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A Bethe Ansatz Study of $N = 4$ Fateev-Zamolodchikov Model [†]

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Abstract: A Bethe Ansatz study of Fateev-Zamolodchikov Model is undertaken. A coupled system of Bethe Equation emerges which decouples in the special case of spin $N = 4$. Closed explicit expression for the ground state energy can be found even for finite lattice. The dispersion relation for the excitation spectrum is also found.

Keywords : Fateev-Zamolodchikov model, Z_N spin model, Bethe ansatz equation, Chiral Potts model, Self-duality.

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I. Introduction

A two dimensional self dual Z_N lattice spin model with nearest neighbor interaction (FZM) was obtained by V.A. Fateev and A.B. Zamolodchikov [1] which will be referred to as the Fateev-Zamolodchikov Model. This was obtained as the self dual [2] solution of the star-triangle relations [3]. We obtained [4, 5] coupled transcendental equations involving the zeroes of the automorphically connected transfer matrices of the model. In

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the same paper the free energy for Ferromagnetic and Anti-Ferromagnetic cases were obtained in the thermodynamic limit $M \rightarrow \infty$ for both odd and even spins. In this paper we present a case in which a complete decoupling of Bethe Equations occur in the ground state [6]. The ground state energy can be written in a closed form even for a finite sized lattice.

The FZM is defined in terms of Boltzmann weights (BW) as

$$\frac{W(n|u)}{W(0|u)} = \prod_{j=1}^{n} \frac{\sin(\pi j/N - \pi/2N - u)}{\sin(\pi j/N - \pi/2N + u)} \tag{1}$$

$$\frac{\overline{W}(n|u)}{\overline{W}(0|u)} = \prod_{j=1}^{n} \frac{\sin(\pi j/N - \pi/N + u)}{\sin(\pi j/N - u)} \tag{2}$$

The normalization is set as $W(0|u) = \overline{W}(0|u) = 1$.
Fateev-Zamolodchikov Model can be obtained as a limit of self dual Chiral Potts (CP) model whose BWs are given by

$$\frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^{n} \frac{b_q - \omega^j a_p}{b_p - \omega^j a_q}. \tag{3}$$

$$\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega a_p - \omega^j a_q}{b_q - \omega^j b_p}. \tag{4}$$

where $\omega = \exp(2\pi i/N)$ and the paired complex variables $(a, b) \in C^2$ satisfy the constraint

$$a_x^N + b_x^N = \kappa \tag{5}$$

$\kappa \in [0, 1]$, and $x = p$ or q. In the non-chiral limit $\kappa \rightarrow 0$, we parametrize (a_x, b_x) in Eq.(5) as:

$$a_x = e^{2ix} \quad b_x = \omega^{1/2} e^{2ix}. \tag{6}$$

Defining $u = q - p$ Eq.(3) and Eq.(4) reduce to Eq.(1) and Eq.(2). We will retain suffixes (p, q) in the boltzmann weights $W_{pq}(n|u)$ and $\overline{W}_{pq}(n|u)$ to signify that these BWs are obtained from the Chiral Potts BWs define in terms of p and q variables.

In order to get a connection formula among the the zeros of the eigenvalues of the transfer matrix T_q, we will use functional equations connecting T_q with its automorphically conjugate partners. For any $(a,b) \in C^2$ satisfying the CP constraint relation there exist other complex pairs connected to them which satisfy the same relation. Two such automorphic relations of importance are,

$$R(a,b) = (b, \omega a) \tag{7}$$
$$U(a,b) = (\omega a, b) \tag{8}$$

It is rather straightforward to check;

$$a_{Rx}^N + b_{Rx}^N = \kappa$$
$$a_{Ux}^N + b_{Ux}^N = \kappa$$

from the relation

$$a_x^N + b_x^N = \kappa$$

If one attempts to go from CP BWs to FZM BWs through the limiting process $\kappa \rightarrow 0$, one gets slightly different expressions for automorphic partners W_{pRq}, $\overline{W}_{pRq}$, W_{pUq} , and $\overline{W}_{pUq}$.

$$\frac{W_{pRq}(n|u)}{W_{pRq}(0|u)} = \prod_{k=1}^{n} \frac{\sin(\pi k/N - \pi/N - u)}{\sin(\pi k/N + u)} \tag{9}$$
$$\frac{\overline{W}_{pRq}(n|u)}{\overline{W}_{pRq}(0|u)} = \prod_{k=1}^{n} \frac{\sin(\pi k/N - \pi/2N + u)}{\sin(\pi k/N - \pi/2N - u)} \tag{10}$$
$$\frac{W_{pUq}(n|u)}{W_{pUq}(0|u)} = e^{-\frac{i\pi n}{N}} \prod_{k=1}^{n} \frac{\sin(\pi k/N - \pi/2N - u)}{\sin(\pi k/N + \pi/2N + u)} \tag{11}$$
$$\frac{\overline{W}_{pUq}(n|u)}{\overline{W}_{pUq}(0|u)} = e^{\frac{i\pi n}{N}} \prod_{k=1}^{n} \frac{\sin(\pi k/N + u)}{\sin(\pi k/N - u)} \tag{12}$$

In the non chiral limit, $T_q \longrightarrow T_q(u)$ and $T_{Rq} \longrightarrow T_q(u + \pi/2N)$. There is no similar relation connecting T_q and T_{Uq}.

We use the set of functional equations for the eigenvalues of transfer matrices of Chiral Potts derived by Baxter, Bazhanov and Perk [7, 8].

This functional relation appears in reference [7] as Eq.(4.40) and has the following form,

$$\tilde{T}_{\bar{q}} = \sum_{m=0}^{N-1} c_{m,q} T_{U^m q}^{-1} T_q T_{U^{m+1} q}^{-1} X^{-m-1} \tag{13}$$

where $\tilde{T} = TS$, $\bar{q} = (a_{\bar{q}}, b_{\bar{q}}) = UR^{-1}(a_q, b_q)$, and
$c_{m,q} =$

$$\left(\left(\prod_{j=0}^{m-1} \frac{b_p - \omega^{j+1} a_q}{a_p - \omega^j a_q} \right) \cdot \left(\prod_{j=m+1}^{N-1} \frac{\omega(a_p - \omega^j a_q)}{b_p - \omega^{j+1} a_q} \right) \cdot \left(\frac{N(b_q - b_p)(b_p - a_q)}{a_p b_p - \omega^m a_q b_q} \right) \right)^M$$

where the shift operator S and the global spin raising operator X are defined as

$$\begin{aligned} S|n_1, n_2, n_3, \ldots n_M > &= |n_M, n_1, n_2, \ldots n_{M-1} > \\ X_k|n_1 \ldots n_k \ldots n_M > &= |n_1 \ldots n_k + 1 \ldots n_M > \bmod N \end{aligned}$$

In some previous work [4, 5] it was shown in detail how to obtain the functional equations for the zeroes of the transfer matrices T_q, i.e. (v_i) and T_{Uq}, i.e. $(\bar{v}_i)$ using eq.(13). For N-even we obtained

$$\prod_{j=1}^{L_{Uq}} \frac{\sin(v_i - \bar{v}_j)}{\sin(v_i - \bar{v}_j - \frac{\pi}{N})} = (-1)^{M+1} \left[\frac{\sin 2(v_i - \frac{\pi}{2N})}{\sin(2v_i)} \right]^{2M} \tag{14}$$

$$\prod_{j=1}^{L_q} \frac{\sin(\bar{v}_i - v_j)}{\sin(\bar{v}_i - v_j + \frac{\pi}{N})} = (-1)^{M+1} \tag{15}$$

II. Simplification of BAE in the case of even spin

One makes the following change of variables to rewrite the BAE's for even case in a simpler (and standard) form,

$$\begin{aligned} v_j &= i\lambda_j + \frac{\pi}{4N} \\ \bar{v}_j &= i\bar{\lambda}_j - \frac{\pi}{4N} \end{aligned} \tag{16}$$

A detailed numerical study of the resulting equations for even spin case shows that the new variables $\lambda_j s$ are related to one another. In fact

$$\forall \ \lambda_j \ \exists \ \lambda_j + i\pi/2 \ mod(\pi) \tag{17}$$

Thus we can group λ_j such that $\lambda_j \in [-\pi/4, \pi/4]$. Using transformation rules for the hyperbolic functions one can rewrite the equations in terms of multiples of $\lambda_j s$ and $\bar{\lambda}_j s$. Define :

$$\chi_j = 2 \cdot \lambda_j \qquad \text{and} \qquad \bar{\chi}_j = 2 \cdot \bar{\lambda}_j \tag{18}$$

Finally the transcendental equations involving the zeros of the transfer matrix are cast in a form similar to the *standard* or *usual* Bethe ansatz equations (BAE).

$$\prod_{k=1}^{\frac{L_{U_q}}{2}} \frac{\sinh(\chi_j - \bar{\chi}_k - i\pi/4)}{\sinh(\chi_j - \bar{\chi}_k + i\pi/4)} = (-1)^{M+1} \left(\frac{\sinh(\chi_j + i\pi/8)}{\sinh(\chi_j - i\pi/8)} \right)^{2M} \tag{19}$$

$$\prod_{k=1}^{\frac{L_q}{2}} \frac{\sinh(\bar{\chi}_j - \chi_k - i\pi/4)}{\sinh(\bar{\chi}_j - \chi_k + i\pi/4)} = (-1)^{M+1} \tag{20}$$

The first equation has the departure from usual BAE in that the signs in front of the phase in the numerator are different on left hand side (LHS) and right hand side (RHS). The second equation is more unique in the sense that it has no spectral variable dependence on the RHS.

III. Ground State for spin 4 FZM

In the case of spin $N = 4$, the BAE's completely decouples for Ferromagnetic case. Moreover the equation simplifies and it is possible to solve it even for a finite sized lattice. This is quite unique.

For $N = 4$, the FM ground state corresponds to a filled band of $(2s)$ strings [9] for T_q and that of $(1-)$ for T_{U_q}. Exploiting these special string structures:

$$\begin{aligned} \chi_j &= \chi_j^r \pm \frac{i\pi}{4} \qquad \text{where} \qquad \chi_j^r \in \mathcal{R} \\ \bar{\chi}_j &= \bar{\chi}_j^r - i\pi \qquad \text{where} \qquad \bar{\chi}_j^r \in \mathcal{R} \end{aligned}$$

and using simple hypergeometric identities, Eq.(20) reduces to an identity and eq.(19) simplifies to

$$\left[\frac{\sinh\left(2\chi_j^r - \frac{\pi i}{4}\right)}{\sinh\left(2\chi_j^r + \frac{\pi i}{4}\right)}\right]^{2M} = \pm 1 \tag{21}$$

the sign on the RHS depends on whether M is even or odd. Taking logarithm of the equation we get

$$2M \cdot \left[2\arctan(\,\cot(-\frac{\pi}{4})\ \tanh(2\,\chi^r)\,)\right] = 2\nu \cdot \pi \tag{22}$$

where ν is an integer or half integer depending on M being odd or even. We can express χ in terms of this integer by simple inversion of the formula. The suffix of χ is omitted, however they are now parametrized by the integer (half-integer) ν.

$$\chi^r_{(\nu)} = \frac{1}{2}\tanh^{-1}\left(\tan\left(\frac{\nu\pi}{2M}\right)\right) \tag{23}$$

This formula is in excellent agreement with finite size numerical computation.

Table I: Comparison of χ as found from formula (23) and from numerical simulation on finite lattices.

M	ν	$\chi_\nu^{formula}$	$\chi_\nu^{num.sim.}$
2	$-\frac{1}{2}$	.215	.216
	$\frac{1}{2}$	−.215	−.216
3	−1	−.329	−.326
	0	0	0
	1	.329	.326
4	$-\frac{3}{2}$	.4004	.4001
	$-\frac{1}{2}$	.10085	.10081
	$\frac{1}{2}$	−.10085	−.10081
	$\frac{3}{2}$	−.4004	−.4001
5	−2	−.460	−.458
	−1	−.169	−.169
	0	0	0
	1	.169	.169
	2	.460	.458

Once the expression of the string centers are found for a finite system, the calculation of energy and central charge is straightforward. Using the expression for energy derived for even spin system we get [5, 6]

$$E = 2\sum_{j=1}^{L} \cot\left(\frac{\pi}{2N} + i\chi_{(\nu)}\right) - 2M\sum_{j=1}^{\frac{N}{2}} \cot\left(\frac{\pi k}{N}\right) \tag{24}$$

The energy values are also verified as having very good agreement with numerical values obtained from numerical simulation.
Let us concentrate on the first term in energy, since the other one is a simple additive linear term.

$$\begin{aligned}
\frac{E}{M} &= \frac{2}{M}\sum_{j=1}^{\frac{NM}{4}} 2\cot\left(2i\chi_j^{r} + \frac{\pi}{2} + \frac{\pi}{N}\right) \\
&= \frac{4}{M}\sum_{j=1}^{\frac{NM}{4}} \frac{1+\tan(2i\chi_j)}{1-\tan(2i\chi_j)} \\
&= \frac{4}{M}\sum_{j=1}^{\frac{NM}{4}} e^{2i\arctan(\tanh(2\chi_j))} \\
&= \frac{4}{M}\sum_{\nu} e^{2i\arctan\left(\tan\left(\frac{\nu\pi}{2M}\right)\right)}
\end{aligned}$$

Using the standard summation formula for a geometric series we get

$$\frac{E}{M} = \frac{2}{M\sin\left(\frac{\pi}{2M}\right)} \tag{25}$$

In the limit, $\lim_{M\to\infty}$ this gives a value $\frac{4}{\pi}$. One can recall the expansion of the inverse of trigonometric sine function.

$$\csc(z) = \frac{1}{z} + \frac{z}{6} + \frac{7z^2}{360} + \dots \tag{26}$$

Thus we get an expansion of $\frac{E}{M}$ in powers of M.

$$\frac{E}{M} = \frac{4}{\pi} + \frac{1}{M^2}\cdot\frac{\pi}{6} + \dots \tag{27}$$

IV. Excitation on Ferromagnetic ground state

In this section we study the excitation spectrum over the Ferromagnetic (FM) ground state. The FM ground state is given by a filled band of $(2s)$ strings.

Let us recall the equation for counting integers.

$$\frac{1}{2\pi}\Theta_j^{(1)}(\lambda_\alpha^j) - \frac{1}{2\pi M}\sum_k \sum_{\beta=1}^{M^{(k)}} \Theta_{jk}^{(2)}(\lambda_\alpha^{(j)} - \lambda_\beta^{(k)}) = \frac{I_\alpha^{(j)}}{M}$$

In the continuum limit this equation takes the form

$$Z_{(2s)}(\chi) = \frac{1}{2\pi}\Theta_{(2s)}^{(1)}(\chi) - \frac{1}{2\pi M}\sum_k \sum_{\beta=1}^{M^{(k)}} \Theta_{(2s,k)}^{(2)}(\chi - \chi_\beta^k) \qquad (28)$$

The density of $(2s)$ vacancies is given by

$$\sigma_{(2s)}(\chi) \doteq -Z'_{(2s)}(\chi) \qquad (29)$$

The vacancy density and the density of (2s) particles is related.

$$\sigma_{(2s)}(\chi) = \rho_{(2s)}(\chi) + \frac{1}{M}\sum_{\beta=1}^{M_h^{(2s)}} \delta\left(\chi - \chi_\beta^{(2s)h}\right) \qquad (30)$$

where $\chi_\beta^{(2s)h}$ are the position of the holes.
Thus

$$-\sigma_{(2s)}(\chi) = \frac{1}{2\pi}\Theta_{(2s)}^{(1)'}(\chi) - \frac{1}{2\pi M}\sum_{k\neq(2s)} \sum_{\beta=1}^{M^{(k)}} \Theta_{(2s,k)}^{(2)'}(\chi - \chi_\beta^k) -$$
$$\frac{1}{2\pi}\int \Theta_{(2s,2s)}^{(2)'}(\chi-\mu)d\mu + \frac{1}{2\pi M}\sum_{h=1}^{M_h^{(2s)}} \Theta_{(2s,2s)}^{(2)'}(\chi - \chi_\beta^{(2s)h}) \qquad (31)$$

The vacancy density has contribution from (2s)- ground state, (2s)-holes and excited particles $\sigma_{(2s)} = \sigma_{(2s)}^{(0)} + \sigma_{(2s)}^{(h)} + \sum_j \sigma_{(2s)}^{(j)}$ where $\sigma_{(2s)}^{(0)}$ is the same as $\rho_{(2s)}$ above.

The expressions for energy (E) and momentum (P) for Spin 4 FZM are given by

$$E = \sum_{k=1}^{\frac{L}{2}} \cot\left(i\chi_k + \frac{\pi}{8}\right) - 2M\sum_{k=1}^{2} \cot\left(\frac{\pi k}{4}\right) \tag{32}$$

$$e^{iP} = \prod_{k=1}^{\frac{L}{2}} \frac{\sinh\left(\chi_k + \frac{i\pi}{8}\right)}{\sinh\left(\chi_k - \frac{i\pi}{8}\right)} \tag{33}$$

Thus the energy of a state designated by a given set of strings

$$\begin{aligned} E &= \sum_{\substack{k \\ strings}} \sum_{\beta=1}^{M^{(k)}} \epsilon_k(\chi_\beta^{(k)}) \\ &= \int d\chi\sigma_{(2s)}(\chi)\epsilon_{(2s)}(\chi) - \sum_{\beta=1}^{M_h^{(2s)}} \epsilon_{(2s)}\left(\chi_\beta^{(2s)h}\right) + \sum_{\beta\neq k}\sum_{\beta=1}^{M^{(k)}} \epsilon\left(\chi_\beta^{(k)}\right) \end{aligned} \tag{34}$$

The bare energies for n-string with parity v is easily obtained

$$\begin{aligned} \epsilon_{(n,v)}(\chi_\alpha) &= \sum_{k=1}^{n} \cot\left(i\chi_{k,\alpha}^{(n,v)} + \frac{\pi}{8}\right) \\ &= \sum_{k=1}^{n} \cot\left(i\chi_\alpha^{(n,v)} - 2\gamma(n+1-2k) - \frac{\pi}{4}(1-v) + \frac{\pi}{8}\right) \end{aligned} \tag{35}$$

One can separate the real and imaginary parts of this expression. The requirement of reality of energy determines the additional constraints on rapidities χ_j.

The real part of the energy is equal to the derivative of the $\Theta_j^{(1)}(\chi)$ function already encountered upto a multiplicative constant.

$$Re\quad\left[\epsilon_{(n,v)}(\chi_\alpha^{(n,v)})\right]$$

$$= \sum_{k=1}^{n} \frac{\sin 2\left(\frac{\pi}{2} + 2\gamma(n+1-2k) + \frac{(1-v)\pi}{4} - \frac{\pi}{8}\right)}{\cos 2\left(\frac{\pi}{2} + 2\gamma(n+1-2k) + \frac{(1-v)\pi}{4} - \frac{\pi}{8}\right) + \cosh 2\left(-\chi_{\alpha}^{(n,v)}\right)}$$

$$= \sum_{k=1}^{n} \frac{-\sin\left(4\gamma(n+1-2k) + \frac{(1-v)\pi}{2} - \frac{\pi}{4}\right)}{\cosh 2\left(-\chi_{\alpha}^{(n,v)}\right) - \cos\left(4\gamma(n+1-2k) + \frac{(1-v)\pi}{2} - \frac{\pi}{4}\right)}$$

The $\Theta_j^{(1)}(\chi)$-function for $j = (n, v)$ is as follows

$$\begin{aligned}\Theta_{(n,v)}^{(1)}(\chi) &= \sum_{l=1}^{n} 2\phi(\chi, 2\gamma(n+\frac{3}{2}-2l), v) \\ &= 4\sum_{l=1}^{n} \arctan\left(\cot\left(2\gamma(n+\frac{3}{2}-2l) + \frac{(1-v)\pi}{4}\right)\tanh(\chi)\right)\end{aligned}$$

differentiating

$$\Theta_{(n,v)}^{(1)\prime}(\chi) = \sum_{l=1}^{n} \frac{\sin 2\left(2\gamma\left(n+\frac{3}{2}-2l\right) + \frac{(1-v)\pi}{4}\right)}{-\cos 2\left(2\gamma\left(n+\frac{3}{2}-2l\right) + \frac{(1-v)\pi}{4}\right) + \cosh(2\chi)} \tag{36}$$

whence

$$Re\left[\epsilon_{(n,v)}(\chi)\right] = -\frac{1}{4}\Theta_{(n,v)}^{(1)\prime}(\chi) \tag{37}$$

Imaginary part of the energy $\epsilon_{(n,v)}(\chi)$

$$Im\left[\epsilon_{(n,v)}(\chi)\right] = \sum_{l=1}^{n} \frac{-\sin 2\left(\chi_{\alpha}^{(n,v)} + \frac{(1-v)\pi}{4}\right)}{-\cos 2\left(2\gamma\left(n+\frac{3}{2}-2l\right) + \frac{(1-v)\pi}{4}\right) + \cosh(2\chi)} \tag{38}$$

Detailed numerical study shows ([4, 5] and to be presented elsewhere) that there exist several spurious solutions and only a subset of them, corresponding to a specific choice of counting numbers $I_{\alpha}^{(j)}$, is admissible. Non-string solutions exist, however they are not as numerous. From the numerical study we find that the elementary excitations over the FM ground state (a sea of (2θ)-strings) are (a) a pair of (1+) strings , and (b) (1+) and (1−) strings.

The FM ground state is a filled band of $(2s)$ strings. The density of ground state energy is

$$e_0 = \lim_{M\to\infty}\frac{E_0}{M} = \int d\chi \rho_{(2s)}(\chi)\epsilon_{(2s)}(\lambda) - 2\sum_{k=1}^{2}\cot\left(\frac{\pi k}{4}\right) \tag{39}$$

The observed correlation between the integers suggest that the rapidities corresponding to $(2s)$ and (a) ought to be connected allowing cancelation of the imaginary part of the total energy.

$$Im\left[\epsilon_{(2s)}\right] = Im\left[\epsilon_{(a)}\right] \tag{40}$$

It is shown in the appendix that $Im\left[\epsilon_{(2s)}\right] = Im\left[\epsilon_{(a)}\right]$ and $Re\left[\epsilon_{(2s)}\right] =$ - $Re\left[\epsilon_{(a)}\right]$.

$$Re\left[\epsilon_{(a)}(\chi)\right] = \frac{4}{\cosh(4\chi)} \tag{41}$$

Similar argument holds for (b)-type excitations, where $Im\left[\epsilon_{(2s)}\right] = Im\left[\epsilon_{(b)}\right.$ and $Re\left[\epsilon_{(2s)}\right] = -Re\left[\epsilon_{(b)}\right]$.

$$Re\left[\epsilon_{(b)}(\chi)\right] = \frac{4}{\cosh(4\chi)} \tag{42}$$

One should note that the dressed energy and bare energy are equal since the function coupling the ground state density to excited states $\Theta^{(2)}_{(j,k)}$ is zero for $j = (2s)$. Thus we arrive at

$$E = E_0 + \sum_{\beta=1}^{M^{(a)}} 2\epsilon_{(a)}\left(\chi^{(a)}_\beta\right) + \sum_{\beta=1}^{M^{(b)}} 2\epsilon_{(b)}\left(\chi^{(b)}_\beta\right) \tag{43}$$

where E_0 is the ground state energy.

We now turn to the calculation of momentum. The momentum associated with a string of length j and parity v is found to be

$$\begin{aligned} p_{(1+)}(\chi) &= -\frac{1}{2}\Theta^{(1)}_{(1+)}(\chi) \\ p_{(j)}(\chi) &= -\frac{1}{2}\Theta^{(1)}_{(j)}(\chi) \quad \text{for } j \neq (1+) \end{aligned} \tag{44}$$

whence

$$p_{(a)}(\chi) = p_{(b)}(\chi) = 2\arctan\left(\tanh\left(\frac{\chi}{2}\right)\right) + \pi \tag{45}$$

we finally derive the dispersion relation,

$$\begin{aligned} \epsilon_{(a)}(p) &= 4\sin\left(\frac{p}{2}\right) \\ \epsilon_{(b)}(p) &= 4\sin\left(\frac{p}{2}\right) \end{aligned} \tag{46}$$

V. Conclusion

The simplification of Bethe Equation in the special case of $N = 4$ was first identified in an earlier paper [6]. The coupled set of Bethe Ansatz Equations completely decouple for the ground state case. The complete classification of states is another very interesting simplification. The simple form of excitation spectrum is also derived.

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