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Fermion-Dyon dynamics in non-Abelian gauge theory

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Abstract : The study of behaviour of a fermion in the field of non-Abelian dyon has been undertaken in Lagrangian and Hamiltonian formulation. Solving Dirac equation, expression for energy eigen value has been obtained and the Hamiltonian of this system has been shown to involve spin as well as contribution of massive fields associated with these particles. By introducing suitable spinors, the Pauli equation for a dyon moving in the field of fermion has been solved in non-Abelian gauge theory and it is shown that introduction of massive fields perceptibly modifies the energy eigen value and eigen function of bound states of the system.

Keywords : Fermion-dyon dynamics, non-Abelian gauge theory, bound states.

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1. Introduction

The renewed interest in past few years in the theory of monopole and dyon is partly due to the work of 't Hooft [1] and Polyakov [2] who embedded U(1) electromagnetic field into SU(2) gauge theory and obtained the numerical solutions of canonical finite mass monopole in the whole space through spontaneous symmetry breaking due to Higg's field which leaves behind unbroken U(1) gauge symmetry. Julia and Zee [3] obtained the corresponding numerical solutions for a dyon and Prasad and Sommerfield [4] derived the analytic stable solutions for the monopoles and dyons of finite mass by keeping the symmetry of vacuum broken but letting the self interaction of Higg's field approaching zero. At present, it is widely recognized [5] that SU(5) grand unified model [6] is a gauge theory that contains magnetic monopole solutions and consequently, the question of existence of monopole and dyon has gathered enormous potential importance in connection with the problem of quark confinement [7], possible magnetic condensation of vacuum [8], their role in catalyzing proton decay [9,10] and the possible explanation of C P violation [11]. Keeping in view all these facts and observation of Cabrera [12], we have constructed [13,14] the manifestly covariant quantum field theory of dyons each carrying the generalized charge as complex quantity with electric and magnetic charges its real and imaginary parts, formulated non-Abelian theory of dyons [15,16] and investigated bound states of dyon and fermion [17–19] in non-relativistic as well as in relativistic framework. We have also shown that [17,18] relativistic bound states of a system of a fermion moving in the field of dyon are not possible due to the presence of a term vanishing more rapidly than r^{-1} in the potential of the system. In order to understand the behaviour of relativistic dyons, we have introduced the spin in the Hamiltonian of the system of two dyons in an *ad hoc* manner [20] and showed that solutions for Pauli equation exist for dyon-dyon and dyon-fermion [21] bound states in relativistic Pauli theory.

Extending this work in the present paper, we have undertaken the study of Pauli equation for a fermion moving in the field of a non-Abelian dyon. It has been shown that the interaction of spin and the generalized potential leads to an extra energy which is expressible in terms of generalized spin momentum of the particle concerned. It has also been demonstrated that massive fields play a major role in determining the interaction of spin and generalized potential. Analyzing Dirac equation, the problem of interaction of spin and orbital angular momentum of this system has been investigated and the expression for Hamiltonian has been derived. We have also undertaken the study of Pauli equation for a dyon moving in the field of fermion by introducing suitable spinors and it has been demonstrated that bound state energy eigen values and eigen functions are perceptibly modified from the bound states thus formed in Abelian gauge theory [17].

2. Behaviour of an extended dyon in the field of fermion

In order to study the behaviour of extended dyon moving in the field of a fermion, we have considered the extended dyon as an isomultiplet interacting with the field of a fermion. The Lagrangian density for such a system is written (in natural units, $\hbar = c = 1$) as

$$£ = £_0 + i\overline{\psi}\gamma_\mu D_\mu \psi - GK\overline{\psi}T^a \phi^a \psi, \tag{2.1}$$

where $K = e_i\, g_j$ and e_i is the electric charge of fermion and g_j is the magnetic charge on dyon. $£_0$ in eq. (2.1) is the 't Hooft-Polyakov Lagrangian [1,2] and

$$iD_\mu = i\delta_\mu - KV^a_\mu T^a. \tag{2.2}$$

V^a_μ in eq. (2.2) is the matrix form of generalized potential of the dyon and T^a are the matrices given by

$$(T^c)_{ab} = i\varepsilon_{abc}. \tag{2.2a}$$

The third term in eq. (2.1) carries the mass of incident dyon in terms of Higg's scalar field, G is the coupling constant between Higg's triplet and iso-spintriplet. Using minimal replacement (2.2) the Lagrangian takes the following form :

$$\pounds = \pounds_0 + i\overline{\psi}\,\dot{\psi} + \overline{\psi}\gamma_k\left(\delta_k - KV_k^a T^a\right)\psi$$
$$-\overline{\psi}KV_0^a T^a\psi - GK\overline{\psi}T^a\phi^a\dot{\psi}. \tag{2.3}$$

Defining the conjugate momenta as $\pi = \delta\pounds/(d\dot{\psi})$ and by using the definition $H = \pi\dot{\psi} - \pounds$, for Hamiltonian, the wave equation for multiplets of spinor field ψ is written as,

$$H\psi = \Big[\hat{\alpha}\cdot\left(\hat{P} - KV_a^T T_a\right) + \beta GK\phi_a T_a + KV_a^0 T_a$$
$$+\left(KV_b^0 T^b\right)^2\Big]\psi = E\psi. \tag{2.4}$$

In the above equation, we have substituted the matrices $\gamma^k = \alpha$ and $\gamma^0 = \beta$. In the first term of this equation V_a^T is transverse vector potential of dyon field and the term $KV_a^0 T_a$, under a suitable gauge transformation when the operator $\hat{r}$. T in iso-spin space goes to an operator T_3, represents the Coulombian type of scalar potential $e_i e_j/r$ which is the first term of interaction potential between fermion and extended dyon [17]. The observation made in non-relativistic formulation that the symmetry of Hamiltonian requires an additional potential term [18] $e_i e_j/(2mr^2)$ in case of interaction of a fermion with dyon justifies the presence of an additional term $(KV_a^0 T_b)^2$ in the wave equation, which under suitable gauge transformation takes the form $K^2/(2mr^2)$. This potential has been introduced in eq (2.3) in an *ad hoc* manner.

Introducing the dyon-fermion ansatz

$$K\phi_a T_a = \phi(r)\boldsymbol{T}\cdot\hat{r}; \qquad KV_a^T T_a = \left(\frac{1-k(r)}{r}\right)\hat{r}\times \boldsymbol{T};$$
$$KV_a^0 T_a = V_1(r)\boldsymbol{T}\cdot\hat{r}; \qquad KV_b^0 T_b = V_2(r)\boldsymbol{T}\cdot\hat{r}; \tag{2.5}$$

the wave equation takes the following form

$$\left[\hat{\alpha}\cdot\left(\hat{P} - \frac{\hat{r}\times\boldsymbol{T}}{r}\right) + \hat{\alpha}\cdot\left(\frac{\hat{r}\times\boldsymbol{T}}{r}\right)k(r) + \beta G\phi(r)\boldsymbol{T}\cdot\hat{r} + V_1(r)\boldsymbol{T}\cdot\hat{r}\right.$$
$$\left.+\left[\frac{V_2(r)\boldsymbol{T}\cdot\hat{r}}{2m}\right]^2 - E\right]\psi = 0. \tag{2.6}$$

This is a very complicated equation, however, we can simplify it by making following short distance simplification as the distances involved are very small

$$k(r) = 1, \tag{2.7}$$
$$G\phi(r)\boldsymbol{T}\cdot\hat{r} = m_0,$$

where m_0 is the mass associated with a particular component of Higg's triplet. Eq. (2.6) then takes the following form for free particle Dirac equation (*i.e.* when $V_1(r)$ and $V_2(r)$ are zero);

$$\left[\hat{\alpha}\cdot\hat{P}+\beta m_0\right]\psi = E\psi, \tag{2.8}$$

where m_0 is the mass of dyon. The complete plane wave solution for bispinor $\psi(x)$ may be written as

$$\psi(x) = \begin{pmatrix}\hat{\xi}\\ \hat{\eta}\end{pmatrix} = \zeta(p^{\mu}, \sigma^{\mu})\exp\left\{-\frac{i}{\hbar}p^{\mu}X_{\mu}\right\}, \tag{2.9}$$

where ζ is a function and the suffix P^{μ} is to tell us that we are dealing with plane wave of momentum P and the suffix σ^{μ} is to identify four solutions of ψ, while $\hat{\xi}$ and $\hat{\eta}$ are two component functions. In order to derive the wave equation of an extended dyon moving in the field of a fermion, let us consider the four-potential of field as $\left\{V_{\upsilon}^{j}\right\} = \left\{A_{\upsilon}^{j} - iB_{\upsilon}^{j}\right\}$ with A_{υ}^{j} and B_{υ}^{j} as electric and magnetic four-potential associated with dyons carrying the generalized charge $q_j = e_j - ig_j$ with e_j and g_j as electric and magnetic charges. The wave equation in this case may be obtained by using the following transformation [22]

$$\nabla_{\upsilon} = \left(\delta_{\upsilon} - iKV_{\upsilon}X\right), \tag{2.10}$$

where vector sign (bold) and cross product (×) are demonstrated in internal group space; $\upsilon = 0, 1, 2, 3$ are indices representing external degrees of freedom. In presence of dyons, the introduction of second four-potential is actually compensated by an enlargement of the group of gauge transformation [16]. SU(3) gauge symmetry spontaneously broken by an octad Higg's field exihibits SU(2) × U(1) symmetry with the non-zero vacuum expectation value of the Higg's field. According to general topological argument [22], the very presence of the U(1) factor in the unbroken gauge group guarantees the existence of smooth, finite energy stable particle like solutions with quantized magnetic charges and chirality quantised dyon. With the help of prescription given by eq. (2.10), we get the following equation for the two component spinor $\hat{\xi}$ and $\hat{\eta}$;

$$\left[\hat{\sigma}\cdot\left(\hat{P} - KV_{aj}^{T}\hat{T}_a\right)\right]\hat{\eta} = \left[\varepsilon - V_{a0}^{T}\hat{T}_a - m\right]\hat{\xi}, \tag{2.11}$$

$$\left[\hat{\sigma}\cdot\left(\hat{P} - KV_{aj}^{T}\hat{T}_a\right)\right]\hat{\xi} = \left[\varepsilon - V_{a0}^{T}\hat{T}_a - m\right]\hat{\eta}. \tag{2.12}$$

Restricting our-self to the case of non-relativistic motion in a weak field and considering only positive energy solutions

$$\varepsilon = E+m; \qquad \left|E - KV_{a0}T_a\right| \langle\langle m, \tag{2.13}$$

we get the following energy eigenvalue equation in term of $\hat{\xi}$;

$$\left[\frac{1}{2m_0}\left\{\hat{P} - KV_{aj}^{T}\hat{T}_a\right\}^2 + KV_{a0}T_a - \frac{|k|}{2m_0}\left(\hat{\sigma}\cdot\hat{g}\right)\right]\hat{\xi} = E\hat{\xi}, \tag{2.14}$$

where $\hat{g} = \hat{\partial}\times\left\{K/|K|\, V^T_{aj}\hat{T}_a\right\} = \hat{\partial}\times\left\{e^{-i\theta}\, V^T_{aj}\hat{T}_a\right\}.$

This equation looks very complicated but it may be simplified by using dyon-fermion ansatz given by eqs. (2.5) and (2.7). As such eq. (2.14) may be written as follows

$$\left[\frac{1}{2m_0}\left\{\hat{P} - KV^T\right\}^2 + KV_0 - \frac{|K|}{2m_0}(\hat{\sigma}\cdot\hat{g})\right]\hat{\xi} = E\hat{\xi}. \tag{2.15}$$

Now
$$\hat{g} = \hat{\partial}\times\left\{\frac{K}{|K|}V^T\right\} = \hat{\partial}\times\left\{Ke^{-i\theta}\,V^T\right\} \tag{2.16}$$

with
$$\tan\theta = \frac{g}{e} = \frac{B_\mu}{A_\mu}, \tag{2.17}$$

where A_μ and B_μ are electric and magnetic four-potentials respectively. Eq. (2.15) is analogous to Pauli equation for a dyon moving in the electromagnetic field of a fermion [18,21]. It has the following modification in the energy gained by a non-Abelian dyon moving in the field of a fermion :

$$E' = \frac{|K|}{2m_0}(\hat{\sigma}\cdot\hat{g}). \tag{2.18}$$

This equation can also be written as

$$E' = \hat{\mu}_D\cdot\hat{g} = -\hat{\mu}_{D'}(\hat{\sigma}\cdot\hat{g}) \tag{2.19}$$

where
$$\mu_{D'} = \frac{|K|}{2m_0}$$

is Bohr magneton for a dyon moving in the field of a fermion and

$$\hat{\mu}_D = \hat{\mu}_{D'}\cdot\hat{\sigma} \tag{2.20}$$

is generalized spin moment of dyon. Eqs. (2.18), (2.19) and (2.20) show that massive field plays major role in describing the dynamics of non-Abelian dyon moving in the field of a fermion. Consequently, extra energy term in the Hamiltonian may be interpreted as the interaction of massive fields associated with non-Abelian dyon with a vector field resulting from the spatial rotation of generalized four potential. The third component of the generalized spin moment operator for dyon may be written as

$$(\hat{\mu}_D)_3 = \frac{K}{2m_0}\hat{\sigma}_3, \tag{2.21}$$

the eigenvalues of which are

$$\pm\frac{|K|}{2m_0} = \pm\mu_D. \tag{2.22}$$

Its value is very large as compared to the value of ordinary Bohr magneton of Abelian electromagnetic theory mainly due to the presence of massive fields associated with non-Abelian dyon and magnetic charge present in the dyon.

3. Spin-orbit coupling for spin-1/2 non-Abelian dyon moving in the field of a fermion

Let us consider the motion of a spin-1/2 non-Abelian dyon moving in the field of a fermion retaining the terms up to order of v^2 (where v is the velocity). Putting $V^T = 0$ and $V_0(r) = V_d(r) = KV_0$ in eqs. (2.11) and (2.12) and applying monopole dyon ansatz given by eqs. (2.5) and (2.7) we get

$$[E - V_d(r)]\hat{\xi} = (\hat{\sigma}\cdot\hat{P})\hat{\eta}, \tag{3.1}$$

$$[E + 2m - V_d(r)]\hat{\eta} = (\hat{\sigma}\cdot\hat{P})\hat{\xi}, \tag{3.2}$$

where we have used $\varepsilon = E + m$. These equations, in the first approximation, yield the following energy eigenvalue equation in terms of spin function $\hat{\xi}$:

$$[E - KV_0]\hat{\xi} = \frac{(\hat{\sigma}\cdot\hat{P})}{2m_0}\left[1 - \frac{1}{2m_0}\left\{E - \mu_{ij}V_0\right\} \times (\hat{\sigma}\cdot\hat{P})\right] \tag{3.3}$$

which on further simplification, gives the following expression for energy operator (Hamiltonian) under first approximation :

$$H = KV_0 + \left[1 - \frac{1}{2m_0}\left\{E - KV_0\right\}\frac{\hat{P}^2}{2m_0} + \frac{1}{4m_0^2}\left[\hat{\sigma}\cdot\left\{K\hat{\partial}V_0 \times \hat{P}\right\}\right]\right.$$
$$\left. - \frac{1}{4m_0^2}\left[K\hat{\partial}V_0 \cdot \hat{P}\right]\right]. \tag{3.4}$$

In order to derive the expression for Hamiltonian in second approximation, we use the following function ϕ in place of $\hat{\xi}$ in eq. (3.3)

$$\hat{\phi} = \hat{u}\hat{\xi} \tag{3.5}$$

the normalization of which up to second order leads to the following value of factor u

$$\hat{u} \approx 1 - \frac{\hat{P}^2}{8m_0}. \tag{3.6}$$

Using this value of $\hat{u}$ (and hence of $\hat{\phi}$), we get the following relativistic expression for corresponding Hamiltonian in second approximation

$$\hat{H}' = \hat{u}\hat{H}\hat{u}^{-1}$$

or

$$\hat{H}' = \left[\frac{\hat{P}^2}{2m_0} + KV_0\right] - \frac{1}{2m_0}[E - KV_0] - \frac{1}{8m_0^2}[K\operatorname{div}\psi_D^*]$$
$$- \frac{1}{4m_0^2}\left[\hat{\sigma}\cdot K\psi_D^* \times \hat{P}\right] = \hat{H}_0 + \hat{H}_1 + \hat{H}_2 + \hat{H}_3$$
$$= \hat{H}_0 + \hat{H}_1, \tag{3.7}$$

where $\psi_D^* = -\hat{\partial}V_0^* = -q\varepsilon$ (ε is the field strength) and $\hat{H}_0$ corresponds to the non-relativistic term of the Hamiltonian while $\hat{H}_I$ is the relativistic correction term to the Hamiltonian various parts of which arise due to different relativistic interactions.

4. Pauli equation for an extended spin-1/2 dyon in the field of a fermion

In order to deal, more accurately, the motion of spin-1/2 extended dyon in the field of a fermion with the inclusion of spin effect we start with following Lagrangian density

$$\pounds = \pounds_0 + i\psi^\dagger \dot{\psi} - \frac{1}{2m}(\nabla\psi^\dagger)(\nabla\psi) + KV_a^0 T_a \psi^\dagger \psi + \frac{1}{2m}\left(KV_b^0 T_b\right)^2 \psi^\dagger \psi, \tag{4.1}$$

where $\pounds_0$ is the 't Hooft-Polyakov [1,2] Lagrangian and the remaining part of this Lagrangian density corresponds to the Schroedinger field in which the interaction with the scalar generalized potential has been written in terms of last two terms and interaction with the vector generalized potential could be obtained by writing

$$\begin{aligned} V_\mu &= V_\mu^a T^a, \\ G_{\mu\nu} &= G_{\mu\nu}^a T^a, \\ D^\mu &= \partial^\mu + iKV^{a\mu} T^a, \end{aligned} \tag{4.2}$$

where matrices T^a satisfy the commutation relation given by eq. (2.2a) and $G_{\mu\nu}$ is the field tensor associated with dyons [22]. In this Lagrangian, the term $KV_a^0 T_a \psi^\dagger \psi$ is the non-Abelian gauge form of the Coulombian type of interaction depending upon electric coupling parameter $e_i e_j$ and the term $1/(2m)$ $(KV_b^0 T_b)^2 \psi^+ \psi$ is the non-Abelian gauge form of scalar interaction depending upon magnetic coupling parameter $(e_i g_j)$ which is introduced in the Hamiltonian of dyon-fermion system to maintain the higher symmetry of the Hamiltonian.

Using field theoretical method to obtain complex conjugate of Schroedinger field and the Hamiltonian, the non-relativistic wave equation for i-th dyon in the field of a fermion with charge e_j in non-Abelian gauge form may be written as

$$\frac{1}{2m}\left[\left(-i\nabla - KV_a^T T_a\right)^2 + \left(2GK\phi_a T_a\right)\cdot\left(KV_a^0 T_a\right) + K\left(V_b T_b\right)^2\right]\psi = E\psi \tag{4.3}$$

which can be simplified in the following form with the help of dyon-fermion ansatz given by eqs. (2.5) and (2.7) :

$$\frac{1}{2m}\left[(-i\nabla)^2 + 2m_0 V_1(r)\left\{T\cdot\hat{r}\right\} + V_2(r)\left\{T\cdot\hat{r}\right\}^2\right]\psi = E\psi, \tag{4.4}$$

where $V_1(r)$ is the scalar potential depending on the electric charges of the dyon and fermion while $V_2(r)$ is the scalar potential depending upon electric charge of the fermion and magnetic charge of dyon. Under a suitable gauge transformation where the operator $\hat{r}.\boldsymbol{T}$ in isospin space goes to an operator $\hat{T}_3$ *i.e.* $\boldsymbol{T}.\hat{r} \rightarrow \hat{T}_3$ and the matrix $\hat{T}_3$ is given as :

$$\hat{T}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so $$(\boldsymbol{T}.\hat{r})^2 = (\hat{T}_3)^2$$

which is a unit matrix. We can write eq. (4.4) as follows

$$\frac{1}{2m}\left[(-i\nabla)^2 + 2m_0 V_1(r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + V_2(r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 + F(r)\boldsymbol{L}\cdot\hat{\sigma}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$= E\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{4.5}$$

This equation splits in to the following equations

$$\frac{1}{2m}\left[(-i\nabla)^2 + 2m_0 V_1(r) + V_2(r) + F(r)\boldsymbol{L}\cdot\hat{\sigma}\right]\psi_1 = E_1\psi_1 \tag{4.6a}$$

and $$\frac{1}{2m}\left[(-i\nabla)^2 - 2m_0 V_1(r) + V_2(r) + F(r)\boldsymbol{L}\cdot\hat{\sigma}\right]\psi_2 = E_2\psi_2 \tag{4.6b}$$

both of these equations are same except for a negative sign which occurs due to the structure of matrix $\hat{T}_3$ implying that energy eigenvalues are further degenerate due to the internal degrees of freedom of non-Abelian dyons.

For solving these equation we shall treat spin-orbit interaction $F(r)\boldsymbol{L}.\hat{\sigma}$ as small perturbation. Though the non-relativistic Pauli equation (4.6) are not sufficiently complex to yield precise values for the fine structure of energy levels of dyon-fermion bound states yet these can be safely taken as a useful guide to an understanding the role of spin in bound states of dyon and fermion.

The unperturbed Hamiltonian

$$H_0 = -\frac{1}{2m}\hat{\nabla}^2 + V(r) \tag{4.7}$$

where $$V(r) = -\frac{e_i e_j}{r} + \frac{e_i g_j}{2mr^2} \quad \text{for (4.6a)}$$

and $$V'(r) = +\frac{e_i e_j}{r} + \frac{e_i g_j}{2mr^2} \quad \text{for (4.6b)}$$

represents central force problem for fermion-dyon dynamics in non-Abelian gauge theory due to the structure of matrix T_3 and the spin-orbit interaction energy $\hat{H}'$ is given by $\hat{H}_3$ of eq. (3.8) which may also be written as [21];

$$\hat{H}_1^1 = -\frac{e_i e_j}{2m_0^2}\left\langle\frac{1}{r^3}\right\rangle\hat{L}\cdot\hat{S} + \frac{e_i g_j}{2m_0^4}\left\langle\frac{1}{r^4}\right\rangle\hat{L}\cdot\hat{S}, \tag{4.8a}$$

and
$$\hat{H}_2' = + \frac{e_i e_j}{2m_0^2}\left\langle\frac{1}{r^3}\right\rangle \hat{L}\cdot\hat{S} + \frac{e_i g_j}{2m_0^4}\left\langle\frac{1}{r^4}\right\rangle \hat{L}\cdot\hat{S}; \tag{4.8b}$$

where the symbols have their usual meaning. Since $\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{L}.\hat{S}$, the Pauli operator for $\hat{H}'$, is given by :

$$\begin{aligned}(H_1')_P = &- \frac{e_i e_j}{4m_0^2}\left\langle\frac{1}{r^3}\right\rangle\left[\left(\hat{J}^2\right)_P - \left(\hat{L}^2\right)_P - \left(\hat{S}^2\right)_P\right] \\ &+ \frac{e_i g_j}{4m_0^4}\left\langle\frac{1}{r^4}\right\rangle\left[\left(\hat{J}^2\right)_P - \left(\hat{L}^2\right)_P - \left(\hat{S}^2\right)_P\right]\end{aligned} \tag{4.9a}$$

and
$$\begin{aligned}(H_2')_P = &\frac{e_i e_j}{4m_0^2}\left\langle\frac{1}{r^3}\right\rangle\left[\left(\hat{J}^2\right)_P - \left(\hat{L}^2\right)_P - \left(\hat{S}^2\right)_P\right] \\ &+ \frac{e_i g_j}{4m_0^4}\left\langle\frac{1}{r^4}\right\rangle\left[\left(\hat{J}^2\right)_P - \left(\hat{L}^2\right)_P - \left(\hat{S}^2\right)_P\right]\end{aligned} \tag{4.9b}$$

and the Pauli equation becomes

$$\left[\hat{H}_{1P}\right]\psi_P = \left[\left(\hat{H}_0\right)_P + \left(\hat{H}_1'\right)_P\right]\psi = W_1(P)\psi, \tag{4.10a}$$

$$\left[\hat{H}_{2P}\right]\psi_P = \left[\left(H_0\right)_P + \left(\hat{H}_2'\right)_{\hat{P}}\right]\psi = W_2(P)\psi, \tag{4.10b}$$

where
$$\left(\hat{H}_0\right)_P = \begin{pmatrix}\hat{H}_{10} & 0 \\ 0 & \hat{H}_{20}\end{pmatrix}_P$$

$$= \begin{pmatrix} -\frac{1}{2m}\hat{\nabla}^2 + \frac{e_i e_j}{r} - \frac{e_i g_j}{2mr^2} & 0 \\ 0 & -\frac{1}{2m}\hat{\nabla}^2 - \frac{e_i e_j}{r} - \frac{e_i g_j}{2mr^2}\end{pmatrix}_P \tag{4.11}$$

and
$$(\psi_P) = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}_P \tag{4.12}$$

represents the Pauli wave function. The Pauli wave equation for unperturbed Hamiltonian is given as :

$$\begin{pmatrix}\hat{H}_{10} & 0 \\ 0 & \hat{H}_{20}\end{pmatrix}\begin{pmatrix}\psi_+^{(0)} \\ \psi_-^{(0)}\end{pmatrix} = W^{(0)}\begin{pmatrix}\psi_+^{(0)} \\ \psi_-^{(0)}\end{pmatrix} \tag{4.13}$$

or
$$\hat{H}_{10}\psi_+^{(0)} = W^{(0)}\psi_+^{(0)}, \tag{4.13a}$$

$$\hat{H}_{20}\psi_-^{(0)} = W^{(0)}\psi_-^{(0)}, \tag{4.13b}$$

where
$$\hat{H}_{10} = -\frac{1}{2m}\hat{\nabla}^2 + \frac{e_i e_j}{r} - \frac{e_i g_j}{2mr^2}$$

$$\hat{H}_{20} = -\frac{1}{2m}\hat{\nabla}^2 - \frac{e_i e_j}{r} - \frac{e_i g_j}{2mr^2},$$

or

$$H_0 \psi_\pm^{(0)} = W^{(0)} \psi_\pm^{(0)}. \tag{4.13c}$$

This wave equation can be solved by introducing the total angular momentum operator as vector sum of the orbital angular momentum and the gauge field isotopic spin and spin as follows

$$\begin{aligned} J &= L + S, \\ L &= M + (T.\hat{r})\hat{r}, \\ M &= \hat{r} \times \left| P - \frac{\hat{r} \times T}{} \right), \end{aligned} \tag{4.14}$$

which satisfy the following eigenvalue equation for the angular momentum eigenfunction

$$\begin{matrix} \hat{J}^2 \\ \hat{L}^2 \\ \hat{J}_3 \\ \hat{T}^2 \end{matrix} \; Y_{k,l',m'}(\theta, \phi) = \begin{bmatrix} J(J+1) \\ l(l+1) \\ m_j \\ t(t+1) \end{bmatrix} Y_{k,l',m'}(\theta, \phi) \tag{4.15}$$

where $Y_{k,l',m'}(\theta, \phi)$ are dyon harmonics [23] and the radial function $\frac{U(r)}{r} = R(r)$ satisfy the equation

$$\psi = \frac{U(r)}{r} Y_{k,l',m'}(\theta, \phi). \tag{4.16}$$

Solving equation (4.13a) and (4.13b), we get the following energy eigenvalue for the system of an extended dyon spinning around another fermion.

$$E_N = -2(m_0^2/m)(e_i e_j)^2 \left[(2N+1) + \left\{ (2l+1)^2 + \left(\frac{e_i g_j}{m} \right)^2 \right\}^{-1/2} \right]^{-2}, \tag{4.17}$$

where $N = 0, 1, 2, \ldots$

and $\psi_\pm^0$ are wave functions simplified to $R_{nl}(r) Y_{k,l'm'}(\theta, \phi)$, where $Y_{k,l',m'}(\theta, \phi)$ are dyon harmonics. Thus, the Pauli wave function for spin up and down states are given by

$$\begin{aligned} (\psi_+^0)_P &= \psi_{(n,l',m_l,m_s=+1/2)} = R_{nl} Y_{k,l',m,} |\uparrow\rangle \\ &= \begin{pmatrix} R_{nl} Y_{k,m'l'} \\ 0 \end{pmatrix} \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} (\psi_-^0)_P &= \psi_{(n,l',m_l,m_s=-1/2)} \\ &= R_{nl} Y_{k,m',l'} |\downarrow\rangle \\ &= \begin{pmatrix} 0 \\ R_{nl} Y_{k,l',m'} \end{pmatrix}. \end{aligned} \tag{4.19}$$

In the absence of spin-orbit interaction both the wave function correspond to the same energy. In order to determine splitting due to spin orbit interaction, we should choose a representation in which $\hat{H}_1$ is diagonal :

$$(\phi_1)_P = \phi_{(n,l;j=1+1/2,m_j)}$$

$$= \sqrt{\frac{l+m_j+1/2}{2l+1}}\,\psi_{(n,l,m_l=m_j+1/2\, m_j=+1/2)}$$

$$+\sqrt{\frac{l-m_j+1/2}{2l+1}}\,\psi_{(n,l,m_l=m_j+1/2,m_j=-1/2)}$$

$$\begin{pmatrix} \sqrt{\frac{l+m_j+1/2}{2l+1}} & R_{nl}\,Y_{k,l,m_j-1/2} \\ \sqrt{\frac{l-m_j+1/2}{2l+1}} & R_{nl}\,Y_{k,l',m_j+1/2} \end{pmatrix} \tag{4.20}$$

Similarly, we can write $(\phi_2)_P = \phi\,(n, l, j = l - 1/2, m_j)$. Then the first order perturbation due to spin-orbit interaction would be given by

$$W_s^{(1)} = \int d\tau \hat{\phi}^+ (\hat{H}_1')_P\, \phi$$

$$= \frac{e_i e_j}{4m_0^2}\int d\tau \frac{1}{r_3}\hat{\phi}^+\left[(\hat{J}^2)_P - (\hat{L}^2)_P - (\hat{S}^2)_P\right]\hat{\phi}$$

$$+\frac{e_i g_j}{8m_0^4}\int d\tau \frac{1}{r^4}\phi^+\left[(\hat{J}^2)_P - (\hat{L}^2)_P - (\hat{S}^2)_P\right]\hat{\phi} \tag{4.21}$$

or

$$W_s^{(1)} = \frac{e_i e_j}{4m_0^2}\left[\{j(j+1) - l(l+1) - 3/4\}\int d\tau(1/r^3)\right.$$

$$\times\left[\frac{l\pm m_j+1/2}{2l+1}\left|R_{nl}\right|^2\left|Y_{k,l,m_j-1/2}\right|^2\right.$$

$$\left.+\frac{l\mp m_j+1/2}{(2l+1)}\left|R_{nl}\right|^2\left|Y_{k,l,m_j+1/2}\right|^2\right]$$

$$+\frac{e_i g_j}{8m_0^4}\left[\{j(j+1)-l(l+1)-3/4\}\int d\tau(1/r^4)\right]\left[\frac{l\pm m_j+1/2}{2l+1}\right.$$

$$\left.\times\left|R_{nl}\right|^2\left|Y_{k,l,m_j-1/2}\right|^2+\frac{l\mp m_j+1/2}{2l+1}\left|R_{nl}\right|^2\left|Y_{k,l,m_j+1/2}\right|^2\right], \tag{4.22}$$

where the upper and lower signs corresponds to $j = l + (1/2)$ and $j = l - (1/2)$ respectively. After integration we get

$$W_r^{(1)} = \begin{cases} \dfrac{e_i e_j}{4m_0^2} l\left\langle\dfrac{1}{r^3}\right\rangle + \dfrac{e_i g_j}{8m_0^4} l\left\langle\dfrac{1}{r^4}\right\rangle & \text{for } j = l+(1/2) \\ -\dfrac{e_i e_j}{4m_0^2}(l+1)\left\langle\dfrac{1}{r^3}\right\rangle + \dfrac{e_i g_j}{8m_0^4} - (l+1)\left\langle\dfrac{1}{r^4}\right\rangle & \text{for } j = l-(1/2) \end{cases} \quad (4.23)$$

where $\left\langle\dfrac{1}{r^3}\right\rangle = \displaystyle\int_0^\infty \dfrac{1}{r^3}|R_{nl}|^2 r^2 dr = \dfrac{1}{n^3 l\{l+(1/2)\}(l+1)}\cdot\dfrac{1}{a_0^3},$

$$\left\langle\frac{1}{r^4}\right\rangle = \int_0^\infty \frac{1}{r^4}|R_{nl}|^2 r^2 dr = \frac{3-5n^3\{l+(1/2)\}a_0^2}{n^5\{l-(1/2)\}\{l+(1/2)\}\{l+(3/2)\}a_0^4}. \quad (4.24)$$

The splitting of energy levels corresponding to equation (4.13a) is

$$W = W^{(0)} + W^{(1)}$$

$$W = \begin{cases} E_N - \dfrac{E_N e_i e_j}{2m_0^2 n^3(l+1)(2l+1)a_0^2} - \dfrac{E_N(e_i g_j)l[3-5n^3\{l+(1/2)\}]a_0^2}{m_0^4 n^5(2l-1)(2l+1)(2l+3)a_0^4} & \text{for } j = l+(1/2) \\ E_N + \dfrac{E_N e_i e_j}{2m_0^2 nl(2l+1)a_0^2} - \dfrac{E_N(e_i g_j)l[3-5n^3\{l+(1/2)\}]a_0^2}{m_0^4 n^5(2l-1)(2l+1)(2l+3)a_0^4} & \text{for } j = l-(1/2) \end{cases} \quad (4.25)$$

Similarly, we get another set of equations describing splitting of energy levels corresponding to eq. (4.10b) with the sign of $e_i e_j$ reversed. E_N for both the cases is given by eq. (4.17) and Bohr radius a_0 for system is given as

$$a_0 = \frac{(e_i g_j)^2 + 1}{m m_0 e_i e_j}. \quad (4.26)$$

Eq. (4.25) gives the splitting in the energy levels corresponding to quantum number n for $j = l + 1/2$ and $j = l - 1/2$ respectively. It shows that *ad hoc* introduction of spin modifies the usual energy eigenvalue and eigenfunction of bound state of a dyon and fermion [20,21]. The bound state of an extended dyon and fermion is further modified due to presence of massive field which play major role in forming the bound state. The *ad hoc* introduction of spin becomes important since the relativistic Dirac equation for bound state of a fermion and extended dyon can not be solved exactly due to the presence of a term vanishing more rapidly than r^{-1} in the interaction potential of the system.

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