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Parameterization of the Coulombic bremsstrahlung spectrum

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Abstract: We show that it is possible to substantially improve the analytical characterization of the Coulombic bremsstrahlung cross section up to 2 MeV for all elements of the periodic table, by modifying the Born approximation with Elwert factor. A simple analytical expression is obtained by modification of the Elwert factor and introduction of an empirical higher-order Born correction. These are used together with the Bethe-Heitler expression to achieve an accuracy of predictions within 10% throughout the spectrum. To verify our analytical expression the bremsstrahlung spectra have also been calculated in the partial-wave approximation for nuclei of Al, Fe, Mo, W, Au and U in this energy range.

Keywords: Coulombic potential, bremsstrahlung, Elwert factor, Born approximation

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1. Introduction

Bremsstrahlung from nuclei (Coulombic bremsstrahlung) is one of the most important radiation process in plasma physics and astrophysics. Cross sections of this process often are used for estimation of bremsstrahlung spectra from ions and even from neutral atoms, since at high plasma temperatures

the error introduced in this way into the calculations is usually very small.

Despite the comparative simplicity of this problem, the analytical solution for point Coulomb bremsstrahlung cross sections (CBS) from nuclei can be obtained only with approximations. A general expression was obtained in [1] within the dipole approximation in the nonrelativistic regime of incident electron energies T_1. At small, "classical", electron energies it reduces to the well known classical electrodynamic expression [2], while at high incident energies one can obtain the nonrelativistic Born-approximation result modified by the Elwert factor [3]. However, at higher energies, relativistic effects should be taken into account, and nonrelativistic Elwert-Born approximation fails. It is not possible to obtain relativistic results for CBS as good as those obtained in the nonrelativistic case. The corresponding relativistic Born approximation, expressed by the Bethe-Heitler formula [4], together with the Elwert factor (EBH), does not give good results for high-Z elements in the relativistic regime [5]. Errors appear to be $O(Z\alpha)$ (order of $Z\alpha$) in the 100-1000 keV range (Z is the atomic number, α is the fine structure constant). (Note that in the high-energy limit the Bethe-Maximon approximation is valid [6].)

The purpose of this paper is to obtain simple but accurate analytical expressions for the bremsstrahlung cross sections from nuclei in the intermediate energy range 2 keV - 2 MeV. We propose a modification of the relativistic Born approximation, which together with classical expressions, gives a parametrization of the Coulombic bremsstrahlung spectra with an accuracy of at least 10% for all elements of the periodic table in this region.

Since, in general, experimental results for the bremsstrahlung spectra are not available, the accuracy of analytic approximations for CBS can be established only by comparison with theoretical numerical results. Partial-wave calculations (PW) [7], which are in good agreement with experiment for the differential (in photon angle) CBS [8], will be used as a standard here for the spectrum, to which we compare our analytical predictions. Such a method had been used in [5] for investigation of the validity of classical, nonrelativistic Elwert-Born (EB), and relativistic Elwert-Bethe-Heitler (EBH) expressions in the energy range 1-500 keV. To consider a wider energy range, we have calculated additional PW CBS data for Al, Fe, Mo, W, Au and U, expanding in energy and fraction of energy radiated, the very restricted set of numerical bremsstrahlung data which had been available previously [5]. Our calculations show that the accuracy of the analytic expressions which had been examined in Ref. [5] becomes worse at higher electron energies. PW results for all elements are considerably bigger than EBH and smaller than EB results.

We find that a modification of the Elwert factor included in the EBH formula improves the results significantly. However, to achieve an accuracy in our analytic parametrization for CBS of better than 10%, for heavy elements as well as for light ones, we also introduce an empirical higher-order Born correction to the EBH formula, independent of the fraction k/T_i of energy radiated and linear in incident energy.

In Sec. 2 we compare different approximations commonly used for CBS in the energy region considered. In Sec. 3 we introduce a modified expression for the Elwert factor, which we recommend using in combination with the Bethe-Heitler formula, and also a simple empirical higher-order Born correction. The accuracy of our parametrization of CBS is demonstrated in Sec. 4.

2. Analytical approximations for Coulombic bremsstrahlung

The nonrelativistic classical and Born approximations are special cases of the more general Sommerfeld result [1]. Obtained in the nonrelativistic dipole approximation, it has the form

$$\sigma_{coul}(k) = \left(\frac{16}{3}\right)\pi^2\alpha^3 \left[\left(\exp\left(2\pi\nu_i\right) - 1\right) \left(1 - \exp\left(-2\pi\nu_f\right)\right) \right]^{-1} \times X_0 \frac{d\left|{}_2F_1(i\nu_i, i\nu_f, 1; X_0)\right|^2}{dX_0} , \quad (1)$$

where $\sigma_{coul}(k) = (k\, d\sigma/dk)\, (\beta_i/Z)^2$ (β_i is the initial velocity of the scattering electron), ${}_2F_1(i\nu_i, i\nu_f, 1; X_0)$ is a hypergeometric function, and α is the fine structure constant. The variable $X_0 = -4\nu_i\nu_f/(\nu_i - \nu_f)$, where $\nu_{i(f)} = Z\alpha/\beta_{i(f)}$, $p_{i(f)} = \sqrt{2T_{i(f)}}$, $T_{i(f)}$ and $p_{i(f)}$ are the initial (final) energy and momenta of the scattering electron. "Natural" units $\hbar = m = c = 1$ are being used.

The classical expression for the bremsstrahlung radiation spectrum associated with motion of an electron on a trajectory in the Coulomb field, can be obtained from Eq. (1) at big $\nu_{i(f)}$ [2]. This gives

$$\sigma_{cl} = \left(\frac{4}{3}\right) \pi^2\alpha^3 i\mu\, H^{(1)}_{i\mu}(i\mu)\, H^{(1)\prime}_{i\mu}(i\mu) \quad (2)$$

where $H^{(1)}_{i\mu}(i\mu)$ and $H^{(1)\prime}_{i\mu}(i\mu)$ are the Hankel function and its derivative, respectively. The parameter $\mu = k\nu_i/(2T_i)$. In the soft photon region of the CBS (small μ), Eq. (2) can be accurately approximated as [9]

$$\sigma_{cl} = \left(\frac{16}{3}\right)\alpha^3 (1 + \mu\pi) \ln\left(\frac{2}{\gamma\pi}\right) + O(\mu^2) . \tag{3}$$

The corresponding approximate expression for the hard photon region of bremsstrahlung spectra from heavy nuclei (large μ) is

$$\sigma_{cl} = \left(\frac{16\pi}{3\sqrt{3}}\right)\alpha^3 [1 + d_1\mu^{-2/3} + d_2\mu^{-4/3} + d_3\mu^{-2} + O(\mu^{-8/3})] , \tag{4}$$

where $d_1 = 0.217747$, $d_2 = -0.0131214$, and $d_3 = -0.0057$, Euler's constant $\gamma \approx 1.78$.

For high energies of incident and outgoing electrons, the Sommerfeld formula reduces to the nonrelativistic Born approximation

$$\sigma_B(k) = \left(\frac{16\alpha^3}{3}\right)\ln\left(\frac{Q_{max}}{Q_{min}}\right). \tag{5}$$

Here $Q_{max(min)} = p_i \pm p_f$. In the tip of the bremsstrahlung spectrum, Eq. (4) is invalid. In this case it was found from Eq. (1) that Eq. (4) should be multiplied by the factor

$$F_{nrel} = \frac{p_i [1 - \exp(-2\pi Z\alpha/p_i)]}{p_f [1 - \exp(-2\pi Z\alpha/p_f)]}, \tag{6}$$

called the Elwert factor [3].

For small ν, i. e. for high incident energies, relativistic effects should be taken into account. The relativistic Born approximation is given by the Bethe-Heitler [4] formula

$$\sigma_{BH}(k) = \alpha^3\beta^2\left(\frac{p_f}{p_i}\right)\left\{\frac{4}{3} - 2E_iE_f\left[\frac{p_f^2 + p_i^2}{p_i^2p_f^2}\right] + \frac{\varepsilon_iE_f}{p_i^3} + \frac{\varepsilon_fE_i}{p_f^3} - \frac{\varepsilon_i\varepsilon_f}{p_ip_f}\right.$$
$$+ L\left[\frac{8E_iE_f}{3p_ip_f} + \frac{k^2(E_i^2E_f^2 + p_i^2p_f^2)}{p_i^3p_f^3} + \frac{k}{2p_ip_f}\left(\left[\frac{E_iE_f + p_i^2}{p_i^3}\right]\varepsilon_i\right.\right.$$
$$\left.\left. - \left[\frac{E_iE_f + p_f^2}{p_f^3}\right]\varepsilon_f + \frac{2kE_iE_f}{p_i^2p_f^2}\right)\right]\Bigg\},$$

$$L = 2\ln\left[\frac{E_i E_f + p_i p_f - 1}{k}\right] \tag{7}$$

$$\varepsilon_{i(f)} = \ln\left(\frac{E_{i(f)} + p_{i(f)}}{E_{i(f)} - p_{i(f)}}\right)$$

Here $E_{i(f)}$ and $p_{i(f)}$ are the relativistic initial (final) total energy and momenta of the electron.

For the tip region this expression can again be improved by multiplying (7) by an Elwert factor, as in the nonrelativistic case,

$$F_{\text{rel}} = \frac{\beta_i [1 - \exp(-2\pi Z\alpha/\beta_i)]}{\beta_f [1 - \exp(-2\pi Z\alpha/\beta_f)]} \tag{8}$$

Note that the Elwert factor (8) differs from the nonrelativistic Elwert factor (6) in replacing nonrelativistic momenta by the initial and final relativistic velocities of the scattering electron $\beta_{i(f)} = [1 - 1/(1 + T_{i(f)})^2]^{1/2}$. In the case of low-energy electron bremsstrahlung, $\beta_{i(f)} = p_{i(f)}$, and thus the nonrelativistic and relativistic Elwert factors are the same.

To find the best parametrization of the bremsstrahlung spectra in the point Coulombic case a comparison of the results of different approximations with numerical results of partial-wave calculations is needed. PW data for bremsstrahlung spectra from the neutral atoms were tabulated extensively [7,10-12], while only very restricted Coulombic PW results were obtained [12,13]. To improve this situation we have used here the numerical code, previously utilized in [7,10-12]. We have substantially extended the Coulombic data which had previously been available in calculating the CBS of Al, Fe, Mo, W, Au and U in the energy region 0.002-2.000 MeV. We show in comparison the regions of 10% accuracy of classical, relativistic EBH and nonrelativistic EB approximations in Fig. 1a (for iron) and in Fig. 1b (for tungsten) [5].

Note from Fig. 1a that only for large ν_i is the difference between classical predictions (3), (4) and partial wave results less than 10%. This is a situation for $\nu_i > 1.5$ which is present for almost the entire spectrum of iron, Fig. 1a, and for $\nu_i > 2$ for the BS of tungsten, Fig. 1b. To obtain accurate results for smaller ν_i we need to use Elwert-Born approximations, either relativistic EBH or nonrelativistic EB. For light elements, relativistic Elwert-Bethe-Heitler approximation can be used with 10% accuracy for a wider region of parameter ν_i than the nonrelativistic Born approximation while

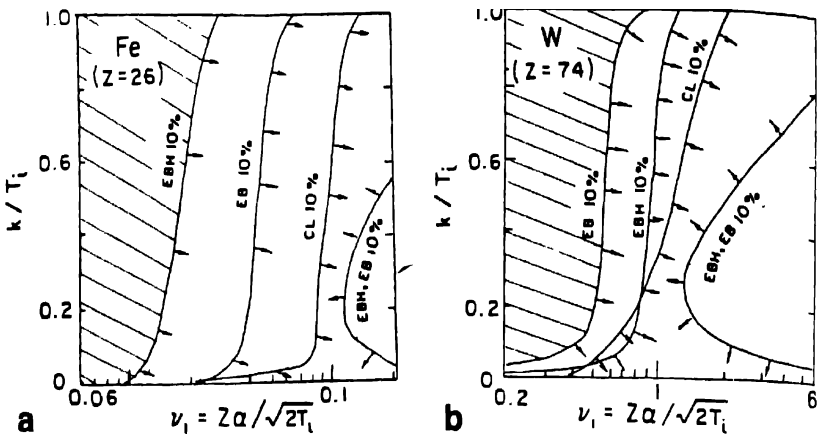

Fig. 1. The regions of 10% accuracy of classical (CL), relativistic EBH and nonrelativistic EB approximations with the respect to numerical partial-wave results a) for iron, b) for tungsten.

(surprisingly) for heavy elements the situation is opposite. The wider region of validity of the nonrelativistic dipole Elwert-Born approximation for heavy elements can be explained by the cancellation among relativistic, retardation, and higher multipole effects. Looking at Fig. 1 we can see, however, that there is a region of small ν_i where no analytical approximation among those we have discussed describes CBS with good accuracy.

To describe the CBS at any energy beyond the classical region we take into account the fact that in the Bethe-Maximon high-energy limit [6] the Bethe-Heitler formula gives 10% accuracy throughout the spectrum except the tip region. We use Bethe-Heitler formula (7) for our analytical approximation of bremsstrahlung cross section, modified, however, by the Elwert factor and introducing a high Born correction.

3. Modification of the Elwert Factor

We propose a relativistic modification of the Elwert factor based on our comparison of EBH and EB approximations, and PW data for bremsstrahlung cross sections. For this purpose we choose the light-Z element aluminum (Z = 13), molybdenum (Z = 42), since this element has an intermediate place in the periodic table, and the high-Z element uranium (Z = 92).

First, we note that the EBH approximation works well for Al in the

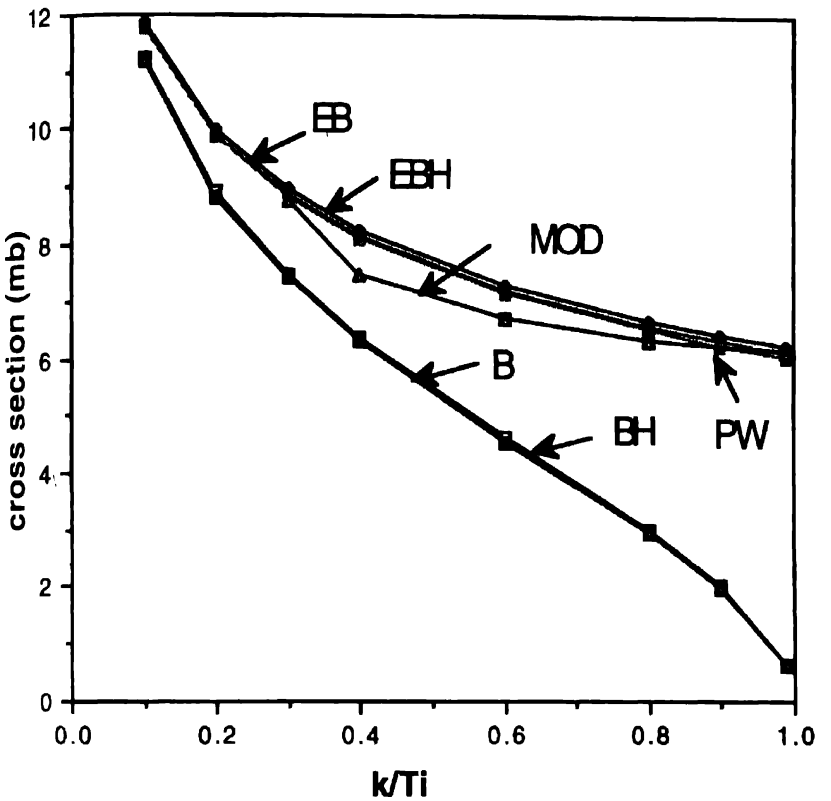

Fig. 2. Comparison of the partial-wave (PW) results with Born (B), Bethe-Heitler (BH), EB and EBH results for Mo at the incident electron energy T_i = 5 keV. The results of BH approximation with modified Elwert factor (mod) coincide with PW results.

energy range we are discussing here. We also note from Fig. 2 that at electron energies less than 10 keV (but at energies not in the range of the validity of the classical approximation) predictions of all these approaches give very similar and good results. The error of the calculations using any of them is no more then 3% according to the partial-wave data available. Only for higher electron energies our modification of the Elwert factor is needed.

We can also note (Fig. 3), that partial wave data lie in-between EBH and EB, closer to the EBH results than to the nonrelativistic Born data. With increasing nuclear charge Z, the PW results become closer to EBH data.

Based on these observations we have tried to improve the Bethe-Heitler approximation by modifying the Elwert factor. Our prescription for CBS is

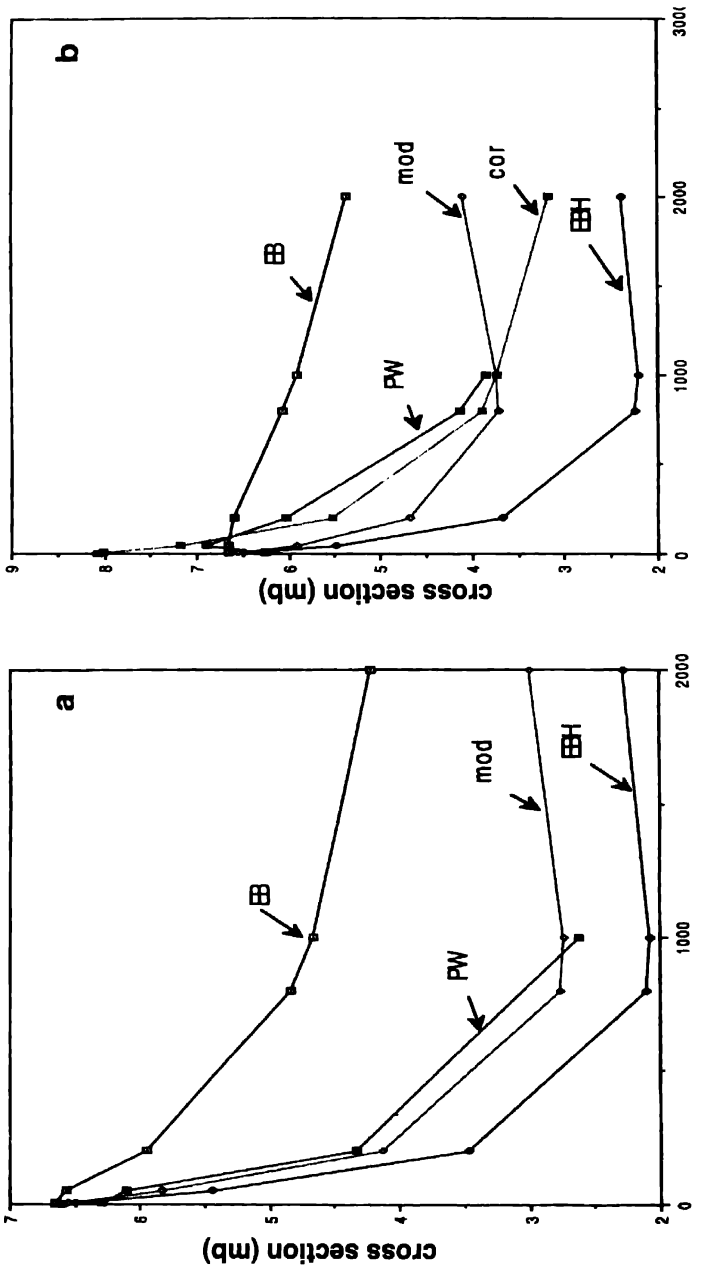

Fig. 3. Dependence of the CBS on energy at $k/T_i = 0.8$ for a) molybdenum, b) uranium. [We use the same notations as in Fig. 2. Results improved by empirical second Born correction (cor) are calculated with expression (7)]

$$\sigma_{\text{mod}}(k) = F_{\text{mod}}\sigma_{\text{BH}}(k), \tag{9}$$

where the modified Elwert factor is taken to have the form (6)

$$F_{\text{mod}} = \frac{p_i[1 - \exp(-2\pi Z\alpha/p_i)]}{p_f[1 - \exp(-2\pi Z\alpha/p_f)]}, \tag{10}$$

but with relativistic $p_{i(f)} = \sqrt{T_{i(f)}(2 + T_{i(f)})}$. In this formula relativistic kinematics are used, while for the nonrelativistic Elwert factor (6) $p_{i(f)}$ are nonrelativistic momenta of incoming and outgoing electrons. At small energies relativistic $p_{i(f)}$ merge to the nonrelativistic values and all Elwert factors (F_{rel}, F_{nrel} and F_{mod}) are equal. Note that F_{mod} satisfies all the properties noted above. It is bigger than F_{rel}, and thus modified σ_{coul} is bigger than Elwert-Bethe-Heitler results. With increasing Z, factor F_{mod} also increases (Figs. 2 and 3).

We have calculated pure Coulombic bremsstrahlung cross sections for Al, Ag, Fe, Mo, W, and U in the partial wave approximation to compare with the results of our new prescription, i.e. with the results of Bethe-Heitler expression modified by the Elwert factor F_{mod}. The results of the EB and EBH approximations as well as $\sigma_{\text{mod}}(k)$ (9) for the bremsstrahlung cross sections from Mo and U are shown in Figs. 2 and 3. The accuracy of the new prescription is much better than both the Elwert-Bethe-Heitler results and Elwert-Born results. However, for heavy nuclei in many cases it is still worse than desirable 10% (Fig. 3b).

To improve our prescriptions and to reach 10% accuracy throughout the nonclassical energy region considered we have made an additional correction to the Bethe-Heitler expression (7), multiplying the cross section $\sigma_{\text{mod}}(k)$ by the empirical factor

$$C(T_i, Z) = 1 + \frac{1}{4}(Z\alpha)^2(2 - T_i), \tag{11}$$

(note that T_i is taken in relativistic units). This reflects our attempt to take into account higher-order Born corrections to the bremsstrahlung cross sections which are of order $(Z\alpha)^2$.

4. Prescription for Coulombic bremsstrahlung cross section

Taking into account the correction (11) and the modification of the Elw-

ert factor (10) we have for the bremsstrahlung cross section

$$\sigma_{cor}(k) = C(T_i, Z)F_{mod}\sigma_{BH}(k) \tag{12}$$

The results of this new prescription σ_{cor} agree very well with the partial-wave calculations (see, for example, Fig. 3b). The accuracy is better than 10% for all cases considered.

The results of our calculations for modified σ_{mod} (9) and corrected σ_{cor} (12) bremsstrahlung cross sections from uranium are presented in Table 1 for T_i = 200 keV, 500 keV, 800 keV, and 1000 keV. These can be compared with the results of the EB approximation, which works for U in the energy range considered much better than EBH approximation. With increasing the energy the error of the EB calculations increases rapidly, approaching more than 50% for the hard photon region at T_i = 500 keV. Our new prescriptions, even without the correction (11), are much better than both EB and EBH approximations. For U at all energies considered, the error of new prescriptions without the correction exceeds 20% for only a few cases and always is less than 23%. The correction improves the results to the 10% level of accuracy for all energies except very low, where a switch to the classical approximation σ_{cl} should be made.

The choice of the approximation we should use [classical with a switch from the soft-photon expression (3) to the hard photon expression (4), or the corrected BH (12)] depends on the parameter $\nu_i = Z\alpha/\beta_i$. From Fig. 1 we saw that for $\nu_i \leq 1$, the classical approximation does not work with the desired accuracy except perhaps in a very narrow soft photon region near $k/T_i = 0$. For this regime expression σ_{cor} (12) can be used (see Fig. 4a for Al with $\nu_i = 0.68$ as an example).

For $\nu_i > 1$, the situation is more complicated. Our prescription here is to start from the classical approximation in the soft-photon region using the formulas (3 and 4) with the switch from soft- to hard-photon classical regime at the cross point of the soft-photon and hard-photon curves. Another switch should be made if a classical curve crosses σ_{cor}. If $\nu_i \gg 1$, this situation does not occur and the second switch is not needed. In Fig. 4b we show the results for Au at T_i = 50 keV ($\nu_i = 1.3$). In this case, the first switch should be done at the point $k/T_i \sim 0.4$. We can see that the hard-photon curve, calculated using the expression (4) crosses the σ_{cor} curve at the point $k/T_i \sim 0.7$. For higher k/T_i expression (12) can be used quite successfully.

Table 1. Coulombic bremsstrahlung cross sections for $_{92}U$. Note that $\sigma_{Kramers} = 561$ mb.

T_i (keV)	k/T_i	σ (in mb) exact PW	prescriptions σ_{mod}	prescriptions σ_{cor}	prescriptions EB	accuracy $\frac{\sigma_{mod}}{PW}$	accuracy $\frac{\sigma_{cor}}{PW}$	accuracy $\frac{EB}{PW}$
200	0.9	5.69	4.37	5.16	6.35	0.77	0.91	1.16
	0.8	6.03	4.71	5.55	6.60	0.78	0.92	1.09
	0.7	6.41	5.08	6.00	6.89	0.79	0.93	1.07
	0.6	6.84	5.52	6.52	6.23	0.81	0.94	1.06
	0.5	7.36	6.04	7.13	7.65	0.82	0.97	1.04
	0.4	7.99	6.69	7.89	8.19	0.84	0.99	1.03
	0.3	8.83	7.53	8.88	8.90	0.85	1.01	1.01
500	0.9	3.83	3.52	3.91	6.09	0.91	1.02	1.74
	0.8	4.70	3.88	4.32	6.34	0.82	0.92	1.35
	0.7	5.18	4.30	4.78	6.63	0.83	0.92	1.28
	0.6	5.73	4.79	5.33	6.99	0.83	0.93	1.22
	0.5	6.37	5.38	5.98	7.44	0.84	0.94	1.17
	0.4	7.16	6.11	6.80	8.00	0.85	0.95	1.12
	0.3	8.23	7.06	7.86	8.74	0.86	0.96	1.06
800	0.9	3.64	3.37	3.53	5.82	0.92	0.97	1.60
	0.8	4.13	3.74	3.90	6.07	0.91	0.94	1.47
	0.7	4.60	4.16	4.35	6.39	0.90	0.95	1.17
	0.6	5.19	4.67	4.88	6.77	0.90	0.94	1.30
	0.5	5.88	5.29	5.53	7.25	0.90	0.94	1.23
	0.4	6.73	6.08	6.35	7.84	0.90	0.94	1.16
	0.3	7.74	7.09	7.41	8.62	0.92	0.96	1.11
1000	0.9	3.39	3.40	3.40	5.65	1.00	1.00	1.67
	0.8	3.86	3.76	3.76	5.92	0.97	0.97	1.53
	0.7	4.38	4.18	4.17	6.25	0.95	0.95	1.43
	0.6	5.00	4.70	4.68	6.65	0.94	0.94	1.33
	0.5	5.68	5.34	5.32	7.15	0.94	0.94	1.26
	0.4	6.50	6.14	6.12	7.76	0.94	0.94	1.19
	0.3	7.47	7.19	7.16	8.55	0.95	0.95	1.14

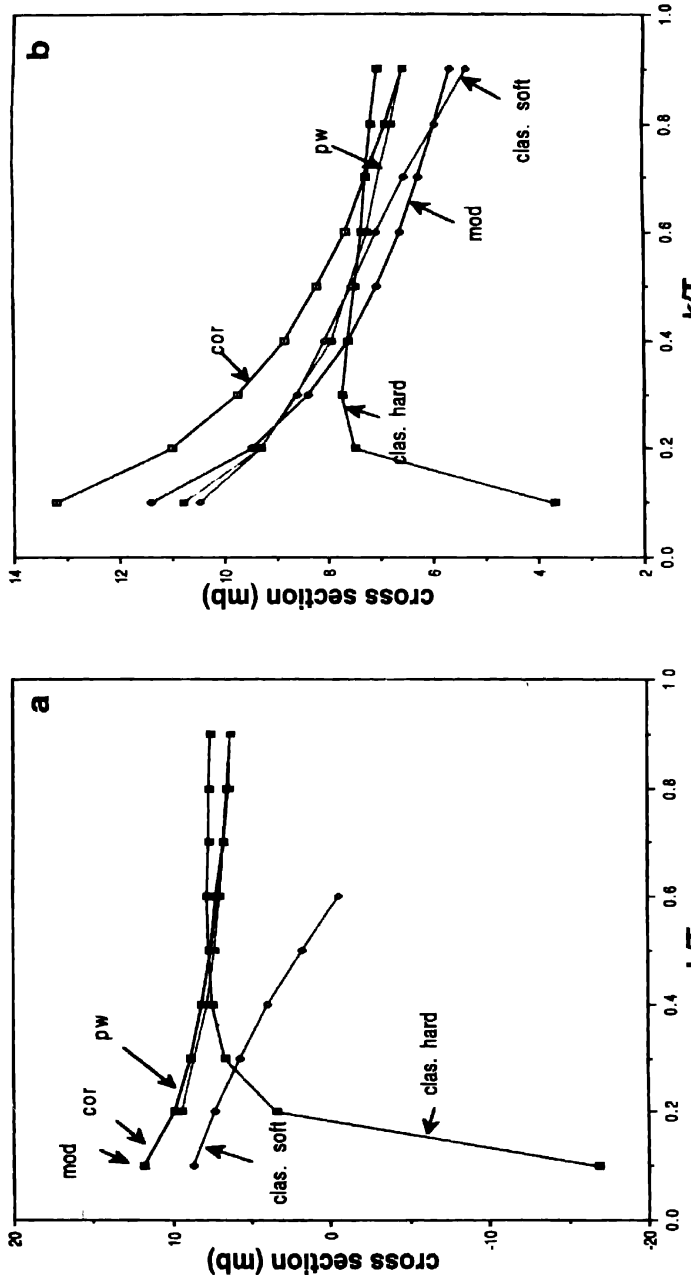

Fig. 4. Comparison of the classical approximation and our modified (mod) and corrected (cor) prescriptions for a) Al at $T_i = 5$ keV and b) Au at $T_i = 50$ keV. Classical results are obtained using soft-photon expression (3) (clas. soft) and hard-photon expansion (4) (clas. hard).

5. Conclusion

We have obtained prescriptions for the Coulombic bremsstrahlung spectrum in the energy range 2 keV - 2 MeV for all elements of the periodic table. Our prescriptions depend on the parameter $\nu_i = Z\alpha/\beta_i$, and thereby on the nuclear charge and the incident electron energy. If $\nu_i \leq 1$, the expression (12) for σ_{cor}, obtained by multiplying the Bethe-Heitler formula by the modified Elwert factor with an empirical higher-order Born correction, gives results accurate within 10%. For other values of T_i and Z, for which $\nu_i > 1$, starting from the soft-photon region, we should use classical approximation (3) with a switch to the expression (4), as it was described in Sec. 3. If the classical curve crosses σ_{cor} with increasing k/T_i another switch to σ_{cor} generally gives the desired 10% level accuracy.

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