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Hydrodynamical approach to quantum physics*

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For convenience and in order to focus your attention to this novel approach to quantum physics, which not only bridges classical and quantum physics in a more satisfactory way, at least from pedagogical standpoint, but has also the potentiality to overcome the limitations of present day quantum theory, I shall at the outset summarise the conclusions arrived at.

Our fundamental hypothesis rests on the fact that the basic *empirical* observation of quantum phenomena is that a particle possesses *simultaneously* both corpuscular and wave properties. Both are physical realities, complimentary physical properties, (in the usual sense and not in the sense of Copenhagen interpretation), of a particle. Consequently, there must be an intimate relation between the Newton-Einstein corpuscular properties and Huygens-Maxwell wave properties of the particle. But this would need an extension of the concept of the dynamical mass of the particle. The dynamical mass of this theory depends also on the space-time curvature of the amplitude of the wavefield. Everything else can be looked upon as a formal development from this basic experience.

I will try to prove here the following results :

1) The wave function $\epsilon(\boldsymbol{x}, t)$, (not Schroedinger's Ψ-functions), is a physical reality. This comes as an inevitable conclusion from Renninger's Gedanken experiment[1]. At the present state of our knowledge we can only guess about the nature of the physical reality. I will talk on that in my last lecture. In anticipating the conclusions, which still are of provisional nature, I would like to put forward the idea that the ultimate physical reality (in so far as it can be inferred from our present day knowledge) is the energy density continuum (in the mathematical sense)—almost akin to the vacuum of modern physics—whose space-time topological distortions and fluctuations give rise to observable phenomena. The relations between mathematical functions and the corresponding physically observable functions are given by the Function Algebra of Hosemann & Bagchi[2–4]**.

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** For a convenient exposition of the concept of Function Complex and its algebra as well as its relation to Schwartz-Temple generalised functions *see* Chapter V and the Appendix of the reference 4.

2) Based on almost self-evident postulates, one can derive a nonlinear equation for this wave field which under restricted conditions gives rise to all the basic relevant equations of quantum mechanics and of classical and relativistic physics.

I must emphasize here that in its methodology and general outlook, this theory has nothing to do with the *hidden variable* theory and is also quite different from that of Bohm, Vigier, Takabyasi and their co-workers. Nevertheless, recent researches of de Broglie School clearly point out that their *subquantum fluid* is the physical reality I am talking of. Following G. Mie, I would like to call it *World Aether.* It appears that this must play a great role in attempts to develop a universal field theory. At the present moment, to understand physically the properties of elementary particles it would be highly desirable to find the connections between various unorthodox causal approaches to *transquantum* physics, and in particular, the most general property of this world aether and its relation to the *corpuscle* and its wave field.

Without going into philosophical discussions,* I would like to remark that all existing discussions on hidden variables and transquantum causal physics rest on the fundamental assumptions of the formalism of linear operators in Hilbert space. As soon as one recognizes the fact that our equation is nonlinear in which the singularity as well as nonanalyticity of the wave field is likely to play an important role, all these criticisms and remarks lose their force as anything binding.

3) We show that Schroedinger equation (for a single particle) comes as a linear nonrelativistic approximation and the operator formalism of quantum theory from the condition that the wave function must satisfy the condition

$$\underline{\boldsymbol{p}}_{\omega}.(\epsilon^*\underline{\boldsymbol{\nabla}}\epsilon+\epsilon\underline{\boldsymbol{\nabla}}\epsilon^*) = 0 \;\ldots\; \text{(Schroedinger condition)}^{**}$$

4) Point mechanics (both relativistic and classical) results from the restriction that the space-time curvature of the amplitude of the wave function is zero. We need not assume that $h \to 0$ to get point mechanics, although we could arrive at it by making this nonpermissible approximation. However, the fact remains that even in the classical domain h is *not* zero. We prove that point mechanics remains *strictly* valid, even if h remains finite, as long as the wave field associated with the particle does not suffer diffraction. The relation between classical and wave mechanics is exactly analogous to that between geometrical and physical optics.

* For such philosophical discussions, *see* my lectures on the Problems of Philosophy of Science, to be published shortly.

** For the meaning of the Symbols, *see* later (section III).

5) Photon corpuscle's velocity in the Fraunhofer zone is c, but in the Fresnel zone it is less than c. There is an intimate relation between the singularity of the field representing the corpuscle and the extended field outside it.

6) Heisenberg's Uncertainity Principle needs a minor correction. It is valid for the average experimental value. It need not be accepted as a fundamental principle of nature for the *description* of the physics of a single particle***.

If we agree that any physical phenomenon must *in principle* be describable as a function of x, y, z, t, we can safely conclude that Heisenberg's Uncertainty Principle is the price of our representation. Instead of talking of momentum directly in space-time coordinates, in quantum theory one is characterising the momentum of the particle in terms of the Fourier components of the wave field. Consequently, Heisenberg's Uncertainty Principle is nothing but the statement of the generally valid mathematical relation between the coordinates of the physical space-time and those of its reciprocal space.

If we accept this point of view, then it is doubtful whether the causality condition of Quantum Field Theory, (namely $[p, q] = 0$, for space-like vectors, assumed on the basis of finite signal velocity), should be of universal applicability.

7) Our work on Kepler problem[9] shows that Schroedinger's ψ-function is the resultant of two partial pilot waves belonging to the pilot wave of a single particle : $\epsilon_1 + \epsilon_2 = \psi$.

Consequently, if one uses ψ as the basis of physical interpretation, one has to fall back upon statistical interpretation and one cannot find a deterministic relation between ψ-function and the actual trajectory of the particle.

It is generally believed, (albeit erroneously, that it is not possible to formulate quantum phenomena on a causal basis. But the work of Hosemann & Bagchi[6-9] on the scalar theory and that of Bagchi[10] on the vector formulation of the theory convincingly disprove this mistaken notion. At least so far as a single particle Schroedinger and Dirac equations are concerned, our work has mathematically and physically proved that it is possible to derive these celebrated equations causally. Only future can show whether this causal theory and the axioms on which this rests can be extended to cover more complicated physical situations.

At the present state of our knowledge, one can safely assume that Einstein's point of view, namely, quantum mechanical formalism must be an ensemble description and it should be possible to discover some causal fundamental equations which would lead to the ensemble description for a collection of particles, is the correct one also physically.

*** It is worthwhile to note that from entirely different considerations Dirac[5] also came to the same conclusion.

I would particuarly like to emphasize here that although our theory is a causal theory and our formalism is closely related to Hamilton-Jacobi formalism of classical mechanics and to Hamilton-de Broglie pilot principle of quantum mechanics, it is, nevertleless, not a march back to classical physics. As will be evident later on, the initial value problem of classical mechanics had been changed to boundary value problems of the mechanics of a particle, just as in wave mechanics.

8) So long we have been talking of a scalar field. Now if we extend these ideas to a vector field in Minkowski space, we again get a nonlinear field equation. This vector field automatically splits up into two parts :

(i) an irrotational part which can be correlated with the linear momentum of the centroid of the particle associated with translational motion; and

(ii) a vortical part which can represent a particle with intrinsic angular momentum. This vortex has the dimension of Compton wavelength in physical space.

We get the Proca equation for any particle with any *spin**, if the particle is uncharged and/or in the absence of an external field. We obtain Dirac equation in the form of a second order linear partial differential equation, the so called *iterated* Dirac equation of Sommerfeld[11], provided we integrate the wave field over its vortical domain. Only the hypercomplex quantities introduced by Sommerfeld have been replaced by the more physically meaningful concept of the components of the four-rotation of the field.

With this vector field and the universal existence of vortical field, one can *infer*** many properties of particles which appear as mysterious both from classical and from quantum mechanical point of view, e.g., the trembling motion (Zitterbewegung) of the electron proposed by Schroedinger, velocity c of the Dirac electron, (see ref. 12), the energy spectrum of the roton of ^{4}He and possibly

* It is desirable to distinguish between the quantum mechanical quantity *spin* and *the intrinsic angular momentum* (in the classical sense) of the particle. For details see later (section XII(iv)).

** Unless and until one gets the singular solution as well as all possible non-analytic solutions of the nonlinear partial differential equation and knows how the quantum vortices interact with one another when they penetrate into the vortical domain of the particles and also the nature of the turbulence created in the resultant wave field due to the interaction of waves of individual particles and the return of this resultant wave field to the equilibrium situation, one cannot hope to predict anything definitely. It is obvious that this project can hardly be carried out at the present stage of our knowledge. Consequently, in order to proceed further, we must try to guess intuitively by positing physically plausible conjectures on the basis of our existing knowledge. I need not therefore apologize for the physical extrapolations to be found at the concluding part of my lecture.

the nature of nuclear forces and the mechanism of the creation of elementary particles.

With these introductory remarks about the aim of those lectures, let me first say a few words on Renninger's work before proceeding to develop the theory on postulatory basis.

II. Renninger's Gedanken Experiment

It is now generally believed that no experiment can *simultaneously* prove the corpuscular and wave aspects of a particle. Renninger wants to prove with the help of a Gedanken experiment—(which he asserts can be realized in practice also)—that each light quantum (or an electron) is a corpuscle of energy which is guided *causally* by its wave field existing outside the domain of the corpuscle. His arguments are based on two experimentally established facts, namely,

1) All interference experiments run in the same way whether many photons appear simultaneously or they appear slowly one by one. That means, each photon interferes with itself, (cf. also Dirac[12]).

2) Many partially coherent beams of light remain coherent when they travel in different and separated paths. Michelson & Gale[13] had established this experimentally for an optical path length of 2 km.

Figure 1 represents a schematic arrangement of Renninger's experiment.

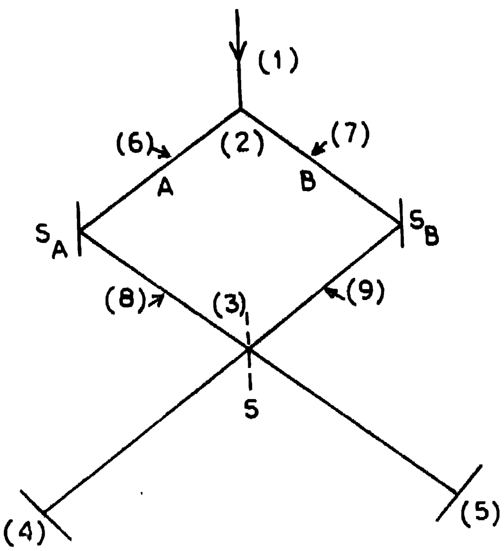

Fig. 1

A parallel beam of monochromatic light (1) is spatially separated at (2) into two beams A and B and after a certain time these two separated beams meet at (3).

Here the two beams are allowed to interfere with each other with the help of a half-silvered plate S. The path difference between A and B can be so arranged (with the help of the mirrors S_A, B_B and/or phase plates) that either the field (4) is bright and the field (5) is dark or *vice-versa*. Further, the light source acts in such a way that not more than one photon enters the path between (2) and (3) simultaneously. Moreover at different places (6) (7) (8) and (9) one can insert detectors and/or $\lambda/2$ plates.

Now, suppose the experimental arrangement is so made that initially we have (4) bright and (5) dark. We can then infer the following experimental observations.

(a) When nothing is inserted in the path, all the photons come to (4) and none in (5).

(b) If we insert a detector in (6), we find the two possibilities with 50% probability, namely,

(i) If the photon is registered in (6), it vanishes and both (4) and (5) remain dark. No experiment can detect the presence of the photon after it has been absorbed by the detector.

(ii) If the photon corpuscle does not pass through (6), obviously it is passing through the path (7). This can be established by the result different from the situation referred to in (i). In this case the photon comes either to (4) or to (5) each with the probability of 50%. That means, by blocking the path A we have changed the experimental outcome.

(c) Now, let us make another experiment in which instead of the absorber we have inserted a perfectly transparent $\lambda/2$-plate in (6). This time all the photons will be registered at (5) and none in (4). That means it is possible through an experiment to guide a photon corpuscle whether they are located in the path A or in the path B *always* to (5). Note that without the $\lambda/2$-plate or the absorber in (6) all the photons went to (4).

Thus, we can unequivocally conclude from this series of experiments that the corpuscle of energy belonging to the photon lying somewhere between (2) and (3) can be guided by tampering with the extended wave field associated with this photon and far outside the domain of this corpuscle of energy. Evidently, this extended wavefield must have some physical reality since one can, by inserting the phase plate suitably, determine the fate of all the photons irrespective of the fact whether the photon's corpuscular energy lies in the path $2 \rightarrow 6 \rightarrow 8 \rightarrow 3$ or in the path $2 \rightarrow 7 \rightarrow 9 \rightarrow 3$. Moreover, one can direct all of them either to (4) or to (5) according to one's convenience.

Now, a serious question arises. One might pertinently ask : What happens to the wave field when the photon vanishes in (6) through an absorber ? We can in no way detect the presence of this wave field after the photon had vanished.

There are two easy solutions (and both are physically untenable) to this dilemma. Either, one can say that the field vanishes instantaneously. The field then must contract with ultraphoton velocity which contradicts the special theory of relativity. Or, the wave field must be energyless which contradicts almost self-evident conclusions arrived at from Renninger's experiment, since causal connection must be due to some interaction of energy.

But both these two unsatisfactory explanations can be avoided if we assume that the physical reality is the continuum of energy density and the corpuscle and its associated wave field are space-time topological distortions and fluctuations of this continuum.

Further, *practically* the entire energy of the photon is concentrated in the singular domain of the topological distortion of the continuum. After the photon is absorbed, the wave field, which must carry slight energy, returns back to the unperturbed state of the continuum, whose properties cannot be measured.

It might be noted that this plausible physical conjecture about the physical reality is perfectly consistent with de Broglie's idea of the double solution of the pilot wave. Renninger himself, however, did not speculate about the nature of the physical reality and its connection with the wave field, although he asserted to have proved that the wave field associated with a particle is a physical reality.

III. Derivation of the Generalized Equation for the Scalar Field of a Single Particle

(i) *Notations* :

Underlined quantities are four-vectors in Minkowski space. The coordinates of this space are $x_1, x_2, x_3, x_0 = ict$.

Signature : $+ + + -$

Examples : Four distance, $\underline{x} = \mathbf{x} + ict.\mathbf{s}_0 = \Sigma x_j \mathbf{s}_j$

$\mathbf{s}_j$'s $(j = 0, 1, 2, 3)$ are unit vectors along the four mutually orthogonal axes and $\mathbf{s}_0$ along the time axis indicate the three vectors in physical space. Four velocity $\underline{v} = d\underline{x}/d\tau = \kappa(\mathbf{v} + ic.\mathbf{s}_0)$,

where the proper time $d\tau$ is related to the local time dt by

$$dt = \kappa d\tau \text{ and } \kappa = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$

The scalar product of $\underline{v}$ with itself is given by $\underline{v}^2 = -c^2$

An arbitrary four-vector $\underline{A}(\underline{x})$ is represented in terms of four components by

$\underline{A} = A_1, \mathbf{s}_1, + A_2\mathbf{s}_2 + A_3\mathbf{s}_3 + A_0\mathbf{s}_0.$

It will also be convenient to write the vector product of two and three four-vectors in the following way : (cf. Sommerfeld [11a])

Vector product of two four-vectors, the socalled six-vector $\widetilde{F}$, is written as

$$\widetilde{F} = [\underline{A}\underline{B}]$$

whose components are

$$F_{kl} = \begin{vmatrix} A_k & A_l \\ B_k & B_l \end{vmatrix} = -F_{lk}, \quad (\kappa, l = 0, 1, 2, 3).$$

Vector multiplication of a six-vector with a four-vector gives again a four-vector.

Thus,

$$\underline{D} = [\underline{A}[\underline{B}\,\underline{C}]]$$

Its jth component is given by

$$D_j = \sum_{k=0}^{3} A_k(B_j C_k - B_k C_j)$$

The four-gradient $\underline{\nabla} = \boldsymbol{\nabla} + \frac{\partial}{ic\partial t} \cdot \boldsymbol{s}_0$.

According to Function algebra, the D'Alembertian,

$\Box = \nabla^2 - \frac{\partial^2}{c^2\partial t^2}$, results from the convolution product of gradient operators.

For any arbitrary four-vector,

$$\frac{d\underline{A}}{d\tau} = \frac{\partial \underline{A}}{\partial \tau} + (\underline{A}\nabla)\underline{A},$$

where

$$(\underline{A}\nabla)\underline{A} \equiv A_1 \frac{\partial \underline{A}}{\partial x_1} + A_2 \frac{\partial \underline{A}}{\partial x_2} + A_3 \frac{\partial \underline{A}}{\partial x_3} + A_0 \frac{\partial \underline{A}}{\partial x_0}$$

We shall write all equations at first as Lorentz-invariant world equations in Minkowski space. Any nonrelativistic equation will be derived from the corresponding relativistic equation by making appropriate approximations.

(ii) *Definitions* :

Let the wave function associated with a particle be represented by the scalar function

$$\epsilon(\underline{x}) = a(\underline{x}) \exp(iW(\underline{x})/\hbar), \tag{1}$$

where a and W are real and $\hbar$ is Planck's constant h divided by 2π. Later on, it will be shown that $W(\boldsymbol{x}, t)$ can be identified with Hamilton's principal function.

The generalized four-momentum of the corpuscle (in the language of point mechanics) is defined as

$$\underline{p} = \underline{p}_N + \underline{p}_e \quad \ldots \quad (2)$$

where

$$\underline{p}_e = e\underline{\Phi} \qquad \dots (3)$$

represents the field momentum due to an external field (of electromagnetic type) whose four-potential is given by Φ and can be expressed as

$$\underline{\Phi} = \boldsymbol{\Phi} + \frac{i}{c}\,\Phi_0 \cdot \boldsymbol{s}_0 \qquad \dots (3a)$$

In electromagnetic theory, the vector potential $\boldsymbol{\Phi}$ and the scalar potential Φ_0 are related to the field quantities $\boldsymbol{E}$ and $\boldsymbol{B}$ by

$$\boldsymbol{B} = \text{curl}\,\boldsymbol{\Phi}; \quad \boldsymbol{E} = -\frac{\partial \boldsymbol{\Phi}}{\partial t} - \text{grad}\,\Phi_0 \qquad \dots (3b)$$

e is the invariant charge of the corpuscle.

$\underline{p}_N$ represents the four-kinetic momentum of the corpuscle, (the subscript N would remind us that this is the momentum of the corpuscle in the sense of Newton and Einstein). But in view of the complementary properties of the particle mentioned in section I, we have to extend the usual definition of this quantity. We define it by the relation

$$\underline{p}_N = M_0 \underline{v} \qquad \dots (4)$$

where

$$M_0 = \mu m_0 \qquad \dots (5)^*$$

and

$$\mu = \left[\, 1 - \frac{\Box a}{a} \cdot \left(\frac{\hbar}{m_0 c}\right)^2 \right]^{1/2} \qquad \dots (6)$$

μ, for reasons discussed later, is called the mass factor and m_0 is the conventional rest mass of the particle.

Consequently, the dynamical mass of the particle

$$M = \mu m_0 \kappa, \qquad \dots (7)$$

depends on the space-time curvature of the amplitude of its associated wave function and, in general, changes if the wave suffers diffraction or the particle finds itself in non-stationary states. The usual expressions of point mechanics as well as of those of geometrical optics are obtained from the conditions

* This relation was obtained first by de Broglie in 1927[14]. In the language of de Broglie school, μ is referred to as Bohm's quantum potential.

$$\Box a = 0; \quad \mu = 1 \qquad \ldots \quad (8)^*$$

Only under this special condition the dynamic mass M as well as its kinetic momentum reduces to the usual expressions of relativistic mechanics. We shall see below that this mass factor μ is very important for the causal description of the trajectory of the particle passing through slits.

(iii) *Postulates* :

As noted previously, corpuscular and wave properties of a particle must be intimately and uniquely connected with one another. We therefore postulate that this connection is given by Hamilton-de Broglie pilot principle. That is,

$$\underline{p} = \underline{\nabla} W = \frac{\hbar}{2i}\left[\frac{\nabla\epsilon}{\epsilon} - \frac{\nabla\epsilon^*}{\epsilon^*}\right], \text{ (Postulate I)} \qquad \ldots \quad (9)^{**}$$

Here,

$$\epsilon^* (\underline{x}) = a(\underline{x}) \exp - \left(\frac{iW(\underline{x})}{\hbar}\right) \qquad \ldots \quad (10)$$

and

$$\underline{\nabla}\epsilon = (\underline{\nabla} a + \frac{i}{\hbar} \underline{p} a) \exp (iW/\hbar) \qquad \ldots \quad (11)$$

* This important special case was first pointed to me by my revered professor (late) S. N. Bose in 1954 while, on a short visit to Calcutta from Fritz Haber Institute, Berlin, I was discussing with him the scalar theory which was still in its nascent stage. In fact, unaware of de Broglie's expression (5) and (6), we had to incorporate the mass factor μ in the space-time dependent mass M_0 for the sake of consistency of the theory. Prof. Bose's remarks had helped us immensely in introducing the factor μ and thus formulating the scalar theory successfully. Of course, all through we had the benefit of the vast scholarship and constructive criticisms of late Prof. Max von Laue. I would like to take this opportunity to express my deep gratitude to these two savants as well as to late Prof. A. Einstein and to Prof. Louis de Broglie for encouragement and certain important remarks before the work on the scalar theory was finally published.

** It is interesting to note that Prof. A. Einsten liked this formulation of the generalized momentum as a function of the wave function. Later on, we found that the exact and complete solution of the corresponding nonlinear differential equation (18) together with (9), (or the corresponding expression (122) of the vector theory), could offer us the mass spectrum of free and relatively stable elementary particles. The masses are given by (cf. eqns. 4, 5, 6 and 20).

$$m_0 = \frac{1}{c}\left[\frac{1}{c^2}\left(\frac{\partial W}{\partial t} + U\right)^2 - (\text{grad } W - \boldsymbol{p}_e)^2 \mp \frac{\Box a}{a} \hbar^2\right]^{\frac{1}{2}} \qquad \ldots \quad (9a)$$

and

$$M_0 = \frac{1}{c}\left[\frac{1}{c^2}\left(\frac{\partial W}{\partial t} + U)^2 - (\text{grad } W - \boldsymbol{p}_e\right)^2\right]^{\frac{1}{2}} \qquad \ldots \quad (9b)$$

It must however be noted that for getting the mass spectrum of unstable elementary particles, which arise from the interaction of wave fields within the vortical domains, (thus generating turbulence), we are faced with insurmountable mathematical difficulties as well as completely unknown physical laws.

Energy-momentum density is therefore given by

$$\epsilon\epsilon^{*}\underline{p} = \frac{\hbar}{2i}\left[\epsilon^{*}\nabla\epsilon - \epsilon\nabla\epsilon^{*}\right]. \quad \ldots \quad (12)$$

In order to get the desired equation for the wave field we postulate further that the energy-momentum is conserved. That is,

$$\nabla\cdot(\epsilon\epsilon^{*}\underline{p}) = 0 \text{ (Postulate II)}. \quad \ldots \quad (13)$$

(iv) *Differential Equations* :

Substituting (12) in (13), we get immediately the differential equation (14) in a very symmetrical form :

$$\epsilon^{*}\Box\epsilon - \epsilon\Box\epsilon^{*} = 0. \quad \ldots \quad (14)$$

One can also arrive at other equvailent expressions for the wave field. Thus, from (11) and (13), we get

$$\Box\epsilon + \left[\frac{p^{2}}{\hbar^{2}} - \frac{\Box a}{a}\right]\epsilon = 0. \quad \ldots \quad (15)$$

Substituting here the expressions for $\underline{p}$ given in (9), we get

$$\Box\epsilon - \left[\frac{1}{4}\left(\frac{\nabla\epsilon}{\epsilon} - \frac{\nabla\epsilon^{*}}{\epsilon^{*}}\right)^{2} + \frac{\Box a}{a}\right]\epsilon = 0. \quad \ldots \quad (16)$$

All these equations (14–16) are generally valid for any particle whose wave field can be represented by a scalar function. They also satisfy the pilot principle, (eq. 9), as well as the principle of energy-momentum conservation, (eq. 13). Nevertheless, they are too general to be of any pracitical use, at least at the present state of our knowledge. Moreover, in order to bridge the gulf between the equations of this causal theory and the existing fundamental equations of classical and quantum physics, we need a differential equation in which the characteristic properties of a particle, (e.g., rest mass, charge) enter into the equation explicitly.

This can be easily achieved if we use the definition (2) of the generalized momentum and its connection with the wave field, (the postulate I, eq. 9). We first note, (cf. eqs. 2, 4, 5, 6),

$$(\underline{p}-\underline{p}_{e})^{2} = \underline{p}_{N}^{2} = -(\mu m_{0}c)^{2}. \quad \ldots \quad (17)$$

Replacing the value of $\underline{p}^{2}$ obtained from this equation in eq. (15) and remembering the value of μ given in (6) we finally obtain our desired equation, namely

$$\Box\epsilon + \frac{1}{\hbar^{2}}\left[2(\underline{p}_{e}\cdot\underline{p}) - \underline{p}_{e}^{2} - m_{0}^{2}c^{2}\epsilon\right] = 0 \quad (18)$$

Inserting the expression (9) for $\underline{p}$ here, and subsequently multiplying with ϵ^* we get a symmetric form of the differential equation (19) of a given particle, which contains besides its wave function only its prescribed properties, namely, the rest mass and the charge, moving under a given external field of electromagnetic type.

$$\epsilon^*\Box\epsilon+\epsilon\Box\epsilon^*-\frac{2i}{\hbar}\,[(\epsilon^*\underline{\nabla}\epsilon-\epsilon\underline{\nabla}\epsilon^*)\cdot\underline{p_e}]-\frac{2}{\hbar^2}[(m_0c)^2+p_e^2]\epsilon\epsilon^* = 0. \qquad (19)$$

We assert that eq. (18) or (19) represents the most general differential equation governing the motion of a given particle, (whose wave field can be represented by a scalar function), under a given external field of electromagnetic type. This assertion gets its full *a posteriori* justification from the fact that it reduces to the well known fundamental equations of classical and quantum physics under well defined restricted conditions.

It might be noted that all the above differential equations are non-linear partial differential equations. Our work on the Kepler[9] problem strongly suggests that nonanalytic and singular solutions of these equations would be of great physical importance. An exact and complete solution of this equation (18) would be needed to understand as yet mysterious properties of a particle*.

IV. Generalized Hamilton-Jacobi Equation

If we use the expression (6) for μ and the postulate I, (eq. 9), in the eq. (17), we get

$$(\nabla W-\underline{p_e})^2+m_0^2c^2 = \hbar^2\frac{\Box a}{a}. \qquad \ldots (20)$$

From (9) it follows

$$\mathbf{p} = \text{grad}\, W; \quad H = -\frac{\partial W}{\partial t}. \qquad \ldots (21)$$

Now, let us rewrite the four-vectors p and p_e in the form

$$\underline{p} = \mathbf{p}+\frac{i}{c}H\mathbf{s}_0, \qquad (22a)$$

$$p_e = \mathbf{p}_e+\frac{i}{c}U\mathbf{s}_0, \qquad (22b)$$

* It might also be noted that all the above equations involve both ϵ and its complex conjugate ϵ^*. What is the significance of the fact that no *general* wave equation can be formulated either with ϵ or with ϵ^* alone ? Mathematically one can conclude from this that the most general wave function is nonanalytic. What is the physical significance of this ? Does it mean that the invariant charge of the particle can perhaps be expressed in terms of the properties of the wave function only without explicitly assuming the existence of the charge as an additional datum ?

Eq. (20) then for the case of point mechanics i.e., $\Box a = 0$ (8) reduces to the well known relativistic Hamilton-Jacobi equation (23) in the field of electromagnetic type :

$$(\text{grad } W - \boldsymbol{p}_e)^2 - \left(\frac{H-U}{c}\right)^2 + (m_0 c)^2 = 0 \quad \ldots \quad (23)$$

Further, if H ia a system constant, one can express W through integration of (21) in the form

$$W(\boldsymbol{x}, t) = S(\boldsymbol{x}) - Ht \; ; \; H - m_0 c^2 = E = \text{constant} \quad \ldots \quad (24)$$

In the nonrelativistic case eq. (23) reduces to the nonrelativistic Hamilton-Jacobi equation (25).

$$\frac{(\text{grad } S - \boldsymbol{p}_e)^2}{2m_0} + U = E \; ; \; \left(\Box a = 0; \; E = \text{constant}; \left|\frac{m - m_0}{m_0}\right| << 1\right) \quad \ldots \quad (25)$$

Thus, we are perfectly justified to characterize point mechanics as well as geometrical optics by the condition (8). It must however be noted that in order to solve the generalized $H-J$ equation (20) and to determine the trajectory of the particle, one needs the amplitude as a function of space and time. That is, the initial value problem of classical mechanics has been changed to the boundary value problem of quantum physics.

It is obvious that the eq. (20) for the phase of the wave function is a differential equation of the first order and cannot represent a wave equation. But in the hands of Schroedinger this was changed into a second order partial differential equation with the help of operator formalism whose *physical* significance, I think, has not yet been fully explored.

In wave mechanics, one defines

$$\frac{\hbar}{i} \nabla = \underline{p}_{op} \quad \ldots \quad (26a)$$

or,

$$\frac{\hbar}{i} \text{ grad} = \boldsymbol{p}_{op}; \quad \frac{\hbar}{i} \frac{\partial}{\partial t} = -H_{op}$$

and

$$\underline{p}_{op}^2 = -\hbar^2 \Box \quad \ldots \quad (26b)$$

But, according to Function Algebra of physically observable functions, the convolution product, (and not the usual product), of the gradient operator with itself yields the D'Alembert's Operator. That is,

$$\underline{p}_{op} * \underline{p}_{op} = -\hbar^2 \Box \quad \ldots \quad (27)$$

(* symbol for the convolution product).

From the general theory of Fourier transformation it therefore follows that $\underline{p}_{op}$ of (26) is not physically identical with the generalized four-momentum of the particle $\underline{p}(\boldsymbol{x}, t)$ but its mathematical representation in the Fourier space. As noted previously, we are therefore justified in concluding that Heisenberg's Uncertainty Principle is the price of the representation of the physical quantity $\underline{p}(\boldsymbol{x}, t)$ by its Fourier transformation.

V. Important Special Cases of the Generalized Wave Equation

(i) *Klein-Gordon Equation* :

The generalized wave equation (18) reduces to the well-known Klein-Gordon equation.

$$\Box\epsilon - \left(\frac{m_0 c}{\hbar}\right)^2 \epsilon = 0, \quad \ldots (28)$$

for the special case $\underline{p}_e = 0$.

(ii) *The Wave Equation of Optics* :

For $m_0 = e = 0$, eq. (18) reduces to the wave equation

$$\nabla^2\epsilon - \frac{1}{c^2}\frac{\partial^2\epsilon}{\partial t^2} = 0 \quad \ldots (29)$$

From the generally valid equation (18), we can at once conclude that eq. (29) governs the motion not only of photons but also of any neutral particle of rest mass zero, provided its wave field can be represented by a scalar function.

(iii) *Equations of Wave Mechanics* :

In wave mechanics the differential equations are obtained from the operator formalism. But in this theory $\underline{p}$ as a function of $\boldsymbol{x}$ and t is directly related with its associated wave function, also a function of $\boldsymbol{x}$ and t, by the postulate I, eq. (9). Now, one can easily establish that the usual operator formalism results when the wave function $\epsilon(\boldsymbol{x}, t)$ is such that it satisfies the Schroedinger condition (30) namely,

$$(\underline{p}_e . \underline{\nabla}\epsilon/\epsilon) = -\left(\frac{\underline{\nabla}\epsilon^*}{\epsilon^*} . \underline{p}_e\right) \quad (30)$$

Using this relation in the generalized wave equation (18) we obtain the relativistic Schroedinger-Gordon equation

$$\Box\epsilon + \frac{1}{\hbar^2}\left[\frac{2\hbar}{i}(\underline{p}_e\underline{\nabla}\epsilon) - \underline{p}_e^2\epsilon - (m_0 c)^2\epsilon\right] = 0 \quad \ldots (31)$$

Let us now look at the important special case, namely, the so called stationary states of the wave field. Then it follows from (1) and (11):

$$\frac{\partial^2 \epsilon}{\partial t^2} = -\left(\frac{H}{\hbar}\right)^2 \epsilon \qquad \ldots \quad (32)$$

provided

$$\frac{\partial a}{\partial t} = 0 \quad \text{and} \quad \frac{\partial H}{\partial t} = 0. \qquad \ldots \quad (33)$$

Using those relations in (18) we get

$$\nabla^2\epsilon + \frac{1}{\hbar^2}\left[\left(\frac{H-U}{c}\right)^2 - \boldsymbol{p}_e^2 - (m_0 c)^2 - 2(\boldsymbol{p}\cdot\boldsymbol{p}_e)\right]\epsilon = 0. \qquad \ldots \quad (34)$$

For an electrostatic potential, i.e., $\boldsymbol{p}_e = 0$, eq. (34) is the relativistic Schroedinger equation used by Sommerfeld in investigating the fine structure of the hydrogen spectrum.

Finally, using the relations (30), (33) and the nonrelativistic approximation $\left|\frac{M-m_0}{m_0}\right| << 1$ as well as the approximation $|\boldsymbol{p}_e| << |\boldsymbol{p}|$, we get from (34) the time-independent Schroedinger equation (35).

$$\nabla^2\epsilon + \frac{2m_0}{\hbar^2}(E-U)\epsilon - \frac{2i}{\hbar}(\boldsymbol{p}_e.\text{grad}\ \epsilon) = 0, \qquad \ldots \quad (35)$$

where

$$E = H - m_0 c^2.$$

VI. Extension of Point Mechanics—Diffraction Force

It is well known that classical as well as relativistic point mechanics cannot explain diffraction phenomena, since they do not take into consideration the amplitude of the wave function. But with the redefinition of the kinetic momentum of the particle as given by eqns. (4–7) one can study the motion of a particle following the methodology closely analogous to that of the usual point mechanics. The factor μ produces a new type of force, the diffraction force, which deviates the trajectory of the particle whenever it passes through a slit and always according to the pilot principle.

In order to see this, let us express the four-force $\underline{F}$ by

$$\underline{F} = \frac{d}{d\tau} M_0 \underline{v}. \qquad \ldots \quad (36)$$

Since the kinetic momentum $\underline{p}_N$ is given by the difference of two field quantities $\underline{p}$ and $\underline{p}_e$, it can itself be looked upon as a field quantity. Consequently, we can write (36) in the form

$$\underline{F} = (\underline{v}\,\underline{\nabla})\underline{p}_N \qquad \ldots \quad (37)$$

From this, using the four-dimensional vector notations, we get

$$\underline{F} = \nabla(v_e.p_N)-[v[\nabla\, p_N]]. \tag{38}$$

(The subscript e denotes that this quantity should be kept constant during the differentiation process.

Further, from (9) and (2) it follows

$$[\,\underline{\nabla}\,\underline{p}\,]=0;$$

or,

$$[\underline{\nabla}\,\underline{p_N}] = -\,[\underline{\nabla}\,\underline{p_e}]. \tag{39}$$

Substituting this in (38) and noting that $(\underline{v}.\underline{F}) = -\frac{d}{d\tau}(M_0c^2)$ we can write the four-force $\underline{F}$ given by (36) as the sum of two types of forces, namely,

$$\underline{F} = \underline{F}_D+\underline{F}_e,$$

where

$$\underline{F}_e = [\underline{v}[\underline{\nabla}\,\underline{p_e}]] \quad \ldots \tag{40}$$

is the well known Lorentz force due to the external electrodynamic field and

$$\underline{F}_D = -m_0c^2\underline{\nabla}\mu. \quad \ldots \tag{41}$$

The spatial components of this four-force $\underline{F}_D$ is given by

$$\boldsymbol{F_D} = -\frac{m_0c^2}{\kappa}\text{ grad }\mu;\ \left(\kappa=\frac{1}{\sqrt{1-v^2/c^2}}\right). \quad \ldots \tag{42}$$

We call this $\boldsymbol{F_D}$ as the diffraction force since it results from the diffraction of the wave field and vanishes in the case of geometrical optics and the usual point mechanics. Since the diffraction force is obtained from the gradient of μ, one can also formally characterise μ as the quantum potential, but it must not be overlooked that it exists only when the wave field suffers diffraction or exists in non-stationary states.

VII. De Broglie Relations

De Broglie had shown that the phase velocity μ of the matter-wave is a space-like vector and the particle velocity $\underline{v}$ is the group velocity.

The phase velocity is given by $dW/d\tau$. One can express this as

$$\frac{dW}{d\tau} = (\underline{u}\underline{\nabla})W \quad \ldots \tag{43}$$

where

$$\underline{u} = \frac{1}{\sqrt{1-u^2/c^2)}}(\boldsymbol{u}+ic.\boldsymbol{s}_0) \quad \ldots \tag{43a}$$

From (21) and (22) one gets

$$(\boldsymbol{u}\,\boldsymbol{p})-H=0. \qquad \ldots (44)$$

Remembering that $\boldsymbol{p}$ is always orthogonal to surfaces of constant phases, one finds that the component u_p of $\boldsymbol{u}$ along this orthogonal direction is given by

$$\underline{u}_p=\frac{H}{|\boldsymbol{p}|}=\frac{\mu m_0\kappa c^2+U}{|\mu m_0\kappa \underline{v}+\boldsymbol{p}_e|}. \qquad \ldots (45)$$

For $\mu=1$, this is the reciprocal relation of de Broglie[14].

Now, rewriting $\underline{p}_N$ given by (4) in the form

$$\left.\begin{array}{l} p_N=\boldsymbol{p}_N+\frac{i}{c}E_N\boldsymbol{s}_0\,; \\ \boldsymbol{p}_N=Mv;\ E_N=mc^2 \end{array}\right\}, \qquad \ldots (46)$$

one gets the generally valid relation

$$\boldsymbol{v}=\frac{c^2\boldsymbol{p}_N}{E_N}=-c^2\frac{\text{grad } W-e\Phi}{\frac{\partial W}{\partial t}+e\Phi_0} \qquad \ldots (47)$$

which is known as *formule du guidage* of de Broglie[14].

Finally, expanding the phase W in power series around a world point $\underline{x}_a$ and neglecting all terms higher than the first order, one gets the famous Einstein-de Broglie relations. Thus,

$$\epsilon(\underline{x}_a+d\underline{x})\approx a(\underline{x}_a)\exp\left[\frac{i}{h}\{(W(\underline{x}_a)+(d_{\underline{x}}\cdot\underline{p}(\underline{x}_a))\}\right]$$

and

$$\lambda_a=\frac{h}{|\boldsymbol{p}(\boldsymbol{x}_a)|}\,;\ \nu_a\approx\frac{H(\boldsymbol{x}_a)}{h}. \qquad \ldots (48)$$

As is evident from above, the relations are strictly valid for plane waves. They connect the wave properties λ,ν with the corpuscular properties $\boldsymbol{p},H$ in the neighbourhood of the centroid of the particle, (i.e., the singularity of the wave field). But the relation (9) connects them everywhere and for all cases.

VIII. Generalized Analytical Mechanics

(i) *Lagrangian Mechanics* :

The rate of change of the generalized momentum $\underline{p}$ defined in (2) can be expressed as, (since it is a field quantity),

$$\frac{d\underline{p}}{d\tau}=\underline{\nabla}(\underline{v}_c\cdot\underline{p}).\ \text{(cf. 39)} \qquad \ldots (49)$$

or

$$\left.\begin{aligned} \frac{d\boldsymbol{p}}{dt} &= \text{grad}\left(\frac{v_c \cdot p}{\kappa_c}\right); \\ \frac{dH}{dt} &= -\frac{\partial}{\partial t}\left(\frac{v_c \cdot p}{\kappa_c}\right) \end{aligned}\right\} \quad \dots (50)$$

Following Schwarzchild[15] (and regarding the Lagrangian as a function of x_j, v_j, t) we define the Lagrangian by (51)

$$\mathrm{L}(x_j, v_j, t) = \left(\frac{v \cdot p}{\kappa}\right) \quad \dots (51)$$

$$= -\frac{\mu m_0 c^2}{\kappa} + (\boldsymbol{v} \cdot \boldsymbol{p}_e) - U \quad \dots (52)$$

Now since

$$\frac{\partial}{\partial v_j}\left(\frac{1}{\kappa}\right) = -\frac{\kappa v_j}{c^2} \quad \dots (53)$$

and the quantities $\boldsymbol{p}_e$, U and μ are functions of $\boldsymbol{x}$ and t, we get from (52), (50) and (2)

$$\frac{\partial \mathrm{L}}{\partial \dot{x}_j} = \mu m_0 \kappa v_j + p_{ej} = p_j; \quad (x_j = v_j; \quad j = 1, 2, 3) \quad \dots (54)$$

$$\frac{\partial p_j}{dt} = \frac{\partial \mathrm{L}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{v \cdot p^e}{\kappa}\right) - \frac{m_0 c^2}{\kappa} \frac{\partial \mu}{\partial x_j} \quad (j = 1, 2, 3) \quad \dots (55)$$

Differentiating (54) with respect to time along the world line of the centroid of the particle we get Lagrangian equations of the second kind (56):

$$\frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{x}} - \frac{\partial \mathrm{L}}{\partial x_j} = 0; \quad (j = 1, 2, 3) \quad \dots (56)$$

From (56) using calculus of variations one gets

$$\int \mathrm{L} dt = \text{Extremum} \quad \dots (57)$$

It is interesting to note that we have not had to postulate the Principle of Least Action. It follows automatically from our two postulates and the definition (51) of the Lagrangian.

(ii) *Hamiltonian Mechanics* :

In section IV we have seen that Hamilton-Jacobi equation follows from our two postulates. Now, from (2–7) and (21–22) we get

$$H = U + c[(\mu m_0 c)^2 + (\text{grad } W - \boldsymbol{p}_e)^2]^{\frac{1}{2}} \quad \dots (58)$$

and from (22) and (52)

$$H = (\boldsymbol{pv}) - L \qquad \dots \quad (59)$$

It is therefore obvious that H represents the Hamiltonian. This can be proved easily by putting $\mu = 1, \Box a = 0$, (the conditions for the validity of point mechanics) and using (54) in (59). We then get the principle of conservation of mechanical energy, namely,

$$H = \sum_j \frac{\partial L}{\partial \dot{x}_j} \dot{x}_j - L = \text{constant} \qquad \dots \quad (60)$$

It must be noted however that in the general case, ($\mu \neq 1$, $\Box a \neq 0$), H need not remain constant. Nevertheless, due to the continuity condition, (Postulate II, eq. (13)), the energy of the particle is not lost. As proved later, under suitable conditions there can be an exchange of energy between the corpuscle and its extended wave field. I would leave to you to guess what its significance might be in the concept of *pseudo particle*, *bare mass* and *renormalised mass* of quantum field theory.

In order to convince you finally that H given by (58) is the generalized Hamiltonian function, we shall prove that H satisfies Hamilton's canonical equations of motion.

Let us consider H as a function of p_j, x_j, v_j, and t. Then from (54) and (59) we get

$$\frac{\partial H}{\partial \dot{x}_j} = 0; \quad (j = 1, 2, 3)$$

Consequently, H can be looked upon as a function of the independent variables x_j, p_j, t.

Moreover, from (54), (55) and (59) we get

$$-\frac{\partial L}{\partial x_j} = \frac{\partial H}{\partial x_j} = -\frac{dp_j}{dt}$$

$$(j = 1, 2, 3) \qquad \dots \quad (61)$$

$$\frac{\partial H}{\partial p_j} = \frac{dx_j}{dt}.$$

These equations are evidently Hamilton's canonical equations of motion. But now they are proved to be valid generally, including diffraction phenomena.

It would be evident from sections IV and (VI) that the *only* drawback of classical mechanics is the fact that it did not take into account the amplitude of a wave, although its phase W provided a powerful tool in the hands of Hamilton and Jacobi. It is unfortunate that these classical giants did not consider the phase W as the phase of a physical wave, but only as a tool for calculations. It is rather surprising that even Hamilton did not pursue his ideas more thoroughly by taking the amplitude also into consideration. Were they too much influenced by the philosophy of Newton that a physicist should not speculate ? Anyway, one cannot but wonder at the beautiful and powerful edifice they had built up. We have seen here that by merely incorporating a factor μ dependent on the amplitude, the entire edifice of classical mechanics retains its general validity and this structure is powerful enough not only to offer an explanation for the diffraction phenomena but also can explain quantum mechanics in the spirit of Newton, Huygens, Maxwell and Einstein.

IX. Classical Wave Phenomena

We have seen that the new theory is capable of deducing all the fundamental equations of point mechanics and wave mechanics of a single particle. Since in this theory the corpuscle and its wave are intimtately connected together, we should expect that this theory should also lead us to the basic wave properties, so far as they can be deduced from a scalar theory.

In pre-quantum physics waves were thought to be generated by the motion of a collection of particles. The waves as such did not constitute a *distinct* physical entity. For a single particle, it was meaningless to talk about waves. In dealing with macroscopic properties it was found to be convenient, instead of considering the motion of individual particles, to forget about the existence of particles and consider only the wave equation with proper boundary conditions.

From Maxwell, Hertz and Lorentz, we have learnt that for electromagnetic waves, only the field quantities are of importance. Though this field is generated by the motion of charged particles, the radiation itself can be studied only by field equations and there is no place for a particle in this radiation.

With the emergence of quantum phenomena the picture has been changed radically. Not only it has become perfectly meaningful to talk about the wave properties of a single particle, radiation itself is considered to be nothing but a collection of particles and pseudo particles. The waves *per se* as a *physical reality* had vanished from our concept. The particles exhibit wave properties because their motion is governed by the so-called wave equations, although the wave function itself is not a physical reality.

In the light of this new theory, however, both the corpuscle and its pilot wave are physical realities. For electromagnetic radiation, we have to explore the corpuscular properties of the photon as well as its relation to its pilot wave and the properties of the pilot wave itself. The trajectory of a single particle is completely determined by eq. (9), if we know its initial location and the wave function for all world points. But, contrary to classical mechanics, the problem involves the solution of the wave equation under specified boundary conditions. Moreover, in order to compare our results with those of classical and quantum physics, we have to consider a collection of particles and various possible normalizations.

(i) *Normalizations*

An ensemble of particles without mutual interactions can be defined in such a way that the pilot wave ϵ is the same for all particles, except for a statistically uncorrelated phase shift δ. That means,

$$\epsilon_k \exp(i\delta_k) = a_k \exp i\left(\frac{W_k}{\hbar} + \delta_k\right) = a \exp\left(i\frac{W}{\hbar}\right), \text{ for all } k. \qquad \dots \quad (62)$$

For this collection of particles one can choose the initial density of corpuscles $\rho(\boldsymbol{x})$ within a time-like 3-dimensional hypersurface of the space-time world in such a way that it is proportional to the square of the amplitude of the pilot wave. That is

$$\rho(\boldsymbol{x}) = \gamma^2 \epsilon\epsilon^*; \quad (\gamma^2 = \text{constant}). \qquad (63)$$

This is the collective normalization. But if one chooses the proportionality constant in such a way that only one corpuscle lies within this hypersurface we have evidently the individual normalization.

In order to choose γ^2 appropriately for different cases, it is necessary to consider the energy-momentum current density of the corpuscles.

From the postulate II, (eq. 13), we get for electromagnetic fields

$$\nabla \cdot (\epsilon\epsilon^* p_N) = 0$$

if

$$\frac{\partial a}{\partial t} = 0, \quad \underline{\nabla} \cdot \underline{p}_e = 0 \quad \text{and} \quad \underline{p}_e = \frac{i}{c} U_{s_0}. \qquad \dots \quad (64)$$

Consequently, the energy-momentum of the corpuscle is conserved only in stationary states and for $\underline{p}_e \neq 0$, in that particular inertial system in which the sources generating the field lie at rest. The invalidity of (64) in the general case does not mean that energy-momentum is lost, but only the fact (cf. eq. 13) that there is an interchange of energy-momentum between the corpuscle and its pilot wave.

From Gauss theorem and eq. (13) we get

$$\int \epsilon\epsilon^* p \, dS = 0.$$

Now choosing this closed hypersurface S as the 3-dimensional physical valume at two different instants, t_0 and t_1, and using (22) we obtain

$$\int_{t_0} \epsilon\epsilon^* H \, dv_x = \int_{t_1} \epsilon\epsilon^* H \, dv_x = C_0 = \text{constant for all } t \tag{65}$$

(dv_x *is a volume element in physical space*).

Let us now choose γ^2 *in such a way that at* t_0 *within each volume element* ΔV_k

$$\int_{\Delta V_k} \gamma^2 \epsilon\epsilon^* H dv_x = 1, \text{ for all } k. \quad \ldots \quad (66)$$

Then within the world tube enclosing only this volume element and lying parallel to $\underline{p}$ the relation (66) remains valid for all time t. Moreover, the density of packing of such world tubes within the hyperspaces S (i.e., physical volume V) at any any time remains constant and is given by

$$\rho(x_k) = \frac{1}{\Delta V_k} = \gamma^2 \epsilon\epsilon^* H(x_k), \tag{67}$$

where x_K is any point within ΔV_K.

We now choose the ensemble normalization by the relation

$$\gamma^2 = \frac{N}{C_0} \quad \text{(Ensemble Normalization)}, \tag{68}$$

where N is the number of such world tubes within V and reamains constant for all time.

For the static and stationary case, (cf. 64), if we construct the woild tubes parallel to p_N, then

$$\rho(\boldsymbol{x}) = \gamma^2 (\epsilon\epsilon^* H)(\boldsymbol{x}) \tag{69}$$

gives the density of corpuscles and remains proportional to $\epsilon\epsilon^* H$ for all time provided it was chosen in such a way at a time $t = t_0$. The inertial system in which this is the case is really the proper system, i.e., the system in which the sources are at rest and the wave field is stationary.

Further, if H is a system constant, then in this particular rest frame, one can choose

$$\gamma^2 = \frac{N}{H} \quad \text{(Maxwell Normalization)}$$

or, $\ldots$ (70)

$$\rho = N\epsilon\epsilon^*.$$

For reasons to be discussed later we call this Maxwell Normalization.

One can then also choose

$$\gamma^2 = \frac{1}{H}$$

or,

$$\rho = \epsilon\epsilon^* \quad \text{(Wave Mechanical Normalization)} \qquad \dots (71)$$

In these two special cases $\epsilon\epsilon^*$ has the dimension of density (cm^{-3}). From (71), we also get

$$\int \epsilon\epsilon^* dv_x = 1. \qquad \dots (72)$$

It is to be noted that though (71) is also valid for nonstationary wave field and nonstatic external fields, the subsidiary condition $H = \text{constant}$ implies that for nonstationary and polychromatic wave fields and nonstatic external field $\epsilon\epsilon^*$ cannot give us the density of corpuscles.

Finally, the individual normalization is obtained when γ^2 vanishes in all volume elements except in one particular one around $x = x_j(t_0)$ which has the value $\frac{1}{\Delta V_j \epsilon\epsilon^*}$. With $\Delta V_j \rightarrow 0$, we have

$$\gamma^2 = \frac{1}{\epsilon\epsilon^*} \, \delta(x - x_j(t_0)) \qquad \dots (73)$$

and

$$\int \gamma^2 \epsilon\epsilon^* H dv_x = H(x_j(t)), \qquad \text{(cf. eq. 67)} \qquad \dots (74)$$

(δ is Dirac's delta-function).

Eq. (74) gives the measured value of the energy of the corpuscale at the point x_j at a time t. In most of the experimental situations we have the ensemble normalization.

(ii) *The properties of the photon and its pilot wave in Fraunhofer and Fresnel Zones*

We have already proved that the pilot wave of a photon satisfies the equation of wave optics (scalar)

$$\Box \epsilon = 0 \qquad \dots (29)$$

From eqns. (4–7) and (46) it follows :

$$\left.\begin{aligned} \underline{p}_N &= \boldsymbol{p}_N + \frac{i}{c} E_N s_0 \\ \boldsymbol{p}_N &= M\boldsymbol{v} \; ; \; E_N = Mc^2 \\ M = \mu m_0 \kappa \; ; \quad M_0 &= \mu m_0 \; ; \quad \kappa = \frac{1}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\} \qquad \dots (75)$$

Since for photons $m_0 = e = 0$, we have

$$M_0 = \mu m_0 = \frac{\hbar}{c} \sqrt{-\Box \frac{a}{a}}. \qquad \dots (76)$$

For the diffraction force on the photon (cf. eq. 42), we have

$$F_D = -\frac{\hbar c}{\kappa} \operatorname{grad} \sqrt{-\frac{\Box a}{a}}. \quad \ldots (77)$$

The velocity v of the photon corpuscle (cf. eq. 47) is given by

$$v = -c^2 \frac{\operatorname{grad} W}{\frac{\partial W}{\partial t}}. \quad \ldots (78)$$

We note that the corpuscular properties of the photon given by (75) can be calculated only with the help of the properties of its pilot wave ϵ. From the known properties of Fraunhofer diffraction we can infer that in this domain.

$$\Box a = 0. \quad \ldots (79)$$

Thus in the Fraunhofer zone both M_0 and F_D vanish and the velocity of the photon corpuscle $|v| = c$, otherwise p_N and E_N would vanish. Consequently, in Fraunhofer zone, the photon corpuscles move in a straight line normal to the surfaces of constant phase of its pilot wave with a constant velocity c. This also follows from (9b) and (20) :

$$0 = M_0 = \frac{1}{c} \sqrt{-(\underline{\nabla} W)^2} \quad \ldots (80)$$

Consequently, $\underline{\nabla} W$ is a light-like vector.

From (9a) we find that the rest mass of the photon corpuscle expressed in terms of its pilot wave, namely,

$$m_0 = \frac{1}{c} \sqrt{\frac{\Box a\, \hbar^2}{a} (\underline{\nabla} W)^2} \qquad \text{(cf. 9}a\text{)}$$

also vanishes identically throughout the Fraunhofer domain. In the general case the phase equation (20) reduces to

$$(\operatorname{grad} W)^2 - \left(\frac{\partial W}{c\partial t}\right)^2 = \frac{\Box a}{a} \hbar^2 \quad \ldots (81)$$

which for geometrical optics, ($\Box a = 0$), is identical with the eikonal equation.

The Lagrangian and the Hamiltonian functions are given by (cf). eqs. (52) and (58)

$$\mathrm{L} = -\frac{\hbar c}{\kappa} \sqrt{-\Box \frac{a}{a}} \quad \ldots (82)$$

$$H = c\sqrt{(\operatorname{grad} W)^2 - \hbar^2 \frac{\Box a}{a}} = Mc^2 \quad \ldots (83)$$

Evidently, in the Fresnel Zone $\Box a$ cannot be equal to zero. From (76) we see that in the Fresnel zone we must have

$$\Box a < 0, \tag{84}$$

otherwise M_0 would be imaginary.

Now, since for photons de Broglie relation (45) becomes

$$u_p |v| = c^2. \tag{85}$$

The phase velocity component u_p must be greater than c, otherwise M would be imaginary.

Consequently, in the Fresenel zone the velocity of the photon corpuscle must be less than c. This does not contradict the second posulate of the special theory of relativity since it deals with point mechanics and the transport of energy takes places through the photon corpuscle moving under conditions where the relation $\Box a = 0$ is fulfilled*.

We must therefore carefully distinguish between the velocity of the photon corpuscle and the constant velocity of the Huygen's elementary waves. That the Huygen's elementary waves propagate with the constant velocity c follows also fro m the wave eqn. (29) and its fundamental solution (see Hosemann and Bagchi[17]).

One can also show, (for details see ref. (8)), that the intensity of the diffracted beam in the Fresnel zone is not proportional to the square of the amplitude of the pilot wave, but is given by :

$$I = \epsilon\epsilon^* |\text{grad } W/\hbar| \qquad \ldots \quad (86)$$

From the foregoing it would be clear that in order to obtain new results out of this scalar theory we have to solve the non-linear differential equation exactly and completely, a task hardly feasible at the moment. But there we have two important results of this new approachk, namely (i) the velocity of the photon corpuscle in the Fresnel zone is less than c; (ii) the intensity of the diffracted light in the Fresenel zone is not proportional to the square of the amplitude of its pilot wave. Both these conclusions can be tested experimentally with the help of microwaves**.

X. Heisenberg's Uncertainty Principle

From the foregoing discussions it will be obvious that in this theory the position as well as the energy and momentum, considered as a function of x and

* Cf. Shapiro's work[16].

** Another experimental proof could be obtained if one czn measure the intrinsic angular momentum of a collection of free electrons from the torque exerted (and not from the energy spectrum which is related to the corresponding *Spin*. For details *see* section XII,(iv)).

t, of a corpuscle are perfectly determined through the definitions (2–7) and the two postulates, (eqns. 9 and 13). Heisenberg's uncertainty relation results from the fact, as stated previously, that in quantum theory one defines the momentum not as a function of coordinates of Minkowski space, but by those if its reciprocal space. Consequently, this uncertainty is the price of representation and follows generally from the theory of Fourier transformation. In actual experiments involving a collection of free particles with ensemble normalization, (cf. eq. 68), we get the uncertainty relation connecting the integral widths of the centroid of the collection and of its average momentum, the latter being expressed as a function of the Fourier space. In no way this should lead to the conclusion that the position and momentum (both considered as functions of physical space and time) cannot be determined, at least in principle, simultaneously. I would like to think that the section III–IX had convincingly proved that it is possilble *in principle* to describe the quantum phenomena causally and deterministically at least for a single particle whose wave field can be represented by a scalar function.

Before we deduce the expression for this uncertainty relation in its more general form, let me state here, for convenience, some of the relevant formulae in the theory of Fourier transformation which we shall have to use to deduce the uncertainty relation. These formulae are generally valid for any function complex. For the proof of these relations one might look into Sections II, III, and V of the reference 4.

Let $E(\underline{b})$ be the Fourier transform of $\epsilon(\underline{x})$.

$$E(\underline{b}) = \mathscr{F}(\epsilon) \equiv \int \epsilon(\underline{x}) \exp\left[-2\pi i(\underline{b}.\underline{x}\right] dv_x \qquad \ldots (87)$$

$$\underline{b} = \boldsymbol{b} + \frac{i}{c}\nu\, \boldsymbol{s}_0\,; \quad |\boldsymbol{b}| = \frac{1}{\lambda} \qquad \ldots (88)$$

$\underline{b}$ is the four-vector reciprocal to the four-vector $\underline{x}$ of the Minkowski space.

Then,

$$\mathscr{F}(\underline{\nabla}) = 2\pi i b; \quad \mathscr{F}(\Box)\, \mathrm{M} -4\pi^2 \underline{b}^2, \qquad \ldots (89)$$

$$\mathscr{F}(\epsilon_1 {}^* \epsilon_2) = E_1.E_2; \quad \mathscr{F}(\epsilon_1.\epsilon_2) = E_1{}^*E_2; \quad \mathscr{F}(\epsilon^*) = E^*(-\underline{b}), \qquad \ldots (90)$$

$$\lim_{b\to 0} \mathscr{F}(\epsilon) = \int \epsilon dv_x = E(0), \qquad \ldots (91)$$

$$\left.\begin{aligned} \mathscr{F}(\epsilon\epsilon^*) &= E(\underline{b}) {}^* E^*(-\underline{b}) \\ \int \epsilon\epsilon^* dv_x &= \int EE^* dv \end{aligned}\right\} \qquad \ldots (92)$$

$$\mathscr{F}(\epsilon\epsilon^*\underline{p}) = \frac{\hbar}{2i}\, \mathscr{F}(\epsilon^*\underline{\nabla}\epsilon - \epsilon\underline{\nabla}\epsilon^*) = \frac{h}{2}[E^*(-\underline{b})^*(\underline{b}E) - E^*(\underline{b}E^*(-\underline{b})], \qquad \ldots (93)$$

$$\int (\epsilon\epsilon^* \underline{p}) dv_x = h \int (EE^* \underline{b}) dv \qquad \ldots (94)$$

$$\mathcal{F}(\epsilon\epsilon^* \underline{p}^2) = -h^2 \mathcal{F}(\epsilon^* \Box \epsilon - a \Box a) = \hbar^2[E^*(\underline{b})^*(\underline{b}^2.E) - A^*(\underline{b}^2 A)] \qquad \ldots (95)$$

$$\int (\epsilon\epsilon^* \underline{p}^2) dv_{\underline{x}} = h^2 [\int (EE^* \underline{b}^2) dv_{\underline{b}} - \int (A^2 \underline{b}^2) dv_{\underline{b}}] \qquad \ldots (96)$$

$[A = \mathcal{F}(a)]$.

Let us now define the mean values of the relevant quantities for the ensemble in the conventional way. Thus,

$$\bar{\underline{p}} = \frac{\int \epsilon\epsilon^* \underline{p} dv_x}{\int \epsilon\epsilon^* dv_{\underline{x}}}; \; \bar{\underline{x}} = \frac{\int \epsilon\epsilon^* x dv_x}{\int \epsilon\epsilon^* dv}; \; \bar{\underline{p}}^2 = \frac{\int \epsilon\epsilon^* \underline{p}^2 dv_x}{\int \epsilon\epsilon^* dv_{\underline{x}}} \qquad \ldots (97)$$

$$\bar{\underline{b}} = \frac{\int EE^* b dv_b}{\int EE^* dv_b}; \; \bar{b}^2 = \frac{\int EE^* b^2 dv_b}{\int EE^* dv_b}; \; \overline{(b')^2} = \frac{\int A^2 b^2 dv_b}{\int A^2 dv_{\underline{b}}} \qquad \ldots (98)$$

From (93) the de Broglie relation for the mean values of the ensemble also follows :

$$\bar{\underline{p}} = h \underline{b} \qquad \ldots (99)$$

Further, let δx_j denote the integral width of the density distribution $\rho(\boldsymbol{x})$ of the ensemble, δp_j, that of the momentum distribution in Minkowski space and δb_j that of its spectrum in Fourier space. We also define them in the conventional way; (j denotes the components 1, 2, 3, 0) :

$$(\delta x_j)^2 = \frac{\int \epsilon\epsilon^* (x_j^2 - \bar{x}_j^2) dv_x}{\int \epsilon\epsilon^* dv_{\underline{x}}} \qquad \ldots (100)$$

$$(\delta p_j)^2 = \frac{\int \epsilon\epsilon^* (p_j^2 - \bar{p}_j^2) dv_x}{\int \epsilon\epsilon^* dv_x} \qquad \ldots (101)$$

$$(\delta b_j)^2 = \frac{\int EE^* (b_j^2 - \bar{b}_j^2) dv_{\underline{b}}}{\int EE^* dv_{\underline{b}}} \qquad \ldots (102)$$

Now, the theory of Fourier transformation shows that in general

$$\delta x_j \delta b_j = \beta; \qquad \ldots (103)$$

(For Gaussian distributions, $\beta = 1$ and for other reasonable distributions $\beta \approx 1$).

Hence from the mathematically valid relations given above, it follows :

$$\delta p_j = h\sqrt{(\delta b_j)^2 - (\bar{b}_j')^2} \qquad \ldots (104)$$

and

$$\delta x_j \delta p_j = \beta h \sqrt{1 - (\bar{b}_j')^2/(\delta b_j)^2} \qquad \ldots (105)$$

It can be proved (see ref. 9) that outside the nuclear domain and the Fresnel zone where one may put $\mu \approx 1$, the correction term whtin the square root is negligible.

For the observable corpuscular momentum $\underline{p}_N$ one can prove similarly

$$\bar{\underline{p}}_N = h\bar{\underline{b}} - \bar{\underline{p}}_e \qquad \dots \quad (106)$$

$$\delta x_j \delta p_{Nj} = \beta h \sqrt{1 - \frac{\bar{b}_j'^2 + [(2\delta^2 p_j p_{ej} - \delta p_{ej})^2]/h^2}{(\delta b_j)^2}} \qquad \dots \quad (107)$$

$$= \beta h \quad \text{for} \quad |\mu - 1| << 1; \quad |p_{oj}| << |\underline{p}| \qquad \dots \quad (108)$$

XI. Current Density $\underline{J}$

(i) *Photons and Classical Electromagnetic Radiation*

From (2), (3) and (13) and $m_0 = e = 0$, it follows generally

$$\text{div.}\, \epsilon\epsilon^* \boldsymbol{p}_N + \frac{1}{c^2} \frac{\partial \epsilon\epsilon^*}{\partial t} E_N = 0 \qquad \dots \quad (109)$$

Now, using the ensemble normalization (68)*, which in this case becomes

$$\gamma^2 = \frac{N}{C_0};$$

where

$C_0 = \int \epsilon\epsilon^* \; E_N dv_x =$ Constant for all t and for any arbitrary inertial frame,

the continuity condition (109) for the collection of photons can be expressed as

$$\text{div}(\rho \boldsymbol{v}) + \frac{\partial \rho}{\partial t} = 0; \quad \rho = \kappa \rho_0; \quad \rho_0 = \gamma^2 \epsilon\epsilon^* \mu m_0 c^2, \qquad \dots \quad (110)$$

Here ρ is the number density of corpuscles in the volume element dv_x; $\rho_0 =$ proper density and $\int \rho dv_x = N$, the total number of corpuscles in the ensemble, (cf. eqs. (67) and (68)).

If instead we choose the normalization (70), we get from (109) the relation

$$(\text{div}\, \boldsymbol{S}) + \frac{1}{c^2} \frac{\partial E}{\partial t} = 0 \qquad \dots \quad (111)$$

where

$$\boldsymbol{S} = \rho \boldsymbol{p}_N; \quad E = \rho E_N; \quad \rho = N\epsilon\epsilon^* \qquad \dots \quad (112)$$

Evidently, eq. (111) is the continuity condition of the classical electromagnetic radiation and $\boldsymbol{S}$ is the Poynting's vector. It must, however, be emphasized that contrary to eq. (109), eq. (111) is valid only in the inertial system in which E is constant for all $\boldsymbol{x}$ and t. Further, we get (111) only by using the relations (112). It is for this reason that we characterized the normalization condition (70) as Maxwell Normalization.

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* We are using here the generally accepted notion that classical electromagnetic theory deals with a collection of photons.

It is perhaps worth noting that the restricted validity of (111) is not really due to the subsidiary condition E = constant, since any polychromatic radiation can be resolved into monochromatic radiations through Fourier analysis. It arises from the subsidiary condition $\rho = N\epsilon\epsilon^*$, which means that the intensity of the radiation is proportional to the square of the amplitude. We have proved before thtat this is not correct in the Fresnel zone (see eq. (86)).

XII. Wave Mechanical Current Density

First, let us note that contrary to the general validity of eq. (109) for photons, this equation, as we have emphasized before (cf. eq. 63), is not generally valid for a particle whose rest mass is not zero. Consequently, we prove now that the usual expression for the quantum mechanical current density, is not correct in the general case. It is valid only if any one of the following conditions is satisfied.

a) There is no external field, i.e., $\Phi = 0$,

b) There exists an inertial system for which $\partial a/\partial t = \Phi$. grad $a = 0$ for all x and t,

c) There exists an inertial system for which $\Phi_0 - |\text{grad}\ \Psi| = 0$ for all x and t.

Let us consider a collection of particles, ($m_0 \neq 0$), with ensemble normalization (68) and put

$$\gamma\epsilon = \Psi$$

(Ψ is not to be identified without further qualifications with the time dependent relativistic quantum mechanical wave function), and

$$\frac{\gamma^2\epsilon\epsilon^* p_N}{m_0} \quad J. \tag{113}$$

From (2, 3 and 9), it then follows

$$\underline{J} = \frac{h}{4\pi i m_0}(\Psi^*\underline{\nabla}\Psi - \Psi\underline{\nabla}\Psi^*) - \frac{e}{m_0}\Psi\Psi^*\Phi = 0. \tag{114}$$

Eq. (114) is nothing but the usual expression for the quantum mechanical four-current density.

Now, the generally valid continuity condition, namely, Postulate II, eq. (13), can be expressed in the form

$$0 = \underline{\nabla}\cdot(\epsilon\epsilon^*\underline{p}) = \underline{\nabla}\cdot(\epsilon\epsilon^* \underline{p}_N) + a^2\underline{\nabla}\cdot\underline{p}_e + 2a\underline{p}_e\underline{\nabla}a. \quad \ldots \tag{115}$$

Consequently, utilizing the Lorentz condition for the four-potential, we see that the devergence of J is given by

$$\underline{\nabla}\cdot\underline{J} = -\frac{2|\Psi|e}{m_0}(\underline{\Phi}\cdot\underline{\nabla}|\Psi|) \tag{116}$$

Hence, $\underline{J}$ is divergence free only in the case when any of the restrictions (a)–(c) mentioned above are fulfilled.

Evidently, $\epsilon\epsilon^* \underline{p}_N$ is a measure of the current density of energy-momentum of a corpuscle registered by the measuring instrument under normal experimental situations. If we construct again a world tube parallel to the $\underline{p}_N$—field in which the particles are subjected to ensemble normalization, we see that only under these restricted conditions $\Psi\Psi^*$ is a measure of the intensity of the current within the ensemble and the density of corpuscles ρ satisfies the continuity condition (110). Further, using the relation (47) for the velocity $\boldsymbol{v}$ of the corpusucle, we get the expression (117) for the intensity I of a beam of particles obeying the ensemble normalization and the condition (*b*).

$$\left.\begin{aligned} I &= \text{constant } \Psi\Psi^* |\boldsymbol{v}| E_N \\ &= \text{constant} |\text{grad } W - e\Phi| \Psi\Psi^*. \end{aligned}\right\} \quad \ldots \quad (117)$$

In the absence of the external field, eq. (117) reduces to the eq. (86) we had obtained previously for photons in the Fresnel zone.

It is to be noted that the above conclusion, namely, the four-current density $\underline{J}$ is not divergence free under more general conditions, does not mean that the number of particles is not conserved, even when annihilation or creation processes cannot take place. This states only that under these circumstances there is an
cannot take place. This states only that under these circumstances there is an
intense interaction and exchange of energy and momentum between the corpuscle and its associated wave. The continuity condition for the generalized four-momentum, postulate II, eq. (13) assures us that everywhere under all circumstances the energy-momentum is conserved and the corresponding four-current density $\underline{J}'$ is always divergence free, not only for an individual particle but also for any ensemble. In order to see this, let us write

$$\underline{J}' = \gamma^2 \epsilon\epsilon^* \underline{p}. \quad \ldots \quad (118)$$

From (13) it immediately follows

$$\underline{\nabla} \cdot \underline{J}' = 0 \quad \ldots \quad (119)$$

and putting $\gamma\epsilon = \Psi'$, we get for ensemble normalization

$$\Psi' \Psi'^* dv_x = \bar{N} = \text{constant for all } t. \quad \ldots \quad (120)$$

But now $\Psi'\Psi^*$ is not a measure of the density of corpuscles (even under ensemble normalizations) which one gets experimentally. Consequently, problems like Klein's Paradox and the positive definite character of density arising out of the discussions of Klein-Gordon equation according to the formalism of quantum mechanics are not relevant here at all.

XIII. THE EQUATION OF THE VECTOR FIELD OF A PARTICLE

So long we have been talking of the scalar wave function associated with a particle. Its logical consistency and pedagogical success in bridging the gulf between classical and present day quantum physics more realistically and its ability to derive an equation based on almost self-evident postulates which under well defined restricted conditions gives rise to almost all the basic and relevant equations of physics compelous to extend to it the more realistic case, namely, that of the vector wave field of a particle. Though as yet due to the unsolved problems, of mathematics and physics, this causal theory could not predict new results which could be easily verified, we have already referred to three conclusions derived from this theory which can be verified experimentally. The subsequent sections, I hope, will convince you that it is worthwhile to overcome these limitations of mathematics and physics for the progress of physics.

(i) *Definitions and Postulates*

Let the wave function associated with a pa ticle be represented by the four vector

$$\underline{\epsilon}(x) = \underline{A}(x) \exp iW(x)/\hbar \tag{121}$$

Now, in order to define generally the four momentum density $\underline{\epsilon}\underline{\epsilon}^*\underline{P}(x)$ we would like to express it as a function of the wave field ϵ, (and *not* by the gradient of the phase W of the wave field). We therefore, define it, (analogous to eq. 12), by relations*

$$\underline{\epsilon}\,\underline{\epsilon}^*\underline{P} = \frac{\hbar}{2i}\,[(\underline{\epsilon}^*\underline{\nabla})\underline{\epsilon} - (\underline{\epsilon}\,\underline{\nabla})\underline{\epsilon}^*] \tag{122}$$

where

$$\underline{\epsilon}^*(x) = \underline{A}(x) \exp -iW/\hbar \tag{122a)**}$$

* We shall presently see that in order to avoid confusion with symbols used previously, we are to express the generalized four momentum by the new symbol $\underline{P}$ (instead of $\underline{p}$ used previously).

** It should be noted that $\underline{\epsilon}^*(x)$ defined in this way in not the usual mathematical complex conjugate of $\underline{\epsilon}(x)$ (contrary to the scalar case). Since the scalar product of two four-vectors must be Lorentz invariant we *cannot* write

$$\underline{\epsilon}^*(x) = \underline{A}^*(x)\,(\exp -iW/h)$$

(where $\underline{A}^*$ is the usual mathematical conjugate of $\underline{A}$) because $\underline{A}(x)$, $\underline{A}^*(x)$ is not Lorentz invariant whereas $\underline{A}(x)$. $\underline{A}(x)$ denoting the square of the amplitude of the wave function is always Lorentz invariant.

Using four dimensional vector calculus, one can write (122) in the form

$$\underline{\epsilon}\,\underline{\epsilon}^{*}\underline{P} = \frac{\hbar}{2i}\,[\underline{\epsilon}^{*}.(\underline{\nabla}.\underline{\epsilon})-\underline{\epsilon}.(\underline{\nabla}.\underline{\epsilon}^{*})] \qquad \ldots \quad (123)$$

$$= \underline{A}^{2}.\underline{p}+\underline{A}^{2}.\underline{\omega} \qquad \ldots \quad (124)$$

or,

$$\underline{P} = \underline{p}+\underline{\omega} \qquad \ldots \quad (125)$$

where

$$\underline{p} = \underline{\nabla}\underline{W} \qquad \ldots \quad (126)$$

and

$$\underline{\omega} = [\underline{s}_{A}(\underline{s}_{A}\underline{p})] \qquad \ldots \quad (127)$$

$\underline{s}_A$ is the unit vector vector in the direction of $\underline{A}$.

The vector $\underline{P}$ is parallel to $\underline{A}$ and from (125) we see at once that the vector $\underline{P}$ is composed of an irrotational part $\underline{p}$ and a rotational part $\underline{\omega}$. As before, we identify $\underline{p} = \underline{\nabla}\underline{W}$ as the four momentum of the corpusale i.e.,

$$\underline{p} = \underline{p}_N+\underline{p}_e \qquad \ldots \quad (128)$$

and

$$\underline{p}_e = e\underline{\Phi} \qquad \ldots \quad (3)$$

$$\underline{p}_N = \mu m_0 \underline{v} \qquad \ldots \quad (129)$$

The scalar mass factor μ should now be expressed as

$$\mu = \left[1-\frac{(\underline{A}.\Box\underline{A})}{\underline{A}^{2}}\cdot\left(\frac{\hbar}{m_0 c}\right)^{2}\right]^{\frac{1}{2}} \qquad \ldots \quad (130)$$

Evidently, the function $\underline{\omega}$ is then to be associated with the linear momentum of the particle due to the vortical motion of its wavefield so that $[\underline{r}_0\underline{\omega}]$ would denote the intrinsic four-angular momentum of the particle, if $\underline{r}_0$ is the radius of a columnar vortex.

We now postulate that the continuity condition is applicable to the entire wave field. That means,

$$\underline{\nabla}.(\epsilon\epsilon^{*}\underline{P}) = 0 \qquad \ldots \quad (131)$$

(ii) *Differential Equations for the Vector Wave Field*

From (131) and again using vector calculus we get various equivalent forms of the differential equation for the wave field of any particle.

For example,

$$\underline{\epsilon}^{*}.\Box\underline{\epsilon}+\underline{\epsilon}^{*}[\underline{\nabla}[\underline{\nabla}\,\underline{\epsilon}]]-\underline{\epsilon}.\Box\underline{\epsilon}^{*}-\underline{\epsilon}.[\underline{\nabla}[\underline{\nabla}\,\underline{\epsilon}^{*}]] = 0 \qquad \ldots \quad (132)$$

or,

$$\underline{\epsilon}^{*}.\Box\underline{\epsilon} = \underline{A}.\Box\underline{A}-\frac{\underline{A}^{2}.\underline{p}^{2}}{\hbar^{2}} \qquad \ldots \quad (133)$$

Substituting in (133) the relation

$$\underline{p}_N^2 = (\underline{p}-\underline{p}_e)^2$$

or,

$$\underline{p}^2 = 2(\underline{p}.\underline{p}_e)-\underline{p}_e^2-\mu^2 m_0^2 c^2$$

and value of μ given by (130), we obtain

$$\underline{\epsilon}^*.\Box\underline{\epsilon}+\underline{\epsilon}\frac{1}{\hbar^2}[2(\underline{p}.\underline{p}_e)-\underline{p}_e^2-m_0^2c^2]\underline{\epsilon}^*.\underline{\epsilon} = 0. \quad \dots \quad (134)$$

Since $\underline{\epsilon}^*$ and $\underline{\epsilon}$ are not mutually (pseudo-) orthogonal, we have the wave equation (135) containing the specific and assigned properties, (e.g. rest mass and charge) of a given particle.

$$\Box\underline{\epsilon}+\frac{1}{\hbar^2}[2(\underline{p}.\underline{p}_e)-\underline{p}_e^2-m_0^2c^2]\underline{\epsilon} = 0. \quad \dots \quad (135)$$

Finally, using the relation (125) we obtain our desired equation for a single particle associated with a vector wave field.

$$\Box\underline{\epsilon}+\frac{1}{\hbar^2}[2(\underline{P}.\underline{p}_e)-\underline{p}_e^2-m_0^2c^2]\underline{\epsilon} = \frac{2}{\hbar^2}(\underline{\omega}.\underline{p}_e).\underline{\epsilon}. \quad \dots \quad (136)$$

We assume that (136) is the most general equation for any single particle (containing its mass and charge), whose associated wave can be represented by a vector field. We justify this by deducing from this equation various known equations of physics.

(iii) *Important Special Cases of the Eq.* (136)

For photons, $m_0 = e = 0$, eq. (136) reduces to

$$\Box\underline{\epsilon} = 0. \quad \dots \quad (137)^*$$

In a force—free field we get the equation of Proca type

$$\Box\underline{\epsilon}-\left(\frac{m_0^2c^2)}{\hbar^2}\right).\underline{\epsilon} = 0, \quad (\text{for } \underline{p}_e = 0). \quad \dots \quad (138)$$

We notice that $\underline{\omega}$ does not enter explicitly into this equation. Consequently, we believe that this equation is valid for all particles (including electrons) in the absence of an external field. In order to prove this we shall convert eq. (136) to the "iterated" Dirac equation, (cf. Sommerfeld[11])**.

* At the present state of our knowledge it would be premature to identify $\underline{\epsilon}$ with the *self four-potential.*

** It should be noted that a vector can be represented in terms of spinors and tensors in terms of vectors.

XIV. Iterated Dirac Equation

We shall assume that the wave function shows a columnar vortex of radius r_0 around the centroid of the particle, (i.e., singularity of the field) at x_a. In the rest frame of the particle and averaged over the dimension of the vortex we can write for the scalar coefficient at the right hand side of eq. (136) the expression

$$\frac{2}{\hbar^2} \cdot \oint (\underline{p}_e.\underline{\omega})ds = \frac{2}{\hbar^2} \oint [\underline{p}_e(\underline{x}_a).\underline{\omega} + \delta \underline{p}_e.\underline{\omega}]ds$$

$$= \frac{1}{\hbar} [\underline{\nabla}\, \underline{p}_e][s_{ij}].$$

Since, as is well known, $\underline{\omega}$ vanishes at x_a and its average value over the area of the vortex is also zero and the circulation is to be quantized, according to the pilot wave theory, in integral units of h, so that the circulation

$$\Gamma = \oint \underline{v}ds = \frac{nh}{m_0} \qquad \ldots (139)$$

where n is an integer and m_0 is the rest mass of the particle.

s_{ij} is the ij-th component of the antisymmetric six-vector representing the four-rotation of the vortex field. It might be noted that at least formally. s_{ij} has the same relevant algebraic properties as Dirac matrices and the hypercomplex quantities introduced by Sommerfeld in Dirac's theory. Consequently, equation (136) under these conditions, i.e., when the field is averaged over the dimension of the vortex, can be expressed as

$$\Box \underline{\epsilon} + \frac{1}{\hbar^2} [(2\underline{P}.\underline{p}_e) - \underline{p}_e^2 - m_0^2c^2]\underline{\epsilon} = \frac{1}{\hbar} [\underline{\nabla}\, \underline{p}_e] \cdot [s_{ij}].\underline{c}$$

$$= \frac{e}{\hbar} \sum_{\alpha,\beta=1}^{3} (F_{\alpha\beta} \cdot s_{\alpha\beta})\, \underline{\epsilon} - \frac{e}{\hbar c}\, i \sum_{\alpha=1}^{3} (F_{\alpha 0} \cdot s_{\alpha 0}) \cdot \underline{\epsilon}. \qquad \ldots (140)$$

Eq. (140) is completely equivalent to the iterated Dirac equation of Sommerfeld provided we replace $\underline{P}$ by the corresponding quantum mechanical operator* and the hypercomplex quantities $\gamma_\alpha\, \gamma_\beta$ by $s_{\alpha\beta}$. $F_{\alpha\beta}$ are the components of the field tensor given by $[\underline{\nabla}\, \underline{\phi}]$.

Sommerfeld had shown that the iterated Dirac equation gives the same eigenvalues as the original linearized Dirac equations. Eq. (140), however,

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* Corresponding to the Schrodinger condition, eq. (30), we have here the Dirac condition for the validity of the operator formalism

$$(\delta\, \underline{p}_e.\underline{\omega}).\underline{\epsilon} - \frac{\hbar}{i} e^{iW/\hbar}\, \underline{p}_e \cdot \underline{\nabla}\underline{A} = 0 \qquad \ldots (140a)$$

being a second order equation, may in more general cases provide solutions which need not be contained in the linearized first order Dirac equations. Further, in our case $\underline{P}$ in (140) must be replaced by (125) so that eq. (140) in itself is non-linear and its singular as well as nonanalytic solutions might be of physical interest. Nevertheless, it should contain atl the solutions, obtainable from the original equations of Dirac as special cases.

It might be noted that we have quantized the circulation in integral units of h, (cf. eq. 139). Consequently, the intrinsic angular momentum of any particle (in the strict sense of classical definition) is $n\hbar$, where n is an integer. Thanks to the differential equation (140) this does not contradict the observable data and we get the energy eigen values in an external field corresponding to this intrinsic angular momentum correctly in units of $\frac{1}{2}\hbar$. Hence, we conclude that this theory and the wave equation (136) or (140) contradict neithter quantum theory nor the observable spectral data. It must be remembered that the quantum mechanical *spin* is independent of the space-time coordinates and is therefore, as is well known, not to be identified with the intrinsic angular momentum of which we are talking here. In quantum mechanics, we only formally associate (on the basis correspondence principle) a canonical dynamical variable with the corresponding energy eigen value. Recalling this basic featu e of quantum mechanics we can therefore as well say without any contradiction whatsoever that whereas the intrinsic angular momentum (in the sense of classical mechanics) of any particle is nh, (n is an integer), its *spin* momentum (in quantum mechanical sense) is $nh/2$. This remarkable feature is essentially due to the expression (122) which contains the quantum mechanical definition of energy-momentum vector as a special case. The wave field associated with a particle at *rest* in general shows vortical nature and we find that the results derived from this theory are perfectly consistent with quantum theory and with experimental observations. The only objection which might be put forward against this theory, is that the gyromagnetic ratio of any charged particle in the *free state* should be 1, but experiments on ferromagnets in the *solid state* definitely show that this value is 2. This is a point which needs further careful investigation. But it should be noted that it is rather risky to *extrapolate* the behaviour of a ferromagnet in the solid state to the region of free elementary particles. Further, that the actual ratio of magnetic moment and intrinsic angular momentum for the case of free electron is 1 (and not 2) is strongly suggested by the value of the effective number (2.221) of Bohr magnetons per unit volume for the case of iron (cf. Dekker[18]).

Once we accept this theory and realize that we can say that the intrinsic angular momentum of a particle is nh, whereas its spin momentum (in the language of quantum mechanics) is $nh/2$, we can also put forward a model for the electron, the so-called, Zitterbewegung model of the electron proposed by Schroedinger.

We can then also explain very naturally the strange property of a Dirac electron, namely, that the electron *at rest* has the velocity c.

XV. The Properties of Free Quantum Vortices Physical Models for Elementary Particles

The detailed and exact behaviour of a single stable elementary particle can be investigated only if we can find all possible solutions of the nonlinear differential (eq. 136). In order to explore the mutual interaction of such stable particles which may generate unstable elementary particles, we have to known how the individual quantum vortices interact when they penetrate into the domains of these vortices. We hve also to know the turbulent behaviour of the resultant wave field and in particular its stability conditions. It is therefore obvious that the present state of our knowledge we are faced with as yet unsolved problems of physics and mathematics. Nevertheless, if we want to guess the physical nature of elementary particles, we will be obliged to make risky extrapolations of our existing knowledge. We shall therefore in this section make some physically plausible conjectures by constructing simplified models and utilising already known physical and mathematical results.

We have seen that the vector field associated with a particle consists of an irrotational part which describes the translational motion of the particle and a rotational part which we have associated with its nonclassical intrinsic angular motion around the centroid of the particle. Consequently, a particle even at rest would posses, because of its intimate association with its own field, an intrinsic angular momentum. The centroid of a moving particle would therefore describe a helical motion of the singularity of the field. Because of simplicity, we shall restrict ourselves mainly to columnar vortices whose radius is of the dimension of the Compton wave-length of the particle and assume at first that these vortices do not interpenetrate into one another. Moreover, we shall apply the well known laws of classical hydrodynamics, although the motion* of the single particle, according to this theory, is governed by eq. (136).

We assume further that these waves represent the topological distortions and fluctuations of the *World Aether* from its state of equilibrium and this world aether always moves with the velocity c, in consonance with the velocity of Huygen's elementary waves. This would at once make the Zitterbewegung of the electron consistent and also make meaningful the strange property of the Dirac electron at rest.

* For an exact theory one should distinguish between the phase velocity of waves in aether and group velocity of the motion of the vortex as well as the density distribution within the vortical element. For simple models, one can also think of vortex ring and spherical vortex.

(i) *Schroedinger's Zitterbewegung Model of the Electron*

It is very likely that the charge could eventually be represented by the wave field alone. But at this moment, one is not quite sure how to represent the charge of a particle as a function of its wave field. Consequently, we shall assume the existence of charge as an external property given empirically.

Let us suppose that the vortex of the world aether which would represent the electron is generated by the circular motion of the aether around the centroid of the electron, (the singularity of the field), having the radius R. The frequency of the vortical motion is therefore

$$\nu = \frac{c}{2\pi R}. \qquad \ldots (141)$$

Now, the rest mass of the entity which we call electron is known to be m_0 and the frequency ν of its de Broglie wave is given by

$$h\nu = m_0 c^2. \qquad \ldots (142)$$

We assume that this frequency of de Broglie wave is really the frequency of the vortical motion of the wave field associated with the electron and the total energy of the vortex represents the rest mass of the electron*. Hence it follows

$$m_0 c^2 = h\nu = \frac{hc}{2\pi R} \quad \text{or,} \quad R = \hbar / m_0 c. \qquad \ldots (143)$$

The corresponding angular momentum is

$$R.\frac{h\nu}{c} = \hbar \qquad (144)^{**}$$

and the value of the magnetic moment is given by the Bohr magneton.

$$\mu_B = \left(\frac{e\nu}{c}\right) . \pi R^2 = \frac{eR}{2} = \frac{e\hbar}{2m_0 c}. \qquad \ldots (145)$$

It might be noted that the value of R is the Compton wavelength of the electron. Consequently, this model shows why we shall have difficulties within this domain,

* This at once clarifies the mysterious relation between the energy of a particle and the frequency of its de Broglie wave and might eventually lead to the physical meanings of "bare mass" and "renormalised mass" of an elementary particle, since some energy resides in the wave field outside the domaiain of the vortex.

** The previous difficulties and inconsistencies of the Zitterbewegung were essentially due to the incorrect assumption that the intrinsic angular momentum (in the classical sense) of the electron is $h/2$. It would be interesting to explore the excited states of the vortical element and its connection, if any, between the electron and the muon. In principle, the nonlinear differential equation can offer such excited unstable states.

since we cannot assume the electron to be equivalent to a point charge for electromagnetic interactions within this region.

One can easily extend this idea to get the equivalent radius of the vortex due to a nucleon. One comes to the right order of magnitude so far as the nuclear radius is concerned, but to get the correct magnetic moment one has to assume specific charge distribution within this region.

(ii) *Rotons of Landau*

Consider a spherical particle of radius r_0 and mass m_0 at rest at absolute zero degree of temperature. According to this theory, it should possess practically only rotational energy due to the vortical motion of the aether. The energy of the vortex at the ground state (if we neglect the slight energy contained in its wave field outside the singularity), is the usual rest energy of the particle. The rotational energy of the particle at higher temperatures is therefore due to the excitation of this vortex.*

Since for quantitative expressions in this section XV, for reasons already mentioned above, we shall use the corresponding relations given by existing theories, we take this rotational energies as given by

$$W_{rot} = \frac{n^2 h^2}{2I}, \qquad (n = 1, 2, 3, \ldots). \qquad \ldots \quad (146)$$

(I = moment of inertia of the particle).

If the particle rotates, we have the lowest rotational energy, W_1 for $n = 1$ in (146). The minimum amount of energy required to excite the particle to its next rotational level is therefore $\delta W = W_2 - W_1$. We assume that this excitation energy of the vortex of the aether corresponds to the energy of exicitation of the rotons postulated by Landau**. In order to account for the observed properties of liquid ^{4}He near absolute zero, Landau suggested that its energy spectrum is given by

$$E(p) = \Delta + \frac{(p - p_0)^2}{2m} \qquad \ldots \quad (147)$$

* For dealing with real particles, columnar, vortex would not be an adequate representation of the particle. Even for the simplest case, we have to consider a spherical vortex. Further, for an atom containing many elementary particles, a single vortex representing the atom cannot be justified without further quantitative investigations. Nevertheless, this simple model seems to be instructive.

**It is interesting to note that at first Landau suggested that the roton should correspond to vortex motion.

Atkins, by comparing (147) with the experimental results obtained the following values.

$$\frac{\Delta}{k} = 8.9 \pm 0.2^{\circ}K$$

$$p_0 = (2.1 \pm 0.05)10^{-19} \text{ erg sec cm}^{-1}$$

$$\frac{p_0}{h} = 1.99 \text{ Å}^{-1}$$

$$m = 0.26\, m_0$$

k = Boltzmann constant.

For our model it is natural to correlate Δ with W_1 and p_0 with $(2m_0\delta W)^{\frac{1}{2}}$. Taking the standard values for ^4He, namely,

$$r_0 = 1.3\text{Å}, \quad m_0 = 6.69\ 10^{-24}\text{ g}$$

we obtain the following results :

$$\frac{W_1}{k} = 8.9^{\circ}\text{K}; \qquad p_0 = 2.2\ 10^{-19}\text{ erg.sec.cm}^{-1}$$

$$p_0/h = 2.106\text{Å}^{-1}.$$

(iii) *Zero-Point Motion*

In a collection of *ideal gas like* particles the vortices ascribed to each particle are free and consequently, according to the well known laws of hydrodynamics, if the neighbouring vortices are antiparallel, the centre of each vortex would move perpendicular to the line joining the two vortices with a velocity.

$$v_1 = \frac{\Gamma}{2\pi d} = \frac{n\hbar}{md}, \qquad \dots \quad (150)$$

where d is the distance between the centres of the two vortices and m the mass of a particle. Hence even at absolute zero the vortices would move and correspondingly the particles would have translational motion. A collection of free particles, therefore, cannot remain at rest.* Thus one can understand the causes for the random motion of ideal gases postulated in an *ad hoc* fashion in classical kinetic theory. It is interesting to recall that Lord Kelvin[20] already suggested that this translatory motion was to be ascribed to the interaction of his *Vertex atoms*. According to our model of the particle, the average energy of an ideal monatomic gas at room temperature (neglecting the proper energy), is principally due to the intrinsic excited rotational energies of the vortical element plus the kinetic energy due to the translational motion of this vortical element as a result of the interaction of the neighbouring vortices.

* Note that this zero-point energy exists even if the particles do not vibrate.

(iv) *Spatial Arrangement of Vortices. Quantum Statistics*

How would the vortices belonging to elementary particles be arranged in space ? Unless we can answer this question, we would not be able to understand why Nature provides us with only two kinds of statistics, symmetric and antisymmetric.

The correct answer to these questions is to be obtained from consideration of stability and mutual interaction of vortices, in analogy to *v.* Karman's vortex streets of classical hydrodynamics. It is obvious that at the present state of our knowledge, we cannot hope to answer these questions satisfactorily. The equation (136) describes the motion of a single particle but as yet we do not know the nature of interactions between quantum vortices even when they do not penetrate into each other. But if we consider a collection of *free* vortices in a force-free field, we can guess how the two states for the collection may arise. In this case, eq. (136) becomes linear and we can apply the principle of superposition in describing the state, at least outside the domain of their singularities. If the neighbouring vortices are arranged antiparallely we see that by superposition they give rise to a symmetric wave function corresponding to a collection of bosons. In the case of fermions, the phases of adjacent antiparallel vortices must then differ by an additional amount of $\pi/2$ to give rise to an antisymmetric wave function.

(v) *Nuclear Forces—Turbulence*

It is known that whereas two antiparallel vortices induce translatory motion on each other, two parallel vortices induce a common rotational motion about the center of gravity of the vortices. Can we not make a bold conjecture that the tremendous energy which binds the nucleons is essentially due to this coupling of parallel vortices ? A simple calculation* shows that an energy of 8.4 MeV is required to separate two neutral nucleons (each of nuclear mass 1) bound in such a fashion that the centre of each vortex rotates with a common radius of $r_0 = 10^{-13}$ cm. Taking the binding energy of the deuteron as 2.226 MeV the distance comes out as 1.94×10^{-13} cm. so that for the distance between the centroids of the proton and the neutron we get a value 3.88×10^{-13} cm. quite

* The linear velocity of rotation is given by

$$V_1 = \frac{\Gamma}{2\pi d} = \frac{h}{2r_0 m} \qquad \text{for } d = 2r_0.$$

The centripetal force is therefore

$$\frac{mv_1^2}{r_0} = \frac{\hbar^2}{4mr_0^3}$$

and the energy required to separate two nucleons is

$$E = \int_{r_0}^{\infty} \frac{\hbar^2}{4mr^3}\, dr = \frac{\hbar^2}{8mr_0^2} = 8.41 \text{ MeV for } r_0 = 10^{-13}\text{ cm}, \quad m = 1.67 \times 10^{-24} \text{ g.} \qquad \dots (149)$$

close to the radius of the deuteron. For atomic distance (10^{-8} cm) this energy already reduces to the order of 10^{-4} eV. Many of the properties of nucleons could be understood qualitatively on the basis of the existence of such quantized vortex couples within the nucleus.

In nuclear bombardment of high energy particles, the vortices within the target would suffer tremendous disturbances due to the known Magnus force as well as due to the interpenetration of the vortices (about which we do not know anything as yet). As a result, it is very likely that the turbulent motion would be generated inside the nucleus of the target. The stability conditions would then decide how many and what type of new eddies disguised as elementary particles would emerge out of this turmoil. The basic troubles of high energy nuclear physics appear to lie in the inherent difficulties of the subject of turbulence and the complete solution of the nonlinear partial differential equation.

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