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Supersymmetrized Schrödinger equation for Fermion–Dyon system

B S Rajput and V P Pandey
Department of Physics, Kumaun University,
Nainital-263 001, India

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Abstract : Supersymmetrized Schrödinger equation for Fermion-Dyon system has been obtained by dimensional reduction of supersymmetrized four-dimensional harmonic oscillator and it has been interpreted as an ensemble of two Schrödinger and one Pauli's equation each describing the motion of an electrically charged particle in the field of a Dyon with different magnetic charges.

Keywords : Schrödinger equation, supersymmetry, Fermion-Dyon system

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1. Introduction

Monopoles and dyons became the intrinsic parts of all current grand unified theories [1] with the enormous potential importance in connection with their role in catalyzing proton dacay [2,3], the quark confinement problem of QCD [4,5] and RCD [6,9] and CP-violation in terms of non-zero vacuum angle [10]. The dyon-fermion dynamics has been worked out by various authors [2, 11–13] and it has been shown that the nature of dyons is strongly perturbed by fermionic sector which couples with them. In our recent paper [13], we have undertaken the study of dyon-fermion bound states and showed that in dyon-fermion system the fermion moves on a cone with its apex at the dyon and axis along its angular momentum. It has also been shown that the exact solution of Dirac equation for such a system is not possible due to the presence of terms vanishing more rapidly than r^{-1} in the potential of the system.

Keeping in view the symmetry of Schrödinger equation for a fermion in the field of monopole [14,15] and the difficulties faced [16] in supersymmetrizing the Pauli's equation of a fermion in the field of dyon, in the present paper, we try to obtain the supersymmetrized solution of Schrödinger equation of fermion in dyonic field by the dimensional reduction of supersymmetrized generalized four-dimensional Harmonic oscillator that we can derive the modifications in our earlier results of eigen values and eigen functions of fermion-dyon system as the result of supersymmetrization. The supersymmetrized Schrödinger equation for this system has been interpreted as describing the quantum dynamics of a supermultplet of two spin-0 and one spin-1/2 particles in a Coulomb field. It has also been interpreted as an ensemble of two Schrödinger and one Pauli's equation each describing the motion of an electrically charged particle in the field of a dyon with different magnetic charges.

2. Dimensional reduction of four-dimensional harmonic oscillator to fermion-dyon system

Let us consider the motion of a fermion of mass ($m = 1/2$) and charge e_1 in the field of a dyon carrying generalized charge

$$q = e_2 - ig_2$$

in terms of electric and magnetic charges e_2 and g_2 respectively. Its Schrödinger equation may be written as [13]

$$\hat{H}_D \psi = E\psi \tag{2.1}$$

where $$\hat{H}_D = \hat{\pi}^2 - \frac{K}{|x|} + \frac{P^2}{|x|^2} \tag{2.2}$$

with $$\hat{\pi}_i = \left(\hat{p}_i - e_i \hat{V}_i\right),$$

$$K = e_1 e_2, \tag{2.3}$$

$$P = e_1 g_2 .$$

In eq. (2.3), $\hat{V}$ is the potential of the field of dyon. Rescaling by $\alpha(r = \alpha x)$, eq. (2.1) may be written as follows :

$$\hat{H}_D^{(\pi)} = \frac{K}{\alpha}\psi, \tag{2.4}$$

with $$H_D^{(\pi)} = \left(rv^2 - r + P^2/r\right), \quad \text{for } \alpha = \sqrt{-E}$$

and $$H_D^{(\pi)} = \left(rv^2 + r + P^2/r\right), \quad \text{for } \alpha = \sqrt{E}, \tag{2.5}$$

where $$v_1 = p_i - e_1 V_i . \tag{2.6}$$

Equation (2.4) may be written in the following specific form :

$$\hat{H}_D^{(\pi)} \psi = \left[\hat{r}\left(\hat{p}_k - e_1 \hat{V}_k\right)^2 + \frac{P^2}{r} + \hat{r}\right]\psi = \frac{K}{\alpha}\psi . \tag{2.7}$$

The corresponding angular momentum operator may be written as follows :

$$\hat{L}_i^{(\pi)} = \varepsilon_{ijk}\,\hat{r}_j\left(\hat{p}_k - e_1\hat{V}_k\right) + P\hat{r}_i, \tag{2.8}$$

with

$$\hat{L}^{2(\pi)} = l(1+l) + e_1^2 g_2^2 = l'(l'+1). \tag{2.9}$$

Solution of eq. (2.7) has been obtained in the following form in our earlier paper [13] :

$$\psi_{(r,\theta,\phi)} = \frac{u(r)}{r}\, Y_{e_1 g_2, l', m'}(\theta, \phi), \tag{2.10}$$

where $Y_{e_1 g_2, l', m'}(\theta, \phi)$ are the spherical harmonics for a fermion-dyon system and $u(r)$ is given as follows in terms of confluent hyper-geometric functions :

$$u(r) = (\alpha r)^a \exp(-\alpha r) F\left(a - \frac{\alpha K}{2}, 2a, 2\alpha r\right) \tag{2.11}$$

with

$$a = \frac{1}{2} + \left[(l' + 1/2)^2 - (e_1 g_2)^2\right]^{1/2}. \tag{2.11a}$$

The corresponding bound state energy of fermion-dyon system has been obtained in the following form [13] :

$$E = \frac{(e_1 e_2)^2}{4\left[\sqrt{l'(l'+1) - e_1^2 g_2^2 + \frac{1}{4}} + \frac{1}{2} + n\right]^2}, \tag{2.12}$$

where n is an integer and we have chosen fermionic mass $m = 1/2$. Equation (2.4) is equivalent to the Schrödinger equation for the four-dimensional oscillator [14]

$$\hat{H}\tilde{\psi} = \frac{K}{\alpha}\tilde{\psi}, \tag{2.13}$$

with the constraint

$$X\tilde{\psi} = -i\frac{\partial}{\partial\omega}\tilde{\psi} = P\tilde{\psi} \tag{2.14}$$

where

$$H = -r\left[\frac{1}{r^2}\left[\frac{\partial}{\partial\gamma} r^2 \frac{\partial}{\partial\gamma}\right] + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right.\right.$$
$$\left.\left. + \frac{1}{\sin^2\theta}\left[\frac{\partial^2}{\partial\omega^2} + 2\cos\theta\frac{\partial^2}{\partial\phi\partial\omega}\right] - 1\right]\right]. \tag{2.15}$$

Solution of eq. (2.14) may be written as

$$\tilde{\psi} = \tilde{\psi}(r, e, \phi, \omega) = e^{i p \omega}(\gamma, \theta, \phi), \tag{2.16}$$

where ω is the angular velocity in the domain $(0, 4\pi)$. The condition for single-valuedness of $\tilde{\psi}$ requires that P should be quantized in half integral units :

$$P = e_1 g_2 = \frac{1}{2} n.$$

The system described by eqs. (2.13) and (2.14) possesses SO(4, 2) spectrum symmetry. The angular momentum operator corresponding to Hamiltonian of eq. (2.15) operates through the following relation

$$\hat{L}_i \tilde{\psi} = \exp\left[iP\omega\right]\hat{L}_i(\pi)\psi, \tag{2.17}$$

where
$$\hat{L}_i^{(\pi)} = \varepsilon_{ijk}\, r_j p_k - P\frac{r_i - \delta_{i_3 r_3}}{r\sin^2\theta}, \tag{2.18}$$

showing that the projected SU(2) generators have a P-dependent contribution.

Choosing

$$V_i = g_2\, \varepsilon_{ij3}\, \frac{r_j r_3}{r\left(r_1^2 + r_2^2\right)}$$

or in spherical coordinates

$$V_r = 0,\ V_\theta = 0,\ V_\phi = -g_2\cos\theta,$$

as the potential for a monopole of strength g_2, this $L_i^{(\pi)}$ of eq. (2.18) may readily be recasted in the form given by eq. (2.8) for fermion-dyon system. Then Hamiltonian of eq. (2.13) for four-dimensional harmonic oscillator reduces to that of eq. (2.7) for the motion of a fermion through the field of a dyon. This reduction takes place under the projection

$$\pi : R^+ \times S^3 \rightarrow R^+ \times S^2 \tag{2.19}$$

imposing the mapping

$$\left(Z_1, Z_2\right) \rightarrow \left(r, \bar{Z}_1/\bar{Z}_2\right), \tag{2.20}$$

$$Z_1 = \sqrt{r_1} + i\sqrt{r_2},\ Z_2 = \sqrt{r_3} + i\sqrt{r_4}, \tag{2.21}$$

such that
$$\bar{Z}_a Z_a = |Z_1|^2 + |Z_2|^2 = r > 0. \tag{2.22}$$

Taking a 2-sphere of radius r and projecting the point $r\frac{\bar{Z}_1}{Z_2}$ of $C \rightarrow S^2$ through north pole, we get

$$\frac{Z_1}{Z_2} = \frac{r_1 - ir_2}{r - r_3}$$

or
$$r_i = \pi_i(Z_a) = \bar{Z}_a\, \sigma_{ab}^i\, Z_b,$$

which is Kustaanheimo-Stiefel (K–S) transformation with σ_i as standard Pauli matrices. This projection defines a principal fibre bundle

$$M \sim R^+ \times S^3$$

with $U(1)$ as the structural group.

3. Supersymmetrized Fermion-Dyon system

In order to supersymmetrize the system described by eq. (2.7), let us start with the supersymmetrization of generalized four-dimensional harmonic oscillator described by eq. (2.13) and then perform the dimensional reduction through eqs. (2.19) and (2.20). To meet this end, let us choose the supercharges

$$Q = \left(\frac{\partial}{\partial Z_a} - \frac{\partial W}{\partial Z_a}\right)\eta_a, \tag{3.1}$$

$$Q^+ = \left(-\frac{\partial}{\partial \bar{Z}_a} - \frac{\partial W}{\partial \bar{Z}_a}\right)\eta_a^+, \tag{3.2}$$

where W is a real function of Z an $\bar{Z}$ and η satisfies the following Clifford algebra :

$$\{\eta_a, \eta_b\} = \{\eta_a^+, \eta_b^+\} = 0, \tag{3.3}$$

$$\{\eta_a, \eta_b^+\} = 2\delta ab \quad \text{for } a, b = 1, 2.$$

Choosing $\quad W(Z, \bar{Z}) = \lambda \ln (\bar{Z}_a Z_a),$ (3.4)

where $\lambda \in R$, the domain of Clifford algebra of eq. (3.3), we get

$$H = \frac{1}{2}\{Q, Q^+\}$$

$$= \frac{-\partial^2}{\partial \bar{Z}_a \partial Z_a} + \frac{\lambda}{\bar{Z}_a z_a}(\lambda - C) - 2\lambda \frac{\bar{Z}_a \sigma^i_{ab} Z_b \hat{S}_i}{(\bar{Z}_a Z_a)^2}, \tag{3.5}$$

where

$$C = Z_a\left(\frac{\partial}{\partial Z_a}\right) - \bar{Z}_a \frac{\partial}{\partial \bar{Z}_a} + \frac{1}{4}[\eta_a, \eta_b^+] \tag{3.6}$$

and

$$\hat{S}^i = \frac{1}{2}\eta_a^+ \sigma^i_{ab} \eta_b. \tag{3.7}$$

This H has a $n = 2$ conformal supersymmetry. In addition to the operators Q, Q^+ and H, let us also construct the following operators :

$$D = \frac{1}{2} i\left[Z_a \frac{\partial}{\partial Z_a} + \bar{Z}_a \frac{\partial}{\partial \bar{Z}_a} + 2\right],$$

$$K = \bar{Z}_a Z_a,$$

$$S = i\bar{Z}_a \eta_a,$$

$$S^+ = -iZ_a \eta_a^+, \tag{3.8}$$

and

$$Y = \frac{1}{2}\left(Z_a \frac{\partial}{\partial Z_a} - \bar{Z}_a \frac{\partial}{\partial \bar{Z}_a}\right) + 1/4\,(\eta_a, \eta_b^+) - \lambda.$$

These generators satisfy the Osp (2.1).structure relation and remain invariant under the SU(2) action generated by the operators

$$J^{i} = -\frac{1}{2}\left[Z_a \sigma^{i}_{ab} \frac{\partial}{\partial Z_b} - \bar{Z}_a \sigma^{i}_{ab} \frac{\partial}{\partial \bar{Z}_b} \right] + \hat{S}^{i} \tag{3.9}$$

Operator C, given by eq. (3.6), commutes with all the Osp (2,1) generators and also with $\hat{J}$. As such, the full invariant algebra of this problem is Osp (2,1) ⊕ SU(2) ⊕ U(1). Within this algebra, the Hamiltonian of eq. (3.5) for supersymmetried four-dimensional harmonic oscillator may be generalized in the form of the following operator

$$R = H + K = H + \bar{Z}_a Z_a. \tag{3.10}$$

The constraint (2.14) may then be written in the following form

$$\chi \tilde{\psi}_\mu = \frac{1}{2}\left[Z_a \frac{\partial}{\partial Z_a} - \bar{Z}_a \frac{\partial}{\partial \bar{Z}_a} \right] \tilde{\psi}_\mu = P \tilde{\psi}_\mu, \ (\mu = 1, 2, 3, 4) \tag{3.11}$$

showing that every component of $\tilde{\psi}$ transforms according to the same U(1) representation But the generators Q, Q^{+}, S and S^{+} do not commute with $\bar{\chi}$ and hence the supersymmetry of R will be lost under the projection involving this constraint. Thus, we modify this constraint to the following equivalent condition

$$C\tilde{\psi} = 2P\tilde{\psi}, \tag{3.12}$$

where the operator C, given by eq. (3.6) commutes with every element of Osp (2,1) ⊕ SU(2) algebra and it will affect the projection without breaking any symmetry of the four-dimensional system. Condition (3.12) may be understood as the supersymmetrized version of the constraint (2.14). Then eq. (2.16) may be obtained in the following supersymmetrized form

$$\tilde{\psi}_{(r,\theta,\phi,\omega)} = \exp i\left[\left(P - \frac{1}{2}\Sigma\right)\omega\right] \psi_{(r,\theta,\phi)}, \tag{3.13}$$

where $$\Sigma = \frac{1}{4}\left[\eta_a, \eta_a^{+}\right]. \tag{3.14}$$

Then eq. (2.17) is supersymmetrized into the following form

$$\hat{J}_i \tilde{\psi}_{(r,\theta,\phi,\omega)} = \exp i\left[\left(P - \frac{1}{2}\Sigma\right)\omega\right] \hat{J}_i^{(\pi)} \psi_{(r,\theta,\phi)}, \tag{3.15}$$

where $\hat{J}_i$ is SU(2) generator given by eq. (3.9) and $\hat{J}_i^{(\pi)}$ is given by the following supersymmetrized form of eq. (2.8) :

$$\hat{J}_i^{(\pi)} = \varepsilon_{ijk} r_j v_k + \left(P - \frac{1}{2}\Sigma\right) r_i + \hat{S}_i \tag{3.16}$$

with $$\nu_k = p_k - \left(P - \frac{1}{2}\Sigma\right)V_k^D \tag{3.17}$$

and V_k^D as the potential of a Dirac monopole of unit strength. In eq. (3.16) $\hat{S}_i$ is the spin matrix given by eq. (3.7) with

$$\eta_1 = \sqrt{\frac{1}{2}}i\left[\gamma^3 + i\gamma^4\right],$$

$$\eta_2 = \sqrt{\frac{1}{2}}i\left[\gamma^1 + i\gamma^2\right]. \tag{3.18}$$

Choosing chiral basis

$$\gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $$\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we get $$(\eta_a, \eta_a^+) = 4\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}$$

and $$\hat{S}^i = \begin{pmatrix} 0 & 0 \\ 0 & 1/2\sigma^i \end{pmatrix}, \tag{3.19}$$

which is fully reducible form showing that the system under consideration comprises two spin-0 and one spin-1/2 particles. Then equation

$$R\overline{\psi} = \frac{K}{\alpha}\tilde{\psi}$$

reduces to the following supersymmetrized version of eq. (2.7)

$$R^{(\pi)}\psi = \frac{K}{\alpha}\psi, \tag{3.20}$$

where $$R^{(\pi)} = r\left[p_i - (P - 1/2\Sigma)V_i^D\right]^2 + r$$
$$+ \frac{(\lambda - P)^2 - P\Sigma + 1/4\Sigma^2}{r} - \frac{2\lambda r^i S^i}{r^2}. \tag{3.21}$$

Rescaling by α, eq. (3.20) may be written into the form given by eq. (2.1) where

$$H_D = \left[p_i - (P - 1/2\Sigma)V_i^D\right]^2 - K/|x|$$
$$\frac{\left[(\lambda - P)^2 - P\Sigma + 1/4\Sigma^2\right]}{|x|^2} \quad \frac{2\lambda x^i \hat{S}^i}{|x|^3}. \tag{3.22}$$

The spectrum of this hamiltonian may be obtained in the form of following supersymmetric generalization of eq. (2.12)

$$E = \frac{e_1^2 e_2^2}{4\left[\sqrt{j(j+1) - e_1^2 g_2^2 + 1/4(q-\lambda)^2} - 1/4(1-\chi)\hat{\alpha} + 1/2 + n\right]^2}$$

$$= E_{n,j,\hat{\alpha},\chi}, \tag{3.23}$$

where χ stands for chirality, $j(j + 1)$ gives the eigen values of the operator J^2 with J given by eq. (3.16) and $\hat{\alpha}$ denotes the eigen values of the operator $\hat{A}$ given by

$$\hat{A} = i[Q, S^+] + i[Q^+, S]. \tag{3.24}$$

The eigen functions corresponding to these energy eigen values (3.23) satisfy the following equations :

$$\begin{aligned}
\hat{H}_D\left|j, m, \hat{\alpha}, \chi, n\right\rangle &= E\left|j, m, \hat{\alpha}, \chi, n\right\rangle, \\
J^2\left|j, m, \hat{\alpha}, \hat{\chi}, n\right\rangle &= j(j+1)\left|j, m, \hat{\alpha}, \chi, n\right\rangle, \\
\hat{J}_3\left|j, m, \hat{\alpha}, \chi, n\right\rangle &= m\left|j, m, \hat{\alpha}, \chi, n\right\rangle, \\
\hat{A}\left|j, m, \hat{\alpha}, \chi, n\right\rangle &= \hat{\alpha}\left|j, m, \hat{\alpha}, \hat{\chi}, n\right\rangle, \\
\gamma^5\left|j, m, \hat{\alpha}, \chi, n\right\rangle &= \chi\left|j, m, \hat{\alpha}, \chi, n\right\rangle.
\end{aligned} \tag{3.25}$$

The eigen states $\left|j, m, \hat{\alpha}, \chi, n\right\rangle$ obviously belong to a representation of Osp (2,1) ⊕ SU(2)

Setting $\lambda = P$, the two lower components of eq. (3.22) become

$$H_p = (p - e_1 V)^2 - \frac{e_1 e_2}{|x|} - \frac{e_1 g_2 x.\sigma}{|x|^3}, \tag{3.26}$$

which is the Hamiltomian of Pauli's equation for the spin –.1/2 particle of charge e_1 (and mass = 1/2) in the field of dyon carrying electric and magnetic charges e_2 and g_2 respectively. For this case, eq. (3.23) reduces to

$$E_{n,j,\hat{\alpha}} = \frac{e_1^2 e_2^2}{4\left[\sqrt{j(j+1) - e_1^2 g_2^2 + 1/4} + 1/2(1-\hat{\alpha}) + n\right]^2}, \tag{3.27}$$

which is Pauli's generalization of eq. (2.12).

For $\hat{H}_D$ given by eq. (3.22), the supersymmetric equation for the fermion moving in the field of a dyon may be written as follows :

$$\hat{H}_D \psi = E\psi, \tag{3.28}$$

where eigen values E are given by eq. (3.23) and the corresponding eigen functions satisfy eq. (3.25).

4. Conclusion

The eq. (3.28) can be interpreted as describing the quantum dynamics of a supermultiplet of two spin-0 and one spin-1/2 particles in a Coloumb field. It can also be looked at as an ensemble of two Schrödinger and one Pauli equation each describing the motion of a particle with electric charge-e_2 in the field of dyon with electric charge e_1 and with magnetic charges respectively equal to $(e_1 g_2 - 1/2)/e_1$, $(e_1 g_2 + 1/2)/e_1$ and g_2. Taking e_1 as the electric charge of the point particle sitting at the origin, we can fix the electric charge of the supermultiplet to be e_2. From the coupling to the potential V_i^D, we see that the spin-0 particles must be assigned magnetic charges equal to $g_2 \pm 1/2e_1$ while the spin-1/2 particle will have its magnetic charge equal to g_2.

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