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Study of the Contribution of Nonlinear Normal Modes (NNMs) in Large Amplitude Oscillations of Simply Supported Beams

Javier González-Carbajal*, Daniel García-Vallejo, Jaime Domínguez

Department of Mechanical and Engineering and Manufacturing, Faculty of Engineering of Seville, C/ Camino de los Descubrimientos s/n, 41092 Seville, Spain

Abstract

This paper focuses on the Nonlinear Normal Modes of simply supported beams with unrestrained axial displacements. Two different configurations are considered, depending on whether longitudinal displacements are allowed at one end of the beam or at both ends. An integro-differential equation is obtained for the transverse displacement of the beam, upon the common assumption of inextensibility. By using a perturbation approach, the NNMs are analytically computed, which yields a frequency-amplitude relation for each NNM. These analytical curves are compared to FE results, showing a remarkable accordance. Noticeably, qualitatively different behaviors are found for the first NNM in both configurations: with one free end, the beam softens; with both ends free, it hardens.

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1. Introduction

The dynamic behaviour of linear elastic beams under hypothesis of small strains and small displacements is well-known. However, in numerous applications, deflections are large enough to make the assumption of small displacements no more suitable. In these cases, the equilibrium needs to be imposed on the deformed configuration of the structure, what makes the system nonlinear.

Simply supported beams are usually classified into two groups, depending on whether longitudinal displacements are restrained or not. This paper focuses on the unrestrained case, under two different configurations: one free end (unsymmetrical case, Fig. 1(a)) or both ends free (symmetrical case, Fig. 1(b)). The objective is two obtain and discuss the nonlinear normal modes of the beam for moderately large deflections, by applying perturbation methods to the equations of motion. In the restrained case (Fig. 1(c)), Mettler showed that the main nonlinearity is geometric hardening due to the stretching of the midline of the beam [1].

Different authors have contributed to the study of the unsymmetrical case. For instance, Thomsen [2] and Han et al. [3], used an Eulerian description of the motion in which time and horizontal position in the deformed beam were utilized as independent variables. In their work, they assumed a priori that longitudinal inertia could be neglected. Also when studying the unsymmetrical case, Lacarbonara et al. [4] obtained an integro-differential equation in terms of the transverse deflection depending both on geometric and inertial terms. On the other hand, the symmetrical case has been less extensively studied in the literature, where the numerical analyses performed by Woodall [5] are remarkable. In his work, Woodall utilized different numerical techniques to obtain models of the simply supported beam with unrestrained ends.

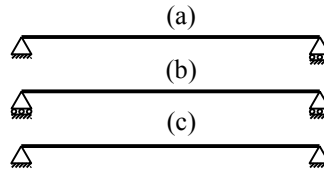


Fig. 1 Configurations of a S-S beam: (a) unsymmetrical axially unrestrained; (b) symmetrical axially unrestrained; (c) Axially restrained

The analytical treatment presented in this paper leads to a relation between the oscillation frequency and the amplitude for each NNM, both for the symmetrical and unsymmetrical cases. The NNM method has been analysed in the work of Kerschen et al. [10] who described in simple terms the properties of such method. These analytical curves are validated by comparing them with finite element results. Other methodologies to study nonlinear oscillation of beams can be found in the literature. Worth of mention is the study by Azrar et al. [11, 12], where a general model based on Hamilton's principle and spectral analysis is utilized to study the non-linear free vibrations occurring at large displacement amplitudes. In addition, the work of Claeys et al. [13], mainly based on the multiple scales method, includes a comparison of measured and simulated nonlinear vibrations of a clamped-clamped steel beam with non-ideal boundary conditions. This work contributes to the study of nonlinear oscillations of simply supported beams by providing results that are validated against numerical results of a FEM model.

2. General Equations

First, it is convenient to state the assumptions on which the presented study is based: an initially straight beam with uniform cross section is considered. The motion is assumed to be planar, with plane sections remaining plane. Shear deformations and moment of inertia of cross sections are assumed to be negligible. The material is assumed to be linear and elastic and the strains are assumed to be small. The deflections are taken to be moderately large. Based on this, only nonlinearities up to order three will be considered. The beam is assumed to be inextensible. This is a usual assumption in the bending analysis of axially unrestrained beams [4–7], based on the fact that the axial stiffness of a slender beam is generally much greater than its bending stiffness. Finally, no damping or external excitation will be considered.

The initial and deformed configurations of the beam are shown in Fig. 2. The longitudinal and transverse forces (according to the direction of the undeformed beam) are represented by H and V , respectively, while M stands for the bending moment. Coordinate X measures the position along the middle line of the undeformed beam, varying from 0 to the total length L . The longitudinal and transverse displacements are u and v , respectively, while ψ represents the angle of rotation of the section. Since shear deformations are not being considered, it coincides with the angle of rotation of the middle line of the beam. Taking a lagrangian description of the motion, a prime will be used for partial derivatives with respect to X , while a dot will stand for partial derivatives with respect to time t .

Since the material is assumed to be elastic and linear, the bending moment is proportional to the curvature of the beam:

$$M = EI\psi', \quad (1)$$

where E stands for the Young modulus of the material and I is the moment of inertia of the cross section around the bending axis. With some simple geometric relations, the curvature can be written in terms of the displacements [7]:

$$\begin{cases} \sin \psi = v' \\ \cos \psi = 1 + u' \end{cases} \Rightarrow \psi' = \frac{v''}{1 + u'} \tag{2}$$

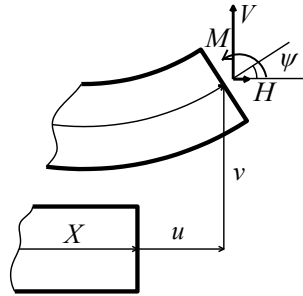


Fig. 2 Forces and displacements

Then, the nonlinear equations of motion of the beam can be written as

$$\left\{ \rho A \ddot{u} = H', \quad \rho A \ddot{v} = V', \quad M' + V(1 + u') - H v' = 0, \quad M = EI \frac{v''}{1 + u'}, \quad u' = -\frac{v'^2}{2} \right\}. \tag{3}$$

The first three of these equations represent the equilibrium of horizontal and vertical forces and the equilibrium of moments, which can be readily obtained by considering the forces and moments acting on a differential element of the beam. The fourth relation represents the constitutive law, obtained by combining (1) and (2). The last of equations (3) corresponds to the inextensibility condition [7].

In order to obtain a single equation for the unknown function $v(X, t)$, the moments equilibrium equation is divided by $(1 + u')$ and differentiated with respect to X . After some manipulations, and retaining nonlinear terms up to order three, the following equation is obtained:

$$\rho A \ddot{v} + EI[v^{IV} + (v'(v'v''))'] - [Hv']' = 0. \tag{4}$$

The only remaining step consists in writing the longitudinal force H in terms of the transverse displacement. In order to do so, consider the subsystem formed by the first of relations (3) and the inextensibility condition, differentiated twice with respect to time:

$$\{H' = \rho A \ddot{u}, \quad \ddot{u}' = -(v'\ddot{v}' + \dot{v}'^2)\} \tag{5}$$

The axial boundary conditions needed to solve system (5) are different for the symmetrical and unsymmetrical cases (letters (a) and (b) label the equations corresponding to the unsymmetrical and symmetrical cases, respectively):

$$(a) u(0, t) = H(L, t) = 0, \quad (b) H(0, t) = H(L, t) = 0. \tag{6}$$

Integrating (5) with boundary conditions (6) leads to

$$H = -\rho A \int_L^X \int_0^z (v'\ddot{v}' + \dot{v}'^2) dy dz \tag{7a}$$

$$H = -\rho A \int_L^X \int_0^z (v'\ddot{v}' + \dot{v}'^2) dy dz + \rho A \frac{(X-L)}{L} \int_0^L \int_0^z (v'\ddot{v}' + \dot{v}'^2) dy dz \tag{7b}$$

Clearly, introducing (7) into (4) yields the desired equation for $v(X, t)$. However, it is convenient to use dimensionless variables:

$$v^* = v/L, u^* = u/L, \xi = X/L, \tau = \sqrt{EI/\rho AL^4} t \quad (8)$$

Omitting the asterisks, and using now a prime and a dot for partial differentiation with respect to ξ and τ , respectively, an integro-differential equation is obtained for function $v(\xi, \tau)$:

$$\ddot{v} + v^{IV} + N_G(v(\xi, \tau)) + N_I(v(\xi, \tau)) = 0, \text{ where } N_G = [v'(v'v'')] \quad (9)$$

$$N_I = \left[v' \int_1^\xi \int_0^\eta (v' \ddot{v}' + \dot{v}'^2) d\gamma d\eta \right]' \quad (10a)$$

$$N_I = \left[v' \int_1^\xi \int_0^\eta (v' \ddot{v}' + \dot{v}'^2) d\gamma d\eta \right]' + [v'(1 - \xi)]' \int_0^1 \int_0^\eta (v' \ddot{v}' + \dot{v}'^2) d\gamma d\eta, \quad (10b)$$

with boundary conditions

$$v(0, \tau) = v''(0, \tau) = v(1, \tau) = v''(1, \tau) = 0 \quad (11)$$

Note the presence of two different nonlinearities in (9). There exists a geometric nonlinear term N_G , due the nonlinear expression for the curvature (2) and the fact that the lever arm associated to vertical force V depends on the beam deformation. On the other hand, an inertial nonlinear term N_I accounts for the bending moment generated by horizontal force H , which in turn is produced by the longitudinal inertia of the beam.

3. Computation of the NNMs

In this section, the NNMs of the beam are computed as particular solutions of the general equation (9). Following the procedure described in [8], the transverse displacement is expanded as

$$v(\xi, \tau) = \sum_{j=1}^{\infty} \phi_j(\xi) q_j(\tau), \quad (12)$$

where $\phi_j(\xi)$ represents the j -th linear mode of the simply supported beam: $\phi_j(\xi) = \sin(j\pi\xi)$. Introducing (12) in (9), multiplying by each $\phi_j(\xi)$ and integrating over the beam length, yields

$$\ddot{q}_j + \omega_j^2 q_j + G_j = 0, \quad j = 1, 2, \dots \quad (13)$$

where $\omega_j = (j\pi)^2$ and

$$G_j = 2 \langle \phi_j, N_G \left(\sum_{m=1}^{\infty} \phi_m q_m \right) \rangle + 2 \langle \phi_j, N_I \left(\sum_{m=1}^{\infty} \phi_m q_m \right) \rangle. \quad (14)$$

The following notation has been used: $\langle f_1(\xi), f_2(\xi) \rangle \equiv \int_0^1 f_1(\xi) f_2(\xi) d\xi$. Note that equation (13), with definition (14), is exactly the same as (9), with the only difference of using modal coordinates. No approximations have been made yet.

In order to compute the k -th NNM, i.e. the nonlinear extension of the k -th linear mode, it is convenient to recall the following property of NNMs, according to the definition of Shaw and Pierre [9]: when the system oscillates along its k -th NNM, coordinates q_j, \dot{q}_j with $j \neq k$ (slaved variables) are functions of order 2 or higher in q_k and \dot{q}_k , (master variables). Thanks to this feature, (14) can be approximated as

$$G_j = 2\langle \phi_j, N_G(\phi_k q_k) \rangle + 2\langle \phi_j, N_I(\phi_k q_k) \rangle + \dots \tag{15}$$

where the dots represent higher order terms. Particularizing (13) and (15) for $j = k$ leads to the governing equation for the master coordinate q_k :

$$\ddot{q}_k + \omega_k^2 q_k + \frac{(k\pi)^6}{2} q_k^3 + \frac{(k\pi)^2}{8} \left[\frac{4(k\pi)^2}{3} - \frac{3}{2} \right] [q_k \dot{q}_k^2 + q_k^2 \ddot{q}_k] + \dots = 0 \tag{16a}$$

$$\ddot{q}_k + \omega_k^2 q_k + \frac{(k\pi)^6}{2} q_k^3 + \frac{(k\pi)^2}{8} \left[\frac{(k\pi)^2}{3} - \frac{3}{2} \right] [q_k \dot{q}_k^2 + q_k^2 \ddot{q}_k] + \dots = 0 \tag{16b}$$

By using a perturbation method, such as a multiple scales approach, equation (16) can be solved:

$$q_k(\tau) = a_k \cos(\omega_{kNL} \tau + \beta_k) + \left(\frac{k\pi}{8} \right)^2 \left[\frac{7}{4} - \frac{2}{3} (k\pi)^2 \right] a_k^3 \cos(3\omega_{kNL} \tau + 3\beta_k) + \dots \tag{17a}$$

$$q_k(\tau) = a_k \cos(\omega_{kNL} \tau + \beta_k) + \left(\frac{k\pi}{8} \right)^2 \left[\frac{7}{4} - \frac{1}{6} (k\pi)^2 \right] a_k^3 \cos(3\omega_{kNL} \tau + 3\beta_k) + \dots \tag{17b}$$

where the nonlinear oscillation frequency ω_{kNL} is given by

$$\frac{\omega_{kNL}}{\omega_k} = 1 + \left[\frac{3}{16} (k\pi)^2 - \frac{(k\pi)^2}{8} \left(\frac{(k\pi)^2}{3} - \frac{3}{8} \right) \right] a_k^2 + \dots \tag{18a}$$

$$\frac{\omega_{kNL}}{\omega_k} = 1 + \left[\frac{3}{16} (k\pi)^2 - \frac{(k\pi)^2}{8} \left(\frac{(k\pi)^2}{12} - \frac{3}{8} \right) \right] a_k^2 + \dots \tag{18b}$$

and $\{a_k, \beta_k\}$ are constants which depend on the initial conditions.

4. Results

In order to see the kind of nonlinear behaviour exhibited by the beam in the two considered configurations, it is interesting to plot the relation between frequency and amplitude, given in equation (18). These curves are represented, for the first NNM, in Fig. 3, where they are compared to FE results obtained with Abaqus®, discretizing the beam in 16 elements with cubic interpolation. In these FE simulations, the beam was discretized in 16 elements with cubic interpolation and the initial conditions were chosen as belonging to the first NNM of the beam. Note the remarkable accordance between numerical and analytical results.

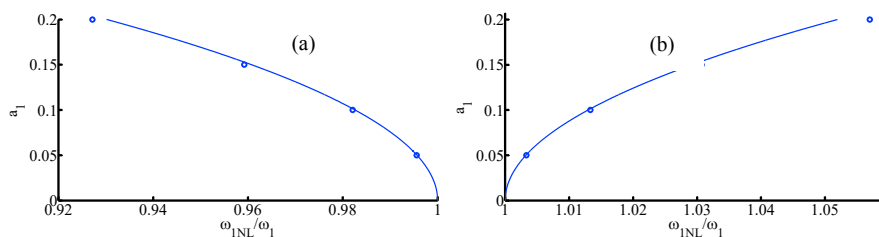


Fig. 3 Backbone curves for the first NNM: (a) unsymmetrical case; (b) symmetrical case

A solid line represents the analytical curves given by (18), while the dots correspond to FE results

5. Conclusions

The main conclusion of this study is the qualitatively different effect of nonlinearities in the two considered cases: for the first NNM, the unsymmetrical beam exhibits a softening behaviour, while a hardening effect is evidenced for the symmetrical configuration. This can be explained by considering that, in the square brackets of equation (18), the first term, which produces hardening, corresponds to the geometric nonlinearity, while the second term, of the softening type, is associated to the inertial nonlinearity. Note that, while the geometric term is the same for both configurations, the inertial term is considerably greater in the unsymmetrical case. Thus, the global behaviour of the beam can be understood as the result of the competition between two opposing nonlinearities: for the first NNM, the inertial term is dominant in the unsymmetrical case, and the geometric term is dominant in the symmetrical case. It is also interesting to note that, from the second NNM on, both configurations exhibit a softening behavior, according to (18). This is due to the fact that the inertial nonlinearity grows with k faster than the geometrical. The reasonably good accordance between analytical and Finite Element results (with axially extensible elements) indicates that the assumption of inextensible middle line, used for the axially unrestrained case, is pertinent –at least for the first two NNMs–.

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