Indian J. Phys. **75B** (6), 505–507 (2001)

I J P B
— an international journal

Charged AdS black holes and test particles

Farook Rahaman* and Mehedi Kalam
Department of Mathematics, Jadavpur University, Kolkata-700 032, India
E-mail : jumath@cal.vsnl.net.in

Received 1 December 2000, accepted 18 July 2001

Abstract : We have presented a detailed analysis of the motion of the test particle around Reissner Nordström-AdS black holes in five dimensional space time. The study of the trajectories of the particles have been done using Hamilton Jacobi (H-J) formalism We have considered test particles with various masses and electric charges and examine its behaviour both in static and non-static cases

Keywords : Charged AdS black hole, test particles, Hamilton-Jacobi formalism

PACS Nos. : 97.60.Lf, 04 20.Jb, 04 70.-s

1. Introduction

Recently, a new family of exact black hole solutions has been discovered both in four and higher dimensional space times [1]. It is interesting to study black holes in anti-de Sitter (AdS) space because the AdS conformal field theory (CFT) duality relates the properties of these black holes to thermal properties of a dual conformal field theory residing on the boundary of AdS space [2]. A five dimensional AdS analogue of Reissner-Nordström AdS (RNAdS) solution of type IIB super gravity was derived by Chamblin *et al* [2]. The action for this model is [2]

$$I = -\frac{1}{16\pi G}\int d^5x\sqrt{-g}\left[R+\frac{12}{l^2}-l^2F^2 -\frac{l^3}{6\sqrt{3}}\cdot \epsilon^{\mu\alpha\beta\gamma\delta}\, A_\mu F_{\alpha\beta}F_{\gamma\delta}\right]. \tag{1}$$

Here, R is the curvature scalar in 5-dimensional space, A_μ is the gauge field, F_{ab} ($F^2 = F_{ab}F^{ab}$) stands for Maxwell field strength. The parameter l measures the size of the S^5. The $AdS_5 \times S^5$ gauged super gravity in five dimension has an S0(6) gauge symmetry, associated with the group of isometries of S^5.

Solutions of type IIB super gravity describing Reissner Nordström AdS black holes with an internal S^5 is [2]

$$ds^2 = -g(r)+\frac{1}{g(r)}dr^2+r^2d\Omega_3^2 \tag{2}$$

with $g(r) = 1-\frac{M}{r^2}+\frac{Q^2}{r^4}+r^2,$ (3)

where M and Q measure the black hole's mass and charge respectively. The electro-magnetic potential in a gauge regular on the outer horizon is

$$A_t = [\phi(r_+)-\phi(r)]dt,$$

where $\phi(r) = \frac{Q}{r^2}$. (4)

Here r_+ is the largest real positive root of $g(r) = 0$.

The aim of the present work is to investigate the motion of the test particles in the gravitational field of a five dimensional RNAdS black hole using Hamilton-Jacobi (H–J) formalism and examine both static and non-static cases and also for charged and uncharged test particles.

2. The Hamilton Jacobi formalism

Let us consider a test particle having mass m and charge e moving in the gravitational field of a five dimensional

*Corresponding Author
Permanent address : Khodar Bazar, Baruipur, 24-Parganas (South), West Bengal-743 302, India

RNAdS black hole. So the H–J equation for the test particle is [3]

$$g^{ik}\left(\frac{\partial S}{\partial x^i}+eA_i\right)\left(\frac{\partial S}{\partial x^k}+eA_k\right)+m^2=0, \qquad (5)$$

where $g_{\mu\gamma}$ and A_μ are the classical background fields (2) and (4) respectively and S is the standard Hamilton's characteristic function. For metric (2), the explicit form of the H–J eq. (5) is [4]

$$-\frac{1}{g}\left[\frac{\partial S}{\partial F}+eQ\left(\frac{1}{r_+^2}\quad 1\right.\right.^{2}+g(r)\left(\frac{\partial S}{\partial r}\right)^2$$

$$\left(\frac{\partial S}{\partial x_1}\right)^2+\left(\frac{\partial S}{\partial x_2}\right)^2+\left(\frac{\partial S}{\partial x_3}\right)^2+m^2=0. \qquad (6)$$

Here, x_1, x_2, x_3 are the coordinates on the surface of the 3-sphere. As there is no explicit dependence of t and coordinates on the 3-sphere, a natural form for the H–J function $S(t, r, x_1, x_2, x_3)$ will be

$$S=-E\cdot t+S_1(r)+p_1x_1+p_2x_2+p_3x_3. \qquad (7)$$

Here, the constant E is identified as the energy of the particle and p_1, p_2, p_3 are the momentum of the particle along different axes on the 3-sphere withp $p=\left(p_1^2+p_2^2+p_3^2\right)^{\frac{1}{2}}$, as the resulting momentum of the particle.

If we substitute the ansatz (7) for S in the H–J equation, then we get the following expression for S_1 as [4]

$$(r)=\epsilon\int\left[\frac{1}{g^2}\left\{E-eQ\left(\frac{1}{r_+^2}-\frac{1}{r^2}\right.\right.\quad\frac{p^2}{rg}\ \frac{m^2}{g}\quad dr. \qquad (8)$$

Here, $\epsilon=\pm1$, the sign changes whenever r passes through a zero of the integrand in (8).

Now, for the trajectory of the particle following the H–J method, we consider,

$$\frac{\partial S}{\partial E}=\text{constant},\quad \frac{\partial S}{\partial p_1}=\text{constant},$$

$$\frac{\partial S}{\partial p_2}=\text{constant},\quad \frac{\partial S}{\partial p_3}=\text{constant}.\ [4].$$

So the path in parametric form gives (choosing constants to be zero without any loss of generality) [4]

$$'=\epsilon\int\left\{E-eQ\left(\frac{1}{r_+^2}-\frac{1}{r^2}\right)\right\}\cdot\frac{1}{g^2}$$

$$\times\left[\frac{1}{g^2}\left\{E-eQ\left(\frac{1}{r_+^2}-\frac{1}{r^2}\right)\right\}^2-\frac{p^2}{r^2g}-\frac{m^2}{g}\right]^{-\frac{1}{2}}dr, \qquad (9)$$

$$x_i=\epsilon\int\frac{p_i}{r^2}\ \frac{1}{g^2}\left\{E-eQ\left(\frac{1}{r_+^2}-\frac{1}{r^2}\right.\right.$$

$$\frac{m^2}{r^2g\quad g}\quad dr. \qquad (10)$$

Hence, the radial velocity of the particle is

$$\frac{dr}{dt}\quad\frac{\frac{1}{g^2}\left\{E-eQ\left(\frac{1}{r_+^2}-\frac{1}{r^2}\right)\right\}^2-\frac{p^2}{r^2g}-\frac{m^2}{g}}{E-eQ\quad 1\quad 1\qquad g^2} \qquad (11)$$

For the turning points of the path of the particle, we have $\frac{dr}{dt}=0$ and consequently the potential curves are [4]

$$\frac{E}{m}=\frac{eQ}{m}\left(\frac{1}{r_+^2}-\frac{1}{r^2}\right)+\sqrt{g}\left(1+\frac{p^2}{r^2m^2}\right)^{\frac{1}{2}}=V(r)$$

$$=\frac{eQ}{m}\left(\frac{1}{r_+^2}-\frac{1}{r^2}\right)+\left(1-\frac{M}{r^2}+\frac{Q^2}{r^4}+r^2\right.$$

$$\times\left|\ 1+\frac{p^2}{r^2m^2}\right. \qquad (12)$$

Since this large expression contains so many arbitrary parameters namely, e, Q, m, p, M, we can not define physical characters of the potential curve. But one can find analytically the extreme points of the potential curve.

In a stationary system, $V(r)$ must have an extremal value given by

$$\frac{dV}{dr}=0. \qquad (13)$$

Hence, we get the following equation

$$-\frac{2eQ}{mr^3}\left(1+\frac{p^2}{m^2r^2}\right)^{\frac{1}{2}}\left(1-\frac{M}{r^2}+\frac{Q^2}{r^4}+r^2\right)^{\frac{1}{2}}$$

$$=\left(1+\frac{p^2}{r^2m^2}\right)\left(\frac{2M}{r^3}-\frac{4Q^2}{r^5}+2r\right)-$$

$$-\frac{2p^2}{m^2r^3}\left(1-\frac{M}{r^2}+\frac{Q^2}{r^4}+r^2\right). \qquad (14)$$

We see that $r=r_+$ is a solution of (14) provided

$r_+^6+Mr_+^2-2Q^2=0$ [since r_+ is a root of $g(r)=0$]

The above equation implies

$r_+^2=S+T$

[where $S=\sqrt[3]{Q^2+\sqrt{\frac{M^3}{27}+Q^4}}$ and $T=\sqrt[3]{Q^2-\sqrt{\frac{M^3}{27}+Q^4}}$]

Hence, trajectory of the test particle is bounded.

3. Test particle in static equilibrium

In static equilibrium, the momentum p must be zero. So from (14), the value of r for which potential will be an extremal, is given by

$$-\frac{2eQ}{mr^3}\left(1-\frac{M}{r^2}+\frac{Q^2}{r^4}+r^2\right)^{\frac{1}{2}}=\left(\frac{2M}{r^3}-\frac{4Q^2}{r^5}+2r\right). \quad (15)$$

From this, we get

$$r^{12}+2Mr^8-\left(4Q^2+\frac{e^2Q^2}{m^2}\right)r^6+\left(M^2-\frac{e^2Q^2}{m^2}\right)r^4$$

$$-MQ^2\left(4-\frac{e^2}{m^2}\right)r^2-\left(\frac{e^2}{m^2}-4\right)Q^4=0 \quad (16)$$

If $\frac{|e|}{m}=2$, we see that it is possible to have bound orbit for the test particle provided $M<\sqrt{2}\,Q$ since this equation has at least two real roots with one positive.

We also see that this is an algebraic equation of even degree with negative last term provided $\frac{|e|}{m}>2$.

This equation has at least two real roots with one is positive, so particle must be trapped by RNAdS black hole.

4. Test particle in non-equilibrium state

Case I . Uncharged test particle (e = 0)

Now the expression (14) simplifies to

$$\frac{2p^2}{m^2r^3}\left(1-\frac{M}{r^2}+\frac{Q^2}{r^4}+r^2\right)=\left(1+\frac{p^2}{m^2r^2}\right)$$

$$\times\left(\frac{2M}{r^3}-\frac{4Q^2}{r^5}+2r\right). \quad (17)$$

Thus, we get the following algebraic equation

$$2m^2r^8+\left(m^2M-2p^2\right)r^4-4Q^2m^2r^2-6p^2Q^2=0. \quad (18)$$

This even-degree algebraic equation has at least two real roots since its last term is negative (with one root positive).

So it is possible to have a bound orbit for the test particle.

Case II : Test particle with charge (i e. e ≠ 0)

From eq. (14), we have the algebraic equation

$$4m^4r^{16}+4m^2\left(2Mm^2+2Mp^2-2p^2\right)r^{12}$$

$$+\left\{4m^2\left(2p^2M-4m^2Q^2\right)-4e^2m^2Q^2\right\}r^{10}$$

$$-\left(4m^2p^2Q^2+4e^2Q^2p^2-2Mm^2-2Mp^2+2p^2\right)r^8$$

$$+\left\{2\left(2Mm^2+2Mp^2-2p^2\right)\left(2p^2M-4m^2Q^2\right)\right.$$

$$\left.+4e^2Q^2m^2M+4e^2Q^2p^2\right\}r^6$$

$$-\left\{12p^2Q^2\left(2Mm^2+2Mp^2-2p^2\right)-4e^2Q^2p^2M\right\}r^4$$

$$+\left\{12p^2Q^2\left(4m^2Q^2-2p^2M\right)-4e^2Q^4p^2\right\}r^2$$

$$+36p^2Q^2=0 \quad (19)$$

Whatever restrictions we impose on the physical parameters, one can see that the above algebraic equation (of degree 16) contains even numbers of variation of signs. So by Descartes rule of sign, this equation has either some real roots or no real roots. So unless we get exact numerical values of the physical parameters, we can not say whether the charged test particle can be trapped by black hole or not.

5. Concluding remarks

In this paper, we have examined the behaviour of a test particle in the gravitational field of a RNAdS black hole in five dimension, using the formalism due to Hamilton and Jacobi. The test particle is considered to be both static and non-static as well as charged or uncharged.

In static case, we have seen that the test particle can be trapped due to some restriction on physical parameters say $\frac{|e|}{m}>2$ (or $\frac{|e|}{m}=2$ and $M<\sqrt{2}\cdot Q$).

For non equilibrium test particle, we have examined the possibility for bound orbit.

We see that uncharged test particle always be trapped by RNAds black hole.

We can not say exactly whether a charged test particle is trapped or not by RNAdS black hole.

For future work, it will be interesting to investigate details how the physical parameters affect on the trajectory of the test particle in the gravitational field of different black holes.

Acknowledgment

Authors are thankful to the referee for his valuable comments and constructive suggestions.

References

[1] D Garfikle, Horowitz and A Strominger *Phys Rev* **D43** 3140 (1991), J Preskill, P Schwarz, A Shapere, S Trivedi and F Wilczek *Mod Phys Lett* **A6** 2353 (1991), E Witten *Phys Rev* **D44** 314 (1991); A Chamblin, R Emparan, C V Johnson and R C Myers *Phys Rev* **D59** 10640 10 (1999), S Sethi and L Suskind *Phys Lett* **B400** 265 (1997), G W Gibbons and K Meada *Nucl Phys* **B298** 741 (1998), M M Caldarelliad and D Klemm *Nucl. Phys* **B545** 434 (1999)

[2] S W Hawking and H S Reall *Phys Rev.* **D61** 0240 14 (1999), A Chamblin, R Emparam, C V Johnson and R C Myers *Phys Rev* **D60** 0640 18 (1999)

[3] L D Landau *The Classical Theory of Fields* (New York Pergamon) (1973)

[4] S Chakraborty *Gen Rel Grav* **28** 1115 (1996)