Indian J. Phys. **75B** (6), 575–577 (2001)

NOTE

I J P B
an international journal

Uniqueness of Schwarzschild solution

V D Deshpande*
B-5, Cosmic Society, Balikashram Road, Ahmednagar-414 001, Maharashtra, India
S V Ingale
76, Sonanagar, Bhistabag Road, Savedi, Ahmednagar-414 001, Maharashtra, India
and
M V Handore
Department of Mathematics, New Arts, Commerce & Science College, Ahmednagar-414 001, Maharashtra, India

Received 16 May 2001, accepted 11 September 2001

Abstract Static metrics different from Schwarzschild could not exist. It confirms the uniqueness of Schwarzschild solution. The geodesics for photons are discussed

Keywords Uniqueness, Schwarzschild solution, geodesics.

PACS No. 04.20 –q

Schwarzschild [1] has solved exactly Einstein's [2] vacuum field equations :

$$G_\mu^\nu \equiv R_\mu^\nu - \frac{1}{2} g_\mu^\nu R = 0, \tag{1}$$

using the spherically symmetric metric

$$ds^2 = -\exp(\lambda(r))dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + \exp(\nu(r))dt^2. \tag{2}$$

The solution given by Tolman [3] is

$$\exp(-\lambda(r)) = \exp(\nu(r)) = 1 - \frac{2m}{r}, \tag{3}$$

where G = gravitational constant = 1 and c = velocity of light = 1. The constant m in eq. (3) can be determined by an appeal to corresponding to Newton's theory of gravitation It is seen that a geometric theory will reduce in the classical limit of weak fields and slowly moving bodies to Newton's theory if and only if

$$g_{oo} \approx (1 + 2\Omega), \tag{4}$$

where Ω is the classical potential for the gravitational field. The quantity m has units of distance and is referred to as the geometric mass of the central body.

There are many applications of the Schwarzschild solution given in the literature [4–6]. In the present work, the authors have shown that the metrics different from Schwarzschild [1] are not spherically symmetric and could not exist. Ultimately, it confirms the uniqueness of Schwarzschild [1] static exterior solution. The geodesics for photons are discussed. The significance of the solution is given in the conclusion.

We consider metrics different from Schwarzschild [1] as

$$ds^2 = -\exp(\lambda(r))dr^2 - \exp(\mu(r))r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + \exp(\nu(r))dt^2 \tag{5}$$

and

$$ds^2 = -\exp(\lambda(r))dr^2 - r^2 d\theta^2 - \exp(\mu(r)) \times r^2 \sin^2 \theta d\phi^2 + \exp(\nu(r))dt^2. \tag{6}$$

We note that these metrics are not spherically symmetric. The field eq. (1) for these metrics can be obtained by using

*Corresponding Author

standard formulae given by Tolman [3]. We see that any exterior solution of static fluid under gravity can be determined on the basis of two boundary conditions . (i) at the boundary of fluid surface and (ii) at infinity

Then it can be easily shown that both metrics (5) and (6) could not exist due to the boundary conditions (i) and (ii). Ultimately, these metrics must reduce to the Schwarzschild metric. This gives the verification of uniqueness of Schwarzschild static exterior solution.

According to the special relativity, the distance between two events along a world line is proportional to the proper time elapsed for any observer moving on the world line. Hence, we have

$$ds^2 = -c^2 d\tau^2 = -c^2\left(1-\frac{2m}{r}\right)dt^2, \tag{7}$$

where c is the velocity of the light in vacuum and τ is the proper time measured by the world line traveller. The minus sign in (7) is due to time-like ds^2.

We consider motion of freely falling material particle or photon in static isotropic gravitational field. The equations of free fall given by Weinberg [7] are

$$\frac{d^2x^\mu}{dp^2} + \{\alpha\beta,\mu\}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} = 0, \tag{8}$$

where p is a parameter describing a trajectory of the particle. For a material particle, we could normalize p so that $p = \tau$. However, for a photon, $(d\tau/dp)$ vanishes.

The metric concerned to the motion of photon is

$$ds^2 = -c^2\left[1-\frac{2m}{r}\right]d\tau^2 + \left[1-\frac{2m}{r}\right]^{-1}dr^2 + r^2 d\phi^2 \tag{9}$$

Here, the calculations deal exclusively with the motion in $\theta = \pi/2$ plane without any loss of generality. The eqs. (8) for the metric (9) are obtained in the form :

$$\frac{d^2r}{dp^2} + \frac{m}{r^2}\left[1-\frac{2m}{r}\right]^{-1}\left[\frac{dr}{dp}\right]^2 - r\left[1-\frac{2m}{r}\right]\left[\frac{d\phi}{dp}\right]^2 + \frac{c^2m}{r^2}\left[1-\frac{2m}{r}\right]\left[\frac{dt}{dp}\right]^2 = 0, \tag{10a}$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\left[\frac{d\phi}{dp}\right]\left[\frac{dr}{dp}\right] = 0, \tag{10b}$$

$$\frac{d^2t}{dp^2} + \frac{2m}{r^2}\left[1-\frac{2m}{r}\right]^{-1}\left[\frac{dt}{dp}\right]\left[\frac{dr}{dp}\right] = 0. \tag{10c}$$

These eqs. (10a–10c) admit circular time-like geodesic orbits around the black hole. Hence, they can be expressed as time-like curves for $p = \tau$ with $r = a$ constant, as :

$$r\left[1-\frac{2m}{r}\right]\left[\frac{d\phi}{d\tau}\right]^2 + \frac{c^2m}{r^2}\left[1-\frac{2m}{r}\right]\left[\frac{dt}{d\tau}\right]^2 = 0, \tag{11a}$$

$$\frac{d^2\phi}{d\tau^2} = 0, \tag{11b}$$

$$\frac{d^2t}{d\tau^2} = 0. \tag{11c}$$

These eqs. (11b) and (11c) evolve simple solutions

$$\phi = b_1\tau + b_2 \quad \text{and} \quad t = b_3\tau + b_4, \tag{12}$$

where (b_1, b_2, b_3, b_4) are arbitrary constants. The eq. (11a) gives for $a > R_s$ (Schwarzschild radius)

$$\frac{d\phi}{dt} = \pm\, c\left[\frac{m}{a^3}\right]^{\frac{1}{2}}. \tag{13}$$

When $t = \tau$, the gravitational field is weak and (13) is equivalent to Newtonian result. Therefore, the circular time-like geodesics exist in Schwarzschild geometry given by

$$r = a, \quad \phi = b_1\tau + b_2 \quad \text{and} \quad t = b_3\tau + b_4, \tag{14}$$

except $(b_1/b_3) = (m/a^3)^{1/2}$.

One can obtain for circular geodesic travellers, proper time elapsed *versus* coordinate time elapsed as

$$\delta\tau = \frac{1}{c}\int\left[c^2\left(1-\frac{2m}{a}\right)dt^2 - a^2d\phi^2\right]^{\frac{1}{2}}. \tag{15}$$

Using (13) and the chain rule for dt, we obtain

$$\delta\tau = \left[1-\frac{3m}{a}\right]^{\frac{1}{2}}\delta t. \tag{16}$$

We should note that $\delta t = 0$ for $a = 3$ m, which is a null circular geodesic. Hence, photons have a circular orbit around the black hole at $a = 3$ m, called 'Photon Sphere'.

On integration, (10b) becomes

$$r^2\frac{d\phi}{dp} = L \text{ (constant)} \tag{17}$$

so that along null geodesics, L corresponds to the angular momentum of photon. Another conserved quantity obtained from (10c) is the total energy of photon given by

$$\left(1-\frac{2m}{r}\right)\left[\frac{dt}{dp}\right] = E \text{ (constant)}. \tag{18}$$

Hence, the eq. (9) reduces for null geodesics, to the form

$$c^2E^2 = \left[\frac{dr}{dp}\right]^2 + \frac{L^2}{r^2}\left[1-\frac{2m}{r}\right]. \tag{19}$$

It may have a unit mass particle of total energy $(c^2E^2/2)$ moving in one dimensional 'effective potential'.

$$V(r) = \frac{L^2}{2r^2}\left(1-\frac{2m}{r}\right). \tag{20}$$

One has to solve the equation

$$\frac{dV}{dr} = \frac{-L^2}{r^3}\left(1-\frac{2m}{r}\right) + \left[\frac{mL^2}{r^4}\right] = 0, \tag{21}$$

to get the maximum potential $V(r)$.

It is seen that the solution r_{max} = 3 m is independent of *L*. This gives the unstable circular orbit for photons as given above. It means that a slight orbital deviation sends the photon into the black hole or to infinity. (For details, see Stuckey [6]).

We start with metrics different from Schwarzschild [1]. These metrics are not spherically symmetric. Any exterior solution of static fluid under its own gravity, can be obtained by applying the boundary conditions : at the boundary of fluid surface and at infinity. On this basis, we have shown that such metrics could not exist at all. Ultimately, they become Schwarzschild form to keep the spherical symmetry and confirm the uniqueness of the solution. It describes the exterior field of any static body. Our sun is a good approximation because the general relativity tests are provided quantitatively by the gravitational fields which occur in the solar system. This solution was first found exactly by Schwarzschild [1]. It is of great significances since (i) it is an exact unique vacuum solution of Einstein's [2] equations, (ii) it is static and spherically symmetric, (iii) it predicts tiny departures from Newton's theory for planetary motions, (iv) it also predicts the bending of light, the redshifts and time delay effects *etc*. and these predictions are accurately confirmed by precise measurements, (v) it describes space time geometry after complete gravitational collapse of massive stars and (vi) it clearly gives non-Euclidean geometry with strong gravitational fields. It is most valuable in the approximately truncated region, as it represents the geometry of collapsing stars, black holes and also worm holes. The uniqueness of the Schwarzschild [1] solution is the Birchoff's theorem [7–9]. It is analogous to the result proved by Newton in the theory of lunar motion. Hence, it applies in general relativity theory as well as Newton's theory. It is also analogous to the well known result of atomic theory. It can be applied to the fields both outside and inside the empty central spherical cavity of a body, but not necessarily a static body. Finally, we say that Schwarzschild [1] exterior solution is very significant because it is most simple and applicable in weak as well as strong gravitational fields and also in the study of entire universe.

References

[1] K Schwarzschild *Berl Ber* **189** 424, 688 (1916)

[2] A Einstein *Ann der Phys* **49** 769 (1916)

[3] R C Tolman *Relativity, Thermodynamics and Cosmology* (Oxford Clarenden) (1934)

[4] J V Narlikar and S V Dhurandhar *Pramana* **6** 688 (1976)

[5] V S Manko *Gen Relat Gravit* **21** 11, 1193 (1989)

[6] W M Stuckey *Am J Phys* **61** 448 (1998)

[7] S Weinberg *Gravitation and Cosmology Principles and Applications of the General Theory of Relativity* (New York John Wiley) (1972)

[8] G D Birkhoff *Relativity and Modern Physics* (Cambridge Harvard University Press) (1923), *Proc Nat Acad Sci US* **29** 231 (1934)

[9] C W Misner, K S Thorne and J A Wheeler *Gravitation* (New York W H Freeman and Company) (1973)