Indian J. Phys. **74B** (5), 375–378 (2000)

I J P B
an international journal

Einstein-Maxwell metrics embedded into E_5

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*Received 15 February 1999, accepted 10 May 1999***

Abstract : It is shown that every Einstein-Maxwell space-time of class one (embedded into E_5) must have $b_i^i \neq 0$, where b_i^j is the corresponding second fundamental form

Keywords : Einstein-Maxwell metrics, general theory of relativity

PACS Nos. : 04 20 –q, 04 90 +e

1. Introduction

We study in this work, the conditions necessarily satisfied by a space-time locally and isometrically embedded into E_5 [1–6], whose curvature is produced by a Maxwell field according to the Einstein equations. In this context, Collinson [7] showed that

$$\text{"... such } R_4 \text{ must have Petrov type } N \text{ with null Faraday tensor } F_{ij} = -F_{ji}\text{".} \qquad (1)$$

We analyze (1) and prove that the corresponding second fundamental form b_i^j possesses a non zero trace, which is a result not found explicitly in [7,8].

2. Einstein-Maxwell R_4 of class one

We employ the notations and quantities from [1,3–6,9–11] This section considers several implications of Theorem (1) which are useful to prove that $b_i^i \neq 0$ which we take up in Section 3.

Eq. (1) establishes that a R_4 of class one, that is, embedded into E_5, has a Weyl tensor C_{ijpq} of type N in the Petrov classification [1,12,13]. Then, according to Sachs [1,14–16], there exists a real null vector (with a four fold degenerate principal direction) $\tilde{k}$, fulfilling the condition

$$C_{ijpq}\tilde{k}^q = 0 \qquad (2)$$

On the other hand, (1) also assures that F_{ij} is a null tensor (its two invariants are equal to zero), which implies that there exists [17,18] a well defined null vector k^r pointing to the future and a non unique space like vector A^r such that :

$$\begin{aligned} &F_{ij} = k_i A_j - k_j A_i, && F_{ab}F^{ab} = {}^*F_{ab}F^{ab} = 0, \\ &F_{ic}k^c = {}^*F_{ic}k^c = 0, && k^r A_r = 0, \\ &A^r A_r = 1, && R_{ic} = k_i k_c, \\ &R_{ic}k^c = R_{ic}A^c = 0, && R = R_c^c = 0, \end{aligned} \qquad (3)$$

where R_{ij} is the Ricci tensor and ${}^*F_{ab}$ is the dual [18,19] of the Faraday tensor. Relations (3) do not change at all if we add to A^r any term of the form Ck^r, with C a constant. This suggests that the vector is not a unique one.

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**The publication of this manuscript has been delayed due to technical problems and considerable time taken by the authors to correct the final version This may kindly be noted —Editor

Further, the Mariot [20,21] and Robinson [22] Theorem (see [23] also) assures us that

"The congruence associated with k^i is geodesic and without deformation". (4)

hence, from a result given by Goldberg Sachs [1,24,25] it is obtained

"k^i is a direction of Debever-Penrose". (5)

which together with (2) leads to the identification :

$$\tilde{k}^i = k^i, \tag{6}$$

which means a notable alignment of the principal directions of the Faraday, Ricci and Weyl tensors.

We now employ the definition of the conformal tensor in terms of the curvature tensor R_{ijkm} :

$$C_{inmj} = R_{inmj} + \frac{1}{2}\left(R_{in}g_{mj} + R_{mj}g_{in} - R_{mn}g_{ij} - R_{ij}g_{mn}\right) - \frac{R}{6}\left(g_{ij}g_{mn} - g_{in}g_{mj}\right) \tag{7}$$

to see that from (2,3,6) follows immediately

$$R_{ijqr}k^r = 0. \tag{8}$$

On the other hand, it is known that the Gauss equation [1,2,31]

$$R_{ijkm} = \varepsilon_1\left(b_{ik}b_{jm} - b_{im}b_{jk}\right), \tag{9}$$

with corresponding Ricci tensor

$$R_{im} = \varepsilon_1\left(b_i^j b_{jm} - b b_{im}\right), \quad b \equiv b_i^i, \tag{10}$$

implies the important identity [3,4,26,27]

$$pb_{ij} = \frac{k_2}{48}g_{ij} - \frac{1}{2}R_{imnj}G^{mn}, \tag{11}$$

where $G_{ac} = R_{ac} - \frac{R}{2}g_{ac}$ is the Einstein tensor, k_2 is the contraction of the Riemann tensor with its double dual [1,5,28-30] given by

$$k_2 = {}^*R^{*ijcd}R_{ijcd} = -24\det\left(b_j^i\right), \tag{12}$$

and p is such that

$$p^2 = -\frac{\varepsilon_1}{6}\left(\frac{R}{24}k_2 + R_{imnj}G^{ij}G^{mn}\right) \geq 0, \tag{13}$$

$\varepsilon_1 = \pm 1$ being the indicator of the normal of R_4 with respect to E_5.

It is not a difficult task to see that (3,8,11,13) imply

$$p = 0, \tag{14}$$

$$k_2 = 0. \tag{15}$$

From (14) we conclude that

"An Einstein-Maxwell field embedded into E_5 is not intrinsically rigid", (16)

and (12,15) imply that the 4×4 matrix $\underset{\sim}{b} = \left(b_j^i\right)$ has no inverse. We point out that the result expressed in (16) is not found explicitly in Collinson [7].

It may also be proved (which will be worked out somewhere else) that our curved space-time has the 14 invariants of second order given by Debever (see refs. [32–34]) equal to zero.

We notice that from (3,8,14,15), the identity (11) takes the trivial form $0 = 0$ and leaves off any information concerning $\underset{\sim}{b}$; however, we can still study its characteristic polynomial in order to improve the analysis, this is the matter of the next section.

3. $\underset{\sim}{b}$ has trace different from zero

Substitution of matrix b_j^i into its characteristic polynomial, gives the equation [35–37]

$$b\underset{\sim}{b}^3 - \frac{\varepsilon_1 R}{2}\underset{\sim}{b}^2 - p\underset{\sim}{b} - \frac{k_2}{24}\underset{\sim}{I} = 0,$$

$$p = \frac{\varepsilon_1}{3}b^{ij}G_{ij} \tag{17}$$

which by use of (3,14,15) noticeably simplifies to

$$\underset{\sim}{b}^4 - b\underset{\sim}{b}^3 = 0, \tag{18}$$

and eq (10) is now rewritten into form

$$b_{ii}b^{ic}R_{cj} = 0 \tag{19}$$

Nevertheless, eq. (10) and the fact that $R_{ic}R_j^c = 0$ (see eq (3)) compels (19) to imply

$$b\left(b_j^c k_c\right)k_j = 0. \tag{20}$$

Then, there are two options .

I. $b \equiv b_i^i = 0$:

We will realize that such possibility should lead to an empty space-time, but this is not our case.

In fact, with $b = 0$ in (10,18) we get

$$\underset{\sim}{b}^4 = 0, \quad R_{ij} = -\varepsilon_1\left(\underset{\sim}{b}^2\right)_{ij}, \tag{21}$$

and because $R = p = k_2 = 0$, it follows that

$$\left(\underset{\sim}{b}^2\right)_i^i = \left(\underset{\sim}{b}^3\right)_i^i = \det\left(b_j^i\right) = 0 \tag{22}$$

therefore, trace of $\underset{\sim}{b}^n = 0$, $n = 1, \ldots, 4$, and Table 1 of Goenner [26] establish the Churchill - Plebański [38–43] type of $\underset{\sim}{b}$:

"b_{ij} is of type $[4N]_{[2]}$ (that is $[(211)]$)". (23)

Also, from Table 2 of [23] we see that a tensor of this type satisfies

$$\underset{\sim}{b}^n = 0, \; n = 2, 3 \tag{24}$$

which when substituted into (21) implies that $R_{ac} = 0$, hence the 4-space is empty (impossible to be embedded into E_5 [1–3,44]) and the Maxwell field does not exist at all. In consequence,

"Every solution of Einstein-Maxwell of

class one has $b \neq 0$". (25)

This result is not derived explicitly in [7,8]. Furthermore, from (3), it follows that R_{ij} is of type [(211)] since it comes from a null electromagnetic field. Then if b were equal to zero, it should be in contradiction to Table 2 of [42].

With (25), we get the main result of this section, however, we have still to consider another option allowed by (20) :

II. $b \neq 0$:

In this case, (20) also implies that k^i is a null principal direction of the second fundamental form,

$$b_{ic} k^c = 0. \tag{26}$$

Further, from (18,19) :

$$\underline{b}^4 - b\, \underline{b}^3 = 0, \qquad \underline{b}^3 - b\, \underline{b}^2 = 0 \tag{27}$$

It follows that the eigenvalues of $\underline{b}$ are b and zero (with three-fold multiplicity); so Table 2 of [42] gives

$$\text{"}b_{ij} \text{ is type } [3N - S]_{1, 3-1} \text{ that is } [(21)1]\text{"}. \tag{28}$$

If we now apply eq. (5) of [42] with $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = b$, it follows that

$$b_{ij} = k_i k_j + b A_i A_j, \quad k_i A^i = 0, \quad A_i A^i = 1,$$
$$b_{ij} A^j = b A_i, \qquad b_{ij} B^j = 0, \quad B_i k^i = B_i A^i = 0. \tag{29}$$

Here, A^i and B^i are simple spacelike eigenvectors and k^i is a two-fold degenerate null eigenvector; we note here that the relations (29) are equivalent to (18) of [8].

From (10,29), it is evident that both R_{mn} and b_{mn} have the same eigenvectors but with different eigenvalues :

$$R_{ij} k^j = 0, \qquad R_{ij} A^j = R_{ij} B^j = 0, \tag{30}$$

which means that R_{ac} is of type [(211)]. Finally, by inserting (29) into (9,10), we obtain $b = -\epsilon_1$, and

$$R_{ijrt} = \epsilon_1 \left(R_{ir} b_{jt} + R_{jt} b_{ir} - R_{it} b_{jr} - R_{jr} b_{it} \right). \tag{31}$$

We finally point out that Stephani [8] proved that the null congruence associated with k^i, does not have either rotation or expansion; it has only the properties expressed in (4). Metrics with such characteristics have been studied by Kundt [45], Kozarzewski [46] and other workers as reported in [1].

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