \frac{c^2 \delta_{jk}}{\sqrt{c^2|\xi|^2 - |\mu|}} - \frac{c^4 \xi_j \xi_k}{(c^2|\xi|^2 - |\mu|)^{3/2}} \right)_{j,k=1}^{n}.$$

  – if, for all $g$, $\operatorname{supp} \hat{g} \subset \mathbb{R}^n \setminus B(0, \frac{\sqrt{|\mu|}}{c})$ , then again we must consider $\delta = 0$ and $\delta > 0$ separately.

If $\delta > 0$, then the root $\tau_-$ comes to the real axis at $|\xi| = \frac{\sqrt{|\mu|}}{c}$, in which case we get decay $(1+t)^{-(\frac{1}{p}-\frac{1}{q})}$: the order with which it meets the axis is $s = 2$ and the codimension of the set $\left\{\xi \in \mathbb{R}^n : |\xi| = \frac{\sqrt{|\mu|}}{c}\right\}$ is $\ell = 1$. If $\delta = 0$, then the roots lie completely on the real axis, and they meet on the sphere $|\xi| = \frac{\sqrt{|\mu|}}{c}$, thus we get decay $1 - (\frac{1}{p} - \frac{1}{q})$.

## 5.3 Higher Order Equations

### 5.3.1 Coefficient of $D_t^{m-1}u$

Let us now derive a simple consequence of the condition that $\operatorname{Im} \tau_k(\xi) \geq 0$, for all $k = 1, \ldots, m$ and $\xi \in \mathbb{R}^n$, for the coefficient of the $D_t^{m-1}u$ term in (1.6) .

Let $L = L(D_x, D_t)$ be an $m^{\text{th}}$ order constant coefficient, linear strictly hyperbolic operator such that $\operatorname{Im} \tau_k(\xi) \geq 0$ for all $k = 1, \ldots, m$ and for all $\xi \in \mathbb{R}^n$. Recall that the characteristic polynomial corresponding to the principal part of $L$ is of the form

$$L_m = L_m(\xi, \tau) = \tau^m + \sum_{k=1}^{m} P_k(\xi)\tau^{m-k} = 0\,,$$

where the $P_k(\xi)$ are homogeneous polynomials of order $k$. Then, by the strict hyperbolicity of $L$, $L_m$ has real roots $\varphi_1(\xi) \leq \varphi_2(\xi) \leq \cdots \leq \varphi_m(\xi)$ (where the inequalities are strict when $\xi \neq 0$). By the Vièta formulae, observe that

$$P_1(\xi) = -\sum_{k=1}^{m} \varphi(\xi) \in \mathbb{R}\,. \tag{5.1}$$

On the other hand, the characteristic polynomial of the full operator is

$$L(\xi, \tau) = \tau^m + \sum_{k=1}^{m} P_k(\xi)\tau^{m-k} + \sum_{j=0}^{m-1} \sum_{|\alpha|+l=j} c_{\alpha,l}\xi^\alpha \tau^l = 0\,. \tag{5.2}$$

In particular, the coefficient of the $\tau^{m-1}$ term is

$$P_1(\xi) + c_{0,m-1} = -\sum_{k=1}^{m} \tau_k(\xi), \tag{5.3}$$

where the $\tau_k(\xi)$, $k = 1, \ldots, m$ are the roots of (5.2). Comparing (5.1) and (5.3), we see that $\operatorname{Im}\left(\sum_{k=1}^{m} \tau_k(\xi)\right) = -\operatorname{Im} c_{0,m-1}$. Therefore, since $\operatorname{Im} \tau_k(\xi) \geq 0$ for all $k = 1, \ldots, m$ and $\xi \in \mathbb{R}^n$, it follows that $\operatorname{Im} c_{0,m-1} \leq 0$, or, equivalently, $ic_{0,m-1} \geq 0$ (note that we are looking at an operator which has real coefficients in the form $L(\partial_t, \partial_x)$). Furthermore, if $\operatorname{Im} c_{0,m-1} = 0$ then it must be the case that $\operatorname{Im} \tau_k(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ and $k = 1, \ldots, m$ since the characteristic roots are continuous. Hence we have shown the following:

**Proposition 5.3.1.** *Let $L = L(D_x, D_t)$ be an $m^{th}$ order linear constant coefficient strictly hyperbolic operator such that all the characteristic roots $\tau_k(\xi)$, $k = 1, \ldots, m$, satisfy $\operatorname{Im} \tau_k(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$. Then the imaginary part of the coefficient of $D_t^{m-1}u$ is non-positive. Furthermore, if in addition the (imaginary part of the) coefficient of $D_t^{m-1}u$ is zero then each of the characteristic roots lie completely on the real axis.* □

**Remark 5.3.1:** If we transform our operator back to the form $L(\partial_x, \partial_t)$, this result tells us that in order for the characteristic polynomial to be stable, that is $\operatorname{Im} \tau_k(\xi) \geq 0$ for all $k = 1, \ldots, m$, $\xi \in \mathbb{R}^n$, it is necessary for the coefficient of $\partial_t^{m-1}u$ to be non-negative; this is the case for the dissipative wave equation. This gives a notion of *higher order dissipation*, since it is necessary for the characteristic roots to behave geometrically like those of the wave equation with a dissipative term, i.e they lie in the half-plane $\operatorname{Im} z \geq 0$ and lie away from $\operatorname{Im} z = 0$ for large $|\xi|$.

In the next section, we look at the case where $c_{0,m-1} = 0$, in which case the characteristic roots must lie completely on the real axis. First, though, let us consider the case where a root lies completely on the real axis but the coefficient $c_{0,m-1} \neq 0$;

Consider a constant coefficient strictly hyperbolic operator of the form

$$L_m(\partial_x, \partial_t) + L_{m-1}(\partial_x, \partial_t) + L_{m-2}(\partial_x, \partial_t) = 0, \tag{5.4}$$

where $L_r = L_r(\partial_x, \partial_t)$ denotes a homogeneous operator of degree $r$ with real coefficients. Furthermore, assume that $L_{m-1}$ is not identically zero. Let $\tau(\xi)$ be a characteristic root of (5.4) which lies completely on the real axis.

So, noting $D_{x_j} = -i\partial_{x_j}$, $D_t = -i\partial_t$, we have that $\tau(\xi)$ is a root of

$$L_m(\xi,\tau) - iL_{m-1}(\xi,\tau) - L_{m-2}(\xi,\tau) = 0.$$

This means that $L_{m-1}(\xi,\tau(\xi)) = 0$, and so $\tau(\xi)$ is homogeneous of order 1, and thus the Hessian of $\tau(\xi)$ is nonsingular, and by Theorem 2.1.1, we obtain decay of order $-n/2$.

### 5.3.2 Hyperbolic triples

We now turn to the case $c_{0,m-1} = 0$ where, by Proposition 5.3.1, all the characteristic roots lie completely on the real axis. In order to study this case we cite some results of Volevich–Radkevich [VR03] on the theory of hyperbolic pairs and triples. Throughout this section, $L_r(\xi,\tau)$ denotes a homogeneous polynomial in $\tau$ and $\xi = (\xi_1,\dots,\xi_n)$ of order $r$ such that $L_r(i\xi,\tau,)$ has real coefficients.

**Definition 5.1.** *Suppose $L_m = L_m(\xi,\tau)$ and $L_{m-1} = L_{m-1}(\xi,\tau)$ are (homogeneous) polynomials as above. Furthermore, assume that the roots of $L_m$, $\tau_1(\xi),\dots,\tau_m(\xi)$, and those of $L_{m-1}$, $\sigma_1(\xi),\dots,\sigma_{m-1}(\xi)$, are real-valued (in which case we say $L_m$ and $L_{m-1}$ are* hyperbolic polynomials*). Then, $(L_m, L_{m-1})$ is called a* hyperbolic pair *if (possibly after reordering)*

$$\tau_1(\xi) \le \sigma_1(\xi) \le \tau_2(\xi) \le \cdots \le \tau_{m-1}(\xi) \le \sigma_{m-1}(\xi) \le \tau_m(\xi). \tag{5.5}$$

*If, in addition, the roots of $L_m, L_{m-1}$ are pairwise distinct for $\xi \neq 0$ (in which case they are called* strictly hyperbolic polynomials*) and the inequalities in (5.5) are all strict, then we say $(L_m, L_{m-1})$ is a* strictly hyperbolic pair*.*

**Definition 5.2.** *Let*

$$L_m = L_m(\xi,\tau)\,,\ L_{m-1} = L_{m-1}(\xi,\tau)\,,\ L_{m-2} = L_{m-2}(\xi,\tau)$$

*be (homogeneous) hyperbolic polynomials. If $(L_m, L_{m-1})$ and $(L_{m-1}, L_{m-2})$ are both hyperbolic pairs then we say that $(L_m, L_{m-1}, L_{m-2})$ is a* hyperbolic triple*. If, in addition, all the polynomials and all the pairs are strictly hyperbolic (in the sense of Definition 5.1) then $(L_m, L_{m-1}, L_{m-2})$ is called a* strictly hyperbolic triple.

**Theorem 5.3.2** ([VR03]). *Suppose that* $(L_m, L_{m-1}, L_{m-2})$ *is a strictly hyperbolic triple. Then* $L_m(\xi,\tau)+L_{m-1}(\xi,\tau)+L_{m-2}(\xi,\tau) \neq 0$ *for all* $\operatorname{Im}\tau \leq 0$. *Furthermore,* $L_m, L_{m-1}, L_{m-2}$ *pairwise have no common purely imaginary zeros.*

We also cite a theorem of Hermite (see, for example, [Nis00]):

**Theorem 5.3.3.** *Suppose* $p_m(z)$, $p_{m-1}(z)$ *are real polynomials of degree* $m, m-1$ *respectively and that all the zeros of* $p(z) = p_m(z) - ip_{m-1}(z)$ *lie in the upper half-plane (that is, if* $p(z) = 0$ *then* $\operatorname{Im} z > 0$*). Then all the zeros of* $p_m(z)$ *and* $p_{m-1}(z)$ *are real and distinct.*

Assume that $L$ is of the form $L_m(D_x, D_t) + L_{m-2}(D_x, D_t)$, where the $L_r$ are as in Definition 5.2 and neither is identically zero. Suppose that there exists a homogeneous operator of order $m-1$, $L_{m-1}(D_x, D_t)$, such that the characteristic polynomials $L_m(\xi,\tau)$, $L_{m-1}(\xi,\tau)$ and $L_{m-2}(\xi,\tau)$ form a strictly hyperbolic triple. Then, by Theorem 5.3.2,

$$L_m(\xi,\tau) + L_{m-1}(\xi,\tau) + L_{m-2}(\xi,\tau) \neq 0 \text{ for } \operatorname{Im}\tau \leq 0\,.$$

Thus, by Theorem 5.3.3, all the zeros of $L_m(\xi,\tau) + L_{m-2}(\xi,\tau)$ are real, but clearly non-homogeneous. So, using this construction, we can obtain examples of operators for which all the characteristic roots lie completely on the imaginary axis, but for which we cannot expect the standard Sugimoto decay rate to hold, where homogeneity is relied upon.

## 5.4 Further Extensions and Applications

### 5.4.1 Inhomogeneous Equations and Application to Semi-linear Equations

So far we have looked at the homogeneous Cauchy problem. In [Sug94] the result is then used to find estimates for the inhomogeneous Cauchy problem:

$$\left.\begin{aligned}&D_t^m u+\sum_{j=1}^{m}P_j(D_x)D_t^{m-j}u+\sum_{l=0}^{m-1}\sum_{|\alpha|+r=l}c_{\alpha,r}D_x^\alpha D_t^r u=f,\quad t>0,\\&D_t^l u(x,0)=0,\quad l=0,\dots,m-1,\ x\in\mathbb{R}^n\,,\end{aligned}\right\}$$

where the right-hand side function $f\in C_0^\infty(\mathbb{R}^n)$.

With the results for the linear Cauchy problem, it is then a matter of routine to show existence and uniqueness for semilinear equations with small data,

$$\left.\begin{aligned}&D_t^m u+\sum_{j=1}^{m}P_j(D_x)D_t^{m-j}u+\sum_{l=0}^{m-1}\sum_{|\alpha|+r=l}c_{\alpha,r}D_x^\alpha D_t^r u=H(u),\quad t>0,\\&D_t^l u(x,0)=f_l(x)\in C_0^\infty(\mathbb{R}^n),\quad l=0,\dots,m-1,\ x\in\mathbb{R}^n\,.\end{aligned}\right\}$$

This was first done for nonlinear wave equations by Strichartz in [Str70b], and has been since made a standard procedure—see, for example, [Rac92], [Mat76] and [Sug94]. For full results, see [Sug94]—there is no change in our situation.

### 5.4.2 Strictly Hyperbolic Systems

Our results can now be used to find $L^p-L^q$ estimates for strictly hyperbolic systems:

Let

$$iU_t=A(D)U\,,\quad U(0)=U_0\,,$$

be an $m\times m$ first order strictly hyperbolic system of PDEs. That is, the associated system of polynomials may be written as $A(\xi)=A_1(\xi)+A_0(\xi)$ and the roots $\varphi_1(\xi),\dots,\varphi_m(\xi)$ of $\det(\varphi I-A_0(\xi))=0$ are all real and distinct away from the origin. Also, $A_0(\xi)\in S^0_{1,0}(\mathbb{R}^n)$. Denote the roots of the equation $\det(\tau I-A(\xi))=0$ (an $m^{\text{th}}$ order polynomial in $\tau$ with smooth coefficients) by $\tau_1(\xi),\dots,\tau_m(\xi)$. Now, by analogy to the case of the $m^{\text{th}}$ order equation, we can, via perturbation methods, show that for large $|\xi|$ the $\tau_k(\xi)$ behave similarly to the $\varphi_k(\xi)$, in that they are distinct, analytic and belong to $S^1_{1,0}(\mathbb{R}^n)$. For bounded $|\xi|$ we will need similar regularity assumptions on the characteristic roots $\tau_k(\xi)$ as for the equations. Furthermore, we must assume that there exists $Q \in S^0_{1,0}(\mathbb{R}^n)$ such that $|\det Q(\xi)| \geq C > 0$ so

$$Q^{-1}AQ = \operatorname{diag}(\tau_1(\xi), \dots, \tau_m(\xi)) =: T\,.$$

Then, we use the transformation $U = QV$; so

$$U_t = QV_t \implies iQV_t = A(D)QV \implies iV_t = TV\,;\; U(0) = QV(0).$$

This is now $m$ independent equations:

$$\partial_t V_k = \tau_k(D)V_k, \quad k = 1, \dots, m, \quad V_k(0) = (Q^{-1}U(0))_k$$

each of which is solved by

$$V_k(x,t) = \int e^{i(\tau_k(\xi)t + x\cdot\xi)}\widehat{V}_k(0,\xi)\,d\xi\,.$$

Now, $Q \in S^0(\mathbb{R}^n)$, so it is a bounded map $L^q \mapsto L^q$, and we can get our estimates for $V_k$ as in the case of $m^{\text{th}}$ order equations; thus

$$\|U\|_{L^q} = \|QV\|_{L^q} \leq C\|V\|_{L^q} \leq CK(t)\|V\|_{L^p} = \\ CK(t)\|Q^{-1}U\|_{L^p} \leq CK(t)\|U\|_{L^p}\,,$$

where $K(t)$ is as in Theorem 2.1.1.

### 5.4.3 Fokker–Planck Equation

In [VR04], there are examples from that arise from the Fokker–Planck equation, $\nabla_{\mathbf{c}} \cdot (\mathbf{c} + \nabla_{\mathbf{c}})f = S(f)$, where $S(f) = \nabla_{\mathbf{c}} \cdot (\mathbf{c} + \nabla_{\mathbf{c}})f$, which give systems where $\operatorname{Im}\tau_j(\xi) \geq 0$ for all $\xi \neq 0$.

Indeed, using standard techniques they reduce the problem to studying the system

$$P(\tau,\xi) \equiv \det(\tau I + \sum_j A_j\xi_j - iB) = 0$$

$$P(\tau,0) = \det(\tau I - iB) = \tau\prod_{j=1}^{N}(\tau - ji)^{\gamma_j}\,,$$

and give conditions and examples for which each of the roots $\operatorname{Im}\tau_j(\xi) \geq 0$ for all $\xi \neq 0$; when the polynomial is stable, i.e. $\operatorname{Im}\tau_j(\xi) \geq 0$ and $\operatorname{Im}\tau_j(\xi) = 0$ implies $\xi = 0$, then we simply must calculate the order with which the root (there is only one such root) meets the axis. This is the multi-index $\alpha$ with smallest $|\alpha|$ such that $\partial_\xi^\alpha P(0,0) \neq 0$. So, by Theorem 2.1.1,

$$\|u(\cdot,t)\|_{L^q} \leq C(1+t)^{-\frac{n}{|\alpha|}(\frac{1}{p}-\frac{1}{q})} \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p-l}} ,$$

where $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q} = 1$, and $N_p$ is some constant which can be calculated. In certain cases—see the paper cited above—we can ensure $|\alpha| = 2$, in which case we have the same decay as for dissipative wave equation.

# Part II

# Second Order Equations with Time Dependent Coefficients

# Chapter 6: Introduction and Main Results

## 6.1 Introduction

In this part, we give a result for second order strictly hyperbolic linear operators with time-dependent coefficients; this work formed a joint paper, [RS05], between the author and Professor Reissig, of TU Bergakademie, Freiberg.

The influence of a time-dependent coefficient on decay estimates for such operators was studied in a series of papers [RY99], [RY00a], [RY00b]. A classification for decay estimates of solutions to the Cauchy problem

$$\left.\begin{aligned} &\partial_t^2 u(t,x) - b(t)^2\lambda(t)^2\Delta_x u(t,x) = 0,\ (t,x)\in[0,\infty)\times\mathbb{R}^n,\\ &u(0,x)=\varphi(x),\ \partial_t u(0,x)=\psi(x),\ x\in\mathbb{R}^n,\ \varphi,\psi\in C_0^\infty, \end{aligned}\right\} \tag{6.1}$$

is given, where $b(t)$ is a bounded function and $\lambda(t)$ is a strictly increasing function which satisfy, for some positive constants $C_0, C_1, C, c, c_k$ ($k = 0,1,2,\dots$),

$$0 < C_0 \le b(t)^2 \le C_1\ ,\ \text{for large } t\,;\quad \Lambda(t) := \int_0^t \lambda(s)\,ds \to \infty \ \text{ as }\ t\to\infty\,;$$
$$c(\log\Lambda(t))^{-c} \le c_0\frac{\lambda(t)}{\Lambda(t)} \le \frac{\lambda'(t)}{\lambda(t)} \le c_1\frac{\lambda(t)}{\Lambda(t)} \le C(\log\Lambda(t))^C\ ,\ \text{ for large } t\,;$$
$$|D_t^k\lambda(t)| \le c_k\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^k,\ \text{ for } k=2,3,\dots \text{ and } t \text{ large}\,.$$

This classification is based on the interplay between $b(t)$ and $\lambda(t)$, that is the so-called *speed of oscillations*; more precisely, the condition

$$|D_t^k b(t)| \le C_{b,k}\left(\frac{\lambda(t)}{\Lambda(t)}(\log\Lambda(t))^\beta\right)^k,\ \text{for all } k\in\mathbb{N} \text{ and large } t\,, \tag{6.2}$$

plays the fundamental rôle: if (6.2) holds for $\beta \in [0,1]$ then estimates of the form

$$\|u_t(t,\cdot)\|_{L^q} + \|\lambda(t)\nabla_x u(t,\cdot)\|_{L^q} \leq C(1+\Lambda(t))^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\beta_0}(\|\varphi\|_{W_p^{L+1}} + \|\psi\|_{W_p^L}) \quad (6.3)$$

hold for the solution $u = u(t,x)$ to (6.1) for some constant $\beta_0$ which depends on $\beta$; here $L = n\left(\frac{1}{p} - \frac{1}{q}\right) + 1$, $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore, if (6.2) does not hold for $\beta = 1$ then no such estimate can be found: a counterexample is constructed in [RY99].

In [RY00c] the Cauchy problem for general second order strictly hyperbolic operators with increasing time-dependent coefficients is studied. That is, the problem

$$\partial_t^2 u(t,x) + \sum_{i=1}^n b_i(t)\partial_{x_i t}^2 u(t,x) - \sum_{i,j=1}^n a_{ij}(t)\partial_{x_i x_j}^2 u(t,x) = 0\,,$$
$$u(0,x) = \varphi(x)\,,\ u_t(0,x) = \psi(x)\,,$$

where the quadratic form $\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j$ satisfies

$$d_0\nu(t)^2|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j \leq d_1\nu(t)^2|\xi|^2\,,$$

for some positive function $\nu \in C^\infty(0,\infty)$ and positive constants $d_0, d_1$. Also, the following conditions, which are analogous to those above for the operator in (6.1), are assumed to hold:

$$c_0\frac{\nu(t)}{N(t)} \leq \frac{\nu'(t)}{\nu(t)} \leq c_1\frac{\nu(t)}{N(t)}$$
$$\text{and}\quad |D_t^k\nu(t)| \leq c_k\left(\frac{\nu(t)}{N(t)}\right)^k \nu(t) \ \text{ for } k = 2,3,\dots \text{ and } t \text{ large,}$$
$$\text{where}\quad N(t) := \int_0^t \nu(s)\,ds \to \infty \quad \text{as} \quad t \to \infty\,.$$

For this problem, only the case which corresponds to that of $\beta = 0$ in (6.2) is studied; that is, if the following conditions are assumed for the coefficients for $\xi \in \mathbb{R}^n$, large $t$, and each $k = 0, 1, 2, \ldots$:

$$\Big|D_t^k \sum_{i=1}^{n} b_i(t)\xi_i\Big| \le C_k \nu(t) \left(\frac{\nu(t)}{N(t)}\right)^k |\xi|\,,$$
$$\Big|D_t^k \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j\Big| \le C_k \nu(t)^2 \left(\frac{\nu(t)}{N(t)}\right)^k |\xi|^2\,,$$

$$\text{stabilization conditions:} \quad \lim_{t\to\infty} \frac{b_i(t)}{\nu(t)}, \quad \lim_{t\to\infty} \frac{a_{ij}(t)}{\nu(t)^2} \quad \text{exist}\,,$$

then an estimate of the form (6.3) with a, in general, nonnegative $\beta_0$ holds. However, in contrast to the problem (6.1), no classification involving the "log-effect" (i.e. an analogue to the condition (6.2) for $\beta \in (0, 1)$) is currently known. A detailed discussion of all these results with proofs can be found in [RY].

In this part, we study the limiting case of (6.1) where $\lambda(t) \equiv 1$, which is not covered by the above results. For this limiting case we will give a more precise classification of oscillations and describe the corresponding more precise classification of decay estimates. Indeed, we consider the following Cauchy problem for $u = u(t, x)$:

$$\left.\begin{aligned} &\partial_t^2 u - a(t)\Delta u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ &u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \quad \varphi, \psi \in C_0^\infty(\mathbb{R}^n), x \in \mathbb{R}^n\,, \end{aligned}\right\} \tag{6.4}$$

where $a = a(t)$ is a bounded, smooth function which satisfies $a(t) \ge C > 0$ for all $t \ge 0$, so the equation from (6.4) is strictly hyperbolic.

**Definition 6.1.** Classification of Oscillations: *Let $a = a(t)$ be a smooth function satisfying*

$$|D_t^k a(t)| \;\le\; C_k \Big(\frac{1}{t+e^3}\,\Big(\log(t+e^3)\Big)^\gamma\Big)^k, \quad k \in \mathbb{N}\,. \tag{6.5}$$

*The parameter $\gamma$ controls the oscillations of $a$. We say that the oscillations of $a$ are* **very slow, slow** *or* **fast** *if $\gamma = 0,\ 0 < \gamma < 1$ or $\gamma = 1$ respectively. If* (6.5) *is not satisfied for $\gamma = 1$, then we say $a$ has* **very fast oscillations**.

We show that if we have very slow, slow or fast oscillations, then $L^p - L^q$

decay estimates can be proved for the solutions of (6.4):

**Theorem 6.1.1.** *Consider the strictly hyperbolic Cauchy problem* (6.4) *where the coefficient* $a = a(t)$ *satisfies* (6.5) *with* $\gamma \in [0, 1]$. *Then there exists a constant* $C$ *such that the following* $L^p - L^q$ *estimate holds for the solution* $u = u(t, x)$:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \le C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+s_0} \|(\nabla_x \varphi, \psi)\|_{W_p^{N_p}},$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \le 2$, $N_p \ge n\left(\frac{1}{p} - \frac{1}{q}\right)$ *and*

- $s_0 = 0$ *if* $\gamma = 0$; *in this case* $C$ *only depends on* $p, n$;
- $s_0 = \varepsilon$ *if* $\gamma \in (0, 1)$ *for all* $\varepsilon > 0$; *in this case* $C$ *depends on* $p, n$ *and* $\varepsilon$;
- $s_0$ *is a fixed constant* (*which can be determined*) *if* $\gamma = 1$; *in this case* $C$ *is independent of* $\varphi, \psi$.

**Example 6.1.1:** Let us consider the Cauchy problem

$$\left.\begin{aligned} &\partial_t^2 u - (2 + \sin(2\pi(\log(t+e^3))^\alpha))\Delta u = 0\ , \\ &u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x)\ . \end{aligned}\right\} \tag{6.6}$$

The coefficient is smooth and bounded. The oscillations are very slow, slow, fast if $\alpha = 1$, $\alpha \in (1, 2)$, $\alpha = 2$, respectively. Consequently, Theorem 6.1.1 can be applied to (6.6).

If $\alpha > 2$, then the oscillations in (6.6) are very fast. We show that, from the point of view of $L^p - L^q$ decay estimates, the behaviour of solutions for (6.6) changes in a rigorous way from $\alpha = 2$ to $\alpha > 2$:

**Theorem 6.1.2.** *Let us consider the Cauchy problem*

$$\left.\begin{aligned} &\partial_t^2 u - (2 + \sin(2\pi(\log(t+e^3))^\alpha))^2\Delta u = 0\ , \\ &u(t_0,x) = \varphi(x),\ \partial_t u(t_0,x) = \psi(x)\ , \end{aligned}\right\} \tag{6.7}$$

*with* $\alpha > 2$. *There do not exist constants* $p, q, M, C_1, C_2$ *such that, for all initial times* $t_0$ *and for all initial data* $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$, *the following* $L^p - L^q$

*estimate holds for all* $t \geq t_0$:

$$E(u)(t)\big|_{L^q} \leq C_1 \ \exp(C_2(\log(t+e^3))^r)E(u)(t_0)\big|_{W_p^M} \ , \qquad (6.8)$$

*where* $r < \alpha - 1$. *Here the (non-standard) energy* $E(u)(t)\big|_{W_p^M}$ *is defined by*

$$E(u)(t)\big|_{W_p^M} := \|\sigma(t)\nabla_x u(t,\cdot)\|_{W_p^M} + \left\|\frac{1}{\sigma(t)^2}\partial_t(u(t,\cdot)\sigma(t))\right\|_{W_p^M}$$

*with* $\sigma(t) := \sqrt{\frac{\alpha(\log(t+e^3))^{\alpha-1}}{t+e^3}}$ .

**Remark 6.1.1:** The heart of the proof of Theorem 6.1.2 is the use of Floquet's theory which is applied to, amongst other things, Hill's equation $w_{tt} + \lambda b(t)^2 w = 0$ (see [MW66]). The function $b = b(t)$ is periodic; $\lambda$ is a constant. In the proof we show that there is a relation between the equation from (6.7) and Hill's equation. For this reason we use the square in the coefficient of (6.7).

# Chapter 7: Proof of Theorem 6.1.1

In order to prove this, we first derive a WKB representation for the solution to the auxiliary problem

$$\partial_t^2 v + a(t)|\xi|^2 v = 0,\ v(0,\xi) = \widehat{\varphi}(\xi),\ \partial_t v(0,\xi) = \widehat{\psi}(\xi), \tag{7.1}$$

which is obtained from (6.4) by partial Fourier transformation with respect to $x$. Then we use standard techniques from the theory of Fourier multipliers to obtain $L^p - L^q$ estimates.

## 7.1 WKB Representation of Solution

### 7.1.1 Division of phase space into zones

To find a WKB representation for the solution of (7.1) we divide the phase space $[0,\infty)\times\mathbb{R}^n_\xi$ into two zones, the *hyperbolic zone* and the *pseudodifferential zone*, denoted $Z_{\rm hyp}(N), Z_{\rm pd}(N)$ respectively. These enable us to use the hyperbolicity of our starting problem (6.4) and tools from hyperbolic theory only in the hyperbolic zone.

**Definition 7.1.** Several Zones*: For a given* $N > 0$*, define the zones* $Z_{\rm hyp}(N)$ *and* $Z_{\rm pd}(N)$ *of the phase space* $[0,\infty)\times\mathbb{R}^n$ *by*

$$Z_{\rm hyp}(N) := \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : |\xi|(t+e^3) \ge N(\log(t+e^3))^\gamma\},$$
$$Z_{\rm pd}(N) := \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : |\xi|(t+e^3) \le N(\log(t+e^3))^\gamma\}.$$

*Here* $\gamma$ *is the parameter from* (6.5)*.*

We shall denote the line that separates these zones by $t_\xi = t(|\xi|)$ which is defined for $\{\xi : |\xi| \le p_0\}$, $p_0 := Ne^{-3}3^\gamma$, implicitly by the formula

$$|\xi|(t_\xi + e^3) = N(\log(t_\xi + e^3))^\gamma.$$

**Lemma 7.1.1.** *For $t_\xi$ as defined above we have, for all multi-indices $\alpha$ with $|\alpha| \geq 1$, the inequality*

$$|\partial_\xi^\alpha t_\xi| \leq C_{\alpha,N}|\xi|^{-1-|\alpha|}(\log(t_\xi + e^3))^\gamma.$$

We also subdivide $Z_{\text{hyp}}(N)$ into two smaller zones, the *oscillations subzone* $Z_{\text{osc}}(N)$ and the *regular subzone* $Z_{\text{reg}}(N)$.

**Definition 7.2.** Several Subzones: *For a given $N > 0$ define the subzones $Z_{\text{osc}}(N)$ and $Z_{\text{reg}}(N)$ of $Z_{\text{hyp}}(N)$ by*

$$Z_{\text{osc}}(N) := \{(t,\xi) : N(\log(t+e^3))^\gamma \leq |\xi|(t+e^3) \leq 2N(\log(t+e^3))^{2\gamma}\}\,,$$
$$Z_{\text{reg}}(N) := \{(t,\xi) : |\xi|(t+e^3) \geq 2N(\log(t+e^3))^{2\gamma}\}\,.$$

We denote the separating line by $\tilde{t}_\xi = \tilde{t}(|\xi|)$ which is defined for $\{\xi : |\xi| \leq p_1\}$, $p_1 := 2Ne^{-3}3^{2\gamma}$, implicitly by the formula

$$|\xi|(\tilde{t}_\xi + e^3) = 2N(\log(\tilde{t}_\xi + e^3))^{2\gamma}.$$

**Lemma 7.1.2.** *For $\tilde{t}_\xi$ as defined above we have, for all multi-indices $\alpha$ with $|\alpha| \geq 1$, the inequalities*

$$|\partial_\xi^\alpha \tilde{t}_\xi| \leq C_{\alpha,N}|\xi|^{-1-|\alpha|}(\log(\tilde{t}_\xi + e^3))^{2\gamma}.$$

### 7.1.2 Representation in the pseudodifferential zone

In $Z_{\text{pd}}(N)$ it is straightforward to get a representation for the solution; observe that (7.1) can be written as the first order system

$$D_t U = \begin{pmatrix} 0 & |\xi| \\ a(t)|\xi| & 0 \end{pmatrix} U, \quad U(0,\xi) = U_0(\xi) := \begin{pmatrix} |\xi|\widehat{\varphi}(\xi) \\ \widehat{\psi}(\xi) \end{pmatrix},$$

where $D_t = \frac{1}{\sqrt{-1}}\partial_t$ and

$$U = U(t,\xi) := \begin{pmatrix} |\xi|v(t,\xi) \\ D_t v(t,\xi) \end{pmatrix}.$$

Hence, we can write $U(t,\xi) = E(t,0,\xi)U_0(\xi)$ where $E = E(t,s,\xi)$, $0 \le s \le t$, solves

$$D_t E = \begin{pmatrix} 0 & |\xi| \\ a(t)|\xi| & 0 \end{pmatrix} E, \quad E(s,s,\xi) = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Naturally, this can be written as an infinite sum via the matrizant representation:

$$E(t,s,\xi) = I + \sum_{j=1}^{\infty} \int_s^t A(t_1,\xi) \int_s^{t_1} A(t_2,\xi) \ldots \int_s^{t_{j-1}} A(t_j,\xi)\, dt_j \ldots dt_1\,, \tag{7.2}$$

where $A(t,\xi) = \sqrt{-1}\left(\begin{smallmatrix} 0 & |\xi| \\ a(t)|\xi| & 0 \end{smallmatrix}\right)$. Observing that $\|A(t,\xi)\| \le Ca(t)|\xi|$, we see

$$\|E(t,s,\xi)\| \le \exp\Big(\int_s^t \|A(r,\xi)\|\, dr\Big) \le e^{C_a N (\log(t+e^3))^\gamma},$$

with $C_a := C \sup_t a(t)$, for $0 \le s \le t \le t_\xi$. Later we need, for special representations of solutions to (7.1) in subzones of the phase space, the behaviour of $\|\partial_\xi^\alpha E(t_\xi,0,\xi)\|$.

**Lemma 7.1.3.** *The following estimates hold for all multi-indices* $\alpha$:

$$\|\partial_\xi^\alpha E(t_\xi,0,\xi)\| \le C_{\alpha,N} |\xi|^{-|\alpha|} (\log(t_\xi + e^3))^{|\alpha|\gamma} e^{C_a N (\log(t_\xi+e^3))^\gamma}.$$

*Proof.* The proof follows from the representation (7.2) and from the statement of Lemma 7.1.1. □

Summarising all the above information we have

**Proposition 7.1.4.** *For* $0 \le t \le t_\xi$ *the following representation holds*:

$$|\xi| v(t,\xi) = E_{11}(t,0,\xi)|\xi|\widehat{\varphi}(\xi) + E_{12}(t,0,\xi)\widehat{\psi}(\xi),$$
$$D_t v(t,\xi) = E_{21}(t,0,\xi)|\xi|\widehat{\varphi}(\xi) + E_{22}(t,0,\xi)\widehat{\psi}(\xi),$$

*and*

$$|\partial_\xi^\alpha E_{kl}(t_\xi,0,\xi)| \le C_{\alpha,N} |\xi|^{-|\alpha|} (\log(t_\xi + e^3))^{|\alpha|\gamma} e^{C_a N (\log(t_\xi+e^3))^\gamma},$$

*for each multi-index* $\alpha$, *for all* $0 \le t \le t_\xi$ *and for all* $k,l = 1,2$.

### 7.1.3 Symbol classes in the hyperbolic zone

The hyperbolic zone $Z_{\rm hyp}(N)$ consists of two parts $Z^{(1)}_{\rm hyp}(N) := \{(t,\xi) \in [t_\xi,\infty) \times \{\xi : |\xi| \le p_0\}\}$ and $Z^{(2)}_{\rm hyp}(N) := \{(t,\xi) \in [0,\infty) \times \{\xi : |\xi| \ge p_0\}\}$. In what follows, we shall restrict our considerations to $Z^{(1)}_{\rm hyp}(N)$. In order to give a representation for the solution to (7.1) in $Z_{\rm hyp}(N)$, we carry out a diagonalisation procedure with suitable remainder at each step. The following definition of symbol classes exactly characterises the necessary properties of the remainders.

**Definition 7.3.** *For each $m_1 \in \mathbb{R}$ and $m_2, N \ge 0$ we define $S_N\{m_1,m_2\}$ to be the set of functions $\sigma = \sigma(t,\xi) \in C^\infty(Z^{(1)}_{\rm hyp}(N))$ such that, for all $(t,\xi) \in Z^{(1)}_{\rm hyp}(N)$, multi-indices $\alpha$ and $k \in \mathbb{N}$*

$$|D_t^k D_\xi^\alpha \sigma(t,\xi)| \le C_{k,\alpha} |\xi|^{m_1-|\alpha|} \left(\tfrac{1}{t+e^3}\big(\log(t+e^3)\big)^\gamma\right)^{m_2+k},$$

*with nonnegative constants $C_{k,\alpha}$ depending only on $k$ and $\alpha$.*

**Lemma 7.1.5.** *The classes $S_N\{m_1,m_2\}$ have the following properties:*

(i) *if $\sigma \in S_N\{m_1,m_2\}$ then*

$$D_\xi^\alpha \sigma \in S_N\{m_1-|\alpha|, m_2\},$$
$$\text{and } D_t^k \sigma \in S_N\{m_1, m_2+k\};$$

(ii) *if $\sigma_1 \in S_N\{m_1,m_2\}, \sigma_2 \in S_N\{p_1,p_2\}$ then $\sigma_1\sigma_2 \in S_N\{m_1+p_1, m_2+p_2\}$;*

(iii) *for all $r \ge 0$ we have $S_N\{m_1,m_2\} \subset S_N\{m_1+r, m_2-r\}$.*

*Proof.* Properties (i) and (ii) are clear by the definition of $S_N\{m_1,m_2\}$. To show (iii), simply observe that, by the definition of $Z_{\rm hyp}(N)$,

$$\frac{(\log(t+e^3))^\gamma}{|\xi|(t+e^3)} \le \frac{1}{N}.$$

Hence, if $\sigma \in S_N\{m_1, m_2\}$ then

$$\begin{aligned}
&|D_t^k D_\xi^\alpha \sigma(t,\xi)| \\
&\qquad \le C_{k,\alpha}|\xi|^{m_1+r-|\alpha|}\left(\frac{1}{t+e^3}(\log(t+e^3))^\gamma\right)^{m_2-r+k}\frac{(\log(t+e^3))^{\gamma r}}{(|\xi|(t+e^3))^r} \\
&\qquad \le C_{k,\alpha}N^{-r}|\xi|^{m_1+r-|\alpha|}\left(\frac{1}{t+e^3}(\log(t+e^3))^\gamma\right)^{m_2-r+k},
\end{aligned}$$

so $\sigma \in S_N\{m_1+r, m_2-r\}$. □

### 7.1.4 Diagonalisation modulo $S_N\{0,1\}$

The equation from (7.1) is equivalent to the first order system

$$D_t V = \begin{pmatrix} 0 & \sqrt{a(t)}|\xi| \\ \sqrt{a(t)}|\xi| & 0 \end{pmatrix} V + \frac{D_t a(t)}{2a(t)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{7.3}$$

where

$$V = V(t,\xi) = \begin{pmatrix} \sqrt{a(t)}|\xi| v(t,\xi) \\ D_t v(t,\xi) \end{pmatrix} \quad \text{for } t \ge t_\xi.$$

Note that the leading matrix is in $S_N\{1,0\}$, and the remainder lies in $S_N\{0,1\}$. In order to have a useful representation for the solution to (7.1) in $Z^{(1)}_{\text{hyp}}(N)$, we must diagonalise this system. Since the eigenvalues of the first matrix are $\tau_1 = \tau_1(\xi) = -\sqrt{a(t)}|\xi|$ and $\tau_2 = \tau_2(\xi) = \sqrt{a(t)}|\xi|$, it is simple to show that

$$M\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}M^{-1} = \begin{pmatrix} 0 & \sqrt{a(t)}|\xi| \\ \sqrt{a(t)}|\xi| & 0 \end{pmatrix},$$

where

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Setting $V_0 = V_0(t,\xi) := M^{-1}V(t,\xi)$, we obtain the following system for $V_0$:

$$D_t V_0 = \begin{pmatrix} \tau_1(\xi) + \frac{D_t a(t)}{4a(t)} & 0 \\ 0 & \tau_2(\xi) + \frac{D_t a(t)}{4a(t)} \end{pmatrix} V_0 + \frac{D_t a(t)}{4a(t)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_0. \tag{7.4}$$

We shall use the notation:

$$\mathcal{D} := \begin{pmatrix} \tau_1(\xi) + \frac{D_t a(t)}{4a(t)} & 0 \\ 0 & \tau_2(\xi) + \frac{D_t a(t)}{4a(t)} \end{pmatrix}, \quad \mathcal{R}_0 := \frac{D_t a(t)}{4a(t)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.5}$$

The system (7.4) has diagonal leading part $\mathcal{D} \in S_N\{1,0\}$ with remainder $\mathcal{R}_0 \in S_N\{0,1\}$. Thus, we have obtained in $Z_{\rm hyp}^{(1)}(N)$ the *diagonalisation of the system* (7.3) *modulo remainder* $\mathcal{R}_0 \in S_N\{0,1\}$.

### 7.1.5 Further considerations in the oscillations subzone: Diagonalisation modulo $S_N\{-1,2\}$

The oscillations subzone $Z_{\rm osc}(N)$ consists of two parts,

$$Z_{\rm osc}^{(1)}(N) := \{(t,\xi) \in [t_\xi, \tilde{t}_\xi] \times \{\xi : |\xi| \le p_0\}\}$$
$$\text{and } Z_{\rm osc}^{(2)}(N) := \{(t,\xi) \in [0, \tilde{t}_\xi] \times \{\xi : p_0 \le |\xi| \le p_1\}\}.$$

In what follows, we restrict our considerations to $Z_{\rm osc}^{(1)}(N)$ if we have in mind the oscillations subzone. In $Z_{\rm hyp}^{(1)}(N)$ we carry out one more step of the diagonalisation procedure. Let

$$\mathcal{N}^{(1)} = \mathcal{N}^{(1)}(t,\xi) := -\frac{D_t a(t)}{4a(t)} \begin{pmatrix} 0 & \frac{1}{\tau_1 - \tau_2} \\ \frac{1}{\tau_2 - \tau_1} & 0 \end{pmatrix} \in S_N\{-1,1\}.$$

Now set $N_1 = N_1(t,\xi) := I + \mathcal{N}^{(1)}$; this is invertible since

$$\|\mathcal{N}^{(1)}\| \le C \frac{(\log(t+e^3))^\gamma}{|\xi|(t+e^3)} \le C/N \qquad \text{by definition of } Z_{\rm hyp}(N),$$

and we choose $N$ in the definition of $Z_{\rm hyp}(N)$, $Z_{\rm pd}(N)$ large enough so that $C/N < 1/2$ here. Let $V_1 = V_1(t,\xi) := N_1^{-1} M^{-1} V$; then we obtain the following equivalent problem to (7.3) for $V_1$ for $t \ge t_\xi$:

$$(D_t - \mathcal{D} - \mathcal{R}_1)V_1 = 0, \quad V_1(t_\xi, \xi) = V_{1,0}(\xi) := N_1^{-1}(t_\xi,\xi) M^{-1} V(t_\xi, \xi), \tag{7.6}$$

where $\mathcal{R}_1 \in S_N\{-1,2\}$. This is a consequence of

**Lemma 7.1.6.** *Let $\mathcal{R}_1 := -N_1^{-1}(D_t\mathcal{N}^{(1)} - \mathcal{R}_0\mathcal{N}^{(1)})$. Then in $Z_{\rm hyp}^{(1)}(N)$ the* *following identity holds:*

$$(D_t - \mathcal{D} - \mathcal{R}_0)N_1 = N_1(D_t - \mathcal{D} - \mathcal{R}_1),$$

*where* $\mathcal{D}, \mathcal{R}_0$ *are as in* (7.5).

*Proof.* Follows immediately from the observation that $[\mathcal{N}^{(1)}, \mathcal{D}] = \mathcal{R}_0$. □

Thus, we have obtained in $Z^{(1)}_{\text{hyp}}(N)$ the *diagonalisation of the system* (7.3) *modulo remainder* $\mathcal{R}_1 \in S_N\{-1, 2\}$.

### 7.1.6 Further considerations in regular subzone: Diagonalisation modulo $S_N\{-m, m+1\}$

In $Z^{(1)}_{\text{osc}}(N)$ there is no point in carrying out any more steps of diagonalisation since there is no useful improvement of regularity between amplitudes from the classes $S_N\{-(l-1), l\}$ and $S_N\{-l, l+1\}$ when $l \geq 2$. However, we get such an improvement of regularity in the regular subzone $Z_{\text{reg}}(N)$. There we carry out $m$ steps of the diagonalisation procedure. The number $m$ is chosen and motivated in Section 7.2. The step depends on the next result, which generalises Lemma 7.1.6.

**Proposition 7.1.7.** *For each* $m \in \mathbb{N}$ *there exist matrix-valued functions*

$$N_m = N_m(t,\xi) \in S_N\{0,0\}, \quad F_m = F_m(t,\xi) \in S_N\{-1,2\}\,,$$
$$\text{and } \mathcal{R}_m = \mathcal{R}_m(t,\xi) \in S_N\{-m, m+1\}$$

*such that* $F_m$ *is diagonal and the following identity holds in* $Z^{(1)}_{\text{hyp}}(N)$:

$$(D_t - \mathcal{D} - \mathcal{R}_0)N_m = N_m(D_t - \mathcal{D} + F_m - \mathcal{R}_m),$$

*where* $\mathcal{D}$, $\mathcal{R}_0$ *are as in* (7.5). *Also, for* $N$ (*in the definition of* $Z_{\text{hyp}}(N)$) *chosen large enough,* $N_m$ *is invertible and its inverse also lies in* $S_N\{0,0\}$.

*Proof.* We seek representations for $N_m$, $F_m$ in the form

$$N_m = \sum_{r=0}^{m} \mathcal{N}^{(r)}, \quad F_m = \sum_{r=0}^{m-1} \mathcal{F}^{(r)}\,.$$

To do this, define inductively the matrix-valued functions $B^{(r)} = \begin{pmatrix} B^{(r)}_{11} & B^{(r)}_{12} \\ B^{(r)}_{21} & B^{(r)}_{22} \end{pmatrix}$, $\mathcal{N}^{(r)}$ and $\mathcal{F}^{(r)}$ in the manner below:

$$\mathcal{N}^{(0)} := I,\ B^{(0)} := \mathcal{R}_0\,,\ \mathcal{F}^{(r)} := \operatorname{diag} B^{(r)},\ \mathcal{N}^{(r+1)} := \begin{pmatrix} 0 & \frac{B^{(r)}_{12}}{\tau_1-\tau_2} \\ \frac{B^{(r)}_{21}}{\tau_2-\tau_1} & 0 \end{pmatrix},$$

$$B^{(r+1)} := (D_1 - \mathcal{D} - \mathcal{R}_0)\Big(\sum_{\rho=0}^{r+1}\mathcal{N}^{(\rho)}\Big) - \Big(\sum_{\rho=0}^{r+1}\mathcal{N}^{(\rho)}\Big)\Big(D_t - \mathcal{D} + \sum_{\rho=0}^{r}\mathcal{F}^{(\rho)}\Big).$$

We claim that $\mathcal{N}^{(r)} \in S_N\{-r, r\}$ and $B^{(r)} \in S_N\{-r, r+1\}$. For $r = 1$ this is clear from Lemma 7.1.6. Assume it holds for $r = k$; then, observing that $\tau_2 - \tau_1 = 2\sqrt{a(t)}|\xi| \in S_N\{1, 0\}$ and noting by the induction hypothesis that $B^{(k)}_{12}, B^{(k)}_{21} \in S_N\{-k, k+1\}$, we see that $\mathcal{N}^{(k+1)} \in S_N\{-(k+1), k+1\}$. Also,

$$\begin{aligned} B^{(k+1)} =& B^{(k)} + (D_t - D - \mathcal{R}_0)\mathcal{N}^{(k+1)} - \sum_{\rho=0}^{k}\mathcal{N}^{(\rho)}\mathcal{F}^{(k)} \\ & - \mathcal{N}^{(k+1)}\Big(D_t - \mathcal{D} + \sum_{\rho=0}^{k}\mathcal{F}^{(\rho)}\Big) = B^{(k)} - \mathcal{F}^{(k)} - [\mathcal{D}, \mathcal{N}^{(k+1)}] + \mathcal{S}, \end{aligned}$$

where $\mathcal{S} = D_t\mathcal{N}^{(k+1)} - \mathcal{R}_0\mathcal{N}^{(k+1)} - \mathcal{N}^{(k+1)}\sum_{\rho=0}^{k}\mathcal{F}^{(\rho)} - \sum_{\rho=1}^{k}\mathcal{N}^{(\rho)}\mathcal{F}^{(k)}$, which lies in $S_N\{-(k+1), k+2\}$ by the induction hypothesis and the rules of the symbolic calculus of Lemma 7.1.5. Furthermore, by definition of $\mathcal{F}^{(k)}$ and $\mathcal{N}^{(k+1)}$,

$$B^{(k)} - \mathcal{F}^{(k)} - [\mathcal{D}, \mathcal{N}^{(k+1)}] = 0\,.$$

Therefore, $B^{(k+1)} = \mathcal{S}$ proving the induction step. So the claim is proved.

Now we claim that $N_m := \sum_{r=0}^{m}\mathcal{N}^{(r)}$ is invertible; this is true because

$$\|\mathcal{N}^{(r)}\| \le C_r\bigg(\frac{\big(\log(t+e^3)\big)^{\gamma}}{(t+e^3)|\xi|}\bigg)^{r} \le \frac{C_r}{N^r},$$

by the definition of $Z_{\text{hyp}}(N)$. Choose $N$ in the definition of $Z_{\text{hyp}}(N)$ so that

$$\frac{C_r}{N^r} \le \frac{1}{2^{r+1}} \text{ for } r = 1, \ldots, m.$$

The value of $m$ shall be chosen later, but since it is fixed, this fixes $N$.

Hence,

$$\|N_m - I\| \le \sum_{r=1}^{m} \|\mathcal{N}^{(r)}\| \le \sum_{r=1}^{m} \frac{1}{2^{r+1}} < \frac{1}{2},$$

thus proving the invertibility of $N_m$. Finally, noting that $\mathcal{F}^{(0)} = 0$, so $\mathcal{F}^{(m)} \in S_N\{-1, 2\}$, and setting $R_m := -N_m^{-1} B^{(m)} \in S_N\{-m, m+1\}$ completes the proof of the proposition. □

The regular subzone $Z_{\text{reg}}(N)$ consists of three parts $Z_{\text{reg}}^{(1)}(N) := \{(t, \xi) \in [\tilde{t}_\xi, \infty) \times \{\xi : |\xi| \le p_0\}\}$, $Z_{\text{reg}}^{(2)}(N) := \{(t, \xi) \in [\tilde{t}_\xi, \infty) \times \{\xi : p_0 \le |\xi| \le p_1\}\}$ and $Z_{\text{reg}}^{(3)}(N) := \{(t, \xi) \in [0, \infty) \times \{\xi : |\xi| \ge p_1\}\}$. In what follows we restrict our considerations to $Z_{\text{reg}}^{(1)}(N)$ if we have in mind the regular subzone. Now we set $V_m = V_m(t, \xi) := N_m^{-1} V_0$ for $t \ge \tilde{t}_\xi$ and see that the system (7.4) for $V_0$ is equivalent in $Z_{\text{reg}}^{(1)}(N)$ to

$$\begin{aligned} &(D_t - \mathcal{D} + F_m - \mathcal{R}_m) V_m = 0, \\ &V_m(\tilde{t}_\xi, \xi) = V_{m,0}(\xi) := N_m^{-1}(\tilde{t}_\xi, \xi) N_1(\tilde{t}_\xi, \xi) V_1(\tilde{t}_\xi, \xi). \end{aligned} \tag{7.7}$$

Thus, we have obtained in $Z_{\text{reg}}^{(1)}(N)$ the *diagonalisation of the system* (7.3) *modulo remainder* $\mathcal{R}_m \in S_N\{-m, m+1\}$, while in $Z_{\text{osc}}^{(1)}(N)$ it is sufficient to carry out the *diagonalisation of the system* (7.3) *modulo remainder* $\mathcal{R}_1 \in S_N\{-1, 2\}$.

### 7.1.7 Fundamental solutions and their properties

Now let us construct representations for the fundamental solutions to the matrix-valued operators appearing in (7.6) and (7.7). First, let $E_2 = E_2(t, s, \xi)$ solve

$$D_t E_2 - \mathcal{D} E_2 = 0, \quad E_2(s, s, \xi) = I,$$

where $t, s \ge t_\xi$ and the matrix $\mathcal{D}$ is as in (7.5). We see that

$$E_2(t, s, \xi) = \left( \frac{a(t)}{a(s)} \right)^{1/4} \begin{pmatrix} e^{i \int_s^t \tau_1(r, \xi)\, dr} & 0 \\ 0 & e^{i \int_s^t \tau_2(r, \xi)\, dr} \end{pmatrix}.$$

Hence, by the strict hyperbolicity of our starting Cauchy problem (6.4) we get

$$\|E_2(t,s,\xi)\| \le \Big(\frac{a(t)}{a(s)}\Big)^{1/4} \le C_a \text{ for all } t,s \ge t_\xi. \tag{7.8}$$

Later we need, for special representations of solutions to (7.1) in $Z^{(1)}_{\text{osc}}(N)$ and in $Z^{(1)}_{\text{reg}}(N)$, the behaviour of $\|\partial_\xi^\alpha E_2(t_\xi,0,\xi)\|$ and of $\|\partial_\xi^\alpha E_2(\tilde{t}_\xi,t_\xi,\xi)\|$.

**Lemma 7.1.8.** *The following estimates hold for all multi-indices $\alpha$:*

$$\|\partial_\xi^\alpha E_2(t_\xi,0,\xi)\| \le C_{\alpha,N}|\xi|^{-|\alpha|}(\log(t_\xi+e^3))^{|\alpha|\gamma},$$
$$\|\partial_\xi^\alpha E_2(\tilde{t}_\xi,t_\xi,\xi)\| \le C_{\alpha,N}|\xi|^{-|\alpha|}(\log(\tilde{t}_\xi+e^3))^{2|\alpha|\gamma}.$$

*Proof.* Follows immediately from Lemmas 7.1.1 and 7.1.2 together with assumption (6.5) and estimate (7.8). □

Now we define $E_{\text{osc}} = E_{\text{osc}}(t,s,\xi)$, $t_\xi \le s \le t \le \tilde{t}_\xi$ to be the fundamental solution to (7.6) in $Z^{(1)}_{\text{osc}}(N)$. This can be written in the form $E_{\text{osc}} = E_2(t,s,\xi)Q_1(t,s,\xi)$ where $Q_1 = Q_1(t,s,\xi)$ solves

$$D_tQ_1 = E_2(s,t,\xi)\mathcal{R}_1(t,\xi)E_2(t,s,\xi)Q_1, \quad Q_1(s,s,\xi) = I.$$

Letting $P_1 = P_1(t,s,\xi) := \sqrt{-1}E_2(s,t,\xi)\mathcal{R}_1(t,\xi)E_2(t,s,\xi)$, we have the matrizant representation for $Q_1$:

$$Q_1(t,s,\xi) = I + \sum_{j=1}^{\infty} \int_s^t P_1(t_1,s,\xi)\dots\int_s^{t_{j-1}} P_1(t_j,s,\xi)\,dt_j\dots dt_1\,.$$

Now $\mathcal{R}_1 \in S_N\{-1,2\}$; therefore, using (7.8), we see that

$$\|P_1(t,s,\xi)\| \le C|\xi|^{-1}\bigg(\frac{(\log(t+e^3))^\gamma}{t+e^3}\bigg)^2.$$

Hence,

$$\|Q_1(t,s,\xi)\| \le \exp\bigg(\int_s^t \|P_1(r,s,\xi)\|\,dr\bigg) \le e^{C_{N,\gamma}(\log(t_\xi+e^3))^\gamma}$$

for all $t_\xi \le s,t \le \tilde{t}_\xi$.

We need, for a special representation of solution to (7.1) in $Z^{(1)}_{\text{osc}}(N)$ and in $Z_{\mathrm{reg}}^{(1)}(N)$, the behaviour of $\|\partial_\xi^\alpha Q_1(t,t_\xi,\xi)\|$ for $t\in[t_\xi,\tilde t_\xi]$.

**Lemma 7.1.9.** *The following estimate holds for all multi-indices $\alpha$ and for all $t\in[t_\xi,\tilde t_\xi]$:*

$$\|\partial_\xi^\alpha Q_1(t,t_\xi,\xi)\| \le C_{\alpha,N}|\xi|^{-|\alpha|}(\log(t+e^3))^{2|\alpha|\gamma}e^{C_{N,\gamma}(\log(t_\xi+e^3))^\gamma}.$$

*Proof.* Follows immediately from the above representation and estimate for $Q_1=Q_1(t,s,\xi)$, and from Lemma 7.1.1 together with assumption (6.5) and estimate (7.8). □

Similarly, in $Z_{\mathrm{reg}}^{(1)}(N)$ we define $E_{\mathrm{reg}}=E_{\mathrm{reg}}(t,s,\xi)$, $\tilde t_\xi\le s,t,$ to be the fundamental solution to (7.7). We write this in the form $E_{\mathrm{reg}}(t,s,\xi)=\tilde E_2(t,s,\xi)Q_m(t,s,\xi)$, where $Q_m=Q_m(t,s,\xi)$ solves

$$D_tQ_m=\tilde E_2(s,t,\xi)\mathcal{R}_m(t,\xi)\tilde E_2(t,s,\xi)Q_m,\quad Q_m(s,s,\xi)=I.$$

Here we define for $\tilde t_\xi\le s,t,$

$$\tilde E_2(t,s,\xi) = \left(\frac{a(t)}{a(s)}\right)^{1/4}\begin{pmatrix} e^{i\int_s^t\tau_1(r,\xi)\,dr-\int_s^t f_m^{(1)}(r,\xi)\,dr} & 0\\ 0 & e^{i\int_s^t\tau_2(r,\xi)\,dr-\int_s^t f_m^{(2)}(r,\xi)\,dr}\end{pmatrix},$$

where $F_m:=\begin{pmatrix} f_m^{(1)} & 0\\ 0 & f_m^{(2)}\end{pmatrix}$. Using $F_m\in S_N\{-1,2\}$ we have $|\int_s^t f_m^{(l)}(r,\xi)\,dr|\le C_m$ for all $\tilde t_\xi\le s,t$ and for $l=1,2$. Then, observing that $\mathcal{R}_m\in S_N\{-m,m+1\}$, we see that the following estimate holds:

$$\|Q_m(t,s,\xi)\|\le C_m \text{ for all } \tilde t_\xi\le s,t.$$

We also need, for a special representation of solution to (7.1) in $Z_{\mathrm{reg}}^{(1)}(N)$, the behaviour of $\|\partial_\xi^\alpha Q_m(t,\tilde t_\xi,\xi)\|$ for $t\ge\tilde t_\xi$.

**Lemma 7.1.10.** *The following estimate holds for all multi-indices $\alpha$ with $|\alpha|\le\frac{m-1}{2}$ and for all $\tilde t_\xi\le t$:*

$$\|\partial_\xi^\alpha Q_m(t,\tilde t_\xi,\xi)\|\le C_{\alpha,m}|\xi|^{-|\alpha|}\quad \textit{for all } \tilde t_\xi\le t\,.$$

*Proof.* It is sufficient to discuss the derivatives with respect to $\xi$ of the term

$$g(t,\xi) := a(\tilde{t}_\xi)^{-\frac{1}{4}}\, e^{i\int_{\tilde{t}_\xi}^{t}\sqrt{a(r)}\,dr|\xi| - \int_{\tilde{t}_\xi}^{t} f_m^{(1)}(r,\xi)\,dr}\, r_m(t,\xi)$$

with $r_m \in S_N\{-m, m+1\}$. Such terms appear in the matrizant representation for $Q_m$. We have

$$|g(t,\xi)| \leq C_m\, |\xi|^{-m} \left( \frac{1}{t+e^3}\, (\log(t+e^3))^{\gamma} \right)^{m+1} .$$

Derivatives of $r_m$ with respect to $\xi$ generate $|\xi|^{-|\alpha|}$. By Lemma 7.1.2, assumption (6.5) and the definition of $Z_{\text{reg}}(N)$ we conclude that

$$|\partial_{\xi_1} a(\tilde{t}_\xi)^{-1/4}| \;\leq\; C\, \frac{(\log(\tilde{t}_\xi + e^3))^{3\gamma}}{|\xi|^2(\tilde{t}_\xi + e^3)} \;\leq\; C_N (\log(\tilde{t}_\xi + e^3))^{\gamma} |\xi|^{-1} .$$

In the same way one can show

$$|\partial_\xi^\alpha a(\tilde{t}_\xi)^{-1/4}| \;\leq\; C_{\alpha,N} (\log(\tilde{t}_\xi + e^3))^{|\alpha|\gamma} |\xi|^{-|\alpha|} .$$

Differentiating $\int_{\tilde{t}_\xi}^{t} f_m^{(1)}(r,\xi)\,dr$ with respect to $\xi_1$ gives

$$\int_{\tilde{t}_\xi}^{t} \partial_{\xi_1} f_m^{(1)}(r,\xi)\,dr - f_m^{(1)}(\tilde{t}_\xi,\xi)\, \frac{\partial \tilde{t}_\xi}{\partial \xi_1} .$$

The integral can be estimated by $C_m|\xi|^{-1}$. Using $f_m^{(1)} \in S_N\{-1,2\}$ gives the estimate

$$\left| f_m^{(1)}(\tilde{t}_\xi,\xi)\, \frac{\partial \tilde{t}_\xi}{\partial \xi_1} \right| \;\leq\; C_m\, \frac{(\log(\tilde{t}_\xi + e^3))^{4\gamma}}{|\xi|^3(\tilde{t}_\xi + e^3)^2} \;\leq\; C_{m,N} |\xi|^{-1} .$$

Higher derivatives of $\tilde{t}_\xi$ give rise to log terms. Thus, we get

$$\left| \partial_\xi^\alpha\, e^{-\int_{\tilde{t}_\xi}^{t} f_m^{(1)}(r,\xi)\,dr} \right| \leq C_{\alpha,N} (\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma} |\xi|^{-|\alpha|} .$$

The main problem arises from $\int_{\tilde{t}_\xi}^{t} \sqrt{a(r)}\,dr|\xi|$. Differentiation $\partial_{\xi_1}$ allows only an estimate like

$$\Big|\partial_{\xi_1}\Big(\int_{\tilde{t}_\xi}^t \sqrt{a(r)}\,dr\,|\xi|\Big)\Big| \le C_a(t+e^3)\,.$$

But now we can use that $r_m \in S_N\{-m, m+1\}$. If we differentiate $\alpha$ times, then for all $t \ge \tilde{t}_\xi$ we have

$$\begin{aligned}
|(t+e^3)^{|\alpha|} r_m(t,\xi)| &\le C_m \frac{(\log(t+e^3))^{\gamma(m+1)}}{|\xi|^m (t+e^3)^{m+1-|\alpha|}} \\
&\le \frac{C_m}{|\xi|^{|\alpha|}} \frac{(\log(t+e^3))^{\gamma(m+1)}}{|\xi|^{m-|\alpha|}(t+e^3)^{m+1-|\alpha|}} \\
&\le \frac{C_m}{|\xi|^{|\alpha|}} \frac{(\log(t+e^3))^{\gamma(m-1)}(\log(t+e^3))^{2\gamma}}{(|\xi|(t+e^3))^{m-1-|\alpha|}|\xi|(t+e^3)^2} \\
&\le \frac{C_{m,N}}{|\xi|^{|\alpha|}} \frac{(\log(t+e^3))^{2\gamma}}{|\xi|(t+e^3)^2}
\end{aligned}$$

if $|\alpha| \le \frac{m-1}{2}$. Consequently we earn $|\xi|^{-|\alpha|}$ and a term which is integrable over $[\tilde{t}_\xi, t]$ for all $t$. It remains to explain how we proceed with the terms $(\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma}$ arising in the above estimates. These terms we couple with $r_m$ also and get, for $|\alpha| \le \frac{m-1}{2}$,

$$\begin{aligned}
|(\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma} r_m(t,\xi)| &\le C_m \Big|(\log(t+e^3))^{(m-1)\gamma} \frac{(\log(t+e^3))^{(m+1)\gamma}}{|\xi|^m (t+e^3)^{m+1}}\Big| \\
&\le C_m \frac{(\log(t+e^3))^{2(m-1)\gamma}}{(|\xi|(t+e^3))^{m-1}} \frac{(\log(t+e^3))^{2\gamma}}{|\xi|(t+e^3)^2}\,.
\end{aligned}$$

Using the definition of $Z_{\text{reg}}(N)$, the first factor is uniformly bounded. The second one is integrable over $[\tilde{t}_\xi, t]$. This completes the proof of our lemma.

□

### 7.1.8 Representation of solutions in subzones

Now we are in position to give representations for the solution to (7.1) in $Z_{\text{osc}}^{(1)}$ and $Z_{\text{reg}}^{(1)}$. The vector-function $V = V(t,\xi)$ is a solution of (7.3).

*In $Z_{\mathrm{osc}}^{(1)}(N)$:* For $t_\xi \le t \le \tilde{t}_\xi$ we have

$$V(t,\xi) = MN_1(t,\xi)E_2(t,0,\xi)E_2(0,t_\xi,\xi)Q_1(t,t_\xi,\xi)N_1(t_\xi,\xi)^{-1}M^{-1} \cdot \begin{pmatrix} \sqrt{a(t_\xi)} & 0 \\ 0 & 1 \end{pmatrix} E(t_\xi,0,\xi)U_0(\xi), \quad (7.9)$$

where we recall $E(t_\xi,0,\xi)$ is obtained in the representation of the solution in the pseudodifferential zone.

*In $Z_{\mathrm{reg}}(N)$:* For $t \ge \tilde{t}_\xi$ we have

$$\begin{aligned} V(t,\xi) &= MN_m(t,\xi)E_2(t,0,\xi)E_2(0,\tilde{t}_\xi,\xi)\tilde{F}_m(t,\tilde{t}_\xi,\xi) \\ &\cdot Q_m(t,\tilde{t}_\xi,\xi)N_m(\tilde{t}_\xi,\xi)^{-1}N_1(\tilde{t}_\xi,\xi)E_2(\tilde{t}_\xi,t_\xi,\xi)Q_1(\tilde{t}_\xi,t_\xi,\xi)N_1(t_\xi,\xi)^{-1} \\ &\cdot M^{-1}\begin{pmatrix} \sqrt{a(t_\xi)} & 0 \\ 0 & 1 \end{pmatrix} E(t_\xi,0,\xi)U_0(\xi), \end{aligned}$$

with

$$\tilde{F}_m(t,\tilde{t}_\xi,\xi) = \begin{pmatrix} e^{-\int_{\tilde{t}_\xi}^t f_m^{(1)}(r,\xi)\,dr} & 0 \\ 0 & e^{-\int_{\tilde{t}_\xi}^t f_m^{(2)}(r,\xi)\,dr} \end{pmatrix},$$

where we have used the representation (7.9) at $t = \tilde{t}_\xi$.

Before we discuss the representation of the solution to (7.1) we collect together some useful estimates.

**Lemma 7.1.11.** *The following estimates hold for all multi-indices $\alpha$:*

$$\begin{aligned} \|\partial_\xi^\alpha N_1(t_\xi,\xi)\| &\le C_\alpha |\xi|^{-|\alpha|}(\log(t_\xi + e^3))^{|\alpha|\gamma}, \\ \|\partial_\xi^\alpha N_m(\tilde{t}_\xi,\xi)\| &\le C_\alpha |\xi|^{-|\alpha|}(\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma}, \\ \|\partial_\xi^\alpha \tilde{F}_m(t,\tilde{t}_\xi,\xi)\| &\le C_\alpha |\xi|^{-|\alpha|}(\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma}, \ \textit{for all } t \ge \tilde{t}_\xi. \end{aligned}$$

*Proof.* Follows from the above representation for $\tilde{F}_m = \tilde{F}_m(t,\tilde{t}_\xi,\xi)$ immediately, by Lemmas 7.1.1 and 7.1.2 together with assumption (6.5) and the definition of zones. ☐

Summarising all the above results we have

**Proposition 7.1.12.** *The following WKB representations hold for the solution to* (7.1):

$$\begin{aligned}|\xi|v(t,\xi) &= b_{11}^{(1)}(t,\xi)e^{-i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}|\xi|\widehat{\varphi}(\xi)+b_{12}^{(1)}(t,\xi)e^{-i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}\widehat{\psi}(\xi)\\ &\quad + b_{21}^{(1)}(t,\xi)e^{i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}|\xi|\widehat{\varphi}(\xi)+b_{22}^{(1)}(t,\xi)e^{i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}\widehat{\psi}(\xi),\\ D_tv(t,\xi) &= b_{11}^{(2)}(t,\xi)e^{-i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}|\xi|\widehat{\varphi}(\xi)+b_{12}^{(2)}(t,\xi)e^{-i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}\widehat{\psi}(\xi)\\ &\quad + b_{21}^{(2)}(t,\xi)e^{i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}|\xi|\widehat{\varphi}(\xi)+b_{22}^{(2)}(t,\xi)e^{i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}\widehat{\psi}(\xi).\end{aligned}$$

*Here the amplitudes* $b_{kl}^{(p)}(t,\xi)$, $p,k,l=1,2$, *satisfy the following estimates*:

- *in* $Z_{\mathrm{pd}}(N)\cup Z_{\mathrm{osc}}^{(1)}(N)$ : $|b_{kl}^{(p)}(t,\xi)|\le C_{N,a,\gamma}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}$;
- *in* $Z_{\mathrm{reg}}^{(1)}(N)$: $|\partial_\xi^\alpha b_{kl}^{(p)}(t,\xi)|\le C_{N,a,\gamma,\alpha}|\xi|^{-|\alpha|}(\log(t+e^3))^{2|\alpha|\gamma}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}$ *for all* $|\alpha|\le\frac{m-1}{2}$, *where* $m$ *is the number of steps of diagonalisation in* $Z_{\mathrm{reg}}^{(1)}(N)$.

We obtain similar representations in the other parts of the phase space. The amplitudes satisfy at worst the estimates above.

## 7.2 $L^p-L^q$ Estimates for Fourier Multipliers[1]

Using Proposition 7.1.12 we can write down the following Fourier multiplier representation for the solution $u=u(t,x)$ to (6.4):

$$\begin{aligned}&|D_x|u(t,x)\\ &\quad=\mathcal{F}^{-1}\Big(b_{11}^{(1)}(t,\xi)e^{-i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}|\xi|\widehat{\varphi}(\xi)+b_{12}^{(1)}(t,\xi)e^{-i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}\widehat{\psi}(\xi)\\ &\qquad+b_{21}^{(1)}(t,\xi)e^{i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}|\xi|\widehat{\varphi}(\xi)+b_{22}^{(1)}(t,\xi)e^{i|\xi|\int\limits_0^t\sqrt{a(s)}\,ds}\widehat{\psi}(\xi)\Big),\end{aligned}\tag{7.10}$$

[1]The results in this section were obtained by the co-author of the joint paper.

$$D_t u(t,x)$$
$$= \mathcal{F}^{-1}\Big(b_{11}^{(2)}(t,\xi)e^{-i|\xi|\int\limits_0^t \sqrt{a(s)}\,ds}|\xi|\widehat{\varphi}(\xi) + b_{12}^{(2)}(t,\xi)e^{-i|\xi|\int\limits_0^t \sqrt{a(s)}\,ds}\widehat{\psi}(\xi)$$
$$+ b_{21}^{(2)}(t,\xi)e^{i|\xi|\int\limits_0^t \sqrt{a(s)}\,ds}|\xi|\widehat{\varphi}(\xi) + b_{22}^{(2)}(t,\xi)e^{i|\xi|\int\limits_0^t \sqrt{a(s)}\,ds}\widehat{\psi}(\xi)\Big), \quad (7.11)$$

where $\mathcal{F}^{-1}$ denotes the inverse to the partial Fourier transform with respect to $x$. Here the amplitudes $b_{kl}^{(p)}(t,\xi)$, $p,k,l = 1,2$, satisfy the following estimates:

- in $Z_{\text{pd}}(N) \cup Z_{\text{osc}}(N)$ : $|b_{kl}^{(p)}(t,\xi)| \le C_{N,a,\gamma} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}$;
- in $Z_{\text{reg}}(N)$: $|\partial_\xi^\alpha b_{kl}^{(p)}(t,\xi)| \le C_{N,a,\gamma,\alpha}|\xi|^{-|\alpha|}(\log(t+e^3))^{2|\alpha|\gamma}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}$ for all $|\alpha| \le \frac{m-1}{2}$, where $m$ is the number of steps of diagonalisation in $Z_{\text{reg}}^{(1)}(N)$.

Our next goal is to estimate these Fourier multipliers. We use the approach from [RY] and from [RY00b].

#### 7.2.1 $L^p - L^q$ estimates for Fourier multipliers with amplitudes vanishing in regular subzone

Let us choose a function $\psi \in C^\infty(\mathbb{R}^n)$ satisfying $\psi(\xi) \equiv 0$ for $|\xi| \le 1$, $\psi(\xi) \equiv 1$ for $|\xi| \ge 2$ and $0 \le \psi(\xi) \le 1$. Further, denote $K(t) := N(\log(t+e^3))^{2\gamma}/(t+e^3)$.

**Theorem 7.2.1.** *Let us consider Fourier multipliers which are defined by*

$$\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}(1-\psi(\xi/K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big).$$

*Suppose that $a = a(t,\xi)$ satisfies the following assumption*:

- *in* $Z_{\text{pd}}(N) \cup Z_{\text{osc}}(N) : |a(t,\xi)| \le C_{N,a,\gamma}\exp(C_{N,a,\gamma}(\log(t+e^3))^\gamma)$.

*Then we have the $L^p - L^q$ estimate*

$$\big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}(1-\psi(\xi/K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\big\|_{L^q}$$
$$\le C_{N,a,\gamma}K(t)^{-2r+n\left(\frac{1}{p}-\frac{1}{q}\right)}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\|\varphi\|_{L^p},$$

*provided that* $0 \le 2r \le n(\frac{1}{p}-\frac{1}{q}),\ 1<p\le 2,\ \frac{1}{p}+\frac{1}{q}=1.$

*Proof.* Let us consider

$$I_0 := \Big\| \mathcal{F}^{-1}\Big( e^{i|\xi|\int_0^t \sqrt{a(s)}ds} (1-\psi(\xi/K(t)))|\xi|^{-2r} a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}^q .$$

Using the transformations $\xi = K(t)\eta$ and $z = K(t)x$ we conclude

$$\begin{aligned} I_0 &= K(t)^{nq-2rq-n} \\ &\Big\| \mathcal{F}^{-1}\Big( e^{i\,K(t)|\eta|\int_0^t \sqrt{a(s)}ds} (1-\psi(\eta))|\eta|^{-2r} a(t,K(t)\eta)\mathcal{F}(\varphi)(K(t)\eta)\Big)\Big\|_{L^q}^q . \end{aligned}$$

The point $(t,K(t)\eta)$ with $|\eta|\le 2$ (support of $1-\psi$) belongs to $Z_{\text{pd}}(N)\cup Z_{\text{osc}}(N)$. Therefore $|a(t,K(t)\eta)| \le C_{N,a,\gamma}\exp(C_{N,a,\gamma}(\log(t+e^3))^\gamma)$. For $I_0$ we obtain

$$\begin{aligned} I_0^{1/q} &\le \\ & K(t)^{n-2r-\frac{n}{q}} \Big\| \mathcal{F}^{-1}\Big( e^{i\,K(t)|\eta|\int_0^t \sqrt{a(s)}ds} (1-\psi(\eta))|\eta|^{-2r} a(t,K(t)\eta)\Big) \\ &\qquad * \mathcal{F}^{-1}(\mathcal{F}(\varphi)(K(t)\eta))\Big\|_{L^q}. \end{aligned}$$

Now let us denote

$$\begin{aligned} T_t &:= \mathcal{F}^{-1}\Big( e^{i\,K(t)|\eta|\int_0^t \sqrt{a(s)}ds} (1-\psi(\eta))|\eta|^{-2r} a(t,K(t)\eta)\Big) \\ &\qquad \exp(-C_{N,a,\gamma}(\log(t+e^3))^\gamma). \end{aligned}$$

We have, together with the estimate for $a(t,K(t)\eta)$,

$$\text{meas}\{\eta : |\mathcal{F}(T_t)| \ge l\} \le \text{ meas}\{\eta : |\eta| \le C\, l^{-\frac{1}{2r}}\} \le C\, l^{-\frac{n}{2r}} .$$

Due to Theorem 1.11 from [Hör60] we have $\mathcal{F}(T_t)\in M_p^q$ for all $2r \le n(\frac{1}{p}-\frac{1}{q})$. Here $M_p^q$ denotes the set of Fourier transforms $\mathcal{F}(T)$ of distributions $T\in L_p^q$, where $L_p^q$ denotes the space of tempered distributions such that $\|T*u\|_{L^q} \le$

$C\|u\|_{L^p}$ with a constant $C$ independent of $u$. Hence $T_t \in L^q_p$ and

$$\|T_t * \mathcal{F}^{-1}(\mathcal{F}(\varphi)(K(t)\eta))\|_{L^q} \leq C_p\, K(t)^{-n+\frac{n}{p}} \|\varphi\|_{L^p} .$$

Thus, we have proved

$$I_0^{1/q} \leq C\, K(t)^{-2r+n(\frac{1}{p}-\frac{1}{q})} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma} \|\varphi\|_{L^p} ,$$

and

$$\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}(1-\psi(\xi/K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}$$
$$\leq C\, K(t)^{-2r+n(\frac{1}{p}-\frac{1}{q})} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma} \|\varphi\|_{L^p} ,$$

respectively. Thus, we have derived the statement of our theorem. □

### 7.2.2 $L^p - L^q$ estimates for Fourier multipliers with amplitudes supported in regular subzone

**Theorem 7.2.2.** *Let us consider Fourier multipliers which are defined by*

$$\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big) .$$

*Suppose that $a = a(t,\xi)$ satisfies the following assumption:*

- *in $Z_{\mathrm{reg}}(N)$:*

$$|\partial_\xi^\alpha a(t,\xi)| \leq C_{N,a,\gamma,\alpha}|\xi|^{-|\alpha|}(\log(t+e^3))^{2|\alpha|\gamma}\exp(C_{N,a,\gamma}(\log(t+e^3))^\gamma)$$

  *for all $|\alpha| \leq \frac{m-1}{2}$ .*

*Then we have the $L^p - L^q$ estimate*

$$\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}$$
$$\leq C\, K(t)^{-2r+n(\frac{1}{p}-\frac{1}{q})}(\log(t+e^3))^{2M\gamma} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma} \|\varphi\|_{L^p}$$

*provided that* $1 < p \le 2,\ \frac{1}{p} + \frac{1}{q} = 1,\ \frac{n+1}{2}\,(\frac{1}{p} - \frac{1}{q}) \le 2r \le n(\frac{1}{p} - \frac{1}{q})$ *and with a suitable positive constant* $M$.

*Proof.* We generalise the proof of [Pec76] to Fourier multipliers depending on a parameter. If $(t,\xi) \in \operatorname{supp}\psi(\xi/(2K(t)))$, then $(t,\xi) \in Z_{\text{reg}}(N)$. We choose a nonnegative function $\phi = \phi(\xi)$ having compact support in $\{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$. We set $\phi_k(\xi) := \phi(2^{-k}\xi)$ for $k \in \mathbb{N}$ while $\phi_0(\xi) := 1 - \sum\limits_{k=1}^{\infty} \phi_k(\xi)$. The function $\phi_0$ has its support in $\{\xi \in \mathbb{R}^n : |\xi| \le 2\}$. The $L^q$-norm of

$$\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))\phi_0(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)$$

can be estimated as in Theorem 7.2.1. Thus we can restrict ourselves to considering the integral

$$\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))\phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}$$

with $k \in \mathbb{N}$. First we study this integral for $t \ge t_0,\ t_0$ large.

a) $L^1 - L^\infty$ *continuity*: To estimate

$$\begin{aligned} I_k := \Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}&\psi(\xi/(2K(t)))\\ &\cdot \phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q} \end{aligned}$$

we set $\xi/(2K(t)) = 2^k\eta$. We are going to use Lemma 3 from [Bre75]. For this reason we use the inequality

$$\begin{aligned} &\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))\phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\Big)\Big\|_{L^\infty}\\ &\quad\le C\,2^{k(n-2r)}(2\,K(t))^{n-2r}\\ &\quad\Big\|\mathcal{F}^{-1}\Big(e^{i2^k 2K(t)|\eta|\int\limits_0^t \sqrt{a(s)}ds}\psi(2^k\eta)\phi(\eta)|\eta|^{-2r}a(t,2^{k+1}K(t)\eta)\Big)\Big\|_{L^\infty}. \end{aligned}$$

Let us denote $v_k(t,\eta) := \phi(\eta)\psi(2^k\eta)a(t,2^{k+1}K(t)\eta)$. These functions have their supports in $\{\eta \in \mathbb{R}^n : 1/2 \le |\eta| \le 2\}$. According to [Lit63] (see also [Bre75] or [Pec76]) we have (here we need $t_0$ large)

$$\Big\|\mathcal{F}^{-1}\Big(e^{i2^{k+1}K(t)|\eta|\int_0^t\sqrt{a(s)}ds}|\eta|^{-2r}v_k(t,\eta)\Big)\Big\|_{L^\infty}$$
$$\leq C\Big(2^{k+1}K(t)\int_0^t\sqrt{a(s)}ds\Big)^{-\frac{n-1}{2}}\sum_{|\alpha|\leq M}\|D_\eta^\alpha(|\eta|^{-2r}v_k(t,\eta))\|_{L^\infty}\,.$$

According to the assumption for $a=a(t,\xi)$ we have

$$\|D_\eta^\alpha(|\eta|^{-2r}v_k(t,\eta))\|_{L^\infty}\leq C_{N,a,\gamma,M}(\log(t+e^3))^{2|\alpha|\gamma}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}$$

for all $|\alpha|\leq\frac{m-1}{2}$. If we use $m=2M+1$ steps in our diagonalisation procedure in $Z_{\text{reg}}(N)$, then the last inequality holds for all $|\alpha|\leq M$. Hence,

$$\Big\|\mathcal{F}^{-1}\Big(e^{i2^{k+1}K(t)|\eta|\int_0^t\sqrt{a(s)}ds}|\eta|^{-2r}v_k(t,\eta)\Big)\Big\|_{L^\infty}$$
$$\leq C_{N,a,\gamma,M}2^{-\frac{k}{2}(n-1)}\Big(K(t)\int_0^t\sqrt{a(s)}ds\Big)^{-\frac{n-1}{2}}$$
$$(\log(t+e^3))^{2M\gamma}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\,.$$

All together we have shown

$$\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t\sqrt{a(s)}ds}\psi\Big(\frac{\xi}{2K(t)}\Big)\cdot\phi_k\Big(\frac{\xi}{2K(t)}\Big)|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^\infty}$$
$$\leq C_{N,a,\gamma,M}2^{k(\frac{n+1}{2}-2r)}K(t)^{n-2r}\Big(K(t)\int_0^t\sqrt{a(s)}ds\Big)^{-\frac{n-1}{2}}$$
$$\cdot(\log(t+e^3))^{2M\gamma}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\|\varphi\|_{L^1}\,.$$

b) $L^2-L^2$ *continuity*: To estimate $L^2$–norms we apply Lemma 3 from [Bre75]. To this end we take into consideration

$$\Big\|e^{i|\xi|\int_0^t\sqrt{a(s)}ds}\psi\Big(\frac{\xi}{2K(t)}\Big)\phi_k\Big(\frac{\xi}{2K(t)}\Big)|\xi|^{-2r}a(t,\xi)\Big\|_{L^\infty}$$
$$\leq\sup_{2^{k-1}\leq\frac{|\xi|}{2K(t)}\leq 2^{k+1}}\frac{|a(t,\xi)|}{|\xi|^{2r}}\leq C_{N,a,\gamma,0}2^{-2kr}K(t)^{-2r}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\,.$$

Hence,

$$\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t\sqrt{a(s)}ds}\psi\Big(\frac{\xi}{2K(t)}\Big)\cdot\phi_k\Big(\frac{\xi}{2K(t)}\Big)|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^2}$$
$$\le C_{N,a,\gamma,0}2^{-2kr}K(t)^{-2r}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\|\varphi\|_{L^2}\,.$$

c) *Interpolation argument*: An interpolation argument between $L^1-L^\infty$ and $L^2-L^2$ estimates from a) and b) yields

$$\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t\sqrt{a(s)}ds}\psi\Big(\frac{\xi}{2K(t)}\Big)\cdot\phi_k\Big(\frac{\xi}{2K(t)}\Big)|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}$$
$$\le C_{N,a,\gamma,M}2^{k(\frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})-2r)}K(t)^{n(\frac{1}{p}-\frac{1}{q})-2r}(\log(t+e^3))^{2M\gamma}$$
$$\cdot\, e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\|\varphi\|_{L^p}$$

for $t\ge t_0,\ t_0$ large, provided $1<p\le 2,\ \frac{1}{p}+\frac{1}{q}=1,\ \frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})\le 2r\le n(\frac{1}{p}-\frac{1}{q})$. Applying Lemma 2 from [Bre75] proves the statement of the theorem for $t\ge t_0,\ t_0$ large.

d) *Estimates for small t*: It remains to estimate the $L^q$–norms of

$$\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t\sqrt{a(s)}ds}\psi(\xi/(2K(t)))\phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)$$

for $t\in[0,t_0]$. Here we will not use the stationary phase method, the key tool to get the above estimates for $t\ge t_0$. Instead we apply the Hardy–Littlewood inequality as we did to get the estimates in Theorem 7.2.1. Let us sketch the differences in the proof. Using the transformations $\xi=2^{k+1}K(t)\eta$ and $z=2^{k+1}K(t)x$ we conclude for $k\in\mathbb{N}$

$$I_k:=(2^{k+1}K(t))^{nq-2rq-n}\Big\|\mathcal{F}^{-1}\Big(e^{i\,K(t)|\eta|\int\limits_0^t\sqrt{a(s)}ds}\psi(2^k\eta)\phi(\eta)|\eta|^{-2r}$$
$$\times\, a(t,2^{k+1}K(t)\eta)\mathcal{F}(\varphi)(2^{k+1}K(t)\eta)\Big)\Big\|^q_{L^q}\,.$$

The point $(t,2^{k+1}K(t)\eta)$ with $|\eta|\in[1/2,2]$ (support of $\phi$) belongs to $Z_{\rm reg}(N)$. Therefore $|a(t,2^{k+1}K(t)\eta)|\le C_{N,a,\gamma,0}\exp(C_{N,a,\gamma}(\log(t+$

$e^3))^\gamma)$. For $I_k$ we obtain

$$I_k^{1/q} \le (2^{k+1}K(t))^{n-2r-\frac{n}{q}} \Big\| \mathcal{F}^{-1}\Big( e^{i\,K(t)|\eta| \int_0^t \sqrt{a(s)}ds} \psi(2^k\eta)\phi(\eta)|\eta|^{-2r} \times a(t,2^{k+1}K(t)\eta)\Big) * \mathcal{F}^{-1}(\mathcal{F}(\varphi)(2^{k+1}K(t)\eta))\Big\|_{L^q}.$$

Now let us denote

$$T_{t,k} := \mathcal{F}^{-1}\Big( e^{i\,K(t)|\eta| \int_0^t \sqrt{a(s)}ds} \psi(2^k\eta)\phi(\eta)|\eta|^{-2r} a(t,2^{k+1}K(t)\eta)\Big) \times \exp(-C_{N,a,\gamma}(\log(t+e^3))^\gamma).$$

Then $T_{t,k}$ has the same properties as described for $T_t$ in the proof to Theorem 7.2.1. Thus we can derive

$$\Big\| \mathcal{F}^{-1}\Big( e^{i|\xi| \int_0^t \sqrt{a(s)}ds} \psi\Big(\frac{\xi}{2K(t)}\Big) \cdot \phi_k\Big(\frac{\xi}{2K(t)}\Big) |\xi|^{-2r} a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q} \le C(2^kK(t))^{-2r+n(\frac{1}{p}-\frac{1}{q})} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma} \|\varphi\|_{L^p}.$$

Lemma 2 from [Bre75] proves with $2r \ge n(\frac{1}{p}-\frac{1}{q})$ the statement of the theorem for $t \in [0,t_0]$. This completes the proof. □

## 7.3 End of the Proof

*Proof.* The statements of Theorems 7.2.1 and 7.2.2 applied to the representations (7.10) and (7.11) enable us to derive the estimates from Theorem 6.1.1. If $t \in (0,t_0]$, then we choose in the estimates from Theorems 7.2.1 and 7.2.2 the parameter $2r = n(\frac{1}{p}-\frac{1}{q})$. This fixes the necessary regularity $N_p = n(\frac{1}{p}-\frac{1}{q})$. If $t \in [t_0,\infty)$, then we choose in Theorem 7.2.2 the parameter $2r = \frac{n+1}{2}(\frac{1}{p}-\frac{1}{q})$. Now let us distinguish the different cases for $\gamma$. If $\gamma = 0$, then we directly obtain the classical Strichartz' $L^p - L^q$ decay estimate from Theorem 6.1.1 with $s_0 = 0$. If $\gamma \in (0,1)$, then the main influence on changes to the classical Strichartz' decay rate comes from the term $\exp(C_{N,a,\gamma}(\log(t+e^3))^\gamma$ in Theorems 7.2.1 and 7.2.2. For each $\varepsilon$ this term can be estimated by $C_\varepsilon(1+t)^\varepsilon$. Thus $s_0 = \varepsilon$ for all $\varepsilon > 0$. Finally, if $\gamma = 1$, then this term produces, together with the log terms, a factor like $(1+t)^{s_0}$, where $s_0$ eventually becomes a large positive constant. $\square$

Let us formulate a corollary of Theorem 6.1.1. At first sight, the statement of this corollary does not seem to be very surprising, but its meaning lies in a comparison of the cases $\gamma \in [0,1]$ and $\gamma > 1$ in (6.5).

**Corollary 7.3.1.** *Consider the strictly hyperbolic Cauchy problem*

$$\left.\begin{aligned} &\partial_t^2 u - a(t)\Delta u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ &u(t_0,x) = \varphi(x),\ \partial_t u(t_0,x) = \psi(x),\ t_0 \geq T, \end{aligned}\right\}$$

*where $T$ is large and the coefficient $a = a(t)$ satisfies (6.5) with $\gamma \in [0,1]$. Then there exists a constant $C$ which is independent of $t_0 \geq T$ and $t \geq t_0$ such that the following $L^p - L^q$ estimate holds for the solution $u = u(t,x)$:*

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \leq C(1+t)^{s_0} \|(\nabla_x \varphi, \psi)\|_{W_p^{N_p}},$$

*where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \leq 2$, $N_p \geq n\big(\frac{1}{p} - \frac{1}{q}\big)$ and*

- *$s_0 = 0$ if $\gamma = 0$; in this case $C$ only depends on $p, n$;*
- *$s_0 = \varepsilon$ if $\gamma \in (0,1)$ for all $\varepsilon > 0$; in this case $C$ depends on $p, n$ and $\varepsilon$;*
- *$s_0$ is a fixed constant (which can be determined) if $\gamma = 1$; in this case $C$ is independent of $\varphi, \psi$.*

*Proof.* The transformation $t := t_0 + \tau$ transfers the above Cauchy problem to

$$\partial_\tau^2 u - a_{t_0}(\tau)\Delta u = 0, \quad u(0,x) = \varphi(x),\ \partial_\tau u(0,x) = \psi(x),$$

where $a_{t_0}(\tau) := a(t_0 + \tau)$. The coefficients $a_{t_0}(\tau)$ satisfy, for all $t_0 \geq T$, the estimates (6.5) with the same constants $C_k$. Thus we can follow the proof of Theorem 6.1.1 and obtain the $L^p - L^q$ estimate

$$\|(u_\tau(\tau,\cdot), \nabla_x u(\tau,\cdot))\|_{L^q} \leq C(1+\tau)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+s_0} \|(\nabla_x \varphi, \psi)\|_{W_p^{N_p}}.$$

Setting $\tau = t - t_0$ in the last inequality gives the statement of the corollary. $\square$

# Chapter 8:
# Proof of Theorem 6.1.2

The proof is based on an application of Floquet's theory, an idea used in [Tar95] to show that the Cauchy problem for $\partial_t^2 - \exp(-2t^{-\alpha})b(t^{-1})\partial_x^2$ is not $C^\infty$ well-posed when $0 < \alpha < 1/2$, where $b = b(t)$ is a positive, smooth, 1-periodic function. A similar idea is used in [RY99] to study $L^p - L^q$ estimates for hyperbolic equations with increasing coefficients.

*Proof.* In order to apply Floquet's theory, it is necessary to first transform (6.7) so that the coefficient is periodic. This idea is used in [Hir03] when studying the $C^\infty$ well-posedness of strictly hyperbolic equations with non-Lipschitz coefficients. Then, a lower bound is found for a suitable energy of the solution of the transformed problem via estimates for an auxiliary problem. Finally, we derive a contradiction to (6.6) by obtaining a lower bound for the non-standard energy of Theorem 6.1.2 of the solution to (6.7).

## 8.1 Transformation of the Cauchy Problem (6.7)

Let

$$s = s(t) := (\log(t + e^3))^\alpha, \text{ (with inverse } t(s) := e^{s^{1/\alpha}} - e^3),$$
$$w = w(s, x) := \sqrt{\tau(s)}u(t(s), x),$$

where $\tau(s) := \frac{ds}{dt}(t(s)) = \alpha s^{1-(1/\alpha)} e^{-s^{1/\alpha}}$, and, instead of (6.7), consider the Cauchy problem obtained after this transformation. By simple calculations, we have

$$w_s(s,x) = \frac{1}{2}\frac{\tau'(s)}{\tau(s)}w(s,x) + \frac{1}{\sqrt{\tau(s)}}u_t(t(s),x);$$

$$\begin{aligned} w_{ss}(s,x) & \\ =&\frac{1}{4}\left(\frac{2\tau''(s)\tau(s)-\tau'(s)^2}{\tau(s)^2}\right)w(s,x) + \frac{1}{\tau(s)^2}(2+\sin(2\pi s))^2\Delta w(s,x) \\ =&\frac{1}{4\alpha^2 s^2}(s^{2/\alpha}-\alpha^2+1)w + \frac{1}{\alpha^2}e^{2s^{1/\alpha}}s^{(2/\alpha)-2}(2+\sin(2\pi s))^2\Delta w, \end{aligned}$$

since $t'(s) = 1/\tau(s)$. Transforming the initial data, we obtain the following conditions for $w(s,x)$ at $s = s_0$:

$$\left.\begin{aligned} w(s_0,x) =&\sqrt{\tau(s_0)}\varphi(x) =: \widetilde{\varphi}(x), \\ w_s(s_0,x) =&\frac{1}{2}\frac{\tau'(s_0)}{\sqrt{\tau(s_0)}}\varphi(x) + \frac{1}{\sqrt{\tau(s_0)}}\psi(x) =: \widetilde{\psi}(x). \end{aligned}\right\} \tag{8.1}$$

The problem is now in the form

$$\left.\begin{aligned} & w_{ss} - \nu(s)^2 b(s)^2\Delta w + \mu(s)w = 0, \\ & w(s_0,x) = \widetilde{\varphi}(x),\ w_s(s_0,x) = \widetilde{\psi}(x), \end{aligned}\right\} \tag{8.2}$$

where

$$\begin{aligned} \nu(s) := e^{s^{1/\alpha}}s^{(1/\alpha)-1},\ b(s) := (2+\sin(2\pi s))/\alpha, \\ \mu(s) := \frac{1}{4\alpha^2 s^2}(\alpha^2-1-s^{2/\alpha}) = O(s^{(2/\alpha)-2}) \text{ as } s\to\infty. \end{aligned}$$

Note that $b(s)$ is a non-constant, smooth, positive, periodic function with period 1. This is now in a form where the application of Floquet's theory is possible.

## 8.2 Application of Floquet's Theory

Consider the second order ordinary differential equation for $v = v(s)$,

$$v_{ss} + \lambda b(s)^2 v = 0.$$

Let $X$ be the fundamental matrix corresponding to this problem. That is, $X = X(s, s_0)$ solves the first order system of ordinary differential equations

$$d_s X = \begin{pmatrix} 0 & -\lambda b(s)^2 \\ 1 & 0 \end{pmatrix} X, \quad X(s_0, s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8.3}$$

We recall the following lemma from Floquet's theory (see, for example, [MW66] and [Tar95]):

**Lemma 8.2.1.** *Suppose $b(s)$ is a 1-periodic, non-constant, positive and smooth function on $\mathbb{R}$ and $s_0 \in \mathbb{N} \cup \{0\}$. Then there exists $\lambda_0 > 0$ such that the fundamental matrix $X(s, s_0)$ corresponding to $v_{ss} + \lambda_0 b(s)^2 v = 0$ evaluated at $s = s_0 + 1$ (i.e. $X(s_0 + 1, s_0)$) has eigenvalues $\mu_0, \mu_0^{-1}$ and $|\mu_0| > 1$.*

We use this to approximate the solution to the ordinary differential equation

$$v_{ss} + \lambda(s, \xi) b(s)^2 v = 0, \tag{8.4}$$

with suitable Cauchy data, where $\lambda(s, \xi) = \lambda_1(s, \xi) + \lambda_2(s)$ and

$$\lambda_1(s, \xi) := |\xi|^2 \nu(s)^2 = |\xi|^2 s^{(2/\alpha)-2} e^{2s^{1/\alpha}}, \quad \lambda_2(s) := \frac{\mu(s)}{b(s)^2} = \frac{\alpha^2 - 1 - s^{2/\alpha}}{4\alpha^2 s^2 b(s)^2}.$$

Observe that $\partial_s \lambda_1(s, \xi) = |\xi|^2 e^{2s^{1/\alpha}} \big( ((2/\alpha) - 2) s^{(2/\alpha)-3} + (2/\alpha) s^{(3/\alpha)-3} \big) > 0$ for $s > T_0$ for large enough $T_0$. Henceforth, we shall always assume $s > T_0$. So, for each $\xi \in \mathbb{R}^n$, $\lambda_1(s, \xi)$ is a monotonically increasing function in $s$ on its domain $[s_0, \infty)$. Also, it is clear that $\lambda_2(s) \to 0$ as $s \to \infty$.

We also define $s_\xi \in \mathbb{N}$ implicitly by the formula $\lambda(s_\xi, \xi) = \lambda_0$, where $\lambda_0$ is from Lemma 8.2.1. In addition, we require that $s_\xi > T$ where $T$ is large enough to ensure that $s_\xi \to \infty$ as $|\xi| \to 0$:

**Lemma 8.2.2.** *There exists $T > 0$ such that for $s_\xi$ as defined*

$$s_\xi \to \infty \text{ as } |\xi| \to 0.$$

*Proof.* Since $\lambda_2(s) \to 0$ as $s \to \infty$, we can choose $T_1 > 0$ such that $s > T_1$

implies that $|\lambda_2(s)| < \lambda_0/2$. Then, by definition, as we insist $s_\xi > T_1$,

$$\frac{\lambda_1(s_\xi, \xi)}{|\xi|^2} = \frac{\lambda_0}{|\xi|^2} - \frac{\lambda_2(s_\xi)}{|\xi|^2} \geq \frac{\lambda_0}{2|\xi|^2} \to \infty \text{ as } |\xi| \to 0.$$

Now, since $\lambda_1(s, \xi)/|\xi|^2$ is monotonically increasing for $s > T_0$, by setting $T := \max\{T_0, T_1\}$ it follows that $\lim_{|\xi|\to 0} s_\xi = \infty$. $\square$

We remark that this result allows us to take $s_\xi \in \mathbb{N}$ for *any* (large) integer—simply choose $|\xi|$ appropriately small enough.

## 8.3 Properties of $\lambda(s, \xi)$ and $X(s_\xi + 1, s_\xi)$

For the function $\lambda(s, \xi)$ we have the following result.

**Lemma 8.3.1.** *There exist constants* $0 < \rho < 1$ *and* $K > 0$ *such that if* $0 \leq \delta \leq \rho s - K$ *then*

$$|\lambda_1(s, \xi) - \lambda_1(s - \delta, \xi)| \leq C\delta\lambda_1(s, \xi)s^{(1/\alpha)-1}$$
$$\text{and } |\lambda_2(s) - \lambda_2(s - \delta)| \leq Cs^{(1/\alpha)-1},$$

*for some positive constant* $C$.

*Proof.* For the first part we apply the mean value theorem; this implies that there exists a constant $\tilde{s} \in (s - \delta, s)$ such that

$$\begin{aligned}
|\lambda_1(s, \xi) - \lambda_1(s - \delta, \xi)| &= |\xi|^2 \big| s^{(2/\alpha)-2} e^{2s^{1/\alpha}} - (s - \delta)^{(2/\alpha)-2} e^{2(s-\delta)^{1/\alpha}} \big| \\
&\leq \frac{2}{\alpha} |\xi|^2 \delta e^{2\tilde{s}^{1/\alpha}} \tilde{s}^{3((1/\alpha)-1)} \big| 1 + (1 - \alpha)\tilde{s}^{-1/\alpha} \big| \\
&\leq C\delta s^{(1/\alpha)-1} \lambda_1(s, \xi) e^{2(\tilde{s}^{1/\alpha} - s^{1/\alpha})} (s/\tilde{s})^{3(1-(1/\alpha))} \leq C\delta s^{(1/\alpha)-1} \lambda_1(s, \xi),
\end{aligned}$$

since $e^{2s^{1/\alpha}} s^{-3(1-(1/\alpha))}$ is monotonically increasing for large $s$ (we define $K$ so that this is for $s > K$) and $s - \delta > K$ by hypothesis.

For the second part, simply observe that, with $b_0 := \min_s b(s)$,

$$
\begin{aligned}
|\lambda_2(s) - &\lambda_2(s-\delta)| \\
&\leq \frac{1}{4\alpha^2 b_0^2}\big((\alpha^2-1)|s^{-2}-(s-\delta)^{-2}| + |s^{(2/\alpha)-2}-(s-\delta)^{(2/\alpha)-2}|\big) \\
&\leq C s^{(2/\alpha)-2}\Big((\alpha^2-1)s^{-2/\alpha}\Big|1-\Big(\frac{s}{s-\delta}\Big)^2\Big| + \Big|1-\Big(\frac{s}{s-\delta}\Big)^{2-(2/\alpha)}\Big|\Big) \\
&\leq C s^{(1/\alpha)-1},
\end{aligned}
$$

when $0 \leq \delta \leq \rho s - K$ for some $0 < \rho < 1$.

□

Now consider the fundamental matrix $X(s, s_0)$, which was defined as the solution of the system of ordinary differential equations (8.3), evaluated at the point $s = s_\xi + 1, s_0 = s_\xi$. We write this matrix as

$$
X(s_\xi + 1, s_\xi) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.
$$

By Lemma 8.2.1 this matrix has eigenvalues $\mu_0, \mu_0^{-1}$ where $|\mu_0| > 1$. Observe that

$$
\mu_0 + \mu_0^{-1} = \operatorname{tr} X(s_\xi + 1, s_\xi) = x_{11} + x_{22},
$$

and so,

$$
|\mu_0^{-1} - \mu_0| \leq |x_{11} - \mu_0| + |x_{22} - \mu_0|.
$$

Hence,

$$
\max\{|x_{11} - \mu_0|, |x_{22} - \mu_0|\} \geq \frac{1}{2}|\mu_0^{-1} - \mu_0| > 0.
$$

The last inequality follows from $|\mu_0| > 1$. We can assume, without loss of generality, that

$$
|x_{11} - \mu_0| \geq \frac{1}{2}|\mu_0^{-1} - \mu_0|, \tag{8.5}
$$

and then we also have

$$
|x_{22} - \mu_0^{-1}| \geq \frac{1}{2}|\mu_0^{-1} - \mu_0|. \tag{8.6}
$$

## 8.4 Auxiliary Family of ODEs

Consider the family of ODEs

$$v_{ss} + \lambda(s_\xi - k + s, \xi) b(s_\xi + s)^2 v = 0, \ k \in \mathbb{N} \cup \{0\},$$

where $s_\xi \in \mathbb{N}$ is as in Section 8.2 and $\lambda_0$ is as given in Lemma 8.2.1. Here we are using the 1-periodicity of $b(s)$.

To each problem associate the fundamental matrix $X_k(s, s_1)$ which satisfies

$$d_s X_k = \begin{pmatrix} 0 & -\lambda(s_\xi - k + s) b(s_\xi + s)^2 \\ 1 & 0 \end{pmatrix} X_k, \quad X_k(s_1, s_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We study these matrices evaluated at $(s, s_1) = (1, 0)$; write

$$X_k(1,0) = \begin{pmatrix} x_{11}(k) & x_{12}(k) \\ x_{21}(k) & x_{22}(k) \end{pmatrix}. \tag{8.7}$$

Denote the eigenvalues of this matrix by $\mu_k, \mu_k^{-1}$ where $|\mu_k| \geq 1$ (in fact, later we see that $|\mu_k| > 1$ for all suitable $k$). That this matrix has determinant 1 is an immediate consequence of the formula for the derivative of the determinant of a matrix and the fact that $\mathrm{tr}(A_k(s,\xi)) = 0$ where

$$A_k(s,\xi) = \begin{pmatrix} 0 & -\lambda(s_\xi - k + s, \xi) b(s_\xi + s)^2 \\ 1 & 0 \end{pmatrix}. \tag{8.8}$$

The matrices $X_k(1,0)$ are uniformly bounded for suitably large $k$:

**Lemma 8.4.1.** *Let $F(s)$ be a function satisfying*

$$\lim_{s\to\infty} s^{(1/\alpha)-1} F(s) = 0. \tag{8.9}$$

*Then we have*

$$\max_{s,s_1 \in [0,1]} \|X_k(s,s_1)\| \leq e^{C\lambda_0}$$

*for $1 \leq k \leq cF(s_\xi)$ and some positive constants $C, c$.*

**Remark 8.4.1:** Note that $F(s) := s^\beta$, where $\beta < 1 - (1/\alpha)$, satisfies requirement (8.9).

*Proof.* We have the following representation for $X_k(s, s_1)$:

$$X_k(s,s_1) = \\ I + \sum_{j=1}^{\infty} \int_{s_1}^{s} A_k(r_1,\xi) \int_{s_1}^{r_1} A_k(r_2,\xi) \dots \int_{s_1}^{r_{j-1}} A_k(r_j,\xi)\, dr_j \dots dr_2 dr_1,$$

where $A_k(s,\xi)$ is as in (8.8). Now, by Lemma 8.3.1,

$$\begin{aligned} \|A_k(s,\xi)\| &\leq 1 + b_1^2 \sup_{s\in[0,1]} |\lambda(s_\xi - k + s, \xi)| \\ &\leq 1 + b_1^2 \Big|\lambda_1(s_\xi - k + 1, \xi) + \sup_{s>s_0} \lambda_2(s)\Big| \\ &= 1 + b_1^2 \Big|\lambda_1(s_\xi - k + 1, \xi) - \lambda_1(s_\xi, \xi) - \lambda_2(s_\xi) + \lambda_0 + \sup_{s>s_0} \lambda_2(s)\Big| \\ &\leq 1 + b_1^2 \Big(C(k-1)\lambda_1(s_\xi,\xi) s_\xi^{(1/\alpha)-1} + \lambda_0 + 2\sup_{s>s_0} |\lambda_2(s)|\Big) \end{aligned}$$

provided $0 \leq k-1 \leq \rho s_\xi - K$; here $b_1 = \max_s b(s)$. So, by (8.9), $\|A_k(s,\xi)\| \leq 1 + C_1 b_1^2 \lambda_0$ for large $s_\xi$ when $1 \leq k \leq cF(s_\xi)$; here $c$ is chosen to ensure that $k-1 \leq \rho s_\xi - K$ is satisfied when $k \leq cF(s_\xi)$. Therefore,

$$\max_{s,s_1\in[0,1]} \|X_k(s,s_1)\| \leq \exp\Big(\int_{s_1}^{s} \|A_k(r,\xi)\| dr\Big) \leq C_0 e^{C_1 b_1^2 \lambda_0} = e^{C\lambda_0},$$

provided $1 \leq k \leq cF(s_\xi)$. The lemma is proved. □

The next lemma shows that in some sense $X(s_\xi + 1, s_\xi)$ is "near" to the $X_k(1,0)$ for suitable $k$.

**Lemma 8.4.2.** *Under the assumptions of Lemma 8.4.1 we have*

$$\|X_k(1,0) - X(s_\xi + 1, s_\xi)\| \leq C\lambda_0 s_\xi^{(1/\alpha)-1} F(s_\xi)$$

*for* $1 \leq k \leq cF(s_\xi)$ *and some positive constants* $C, c$.

*Proof.* First, note that $X(s_\xi + s, s_\xi) = X(s,0)$, since $s_\xi \in \mathbb{N}$ and $b(s)$ is

1-periodic. Now, $X_k(s,0)$ satisfies

$$d_s X_k(s,0) = \begin{pmatrix} 0 & -\lambda(s_\xi,\xi)b(s)^2 \\ 1 & 0 \end{pmatrix} X_k(s,0) + \begin{pmatrix} 0 & \big(\lambda(s_\xi,\xi) - \lambda(s_\xi - k + s,\xi)\big)b(s)^2 \\ 0 & 0 \end{pmatrix} X_k(s,0)\,,$$

with $X_k(0,0) = I$. Thus,

$$d_s\big(X_k(s,0) - X(s,0)\big) = \begin{pmatrix} 0 & -\lambda(s_\xi,\xi)b(s)^2 \\ 1 & 0 \end{pmatrix} \big(X_k(s,0) - X(s,0)\big) + \begin{pmatrix} 0 & \big(\lambda(s_\xi,\xi) - \lambda(s_\xi - k + s,\xi)\big)b(s)^2 \\ 0 & 0 \end{pmatrix} X_k(s,0)\,,$$

with initial data $X_k(0,0) - X(0,0) = 0$; here 0 denotes the zero matrix. Now, by Lemma 8.3.1,

$$\begin{aligned} |\lambda_1(s_\xi,\xi) - \lambda_1(s_\xi - k + s,\xi)| &\le C(k-s)\lambda_1(s_\xi,\xi)s_\xi^{(1/\alpha)-1} \\ \text{and } |\lambda_2(s_\xi) - \lambda_2(s_\xi - k + s)| &\le C s_\xi^{(1/\alpha)-1} \le C\lambda_0^{-1} k s_\xi^{(1/\alpha)-1}\lambda(s_\xi,\xi) \end{aligned}$$

for $0 \le k - s \le \rho s_\xi - K$. Therefore,

$$|\lambda(s_\xi,\xi) - \lambda(s_\xi - k + s,\xi)| \le Ck\lambda_0 s_\xi^{(1/\alpha)-1}$$

for $0 \le k - s \le \rho s_\xi - K$. Hence,

$$\begin{aligned} &\|X_k(s,0) - X(s,0)\| \\ &\quad \le \int_0^s C\lambda_0 \|X_k(r,0) - X(r,0)\|\,dr + \int_0^s Ck\lambda_0 s_\xi^{(1/\alpha)-1}\|X_k(r,0)\|\,dr\,. \end{aligned}$$

So, by Lemma 8.4.1, Gronwall's inequality and the hypotheses on $k$,

$$\|X_k(1,0) - X(1,0)\| \le Ck\lambda_0 s_\xi^{(1/\alpha)-1} e^{C\lambda_0} \le C\lambda_0 s_\xi^{(1/\alpha)-1} F(s_\xi)\,,$$

where $1 \le k \le cF(s_\xi)$; $c$ here is chosen as in the proof of Lemma 8.4.1 This completes the proof of the lemma. □

Also, the $X_k(1,0)$ are, in a similar sense, "near" to each other.

**Lemma 8.4.3.** *The following inequality holds for all* $1 \le k \le cF(s_\xi)$, *with* $c$ *as in Lemma 8.4.1,*

$$\|X_{k+1}(1,0) - X_k(1,0)\| \le C\lambda_0 s_\xi^{(1/\alpha)-1},$$

*where* $C$ *is a positive constant and* $F(s)$ *satisfies* (8.9).

*Proof.* Observe

$$\begin{aligned} &d_s\big(X_k(s,0) - X_{k+1}(s,0)\big) \\ &\quad = \begin{pmatrix} 0 & -\lambda(s_\xi - (k+1) + s, \xi)b(s)^2 \\ 1 & 0 \end{pmatrix} \big(X_k(s,0) - X_{k+1}(s,0)\big) \\ &\quad + \begin{pmatrix} 0 & \big(\lambda(s_\xi - (k+1) + s, \xi) - \lambda(s_\xi - k + s, \xi)\big)b(s)^2 \\ 0 & 0 \end{pmatrix} X_k(s,0)\,, \end{aligned}$$

By Lemma 8.3.1,

$$\begin{aligned} &|\lambda_1(s_\xi - (k+1) + s, \xi) - \lambda_1(s_\xi - k + s, \xi)| \\ &\qquad\qquad \le C\lambda_1(s_\xi - k + s, \xi)(s_\xi - k + s)^{(1/\alpha)-1} \end{aligned}$$

and

$$|\lambda_2(s_\xi - (k+1) + s) - \lambda_2(s_\xi - k + s)| \le C(s_\xi - k + s)^{(1/\alpha)-1}$$

for $1 \le \rho(s_\xi - k + s) - K$. This latter condition is satisfied when $1 \le k \le cF(s_\xi)$, where $c$ is as in Lemma 8.4.1, for $s_\xi$ chosen large enough. Now, $\lambda_1(t,\xi)t^{(1/\alpha)-1}$ is increasing for large $t$ (see the proof of Lemma 8.3.1); in particular, it is increasing for $t \ge s_\xi - k + s$ when $s_\xi$ is chosen large enough since

$$s_\xi - k + s \ge s_\xi - cF(\xi) \ge s_\xi - \frac{1}{2}s_\xi^{1-(1/\alpha)} \ge \frac{1}{2}s_\xi$$

by the hypotheses on $k$ and $F(s)$. Also, using the argument above and the hypothesis on $F(s)$,

$$(s_\xi - k + s)^{(1/\alpha)-1} \leq \left(\frac{2}{s_\xi}\right)^{1-(1/\alpha)} \leq C\lambda_0 s_\xi^{(1/\alpha)-1}$$

when $k \leq cF(s_\xi)$. Thus,

$$|\lambda(s_\xi - (k+1) + s, \xi) - \lambda(s_\xi - k + s, \xi)| \leq C\lambda_0 s_\xi^{(1/\alpha)-1}$$

for $1 \leq k \leq cF(s_\xi)$. So, by a similar argument to that used in the proof of Lemma 8.4.2,

$$\|X_{k+1}(1,0) - X_k(1,0)\| \leq C\lambda_0 s_\xi^{(1/\alpha)-1} e^{C\lambda_0} \leq C\lambda_0 s_\xi^{(1/\alpha)-1}$$

for $1 \leq k \leq cF(s_\xi)$, as required. □

These "nearness" lemmas give information about the relations of the eigenvalues of these matrices.

**Corollary 8.4.4.** *For* $1 \leq k \leq cF(s_\xi)$, *where* $F(s)$ *satisfies* (8.9) *and* $c > 0$ *is as in Lemma 8.4.1, the following relations hold for suitably large* $s_\xi$:

1. *for each* $\varepsilon > 0$, *we can choose* $s_\xi$ *large enough so that*

$$|\mu_k - \mu_0| < \varepsilon, \tag{8.10}$$

   *hence, for suitably chosen* $\varepsilon$,

$$|\mu_k| \geq |\mu_0| - \varepsilon > 1; \tag{8.11}$$

2. *there exists* $C > 0$ *such that*

$$|\mu_k - \mu_{k-1}| \leq C\lambda_0 s_\xi^{(1/\alpha)-1}; \tag{8.12}$$

3. *there exists* $C > 0$ *such that*

$$|\mu_k - \mu_k^{-1}| \geq C; \tag{8.13}$$

*4. there exist constants* $C_1, C_2 > 0$ *such that*

$$|\mu_k - x_{11}(k)| \geq C_1, \tag{8.14}$$

$$|\mu_k^{-1} - x_{22}(k)| \geq C_2. \tag{8.15}$$

*Proof.* 1. By Lemma 8.4.2,

$$\begin{aligned}|\mu_k + \mu_k^{-1} - \mu_0 - \mu_0^{-1}| &= |x_{11}(k) + x_{22}(k) - x_{11} - x_{22}| \\ &\leq |x_{11}(k) - x_{11}| + |x_{22}(k) - x_{22}| \leq C\lambda_0 s_\xi^{(1/\alpha)-1} F(s_\xi) \to 0 \text{ as } s_\xi \to \infty.\end{aligned}$$

On the other hand

$$|\mu_k + \mu_k^{-1} - \mu_0 - \mu_0^{-1}| = |(\mu_k - \mu_0)(1 - (\mu_k\mu_0)^{-1})| \geq C|\mu_k - \mu_0|,$$

where $C > 0$, since $|\mu_0| > 1$. Combining these two observations proves (8.10).

2. By Lemma 8.4.3

$$\begin{aligned}&|\mu_k - \mu_{k-1} + \mu_k^{-1} - \mu_{k-1}^{-1}| \\ &\qquad = |x_{11}(k) - x_{11}(k-1) + x_{22}(k) - x_{22}(k-1)| \leq C\lambda_0 s_\xi^{(1/\alpha)-1}.\end{aligned}$$

Choosing $s_\xi$ large enough so that (8.11) holds, we see, by a similar argument to *1.*, that (8.12) also holds.

3. This is clear since (8.11) holds for large enough $s_\xi$.

4. By (8.5), Lemma 8.4.2 and part *1.* of this corollary we have the following for large enough $s_\xi$:

$$\begin{aligned}|\mu_k - x_{11}(k)| &= |\mu_k - \mu_0 + \mu_0 - x_{11} + x_{11} - x_{11}(k)| \\ &\geq |\mu_0 - x_{11}| - |\mu_k - \mu_0| - |x_{11} - x_{11}(k)| \\ &\geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| - \frac{1}{8}|\mu_0 - \mu_0^{-1}| - \frac{1}{8}|\mu_0 - \mu_0^{-1}| = \frac{1}{4}|\mu_0 - \mu_0^{-1}| > 0.\end{aligned}$$

This proves (8.14). The proof of (8.15) is similar, but we use (8.6) in place of (8.5). □

Henceforth, we assume that $s_\xi$ is chosen large enough so that all of the inequalities in Corollary 8.4.4 hold.

## 8.5 Lower Bound for Solution to Auxiliary Cauchy Problem

We are now in a position to give a lower bound for the solution to a Cauchy problem for (8.4).

**Proposition 8.5.1.** *Consider the Cauchy problem*

$$\left.\begin{aligned} &v_{ss} + \lambda(s,\xi)b(s)^2 v = 0, \\ &v(s_\xi - n_0, \xi) = 1,\ v_s(s_\xi - n_0, \xi) = \frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)}, \end{aligned}\right\} \tag{8.16}$$

*where* $cs_\xi^\beta - 1 \le n_0 \le cs_\xi^\beta$ *for* $1/\alpha < \beta < 1 - 1/\alpha$ *and* $c$ *is some positive constant. Then the following estimate holds for the solution* $v = v(s,\xi)$ *at* $s = s_\xi$:

$$|v(s_\xi,\xi)| + |v_s(s_\xi,\xi)| \ge C\exp\big(a(\log|\tfrac{1}{\xi}|)^\gamma\big),$$

*where* $\gamma = \alpha\beta \in (1, \alpha - 1)$ *and* $C, a$ *are positive constants.*

*Proof.* Throughout this proof we assume that $k \le n_0$ at each occurrence of $k$.

Observe that

$$\begin{pmatrix} v_s(s_\xi,\xi) \\ v(s_\xi,\xi) \end{pmatrix} = X_1(1,0)X_2(1,0)\ldots X_{n_0}(1,0)\begin{pmatrix} v_s(s_\xi - n_0,\xi) \\ v(s_\xi - n_0,\xi) \end{pmatrix}, \tag{8.17}$$

where $X_k(1,0)$ is as in (8.7). Now,

$$B_k = \begin{pmatrix} \frac{x_{12}(k)}{\mu_k - x_{11}(k)} & 1 \\ 1 & \frac{x_{21}(k)}{\mu_k^{-1} - x_{22}(k)} \end{pmatrix},$$

is a diagonaliser for $X_k(1,0)$. This is a consequence of the facts that $\det X_k(1,0) = 1$ and $\operatorname{tr} X_k(1,0) = x_{11}(k) + x_{22}(k) = \mu_k + \mu_k^{-1}$.

Observe that

$$\|B_k\| \le C, \tag{8.18}$$

for some constant $C$ independent of $k$; this follows from Lemma 8.4.1 and inequalities (8.14) and (8.15). Furthermore, $B_k$ is invertible for each $k$ since

$$\det B_k = \frac{\mu_k - \mu_k^{-1}}{\mu_k^{-1} - x_{22}(k)},$$

and (8.13) ensures that this is non-zero, together with Lemma 8.4.1 and $|\mu_k^{-1}| < 1$. From this and (8.18), it follows that, in addition, $\|B_k^{-1}\| \leq C$ for some constant independent of $k$.

Also, by (8.14), (8.15), Lemma 8.4.1 and (8.12),

$$\begin{aligned} &\|B_{k+1} - B_k\| \\ &\leq \max_{1\leq k\leq n_0} \|X_k(1,0)\|(C_1|\mu_k - \mu_{k+1}| + C_2|\mu_k^{-1} - \mu_{k+1}^{-1}|) \leq C\lambda_0 s_\xi^{(1/\alpha)-1}. \end{aligned} \tag{8.19}$$

Hence, (8.17) can be rewritten as

$$\begin{aligned} &\begin{pmatrix} v_s(s_\xi,\xi) \\ v(s_\xi,\xi) \end{pmatrix} \\ &= B_1 \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} B_1^{-1} B_2 \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} B_2^{-1} \dots B_{n_0} \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= B_1 \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Set $B_k^{-1} B_{k+1} = I + G_k$. So,

$$\begin{aligned} &\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \\ &= \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} (I + G_1) \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} \dots (I + G_{n_0-1}) \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix} \qquad (8.20) \\ &= \begin{pmatrix} \prod_{k=1}^{n_0} \mu_k & 0 \\ 0 & \prod_{k=1}^{n_0} \mu_k^{-1} \end{pmatrix} + M_1 + \dots + M_{n_0-1}, \end{aligned}$$

where $M_l$ is the matrix which is the sum of all the products of matrices from

(8.20) containing exactly $l$ of the $G_k$ matrices; observe

$$\|M_l\| \le \left(\prod_{k=1}^{n_0}|\mu_k|\right)\left(\sum_{1\le i_1<\cdots<i_l\le n_0-1}\prod_{j=1}^{l}\|G_{i_j}\|\right).$$

By (8.18) and (8.19)

$$\begin{aligned}\|G_k\| = \|B_k^{-1}B_{k+1} - I\| &= \|B_k^{-1}(B_{k+1}-B_k)\| \\ &\le \|B_k^{-1}\|\|B_{k+1}-B_k\| \le C\lambda_0 s_\xi^{(1/\alpha)-1}.\end{aligned}$$

Therefore,

$$\|M_l\| \le \left(\prod_{k=1}^{n_0}|\mu_k|\right)\binom{n_0-1}{l}(C\lambda_0 s_\xi^{(1/\alpha)-1})^l.$$

Thus,

$$|y_{11}| \ge \left(\prod_{k=1}^{n_0}|\mu_k|\right)\left(2-\left(1+C\lambda_0 s_\xi^{(1/\alpha)-1}\right)^{cs_\xi^\beta}\right).$$

Taking account of $\beta < 1 - 1/\alpha$ gives immediately

$$|y_{11}| \ge \frac{1}{2}\left(\prod_{k=1}^{n_0}|\mu_k|\right).$$

On the other hand, $|y_{21}|$ is very small—it is less than $\nu\prod_{k=1}^{n_0}|\mu_k|$, where we can take $\nu$ as small as we like. Hence, using

$$\begin{pmatrix} v_s(s_\xi,\xi) \\ v(s_\xi,\xi)\end{pmatrix} = \begin{pmatrix} \frac{x_{12}(1)}{\mu_1 - x_{11}(1)} & 1 \\ 1 & \frac{x_{21}(1)}{\mu_1^{-1}-x_{22}(1)}\end{pmatrix}\begin{pmatrix} y_{11} \\ y_{21}\end{pmatrix}$$

and (8.11), it follows that

$$|v_s(s_\xi,\xi)| + |v(s_\xi,\xi)| \ge C(|\mu_0|-\varepsilon)^{n_0} \ge Ce^{as_\xi^\beta},$$

for some positive constants $a, C$. Finally, for large $s_\xi$,

$$s_\xi \sim \left(\log\frac{1}{|\xi|}\right)^\alpha, \tag{8.21}$$

and so we have the desired inequality. The proposition is proved. □

## 8.6 Lower Bound for the Energy of $w(s_\xi, x)$

We return to the transformed Cauchy problem (8.2) with initial time chosen as $s_0 = s_\xi - n_0$ and seek a representation for the solution at time $s = s_\xi$ in the unit ball $B_1(0)$. By the existence of a cone of dependence, this only depends on the initial data in the ball $B_R(0)$ at $s = s_\xi - n_0$, where $R = R(n_0, b) \leq C n_0 \min_s b(s)$. Set

$$\widetilde{\varphi}(x) = e^{ix\cdot\xi}\chi(x/R^2), \quad \widetilde{\psi}(x) = \frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)} e^{ix\cdot\xi}\chi(x/R^2) \tag{8.22}$$

to be the data at $s = s_0$, where $\chi(x)$ is a smooth cut-off function which is identically 1 on $|x| < 1$. By the uniqueness of solutions to strictly hyperbolic equations, the solution can be represented in the cone of dependence, and therefore in $B_1(0)$ at $s = s_\xi$, by

$$w = w(s,x) = e^{ix\cdot\xi}v(s,\xi);$$

here $v(s,x)$ is the solution to (8.16) at time $s$. Use $w(s_\xi, x, \xi) = e^{ix\cdot\xi}v(s_\xi,\xi)$ to denote this solution. Then the following lower bound holds for $w$:

$$\begin{aligned} &\|\nabla_x w(s_\xi,\cdot)\|_{L^q} + \|w_s(s_\xi,\cdot)\|_{L^q} \geq \|\nabla_x w(s_\xi,\cdot)\|_{L^q(B_1(0))} + \|w_s(s_\xi,\cdot)\|_{L^q(B_1(0))} \\ &= (|\xi||v(s_\xi,\xi)| + |v_s(s_\xi,\xi)|)\,\text{meas}(B_1(0))^{1/q} \geq C|\xi|\exp\left(a(\log \tfrac{1}{|\xi|})^\gamma\right), \end{aligned} \tag{8.23}$$

where $L^q = L^q(\mathbb{R}^n)$.

## 8.7 Lower Bound for the Energy of $u(\tau_\xi, x)$

Finally, we return to the original problem (6.7).

Set $t_0 = t(s_0) = t(s_\xi - n_0) = e^{(s_\xi - n_0)^{1/\alpha}} - e^3$ and choose the following initial data:

$$\varphi(x) = \frac{1}{\sqrt{\tau(s(t_0))}}\, e^{ix\cdot\xi}\chi(x/R^2) = \frac{1}{\sigma(t_0)}\, e^{ix\cdot\xi}\chi(x/R^2)\,, \tag{8.24}$$

$$\psi(x) = \left(\sqrt{\tau(s(t_0))}\frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)} - \frac{\tau'(s(t_0))}{2\sqrt{\tau(s(t_0))}}\right)e^{ix\cdot\xi}\chi(x/R^2)$$
$$= \left(\sigma(t_0)\frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)} - \frac{\sigma'(t_0)}{\sigma(t_0)^2}\right)e^{ix\cdot\xi}\chi(x/R^2)\,, \tag{8.25}$$

where $\sigma(t)$ is as in Theorem 6.1.2. Here we have taken into account (8.1) and (8.22). Now, by (8.23), the energy defined in Theorem 6.1.2 for $u = u(t,x)$ at $t = \tau_\xi := t(s_\xi) = e^{s_\xi^{1/\alpha}} - e^3$ can be estimated as follows:

$$\begin{aligned}
E(u)(\tau_\xi)\big|_{L^q} &= \Big\|\sigma(\tau_\xi)\nabla_x u(\tau_\xi,\cdot)\Big\|_{L^q} + \Big\|\frac{1}{\sigma(\tau_\xi)^2}\partial_t\big(u(t,\cdot)\sigma(t)\big)\big|_{t=\tau_\xi}\Big\|_{L^q} \\
&= \|\nabla_x w(s_\xi,\cdot)\|_{L^q} + \|w_s(s_\xi,\cdot)\|_{L^q} \ge C|\xi| \exp\big(a(\log\tfrac{1}{|\xi|})^\gamma\big) \\
&\ge C\exp\big[-c_1 s_\xi^{1/\alpha} + ac_2 s_\xi^\beta\big] \\
&= C\exp\big[-c_1\log(\tau_\xi + e^3) + ac_2\big(\log(\tau_\xi + e^3)\big)^\gamma\big],
\end{aligned} \tag{8.26}$$

and $1 < \gamma < \alpha - 1$; here we have used (8.21) in the final equality. If we now assume (6.8) holds with the initial data (8.24), (8.25), then, for $1 < r < \gamma < \alpha - 1$,

$$\begin{aligned}
E(u)(\tau_\xi)\big|_{L^q} &\le C_1\exp\big(C_2(\log(\tau_\xi + e^3))^r\big)E(u)(t_0)\big|_{W_p^M} \\
&= C_1 e^{C_2(\log(\tau_\xi+e^3))^r}\left(1 + \frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)}\right)\|e^{ix\cdot\xi}\chi(x/R^2)\|_{W_p^{M+1}},
\end{aligned}$$

which contradicts (8.26) since $r < \gamma$. The proof of Theorem 6.1.2 is complete. □

**Remark 8.7.1:** We observe that the $L^p - L^q$ estimate from Corollary 7.3.1 derived for the cases very slow, slow, fast oscillations is of the form

$$E(u)(t)\big|_{L^q} \le C\,(1+t)^{s_0}E(u)(t_0)\big|_{W_p^{N_p}} \tag{8.27}$$

with a positive constant $C$ independent of $t_0 \ge T$ and $t \ge t_0$. However, Theorem 6.1.2 states that in the case of very fast oscillations we cannot have an estimate of the form

$$E(u)(t)\big|_{L^q} \leq C_1 \exp(C_2(\log(t+e^3))^r)E(u)(t_0)\big|_{W_p^{N_p}} \tag{8.28}$$

for $1 < r < \alpha - 1$ with positive constants $C_1$ and $C_2$ independent of $t_0 \geq T$ and $t \geq t_0$. Comparing (8.27) with (8.28) we have indeed an essential change in the behaviour of solutions to (6.4) from fast to very fast oscillations.

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