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Rayleigh-Taylor instability of Rivlin-Ericksen elastico-viscous fluid through porous medium

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Abstract Rayleigh-Taylor instability of Rivlin-Ericksen elastico-viscous fluid through porous medium is considered. We have studied the stability aspects of the system. The effects of a uniform horizontal magnetic field and a uniform rotation on the problem are also considered separately. The uniform magnetic field is found to stabilize certain wave-number band whereas the system is unstable for all wave numbers in the absence of magnetic field.

Keywords Rayleigh-Taylor instability, Porous medium, Rivlin-Ericksen fluid

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1. Introduction

The instability of the plane interface separating two fluids when one is accelerated towards the other or when one is superposed over the other, has been studied by several authors. Chandrasekhar [1] has given a detailed account of these investigations. Bhatia [2] has considered the Rayleigh-Taylor instability of two viscous superposed conducting fluids in the presence of a uniform horizontal magnetic field. The stability of superposed fluids in the presence of a variable horizontal magnetic field has been studied by Sharma and Thakur [3].

With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. Oldroyd [4] proposed a theoretical model for a class of viscoelastic fluids. Toms and Strawbridge [5] showed that a dilute solution of methyl methacrylate in n-butyl acetate agrees well with the theoretical model of Oldroyd fluid. Sharma and Sharma [6] have studied the stability of the plane interface separating two Oldroydian visco-elastic superposed fluids of uniform densities. There are many elastico-viscous fluids that can not be characterized by Oldroyd's constitutive relations. One such class of elastico-viscous fluids is Rivlin-Ericksen fluid. Garg *et al* [7] have studied the drag on a sphere oscillating in conducting dusty Rivlin-Ericksen elastico-viscous liquid. Thermal instability in Rivlin-Ericksen elastico-viscous fluid in presence of magnetic field and rotation, separately, has been investigated by Sharma and Kumar [8,9]. Sharma *et al* [10] have considered the instability of streaming Rivlin-Ericksen viscoelastic fluid in porous medium.

In the present paper, we have studied the stability of two superposed Rivlin-Ericksen elastico-viscous fluids in porous medium. The effects of a uniform magnetic field and uniform rotation, having relevance in geophysics, are also considered.

2. Perturbation equations

Here, we consider a static state in which an incompressible Rivlin-Ericksen elastico-viscous fluid is arranged in horizontal strata in porous medium and the pressure p and the density ρ are functions of the vertical co-ordinate z only. The character of the equilibrium of this initial static state is determined, as usual, by assuming that the system is slightly disturbed and then by following its further evolution.

Let T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, v_i and x_i denote the stress tensor, shear stress tensor, rate of strain tensor, Kronecker delta, velocity

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vector and position vector respectively. The constitutive relations for the Rivlin-Ericksen viscoelastic fluid are

$$T_{ij} = -p\delta_{ij} + \tau_{ij},$$

$$\tau_{ij} = 2\left(\mu + \mu'\frac{\partial}{\partial t}\right)e_{ij}, \tag{1}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Let $\boldsymbol{v}(u,v,w), \rho$ and p denote respectively the velocity of fluid, the density and the pressure respectively. Let ε and k_1 stand for medium porosity and medium permeability, respectively.

Then the equations of motion and continuity for Rivlin-Ericksen incompressible viscoelastic fluid in a porous medium are

$$\left[\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{\varepsilon}(\boldsymbol{v}.\nabla)\boldsymbol{v}\right]$$

$$= [-\nabla p + \rho\boldsymbol{g}] - \frac{\rho}{k_1}\left[v + v'\frac{\partial}{\partial t}\right]\boldsymbol{v}, \tag{2}$$

$$\nabla.\boldsymbol{v} = 0, \tag{3}$$

where $\boldsymbol{g}(0,0,-g)$ is the acceleration due to gravity, $v\left(=\frac{\mu}{\rho}\right)$ is kinematic viscosity of the fluid and $v'\left(=\frac{\mu'}{\rho}\right)$ is kinematic viscoelasticity of the fluid.

Since the density of the moving fluid remains unchanged, we have

$$\varepsilon\frac{\partial \rho}{\partial t} + (\boldsymbol{v}.\nabla)\rho = 0. \tag{4}$$

Let $\boldsymbol{v}(u,v,w), \delta\rho$ and δp denote respectively the perturbations in fluid velocity (0,0,0), fluid density ρ and fluid pressure p. Then the linearized perturbation equations [1,10] relevant for the porous medium are

$$\frac{\rho}{\varepsilon}\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\delta p + \boldsymbol{g}\delta\rho - \frac{\rho}{k_1}\left(v + v'\frac{\partial}{\partial t}\right)\boldsymbol{v} \tag{5}$$

$$\nabla.\boldsymbol{v} = 0, \tag{6}$$

$$\varepsilon\frac{\partial}{\partial t}(\delta\rho) = -wD\rho. \tag{7}$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on space coordinates x, y, z and time t is of the form

$$\exp(ik_x x + ik_y y + nt), \tag{8}$$

where k_x, k_y are horizontal wave numbers, $k^2 = k_x^2 + k_y^2$ and n is the rate at which the system departs from the equilibrium. For perturbations of the form (8), eqs. (5)–(7) give

$$\frac{1}{\varepsilon}\rho n u = -ik_x\delta p - \frac{\rho}{k_1}(v + v'n)u, \tag{9}$$

$$\frac{1}{\varepsilon}\rho n v = -ik_y\delta p - \frac{\rho}{k_1}(v + v'n)v, \tag{10}$$

$$\frac{1}{\varepsilon}\rho n w = -D\delta p - g\delta\rho - \frac{\rho}{k_1}(v + v'n)w, \tag{11}$$

$$ik_x u + ik_y v + Dw = 0, \tag{12}$$

$$\varepsilon n\delta\rho = -wD\rho. \tag{13}$$

Eliminating δp between eqs. (9)–(11) with the help of eqs. (12) and (13), we obtain

$$\frac{n}{\varepsilon}\left[D(\rho Dw) - k^2\rho w\right] + \frac{1}{k_1}\left[D\{\rho(v + v'n)Dw\}\right.$$

$$\left. - k^2\rho(v + v'n)w\right] + \frac{gk^2}{\varepsilon n}(D\rho)w = 0. \tag{14}$$

3. Two uniform Rivlin-Ericksen elastico-viscous fluids separated by a horizontal boundary

Consider the case when two superposed Rivlin-Ericksen elastico-viscous fluids of uniform densities ρ_1 and ρ_2, uniform kinematic viscosities v_1 and v_2 and uniform kinematic viscoelasticities v_1' and v_2' are separated by a horizontal boundary at $z = 0$. The subscripts 1 and 2 indicate the lower and upper fluid. Then in each region of constant ρ, constant v and constant v', eq. (14) reduces to

$$(D^2 - k^2)w = 0. \tag{15}$$

The general solution of (15) is

$$w = Ae^{+kz} + Be^{-kz}, \tag{16}$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied in the present problem are :

(i) The velocity should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).

(ii) $w(z)$ is continuous at $z = 0$.

(iii) The jump condition at the interface $z = 0$ between the fluids. This jump condition is obtained by integrating eq. (14) over an infinitesimal element of z including 0, and is

$$\frac{n}{\varepsilon}\left[\rho_2 Dw_2 - \rho_1 Dw_1\right]_{z=0} + \frac{1}{k_1}\left[(\mu_2 + \mu_2'n)Dw_2\right.$$

$$\left. - (\mu_1 + \mu_1'n)Dw_1\right]_{z=0} = -\frac{gk^2}{\varepsilon n}[\rho_2 - \rho_1]w_0, \tag{17}$$

where w_0 is the common value of w at $z = 0$.

Applying the boundary conditions (i) and (ii), we can write

$$w_1 = Ae^{+kz} \quad (z < 0), \tag{18}$$

$$w_2 = Ae^{-kz} \quad (z > 0), \tag{19}$$

where the same constant A has been chosen to ensure the continuity of w at $z = 0$.

Applying the condition (17) to the solutions (18) and (19), we obtain

$$\left[1+\frac{\varepsilon}{k_1}(v_2'\alpha_2+v_1'\alpha_1)\right]n^2 +\frac{\varepsilon}{k_1}(v_2\alpha_2+v_1\alpha_1)n-gk[\alpha_2-\alpha_1]=0. \tag{20}$$

where $\alpha_{1,2}=\frac{\rho_{1,2}}{\rho_1+\rho_2}$, $v_{1,2}=\frac{\mu_{1,2}}{\rho_{1,2}}$, $v'_{1,2}=\frac{\mu'_{1,2}}{\rho_{1,2}}$.

(a) Stable case ($\alpha_2 < \alpha_1$) :

Now for the potentially stable case ($\alpha_2 < \alpha_1$), eq. (20) does not admit of any change of sign and so has no positive root. The system is therefore stable.

(b) Unstable case ($\alpha_2 < \alpha_1$) .

For the potentially unstable case ($\alpha_2 > \alpha_1$), the constant term in (20) is negative. Eq. (20), therefore, allows one change of sign and so has one positive root and hence the system is unstable.

Therefore, the system is unstable for unstable configuration.

4. Effect of a horizontal magnetic field

Here, we consider the motion of an incompressible, infinitely conducting Rivlin-Ericksen elastico-viscous fluid in porous medium in the presence of a uniform horizontal magnetic field $H(H,0,0)$. Let $h(h_x, h_y, h_z)$ denote the perturbation in magnetic field, then the linearized perturbation equations are

$$\frac{\rho}{\varepsilon}\frac{\partial v}{\partial t}=-\nabla\delta p+g\delta\rho+\frac{\mu_e}{4\pi}(\nabla\times h)\times H-\frac{\rho}{k_1}\left(v+v'\frac{\partial}{\partial t}\right)v, \tag{21}$$

$$\nabla\cdot h=0, \tag{22}$$

$$\varepsilon\frac{\partial h}{\partial t}=\nabla\times(v\times H). \tag{23}$$

together with eqs. (6) and (7). Assume that the perturbation $h(h_x, h_y, h_z)$ in magnetic field has also a space and time dependence of the form (4). Here, μ_e stands for magnetic permeability. Following the procedure as in Section 3, we obtain

$$\left[1+\frac{\varepsilon}{k_1}(v_2'\alpha_2+v_1'\alpha_2)\right]n^2+\frac{\varepsilon}{k_1}(v_2\alpha_2+v_1\alpha_1)n + \left[2k_x^2V_A^2-gk(\alpha_2-\alpha_1)\right]=0, \tag{24}$$

where $V_A=\sqrt{\frac{\mu_e H^2}{4\pi(\rho_1+\rho_2)}}$ is the Alfvén velocity.

For the potentially stable case ($\alpha_2 < \alpha_1$), it is evident from eq. (24), that the system is stable.

For the potentially unstable case ($\alpha_2 > \alpha_1$), if

$$2k_x^2V_A^2 > gk(\alpha_2-\alpha_1),$$

i.e. if

$$\rho_2-\rho_1<\frac{\mu_e H^2 k_x^2}{2\pi gk} \tag{25}$$

Eq. (24) does not admit of any change of sign and so has no positive root. The system is therefore stable.

But if $2k_x^2V_A^2 < gk(\alpha_2-\alpha_1)$, (26)

the constant term in (24) is negative. Eq. (24) therefore, allows at least one change of sign and so has at least one positive root The occurrence of a positive root implies that the system is unstable.

Thus for the potentially unstable configuration, the presence of magnetic field stabilizes certain wave-numbers band whereas system was unstable for all wave numbers in the absence of magnetic field.

5. Effect of uniform rotation

Consider the motion of an incompressible Rivlin-Ericksen elastico-viscous fluid in porous medium in the presence of a uniform rotation $\Omega(0,0,\Omega)$. Then the linearized perturbation equations are

$$\frac{\rho}{\varepsilon}\frac{\partial v}{\partial t}=-\nabla\delta p+g\delta\rho+\frac{2\rho}{\varepsilon}(v\times\Omega)-\frac{\rho}{k_1}\left(v+v'\frac{\partial}{\partial t}\right)v, \tag{27}$$

together with eqs. (6) and (7).

Following the same procedure as in Section 3 (and Ref. [1], p 453), we obtain

$$1+\frac{4\Omega^2}{\left[n+\frac{\varepsilon}{k_1}(v+v'n)\right]^2}+\frac{gk^2(\alpha_1-\alpha_2)}{n\left[n+\frac{\varepsilon}{k_1}(v+v'n)\right]\kappa}=0, \tag{28}$$

$$\text{where } \kappa = 1+\frac{2\Omega^2}{\left[n+\frac{\varepsilon}{k_1}(v+v'n)\right]^2} \tag{29}$$

for highly viscous fluid and viscoelastic fluids and for $v_1 = v_2 = v$, $v_1' = v_2' = v'$.

Eq. (28), after substituting the value of κ from (29) and simplification, yields

$$\left[\left(1+\frac{\varepsilon v'}{k_1}\right)^3\right]n^4+\left[\frac{3\varepsilon v}{k_1}\left(1+\frac{\varepsilon v'}{k_1}\right)^2\right]n^3+\left(1+\frac{\varepsilon v'}{k_1}\right)\times\left[\left(\frac{3\varepsilon^2v'^2}{k_1^2}+4\Omega^2\right)+\left(1+\frac{\varepsilon v'}{k_1}\right)gk(\alpha_1-\alpha_2)\right]n^2$$
$$+\left[\frac{\varepsilon^3v^3}{k_1^3}+4\Omega^2\frac{\varepsilon v}{k_1}+2\frac{\varepsilon v}{k_1}\left(1+\frac{\varepsilon v'}{k_1}\right)gk(\alpha_1-\alpha_2)\right]n+\left[\left(\frac{\varepsilon^2v^2}{k_1^2}+2\Omega^2\right)gk(\alpha_1-\alpha_2)\right]=0. \tag{30}$$

For the potentially *stable case* ($\alpha_2 > \alpha_1$), all the coefficients of eq. (30) are positive. So, all the roots of eq. (30) are either real and negative, or there are complex roots (which occur in pairs) with negative real parts and the rest negative real roots. The system is therefore stable in each case.

However, for the potentially *unstable case* ($\alpha_2 > \alpha_1$), the constant term in eq. (30) is negative. Eq. (30) therefore, allows at least one change of sign and so has at least one positive root. The system is therefore unstable for potentially unstable case.

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