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Equilibrium forms of two uniformly charged drops

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Abstract : The equilibrium form of two separate drops, assuming their forms to be deformed spheroids, is considered. The saddle point shapes of a single drop, assuming it to be a deformed form of two touching equal spheroids, are obtained. Numerical computations to get the equilibrium form are carried out as illustrative example.

Keywords : Deformed ellipsoids, saddle point shapes, equilibrium forms

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1. Introduction

In the context of fission and fusion of different nuclei, the problem of finding the equilibrium forms of charged drops and description of the saddle point shapes have been the subject of many authors [1–15]. In the field of heavy ion physics, this work based on macroscopic models, (such as the liquid drop model), makes it possible to determine the energy needed to overcome the interaction barrier between nuclei.

The description of saddle point shapes of a uniformly charged drop or rotating by a deformed ellipsoid of revolution has been considered in previous works [1–3] by using a number of deformation parameters about an ellipsoid of revolution. For small values of the deformation parameter, when the neck thickness of the saddle point shape is small, the description of the drop by one deformed ellipsoid fails and even the Swiatecki results [4] are doubtful.

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As an alternative one should consider two touching deformed ellipsoids to describe the saddle point shape.

In this paper we shall first consider the equilibrium form of two separate drops, assuming their forms to be deformed spheroids. Next we follow a similar method to find the saddle point shape of a single drop by taking it to be a deformed form of two touching spheroids. The trial is made taking into consideration all the possible deformation parameters expressed through the two parameters α_0 and α_1 defined in the text. Moreover, the mutual interaction between the distortions of the two neighbouring nuclei, and that between the distortion of nuclei and the original ellipsoide representing the other nuclei are considered. This in turn, is expected to give better results in determining the equilibrium form for the considered system.

2. Description of the method

It is required to find the equilibrium forms of two separated uniformly charged drops of the same charge density ρ, and distance h between their mass centers (Figure 1). We consider for simplicity the forms to be axially symmetric about the line joining their mass centers.

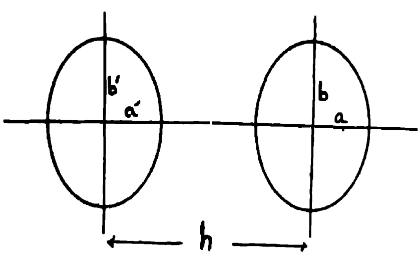

Figure 1. Diagram for the two uniformly charged drops in the form of spheroids having a common symmetry axis.

Since for separated drops, they approximately take the forms of oblate spheroids, when they are far enough, we shall consider the forms to be slightly deformed ellipsoids of revolution. This is also owing to the fact that this approximation worked well in finding the saddle point shapes of a single drop [1] and for a rotating drop [2].

Using elliptic coordinates (u, v) to express the position of any point with respect to either ellipsoids, the deformation of the surfaces is expressed by the following relations :

$$\left.\begin{aligned} u_1 &= u_0\left(1 + \frac{u_0^2}{\left(u_0^2 - v^2\right)}\sum_n \alpha_n P_n(v)\right) = u_0(1 + \Delta(v)), \\ u_1' &= u_0'\left(1 + \frac{u_0'^2}{\left(u_0'^2 - v^2\right)}\sum_n \alpha_n' P_n(v)\right) = u_0'(1 + \Delta'(v)). \end{aligned}\right\} \tag{1}$$

The parameters α_n in eq. (1) are considered small.

We shall express all energies in terms of the surface energy of a sphere having the same volume as the sum of volumes of the two drops. Also we express the dimensions of the length in terms of the radius R of such a sphere. If a, b; a', b' are axes of the original ellipsoids (a, a' along the symmetry axis), then we have :

$$\begin{aligned} ab^2 &= V, \\ a'b'^2 &= V' = 1 - V. \end{aligned} \tag{2}$$

(Note that $1/u_0^2 = 1 - b^2/a^2 = 1 - V/a^3$; $1/u_0'^2 = 1 - V'/a'^3$).

From the constancy of volume, and the position of the center of mass of each drop, the deformation parameters α_0, α_1 (or α_0', α_1') can be expressed in terms of α_2, α_3, α_4, ... (or $\alpha_2', \alpha_3', \alpha_4', \ldots$) as follows :

$$\left.\begin{aligned} \alpha_0 &= -\frac{1}{4} \sum_{n,n'=2} E_{n,n'} \alpha_n \alpha_{n'}, \\ \alpha_1 &= -\frac{3}{4} u_0^2 \sum_{n,n'=2} F_{n,n'} \alpha_n \alpha_{n'}, \end{aligned}\right\} \tag{3}$$

where

$$\begin{aligned} E_{n,n'} &= 2 \int_{-1}^{+1} \frac{P_n(v) P_{n'}(v)}{(1 - v^2/u_0^2)^2} dv \\ &= \sum_{m=0} (4m+1) \langle n, n', 2m \rangle \left(u_0 Q_{2m}(u_0) - u_0^2 Q_{2m}'(u_0) \right), \end{aligned}$$

$$\begin{aligned} F_{n,n'} &= \int_{-1}^{+1} \frac{v(3u_0^2 - v^2)}{(u_0^2 - v^2)^2} P_n(v) P_{n'}(v)\, dv \\ &= \sum_{m=0} (4m+3)) \langle n, n', 2m+1 \rangle \left(Q_{2m+1}(u_0) - u_0 Q_{2m+1}'(u_0) \right). \end{aligned}$$

Here the bracket $\langle n, m, 1 \rangle$ stands for the integral

$$\langle n, m, 1 \rangle = \int_{-1}^{+1} P_n(v) P_m(v) P_1(v) dv, \tag{4}$$

and Q_n, Q_n' are the Legendre function of the second kind and their first derivative.

3. The total surface energy of the drops

The total energy ξ of the two deformed spheroids can be expressed as the sum of several contributions (*i–iii*).

i. The total surface energy $\xi_s + \xi_s'$ *of the two drops* :

ξ_s can be expressed up to the second power in $\Delta(v)$ as :

$$\xi_s = \xi_s^{(0)} + \sum CS(n) \alpha_n + \frac{1}{2} \sum DS(n, n') \alpha_n \alpha_{n'}, \tag{5}$$

where
$$\xi_s^{(0)} = \frac{1}{2}a^2 \int_{-1}^{+1} \frac{dv}{u_0^2}\sqrt{(u_0^2-1)(u_0^2-v^2)} = \frac{1}{2}(b^2 + a^2 t_0 Q_0(t_0)), \tag{6}$$

$$CS(n) = \frac{1}{2}a^2 \int_{-1}^{+1} \frac{u_0^2\,dv}{\sqrt{(u_0^2-1)(u_0^2-v^2)^3}}(2u_0^2 - 1 - v^2)P_n(v)$$
$$= \frac{a^4}{b^2}P_n(0)\big(t_0 Q_n(t_0) - t_0^2 Q_n'(t_0)\big), \tag{7}$$

$$DS(n,n') = \frac{1}{2}a^2 \int_{-1}^{+1} \frac{u_0^4\,dv}{\sqrt{(u_0^2-1)(u_0^2-v^2)^1}}\left[\left(\frac{3t_0^2}{u_0^2-v^2} + \frac{1}{t_0^2}\right.\right.$$
$$\left.\left. - \frac{5t_0^2(1-t_0^2)}{(u_0^2-v^2)^2}\right)P_n(v)P_{n'}(v) + (1-v^2)P_n'(v)P_{n'}'(v)\right], \tag{8}$$

and
$$t_0^2 = 1 - u_0^2. \tag{9}$$

ii. The self coulomb energy of the two spheroids $(\xi_c + \xi_c')$:

ξ_c can be expressed up to second power in α_n as :

$$\xi_c = \xi_c^{(0)} + \sum_n CC(n)\alpha_n + \frac{1}{2}\sum_{n,n'} DC(n,n')\alpha_n\alpha_{n'}, \tag{10}$$

where
$$\xi_c^{(0)} = \frac{(ab^2)^2}{a}u_0 Q_0(u_0), \tag{11}$$

$$CC(n) = \frac{5}{2}a^3b^2\left[2u_0Q_0(u_0)\delta_{n,0} - u_0Q_2(u_0)\frac{2}{5}\delta_{n,2}\right], \tag{12}$$

$CC(n) = 0$ for $n = 1$, $n > 2$,

$$DC(n,n') = \frac{5}{2}a^3b^2\left[u_0Q_0(u_0) - u_0Q_2(u_0)P_2(u_0)\right]E_{n,n'}$$
$$+ \frac{15}{2n+1}a^5\left[u_0P_n(u_0)Q_n(u_0) + \frac{1}{2}\frac{b^2}{a^2}u_0^4Q_2'(u_0)\right]\delta_{n,n'}. \tag{13}$$

iii. The mutual potential energy between the two drops:

This is the sum of three contributions $(a+b+c)$:

(*a*) *The mutual potential energy* $\xi_m^{(0)}$ *between the original spheroids.*

This is already given as :

$$\xi^{(0)} = \frac{VV'}{8hzz'}\left[1 + 11(z+z') + \frac{1-10(z+z')-15(z-z')^2}{z+z'+1}w_2Q_0(w)\right.$$
$$\left. - \frac{4z'(5+z'-5z)}{1+z-z'}wQ_0(w) - \frac{4z(5+z-5z')}{1+z'-z}w'Q_0(w')\right], \tag{14}$$

where
$$\left.\begin{aligned} z &= x^2 = \frac{\gamma^2}{h^2}, \qquad z' = x'^2 = \frac{\gamma'^2}{h^2} \\ w &= \frac{1+z-z'}{2x},\ w' = \frac{1+z'-z}{2x'},\ w_2 = \frac{1+z-z'}{2xx'} \end{aligned}\right\} \tag{15}$$

and
$$\gamma^2 = a^2 - b^2,\ \gamma'^2 = a'^2 - b'^2.$$

All the obtained expressions are functions of u_0^2 and γ^2 where $\gamma^2 u_0^2 = a^2$ is always positive.

(*b*) *The mutual potential energy $\xi_{md} + \xi'_{md}$ between an original ellipsoide and the distortion of the other*:

For example, between the left ellipsoide (a', b', b') and the distortion of the right ellipsoide we have :

$$\xi_{md} = 2\pi\gamma^3 \int_{-1}^{+1} \Big[(u_0^2 - v^2)(\phi(u',v'))_{u=u_0}\, u_0 \Delta + \Big\{2u_0(\phi(u',v'))_0 + (u_0^2 - v^2)\left(\frac{\partial\phi(u',v')}{\partial u}\right)_0\Big\}\frac{1}{2}u_0^2\Delta^2\Big]dv, \tag{16}$$

where ϕ (u', v') is the potential of the left spheroid at any point (u', v') outside, and is expressed as :

$$\begin{aligned}\phi(u',v') &= \frac{5}{4\pi}\frac{a'b'^2}{\gamma'}\left(Q_0(u') - Q_2(u')P_2(v')\right) \\ &= \frac{5}{4\pi}a'b'^2 f(u_0, v).\end{aligned} \tag{17}$$

The $f(u_0, v)$ is the expression after transforming u', v' to $u = u_0$, v, the surface of the right spheroid. Similarly,

$$\begin{aligned}\frac{\partial\phi(u',v')}{\partial u} &= \frac{5}{4\pi}\frac{a'b'^2}{\gamma'}\left(Q_0'(u') - Q_2'(u')P_2(v')\right)\frac{\partial u'}{\partial u} \\ &\quad - 3v'Q_2(u')\frac{\partial v'}{\partial u} = \frac{5a'b'^2}{8\pi\gamma'^2}g(u_0, v).\end{aligned} \tag{18}$$

In order to compute the integrals in eq. (16), we first use the relations between u', $v'$with respect to the left ellipsoid and the coordinates $u = u_0$, v with respect to the right spheroid :

$$\begin{aligned} v' &= \frac{1}{\gamma' u'}(h + av), \\ \gamma'^2 u'^2 &= \frac{1}{2}(r+s), \\ r &= (h+av)^2 + b^2(1-v^2) + \gamma'^2, \end{aligned} \tag{19}$$

and
$$s^2 = r^2 - 4\gamma'^2(h+av)^2.$$

Thus,
$$f(u_0,v) = \frac{1}{\gamma' u'}\left(u'Q_0(u') - u'Q_2(u')P_2(v')\right),$$

$$g(u_0,v) = \frac{1}{\gamma' u'}\Big[(Q_0'(u') - Q_2'(u')P_2(v') - 3u'Q_2(u'))(hv+a)$$
$$+ \frac{1}{s}(Q_0'(u') - Q_2'(u')P_2(v') + 3u'Q_2(u'))(r(a+hv)$$
$$- 2\gamma'^2 v(h+av))\Big]. \tag{20}$$

Now expressing ξ_{md} as an expansion in α_n, we obtain

$$\xi_{md} = \sum_n Cm(n)\alpha_n + \frac{1}{2}\sum_{n,n'} Dm(n,n')\alpha_n \alpha_{n'}, \tag{21}$$

$$Cm(n) = \frac{5}{2}a^3 a' b'^2 \int_{-1}^{+1} f(u_0,v)P_n(v)dv,$$

$$Dm(n,n') = \frac{5}{2}a^3 a' b'^2 \int_{-1}^{+1} \frac{u_0^4 dv}{(u_0^2 - v^2)^2}\Big(2f(u_0,v)$$
$$+ \frac{a}{2\gamma'}\left(1 - \frac{v^2}{u_0^2}\right) g(u_0,v)\Big)P_n(v)P_{n'}(v). \tag{22}$$

On the other hand, the mutual potential energy between the right spheroid (a, b, b) and the distortion of the left spheroid is

$$\xi'_{md} = \sum_n C'_{md}(n)\alpha'_n + \frac{1}{2}\sum_{n,n'} D'_{md}(n,n')\alpha'_n \alpha'_{n'}, \tag{23}$$

where $C'_{md}(n), D'_{md}(n,n')$ are given by the same expressions (22), on replacing u_0, a, b, h by $u'_0, a', b', -h$.

(*c*) *The third contribution will be the mutual potential energy between the two distortions.*

This is obtained in the form :

$$\xi^{(2)}_{md} = \int_{-1}^{+1} \phi_s(u',v')2\pi\gamma^3 u_0^3 \sum_n \alpha_n P_n(v)dv$$
$$= \frac{15}{2}a'^2 a^3 \sum_{n,n'} \alpha'_{n'}\alpha_n u'_0 P_{n'}(u'_0)\int^{1} Q_{n'}(u')\, P_{n'}(v')Pn(v)dv. \tag{24}$$

where $\phi_s(u', v')$ is the potential of a surface deformation defined by eq. (1) of the left spheroid at any point outside and is given as

$$\phi_s(u',v') = \frac{15a'^2}{4\pi}\Sigma_n \alpha'_n u'_0 p_n(u')Q_n(u')p_n(v'). \tag{25}$$

Expressing $$\xi_{md} = \frac{1}{2}\sum D_{md}(n,n')\alpha_n\alpha'_{n'}, \tag{26}$$

we find $$D_{md}(n,n') = 15a'^2a^3u'_0P_{n'}(u'_0)\int_{-1}^{+1} Q_{n'}(u')P_{n'}(v')P_n(v)dv. \tag{27}$$

Thus for even n',

$$D_{md}(n,n') = 15a^3a'^3u'_0P_{n'}(u'_0)\int_{-1}^{+1}\frac{1}{\gamma'u'}u'Q_{n'}(u')P_{n'}(v')P_n(v)dv$$

and for odd n',

$$D_{md}(n,n') = 15a^3a'^2u'_0P_{n'}(u'_0)\int_{-1}^{+1} Q_{n'}(u')P_{n'}(v')P_n(v)dv. \tag{28}$$

Thus, the total energy of the considered system of two deformed spheroids expressed in terms of α_2, α_3, ... can be written as :

$$\xi = \xi^{(0)} + \sum_{n=2} C(n)\alpha_n + \sum_{n=2} C'(n)\alpha'_n + \frac{1}{2}\sum_{n,n'=2} D(n,n')\alpha_n\alpha_{n'}$$
$$+\frac{1}{2}\sum_{n,n'=2} D'(n,n')\alpha'_n\alpha'_{n'} + \frac{1}{2}\sum_{n,n'=2} D''(n,n')\alpha_n\alpha'_{n'}, \tag{29}$$

where $$\xi^{(0)} = \xi_s^{(0)} + \xi_s^{(0)'} + 2X\left(\xi_c^{(0)} + \xi_c^{(0)'} + \xi_m^{(0)}\right), \tag{30}$$

X being the fissionality parameter, defined as half the ratio between the coulomb energy to the surface energy of a sphere of volume equal the sum of volumes of the two drops and of the same charge density.

Also $$C(n) = CS(n) + 2X(CC(n) + Cm(n)),$$
$$C'(n) = CS'(n) + 2X(CC'(n) + Cm'(n)), \tag{31}$$
$$D(n,n') = DSM(n,n') + 2X\left(DCM(n,n') + Dmm(n,n')\right),$$
$$D'(n,n') = DSM'(n,n') + 2X\left(DCM'(n,n') + D'mm(n,n')\right), \tag{32}$$

and $$D''(n,n') = 2XD_{md}(n,n'). \tag{33}$$

The equilibrium form of the drop is obtained by minimizing the total energy ξ as given by eq. (29) with respect to all its parameters. First we minimize with respect to the small deformation parameters $\alpha_2, \alpha_3, \cdots, \alpha'_2, \alpha'_3, \cdots$.

For this we write eq. (29) including a domain comprising of all α_1, α'_1 which we (in short) express as α_1^T. Eq. (36) is then equivalent to

$$\xi = \xi^{(0)} + \sum C^T(n)\alpha_n^T + \frac{1}{2}\sum D^T_{n,n'}(n,n')\alpha_n^T\alpha_{n'}^T. \tag{34}$$

The values of α_n^T corresponding to the minimum are obtained from the relations

$$\frac{\partial \xi}{\partial \alpha_i^T} = C_i^T(i) + \sum_j D^T(i,j)\alpha_j^T = 0,$$

or
$$\alpha_j^T = -\sum_j \left(D^T(i,j)\right)^{-1} C_j^T(j), \tag{35}$$

where $(D^T)^{-1}$ is the reciprocal of the total matrix :

$$D^T = \begin{pmatrix} D & D'' \\ D'' & D' \end{pmatrix}, \tag{36}$$

where the blocks D, D', D'' are matrices whose elements are given in eqs. (31)–(33). Substituting for these values of α_i^T in eq. (34) we obtain the values of ξ corresponding to the equilibrium values of α_i^T as

$$\xi_{\mathrm{eq}} = \xi^{(0)} - \frac{1}{2}\sum_{n,n'} \left(D^T(n,n')\right)^{-1} C^T(n) C^T(n'). \tag{37}$$

$\xi_{eq}(a, a')$ is thus a function of only two parameters a, a' and thus the equilibrium (or saddle point shapes) corresponding to given values of a, a' are obtained by finding two values of a, a' which make $\xi_{eq}(a, a')$ minimum or maximum.

4. Computations for mirror symmetrical drops

In this paper we carry numerical computations to find the equilibrium shape of two separate equal drops having mirror symmetry with respect to a plane perpendicular to the common symmetry axis and bisecting the distance h between their main centers. In this special case we set :

$$a = a',\ b = b',\ \alpha_n' = (-)^n \alpha_n. \tag{38}$$

We thus have

$$\xi = \xi^{(0)} + \sum_{n=2} C(n)\alpha_n + \frac{1}{2}\sum_{n,n'=2} D(n,n')\alpha_n \alpha_{n'}, \tag{39}$$

where
$$\xi^{(0)} = 2\xi_s^{(0)} + 2X\left(2\xi_c^{(0)} + \xi_m^{(0)}\right), \tag{40}$$

$$C(n) = 2CS(n) + 2X(2CC(n) + 2Cm(n)),$$

$$D(n,n') = 2DSM(n,n') + 2X\left(2DCM(n,n') + 2Dmm(n,n')\right)$$

$$+ \frac{(-)^{n'}}{2} D_{\mathrm{md}}(n,n') + \frac{(-)^n}{2} D_{\mathrm{md}}(n',n). \tag{41}$$

The obtained results from which the equilibrium form can be drawn, taking into consideration that the two drops are mirror symmetrical drops—are given in Table 1. The two coordinates indicated in the table are x and y where x is the distance measured from the center of mass of one of the two equal mirror symmetrical drops along the symmetry axis joining their centers and y is the corresponding distance measured in a perpendicular direction to the symmetry axis. The obtained equilibrium form is as shown in Figure 2.

Table 1. Computations for mirror symmetrical drops.

x_1	y_1
−0.5621084 D + 00	0.0000000 D + 00
−0.5337352 D + 00	0.4007342 D + 00
−0.4887200 D + 00	0.5585813 D + 00
−0.4327050 D + 00	0.6682536 D + 00
−0.3702614 D + 00	0.7480404 D + 00
−0.3049863 D + 00	0.8057159 D + 00
−0.2395696 D + 00	0.8460850 D + 00
−0.1758252 D + 00	0.8727363 D + 00
−0.1146992 D + 00	0.8885413 D + 00
−0.5629797 D − 01	0.8957376 D + 00
0.0000000 D + 00	0.8958506 D + 00
0.5530602 D − 01	0.8895987 D + 00
0.1108934 D + 00	0.8768694 D + 00
0.1678863 D + 00	0.8567840 D + 00
0.2270633 D + 00	0.8277811 D + 00
0.2887884 D + 00	0.7876066 D + 00
0.3530164 D + 00	0.7330691 D + 00
0.4193035 D + 00	0.6593016 D + 00
0.4867865 D + 00	0.5576308 D + 00
0.5541256 D + 00	0.4071745 D + 00
0.6194200 D + 00	0.0000000 D + 00

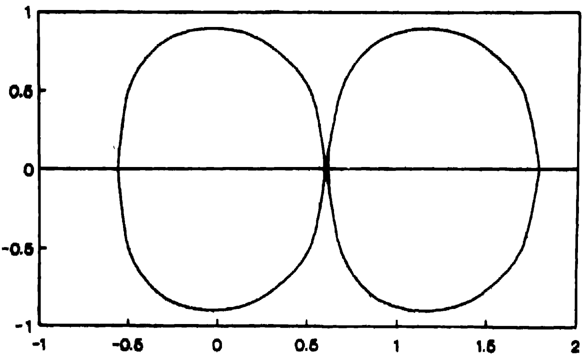

Figure 2. The form of separate drops at a distance $h = 1.185$ between the centers of mass.

5. Two touching drops

In order to express the saddle point shape for a single drop when fissionality parameter X is small, we consider the single drop as a deformation of two touching equal ellipsoids of revolution and having mirror symmetry (for $X = 0$, the saddle point shape is two touching equal spheres).

In this case, instead of applying the condition of constancy of the position of the center of mass we apply the condition

$$\left.\begin{array}{ll} \Delta(-1) = 0 & \text{for the right ellipsoid,} \\ \Delta(+1) = 0 & \text{for the left ellipsoid.} \end{array}\right\} \tag{42}$$

This can be achieved on using the expansion

$$\Delta(\nu) = \frac{u_0^2}{u_0^2 - \nu^2} \sum_{n=1} \beta_n S_n(\nu), \tag{43}$$

where
$$\left.\begin{aligned} S_n(\nu) &= \frac{1}{2}\left(P_n(\nu) + P_{n-1}(\nu)\right) \quad \text{for the right spheroid,} \\ &= \frac{1}{2}\left(-P_n(\nu) + P_{n-1}(\nu)\right) \quad \text{for the left spheroid.} \end{aligned}\right\} \tag{44}$$

The condition for the invariance of volume then becomes

$$\beta_1 = -\frac{1}{4} \sum_{n,n'=2} em(n,n')\beta_n \beta_{n'},$$

$$em(n,n') = 4\int_{-1}^{+1} \frac{u_0^4 d\nu}{\left(u_0^2 - \nu^2\right)^2} S_n(\nu) S_{n'}(\nu). \tag{45}$$

Evaluating as before, the surface energy, the coulomb energy, and the mutual potential energy between the two deformed touching spheroid drops one can get finally these quantities expressed in terms of $\beta_2, \beta_3, \ldots\ldots\ldots$ as :

$$\xi_s = \xi_s^{(0)} + \sum_{i=2} \beta_i C_s(i) + \frac{1}{2}\sum_{i,j} \beta_i \beta_j DSM(I,J), \tag{46}$$

where
$$DSM(i,j) = DS(i,j) - \frac{1}{2} C_s(i)\, em(i,j). \tag{47}$$

In this case (for each spheroid)

$$CS(n) = \frac{1}{2} a^2 \int_{-1}^{+1} \frac{u_0^2 d\nu}{\sqrt{\left(u_0^2 - 1\right)\left(u_0^2 - \nu^2\right)^3}} \left(2u_0^2 - 1 - \nu^2\right) S_n(\nu), \tag{48}$$

$$DS(n,n') = \frac{1}{2} a^2 u_0^4 \int_{-1}^{+1} \left[\left(\frac{1}{t_0^2} + \frac{3t_0^2}{u_0^2 - \nu^2} - \frac{5t_0^2 u_0^2}{\left(u_0^2 - \nu^2\right)^2}\right) S_n(\nu) S_{n'}(\nu) \right.$$

$$\left. + \left(1 - \nu^2\right) S_n'(\nu) S_{n'}'(\nu)\right] \frac{d\nu}{\sqrt{\left(u_0^2 - 1\right)\left(u_0^2 - \nu^2\right)^3}}, \tag{49}$$

And *em* (i, j) is as given by eq. (45).

Also we have, $$\xi_c = \xi_c^{(0)} + \sum_{n=2} CC(n)\beta_n + \frac{1}{2}\sum_{n,n'} DCM(n,n')\beta_n\beta_{n'}, \tag{50}$$

where $$DCM(n,n') = DC(n,n') - \frac{1}{2}CC(1)em(n,n'). \tag{51}$$

The coefficients of expansions are in this case :

$$CC(n) = \frac{5}{2}a^3b^2\int_{-1}^{+1}\left[u_0Q_0(u_0) - u_0Q_2(u_0)P_2(v)\right]S_n(v)dv, \tag{52}$$

$$\begin{aligned}
DC(n,n') = {} & \frac{5}{4}a^3b^2\left[u_0Q_0(u_0) - u_0P_2(u_0)Q_2(u_0)\right]em(n,n') \\
& + \frac{15}{4}u_0^4Q_2'(u_0)a^3b^2\int_{-1}^{+1}S_n(v)S_{n'}(v)dv \\
& + \frac{15}{4}a^5\left[u_0P_n(u_0)Q_n(u_0)\int_{-1}^{+1}P_n(v)S_{n'}(v)dv\right. \\
& \left. + u_0P_{n-1}(u_0)Q_{n-1}(u_0)\int_{-1}^{+1}P_{n-1}(v)S_{n'}(v)dv\right].
\end{aligned} \tag{53}$$

6. The mutual potential energy between an original spheroid (the left one) and the distortion of the right touching spheroid

Following eq. (21), we obtain in this case,

$$\begin{aligned}
\xi_{md} = {} & 2\pi a^3\int_{-1}^{+1}dv\left[\sum_n\beta_nS_n(v)\left(\phi(u',v')\right)_0 + \left\{2\phi(u',v')_0\right.\right. \\
& \left.\left. + \frac{\left(u_0^2 - v^2\right)}{u_0}\left(\frac{\partial\phi(u',v')}{\partial u}\right)_0\right\}\frac{1}{2}\frac{u_0^2}{\left(u_0^2 - v^2\right)^2}\sum_{n,n'}\beta_n\beta_{n'}S_n(v)S_{n'}(v)\right] \\
= {} & \sum_n Cm(n)\beta_n + \frac{1}{2}\sum_{n,n'}Dm(n,n')\beta_n\beta_{n'},
\end{aligned} \tag{54}$$

where $$Cm(n) = 2\pi a^3\int_{-1}^{+1}\left(\phi(u',v')\right)_0S_n(v)dv,$$

$$\begin{aligned}
Dm(n,n') = {} & 2\pi a^3\int_{-1}^{+1}\left\{2\phi(u',v')_{u=u_0} + \frac{\left(u_0^2 - v^2\right)}{u_0}\left(\frac{\partial\phi(u',v')}{\partial u}\right)_0\right\} \\
& \times \frac{u_0^4}{\left(u_0^2 - v^2\right)^2}S_n(v)S_{n'}(v)dv.
\end{aligned} \tag{55}$$

As $\phi(u', v')$ and $\dfrac{\partial\phi(u', v')}{\partial u}$ are given by eqs. (17), (18) we find in this case :

$$Cm(n) = \frac{5}{2}a^3 a' b'^2 \int_{-1}^{+1} f(u_0, v) S_n(v)\, dv,$$

$$Dm(n, n') = \frac{5}{2}a^3 a' b'^2 \int_{-1}^{+1} \left(2f(u_0, v) + \frac{\left(u_0^2 - v^2\right)}{2u_0^2 \gamma'^2} \gamma u_0 g(u_0, v) \right) \times \frac{u_0^4}{\left(u_0^2 - v^2\right)^2} S_n(v) S_{n'}(v)\, dv. \quad (56)$$

Finally, the mutual potential energy between the two distributions in this case is

$$\xi_{\mathrm{md}}^{(2)} = 2\pi a^3 \int_{-1}^{+1} \phi_s(u', v') \sum_n \beta_n S_n(v)\, dv, \quad (57)$$

where $\phi_s(u', v')$ (for left spheroid) in the considered case of two touching spheroids is represented as :

$$\phi_s(u', v') = \frac{15a'^2}{4\pi} \sum_n u_0' P_n(u_0') \frac{1}{2}(-\beta_n + \beta_{n+1}) Q_n(u') P_n(v'). \quad (58)$$

Thus, $$\xi_{\mathrm{md}}^{(2)} = \frac{1}{2}\sum_{n,n'} DMD(n, n')\beta_n \beta_{n'}, \quad (59)$$

where $$DMD(n, n') = \frac{15a^3 a'^2}{2} \sum_{n,n'} \int_{-1}^{+1} S_n(v)\, dv \left[-u_0' P_{n'}(u_0') Q_{n'}(u') P_{n'}(v') + u_0' P_{n'-1}(u_0') Q_{n'-1}(u_0') P_{n'-1}(v') \right]. \quad (60)$$

Similarly, on substituting for β_1 (or β_1') in terms of $\beta_2, \beta_3, \cdots$ (or $\beta_2', \beta_3', \cdots$) [eq. (54)], the following expression in terms of $\beta_2, \beta_3, \cdots$:

$$\xi_m = \sum_{n=2} Cm(n)\beta_n + \frac{1}{2}\sum_{n,n'} DMM(n, n')\beta_n \beta_{n'}, \quad (61)$$

where now $$DMM(n, n') = DM(n, n') - \frac{1}{2}Cm(1)\, em(n, n'). \quad (62)$$

Adding the expressions for ξ_s, ξ_c and ξ_m, one gets the total energy of the system.

The case of mirror reflection is obtained by setting as before :

$$a = a', \; b = b', \; \beta_n' = (-)^n \beta_n.$$

In order to find in this case the saddle point shape, the same procedure as described in the case of two separate drops will be followed.

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