NUCLEAR RADIUS, PROTON CONTENT, ELECTRON SCATTERING, CHARGE DENSITY, MAGIC NUMBERS AND BORN-APPROXIMATION MODIFICATION

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ABSTRACT. The r.m.s radii of nuclei, with z protons are determined closely in terms of $E_n^\circ(A)$, the binding energy per nucleon without coulomb and asymmetry energies. Probing from the other side, it is observed that the form factor $F(q)$ for light or heavy nuclei, determined from experimental $\sigma(\theta)$ values for high energy electrons (Hahn et al 1956) and point charge scattering $\sigma_M(\theta)$, is representable by a simple relation, superposed with small fluctuations. With the help of Born's approximation the $F(q)$ relation is transformed into an expression for ρ, which gives a simple relation for r.m.s radii of nuclei, consistent with that in terms of E°_n. The expression for ρ, so dictated by form-factor data and Born's approximation requires a normalising factor N, whose reciprocal must necessarily measure the factor to be associated with Born's approximation, when applied to nuclei. The fluctuations superposed on the basic relation for the form-factor gives rise to variations of charge density, with maxima at proton contents which are magic numbers. The relations determining basic nuclear characteristics are all expressible in terms of E_n°.

It has been shown (Dutta 1966) that the binding energy per nucleon $E_n^0(A)$, without columb and asymmetry energy corrections, determines many nuclear characteristics satisfactorily. It is considered to be a measure of the internucleonic force and is expressed as :

$$E_n^0(A) = 0.399 \ln (4.654\times 10^4.A^4) \text{ Mev.} \qquad (1)$$

The optimum proton number $Z_0(A)$ in isobaric nucleii is determined by the relation,

$$Z_0/A = 0.842 \exp(-0.059\, E_n^0) \qquad (2)$$

It may be compared with the mass-formula relation (De Bendetti, 1964),

$$Z_0/A = (1.98+0.015A^{2/3})^{-1} \qquad (2a)$$

Similarly, the coulomb energy $U_{cn}^0(Z_0, A)$ per nucleon, of strongly bounded nuclei corresponding to equivalent uniform radius, is determined by the relation,

$$U_{cn}^0/E_n^0 = 1.524\times 10^{-2} \exp (0.236 E_n^0) \qquad (3)$$

The expression for r.m.s. radius 'a' as $(3/5)^{3/2}e^2.Z_0^2/(A.U_{cn}^0)$, gives us, from eqns. (2) and (3),

$$\text{'}a\text{'} = 37(Z/E_n^0) \exp(-0.295\, E_n^0) fm, \qquad (4)$$

In table I, the experimental values of Z_0 estimated from tables (Konig *et al*, 1962) are compared with the values calculated from equations (2) and (2a). Table II compares the calculated and experimental r.m.s radius '*a*' (Hahn, *et al* 1956).

TABLE I

(Z_0 in isobaric nuclei)

A	13	21	31	51	73	111	139	175	197	209	235	245
Z_0(expt)	6.4	10.1	14.6	23.0	31.9	47.5	57.1	70.3	78.4	82.4	91.7	95.6
Z_0(eqn2)	6.7	10.3	14.7	23.0	31.9	46.6	57.1	70.3	78.3	82.6	91.9	95.4
Z_0(eq2a)	6.3	10.0	14.6	23.5	32.6	47 7	58 3	71.5	79.2	83.3	92.1	95.5

TABLE II

(r.m.s. radius 'a' in fm)

nucleus	He^{4}*	C^{12}	Mg^{24}	Si^{28}	S^{32}	Ca^{40}	V^{51}	Co^{59}	In^{115}	Sb^{123}	Au^{197}	Pb^{208}	Bi^{209}
'a' Hofst.	1.61	2.37	2.93	3 04	3.19	3.52	3.59	3 83	4.50	4.63	5 32	5 42	5.52
a(eqn 4)	1.49	2.35	3.00	3.17	3.32	3.61	3.57	3.83	4.63	4.62	5 32	5.41	5 45

*$E_n°$ is calculated by modified relation (Dutta 1966) for small nuclei ; Zo value is used.

Validity of eqn. (4) for all nuclei, as seen in Table II, implies a general form of expression for nuclear charge density and electron scattering. To arrive at the charge density expression search was made for a suitable expression for the form factor $F(q)$ tentatively calculated from experimental $\sigma(\theta)$. (Hahn *et al*, 1956) by relation,

$$|F(q)|^2 = \sigma(\theta)/\sigma_M(\theta); \quad \sigma_M(\theta) = \left(\frac{Ze^2}{2E}\right)^2 \cdot \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} \quad \quad (5)$$

It was observed that the form factor for different nuclei decrease either exponentially with q or by the relation $(\alpha+\beta q^2)^{-2}$, with an apparently periodic function, superposed on the decrease. The exponential form of $F(q)$ would require by Born's approximation, the charge density to be determined by $(\alpha+\beta r^2)^{-2}$. It would give us a divergent expression for r.m.s. radius 'a'. It is therefore, considered that the basic expression for $F(q)$ is of the form $(\alpha+\beta q^2)^{-2}$ which obtains $\rho(r)$ as an exponentially decreasing function of r. The values of $F(q)^{\frac{1}{2}}$ for Co, In, and Au plotted against q^2, in Fig. 1, justifies the consideration. It is also considered that the term in addition to $(\alpha+\beta q^2)^{-2}$ to determine $F(q)$ is of the form $\phi(q)/q$. The $\phi(q)$ values are plotted in Fig. 2 and show Gaussian distribution of points at the positions of maxima and minima in Fig. 1.

The cause and nature of the superposed maxima and minima, may be correlated with the fact that some nuclei, with particular charges, which are unrelated,

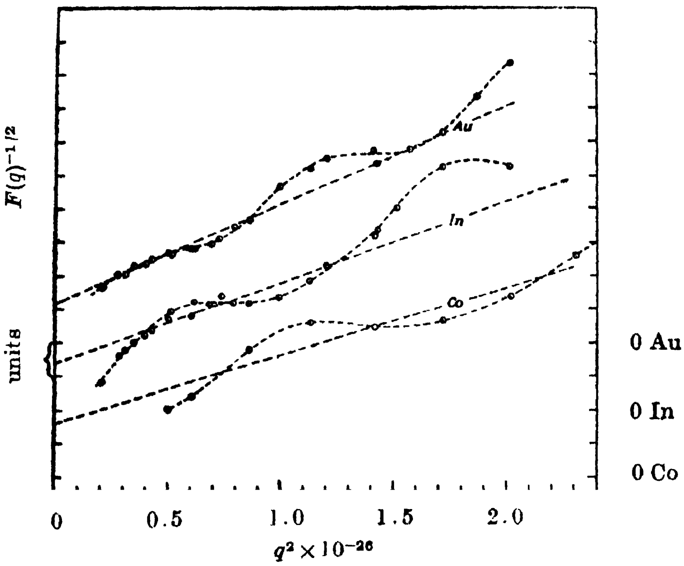

Fig. 1. $F(q)^{-1/2}$ against q^2.

are comparatively more strongly bound. The underlying principle of the shell model is that in heavy nuclei, subcompositions with these charges are more strongly bound. By implication they should have larger charge densities at unrelated values of r. This is expected to be reflected on the general nature of the form factor expression as observed.

For correspodence with the nature of the form-factor characteristics, we use Born approximation at this stage and put the charge density expression, as

$$\rho(r) = \rho^*\exp(-r/r_0) \pm (\rho^*/r) \sum_i k_i\pm \exp(-r^2/4r_i^2).\sin(q_i r). \quad \ldots (6)$$

$$= \rho^*\exp(-r/r_0)[1+\delta(r)].$$

$$= \rho_I + \rho_{II}.$$

where r_0, r_i and $k_i\pm$ have the dimensions of $r(fm)$ and q that of $r^{-n}(fm)$. Hence

$$`a^2\text{'}(r.m.s) = \frac{24r_0^5 \pm \Sigma_i 8\sqrt{\pi}(k_i\pm .q_i).r_i^5(3/2-q_i r_i)\exp(-q_i^2 r_i^2)}{2r_0^3 \pm \Sigma_i 2\sqrt{\pi}(k_i\pm .q_i)r_i^3 \exp(-q_i^2 r_i^2)}. \quad \ldots (7)$$

$$F(q) = \frac{8\pi\rho^* r_0^3}{(1+r_0^2 q^2)^2} \pm \frac{2\pi^{3/2}.\rho^*}{q} . \sum_i k_i \pm r_i \; [\exp\{-r_i^2(q-q_i^2)\}.-\exp\{-r_i^2(q+q_i)^2\}].$$

$$\frac{1}{(\alpha+\beta q^2)^2} \pm \frac{\pi^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\beta^{-3/2}}{4q} . \Sigma_i k_i \pm . r_i \exp\{-r_i^2(q-q_i)^2\} \quad \ldots (8)$$

where, $\beta/\alpha = r_0^2$; $8\pi\alpha^{\frac{1}{2}}\beta^{3/2}\rho^* = 1$, ρ^* is not normalised, .. (8a)
and $\exp\{-r_i^2(q+q_i)^2\}$, is neglected in comparison with $\exp\{-r_i^2(q-q_i)^2\}$.

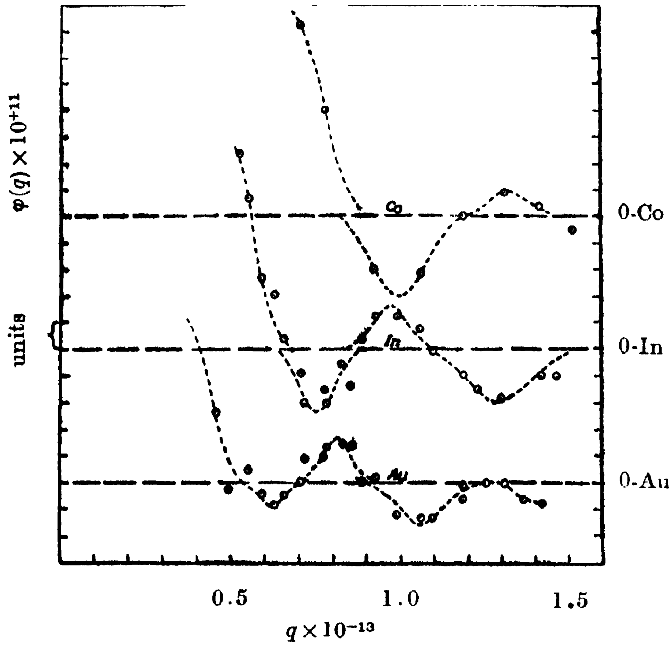

Fig. 2. $\varphi(q)$ against q.
$\varphi(q) = \{F(q)-(\alpha+\beta p^2)^{-2}\}q$

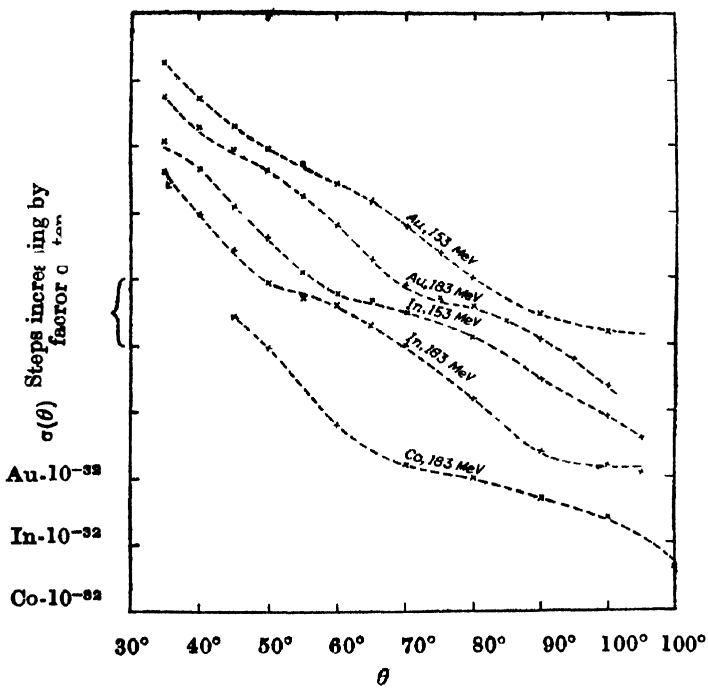

Fig. 3. $\sigma(\theta)$ **against** θ **(Hahn et al 1956)**

The parameters α and β and hence r_0 and ρ^* and the sets of parameters k_i, q_i and r_i, in table III, are determined from Figs (1) and (2). They give us, in accordance with equations (5) and (8), the $\sigma(\theta)$ values for electron scattering. The calculated values of $\sigma(\theta)$ are shown in Fig. 3 by continuous lines. The experimental points on it agree well in all cases. The evaluated parameters $k_i\pm$, q_i, r_i make the second terms in the numerator and denominator of eqn(7) for 'a^2' insignificantly small. Thus, in view of equns (7) and (4)

$$`a' = \sqrt{12}.r_0; \quad r_0 = 10.68\ Z/E_n{}^0 \exp(-0.295E_n{}^0) fm. \tag{9}$$

TABLE III

Parameters

Co			In			Au		
$\rho^* \times 10^{-36}$	r_0(fm)	a(fm)	$\rho^* \times 10^{-36}$	r_0(fm)	a(fm)	$\rho^* \times 10^{-36}$	r_0(fm)	a(fm)
11.13	1.118	3.873	8.065	1.309	4.533	7.382	1.560	5.404
$k_i \pm \times 10^{+15}$	q_i(fm)$^{-1}$	r_i(fm)	$k_i \pm \times 10^{15}$	q_i(fm)$^{-1}$	r_i(fm)	$k_i \pm \times 10^{15}$	q_i(fm)$^{-1}$	r_i(fm)
+8.39	0.675	7.50	+6.88	0.512	12.16	+7.155	0.372	10.38
−2.76	1.000	8.78	−1.72	0.760	15.59	−0 005	0.620	24.46
+0.71	1.324	11.43	+1.43	0.980	14.06	+1.420	0.816	14.55
			−3.21	1.312	6.94	−1.920	1.064	10.14
						−0.683	1.416	15.50

The charge densities $\rho(r)$ and $\rho_I(r)$ as also the variations from basic density $\rho_I(r)$ measured by $\delta(r)$ and $\delta_{III}\pm(r)$, a constituent of $\delta(r)$, have been calculated by equation (6) and are shown in Fig. 4. The $\delta(r)$ and $\delta_{III}\pm(r)$ values show maxima at

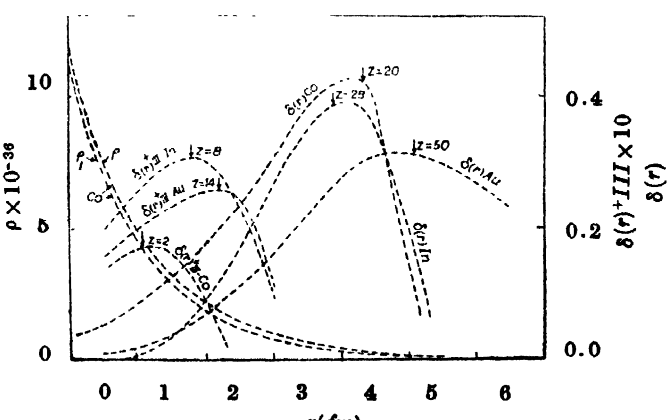

Fig. 4. $\rho(r)$, $\rho_1(r)$, $\delta(r)$, $\delta_m+(r)$ against r.

particular values of r_m for different nuclei. The expression for basic charge density $\rho_I(r)$ determines the average charge $Z(r_m)$, contained upto r_m, by the relation

$$Z(r_m)/Z = \int_0^{r_m} r^2 \exp(-r/r_0)dr / \int_0^{\infty} r^2 \exp(-r/r_0)dr.$$

$$= 1-(1+r_m/r_0+r_m^2/2r_0^2)\exp(-r_m/r_0) \quad \text{.. (10)}$$

The calculated values of Z at r_m corresponding to $\delta(r)$ and $\delta_{III}^{+}(r)$ maxima are near 20, 28, 50 and 2,8, 14 respectively, for the nuclei Co, In and Au, indicated in Fig. 4. The charge values are mostly the magic numbers, where stronger binding was expected.

To normalise the charge density, we have

$$1 = N.4\pi \int_0^{\infty} r^2\rho(r)dr = 8\pi N\rho^* r_0^3, \quad \text{.. (11)}$$

since the 2nd term in equation (6) has insignificant contribution. Thus in view of equation (8a) the normalising factor $N = \alpha^2$ and the normalised charge density at $r = 0$, is

$$\rho_{ON} = N.\rho^* = \frac{1}{8\pi r_0^3} \quad \text{.. (12)}$$

it implies that the Born approximation value of $F(q)_B$ obtained from $\rho(r)$ requires to be multiplied by a numerical factor $1/\alpha^2$ to correspond to $F(q)_\sigma$ calculated by equation (5), where,

$$F(q)_\sigma/F(q)_B = 1/\alpha^2 = 1.618\times10^{-2}\exp(0.295\, E_n^0) = 0.894.\, A.U_{cn}^0/(Z.E_n^0) \quad \text{... (13)}$$

Nuclear characteristics are, thus, determined by the relations :

$$\rho(r) = \rho_{0N}[\exp(-r/r_0) \pm \sum_i (k_i\pm/r)\,.\exp(-r^2/4r_i^2)\,.\sin(q_i r)]$$

$$\rho_{0N} = 1/(8\pi r_0^3); \quad a^2(r.m.s.) = 12.r_0^2.$$

$$F(q) = (4\pi/\alpha^2 q).\int_0^{\infty} r.\rho(r)\sin(qr)dr; \quad \sigma(\theta) = \sigma_M(\theta).F(q)^2.$$

$$1/\alpha^2 = 1.618\times10^{-2}.\exp(0.295\, E_n^0) = 0.984.A.U_{cn}^0/(Z.E_n^0).$$

$$r_0 = 10.68(Z/E_n^0)\exp(-0.295E_n^0)fm.$$

$$U_{en}^0/E_n^0 = 1.524\times 10^{-2}\exp(0.236E_n^0).$$

$$Z/A = 0.842\exp(-0.059\,E_n^0).$$

$$E_n^0 = 0.399\ln(4.654\times 10^4.A^4).\text{Mev}.$$

The irregularities determined by k_i, r_i, q_i's are unpredictable.

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