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Codazzi's equation in four dimensions

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Abstract : We obtain for R_4 the most general form of the tensor $A^{ij} = A^{ij}(g_{ab}, g_{ab,c}, g_{ab,cd})$ satisfying the Codazzi's equation $A_{ij;r} = A_{ir;j}$. We find that this solution exists only if the Weyl tensor fulfils the Bianchi identities. The results are immediately applied to the local and isometric embedding of the space-time into E_5

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We consider Codazzi's equation in a Riemannian 4-space R_4 :

$$A_{ij;r} = A_{ir;j}, \tag{1}$$

with ; denoting covariant derivative for the case of intrinsic rigidity when A_{ij} depends only on the metric tensor g_{ic} and on its first and second partial derivatives (denoted by r) :

$$A^{ij} = A^{ij}(g_{ab}, g_{ab,c}, g_{ab,cd}). \tag{2}$$

By means of Lovelock's theorem [1–3] we will find in Section 2 that the most general form of the tensor A_{ic} fulfilling (1,2), is a linear combination of the metric and Ricci tensors R_{ic}. It will turn out that the conformal tensor C_{ijrq} of the corresponding space-time will satisfy the Bianchi identities [4], *i.e.* R_4 is a C-space [5–7] :

$$C^{r}_{jiq;r} = 0 \tag{3}$$

It means that eq. (3) is a necessary condition for eqs. (1) and (2) to have simultaneous solution in four dimensions.

The results that we obtain are then applied in Section 3 to space-times embedded into E_5, *i.e.* to 4-spaces of class one [4,8–10]. This is so, because in them, there exists the second fundamental form tensor b_{ac} fulfilling (1) [11–13].

Our task is to determine, in the four-dimensional space-time, the general structure of every tensor A_{ij} with the properties (1) and (2). We note that the deduction neither requires that A_{ij} be symmetric nor uses its linear dependence on the second derivatives $g_{ab,cd}$.

Lovelock [1] proved the powerful theorem :

"*The most general tensor B^{ic} in R_4 satisfying*

$$B^{ij} = B^{ij}(g_{ab}, g_{ab,c}, g_{ab,cd}) \text{ and } B^{ij}{}_{;i} = 0, \tag{4}$$

is given by

$$B^{ij} = \alpha G^{ij} + \beta g^{ij}, \tag{5}$$

α and β are constants, and $G_{ic} = R_{ic} - \frac{R}{2} g_{ic}$ is the Einstein tensor [4]"

On the other hand eq. (1) implies the relation

$$A^{ij}{}_{;i} = A^{;j} \text{ with } A \equiv A^{r}_{r}, \tag{6}$$

which together with (2) shows that the tensor $B^{ij} = A^{ij} - Ag^{ij}$ satisfies (4) and, in consequence, (5) leads to

$$A^{ij} - Ag^{ij} = \alpha G^{ij} + \beta g^{ij}. \tag{7}$$

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—Editor

Our final expression is now

$$A^{ij} = \alpha R^{ij} - \frac{1}{6}(\alpha R + 2\beta) g^{ij}, \tag{8}$$

with $A = \frac{1}{3}(\alpha R - 4\beta)$, (9)

which solves the Codazzi's equation with intrinsic rigidity in R_4. We now see that the tensor A_{ij} is symmetric and linear in the $g_{ab,cd}$.

If (8) is substituted into (1) it follows (when $\alpha \neq 0$) that :

$$\left(R_{ij} - \frac{R}{6} g_{ij}\right)_{,k} = \left(R_{ik} - \frac{R}{6} g_{ik}\right)_{,j} \tag{10}$$

which is equivalent to (3) [4,5] and therefore, the Weyl tensor satisfies the Bianchi identities. If $\alpha = 0$, we have the trivial case when A_{ij} is proportional to g_{ij} and C_{ijkq} is not required to satisfy identities (3).

We apply the result of the previous section to the local and isometric embedding of R_4 into E_5, just the case of intrinsic rigidity, *i.e.* when the second fundamental form b_{ac} depends only on the internal geometry of the space-time.

In fact, if R_4 is of class one, the Gauss-Codazzi equations [4,8-13]

$$R_{ijrt} = \epsilon (b_{ir} b_{jt} - b_{it} b_{jr}) \tag{11}$$

$$b_{ij,r} = b_{ir,j} \tag{12}$$

are verified, where $\epsilon = \pm 1$ and R_{aijc} is the corresponding curvature tensor. If we further impose intrinsic rigidity we have

$$b_{ij} = b_{ij}(g_{ab}, g_{ab,c}, g_{ab,cd}) \tag{13}$$

and eqs. (12) and (13) assure that b_{ij} satisfies (1), (2). Hence, the second fundamental form will have the structure [eqs. (8) and (9)] with the restriction (3)

$$b_{ij} = \alpha R_{ij} - \frac{1}{6}(\alpha R + 2\beta) g_{ij},$$

$$b = b^i_i = \frac{1}{3}(\alpha R - 4\beta), \tag{14}$$

where α and β are constants (with values determined by the proper R_4 in question).

The trivial case $\alpha = 0$ implies

$$b_{ij} = -\frac{\beta}{3} g_{ij} = \frac{b}{4} g_{ij}, \tag{15}$$

which after substitution in (11), leads to an space of constant curvature (DeSitter model) [4,14].

If $\alpha \neq 0$, eqs. (11) and (14) show that the metric and Ricci tensors are adequate to express the Riemann tensor in a simple way. We consider the task to find the Petrov type [4,15,16] of R_4 admissible in this $\alpha \neq 0$ case as an open problem. We hope to report a careful analysis somewhere else in the future. Our preliminary calculations suggest that there are no 4-spaces of class one with Petrov types III or N that fulfil (12,13); neither they fulfil (14), in consequence. However, other Petrov types may occur; in fact, the metric

$$ds^2 = \frac{1}{2\phi^2}(d\theta^2 + d\phi^2) - 2dr du \tag{16}$$

corresponds to an space-time of type D which satisfies (14) with $\beta = -2\alpha = -\sqrt{2}$ and $\epsilon = -1$.

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