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# Non-null electromagnetic fields and compacted spin coefficient formalism in general relativity

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**Abstract** The non-null electromagnetic fields have been studied through the compacted spin coefficient formalism due to Geroch, Held and Penrose (GHP) The propagation equations for the shear, twist and expansion of the congruences have been obtained and the conditions are given under which the coupling of twist and expansion is possible The behaviour of the modified Lie derivative operator on the electromagnetic bivector, Ricci tensor and the metric tensor has also been studied

**Keywords** Non-null electromagnetic fields, GHP formalism, modified Lie derivative

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## 1. Introduction

It is known that the spin-coefficient formalism due to Newman and Penrose [1] can successfully be used in treating many problems of general relativity. An extension to this formalism is given by Geroch, Held and Penrose [2]. This formalism is more concise and efficient than the widely known NP formalism. However, Geroch-Held-Penrose formalism (abbreviated as GHP-formalism) has failed to develope to its full potential to the extent to which the NP formalism has. About twentyfive years ago, soon after the appearence of GHP-formalism, Held [3,4] proposed a simple procedure for integration within this formalism and applied it to Petrov type D vacuum metrics. The geometrical meanings of the spin coefficients appearing in this formalism have been given by Ahsan and Malik [5]. Recently, GHP-formalism has again attracted the attention of several workers and in this connection, Ludwig [6] has given an extension to this formalism by considering only quantities that transform properly under all diagonal transformations of the underlying spin-frame, *i e.*, not only under boost-rotation but also under conformal rescaling. The role of commutator relations in this extended formalism has been explored by Edgar [7]. On the other hand, Kolassis and Ludwig [8] have studied the space-times which admit a two dimensional group of conformal motion (and in particular homothetic motion). The so called post Bianchi identities, which play a crucial role in search of Petrov type I solutions of Einstein field equations, have been studied by Ludwig [9] through GHP-formalism. More recently, a procedure for integration within this formalism has been given by Edgar and coworkers [10–15].

Motivated by these applications of GHP-formalism, the non-null electromagnetic fields have been studied in this paper using this formalism. In Section 2, the Maxwell equations for an electromagnetic field of arbitrary type and also for non-null electromagnetic field are given. A study of various properties of the congruences has been made and it is seen that the expansion and twist of the congruences can be coupled together for a non-null electromagnetic field. The behaviour of the modified Lie derivative operator on the electromagnetic field bivector, Ricci tensor and the metric tensor is the subject of study of Section 3, while a discussion of the results has been made in Section 4. Some of the important results concerning GHP-formalism are given in the appendix. A detailed account of such and other related results can be found in [16].

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## 2. The Maxwell's equations and the non-null electromagnetic fields

Let $M$ be a four dimensional Lorentzian manifold that admits a Lorentzian metric of signature $(- - - +)$. Let $Z^a_\mu = \{l^a, n^a, m^a, \overline{m}^a\}$, $(\mu = 1, 2, 3, 4)$, be the complex null tetrad, where $l^a$, $n^a$ are real null vectors and $m^a$, $\overline{m}^a$ are the complex null vectors. All the inner products between the tetrad vectors vanish except $l_a n^a = 1 = -m_a \overline{m}^a$ With these orthogonality properties and the nullity of the tetrad, $g_{ij}$ can be written as

$$g_{ij} = 2l_{(i}n_{j)} - 2m_{(i}\overline{m}_{j)}. \qquad (1)$$

In terms of the complex null tetrad $Z^a_\mu$, the electromagnetic bivector $F_{ij}$ has the following form [1,17]

$$F_{ij} = -2\Re\phi_1 l_{[i}n_{j]} + 2i\Im\phi_1 m_{[i}\overline{m}_{j]} + \phi_2 l_{[i}m_{j]} + \overline{\phi}_2 l_{[i}\overline{m}_{j]} - \overline{\phi}_0 n_{[i}m_{j]} - \phi_0 n_{[i}m_{j]}, \qquad (2)$$

where $\phi_0 = 2F_{ij}l^i m^j$, $\quad \phi_1 = F_{ij}\left(l^i n^j + \overline{m}^i m^j\right)$,

$$\phi_2 = 2F_{ij}\overline{m}^i n^j \qquad (3)$$

are the complex scalars, $\Re\phi_1$ and $\Im\phi_1$, respectively, denote the real and imaginary parts of $\phi_1$. The quantity $\phi_1$ describes the Coulomb component of the field, while the component $\phi_2$ arises from the electric dipole radiation of an accelerated charge. If acceleration is absent then $\phi_2 = 0$.

Depending upon the vanishing of the Maxwell's scalars (3), the electromagnetic field can be classified as [1,17]

Type A : non null (non-singular) : $\phi_0 = \phi_2 = 0$, $\phi_1 \neq 0$,

Type B : null (singular) : $\phi_0 = \phi_1 = 0$, $\phi_2 \neq 0$, (4)

Type C : null (singular) : $\phi_1 = \phi_2 = 0$, $\phi_0 \neq 0$.

It may be noted that in fact, there are just two types (types A and B). Types B and C can be transformed into each other by switching $l^a$ and $n^a$ in the null basis $\{l^a, n^a, m^a, \overline{m}^a\}$. [1]For the sake of completeness, we have mentioned here all the three types. The propagation vector for type B is $l^a$ while for type C it is $n^a$.

The source-free Maxwell's equations

$$\nabla_j F^{ij} = 0,\ \nabla_j F^{*ij} = 0, \qquad (5)$$

[1]The electromagnetic tensor field $F_{ab}$ (in spinor language) is determined by a symmetric spinor $\Phi_{AB}$ and one can write

$$\Phi_{AB} = \alpha_A \beta_B + \alpha_B \beta_A,$$

where $\alpha$ and $\beta$ are spinors If $\alpha$ and $\beta$ are linearly independent, the electromagnetic field is said to be *algebraically general*, otherwise it is *algebraically special* According to this terminology, in fact we are studying the algebraically general electromagnetic fields in this paper However, in the literature the terms 'non-null' and 'null' are commonly used for algebraically general and algebraically special electromagnetic fields, respectively

*For typgraphical reasons these derivatives are denoted by $P$, $P'$, $D$ and $D'$. $P$ and $D$ are pronounced as throu and edth

where $F^{ij}$ is a real bivector and $F^{*ij}$ is its dual, can be expressed as [1,17]

$$\nabla_j N^{ij} = 0, \qquad (6)$$

where $N^{ij} = \frac{1}{2}\left(F^{ij} + iF^{*ij}\right) = \phi_2 l^{[i}m^{j]} - \phi_1\left\{l^{[i}n^{j]} - m^{[i}\overline{m}^{j]}\right\} - \phi_0 n^{[i}\overline{m}^{j]}. \qquad (7)$

It is known [2] that GHP-formalism deals with scalars associated with a tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$ where the scalars undergo transformation

$$\eta \to \lambda^p \overline{\lambda}^q \eta, \qquad (8)$$

whenever the tetrad is changed according to

$$l^a \to \lambda\overline{\lambda} l^a,\ n^a \to \lambda^{-1}\overline{\lambda}^{-1} n^a,\ m^a \to \lambda\overline{\lambda}^{-1} m^a. \qquad (9)$$

Such a scalar is called a spin and boost weighted scalar of type $\{p, q\}$. The spin weight is $\frac{1}{2}(p-q)$ and the boost weight is $\frac{1}{2}(p+q)$.

Out of the twelve spin coefficients appearing in GHP-formalism, only eight are found to be of good spin and boost; the remaining four appear in the definition of the derivatives so that the derivatives may not behave badly under spin and boost transformations. For a scalar $\lambda$ of type $\{p, q\}$, these derivatives are defined as ([2],[16])*

$$\mathcal{P}\eta = (D - p\epsilon - q\overline{\epsilon})\eta,$$
$$\mathcal{P}'\eta = (D' + p\epsilon' + q\overline{\epsilon}')\eta, \qquad (10a)$$
$$\mathcal{D}\eta = (\delta - p\beta - q\overline{\beta}')\eta,$$
$$\mathcal{D}'\eta = (\delta' + p\beta' - q\overline{\beta})\eta. \qquad (10b)$$

From eqs. (2), (3), (6), (7) and (10), the GHP version of the source-free Maxwell equations for an electromagnetic field of arbitrary type are equivalent to

$$\mathcal{P}\phi_1 - \mathcal{D}'\phi_0 = 2\rho\phi_1 - \tau'\phi_0 - \kappa\phi_2, \qquad (11a)$$
$$\mathcal{D}\phi_1 - \mathcal{P}'\phi_0 = 2\tau\phi_1 - \rho'\phi_0 - \sigma\phi_2, \qquad (11b)$$
$$\mathcal{P}\phi_2 - \mathcal{D}'\phi_1 = \rho\phi_2 - 2\tau'\phi_1 + \sigma'\phi_0, \qquad (11c)$$
$$\mathcal{D}\phi_2 - \mathcal{P}'\phi_1 = \tau\phi_2 - 2\rho'\phi_1 + \kappa'\phi_0, \qquad (11d)$$

where $\phi_0 = -\phi'_2 : \{2, 0\}$,

$$\phi_1 = -\phi'_1 : \{0, 0\}, \qquad (12)$$
$$\phi_2 = -\phi'_0 : \{-2, 0\}$$

are the Maxwell scalars defined by eq. (3); and $\{p, q\}$ denotes the spin and boost weight of these scalars.

From equs. (4) and (11), the source-free Maxwell equations for a non-null electromagnetic field are equivalent to

$$\mathcal{P}\phi = 2\rho\phi,\ \mathcal{D}\phi = 2\tau\phi,\ \mathcal{D}'\phi = 2\tau'\phi,\ \mathcal{P}'\phi = 2\rho'\phi, \qquad (13)$$

where $\phi = \phi_1$, while for a null electromagnetic field of type B, the source-free Maxwell equations are equivalent to

$$\mathcal{P}\phi = \rho\phi,\ \mathcal{D}\phi = \tau\phi,\ \kappa = \sigma = 0, \qquad (14)$$

where $\phi = \phi_2$ and $l^a$ is the principal null direction; for a null electromagnetic field of type C, the source-free Maxwell equations are equivalent to

$$\text{Þ}'\phi = \rho'\phi,\ \text{ð}'\phi = \tau'\phi,\ \kappa' = \sigma' = 0, \tag{15}$$

where $\phi = \phi_0$ and $n^a$ is the principal null direction.

For the existence of a solution $\phi$ of a non-null electromagnetic field, the necessary and sufficient condition is that the commutators $[\text{Þ}, \text{ð}], [\text{Þ}, \text{ð}'], [\text{Þ}', \text{ð}], [\text{Þ}', \text{ð}'], [\text{Þ}, \text{Þ}']$ and $[\text{ð}, \text{ð}']$ as computed from GHP-commutators (A11–A13) agree with the commutators obtained from GHP-field equations (A5–A10). The agreement between the commutators exists if and only if the following equations are satisfied.

$$\text{Þ}'\kappa - \text{ð}'\sigma = (2\tau' - \bar{\tau})\sigma - \bar{\rho}'\kappa - 2\Psi_1, \tag{16a}$$

$$\text{Þ}\tau' - \text{ð}'\rho = \bar{\rho}\tau' + \sigma\tau - \bar{\tau}'\rho - \kappa\rho', \tag{16b}$$

$$\text{Þ}'\tau - \text{ð}\rho' = \bar{\rho}\tau + \sigma\tau' - \bar{\tau}'\rho' - \kappa\rho, \tag{16c}$$

$$\text{Þ}\kappa' + \text{ð}\sigma' = \bar{\rho}(\tau' - \kappa') + \bar{\rho}'(\bar{\tau} - \bar{\tau}') + \sigma(2\tau - \bar{\tau}') + \rho(\kappa' - \kappa) - \bar{\rho}'\tau' - 2\Psi_3, \tag{16d}$$

$$D\tau' \quad \text{ð}'\tau = \rho\bar{\rho}' - \rho'\bar{\rho}, \tag{16e}$$

$$\text{Þ}\rho' - \text{Þ}'\rho = \tau\bar{\tau} - \tau'\bar{\tau}'. \tag{16f}$$

The set of eqs. (16) has been obtained by using GHP-commutators (A11–A13) and GHP-field equations (A5–A10), *e.g.*, eq (16a) can be obtained by using the definition of $[\text{Þ}, \text{ð}]\phi$, eq. (A12) and GHP-field eqs. (A7) and (A8). Although the set of eqs. (16) appears to be a complicated one but important conclusions can be made under some special choices of the spin coefficients and we have

*Theorem 1* · Let a non-null electromagnetic field satisfies the source-free Maxwell equations. Suppose it is possible to propagate the complex null tetrad parallelly along the null geodesic congruences $C(l^a)$ and $C(n^a)$ then the set of eq. (16) reduces to

$$\text{ð}'\sigma = 2\Psi_1,\ \text{ð}\sigma' = -2\Psi_3, \tag{17}$$

$$\text{ð}'\rho = 0 = \text{ð}\rho', \tag{18}$$

$$\rho\bar{\rho}' = \rho'\bar{\rho}, \tag{19}$$

$$\text{Þ}\rho' = \text{Þ}'\rho. \tag{20}$$

*Remark :* Eq. (17) describe the propagation of the shear of the congruences $C(l^a)$ and $C(n^a)$. The propagation of expansion and twist is given by eq. (18), while eqs. (19) and (20) describe the coupling of the expansion and twist.

This theorem can easily be proved by using eq. (16) under the hypothesis of the theorem.

The above coupling of expansion and twist do exist even under weaker conditions as described by the following theorem.

*Theorem 2* · Let a non-null electromagnetic field satisfies the source-free Maxwell equation and suppose that the tetrad $Z^a_\mu$ can be chosen such that $\tau$ and $\tau'$ are constant then the set of eq. (16) reduces to

$$\text{ð}'\sigma = -\tau'\sigma + (\bar{\tau}' - \tau)\rho + \bar{\rho}'\kappa + \Psi_1 - \Phi_{01}, \tag{21}$$

$$\text{ð}'\rho = -\bar{\rho}\tau' - \sigma\tau + \bar{\tau}'\rho + \kappa\rho', \tag{22}$$

$$\text{ð}\rho' = -\bar{\rho}\tau - \sigma\tau' - \bar{\tau}'\rho' + \kappa\rho, \tag{23}$$

$$\text{ð}\sigma' = \bar{\rho}(\tau' - \kappa') + \tau'(\rho' - \bar{\rho}') + \rho(\kappa' - \kappa) - \rho'\bar{\tau}' + \sigma\tau - \Psi_3 + \Phi_{21}, \tag{24}$$

$$\rho\bar{\rho}' - \rho'\bar{\rho} = 0, \tag{25}$$

$$\text{Þ}\rho' - \text{Þ}'\rho = \tau\bar{\tau} - \tau'\bar{\tau}'. \tag{26}$$

Here, eqs. (21) and (24) represent the propagation of shear of the congruence $C(l^a)$ and $C(n^a)$, eqs. (22) and (23) describe the propagation of expansion and twist while the coupling of the expansion and twist, as illustrated in Theorem 1, is described by eq. (25). Eq. (26) is identically satisfied in view of the GHP-field equations (A10) and (A10'). The proof of this theorem follows when the hypothesis of Theorem 2 and GHP-field equations (A5–A10) are applied to eq. (16).

## 3. Modified Lie derivative operator

In GHP-formalism [2], the operator

$$\Theta_a = \nabla_a - \frac{1}{2}(p+q)n^b\nabla_a l_b + \frac{1}{2}(p-q)\bar{m}^b\nabla_a m_b \tag{27}$$

plays a crucial role and takes the place of covariant derivative $\nabla_a$. Here $\{p,q\}$ are the GHP-weights of a quantity acted upon by $\Theta_a$. The GHP-differential operators (10) are related to $\Theta_a$ by the equation

$$\Theta_a = l_a\text{Þ}' + n_a\text{Þ} - m_a\text{ð}' - \bar{m}_a\text{ð}. \tag{28}$$

From the properties of the tetrad vectors and the use of eq. (28), it is not hard to find

$$\begin{aligned}
\Theta_a l^b &= (-\bar{\tau}l_a - \bar{\kappa}n_a + \bar{\sigma}m_a + \bar{\rho}\bar{m}_a)m^b \\
&\quad + (-\tau l_a - \kappa n_a + \rho m_a + \sigma\bar{m}_a)\bar{m}^b, \\
\Theta_a n^b &= (-\kappa' l_a - \tau' n_a + \sigma' m_a + \rho'\bar{m}_a)m^b \\
&\quad + (-\bar{\kappa}' l_a - \bar{\tau}' n_a + \bar{\rho}' m_a + \bar{\sigma}'\bar{m}_a)\bar{m}^b, \\
\Theta_a m^b &= (-\bar{\kappa}' l_a - \bar{\tau}' n_a + \bar{\rho}' m_a + \bar{\sigma}'\bar{m}_a)l^b \\
&\quad + (-\tau l_a - \kappa n_a + \rho m_a + \sigma\bar{m}_a)n^b,
\end{aligned} \tag{29}$$

so that

$$\begin{aligned}
\Theta_a l^a &= -(\rho + \bar{\rho}), \\
\Theta_a n^a &= -(\rho' + \bar{\rho}') \\
\Theta_a m^a &= -(\tau + \bar{\tau}').
\end{aligned} \tag{30}$$

The use of $\Theta_a$ is place of $\nabla_a$ enable us to eliminate the spin coefficients $\alpha$, $\beta$, $\varepsilon$ and $\gamma$, which behave badly under boost-rotations. For the same reason the modified Lie differentiation operator $L_\xi$ is defined in which $\nabla_a$ is replaced by $\Theta_a$ and thus the modified Lie derivative of a vector $u^a$ is given by

$$L_\xi u^a = \xi^b \Theta_b u^a - u^b \Theta_b \xi^b. \tag{31}$$

Since $\Theta_a$ may be written as

$$\Theta_a = \nabla_a - p\zeta_a - q\bar{\zeta}_a, \tag{32}$$

$$\text{where } \zeta_a = \gamma l_a + \varepsilon n_a - \alpha m_a - \beta \bar{m}_a \tag{33}$$

the modified Lie derivative $L_\xi$ and the Lie derivative $\mathcal{L}_\xi$ are related by

$$L_\xi = \mathcal{L}_\xi - \xi^a\left(p\zeta_a + q\bar{\zeta}_a\right). \tag{34}$$

The action of this operator $L_\xi$ on the tetrad vectors may be found in [18].

In this section, we shall find the action of this modified Lie derivative operator of the electromagnetic field tensor $F_{ij}$, the Ricci tensor $R_{ij}$ and the metric tensor $g_{ij}$ for the non-null electromagnetic fields.

From eqs. (32) and (34), the modified Lie derivative of $F_{ij}$ with respect to the principal null direction $l^a$ is

$$L_l F_{ij} = \Theta_a F_{ij} l^a + F_{ia}\Theta_j l^a + F_{aj}\Theta_i l^a, \tag{35}$$

where $F_{ij}$ is the electromagnetic bivector given by eq. (2) provided that $\phi_0 = \phi_2 = 0$, $\phi_1 \neq 0$.

Now using eqs. (28), (A1–A4), (29) and (13), after a lengthy calculation, eq. (35) takes the form

$$\begin{aligned} L_l F_{ij} &= 2\{\Re(\rho+\bar{\rho}) - 2\Re(\rho) l_{[i} n_{j]} + 2i\Im(\rho) m_{[i}\bar{m}_{j]}\}\phi \\ &\quad - 2\Re\phi\{\kappa n_{[i}\bar{m}_{j]} + \bar{\kappa} n_{[i} m_{j]}\} + 2i\Im\phi\{-\bar{\tau} l_{[i} m_{j]} \\ &\quad + \tau l_{[i}\bar{m}_{j]} - 2(\rho+\bar{\rho}) m_{[i}\bar{m}_{j]} + \bar{\sigma} m_{[i} m_{j]} + \sigma \bar{m}_{[i}\bar{m}_{j]}\} \\ &\quad + 2(\Re\phi + i\Im\phi)\tau' l_{[i} m_{j]} + 2(\Re\phi + i\Im\phi)\bar{\tau}' l_{[i}\bar{m}_{j]}, \end{aligned} \tag{36}$$

which is non-zero for non-null electromagnetic fields. However, a considerable amount of simplification results in eq. (36) under the hypothesis of theorem 1, if we take the congruence $C(l^a)$ to be expansion-free, and we have

*Theorem 3* : Let the null geodesic congruence $C(l^a)$ and $C(n^a)$ satisfy the Maxwell equations for a non-null electromagnetic field and the tetrad is parallelly propagated along them. If $C(l^a)$ is expansion-free, then

$$L_l F_{ij} = -2i\Im\phi\{\bar{\sigma} m_{[i} m_{j]} + \sigma \bar{m}_{[i}\bar{m}_{j]}\}. \tag{37}$$

In the spin coefficient formalism [1], the field equations

$$R_{ij} = -\frac{1}{4\pi}\left(F_{ik}F_j^k - \frac{1}{4} g_{ij} F^{rs} F_{rs}\right)$$

for a purely electromagnetic distribution takes the following form [19] for different types

$$\text{Type A}: \; R_{ij} = \chi\phi_1\bar{\phi}_1\{l_{(i}n_{j)} + m_{(i}\bar{m}_{j)}\}, \tag{38a}$$

$$\text{Type B}: \; R_{ij} = \frac{1}{2}\chi\phi_2\bar{\phi}_2 l_i l_j, \tag{38b}$$

$$\text{Type C}: \; R_{ij} = \frac{1}{2}\chi\phi_0\bar{\phi}_0 n_i n_j. \tag{38c}$$

It may be noted that eqs. (38b) and (38c) are the well known Lichnerowicz conditions [20] for total gravitational radiation having $l_i$ and $n_i$, respectively, as propagation vectors.

Now using eqs. (28), (A1–A4), (29), (38a) and (13), we have for the non-null electromagnetic fields

$$\begin{aligned} L_l R_{ij} &= \chi\big[2(\rho+\bar{\rho}) l_{(i}n_{j)} + (\bar{\tau} - 2\tau') l_{(i} m_{j)} \\ &\quad + (\tau - 2\bar{\tau}') l_{(i}\bar{m}_{j)} - \bar{\kappa} n_{(i} m_{j)} - \kappa n_{(i}\bar{m}_{j)} - \bar{\sigma} m_{(i} m_{j)} \\ &\quad - \sigma \bar{m}_{(i}\bar{m}_{j)} + (\rho+\bar{\rho}) m_{(i}\bar{m}_{j)}\big]\phi\bar{\phi}. \end{aligned} \tag{39}$$

Under some special circumstances, eq. (39) do admit a simpler form and we have

*Theorem 4* : Let the null geodesic congruence $C(l^a)$ and $C(n^a)$ satisfy the Maxwell equations for non-null electromagnetic fields. If the tetrad is parallelly propagated along $C(l^a)$ and $C(n^a)$, and $C(l^a)$ is expansion-free, then

$$L_l R_{ij} = -\chi\big[\bar{\sigma} m_{(i} m_{j)} + \sigma \bar{m}_{(i}\bar{m}_{j)}\big]\phi\bar{\phi} \tag{40}$$

which is non-zero as $\sigma \neq 0$.

Finally, from the definition of the modified Lie derivative, we have

$$L_l g_{ij} = \Theta_i l_j + \Theta_j l_i$$

which on using eq. (28) and (A1–A4) reduces to

$$\begin{aligned} L_l g_{ij} &= 2\{-\bar{\tau} l_{(i} m_{j)} - \tau l_{(i}\bar{m}_{j)} - \bar{\kappa} n_{(i} m_{j)} - \kappa n_{(i}\bar{m}_{j)} \\ &\quad + \bar{\sigma} m_{(i} m_{j)} + \sigma \bar{m}_{(i}\bar{m}_{j)} + \rho m_{(i}\bar{m}_{j)} + \bar{\rho}\bar{m}_{(i} m_{j)}\} \end{aligned} \tag{41}$$

so that we have

*Theorem 5* : Let the null geodesic congruence $C(l^a)$ and $C(n^a)$ satisfy the Maxwell equations for non-null electromagnetic fields and the tetrad is parallelly propagated along them, then

$$L_l g_{ij} = 2\{\bar{\sigma} m_{(i} m_{j)} + \sigma \bar{m}_{(i}\bar{m}_{j)} + \rho m_{(i}\bar{m}_{j)} + \bar{\rho}\bar{m}_{(i} m_{j)}\}. \tag{42}$$

*Remark* : It is interesting to note that for the Reissner Nördstrom black hole [21], eqs. (36), (39) and (41) take the following forms

$$\begin{aligned} L_l F_{ij} &= 2\{\Re(\rho+\bar{\rho}) - 2\Re(\rho) l_{[i} n_{j]} + 2i\Im(\rho) m_{[i}\bar{m}_{j]}\}\phi \\ &\quad - 4i\Im\phi(\rho+\bar{\rho}) m_{[i}\bar{m}_{j]}, \end{aligned} \tag{43a}$$

$$L_l R_{ij} = \chi(\rho+\bar{\rho})(2l_{(i}n_{j)} + m_{(i}\bar{m}_{j)})\phi\bar{\phi}, \tag{43b}$$

$$L_l g_{ij} = 2(\rho+\bar{\rho}) m_{(i}\bar{m}_{j)}. \tag{43c}$$

These equations suggest that for Reissner Nördstrom black hole the modified Lie derivative of the electromagnetic field tensor, the Ricci tensor and the metric depend on the radial coordinate (as $\rho = -1/r$) and thus for large $r$, $L_l F_{ij} = 0$, $L_l R_{ij} = 0$, and $L_l g_{ij} = 0$.

### 4. Conclusion

The non-null electromagnetic fields have been studied using the compacted spin coefficient formalism due to Geroch, Held and Penrose. The Maxwell equations have been translated in the language of GHP-formalism (*cf.* eqs. (11)–(15)). The equations describing the propagation of shear (eqs. (17), (21) and (24)), expansion and twist (eqs. (18), (22) and (23)) of the null congruences $C(l^a)$ and $C(n^a)$ associated with the non-null electromagnetic field have been obtained and the conditions (eqs. (19), (20) and (25)) under which the expansion and twist of the congruence can be coupled together have also been given. Moreover, the propagation of the shear (eq. (17)) is seen to be related with the longitudinal wave component of the gravitational field in $n^a$ and $l^a$ directions. The role of the modified Lie derivative operator on the electromagnetic field tensor, Ricci tensor and metric tensor has been explored. For Reissner Nördstrom black hole, these derivatives are seen to depend only on one spin coefficient $\rho$ (*cf.* eq. (43)).

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## Appendix

A familiarity with GHP-formnalism is assumed. Here, we shall mention only those results which are used in the present paper. When GHP-derivatives (10) act on tetrad vectors, they give rise to [5]

$$\begin{aligned}
&(a)\quad \mathcal{P} l^a = -\kappa \bar{m}^a - \bar{\kappa} m^a,\\
&(b)\quad \mathcal{P}' l^a = -\bar{\tau} m^a - \tau \bar{m}^a,\\
&(c)\quad \mathcal{D} l^a = -\bar{\rho} m^a - \sigma \bar{m}^a,\\
&(d)\quad \mathcal{D}' l^a = -\bar{\sigma} m^a - \rho \bar{m}^a,
\end{aligned} \tag{A1}$$

$$\begin{aligned}
&(a)\quad \mathcal{P} n^a = -\tau' m^a - \bar{\tau}' \bar{m}^a,\\
&(b)\quad \mathcal{P}' n^a = -\kappa' m^a - \bar{\kappa}' \bar{m}^a,\\
&(c)\quad \mathcal{D} n^a = -\rho' m^a - \bar{\sigma}' \bar{m}^a,\\
&(d)\quad \mathcal{D}' n^a = -\bar{\rho}' \bar{m}^a - \sigma' m^a.
\end{aligned} \tag{A2}$$

$$\begin{aligned}
&(a)\quad \mathcal{P} m^a = -\bar{\tau}' l^a - \kappa n^a,\\
&(b)\quad \mathcal{P}' m^a = -\bar{\kappa}' l^a - \tau n^a,\\
&(c)\quad \mathcal{D} m^a = -\bar{\sigma}' l^a - \sigma n^a,\\
&(d)\quad \mathcal{D}' m^a = -\bar{\rho}' l^a - \rho n^a.
\end{aligned} \tag{A3}$$

$$\begin{aligned}
&(a)\quad \mathcal{P} \bar{m}^a = -\tau' l^a - \bar{\kappa} n^a,\\
&(b)\quad \mathcal{P}' \bar{m}^a = -\kappa' l^a - \bar{\tau} n^a,\\
&(c)\quad \mathcal{D} \bar{m}^a = -\sigma' l^a - \bar{\sigma} n^a,\\
&(d)\quad \mathcal{D}' \bar{m}^a = -\rho' l^a - \bar{\rho} n^a.
\end{aligned} \tag{A4}$$

GHP-field equations [2], [16]

$$\mathcal{P}\rho - \mathcal{D}\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \tau'\kappa + \Phi_{00}, \tag{A5}$$

$$\mathcal{P}'\rho' - \mathcal{D}'\kappa' = \rho'^2 + \sigma'\bar{\sigma}' - \bar{\kappa}'\tau' - \tau\kappa' + \Phi_{22}. \tag{A5'}$$

$$\mathcal{P}\sigma - \mathcal{D}\kappa = (\rho + \bar{\rho})\sigma - \kappa(\tau + \bar{\tau}') + \Psi_0, \tag{A6}$$

$$\mathcal{P}'\sigma' - \mathcal{D}'\kappa' = (\rho' + \bar{\rho}')\sigma' - \kappa'(\tau' + \bar{\tau}) + \Psi_4. \tag{A6'}$$

$$\mathcal{P}\tau - \mathcal{P}'\kappa = (\tau - \bar{\tau}')\rho + \sigma(\bar{\tau} - \tau') + \Psi_1 + \Phi_{01}, \tag{A7}$$

$$\mathcal{P}'\tau' - \mathcal{P}\kappa' = (\tau' - \bar{\tau})\rho' + \sigma'(\bar{\tau}' - \tau) + \Psi_3 + \Phi_{21}. \tag{A7'}$$

$$\mathcal{D}\rho - \mathcal{D}'\sigma = (\rho - \bar{\rho})\tau + (\rho' - \bar{\rho}')\kappa - \Psi_1 + \Phi_{01}, \tag{A8}$$

$$\mathcal{D}'\rho' - \mathcal{D}\sigma' = (\rho' - \bar{\rho}')\tau' + (\rho - \bar{\rho})\kappa' - \Psi_3 + \Phi_{21}. \tag{A8'}$$

$$\mathcal{D}\tau - \mathcal{P}'\sigma = -\rho'\sigma - \bar{\sigma}'\rho + \tau^2 + \kappa\bar{\kappa}' + \Phi_{02}, \tag{A9}$$

$$\mathcal{D}'\tau' - \mathcal{P}\sigma' = -\rho\sigma' - \bar{\sigma}\rho' + \tau'^2 + \kappa'\bar{\kappa}' + \Phi_{20}. \tag{A9'}$$

$$\mathcal{P}'\rho - \mathcal{D}'\tau = \rho\bar{\rho}' + \sigma\sigma' - \tau\bar{\tau} - \kappa\kappa' - \Psi_2 - 2\Lambda, \tag{A10}$$

$$\mathcal{P}\rho' - \mathcal{D}\tau' = \rho'\bar{\rho} + \sigma'\sigma - \tau'\bar{\tau}' - \kappa'\kappa - \Psi_2 - 2\Lambda. \tag{A10'}$$

The above list does not completely exhaust all NP field equations. The remaining equations refer to derivatives of spin coefficients which are spin and boost weighted quantities and cannot therefore, be written like above equations, in GHP-formalism. Instead, they play their role as part of the commutator equations for the differential operators defined by eq. (10). The commutators when applied to a spin and boost weighted quantity $\eta$ of type $\{p,q\}$, are given as follows :

GHP-commutator relation [2,16]

$$[\mathcal{P},\mathcal{P}']\eta=\{(\bar{\tau}-\tau')\mathcal{D}+(\tau-\bar{\tau}')\mathcal{D}'-p(\kappa\kappa'-\tau\tau'+\Psi_2+\Phi_{11}-\Lambda)-q(\bar{\kappa}\bar{\kappa}'-\bar{\tau}\bar{\tau}'+\bar{\Psi}_2+\Phi_{11}-\Lambda)\}\eta, \quad \text{(A11)}$$

$$[\mathcal{P},\mathcal{D}]\eta=\{\bar{\rho}\mathcal{D}+\sigma\mathcal{D}-\bar{\tau}'\mathcal{P}-\kappa\mathcal{P}'-p(\rho'\kappa-\tau'\sigma+\Psi_1)-q(\bar{\sigma}'\bar{\kappa}-\overline{\rho\tau}'+\Phi_{01})\}\eta, \quad \text{(A12)}$$

$$[\mathcal{D},\mathcal{D}']\eta=\{(\bar{\rho}'-\rho')\mathcal{P}+(\rho-\bar{\rho})\mathcal{P}'+p(\rho\rho'-\sigma\sigma'+\Psi_2-\Phi_{11}-\Lambda)-q(\bar{\rho}\bar{\rho}'-\bar{\sigma}\bar{\sigma}'+\Psi_2-\Phi_{11}-\Lambda)\}\eta, \quad \text{(A13)}$$

together with the remaining commutator relations obtained by applying prime, complex conjugation and both to relation (A12). Care must be taken when applying primes and bars to these equations, as $\eta'$, $\bar{\eta}$ and $\bar{\eta}'$ have types different to that of $\eta$. Under the prime, $p$ becomes $-p$ and $q$ becomes $-q$; under bar $p$ becomes $q$ and $q$ becomes $p$; under both bar and prime $p$ becomes $-q$ and $q$ becomes $-p$.