Indian J. Phys. 52B, 19–23 (1978)

Large coupling expansions for eigenenergies of superposition of inverse square and Yukawa potentials

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(*Received* 25 *April* 1977)

The perturbation technique for large coupling constant has been applied to evaluate the eigenenergies for the potential which is the superposition of inverse square and Yukawa potentials. The expansions of eigenenergies obtained converge for large values of coupling constant. This method may be used for superposition of other potentials also.

1. Introduction

Considerable attention has recently been paid to the large coupling solutions to various problems in particle physics with a view to understand strong interactions in field theory. Physicists are no doubt interested in a fully relativistic particle theory but due to insurmountable mathematical difficulties experienced in solving them, the potential theory which can either give exact or approximate solutions, serves as an invaluable simple prototype of model theory in most of the cases.

Several authors have solved non-relativistic Schrödinger equation for large coupling constants. Approximate behaviour of Regge trajectories for the Yukawa potential was derived by Cheng & Wu (1971). Large coupling solutions for the Gauss potential (Müller 1970) and its generalized form (Sharma 1971) have also been discussed. Müller & Faridi (1973) derived large coupling expansion of the eigenenergies for a generalized Yukawa potential.

In the present investigations, the superposition of inverse square and generalised Yukawa potentials has been considered. In section 2, the eigenenergy expansion of the superposed potential has been derived. The expansion converges for large values of coupling constant. In section 3, a brief discussion on the utility of the method has been given.

2. Solution of the Radial Schrödinger Equation

We consider the radial Schrödinger equation in the form

$$-\frac{1}{2}\frac{d^2\psi(r)}{dr^2}-\frac{1}{r}\frac{d\psi(r)}{dr}+V(r)\,\psi(r)+\frac{l(l+1)}{r^2}\,\psi(r=E\psi(r) \qquad (2{\cdot}1)$$

$(\hbar = c = m = 1)$ for the superposition of inverse square and generalized Yukawa potential of the form

$$V(r) = -\left\{\frac{ZB_{-1}}{\lambda r^2} + \frac{Z}{r}\sum_{j=0}^{\infty} B_j(\lambda r)^j\right\} \qquad \dots (2.2)$$

The potential (2.2) may be written in the generalized form as

$$V(r) = -\frac{Z}{r}\sum_{j=-1}^{\infty} B_j(\lambda r)^j, \quad B_0 = 1. \qquad \dots (2.3)$$

In the limit $\lambda \to 0$ and with a large positive value of coupling constant Z, $V(r)$ reduces to the following form

$$V(r) = -\frac{Z}{r^2}\frac{B_{-1}}{\lambda} - \frac{Z}{r} \qquad \dots (2.4)$$

This is a well known molecular potential used by Kratzer for investigating the rotation-vibration spectrum of diatomic molecules.

Setting $l(l+1)-2\left(\frac{ZB_{-1}}{\lambda}\right) = \theta(\theta+1)$ transforms (2.1) with potential (2.4) as

$$-\frac{1}{2}\frac{d^2\psi(r)}{dr^2} - \frac{1}{r}\frac{d\psi(r)}{dr} + \frac{\theta(\theta+1)}{2r^2}\psi(r) - \frac{Z}{r}\psi(r) = E\psi(r). \qquad \dots (2.5)$$

Now for $E < 0$, the solution (Flügge (1974)) of eq. (2.5) is given as

$$\frac{E}{Z^2} = -\frac{1}{2(N+\theta+1)^2} \qquad \dots (2.6)$$

Hence the eigenenergies for the general potential will have the form

$$\frac{E}{Z^2} = -\frac{1}{2(N+\theta+1)^2} + \xi\Delta, \quad \xi = \frac{\lambda}{Z}, \qquad \dots (2.7)$$

where the principal quantum number $n = (N+\theta+1)$, $N = 0, 1, 2, \dots$.

Changing the independent variable of the radial wave equation to y where

$$r = \frac{n}{2Z}\, y$$

and setting

$$\psi(y) = e^{-y/2}y^l u(y),$$

we obtain the equation

$$D_a u(y) = [\xi a y + n\sum_{t=2}^{\infty} \xi^t B_t(n^y/_2)^t]u(y) \qquad \dots (2.8)$$

where

$$D_a = y\,\frac{d^2}{dy^2} + (b-y)\,\frac{d}{dy} - a,$$

$$\alpha = n^2/_2(B_1-\Delta),\; a = -N,\; b = 2\theta+2 = 2l+2+2c \quad \ldots \tag{2.9}$$

and

$$c = (\theta - l).$$

Now

$$\theta = \sqrt{(l+\tfrac{1}{2})^2+(2ZB-1)/\lambda} - \tfrac{1}{2}. \tag{2.10}$$

Also

$$\theta(\theta+1) = (l+c)(l+c+1). \tag{2.11}$$

We find from eq. (2.8) that to order zero in ξ

$$u = u^{(0)} = \phi(a, b; y) \equiv \phi(a), \tag{2.12}$$

where ϕ is a confluent hypergeometric function. ϕ is normalizable only when $a = -N$. Following Müller & Faridi (1973), eq. (2.8) can be solved by using the following recurrence relation of the confluent hypergeometric functions

$$y\phi(a) = (a, a+1)\phi(a+1)+(a, a)\phi(a)+(a, a-1)\phi(a-1) \tag{2.13}$$

where,

$$(a, a+1) = a = 1+\theta-n$$

$$(a, a) = b-2a = 2n$$

$$(a, a-1) = (a-b) = -(n+l+c+1) = -(n+\theta+1) \tag{2.14}$$

In general,

$$y^m\phi(a) = \sum_{j=m}^{\cdots} S_m(a, j)\phi(a+j). \tag{2.15}$$

The coefficients S_m satisfy the following recurrence relation

$$S_m(a, r) = S_{m-1}(a, r-1)(a+r-1, a+r)+S_{m-1}(a, r)(a+r, a+r) + S_{m-1}(a, r+1)(a+r+1, a+r) \quad \ldots \tag{2.16}$$

with the boundary conditions,

$$S_0(a, a) = 1,\; S_0(a, i) = 0 \quad \text{for } i \neq 0;$$

$$S_m(a, r) = 0 \quad \text{for } |r| > m.$$

The first approximation of eq. (2.12) leaves uncompensated

$$R^0(a) \equiv \left[\sum_{i=1}^{\infty} \xi^i \sum_{j=-i}^{i} [a, a+j]_i \phi(a+j) \right] \tag{2.17}$$

where,

$$\begin{cases} [a, a+1]_1 = \alpha a \\ [a, a]_1 = \alpha(b-2a) \\ [a, a-1]_1 = \alpha(a-b) \text{ and for } i > 1 \\ [a, a+j]_j = n(n/2)^i B_i S_i(a, j). \end{cases} \tag{2.18}$$

Adopting the procedure of Müller & Faridi (1973) the value of Δ given in eq. (2.7) can be writtten as

$$O = \xi[a, a]_1 + \xi^2 \left\{ [a, a]_2 + \frac{[a, a+1]_1}{1} [a+1, a]_1 + \frac{[a, a-1]_1}{-1} [a-1, a]_1 \right\} + O(\xi^3). \quad \ldots \quad (2.29)$$

Evaluating in eq. (2.19) terms upto order (ξ^3) in terms of Δ, we have

$$\xi\Delta(n^3 + n^2 c) = \xi(n^3 B_1 + n^2 c B_1^2) + \xi^2[y_1 + y_2 \alpha^2] + \\ + \xi^3[y_3 + \alpha^2 y_4 + \alpha y_5 - \alpha y_6 - \alpha y_7 + \alpha^3 y_8] + O(\xi^4) \quad \ldots \quad (2.20)$$

where

$$y_1 = \{(a-1, a)(a, a-1) + (a, a)^2 + (a+1, a)(a, a+1)\} \frac{n^3 B_2}{4}$$

$$y_2 = (a, a+1)(a+1, a+1) - (a, a-1)(a-1, a-1) \text{ etc.}$$

Putting $C = 0$, since coefficient B_{-1} becomes zero, the potential (2.3) reduces to potential considered by Müller & Faridi (1973). The eigenenergy equation reduces to the form

$$\frac{E}{Z^2} = -\frac{1}{2n^2} + \xi B_1 + \frac{\xi^2 B_2}{2} [3n^2 - l^2 - l] + O(\xi^3). \quad \ldots \quad (2.21)$$

This result agrees with the expansion obtained by Müller & Faridi (1973).

3. Conclusions

In this paper, we have investigated the radial Schrödinger equation for a general potential i.e. superposition of inverse square and generalized Yukawa potentials with large values of the coupling constant. In particular, we obtained the binding energies of all angular momenta in the form of an asymptotic expansion.

This method is quite useful for solving a few other problems also for which bound state eigenenergies can not be exactly evaluated. But by employing this technique if two protentials are superposed, out of which if the eigenenergies of one of them are known, then, it is possible to evaluate the eigenenergies of the other potential by the method used by us. Since in our case, the eigenenergies of Kratzer molecular potential were known, therefore, the eigenenergies of the remaining potential has been evauluated by this perturbation method.

It may be of interest to note that for large values of coupling constant since the coefficient ξ, ξ^2, etc., decrease successively, therefore, the asymptotic expansion for the eigenenergies converges.

ACKNOWLEDGMENT

The authors are grateful to Professor O. N. Kaul for encouragement and association with this work. One of the authors (S.S.R.) is grateful to C.S.I.R. for the award of Junior Fellowship.

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