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Disturbances in a piezo-quartz cantilever under electrical, mechanical and thermal fields

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Abstract : An attempt has been made in this paper to investigate analytically the disturbances produced in a piezo-quartz cantilever under the influence of three different fields, viz., electrical, mechanical and thermal. The cantilever is considered to have certain width, the upper and lower edges of which are free from load and the shearing forces having certain resultant are distributed along $x = 0$. The expression for the electric potential function is taken such that it is constant on $y = \pm c$ and also make the intensity = 0 along the length. We restrict ourselves to the xy plane and components of elastic displacements along x and y directions have been illustrated.

Keywords : Mechanical disturbances, piezo-electricity, cantilever

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The problem of investigating disturbances in piezo-electric media under different inputs have been studied in the literature [1–4] from the point of view of circuit theory. Several other researchers [5–9] have extended the work to find out the disturbances in piezo-electric media. In most of the works on the piezo-electricity, there is a trend to extend the elastic problems to corresponding piezo-electric problems and most of the workers have made use of classical solutions in purely elastic material. A particular area of piezo-electric problem is on bending of piezo-electric material and the problem of disturbances of piezo-quartz

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cantilever are of utmost importance to the physicists as well as to the Engineers for various practical uses in the field of science and technology.

The present work is confined to study the displacement of a cantilever of piezo-electric material under electrical, mechanical and thermal fields. The cantilever, in the present investigation, is considered to have certain width, the upper and lower edges of which are free from load [10]—the shearing forces having certain resultant are distributed along $x = 0$. We restrict ourselves to the xy plane and finally the components of elastic displacements along x and y directions have been illustrated. It is found that the displacements are partly linear, hyperbolic and exponential in nature.

As piezo-electricity is essentially the interaction of the electric and elastic fields in a crystal, we must therefore, define the electrical as well as the elastic state of the crystal and specify its electrical state by two variables—the electric field $\overline{E}$ and the electric displacement $\overline{D}$ and we specify its elastic state by two elastic variables—the stress T and the strain S. Referred to a rectangular system of axes X, Y, Z, the components of the electric field and the electric displacement are supposed to be E_i, D_i $(i = 1\text{-}3)$. Denoting the elastic stress and strain components by T_i, S_i $(i = 1\text{–}6)$ respectively, the stress equations of motion are

$$\frac{\partial T_1}{\partial x} + \frac{\partial T_6}{\partial y} + \frac{\partial T_5}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial T_6}{\partial x} + \frac{\partial T_2}{\partial y} + \frac{\partial T_4}{\partial z} = \rho \frac{\partial^2 v}{\partial t^2}, \qquad (1)$$

$$\frac{\partial T_5}{\partial x} + \frac{\partial T_4}{\partial y} + \frac{\partial T_3}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2},$$

where ρ is the mass per unit volume and u, v, w are the components of elastic displacement.

In free space, the electric displacement D_i satisfies the Gauss's divergence equation

$$\operatorname{div} D = \frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z} = 0 \qquad (2)$$

The linear piezo-electric constitutive relations between T, S, E and D which describe the interrelation among the electrical and elastic variables for piezo-electric materials are

$$S_k = \sum_{i=1}^{6} s_{ki} T_i + \sum_{j=1}^{3} d_{jk} E_j, \qquad k = 1, 2, 3, 4, 5, 6. \qquad (3)$$

$$D_j = \sum_{i=1}^{6} d_{ji} T_i + \sum_{l=1}^{3} e^T_{jl} E_l, \qquad j = 1, 2, 3, \qquad (4)$$

where the constant S^E_{ij} $(= S^E_{ji})$ is the elastic compliances at constant electric field strength E, $d_{ij} = d_{ji}$ is the piezo-electric strain constant while e^T_{ij} $(= e^T_{ji})$ is the dielectric permittivity at constant stress.

Besides these, we have the relations connecting the strain components and displacement components given by

$$S_1 = \frac{\partial u}{\partial x},\ S_2 = \frac{\partial v}{\partial y},\ S_3 = \frac{\partial w}{\partial z}, \tag{5}$$

$$S_4 = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},\ S_5 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x},\ S_6 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \tag{6}$$

In deriving the plane equations, we restrict ourselves to the xy plane and we represent the stress components by

$$T_1 = \frac{\partial^2 \phi}{\partial y^2},\ T_2 = \frac{\partial^2 \phi}{\partial x^2},\ T_6 = \frac{\partial^2 \phi}{\partial y \partial x},\ T_3 = T_4 = T_5 = 0, \tag{7}$$

where ϕ is the stress function. The components of electric field are represented by

$$E_1 = \frac{\partial V}{\partial x},\ E_2 = \frac{\partial V}{\partial y},\ E_3 = 0, \tag{8}$$

where V is the electric potential function.

For plane problems, we assume the piezoelectric relations to be

$$\begin{aligned} S_1 &= s_{11}^E T_1 + s_{12}^E T_2 + d_{11} E_1 + \mu_1^E \theta, \\ S_2 &= s_{12}^E T_1 + s_{11}^E T_2 - d_{11} E_1 + \mu_2^E \theta, \\ S_6 &= 2(s_{11}^E - s_{12}^E) T_6 - 2 d_{11} E_2, \\ D_1 &= d_{11}(T_1 - T_2) + e_{11}^T E_1 + p_1^S \theta, \\ D_2 &= -2 d_{11} T_6 + e_{11}^T E_2 + p_2^S \theta, \end{aligned} \tag{9}$$

where μ^E's are the thermo-elastic compliances at constant electric field strength E, p^S's are the thermo-piezo-electric moduli at constant stress and θ is the input temperature. From eqs. (7)–(9), we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{d_{11}}{e_{11}}\left(\frac{\partial^3 \phi}{\partial x^3} - 3\frac{\partial^3 \phi}{\partial y^2 \partial x}\right) - \left(p_1 \frac{\partial \theta}{\partial x} + p_2 \frac{\partial \theta}{\partial y}\right). \tag{10}$$

Here we choose θ [11], tne temperature input as a linear function of x and y as

$$\theta = \theta_0\left(e^{-\alpha x} + e^{-\beta y}\right),$$

where α and β are arbitrary constants. Eq. (10) becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{d_{11}}{e_{11}}\left(\frac{\partial^3 \phi}{\partial x^3} - 3\frac{\partial^3 \phi}{\partial y^2 \partial x}\right) + \theta_0\left(p_2 \beta e^{-\beta y} + p_1 \alpha e^{-\alpha x}\right). \tag{11}$$

For the problem of cantilever of width $2c$, the upper and lower edges of which are free from load and in which the shearing forces having a resultant P, are distributed along the $x = 0$, we assume for ϕ [7,8,10] as

$$\phi = Axy^3 + Bxy,$$

where the constants A and B are to be determined from the mechanical boundary conditions referred to above, *viz*.,

$$\left[T_6\right]_{y=\pm c} = 0 \quad \text{and} \quad -\int_{-c}^{+c} T_6 dy = P.$$

Applying boundary conditions we get,

$$\phi = -P/4c^3(xy^3 - 3c^2xy) \tag{12}$$

A form of V for which the electrical boundary condition can be reasonably satisfied [8], is given by

$$V = V_0 y(y^2 - c^2). \tag{13}$$

The expression for V leads to the condition that the potential is constant on $y = \pm c$ and also makes $E_1 = 0$ along the length. The constant V_0 is determined from the eqs. (11) and (12) as

$$V_0 = \frac{d_{11}}{e_{11}}\left(-P/4c^3\right) + \frac{\theta_0}{6y}\left(p_2\beta e^{-\beta y} + p_1\alpha e^{-\alpha x}\right). \tag{14}$$

The expression for V becomes

$$V = \left[\frac{d_{11}}{e_{11}}\left(-P/4c^3\right) + \frac{\theta_0}{6y}\left(p_2\beta e^{-\beta y} + p_1\alpha e^{-\alpha x}\right)\right]y(y^2 - c^2). \tag{15}$$

We get from eqs. (7) and (12)

$$T_1 = \partial^2\phi/\partial y^2 = -\frac{3P}{2c^3}xy, \quad T_2 = 0,$$

$$T_6 = -\partial^2\phi/\partial y\partial x = -\frac{3P}{4c^3}\left(c^2 - y^2\right). \tag{16}$$

Finally, to calculate the displacement components (u,v), we start with the eqs. (9) and (5), namely,

$$\frac{\partial u}{\partial x} = -\frac{3P}{2c^3}s_{11}^E xy + \mu_1^E\theta_0(e^{-\alpha x} + e^{-\beta y}), \tag{17.1}$$

$$\frac{\partial v}{\partial y} = -\frac{3P}{2c^3}s_{12}^E xy + \mu_2^E\theta_0(e^{-\alpha x} + e^{-\beta y}), \tag{17.2}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -2(s_{11}^E - s_{12}^E)\frac{3P}{4c^3}(c^2 - y^2) - 2d_{11}^2(-p/4c^3)(3y^2 - c^2)/e_{11}$$
$$- K\beta e^{-\beta y}(y^2 - c^2) + (Ke^{-\beta y} + Le^{-\alpha x})2y, \tag{17.3}$$

where $K\ (= d_{11}\theta_0 p_2\beta/3)$ and $L\ (= d_{11}\theta_0 p_1\alpha/3)$ are constants.

Integrating eqs. (17.1) and (17.2) we get,

$$u = -\frac{3P}{2c^3}s_{11}^E x^2 y + \mu_1^E\theta_0(e^{-\alpha x}/\alpha + e^{-\beta y}x) + f(y), \quad (18.1)$$

$$v = -\frac{3P}{2c^3}s_{12}^E xy^2 + \mu_2^E\theta_0(e^{-\alpha x}y - e^{-\beta y}/\beta) + f(x), \quad (18.2)$$

where $f(x)$ and $f(y)$ are functions of x and y respectively. Differentiating eq. (18.1) with respect to y and eq. (18.2) with respect to x and summing and using eq. (17.3) we get

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{3P}{2c^3}s_{11}^E x^2 - \mu_1^E\theta_0 x\frac{e^{-\beta y}}{\beta} + \frac{df(y)}{dy} - \frac{3P}{2c^3}s_{12}^E y^2$$

$$-\mu_2^E\theta_0 y\frac{e^{-\alpha x}}{\alpha} + \frac{df(x)}{dx}$$

$$= -2(s_{11}^E - s_{12}^E)\frac{3P}{4c^3}(c^2 - y^2) - 2d_{11}^2(-P/4c^3)(3y^2 - c^2)/e_{11}$$

$$-K\beta e^{-\beta y}(y^2 - c^2) + (Ke^{-\beta y} + Le^{-\alpha x})2y,$$

which can be written as

$$\frac{df(x)}{dx} - \frac{3P}{2c^3}s_{11}^E x^2 = m,$$

$$\frac{df(y)}{dy} - \frac{3P}{2c^3}s_{11}^E y^2 - 2\frac{d_{11}^2}{e_{11}}\frac{P}{4c^3}3y^2 + K\beta e^{-\beta y}(y^2 - c^2) - Ke^{-\beta y}2y = e,$$

where m and e are functions of x and y respectively.

Integrating again, we get

$$f(x) = \frac{P}{2c^3}s_{11}^E x^3 + \int m\,dx + h,$$

$$f(y) = \frac{P}{2c^3}\left(s_{11}^E + \frac{d_{11}^2}{e_{11}}\right)y^3 - Ke^{-\beta y}(y^2 + c^2) + \int e\,dy + g,$$

where h and g are the integrating constants. Putting the values of $f(x)$ and $f(y)$ into the eqs. (18.1) and (18.2), we get the components of elastic displacement u and v as

$$u = -\frac{3P}{2c^3}s_{11}^E x^2 y + \mu_1^E\theta_0\left(-\frac{e^{-\alpha x}}{\alpha} + e^{-\beta y}x\right) + \frac{P}{2c^3}\left(s_{11}^E + \frac{d_{11}^2}{e_{11}}\right)y^3$$

$$-d_{11}\theta_0 P_2\beta e^{-\beta y}(y^2 + c^2)/3 + \int e\,dy + g, \quad (19)$$

$$v = -\frac{3P}{2c^3}s_{12}^E xy^2 + \frac{P}{2c^3}s_{11}^E x^3 + \mu_2^E\theta_0(e^{-\alpha x}y - e^{\beta y}/\beta) + \int m\,dx + h. \quad (20)$$

Eqs. (19) and (20) give the components of elastic displacement along x and y direction respectively.

The variation of the component of the elastic displacements along x and y directions, have been shown in eqs. (19) and (20) respectively. As the term e is a function of y, so the expression $\int e dy$ in eq. (19) contains y-term only. We have investigated the variation of the component of the elastic displacement u along x direction, keeping y as constant, so the term $\int e dy$ in eq. (19) will behave as a constant part. Similarly, the term m is a function of x, so the expression $\int m dx$ in eq. (20) contains only x-term and it will give a constant contribution in the variation of the component of the elastic displacement v along y direction, keeping the value of x as constant. It is found from eqs. (19) and (20) that both the disturbances consist of some linear, hyperbolic, exponential and constant part with different coefficients

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