Indian J. Phys. **72A** (4), 331–335 (1998)

IJP A
— an international journal

Matching of Friedmann-Lemaitre-Robertson-Walker and Kantowski-Sachs Cosmologies

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Received 13 January 1998, accepted 26 March 1998

Abstract : The matching of Friedmann-Lemaitre-Robertson-Walker space-times onto Kantowski-Sachs space-times with strings is investigated Employing Darmois junction conditions, a spherically symmetric homogeneous anisotropic Kantowski-Sachs metric with strings can be joined smoothly to the present day universe represented by FLRW space-times. This cosmological model is expected to be an important tool for studying the early stage of the

Keywords : Space-time geometry, strings, Darmois junction conditions, parametrization

PACS No. : 98.80 Hw

1. Introduction

The space-time geometry of the present day universe is believed to be described by FLRW type of metric. But the universe did not have the same type of space-time geometry just after its birth and has passed through a number of different phases before it reached the present day form. Different space-time metrics are developed to describe such different phases and we have the problem of matching of such metrics which occur during the phase change. While the formalism for joining two different space-times is well developed, successful examples of its application are very few. The reason is that since the matching of two solutions usually takes place on a surface sharing some of the symmetries, both of the two matched solutions must come from a restricted subset of all solutions, which is determined by their shared symmetries—this restriction makes the problem of matching a difficult one. The best known examples of matching is probably the matching of FLRW dust space-times with Schwarzschild interior or exterior spacetimes [1–4]. A second example is the matching of FLWR metric with the Kasner metric [5]. In this paper, we will present another example of matching of FLRW space-times with Katowski-Sachs space-time with strings.

2. FLRW metric and Kantowski-Sachs metric with string

The general FLRW metric in its usual spherically symmetric form, can be written as

$$ds^2 = dt^2 - R^2(t)\left[dr^2/(1-kr^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{1}$$

The Kantowski-Sachs metric for spherically symmetric homogeneous anisotropic space-time in presence of strings is of the form

$$ds^2 = dT^2 - b^2(T)\left[d\Theta^2 + \sin^2\Theta d\Phi^2\right] - a^2(T)d\rho^2. \tag{2}$$

With its solutions for geometric strings [6],

$$\begin{aligned} & a(T)\,\alpha(T-T_0)^{-1/3} \\ \text{and} \quad & b(T)\,\alpha(T-T_0)^{2/3}. \end{aligned} \tag{3}$$

3. The matching

From now on, we will use the symbols F and K to denote indexed quantities associated with FLRW and Kantowski-Sachs metrics respectively. Hence, the coordinates of the corresponding metrics can be represented by

$$\begin{aligned} & X_F^i = [t, r, \theta, \varphi], g_{\mathrm{Fab}} \\ & \qquad\qquad\qquad\qquad a, b = 1, 2, 3, 4 \\ \text{and} \quad & X_k^i = [T, \rho, \Theta, \Phi], g_{\mathrm{Kab}}. \end{aligned}$$

We will apply the Darmois set of junction conditions since it does not require the use of the same coordinate systems on both sides of the hyper surface Σ [7]. The two regions of space-times are said to match across Σ, if the first and the second fundamental forms calculated in terms of the coordinates on Σ, are identical. The first and the second fundamental forms are expressed as

$$\gamma_{\alpha\beta} = g_{ij}\, \partial x^i/\partial u^\alpha\, \partial x^j/\partial u^\beta, \qquad i,j = 1, 2, 3, 4 \quad \alpha, \beta = 1, 2, 3 \tag{4}$$

and

$$\Omega_{\alpha\beta} = \left(\Gamma_{ij}^k n_k - n_{i,j}\right) \partial x^i/\partial u^\alpha\, \partial x^j/\partial u^\beta, \tag{5}$$

where $u^\alpha = [u^1 = u, u^2 = v, u^3 = w]$ is the coordinate system on the hypersurface and n_i is its unit normal. Let Σ be given by the functions $f_F[x_k^i(u^\alpha)] = 0$, $f_k[x_k^i(u^\alpha)] = 0$ and two parametric representations $x_F^i = h_F^i(u^\alpha)$, $x_k^i = h_k^i(u^\alpha)$. Then n_i can be calculated by using the relation

$$n_i = f_{,i}\Big/\left(\left|g^{ab} f_{,a}\, f_{,b}\right|\right)^{1/2}, \tag{6}$$

where i denotes $\partial/\partial x^i$.

We now consider a surface represented by the function $f_F(x_F^i) = r - r_0 = 0$ where r_0 is a constant and parametrised by $x_F^1 = t = u$, $x_F^2 = \theta = v$, $x_F^3 = \varphi = w$ and $x_F^4 = r = r_0$. In K–S frame we donot know the form of f_K, however, we will use $x_K^1 = T = T(u)$, $x_K^2 = \Theta = \Theta(u,v)$, $x_K^3 = \Phi = w$ and $x_K^4 = \rho = \rho(u)$ as its parametrization.

Now the condition $\gamma_{F\alpha\beta} = \gamma_{K\alpha\beta}$ implies that

$$1 = (dT/du)^2 - b^2(\partial\Theta/\partial u)^2 - a^2(d\rho/du)^2, \tag{7}$$

$$(\partial\Theta/\partial v)^2 = R^2 r_0^2/b^2, \tag{8}$$

$$R^2 r_0^2/b^2 = \sin^2\Theta/\sin^2\theta, \tag{9}$$

$$b^2\partial\Theta/\partial u\ \partial\Theta/\partial v = 0. \tag{10}$$

From [10] we find that at least one of the terms b^2, $\partial\Theta/\partial u$ or $\partial\Theta/\partial v$ must vanish. However, if $b^2 = 0$ or $\partial\Theta/\partial v = 0$, then according to [8] and [9] we have $R(u) = 0$ which is not allowed. Hence, we are left with the only condition $\partial\Theta/\partial u = 0$ *i.e.* Θ is a function of v only. Eqs. (7) to (10) then reduce to

$$1 = (dT/du)^2 - a^2(d\rho/du)^2, \tag{11}$$

$$(d\Theta/dv)^2 = R^2 r_0^2/b^2, \tag{12}$$

and

$$R^2 r_0^2/b^2 = \sin^2\Theta/\sin^2\theta. \tag{13}$$

From (12) and (13), we have $\Theta = \theta$ by adjusting the constant of integration. Thus (12) and (13) give us

$$R^2 r_0^2/b^2 = 1, \quad i.e. \quad R \propto b. \tag{14}$$

Let us now compute the second fundamental forms. The unit normal in the FLRW space-time can be calculated by using the eq. (6) and also using $f_F(x_F^i) = r - r_0 = 0$, we get

$$n_{Fi} = \delta_i^4 n_{F4}.$$

As can be seen, the normal is space like, *i.e.* $n_F^i n_{Fi} = -1$. Further, we have $\Omega_{F\alpha\beta} = 0\ \forall \alpha, \beta$ since $\partial x_F^4/\partial u^\alpha = \partial r_0/\partial u^\alpha = 0$.

The unit normal in the Kantoswki-Sachs space-time in presence of string is more complicated to obtain since we donot know the explicit form of $f_k(x_k^i)$ except that it should not depend on Θ and Φ. However, n_{ki} must satisfy the two conditions

$$n_K^i n_{Ki} = n_F^i n_{Fi} = -1$$

and

$$n_{Ki}\partial x_K^i/\partial u^\alpha = 0. \tag{15}$$

Thus, we obtain a set of two equations for two unknowns which enable us to derive n_{Ki} as a function of u^α. We have

$$n_{Ki} = [\pm a\, d\rho/du, 0, 0 \mp a\, dT/du]. \tag{16}$$

Now differentiating (15) w.r.t. u^α, we get

$$n_{Ki}\partial^2 x^i_K/\partial u^\beta \partial u^\alpha = -n_{Ki,j}\ \partial x^j_K/\partial u^\beta\ \partial x^i_K/\partial u^\alpha \tag{17}$$

and finally, $$\Omega_{K\alpha\beta} = \Gamma^i_{Khj}\, n_{Ki}\, \partial x^h_K/\partial u^\alpha\ \partial x^j_K/\partial u^\beta + n_{Ki}\,\partial^2 x^i_K/\partial u^\beta \partial u^\alpha . \tag{18}$$

From eq. (16) and noting that $\Gamma^1_{K22}, \Gamma^1_{K33}, \Gamma^1_{K44}$ and Γ^4_{K14} are the only non-zero Christoffel symbols of interest, the condition $\Omega_{F\alpha\beta} = 0 = \Omega_{K\alpha\beta}$ is already satisfied except for $\Omega_{K\alpha\beta}$'s diagonal terms. These three remaining terms are

$$\Omega_{K11} = \Gamma^i_{K44} n_{Ki}(d\rho/du)^2 + \Gamma^i_{K14} n_{K4}\, dT/du\ d\rho/du + n_{K1}\, d^2T/du^2$$
$$+ n_{K4}\, d^2\rho/du^2 = 0, \tag{19}$$

$$\Omega_{K22} = \Gamma^1_{K22} n_{K1} = 0, \tag{20}$$

and $$\Omega_{K33} = \Gamma^1_{K33} n_{K1} = 0. \tag{21}$$

From eqs. (20) and (21), we find that $n_{K1} = 0$. Then from (16) we have

$$d\rho/du = 0.$$

and from (11),

$$dT = \pm du = \pm dt \tag{22}$$

Notice that eq. (19) is automatically satisfied. Now from (14) we get

$$R = b/r_0 = 1/r_0\,(T - T_0)^{2/3} \tag{23}$$

4. Discussion

From (23) we find that the FLRW region has a scale factor $R = 1/r_0\,(t - t_0)^{2/3}$ with consequence that the space-time is Einstein-de Sitter type (a trivial displacement in t makes the argument more evident). Thus, we would show that the spatially flat Einstein-de Sitter space-time can be joined smoothly to a Kantowski-Sachs space time with strings. The presence of the strings in K-S space-time allows the matching of the two space-times smoothly. Moreover, such a matching can be considered only at the very early stages of the universe during which, it is believed the universe passed through a series of phase transitions along with spontaneous breaking of symmetry. Such a symmetry breaking may give rise to topologically stable defects such as appearance of domain walls, strings and monopoles. Out of these three only strings can lead to a very interesting cosmological consequence as can be seen from the following.

We have seen that at a surface defined by $r = r_0$ = constant and ρ = constant the two space-times can be joined smoothly with $R = b/r_0 = 1/r_0\,(t - t_0)^{2/3}$. This can also be seen by noting the forms of the two metrics at this surface.

$$ds^2_{\mathrm{FLRW}} = dt^2 - 1/r_0^2\,(t - t_0)^{4/3}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \tag{24}$$

and $$ds^2_{\mathrm{KS}} = dT^2 - (T - T_0)^{4/3}\left[d\Theta^2 + \sin^2\Theta d\Phi^2\right] - (T - T_0)^{-2/3} d\rho^2 \tag{25}$$

we find from (24) and (25) that the two space time are identified on surfaces r = constant = r_0 and ρ = constant if we simply assume $T = t$, $\theta = \Theta$ and $\varphi = \Phi$. It is also interesting to note that since the space times ae matched across surfaces with r = constant and ρ = constant one can construct a universe of alternating layers of FLRW and K-S regions. In this scenario, the thickness of the K-S layers would be decreasing as $(t - t_0)^{-1/3}$ so that FLRW regions grow with time and at a certain time the K-S region is completely wiped out and the universe becomes FLRW type.

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