Normal Approximations of Regular Curves and Surfaces

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Abstract. Bézier curves and surfaces are two very useful tools in Geometric Modeling, with many applications. In this paper, we will offer a new method to provide approximations of regular curves and surfaces by Bézier ones, with the corresponding examples.

1. Introduction

The classic theory of curves and surfaces is one of the most famous areas in Differential Geometry. It combines very visual and intuitive elements with really deep results and techniques. This theory was mainly developed in the 18th and 19th centuries, by mathematicians like Gaspard Monge (1746-1818), who is considered the father of Differential Geometry because of his work *Application de l'Analyse à la Géométrie*, or Carl Friedrich Gauss (1777-1855), among many others.

But the direct use of regular curves and surfaces in applied areas such as engineering or architecture is not easy in general. There, some other types of geometrical tools have been developed. For example, Bézier curves and surfaces, named after the French Engineer and Mathematician Pierre Bézier (1910-1999). They have proved to be two very useful tools in Geometric Modeling, with many applications.

In this paper, we want to strengthen the link between these two geometrical theories, offering an answer to this main question: given a regular curve or surface, is it possible to approximate it by a Bézier one? Of course, this question is not new, and have been deeply studied in some particular cases (see, for instance, [1] and [2]). But we offer a different approach, with a method which can be used for any kind of regular curve or surface. In fact, our method is somehow inspired by a classical construction in Differential Geometry: the orthogonal variations. Roughly speaking, an orthogonal variation of a curve is produced when the curve is considered as a cord, vibrating in a direction which is orthogonal to its tangent vector at any point. Variations of curves are used, for example, to characterize geodesics as the solutions of a variational problem. For more details and the exact definitions about this procedure, we recommend [3]. Our method can be seen as a kind of discretization of that construction.

2. Preliminaries

In this section, we recall some definitions and basic formulas which we will use later.

A *regular parametrized plane* (resp. *spatial*) *curve* is a differentiable map $\alpha : I \to \mathbb{R}^2$ (resp. $\mathbb{R}^3$) of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^2$ (resp. $\mathbb{R}^3$), such that $\alpha'(t) \neq 0$ for any $t \in I$. We can consider also a curve α defined on a closed interval $[c, d]$ if there exist an open interval (a, b) and a curve $\bar\alpha$ defined on (a, b) such that $[c, d] \subset (a, b)$ and $\alpha(t) = \bar\alpha(t)$, for any $t \in [c, d]$ (i.e., α is the restriction of $\bar\alpha$).

Given a regular parametrized plane curve $\alpha : I \to \mathbb{R}^2 : t \mapsto \alpha(t) = (x(t), y(t))$, we can consider a moving orthonormal dihedron $\{\mathbf{t}, \mathbf{n}\}$ along α, called the Frenet dihedron, defined by:

$$\mathbf{t}(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{1}{|\alpha'(t)|}(x'(t), y'(t)), \quad \mathbf{n}(t) = \frac{1}{|\alpha'(t)|}(-y'(t), x'(t)).$$

The vector $\mathbf{t}$ is called the *unit tangent vector* and $\mathbf{n}$ is known as the *normal vector*.

For a spatial curve α, we first recall that its curvature is given by

$$k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.$$

Therefore, at every point $\alpha(t)$ such that $k(t) \neq 0$, we can consider an orthonormal trihedron $\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$, called the Frenet trihedron, defined by:

$$\mathbf{t}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}, \quad \mathbf{b}(t) = \frac{\alpha'(t) \times \alpha''(t)}{|\alpha'(t) \times \alpha''(t)|}, \quad \mathbf{n}(t) = \mathbf{b}(t) \times \mathbf{n}(t).$$

The vector $\mathbf{t}$ is called the *unit tangent vector*, $\mathbf{n}$ is the *principal normal vector* and $\mathbf{b}$ is known as the *binormal vector*.

A *regular parametrized surface* is a differentiable map $\mathbf{x} : U \to \mathbb{R}^3 : (u, v) \mapsto \mathbf{x}(u, v)$ of an open set U of the real plane $\mathbb{R}^2$ into $\mathbb{R}^3$, such that $\partial\mathbf{x}/\partial u \times \partial\mathbf{x}/\partial v \neq 0$ at any point (some authors also ask such a surface to be injective). This regularity condition allows us to define the *normal vector* at any point of the surface, given by:

$$N = \frac{\partial\mathbf{x}/\partial u \times \partial\mathbf{x}/\partial v}{|\partial\mathbf{x}/\partial u \times \partial\mathbf{x}/\partial v|}.$$

For more details about regular curves and surfaces, we recommend the classical reference [3].

Let us recall now the definitions of Bézier curves and surfaces. Given $n + 1$ points $P_0, \dots, P_n$, they determine the *Bézier curve* $B : [0, 1] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) given by

$$B(t) = \sum_{i=0}^{n} B_i^n(t) P_i,$$

where

$$B_i^n(t) = \binom{n}{i} t^i (1 - t)^{n-i}, \qquad 0 \leq i \leq n$$

is the *Bernstein polynomial* of degree n. Given a grid of $m \cdot n$ points P_{ij}, $0 \leq i \leq m, 0 \leq j \leq n$, it determines the *Bézier surface* $B : [0, 1] \times [0, 1] \to \mathbb{R}^3$ given by

$$B(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) B_j^n(v) P_{ij},$$

where B_i^m and B_j^n are Bernstein polynomials of degree m and n, respectively.

For more details about Bézier curves and surfaces, we recommend the reference [4].

3. Normal approximation of regular curves

Let $\alpha : [0,1] \longrightarrow \mathbb{R}^2$ be a regular parametrized plane curve. We want to approximate it by a Bézier curve, which will be determined by $n + 1$ control points $P_0, P_1, \ldots, P_n$ in $\mathbb{R}^2$. We now describe our method to choose these points:

- **Step 1:** We choose $n + 1$ values $t_0, \ldots, t_n$ in $[0,1]$ such that $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$. We directly put $P_0 = \alpha(0)$, $P_n = \alpha(1)$. For every $i = 1, \ldots, n - 1$, we consider the point

$$P_i(\lambda_i) = \alpha(t_i) + \lambda_i \mathbf{n}(t_i), \tag{1}$$

where λ_i is a real parameter and $\mathbf{n}(t_i)$ is the normal vector to α at $\alpha(t_i)$. This means that the point $P_i(\lambda_i)$ lies on the normal line to α at $\alpha(t_i)$ (see Figure 1)
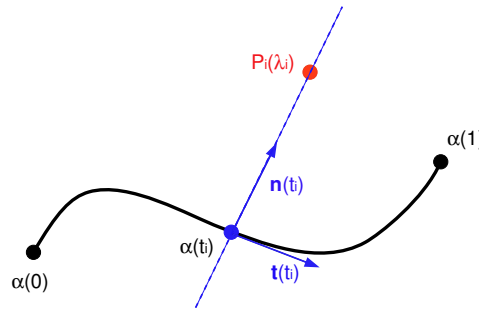

Figure 1: Point $P_i(\lambda_i)$ on the normal line to α at $\alpha(t_i)$.

- **Step 2:** We construct the Bézier curve

$$B(\lambda_1, \ldots, \lambda_{n-1}, t) = B_0^n(t)P_0 + \sum_{i=1}^{n-1} B_i^n(t)P_i(\lambda_i) + B_n^n(t)P_n, \tag{2}$$

depending on the parameters $\lambda_1, \ldots, \lambda_{n-1}$, where $B_i^n(t)$ are the usual Bernstein polynomials.

- **Step 3:** We define the function:

$$F(\lambda_1, \ldots, \lambda_{n-1}) = \int_0^1 |\alpha(t) - B(\lambda_1, \ldots, \lambda_{n-1}, t)|^2 dt. \tag{3}$$

This function is, in fact, the squared semimetric in $L^2([0,1])$. It represents the error between $\alpha(t)$ and $B(\lambda_1, \ldots, \lambda_{n-1}, t)$.

- **Step 4:** In order to obtain the best possible approximation with this method, we minimize the above function, finding values $\lambda_1^0, \ldots, \lambda_{n-1}^0$. Then, the Bézier curve we are looking for is $B(\lambda_1^0, \ldots, \lambda_{n-1}^0, t)$, which is determined by the control points $P_0, P_1(\lambda_1^0), \ldots, P_{n-1}(\lambda_{n-1}^0), P_n$.

  Let us notice how this can be done just by solving a linear equations system given by $\partial F / \partial \lambda_j = 0$, for any $j = 1, \ldots, n - 1$. In fact, by applying the usual "differentiation under the integral sign" criterion, it follows from (3) that

$$\frac{\partial F}{\partial \lambda_j} = \int_0^1 \frac{\partial G}{\partial \lambda_j} dt,$$

where

$$G(\lambda_1, \ldots, \lambda_{n-1}, t) = |\alpha(t) - B(\lambda_1, \ldots, \lambda_{n-1}, t)|^2$$

$$= \sum_{k=1}^{2} (\alpha^k(t) - B^k(\lambda_1, \ldots, \lambda_{n-1}, t))^2,$$

$\alpha^k(t)$ (resp. $B^k(\lambda_1, \ldots, \lambda_{n-1}, t)$) being the k-th component of $\alpha(t)$ (resp. $B(\lambda_1, \ldots, \lambda_{n-1}, t)$). Therefore:

$$\frac{\partial G}{\partial \lambda_j} = 2 \sum_{k=1}^{2} (\alpha^k(t) - B^k(\lambda_1, \ldots, \lambda_{n-1}, t)) \frac{\partial(\alpha^k(t) - B^k(\lambda_1, \ldots, \lambda_{n-1}, t))}{\partial \lambda_j}.$$

But it easily follows from (1) and (2) that

$$\frac{\partial(\alpha^k(t) - B^k(\lambda_1, \ldots, \lambda_{n-1}, t))}{\partial \lambda_j} = -B_j^n(t)\mathbf{n}^k(t_j),$$

where $\mathbf{n}^k(t_j)$ denotes the k-th component of $\mathbf{n}(t_j)$. Thus,

$$\frac{\partial F}{\partial \lambda_j} = -2 \sum_{k=1}^{2} \mathbf{n}^k(t_j) \int_0^1 (\alpha^k(t) - B^k(\lambda_1, \ldots, \lambda_{n-1}, t))B_j^n(t)dt$$

$$= -2 \sum_{k=1}^{2} \mathbf{n}^k(t_j) \left(\int_0^1 \alpha^k(t)B_j^n(t)dt - P_0^k \int_0^1 B_0^n(t)B_j^n(t)dt \right.$$

$$\left. - \sum_{i=1}^{n-1} \alpha^k(t_i) \int_0^1 B_i^n(t)B_j^n(t)dt - P_n^k \int_0^1 B_n^n(t)B_j^n(t)dt \right)$$

$$+ 2 \sum_{i=1}^{n-1} \left(\sum_{k=1}^{2} \mathbf{n}^k(t_i)\mathbf{n}^k(t_j) \int_0^1 B_i^n(t)B_j^n(t)dt \right) \lambda_i,$$

where we have used (1) and (2) again. This expression is clearly linear in $\lambda_1, \ldots, \lambda_{n-1}$.

Example 3.1. *Let $\alpha : [0, 1] \longrightarrow \mathbb{R}^2$ be the parabola given by $\alpha(t) = (t, t^2)$.*

If we want to approximate α with just an intermediate point, we put $n = 2$ and we directly have $P_0 = \alpha(0) = (0, 0)$ and $P_2 = \alpha(1) = (1, 1)$. We choose $t_1 = 1/2$ and construct

$$P_1(\lambda_1) = \alpha(1/2) + \lambda_1 \mathbf{n}(1/2) = \left(\frac{1}{2} - \frac{\sqrt{2}}{2}\lambda_1, \frac{1}{4} + \frac{\sqrt{2}}{2}\lambda_1 \right).$$

Therefore, the Bézier curve depending on parameter λ_1 is given by:

$$B(\lambda_1, t) = B_0^2(t)P_0 + B_1^2(t)P_1(\lambda_1) + B_2^2(t)P_2$$

$$= \left(2(1-t)t \left(\frac{1}{2} - \frac{\sqrt{2}}{2}\lambda_1 \right) + t^2, 2(1-t)t \left(\frac{1}{4} + \frac{\sqrt{2}}{2}\lambda_1 \right) + t^2 \right).$$

Now, we calculate the function appearing in Step 3, resulting:

$$F(\lambda_1) = \int_0^1 |\alpha(t) - B(\lambda_1, t)|^2 dt = \frac{2}{15}\lambda_1^2 + \frac{\sqrt{2}}{30}\lambda_1 + \frac{1}{120}.$$

To minimize this function, we impose $\partial F/\partial \lambda_1 = 0$, and we obtain the linear equation

$$\frac{4}{15}\lambda_1 + \frac{\sqrt{2}}{30} = 0,$$

whose solution is $\lambda_1^0 = -\sqrt{2}/8$. Therefore, our control point is $P_1(\lambda_1^0) = (5/8, 1/8)$ and the corresponding Bézier curve is:

$$B(\lambda_1^0, t) = \left(\frac{5}{4}(1-t)t + t^2, \frac{1}{4}(1-t)t + t^2\right).$$

The error is given by $F(\lambda_1^0) = 0.004166666667$. In Figure 2 we can see both the real curve $\alpha(t)$ (in blue) and its approximation $B(\lambda_1^0, t)$ (in red). We represent in green the control points $P_0, P_1(\lambda_1^0), P_2$.
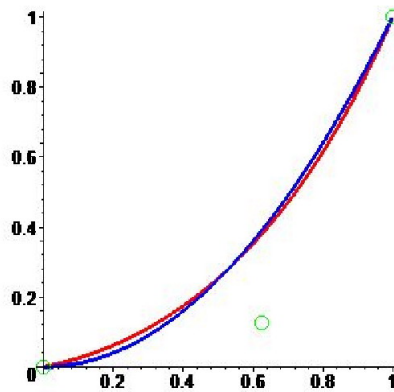

Figure 2: Approximation of a parabola.

If we choose more intermediate control points, we obtain better approximations. The error estimates can be seen in the following table. In all cases, we have chosen the values t_i uniformly distributed.

n	error estimate
2	0.004166666667
3	0.001459250709
4	0.0008372327736

We can improve our method if we take into account a well-known fact for Bézier curves: the derivatives at the endpoints are completely determined. In fact, if $B(t)$ is a Bézier curve given by control points $P_0, P_1, \ldots, P_n$, then

$$B'(0) = n(P_1 - P_0), \quad B'(1) = n(P_n - P_{n-1}).$$

In our case:

$$B'(\lambda_1, \ldots, \lambda_{n-1}, 0) = n(\alpha(t_1) + \lambda_1 \mathbf{n}(t_1) - \alpha(0)),$$
$$B'(\lambda_1, \ldots, \lambda_{n-1}, 1) = n(\alpha(1) - \alpha(t_{n-1}) - \lambda_{n-1} \mathbf{n}(t_{n-1})).$$

If we impose for the Bézier curve $B(\lambda_1, \ldots, \lambda_{n-1}, t)$ of Step 2 that its tangent directions at the endpoints P_0 and P_n are the same as those of the curve $\alpha(t)$, i.e.,

$$B'(\lambda_1, \ldots, \lambda_{n-1}, 0) \cdot \mathbf{n}(0) = 0, \quad B'(\lambda_1, \ldots, \lambda_{n-1}, 1) \cdot \mathbf{n}(1) = 0, \tag{4}$$

then we obtain two linear equations easily determining $\lambda_1^0, \lambda_{n-1}^0$. We follow our method to obtain the other parameters $\lambda_2^0, \ldots, \lambda_{n-2}^0$ and we say that the Bézier curve $B(\lambda_1^0, \ldots, \lambda_{n-1}^0, t)$ is "clamped" at the endpoints P_0, P_n.

Example 3.2. *We apply the above improvement to the parabola of Example 3.1. As we will easily determine control points $P_1(\lambda_1)$ and $P_{n-1}(\lambda_{n-1})$, we put $n = 4$ and we directly have $P_0 = (0,0)$ and $P_4 = (1,1)$. We choose $t_1 = 1/4, t_2 = 1/2, t_3 = 3/4$ and we construct $P_1(\lambda_1), P_2(\lambda_2), P_3(\lambda_3)$ in our usual way.*

Now, we write equations from (4):

$$\frac{1}{4} + \frac{4\sqrt{20}}{5}\lambda_1 = 0, \quad -\frac{\sqrt{5}}{20} - \frac{32\sqrt{65}}{65}\lambda_3 = 0.$$

The solutions are $\lambda_1^0 = -\sqrt{5}/32$ and $\lambda_3^0 = -\sqrt{13}/128$. They completely determine $P_1(\lambda_1^0)$ and $P_3(\lambda_3^0)$. Therefore, we only have to obtain the value for the parameter λ_2 of the Bézier curve $B(\lambda_1^0, \lambda_2, \lambda_3^0, t)$ by minimizing the function $F(\lambda_1^0, \lambda_2, \lambda_3^0)$. The result is $\lambda_2^0 = -59\sqrt{2}/1536$ and the error is now $F(\lambda_1^0, \lambda_2^0, \lambda_3^0) = 0.000873819987$, which is smaller than $F(\lambda_1^0)$ from Example 3.1.

This method to approximate plane curves can also be adapted for spacial curves, i.e., curves in $\mathbb{R}^3$. In such a case, $\mathbf{n}(t_i)$ would be the principal normal vector to α at $\alpha(t_i)$ (see Figure 3). To do so, we need $\kappa(t_i) \neq 0$, where $\kappa(t)$ denotes the curvature of $\alpha(t)$ (if not, $\mathbf{n}(t_i)$ is not defined). Therefore, we should be more careful with the choosing of the values of $t_1, \ldots, t_{n-1}$.
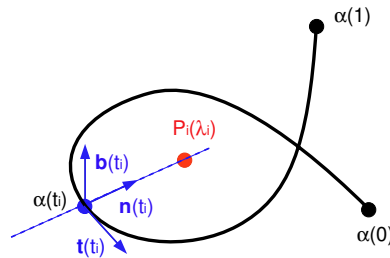

Figure 3: Point $P_i(\lambda_i)$ on the principal normal line to α at $\alpha(t_i)$.

Example 3.3. *Let $\alpha : [0,1] \longrightarrow \mathbb{R}^3$ be the circular helix given by $\alpha(t) = (\cos(2\pi t), \sin(2\pi t), t)$.*

In Figure 4 we show the approximations (in red) of α (in blue) for $n = 2, 3, 4, 5$, i.e., with $3, 4, 5, 6$ control points (in green), respectively. The error estimates can be seen in the following table:

n	error estimate
2	0.5830658815
3	0.1466088404
4	0.04716853518
5	0.02161578330

Let us notice how the approximation for $n = 2$ is very bad, because a Bézier curve determined by 3 control points is always a plane curve.
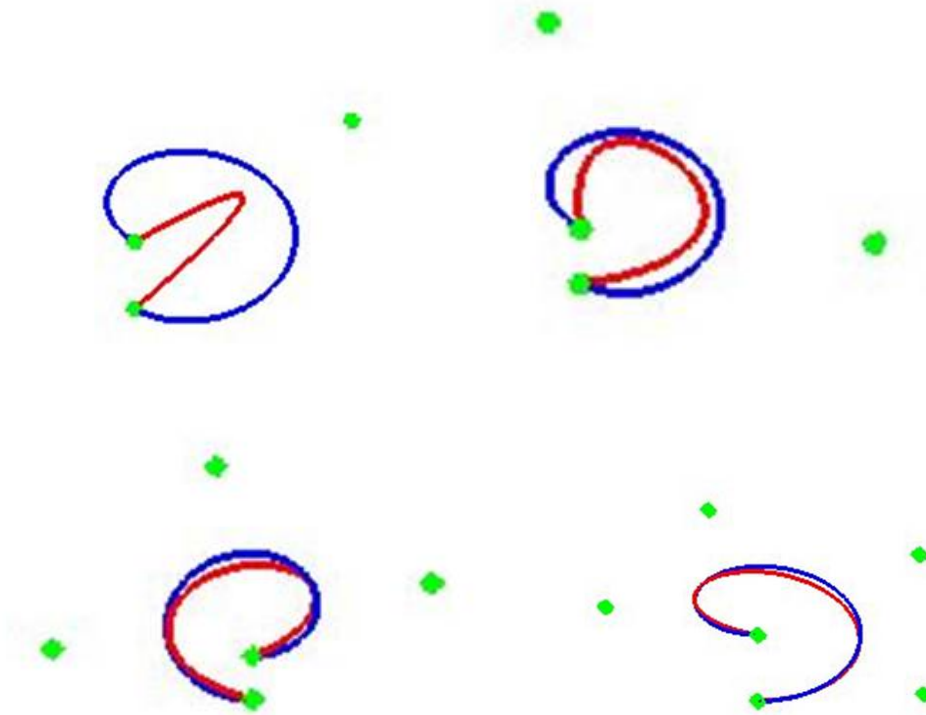
Figure 4: Approximations of a circular helix.

4. Normal approximation of regular surfaces

Let $\mathbf{x} : [0,1] \times [0,1] \longrightarrow \mathbb{R}^3$ be a regular parametrized surface. We want to approximate it by a Bézier surface, which will be determined by a grid of $m \cdot n$ control points P_{ij} in $\mathbb{R}^3$. We follow these steps to choose the points:

- **Step 1:** We approximate the border curves

$$\alpha_1(t) = \mathbf{x}(t,0), \quad \alpha_2(t) = \mathbf{x}(0,t), \quad \alpha_3(t) = \mathbf{x}(t,1), \quad \alpha_4(t) = \mathbf{x}(1,t),$$

by Bézier curves $B_1(t), \ldots, B_4(t)$, by following the method described in Section 3. Therefore, we determine the exterior points of the control grid.

For the interior points of the control grid, we write

$$P_{ij}(\lambda_{ij}) = \mathbf{x}(u_i, v_j) + \lambda_{ij}\mathbf{N}(u_i, v_j),$$

where λ_{ij} is a real parameter and $\mathbf{N}(u_i, v_j)$ is the normal vector to $\mathbf{x}$ at $\mathbf{x}(u_i, v_j)$. This means that the point $P_{ij}(\lambda_{ij})$ lies on the normal line to $\mathbf{x}$ at $\mathbf{x}(u_i, v_j)$.

- **Step 2:** We construct the Bézier surface $B(\lambda_{ij}, u, v)$, depending on the parameters λ_{ij}.

- **Step 3:** We define the function:

$$F(\lambda_{ij}) = \int_0^1 \int_0^1 |\mathbf{x}(u,v) - B(\lambda_{ij}, u, v)|^2 dudv.$$

- **Step 4:** We minimize the above function, finding values λ_{ij}^0. Then, the Bézier surface we are looking for is $B(\lambda_{ij}^0, u, v)$, and the interior points of the control grid are $P_{ij}(\lambda_{ij}^0)$.

Example 4.1. *Let $x : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^3$ be the piece of a circular cylinder given by $x(u, v) = (u, \cos(\pi v), \sin(\pi v))$ and shown in the first frame of Figure 5. The border curves are painted in blue.*

To follow the above described method, we begin by approximating the border curves by Bézier curves with $n = 3$. This will produce a grid of 16 control points (with 4 interior points). The obtained Bézier curves are shown in the second frame of Figure 5. Let us notice how, given that α_1 and α_3 are just line segments, $B_1(t) = \alpha_1(t)$ and $B_3(t) = \alpha_3(t)$, and the corresponding control points lie on these line segments. We then follow our method to determine the interior control points. The resulting Bézier surface is shown in the third frame of Figure 5. Its border is given by Bézier curves $B_1(t), \dots, B_4(t)$.

Figure 5: Approximation of a piece of circular cylinder.

With this method we can approximate any regular parametrized surface. For example, approximations of a piece of a helicoid, a catenoid and a torus are shown in Figure 6.
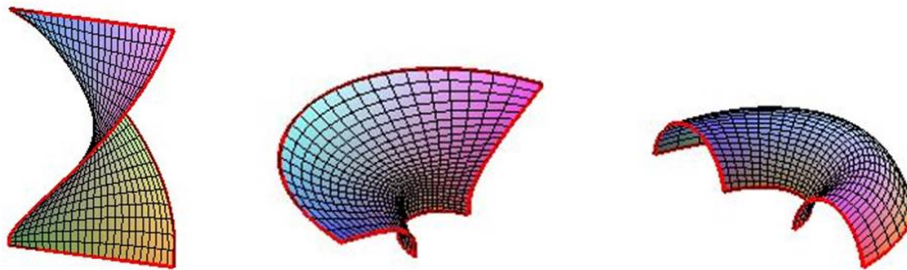

Figure 6: Approximations of helicoid, catenoid and torus.

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