*Indian J. Phys.* **73A** (4), 481–491 (1999)

**IJP A**
— an international journal

# An exact solution of the Ginzburg-Landau equation for superconductivity

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*Received 13 November 1998, accepted 10 December 1998*

**Abstract** : An exact solution is found for the Ginzburg-Landau equations for zero magnetic field and at the critical temperature, with the superconductor being a long cylinder with a 'hole' around the axis. Some properties of the solution are discussed.

**Keywords** : Ginzburg-Landau equations, superconductivity

**PACS No.** : 74.20.De

## 1. Introduction

The Ginzburg-Landau equations for superconductivity [1,2] can be derived by varying the action

$$S = \int d^3 x G_s, \tag{1.1}$$

where $G_s$ is the microscopic Gibbs free energy density given by

$$G_s = F_s - (4\pi)^{-1} \boldsymbol{h} \cdot \boldsymbol{H} \tag{1.2}$$

with $F_s$ as the free energy density given by

$$F_s = F_{n_0} + a|\psi|^2 + \frac{1}{2}b|\psi|^4 + (2m^*)^{-1}\left|\left(-i\hbar\nabla + \frac{e^*}{c}A\right)\psi\right|^2 + \frac{1}{8\pi}\boldsymbol{h}^2. \tag{1.3}$$

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Here, $F_{n_0}$ is the free energy density in the normal state in zero magnetic field, $\psi(x)$ is a complex order parameter, $h$ is the magnetic field with $A$ the corresponding vector potential given by

$$h = \text{curl}\, A. \tag{1.4}$$

$H$ is a uniform external magnetic field. The constants $a$ and $b$ are temperature-dependent phenomenological constants, and $e^*$, $m^*$ are related to the charge $e$ and mass $m$ of the electron as follows :

$$e^* = 2e, \quad m^* = 2m. \tag{1.5}$$

The requirement that the variations of $S$ given by (1.1) with respect to $A$ and $\psi$ vanish leads to the following two equations [1,3] :

$$(2m^*)^{-1}\left(-i\hbar\nabla + \frac{e^*}{c}A\right)^2 \psi + a\psi + b|\psi|^2 \psi = 0, \tag{1.6a}$$

$$(c/4\pi)\,\text{curl}\, h = -(e^*\hbar/2m^* i)(\psi^*\nabla\psi - \psi\nabla\psi^*) - (e^{*2}/m^* c)|\psi|^2 A, \tag{1.6b}$$

provided that $A$, $\psi$ satisfy the following boundary conditions :

$$\hat{n}.\left(-i\hbar\nabla + \frac{e^*}{c}A\right)\psi = 0, \tag{1.7a}$$

$$\hat{n}.(h - H) = 0, \tag{1.7b}$$

where $\hat{n}$ is the normal to the surface of the superconductor. The supercurrent $j$ is given as follows :

$$j = -(e^*\hbar/2m^* i)(\psi^*\nabla\psi - \psi\nabla\psi^*) - (e^{*2}/m^* c)|\psi|^2 A. \tag{1.8}$$

The physical interpretation of the complex order parameter $\psi$ is that $|\psi|^2$ is proportional to the number density of the superconducting electrons, so that the current $j$ derived from $\psi$ given by (1.8) is that of the superconducting electrons. When $\psi \equiv 0$, the material is in the normal state.

## 2. The new solution

We assume magnetic fields to be absent :

$$h = H = 0 \tag{2.1}$$

and set $a = 0$ (the meaning of the latter will be explained later). We then get the following single equation for the order parameter $\psi$ :

$$-\sigma\nabla^2\psi + b|\psi|^2\psi = 0 \tag{2.2}$$

with $\sigma = \hbar^2/2m^*$. Consider now cylindrical polar coordinates $(r, \theta, z)$ and let $\psi$ have the following form :

$$\psi(r, \theta) = u(r)e^{in\theta} \tag{2.3}$$

with $n$ a constant to be determined, so that $\psi$ is independent of $z$. With the latter property we get

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi$$

$$= \left(u'' + \frac{1}{r}u' - \frac{n^2}{r^2}u\right)e^{in\theta}, \qquad (2.4)$$

where $u' \equiv du/dr$, *etc*. Eq. (2.2) then reduces to the following :

$$-\sigma\left(u'' + \frac{1}{r}u' - \frac{n^2}{r^2}u\right) + bu^3 = 0 \qquad (2.5)$$

which is a nonlinear differential equation. There exist no standard methods for solving such equations. We proceed to demonstrate that an exact solution of (2.5) can be found of the following form :

$$u(r) = u_0\left\{u_1 r^{-1} + r^{-\alpha}(\eta + r^{\beta})^{-1}\right\}, \qquad (2.6)$$

where $u_0$, $u_1$, $\alpha$, $\beta$ and $\eta$ are suitable constants. From (2.6) we get

$$u'(r) = u_0\left\{-u_1 r^{-2} - \alpha r^{-\alpha-1}\left(\eta + r^{\beta}\right)^{-1} - \beta r^{-\alpha+\beta-1}\left(\eta + r^{\beta}\right)^{-2}\right\}, \qquad (2.7a)$$

$$u''(r) = u_0\Big\{2u_1 r^{-3} + \alpha(\alpha+1)r^{-\alpha-2}\left(\eta + r^{\beta}\right)^{-1}$$

$$+\beta(2\alpha-\beta+1)r^{-\alpha+\beta-2}\left(\eta + r^{\beta}\right)^{-2} + 2\beta^2 r^{-\alpha+2\beta-2}\left(\eta + r^{\beta}\right)^{-3}\Big\}, \qquad (2.7b)$$

$$u^3(r) = u_0^3\Big\{u_1^3 r^{-3} + 3u_1^2 r^{-\alpha-2}\left(\eta + r^{\beta}\right)^{-1}$$

$$+ 3u_1 r^{-2\alpha-1}\left(\eta + r^{\beta}\right)^{-2} + r^{-3\alpha}\left(\eta + r^{\beta}\right)^{-3}\Big\}. \qquad (2.7c)$$

We substitute from (2.6), (2.7a–c) into (2.5) and set $\alpha + \beta = 1$ (this follows from an intermediate step which we omit) to get the following equation, after cancelling a factor $u_0$ :

$$u_1\left(-\sigma + \sigma n^2 + u_0^2 u_1^2 b\right)r^{-3} + \left(-\sigma\alpha^2 + \sigma n^2 + 3u_0^2 u_1^2 b\right)r^{-\alpha-2}\left(\eta + r^{\beta}\right)^{-1}$$

$$+\left\{-\sigma(1-\alpha)(3\alpha-1) + 3u_1 b\right\}r^{-2\alpha-1}(\eta + r^{\beta})^{-2}$$

$$+\left\{-2\sigma(1-\alpha)^2 + b\right\}r^{-3\alpha}(\eta + r^{\beta})^{-3} = 0, \qquad (2.8)$$

where for convenience of writing we have retained $\beta = 1 - \alpha$ in the expressions $(\eta + r^{\beta})$. It is clear that (2.8) will be satisfied if the constants $\sigma, n, u_0, u_1, \alpha, b$ satisfy the following equations :

$$\sigma(n^2 - 1) + u_0^2 u_1^2 b = 0, \qquad (2.9a)$$

$$\sigma(n^2 - \alpha^2) + 3u_0^2 u_1^2 b = 0, \qquad (2.9b)$$

$$\sigma(\alpha-1)(3\alpha-1) + 3u_1 b = 0, \tag{2.9c}$$

$$2\sigma(1-\alpha)^2 - b = 0. \tag{2.9d}$$

Eqs. (2.9a,b) imply

$$2n^2 + \alpha^2 = 3, \tag{2.10}$$

while (2.9c,d) imply

$$b = 2\sigma(1-\alpha)^2, \quad 6u_1 = (3\alpha-1)/(1-\alpha). \tag{2.11}$$

The constant $u_0$ can be determined from (2.9a) or (2.9b). Although the values $n^2 = 1 = \alpha^2$ satisfy (2.10), this implies through (2.9a), that one of $u_0 . u_1, b$ must be zero and the corresponding solution is not of interest. We therefore choose the following solution of (2.10) :

$$n = \pm\frac{1}{3}, \quad \alpha = \pm 5/3. \tag{2.12}$$

We get two cases which we call Case *I* and *II*, as follows :

*Case I :* $\alpha = -5/3$. This leads to the following values of $b, u_1, \beta$ :

$$b = 128\sigma/9, \quad u_1 = -3/8, \quad \beta = 8/3, \quad u_0 = \pm 2/3 \tag{2.13}$$

so that $u(r)$ is given by

$$u(r) = \pm(2/3)\left\{-(3/8)r^{-1} + r^{5/3}(\eta + r^{8/3})^{-1}\right\}. \tag{2.14}$$

We shall see below that the boundary condition (1.7a) (with $A = 0$) is satisfied in this case if the material of the superconductor lies between $r = r_1$ and $r = r_2$ such that the derivative of $u(r)$ vanishes at these two values. From (2.14) this means that $r_1$, $r_2$ are roots of

$$(3/8)r^{-2} + (5/3)r^{2/3}\left(\eta + r^{8/3}\right)^{-1} - (8/3)r^{10/3}\left(\eta + r^{8/3}\right)^{-2} = 0. \tag{2.15}$$

The only positive solution of this equation is the following one, assuming $\eta$ to be positive :

$$r = r_1 = \left\{\left(29 + 2\sqrt{244}\right)\eta/15\right\}^{3/8} \tag{2.16}$$

which can therefore be taken as the value of $r_1$. It is also clear that $u'(r)$ vanishes for very large $r$, so that $r_2$ can be taken as some very large value of $r$.

*Case II :* $\alpha = +5/3$. In this case the values of $b$, $u_1$, $\beta$ are as follows :

$$b = 8\sigma/9, \quad u_1 = -1, \quad \beta = -2/3, \quad u_0 = \pm 1 \tag{2.17}$$

and the corresponding solution for $u(r)$ is as follows :

$$u(r) = \pm\left\{-r^{-1} + r^{-5/3}\left(\eta + r^{-2/3}\right)^{-1}\right\}$$

$$= \mp\, \eta r^{-1/3}\left(1 + \eta r^{2/3}\right)^{-1}. \tag{2.18}$$

In this case, one does not get a positive value of $r$ for which $u'(r)$ vanishes, unless $\eta$ is negative; but in the latter case, $u(r)$ is infinite for a value of $r$ which is greater than that for which $u'(r)$ is zero, implying that $\psi$ has a singularity in the material of the superconductor, which is unphysical. However, a somewhat different interpretation of the configuration, *etc*., may be possible which will rectify the situation.

The boundary condition (1.7a) in this case reads as follows :

$$(\boldsymbol{n}.\nabla)\psi = 0. \tag{2.19}$$

If $\boldsymbol{i}_r$ and $\boldsymbol{i}_\theta$ are unit vectors in the direction of increasing $r$ and $\theta$ respectively, we have

$$\nabla = \boldsymbol{i}_r \frac{\partial}{\partial r} + \boldsymbol{i}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{k}\frac{\partial}{\partial z}, \tag{2.20}$$

where $\boldsymbol{k}$, as usual, is a unit vector in the $z$-direction. Let the unit normal $\hat{\boldsymbol{n}}$ to the surface be given by

$$\hat{\boldsymbol{n}} = n_r \boldsymbol{i}_r + n_\theta \boldsymbol{i}_\theta + n_z \boldsymbol{k}. \tag{2.21}$$

Then (2.19) reads as follows :

$$\left(n_r \frac{\partial}{\partial r} + \frac{1}{r} n_\theta \frac{\partial}{\partial \theta} + n_z \frac{\partial}{\partial z}\right)\left(u(r)e^{in\theta}\right) = 0 \tag{2.22}$$

yielding the following relation :

$$n_r u'(r) + inr^{-1} n_\theta u = 0. \tag{2.23}$$

This is satisfied if $n_\theta = 0$ and if $u'(r) = 0$; we have used the latter condition in (2.15). Thus, we envisage the material of the superconductor to have a cylindrical 'hole' from $r = 0$ to $r = r_1$ where $r_1$ is given by (2.16). This also avoids the singularity in $u(r)$ given by (2.14) at $r = 0$.

## 3. Discussion of the solution

The full solution is given as follows :

$$\psi(r,\theta) = u(r)e^{in\theta} = \pm(2/3)\left\{-(3/8)r^{-1} + r^{5/3}(\eta + r^{8/3})^{-1}\right\}\exp\left(\pm\frac{1}{3}i\theta\right). \tag{3.1}$$

This satisfies the Ginzburg-Landau equation exactly for $\boldsymbol{A} = \boldsymbol{0} = \boldsymbol{H}$ and for $a = 0$, as well as the boundary condition. However, although $|\psi|^2 = u^2(r)$ is well-defined and single valued in the material of the superconductor, $\psi(r,\theta)$ itself is not single-valued, as is evident from (3.1). We shall come back to this point.

It is well known [1] that the constant $a$ in (1.3) and (1.6a) is in general dependent on the temperature $T$ and has the behaviour (with $a' \neq 0$ at $T = T_c$)

$$a(T) = (T - T_c)a' \tag{3.2}$$

near the critical temperature $T_c$. Moreover, by considering the case of a one-dimensional geometry, one can derive a natural scale of length for spatial variation of the order parameter given by

$$\xi(T) = \left[\hbar^2/2m^*|a(T)|\right]^{1/2} = \hbar/\left[2m^*(T_c - T)a'\right]^{1/2}, \tag{3.3}$$

the second expression obtaining near and below the critical temperature. This length is the so-called Ginzburg-Landau coherence length [1]. Thus this coherence length tends to infinity as $T \to T_c$, while $a(T)$ approaches zero. Therefore the above solution describes a situation very near or at the critical temperature $T = T_c$.

The constant $b$ can be related to the Ginzburg-Landau parameter $\kappa$ given by

$$\kappa = \lambda(T)/\xi(T), \tag{3.4}$$

where $\lambda(T)$ is the penetration length, so that $\kappa$ is independent of the temperature near $T_c$, since $\lambda(T)$ also varies as $(T_c - T)^{-1/2}$ near $T = T_c$. The relation between $\kappa$ and $b$ is as follows :

$$\kappa = \left(m^*c/\hbar e^*\right)\left(b/2\pi\right)^{1/2} \tag{3.5}$$

We next determine the supercurrent $J$ corresponding to the above solution. With $A = 0$ and the operator $\nabla$ given by (2.20) for cylindrical polar coordinates, we get from (1.8) :

$$\begin{aligned} J = & -\left(e^*\hbar/2m^*i\right)\Big\{u(r)e^{\mp i\theta/3}\left(i_r\frac{\partial}{\partial r} + i_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + k\frac{\partial}{\partial z}\right)\left(u(r)e^{\pm i\theta/3}\right) \\ & \qquad -u(r)e^{\pm i\theta/3}\left(i_r\frac{\partial}{\partial r} + i_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + k\frac{\partial}{\partial z}\right)\left(u(r)e^{\mp i\theta/3}\right)\Big\} \\ = & \mp\left(e^*\hbar/3m^*\right)r^{-1}u^2(r)i_\theta. \end{aligned} \tag{3.6}$$

The current, therefore, as expected, is circular with the axis $r = 0$ as centre, the solutions in (3.1) with factors $\exp\left(\pm\frac{1}{3}i\theta\right)$ corresponding to equal and opposite currents. We shall consider later the question of flux quantization.

We consider now the total free energy per unit length in the $z$-direction. Setting $a' = 0$ and $F_{n0} = 0$ (so that the normal energy is ignored) in (1.3), we have

$$\begin{aligned} F_s = & \frac{1}{2}b|\psi|^4 + \left(\hbar^2/2m^*\right)|\nabla\psi|^2 \\ = & \frac{1}{2}bu^4 + \sigma\left|\left(i_r\frac{\partial}{\partial r} + i_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)ue^{\pm i\theta/3}\right|^2 \\ = & \frac{1}{2}bu^4 + \sigma\left(u'^2 + \frac{1}{9r^2}u^2\right). \end{aligned} \tag{3.7}$$

For convenience we write $u(r)$, $u'(r)$ given by (3.1) as follows :

$$u(r) = \pm(2/3)r^{-1}\left\{-(3/8) + r^{8/3}(\eta + r^{8/3})^{-1}\right\}, \tag{3.8a}$$

$$u'(r) = \pm(2/3)r^{-2}\left\{(3/8) + (5/3)r^{8/3}(\eta + r^{8/3})^{-1} - (8/3)r^{16/3}(\eta + r^{8/3})^{-2}\right\}, \tag{3.8b}$$

so that the total free energy $E$ per unit length in the $z$-direction is given by (substituting for $b$ from (3.5)) :

$$\begin{aligned} E &= 2\pi\int rF_s\,dr \\ &= 2\pi\int r\left\{\frac{1}{2}bu^4 + \sigma\left(u'^2 + \frac{1}{9r^2}u^2\right)\right\}dr \\ &= \frac{8\pi\sigma}{9}\int\frac{1}{r^3}\left\{\frac{256}{81}\left(-\frac{3}{8}+\frac{r^{8/3}}{\eta+r^{8/3}}\right)^4 + \left(\frac{3}{8}+\frac{5r^{8/3}}{3(\eta+r^{8/3})}-\frac{8r^{16/3}}{3(\eta+r^{8/3})^2}\right)^2\right. \\ &\quad \left. + \frac{1}{9}\left(-\frac{3}{8} + \frac{r^{8/3}}{\eta+r^{8/3}}\right)^2\right\}dr. \end{aligned} \tag{3.9}$$

In Appendix A, we show how to calculate this expression explicitly. The expression itself is quite complicated and can be written down explicitly if desired, but we have omitted it. The important point to notice, however, is that this expression is finite when the limits are taken in (3.9) as $r_1$ given by (2.16) and another larger value of $r$, no matter how large, so that the energy per unit length is finite, no matter how large the outer radius. We thus have here an exact solution of the Ginzburg-Landau equation satisfying the appropriate boundary condition and with finite energy per unit length of the 'hollow' cylinder that is required by the geometry.

We now come to the 'unphysical' aspect of the solution mentioned earlier, namely, that the order parameter $\psi(r, \theta)$ given by (3.1) is not single valued. We make some remarks in this connection. Sometimes a suitable physical interpretation of a solution emerges much later; such is the case, for example, of several solutions in general relativity. A possible physical interpretation of the solution found here may be connected with flux quantization. This can probably be seen more clearly in the presence of a magnetic field, when the number $n$ in (2.3), in suitable circumstances, may be associated with the quantization of charge. The solution (3.1) would then seem to imply that charge quantization may occur in one-third integral values. Even if such a connection can be established, there is no reason this has anything to do with quarks but such a possibility cannot be ruled out. The lack of single-valuedness might possibly be acceptable because of the geometry, in that the region occupied by the superconductor is not simply connected.

This also may have connections with the Bohm-Aharonov effect. We believe these questions are worth investigating, and may give the new solution more interest than is at present apparent. A preliminary investigation with magnetic field has been carried out [4] which, although of some mathematical interest, has not yielded results of sufficient physical interest. Recent detailed studies of magnetic fields in superconductors [5] may possibly be examined, among other methods, through solutions of the Ginzburg-Landau equation such as the one found here (see also Ref. 6).

We note that the solution is valid for $a = 0$, that is, *at* the critical temperature $T = T_c$. One can investigate the behaviour of the solution near the critical temperature, that is, for small $a$, using a suitable expansion. Referring to the new solution (3.1) as $\psi_0\,(r,\,\theta)$, and the solution for non-zero $a$ as $\psi_a\,(r,\,\theta)$ one can expand the latter as follows [3] :

$$\psi_a(r,\theta) = \psi_0(r,\theta) + a\psi_1(r,\theta) + a^2\psi_2(r,\theta) + \cdots. \tag{3.10}$$

The terms proportional to $a$, $a^2$, *etc.* can be obtained, in principle, in terms of $\psi_0\,(r,\,\theta)$, and one can examine the behaviour of the corresponding solution near the critical point.

Another possible difficulty with the new solution is that there may be some simple situations, such as $\psi(r,\,\theta) =$ constant, for which the energy is lower than that found in the new solution, resulting in instability. However, because of the special geometrical configuration, and the manner in which the state corresponding to the new solution is attained, the simpler states with less energy may not be accessible in some circumstances, thus providing at least quasi-stability to the state given by the new solution. These questions are under investigation.

In Appendix A, we carry out the evaluation of the energy mentioned earlier, and in Appendix B, we point out a similarity of the present problem with that of a static solution of a relativistic massless complex scalar field with quartic self-interaction.

**Acknowledgment**

I am grateful to Prof. J C Taylor of DAMTP, Cambridge, for helpful comments on an earlier version of the paper. I thank Prof. H Araki, Prof. N Nakanishi, Dr. I Ojima and their colleagues for their kind hospitality at the Research Institute for Mathematical Sciences at Kyoto, where a first draft of the paper was prepared during a visit in 1994, and the paper completed during a second visit in 1996. I am grateful to Prof. H Ezawa of Gakushuin University, Tokyo, for pointing out the relevance of Ref. 5; I also thank him and Prof. Araki for their interest in this problem. Some preliminary work on this problem was done while I was visiting ICTP, Trieste, Italy, during 1991. I thank Prof. Abdus Salam, Prof. L Bertocchi and Prof. G Denardo for hospitality at ICTP. This paper is dedicated to the memory of Prof. Abdus Salam, an outstanding scientist and a great philanthropist.

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## Appendix A

In this Appendix we show how to evaluate the expression on the right hand side of (3.9). Expanding the powers and simplifying, we get

$$E = (8\pi\sigma/9)\int \frac{1}{r^3}\left\{\frac{7}{32} + \frac{r^{8/3}}{2(\eta + r^{8/3})} + \frac{32r^{16/3}}{9(\eta + r^{8/3})^2} - \frac{368r^8}{27(\eta + r^{8/3})^3} + \frac{832r^{32/3}}{81(\eta + r^{8/3})^4}\right\}dr. \quad \text{(A1)}$$

We next calculate the various integrals in (A1). Consider the indefinite integral

$$I = \int \frac{r^{-1/3}dr}{(\eta + r^{8/3})} \quad \text{(A2)}$$

Write $\eta \equiv \zeta^4$ and transform to $R = r^{2/3}$ to get $I = (3/2)I'$, with

$$I' = \int \frac{dR}{\zeta^4 + R^4} = \int \frac{dR}{(\zeta^2 + \sqrt{2}\zeta R + R^2)(\zeta^2 - \sqrt{2}\zeta R + R^2)}. \quad \text{(A3)}$$

Resolving into partial fractions and integrating, we get

$$I' = \frac{1}{4\sqrt{2}\zeta^3}\log\frac{\zeta^2 + \sqrt{2}\zeta R + R^2}{\zeta^2 - \sqrt{2}\zeta R + R^2} + \frac{1}{2\sqrt{2}\zeta^3}\left\{\tan^{-1}\left(\frac{R + \zeta/\sqrt{2}}{\zeta/\sqrt{2}}\right) + \tan^{-1}\left(\frac{R - \zeta/\sqrt{2}}{\zeta/\sqrt{2}}\right)\right\}. \quad \text{(A4)}$$

In terms of $\eta$, this result can be written as follows

$$I' = \frac{1}{4\sqrt{2}\eta^{3/4}}\log\frac{\eta^{1/2} + \sqrt{2}\eta^{1/4}R + R^2}{\eta^{1/2} - \sqrt{2}\eta^{1/4}R + R^2} + \frac{1}{2\sqrt{2}\eta^{3/4}}\left\{\tan^{-1}\left(\frac{\sqrt{2}R + \eta^{1/4}}{\eta^{1/4}}\right) + \tan^{-1}\left(\frac{\sqrt{2}R - \eta^{1/4}}{\eta^{1/4}}\right)\right\}. \quad \text{(A5)}$$

The other integrals in (A1) can be written as follows

$$I_1 = \int \frac{r^{7/3}dr}{\left(\eta + r^{8/3}\right)^2}, \quad I_2 = \int \frac{r^5 dr}{\left(\eta + r^{8/3}\right)^3}, \quad I_3 = \int \frac{r^{23/3}dr}{\left(\eta + r^{8/3}\right)^4}. \tag{A6}$$

The same transformations as above yield the following integrals :

$$I_1 = \int \frac{R^4 dR}{\left(\eta + R^4\right)^2}, \quad I_2 = \int \frac{R^8 dR}{\left(\eta + R^4\right)^3}, \quad I_3 = \int \frac{R^{12} dR}{\left(\eta + R^4\right)^4}. \tag{A7}$$

Noting that

$$\int \frac{dR}{\left(\eta + R^4\right)^2} = -\frac{\partial I'}{\partial \eta}, \tag{A8a}$$

$$\int \frac{dR}{\left(\eta + R^4\right)^3} = \frac{1}{2}\frac{\partial^2 I'}{\partial \eta^2}, \tag{A8b}$$

$$\int \frac{dR}{\left(\eta + R^4\right)^4} = -\frac{1}{6}\frac{\partial^3 I'}{\partial \eta^3}, \tag{A8c}$$

we find that

$$I_1 = I' + \eta \frac{\partial I'}{\partial \eta}, \tag{A9a}$$

$$I_2 = I' + 2\eta \frac{\partial I'}{\partial \eta} + \frac{1}{2}\eta^2 \frac{\partial^2 I'}{\partial \eta^2}, \tag{A9b}$$

$$I_3 = I' + 3\eta \frac{\partial I'}{\partial \eta} + \frac{3}{2}\eta^2 \frac{\partial^2 I'}{\partial \eta^2} + \frac{1}{6}\eta^3 \frac{\partial^3 I'}{\partial \eta^3}. \tag{A9c}$$

Thus the integrals $I_1$, $I_2$, $I_3$ can be worked out explicitly with the aid of (A9a–c) and (A5) and so $E$ can be calculated. As these expressions are quite complicated, we omit these, except to note that these integrals tend to zero faster than $I'$ as $R$ (or $r$) tends to infinity. This property leads to the energy being finite in the sense mentioned earlier.

## Appendix B

Consider a massless complex scalar relativistic field interacting with itself through the following Lagrangian density :

$$\pounds = \partial_\mu \phi \partial^\mu \phi^* - \lambda(\phi^* \phi)^2. \tag{B1}$$

The equations of motion are as follows :

$$\Box \phi + 2\lambda(\phi^* \phi)\phi = 0 \tag{B2}$$

and its complex conjugate, where $\Box \equiv \partial_\mu \partial^\mu$ is the d'Alembertian operator. Suppose we consider a static situation where the fields are time-independent. Eq (B2) then reduces to

$$\nabla^2 \phi - 2\lambda(\phi^* \phi)\phi = 0. \tag{B3}$$

Consider now the field configuration to be such that $\phi$ has the following form

$$\phi(r, \theta) = v(r) e^{in'\theta} \tag{B4}$$

where, as before, $(r, \theta, z)$ are cylindrical polar coordinates, $\phi$ is independent of $z$ and $n'$ is a constant, and $v$ is a function of $r$ only. With the use of (2.4) we see that (B3) reduces to the following equation :

$$v'' + \frac{1}{r} v' - \frac{n'^2}{r^2} v - 2\lambda v^3 = 0. \tag{B5}$$

This is the same equation as (2.5) if we identify $2\lambda$ with $b$, and so solutions can be found as above. In this case, the solution corresponding to (2.18) has some relevance; we write this solution as follows :

$$v(r) = kr^{-1/3}\left(1 + kr^{2/3}\right)^{-1} \tag{B6}$$

where we take the constant $k$ to be positive. For this solution if the energy per unit length in the $z$-direction is calculated, including the region around $r = 0$, it is found to be infinite (singular). However, this energy is *less* singular than the electrical energy per unit length of an infinite line electric charge, so that this property itself of the new solution need not deny its use in some circumstances, somewhat like the line charge.