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Note

Some comments on fractality of proton at small x

D K Choudhury[*] and Rupjyoti Gogoi

Department of Physics, Gauhati University,
Guwahati – 781 014, Assam, India

E-mail dkc_phys@yahoo.co.in

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Abstract : Using the concept of self-similarity in the structure of the proton at small x, we comment on possibility of a single positive fractal dimension of proton in analogy with classical monofractals. Plausible dynamics and physical interpretation of fractal dimension are also discussed.

Keywords : Self-similarity, fractal dimension, deep inelastic scattering, structure function, low x

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Self-similarity is a familiar property in nature [1,2]. Many of the seemingly irregular shapes of nature have hidden self-similarity in them. It is not the usual symmetry with respect to rotation or translation, but symmetry with respect to scale or size: a small part of a system is self-similar to the entire system. Such a system is defined through its self-similar dimension, which is in general fraction, hence called fractal dimension. Classical fractals discussed in standard references [1,2] are Cantor dust, Koch curve and Sierpinski gasket whose fractional dimensions are 0.63, 1.26 and 1.585, respectively, which lie between Euclidean point and surfaces.

Notion of self-similarity and fractal dimensions are being used in the phase spaces of hadron multiparticle production processes since nineteen eighties [3-7]. However, these ideas did not attract much attention in contemporary physics of deep inelastic lepton-hadron scattering till 2002 when Lastovicka [8] developed relevant formalism and proposed a functional form of the structure function $F_2(x,Q^2)$ at small x. Specifically, a description of $F_2(x,Q^2)$ reflecting self-similarity is proposed with a few parameters which are fitted to recent HERA data [9,10]. The specific parameterization is claimed to provide an excellent description of the data which covers a region of four momentum transferred squared $0.045 \le Q^2 \le 150 GeV^2$ and of Bjorken x, $6.2\times10^{-7} \le x \le 0.2$.

More recently, it was observed [11-13] that the positivity of fractal dimensions prohibits some of the fitted parameters of the structure function of Ref. [8]. Specifically, out of the fractal dimensions D_1, D_2 and D_3, one is negative ($D_3 = -1.3$). However, the positivity of fractal dimension forbids such negative value. In order to avoid such possibility, it is suggested that the proton is described by the single self-similarity dimension D. This then facilitates one to compare the self-similarity nature of the proton at small x with the classical monofractals which is the aim of the present note.

Under the hypothesis of self-similarity of the proton structure at small x, Lastovicka [8] obtained the following form of the structure function $F_2(x,Q^2)$.

$$F_2(x,Q^2) = \frac{(\exp D_0) Q_0^2 x^{-D_2+1}}{1 + D_3 + D_1 \log 1/x}$$

$$\times \left(x^{-D_1 \log\left(1+Q^2/Q_0^2\right)} \left(1 + Q^2/Q_0^2\right)^{D_3+1} - 1 \right) \quad (1)$$

by using the following form of un-integrated quark density $f_i(x,Q^2)$ of i-th quark flavor :

$$\log f_i(x, Q^2) = D_1 \log 1/x \log\left(1 + Q^2/Q_0^2\right)$$

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[*] Corresponding Author

$$+ D_2 \log {}^{1}\!/_{x} + D_3 \log\left(1 + {}^{Q^2}\!/_{Q_0^2}\right) + D_0' . \quad (2)$$

In eqs. (1) and (2), D_1 is the dimensional correlation relating the two magnification factors $1/x$ and $1+Q^2/Q_0^2$, while D_2 and D_3 are the self-similarity dimensions associated with them, being the normalisation constant. Since the magnification factors should be positive, non-zero and dimensionless, a choice $1+Q^2/Q_0^2$, rather than Q^2 has been made, while Q_0^2 is arbitrary small virtuality, $Q^2 > Q_0^2$. Explicit confrontation with HERA data [9, 10] yields:

$$\begin{aligned} D_0 &= 0.339 \pm 0.145 , \\ D_1 &= 0.073 \pm 0.001, \\ D_2 &= 1.013 \pm 0.01 , \\ D_3 &= -1.287 \pm 0.01 , \\ Q_0^2 &= 0.062 \pm 0.01\, GeV^2 . \end{aligned} \quad (3)$$

As the self-similarity dimensions of fractals are positive [1,2] by its definitions, one expects $D_1 \geq 0$, $D_2 \geq 0$, $D_3 \geq 0$, a feature absent in the empirical fit of [8] as far as D_3 is concerned. In analogy with other classic fractals [1,2], we therefore assume that proton at small x is a monofractal with just one single fractal dimension, so that

$$D_1 = D_2 = D_3 = D . \quad (4)$$

Under such a hypothesis, eq. (1) is rewritten as

$$F_2(x, Q^2) = \frac{(\exp D_0)\, Q_0^2 x^{-D+1}}{1 + D + D \log {}^{1}\!/_{x}}$$

$$\times \left(x^{-D\log\left(1+{}^{Q^2}\!/_{Q_0^2}\right)} \left(1 + {}^{Q^2}\!/_{Q_0^2}\right)^{D+1} - 1 \right) \quad (5)$$

Alternately, monofractality is attainable also for $D_1 = 0$ (zero dimensional correlation) and $D_2 = D_3 = D$, so that eq. (1) takes the alternate form:

$$F_2(x,Q^2) = \frac{(\exp D_0)\, Q_0^2 x^{-D+1}}{1+D} \left(\left(1 + \frac{Q^2}{Q_0^2}\right)^{D+1} - 1 \right) \quad (6)$$

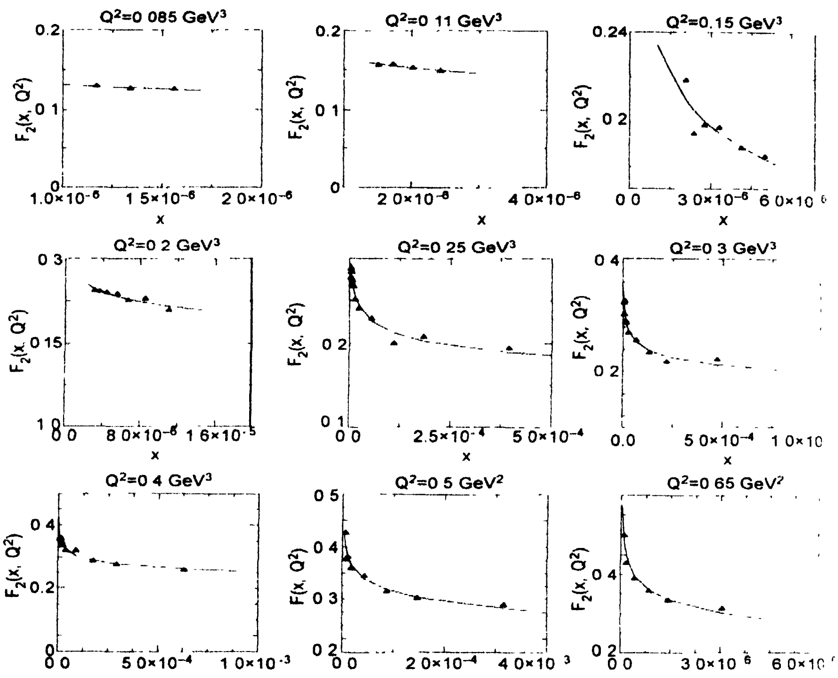

Figure 1. $F_2(x,Q_2)$ *versus* x in bins of Q^2 with $D_1 \neq 0$ (eq (5))

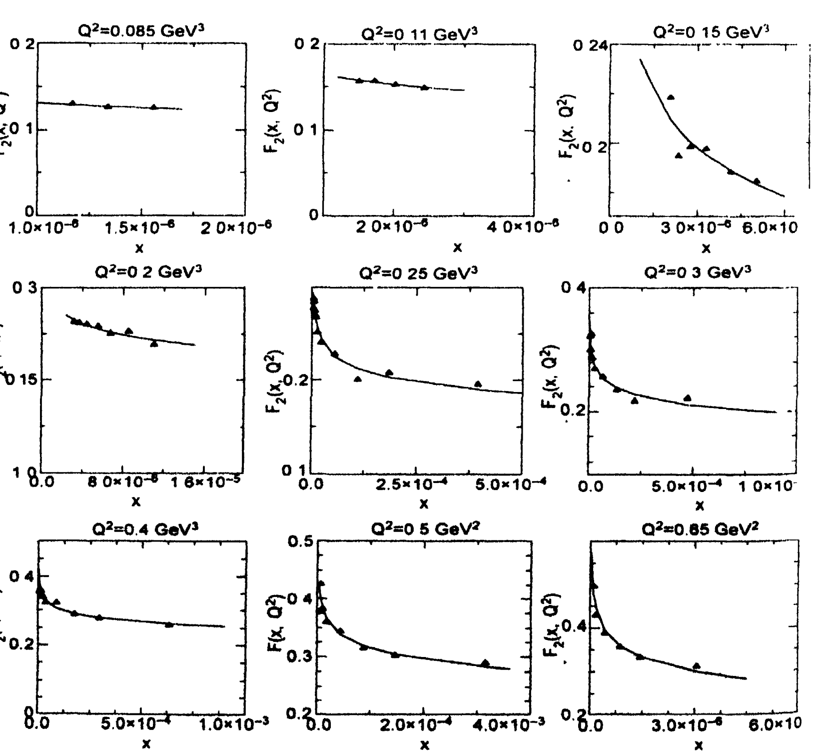

Figure 2. $F_2(x,Q^2)$ *versus* x in bins of Q^2 with $D_1 = 0$ (eq.(6)).

In Figure 1, we plot $F_2(x, Q^2)$ *versus* x in bins of Q^2 as measured by low Q^2 data of ZEUS [10] using eq. (5). Results of the fit yields:

$$D_0 = -1.692 \pm 0.14\,,$$

$$D = 0.653 \pm 0.029\,, \qquad (7)$$

$$Q_0^2 = 0.0449 \pm 0.0003\, GeV^{-2}.$$

However, this fit (eq. (7)) can not be extrapolated to higher Q^2 range of H1 [9]. Even for $Q^2 > 0.4$ GeV^2 of ZEUS [10] data, χ^2 becomes large.

In Figure 2, we show the similar analysis using eq. (6) for ZEUS data [10]. Results of the fit yields:

$$D_0 = -2.713 \pm 0.231,$$

$$D_1 = 0,$$

$$D = 1.107 \pm 0.008,$$

$$Q_0^2 = 0.045 \pm 0.00012\, GeV^{-2}. \qquad (8)$$

This fit (eq. (8)) can be extrapolated to higher Q^2 range of H1 [9] upto $Q^2 = 12$ GeV^2. The χ^2 for eqs. (7) and (8) are recorded in Table 1.

Table 1. The χ^2 for eqs (7) and (8)

Fit		χ^2/dof
eq.(7) ($0.045 = Q^2 = 0\,4$GeV^2)	78 504	1.402
eq.(8) ($0.045 = Q^2 = 12$GeV^2)	83 156	0 621

Our analysis thus indicates that only in the limited $x - Q^2$ range ($Q^2 \leq 12$ GeV^2), the notion of monofractality of proton holds. In that range, dimensional correlation (D_1) vanishes and the proton possesses fractality ($D \approx 1.107$) close to Koch curve ($D \approx 1.26$). Description of $F_2(x, Q^2)$ in the entire small x range in terms of monofractal will result in a continuous x, Q_2 dependent fractal dimension [13] which is a considerable extension of parameter space and contrary to the usual notion of fractal.

It is also instructive at this stage to ascertain the physical interpretation of fractal dimension of proton, since the notion is rather recent in literature. As is well known [14], the fractal dimension measures the way, in which distribution of points fill a geometric space on the average. If the distribution is highly inhomogeneous, the set of points have a distribution of fractal dimensions leading to multifractality. Extending the notion to the $x - Q^2$ plane of the unintegrated quark density, fractal dimension tells, how densely small x partons fill the proton in self-similar way on the average. In the special case of $D \approx D_2 >> D_1$, D_3, unintegrated quark density takes the simple form:

$$f\left(x,Q^2\right) \approx \left(\tfrac{1}{x}\right)^{L} \qquad (9)$$

and fractal dimension is essentially close to x-slope [15] or Pomeron intercept [16-18].

Let us conclude this note with a few comments. The limitation of the present approach as indeed with that proposed by Lastovicka [8] is that it provides merely a parameterization of structure functions based on fractality in terms of a few parameters to be determined from data and identify them later as fractal dimensions. Because of the availability of extremely high quality data [9,10] and accurate parameterizations like CTEQ [19] and MRST [20], such new approach to parametrizing the proton structure function must have strong physical reason for it.

In the present note, we have shown that in the monofractal limit, fractal dimension is closely related to more familiar x-slope [15] or Pomeron intercept [16-18] as is evident from eq. (9). As fractals can be seen in the context of renormalization group (RG) [21,22], as well, eq. (9) can be interpreted as a solution of RG type of equation for the self-similar structure function F_2 itself:

$$\frac{dF_2(x)}{d\ln\frac{1}{x}} = \beta(F_2)\,, \qquad (10)$$

where the β function obeys a power series [21, 22] in F_2,

$$\beta(F) = a_0 + a_1 F_2 + a_2 F_2^2 + \ldots\ldots\ldots \qquad (11)$$

with a_0, a_1, a_2 being constants. Retaining terms upto linear in F_2 and choosing the boundary condition $F_2(x) = 0$ at $x = 1$, eq. (9) immediately follows if the coefficient a_1 of the β -function is identified as the monofractal dimension D.

It is also noteworthy that similar to the present approach, fractal characters of hadrons have been pursued in Refs. [22,23] within a statistical quark model with considerable success. Our results compliment such notion in deep inelastic regime.

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