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A perturbation method for Maxwell's equation in a *pumped medium* II. unstable solutions

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Maxwell's wave equation in a medium whose permittivity undergoes a one dimensional space time variation by the action of a pump wave of angular frequency Ω is solved by a perturbation technique based upon the methods of Bugoliubov and Mitropolsky for non linear oscillations The solution is confined to the special case where unstable solutions are possible for certain frequency bands or wave number bands It is found that for $C > V$ (C the velocity of the wave in the unmodulated medium, V the velocity of the pump wave) there is a frequency band centered at $N\frac{\Omega}{2}\left(\frac{C}{V}+1\right)$, N an integer for which the propagation constant is complex and for $C < V$ there is a wave number band for which the frequencies are complex. The dispersion relation and expression for the amplitude of the associated harmonics are derived

1. Introduction

In the first part of the paper (Mathew 1974) the general solution of Maxwell's wave equation in a medium whose permittivity is modulated by a pump wave of angular frequency Ω progressing with a velocity V along the x direction was discussed. The assumed variation was

$$\epsilon(x_1t) = \epsilon_1\left[1+h\cos\Omega\left(t-\frac{x}{V}\right)\right], \tag{1}$$

where ϵ_1 is the permittivity in the absence of the modulation and h the modulation index which is much less than one. It was found that the wave equation was separable in the variables $\tau = t-\frac{x}{V}$, $X = x$ by the introduction of a separation constant β. Stable solutions occur for $\beta \simeq N\frac{\Omega}{2C}\left(\frac{C^2}{V^2}-1\right)$, where N is an integer, C the velocity of an electromagnetic signal in the immodulated medium given by $C^2 = \frac{1}{\mu_0\epsilon_1}$, μ_0 the permeability and this case was discussed in the first part of the paper

In several situations of physical interest for certain frequency or wave number bands unstable solutions occur where a signal in the medium will exponentially increase or decrease with distance, time or both. When the solution is expressed in terms of β these unstable solutions correspond to β in the neighbourhood of $N \frac{\Omega}{2}\left(\frac{C^2}{V^2}-1\right)$ and this part of the paper deals with this case.

Simon (1960) has discussed the aspect of waves growing or decaying with distance for a particular frequency, but has not given a method of solution which can be applied to all frequencies at which such instabilities occur. Further, the frequency range over which this occur is not given. Cassedy & Oliner (1963) have discussed the case of waves unstable in space, time or both, but analytical expressions are not developed for the frequency and wave number bands. Holbery & Kunz (1966) have treated the unstable solutions for a purely time varying permittivity. In this paper a general perturbation method is developed based on the methods of Bugoliubov & Mitropolsky (1961) for nonlinear oscillations This method can be applied to find all the frequency and wave number bands where the unstable solutions occur.

In this connection, it is of interest to discuss some of the physical possibilities of achieving the permittivity variation given by eq. (1). One method is by acoustic pumping. An acoustic wave propagating in the medium will produce periodic variation in the density of the medium. Since the electrical permittivity is a function of the density (Jackson 1962) the sound wave of frequency Ω can perturb the dielectric constant and effect the assumed permittivity variation. Such processes have been discussed by Slater (1958), Yariv (1965).

Another method is by electromagnetic pumping. The macroscopic permittivity of a medium is a consequence of polarizability of the molecules For ferroelectrics the polarizability depend on the electric field.' For ferro-electrics like barium-strontium titanate mixtures the permittivity (about 10^3) can be reduced by thirty to fifty percent by electric fields of the order of 10^6 volts/meter. For such nonlinear dielectrics the electric displacement D can be written approximately as a function of the electric field E as $D = \epsilon_1 E + \epsilon_2 E^2 + \ldots$ (Zernike & Midwinter 1973) where ϵ_2 is a constant which is small Taking the permittivity $\epsilon = \frac{dD}{dE}$ and assuming the field $E = E_0 \cos \Omega\left(t - \frac{x}{V}\right)$, we get the permittivity given by eq (1). A mechanism for producing the dielectric modulation by passing an intense laser beam through the medium has been discussed by Kroll (1962).

2. The Wave Equation and its Solutions

It was shown (Mathew 1974) that in the one dimensional case the wave equation with the permittivity variation given by eq. (1) is

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \mu_0 \frac{\partial^2}{\partial t^2} [\epsilon(x,t)E(x,t)] = 0, \tag{2}$$

with the change of variables $X = x, \tau = t - \frac{x}{V}$ the electric field $E(X,\tau)$ was expressed as

$$E(X_1\tau) = (\exp i\beta X)(\exp \int \eta(\tau)d\tau)G(\tau). \quad \dots (3)$$

In eq. (3), β is a separation constant,

$$\exp \int \eta(\tau)d\tau = \frac{T_0}{(1-\alpha h \cos\Omega\tau)} \exp i\alpha_1 \left[\frac{2}{(1-h^2\alpha^2)^{\frac{1}{2}}} \tan^{-1} \left\{ \left(\frac{1+h\alpha}{1-h\alpha} \right)^{\frac{1}{2}} \tan \frac{\Omega}{2}\tau \right\} \right], \quad \dots (4)$$

and the function $G(\tau)$ satisfies the differential equation

$$\frac{d^2G}{d\tau^2} + \gamma^2 G = -\gamma^2[h(\alpha_3 \cos \Omega\tau + i\alpha_2 \sin \Omega\tau) + h^2(\alpha_4 \cos^2\Omega\tau + i\alpha_2\alpha \sin 2\Omega\tau) + \dots]G. \quad \dots (5)$$

In eqs. (4) and (5), T_0 is an arbitrary constant,

$$\alpha = \frac{1}{\frac{C^2}{V^2} - 1}, \quad \alpha_1 = \frac{C^2\beta\alpha,}{V\Omega}, \quad \alpha_2 = \frac{\Omega}{V\beta} \quad \dots (6)$$

$$\alpha_3 = 1 + 2\alpha, \qquad \alpha_4 = 2\alpha + 3\alpha^2, \quad \gamma^2 = \frac{(\beta C)^2}{\left(\frac{C^2}{V^2} - 1\right)^2}$$

It was found that for $\gamma \neq p/q\Omega$, (p and q are mutually prime numbers) the solution of eq. (5) was stable with an amplitude which does not vary exponentially with τ.

Now the solutions of eq. (5) when γ is in the neighbourhood of $p/q\Omega$ and equal to $p/q\Omega$ are to be developed. It will be found that for certain values of p/q the solution is unstable with G growing exponentially with τ The cases where unstable solutions are possible are referred to as resonances. These resonances occur for values of $p/q = N/2$, where $N = 1, 2 \dots$ and in these cases the amplitude of an electromagnetic signal excited in the medium will grow or decay with distance, or time or both.

When we consider γ in the vicinity of $\gamma_0 = p/q\Omega$ we can write

$$\gamma^2 = \gamma_0^2 + h\Delta, \quad \dots (7)$$

where Δ represents a *detuining* due to the perturbation h. From eq. (6) using eq. (7), we get to the first order in h

$$\alpha_2 = \alpha_5 - hp, \quad \dots (8)$$

where

$$\alpha_5 = \frac{\Omega C}{\gamma_0 V\left(\frac{C^2}{V^2}-1\right)}, \text{and } \rho = \frac{\Delta\alpha_5}{2\gamma_0^2}, \quad \ldots \quad (9)$$

with these we can write eq. (5) in the form

$$\frac{d^2G}{d\tau^2}+\gamma_0^2G = -[h\{\gamma_0^2(\alpha_3 \cos\Omega\tau+i\alpha_5 \sin\Omega\tau)+\Delta\}$$

$$+h^2\gamma_0^2(\alpha_4 \cos^2\Omega\tau+i\alpha_5\alpha \sin 2\Omega\tau-i\rho \sin\Omega\tau+\ldots]G \quad \ldots \quad (10)$$

Eq. (10) represents the oscillation of a system with natural frequency γ subjected to a perturbing influence given by the terms on the right hand side. To solve this, a method similar to the one used in the first part of the paper can be developed with certain vital modifications. The solutions can be assumed in the form (Bugoliubov & Motropolsky 1961)

$$G = f \cos\psi+h\mathbf{u}_1(f, \psi, \Omega\tau)+h^2\mathbf{u}_2(f, \psi, \Omega\tau), \quad \ldots \quad (11)$$

where u_1, u_2 etc. are periodic functions of the two angular variables ψ and $\Omega\tau$, f and ψ are some functions of τ which can be determined from certain differential equations It can be naturally assumed that

$$\psi = \gamma_0\tau+\phi \quad \ldots \quad (12)$$

where ϕ is a phase difference which in the resonance case may exert vital influence on the amplitude and frequency of the oscillations The amplitude of the oscillation can be assumed to vary with τ as

$$\frac{df}{d\tau} = hR_1(f, \phi)+h^2R_2(f, \phi) \quad \ldots \quad (13)$$

Similarly the *frequency* $\frac{d\psi}{d\tau}$ can be expressed as

$$\frac{d\psi}{d\tau} = \gamma_0+\frac{d\phi}{d\tau} =\gamma_0+hS_1(f,\phi)+h^2S_2(f,\phi)+\ldots \quad \ldots \quad (14)$$

In the nonresonance case $df/d\tau$ and $d\psi/d\tau$ were functions of f alone. Using eqs. (11) through (14) we can express $d^2G/d\tau^2$ in terms of the new variables and the coefficients of like powers of h can be equated in the resulting equation Giving Fourier expansion to the associated functions we can get $U_1, U_2, R_1, S_1, R_2, S_2$ etc. It is found that when $p/q = \frac{1}{2}$ only R_1 and S_1 are nonzero and when $p/q = 1$ only R_2 and S_2 are nonzero. Since R_1 and S_1 are the terms first order in h, for small values of h the dependence of amplitude and frequency no the phase occurs only for $\gamma = \Omega/2$, i.e., resonance occurs only in this case. But for higher values of h the resonances will occur for $\gamma \approx \Omega, 3/2\Omega$ etc Thus the method developed is applicable for all resonances if higher order terms are considered. Considering the first order terms we get

$$\therefore \; U_1 = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\exp i(n\Omega\tau + m\psi)}{\Omega^2 \left[\left(\frac{p}{q}\right)^2 - \left(n + m\frac{p}{q}\right)^2 \right]} \int_0^{2\pi} \int_0^{2\pi} a_0 \exp - i(n\theta + m\psi) d\theta d\psi, \quad \ldots \; (15)$$

where $nq + (m \pm 1)p \neq 0$, $\theta = \Omega\tau$,

$$R_1 = \frac{-q}{4\pi^2 \Omega p} \sum_{\sigma=-\infty}^{\infty} \exp iq\sigma\phi \int_0^{2\pi} \int_0^{2\pi} a_0 \exp - iq\sigma\left(\psi - \frac{p}{q}\theta\right) \sin\psi \, d\psi \, d\theta \quad \ldots \; (16)$$

$$S_1 = \frac{q\Delta}{2\Omega p} - \frac{q}{4\pi^2 f \Omega p} \sum_{\sigma=-\infty}^{\infty} \exp iq\sigma\phi \int_0^{2\pi} \int_0^{2\pi} a_0 \exp - iq\sigma\left(\psi - \frac{p}{q}\theta\right) \cos\psi \, d\psi \, d\theta \; \ldots \; (17)$$

In the above equations,

$$a_0 = -\gamma_0^2 f(\alpha_3 \cos \Omega\tau + i\alpha_5 \sin \Omega\tau) \cos \psi \quad \ldots \; (18)$$

and

$$\Delta = \frac{\gamma^2 - \gamma_0^2}{h} = \frac{2\gamma_0(\gamma - \gamma_0)}{h}. \quad \ldots \; (19)$$

Thus evaluating U_1, R_1 and S_1 for particular p/q, U_2, R_2 and S_2 can be evaluated (Bugoliubov & Mitropolisky, 1961) and hence U_3, R_3, S_3 etc. Thus the method can be applied to higher orders of perturbation

3. Electric Field at the First Resonance

The first resonance is characterized by $\gamma \approx \frac{\Omega}{2}$ and since R_1 and S_1 are nonzero the amplitude and frequency have dependence on phase even for small values of h. The effect of this on the stability of the solutions and hence on the nature of the electric field can be investigated

With $p = 1$, $q = 2$, on evaluating the appropriate integrals in eqs. (15), (16) and (17) and by the use of eqs. (12), (13) and (14), we get

$$U_1 = \frac{f}{32} [(\alpha_3 + \alpha_5) \exp i(\psi + \theta) + (\alpha_3 - \alpha_5) \exp - i(\psi + \theta)], \quad \ldots \; (20)$$

$$\frac{df}{d\tau} = \frac{hf\Omega}{8} (\alpha_3 \sin 2\phi + i\alpha_5 \cos 2\phi), \quad \ldots \; (21)$$

$$\frac{d\phi}{d\tau} = \left(\gamma - \frac{\Omega}{2}\right) + \frac{h\Omega}{8} (\alpha_3 \cos 2\phi - i\alpha_5 \sin 2\phi). \quad \ldots \; (22)$$

To solve differential equations (21) and (22) we can introduce change of variables given by

$$L = f\frac{\Omega}{8}(\alpha_3+\alpha_5)\cos\phi, \qquad \dots \quad (23)$$

$$M = f\frac{\Omega}{8}(\alpha_3-\alpha_5)\sin\phi. \qquad \dots \quad (24)$$

From these using eqs. (21) and (22) we get a differential equation in L and M the solution of which can be assumed to be $L = L_0 \exp \lambda\tau$ and $M = M_0 \exp \lambda\tau$. Substituting these solutions we got two algebraic equations in L_0 and M_0 in terms of λ. For L_0 and M_0 to have a nontrivial solution the determinant of their coefficients in the equation should be zero. This determinental equation gives on substituting for α_3 and α_5 from eqs. (6) and (9)

$$\lambda = \pm\left[\frac{h^2\Omega^2}{64}-\left(\gamma-\frac{\Omega}{2}\right)^2\right]^{\frac{1}{2}}. \qquad \dots \quad (25)$$

From eq. (11), to the first order in h

$$G = f\cos\psi+hU_1 \qquad \dots \quad (26)$$

From eq. (19) the expression for U_1 contains the terms $f\exp i\psi$ and $f\exp-i\psi$. Since $\psi = \gamma_0\tau+\phi$, the expression for G involves terms in $f\exp i\phi$ and $f\exp-i\phi$. Hence it is not necessary to have explicit expression for f and ϕ and we can use eqs. (23) and (24) and the algebraic equation for L_0, M_0 to find the values of $f\cos\phi$ and $f\sin\phi$. Thus evaluating $f\exp i\phi$ and $f\exp-i\phi$ we get

$$G(\tau) = G_0\exp\lambda\tau\left[\exp i\frac{\Omega\tau}{8}\left\{1+\frac{\left(i\lambda+h\frac{\Omega}{2}\alpha_5\right)}{\frac{h\Omega}{8}\alpha_3-\left(\gamma-\frac{\Omega}{2}\right)}\right\}\left\{1+\frac{h}{16}(\alpha_3+\alpha_5)\exp(i\Omega\tau)\right\}\right.$$

$$\left.+\exp-i\frac{\Omega}{2}\tau\left\{1-\frac{\left(i\lambda+h\frac{\Omega}{8}\alpha_5\right)}{h\frac{\Omega}{\gamma}\alpha_3-\left(\gamma-\frac{\Omega}{2}\right)}\right\}\left\{1+\frac{h}{16}(\alpha_3-\alpha_5)\exp(-i\Omega\tau)\right\}\right], \quad \dots \quad (27)$$

where G_0 is an arbitrary constant.

Since λ has two values there are two solutions for G as given by eq. (27). Further, the amplitude varies exponentially with τ if λ is real and is oscillatory if λ is imaginary. By eq. (25), λ is real or imaginary according as $(\gamma-\Omega/2)^2$ is less or greater than $h^2\Omega^2/64$. The amplitude thus varies exponentially with τ for those values of γ for which

$$\frac{\Omega}{2}-h\frac{\Omega}{8}<\gamma<\frac{\Omega}{2}+h\frac{\Omega}{8}$$

and thus in this case the solutions are unstable. Using eqs. (3), (4), and (27), we get the electric field as

$$E(x,t) = A\exp\lambda\Big(t-\frac{x}{V}\Big)\Big[\exp i\Big\{Wt-\Big(\frac{W}{V}-\beta\Big)x\Big\}\times\Big\{1+hC_1\exp i\Omega\Big(t-\frac{x}{V}\Big)$$

$$+hC_2\exp-i\Omega\Big(t-\frac{x}{V}\Big)\Big\}+D\exp\Big\{(W-\Omega)t-\Big(\frac{W-\Omega}{V}-\beta\Big)x\Big\}\times\Big\{1$$

$$+hC'_1\exp i\Omega\Big(t-\frac{x}{V}\Big)+hC'_2\exp-i\Omega\Big(t-\frac{x}{V}\Big)\Big\}\Big], \quad \ldots (28)$$

where

$$W = (\alpha_1+\tfrac{1}{2})\Omega, \quad \ldots (29)$$

$$D = \frac{\frac{h\Omega}{8}(\alpha_3-\alpha_5)-i\lambda-\Big(\gamma-\frac{\Omega}{2}\Big)}{\frac{h\Omega}{8}(\alpha_3+\alpha_5)+i\lambda-\Big(\gamma-\frac{\Omega}{2}\Big)} \quad \ldots (30)$$

$$\left.\begin{aligned} C_1 &= \frac{\Big(\frac{C}{V}+3\Big)}{16\Big(\frac{C^2}{V^2}-1\Big)}, & C_2 &= \frac{1-\frac{C}{2V}}{2\Big(\frac{C^2}{V^2}-1\Big)} \\ C'_1 &= \frac{1+\frac{C}{2V}}{2\Big(\frac{C^2}{V^2}-1\Big)}, & C'_2 &= \frac{\Big(\frac{C}{V}-3\Big)^2}{16\Big(\frac{C^2}{V^2}-1\Big)} \end{aligned}\right\} \quad \ldots (31)$$

and A is an arbitrary Constant.

4. Discussion of Special Cases

Eq. (28) represents a wave with a fundamental frequency W with a propogation constant $(W/V)-\beta$ and with the harmonics of frequencies $W-\Omega$, $W+\Omega$, $W-2\Omega$ with different propagation constants and amplitudes. But we note from eq. (30) that each term is of the order of h and hence D which is the relative amplitude of the harmonic of frequency $W-\Omega$ is independent of h. Thus the amplitude of this harmonic is of the same order as that of W. Thus there are two dominant frequencies W and $W-\Omega$ having propagation constants $(W/V)-\beta$ and $[(W-\Omega)/V]-\beta$ respectively. Further, if λ is real these amplitudes will exponentially increase or decrease with time and distance. The other harmonics are insignificant because they are of the order of h.

We see from eqs. (6) and (29) that the frequency W is related to β and hence to Ω in the resonance case. From eq. (6) the value of β corresponding to $\gamma \approx \frac{\Omega}{2}$ is obviously near $\frac{\Omega}{2C}\left(\frac{C^{\gamma}}{V^2}-1\right)$. It is of interest to discuss some important special cases corresponding to the values of γ in the neighbourhood of $\Omega/2$.

Case (*i*) $\lambda = 0$.

This case corresponds to $\gamma = \frac{\Omega}{2} \pm h\frac{\Omega}{8}$. The corresponding value of β from eq. (6) is

$$\beta = \frac{\Omega}{2C}\left(\frac{C^2}{V^2}-1\right)\left(1 \pm \frac{h}{4}\right) \qquad \ldots (32)$$

Hence from eq. (28) we get the electric field as

$$F(x_1 t) = A \exp \pm ih \frac{\Omega}{8} \frac{C}{V}\left(t-\frac{Vx}{C^2}\right)\Big[(1+hC'_1D)\exp i\omega_0\left(t-\frac{x}{C}\right)$$
$$+D\left(1+h\frac{C_1}{D}\right)\exp i(\omega_0-\Omega)\left(t+\frac{x}{C}\right)$$
$$+\ldots\text{terms of the order of } h\Big] \qquad \ldots (33)$$

where from eq. (30),

$$D = \mp \frac{\left(\frac{C}{V}-1\right)}{\left(\frac{C}{V}+1\right)} \qquad \ldots (34)$$

and

$$\omega_0 = \frac{\Omega}{2}\left(\frac{C}{V}+1\right). \qquad \ldots (35)$$

Eq. (33) can be interpreted as a wave with a fundamental frequency

$$\omega = \omega_0 \pm \frac{h\Omega}{8} \frac{C}{V}$$

with the harmonic of dominant frequency $\omega-\Omega$. The fundamental of frequency ω propagates in the direction of the pump wave while the dominant harmonic of frequency $\omega-\Omega$ propagates in the opposite direction and their propagation constants being

$$\frac{\omega_0}{C} \pm \frac{h\Omega}{8C} \text{ and } \frac{\omega_0-\Omega}{C} \mp \frac{h\Omega}{8C}.$$

Thus we obtain the dispersion relation and see that the waves propagate without any attenuation in space and time.

Case (*ii*) $\gamma = \Omega/2$ which corresponds to $\lambda = \pm h\Omega/8$. In this case

$$\beta = \frac{\Omega}{2C}\left(\frac{C^2}{V^2}-1\right)$$

and from eq. (28) the electric field is approximately

$$E(x,t) = A \exp \pm \frac{h\Omega}{8}\left(t-\frac{x}{V}\right)\left[\exp iw_0\left(t-\frac{x}{C}\right)+D\exp i(w_0-\Omega)\left(t+\frac{x}{C}\right)\right.$$

$$\left.+\ldots\text{terms of the orders of } h\right]. \quad \ldots \quad (36)$$

where the harmonics of order h have been neglected. Eq. (36) represents a wave with dominant frequencies ω_0 and $\omega_0-\Omega$ propagating in opposite directions with the same phase velocity. But their amplitudes grow or decay both in space and time and hence in this case the waves are unstable. We can say that both the frequency and propagation constants are complex.

Case (*iii*) : Now we can consider the propagation constant corresponding to a real frequency $\omega = \frac{\Omega}{2}\left(\frac{C}{V}+1\right)$. This is the most important case since we can know the features of propagations of a signal of frequency ω in the medium.

Considering eq. (28), let us put $iW+\lambda = i\omega$, where ω is real so that

$$\omega = W - i\lambda \quad \ldots \quad (37)$$

We have from eqs (29) and (6)

$$W = \frac{\Omega}{2}+\frac{C^2\beta}{V\left(\frac{C^2}{V^2}-1\right)} = \frac{\Omega}{2}+\frac{C}{V}\gamma, \quad \ldots \quad (38)$$

so that

$$\gamma-\frac{\Omega}{2} = \frac{V}{C}\left[W-\frac{\Omega}{2}\left(\frac{C}{V}+1\right)\right] \quad \ldots \quad (39)$$

Hence from eqs. (37), (39) and (25) we get

$$W = \frac{\left[w-\frac{\Omega}{2}\frac{V^2}{C^2}\left(\frac{C}{V}+1\right)\right]\pm\left[\frac{V^2}{C^2}\left\{w-\frac{\Omega}{2}\left(\frac{C}{V}+1\right)\right\}^2-\left(1-\frac{V^2}{C^2}\right)\frac{h^2\Omega^2}{64}\right]^{\frac{1}{2}}}{\left(1-\frac{V^2}{C^2}\right)}. \quad \ldots \quad (40)$$

Since we are considering a real frequency ω, the quantity W (which will be shown to be related to the propagation constant) is real if

$$\left[w-\frac{\Omega}{2}\left(\frac{C}{V}+1\right)\right]^2 \geqslant \left(\frac{C^2}{V^2}-1\right)\frac{h^2\Omega^2}{64}. \qquad \text{... (41)}$$

But according to eq. (28), the propagation constant corresponding to the frequency $w = W-i\lambda$ is

$$k = \frac{W-i\lambda}{V}-\beta = \frac{w}{V}-\left(W-\frac{\Omega}{2}\right)\frac{V}{C^2}\left(\frac{C^2}{V^2}-1\right). \qquad \text{... (42)}$$

From eq. (42) we see that k is real if ω is real and this in turn hold when condition (41) is satisfied. It is seen that (41) is satisfied always when $C > V$. For $C > V$ it is not satisfied for particular values of w given by

$$\frac{\Omega}{2}\left(\frac{C}{V}+1\right)-\frac{h\Omega}{8}\left(\frac{C^2}{V^2}-1\right)^{\frac{1}{2}} < w < \frac{\Omega}{2}\left(\frac{C}{V}+1\right)+\frac{h\Omega}{8}\left(\frac{C^2}{V^2}-1\right)^{\frac{1}{2}}. \qquad \text{... (43)}$$

Hence we come to the important conclusion that when $C > V$, there is a frequency band centered round $\frac{\Omega}{2}\left(\frac{C}{V}+1\right)$ for which the propagation constant is complex. Such a frequency band is called a stop band. For the central frequency $w = \frac{\Omega}{2}\left(\frac{C}{V}+1\right)$ the propagation constant can be evaluated. From eqs. (40) and (42) with $w = \frac{\Omega}{2}\left(\frac{C}{V}+1\right)$, we get

$$k = \frac{\Omega}{2C}\left(\frac{C}{V}+1\right)\mp\frac{ih\Omega}{2C}\left(\frac{C^2}{V^2}-1\right)^{\frac{1}{2}}. \qquad \text{... (44)}$$

From eqs. (30), (37), (39) and (40) we get

$$D = \pm i\,\frac{\left(\frac{C}{V}-1\right)^{\frac{1}{2}}}{\left(\frac{C}{V}+1\right)^{\frac{1}{2}}},\ \gamma = \frac{\Omega}{2}\pm\frac{h\Omega}{8\left(1-\frac{C^2}{V^2}\right)^{\frac{1}{2}}}. \qquad \text{... (45)}$$

Hence the electric field to the first order in h is

$$E(x,t) = A\exp\mp\frac{h\Omega}{8C}\left(\frac{C^2}{V^2}-1\right)^{\frac{1}{2}}x\left[(1+hC'_1D)\exp iw\left(t-\frac{x}{C}\right)\right.$$

$$\left.+D\left(1+h\frac{C_2}{D}\right)\exp i(w-\Omega)\left(t+\frac{x}{C}\right)+\ldots\right] \qquad \text{... (46)}$$

In this case the dominant frequencies are ω and $\omega-\Omega$. Further it can be noted that for $C > V$, a wave of frequency $\frac{\Omega}{2}\left(\frac{C}{V}+1\right)$ propagating in the direction of the pump wave or one of frequency $\frac{\Omega}{2}\left(\frac{C}{V}-1\right)$ propagating in the opposite direction will exponentially grow or decay with distance. This happens in practical situations when the permittivity modulation of the medium is effected by an acoustic wave whose velocity V is always less than C. In this case the particular electromagnetic signal is attenuated in the medium.

Case (*iv*) : Another interesting case is to find the frequency corresponding to a real propagation constant $k = \frac{\Omega}{2C}\left(\frac{C}{V}+1\right)$. Considering eq. (28) let us put

$$K = \frac{(\alpha_1+\frac{1}{2})\Omega}{V}-\beta = \frac{\Omega}{2V}+\frac{\gamma}{C}, \qquad \ldots \quad (47)$$

so that we can write the propagation constant

$$k = K-\frac{i\lambda}{V} \qquad \ldots \quad (48)$$

Again using eqs. (6), (25) and (47) in (48) we get

$$K = \left[k-\frac{\Omega C}{2V^2}\left(\frac{C}{V}+1\right)\right] \pm \left[\frac{1}{V^2}\left\{Ck-\frac{\Omega}{2}\left(\frac{C}{V}+1\right)\right\}^2-\frac{h^2\Omega^2}{64V^2}\left(1-\frac{C^2}{V^2}\right)\right]^{\frac{1}{2}}. \qquad (49)$$

From eq. (28), the frequency corresponding to the propagation constant K is

$$\omega = (\alpha_1+\tfrac{1}{2})\Omega-i\lambda = kV+V\left(\frac{C^2}{V^2}-1\right)\left(K-\frac{\Omega}{2V}\right)$$

Since we are considering a real k, the frequency ω is real if K is real. For this, we see from eq. (49) that

$$\left[Ck-\frac{\Omega}{2}\left(\frac{C}{V}+1\right)\right]^2 > \frac{h^2\Omega^2}{64}\left(1-\frac{C^2}{V^2}\right). \qquad \ldots \quad (50)$$

Condition (50) is always satisfied when $C > V$. Thus for any real k, ω is always real for $C > V$, but not vice versa. This was discussed in case (iii). For $C < V$, condition (50) is not satisfied for those values of k for which

$$\frac{\Omega}{2C}\left(\frac{C}{V}+1\right)-\frac{h\Omega}{2C}\left(1-\frac{C^2}{V^2}\right)^{\frac{1}{2}} < k < \frac{\Omega}{2C}\left(\frac{C}{V}+1\right)+\frac{h\Omega}{8C}\left(1-\frac{C^2}{V^2}\right)^{\frac{1}{2}}. \qquad \ldots \quad (51)$$

For $C < V$, the frequency is complex for a band of propagation constants centered at $\frac{\Omega}{2C}\left(\frac{C}{V}+1\right)$ of width $\frac{h\Omega}{4C}\left(1-\frac{C^2}{V^2}\right)^{\frac{1}{2}}$. Thus in this case the role of wave propagation constant and frequency is reversed from that for $C > V$. The electric field for $k = \frac{\Omega}{2C}\left(\frac{C}{V}+1\right)$ can be shown to be

$$E(x,t) = A \exp \mp \frac{h\Omega}{8}\left(1-\frac{C^2}{V^2}\right)^{\frac{1}{2}} t\left[(1+hC'_1D)\exp iCk\left(t-\frac{x}{C}\right)\right.$$

$$\left.+D\left(1+h\frac{C_2}{D}\right)\exp i(Ck-\Omega)\left(t+\frac{x}{C}\right)+\ldots\right], \qquad \ldots (52)$$

where

$$D = \pm\frac{i\left(1-\frac{C}{V}\right)^{\frac{1}{2}}}{\left(1+\frac{C}{V}\right)^{\frac{1}{2}}} \qquad \ldots (53)$$

The corresponding value of

$$\gamma = \frac{\Omega}{2}\pm\frac{h\Omega}{8\left(1-\frac{V^2}{C^2}\right)^{\frac{1}{2}}}.$$

When $C < V$, the waves corresponding to a signal of propagation constant $\frac{\Omega}{2C}\left(\frac{C}{V}+1\right)$ propagating along the direction of the pump wave are exponentially growing or decaying with time.

5. Conclusion

The solution of the wave equation was saught by the introduction of a separation constant β which is related to the quantity γ through eq. (6). When the solution to the first order in h was considered, unstable solutions (*i.e.*, waves which grow or decay in space and time) occured for $\gamma \approx \Omega/2$ which is termed as first resonance. In this case the wave consists of two dominant frequencies ω and $\omega-\Omega$ where $\omega = \frac{\Omega}{2}\left(\frac{C}{V}+1\right)$, the former propagating in the direction of the pump wave and the latter in the opposite direction with the same velocity. The nature of the waves and the relative amplitudes of the harmonics depend on the value of γ sufficiently near $\Omega/2$.

(a) For $\gamma = \frac{\Omega}{2}\pm\frac{h\Omega}{8}$, the frequency and propagation constant are real so that the solutions are stable.

(b) For $\gamma = \Omega/2$ the frequency and the propagation constant are complex so that the waves are exponentially grawing or decaying with time and position.

(c) For $\gamma = \frac{\Omega}{2}\pm\frac{h\Omega}{8\left(1-\frac{C^2}{V^2}\right)^{\frac{1}{2}}}$, if $C > V$, the propagation constant corresponding to a real frequency $\omega = \frac{\Omega}{2}\left(\frac{C}{V}+1\right)$ is complex so that the waves grow or decay exponentially with distance. This happens for a frequency band contened at $\frac{\Omega}{2}\left(\frac{C}{V}+1\right)$ of width $\frac{h\Omega}{4}\left(\frac{C^2}{V^2}-1\right)^{\frac{1}{2}}$. It means that an incident signal on the medium in this frequency range will exponentially grow or decay with distance.

(d) For $\gamma = \frac{\Omega}{2} \pm \frac{h\Omega}{8\left(1-\frac{V^2}{C^2}\right)^{\frac{1}{2}}}$, if $C < V$ the frequency compounding to a real propagation constant is complex so that the waves grow or decay exponentially with time. Thus happens for a wave number band centered at $\frac{\Omega}{2C}\left(\frac{C}{V}+1\right)$ and width $\frac{h\Omega}{4C}\left(1-\frac{C^2}{V^2}\right)^{\frac{1}{2}}$. A signal incident on the medium in this wave number region will grow or decay in time.

The perturbation has to be extended to higher orders for higher resonances. For $\gamma \approx \Omega$, it has to be extended to second order in which case the dominant frequencies are ω and $\omega - 2\Omega$, where $\omega = \Omega\left(\frac{C}{V}+1\right)$. In general for the Nth resonance where $\gamma \simeq N\frac{\Omega}{2}$ the perturbation has to be extended to Nth order and the dominant frequencies are ω and $\omega - N\Omega$ where $\omega = N\frac{\Omega}{2}\left(\frac{C}{V}+1\right)$. The width of the frequency band (wave number band) for which the waves are unstable in space (time) is proportional to h^N

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