ON A REPRESENTATION OF THE DIRAC MATRICES

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ABSTRACT. The object of the paper is to construct a representation of the Dirac matrices adaptable to the free particle spinor.

INTRODUCTION

In order to obtain explicitly the solutions of the Dirac equation and to study their properties, it is the usual practice to start with suitable representation of the Dirac matrices, satisfying the anti-commutation relations. Conversely, one can ask the question whether one can build up a representation of the Dirac matrices adapted to the solution of the Dirac equation for a particular problem. The object of this short paper is to construct a representation of the matrices, in the spinor space of free particle. The construction of the matrices, in this way, is very instructive as it gives better insight to their characteristic properties related to the splitting up and mixing of the positive and negative energy states, as well as those of the spin states. The process is also interesting as it shows clearly where is the liberty of the choice and how the elements are determined. Finally, it also exhibits the natural role of the Pauli matrices in the representation and their invariance properties, which have been recently studied by the author (1965).

In the next section, the representation of the matrices in the space of free particle spinor has been developed. In the last section some of their properties, e.g. transposition operation, are discussed.

We will need frequently the Pauli matrices which we take as

$$\sigma_1 = \begin{vmatrix} 0, & 1 \\ 1, & 0 \end{vmatrix}, \quad \sigma_2 = \begin{vmatrix} 0, & -i \\ i, & 0 \end{vmatrix}, \quad \sigma_3 = \begin{vmatrix} 1, & 0 \\ 0, & -1 \end{vmatrix}.$$

The very nature of the problem proposed will lead us to hermitian matrices. However, we do not loose any generality, as it is well-known that any non-hermitian representation may be made equivalent to a hermitian one by suitable similarity transformation (Pauli (1936)).

For any space vector $\vec{q}$ we will use the notation α_q to denote $(\vec{\alpha}.\ \vec{q})/|\vec{q}|$; so that $\alpha_q^2 = 1$. $\sim$ over a matrix denotes its hermitian conjugate.

REPRESENTATION OF THE MATRICES

Let the free particle spinors be $U_+(\vec{p})$ and $U_-(\vec{p})$ corresponding to the positive and negative energies respectively. Hence

$$\left.\begin{aligned}\{(\vec{\alpha}.\vec{p})+\beta mc\}U_+(\vec{p}) &= -p_0U_+(\vec{p})\\ \{(\vec{\alpha}.\vec{p})+\beta mc\}U_-(\vec{p}) &= p_0U_-(\vec{p}),\end{aligned}\right\} \quad \ldots \quad (1)$$

where $p_0 = +\{(\vec{\alpha}.\vec{p})+\beta mc\}^{1/2}$. Let us introduce the matrices α_+ and α_- defined by

$$\alpha_+ = -\frac{1}{p_0}\{(\vec{\alpha}.\vec{p})+\beta mc\} \quad \ldots \quad (2)$$

and

$$\alpha_- = \frac{1}{p_0}\{m\alpha_p-\beta|\vec{p}|\}; \quad \ldots \quad (3)$$

such that

$$\alpha_+^2 = 1, \quad \alpha_-^2 = 1 \quad \ldots \quad (4)$$

and

$$\alpha_+\alpha_-+\alpha_-\alpha_+ = 0. \quad \ldots \quad (5)$$

Let $\vec{j}$ and $\vec{k}$ be two mutually orthogonal unit vectors in the plane perpendicular to $\vec{p}$, i.e.

$$(\vec{p}.\vec{j}) = 0, \quad (\vec{p}.\vec{k}) = 0 \quad \text{and} \quad (\vec{j}.\vec{k}) = 0. \quad \ldots \quad (6)$$

It is clear that $\vec{j}, \vec{k}$ are arbitrary up to a rotation in their plane.

Since α_+ is an involution and traceless one, it has two distinct pairs of eigen-vectors with eigen-values $+1, -1$. Let U_+^1 and U_-^2 be two mutually orthogonal eigen-vectors of α_+ with eigen-values $+1$ and similarly U_-^1 and U_-^2 be those with eigen-values -1. These four vectors are mutually orthogonal and form the basis of our representation. Clearly in this basis α_+ is a diagonal matrix with elements $(+1, +1, -1, -1)$.

The matrices which commute with α_+ leave the subspaces of positive and negative energy states invariant. We can take any of the matrices which anti-commutes with α_+ to connect the positive energy states with those of negative energy and vice versa. In order that the representation is symmetric, with respect to positive and negative energy states, we take α_- as the matrix which just connects positive and negative energy states (without changing spin states) and conversely, as it is an involution. So that,

$$\left.\begin{aligned}\alpha_-U_+^s &= U_-^s\\ \text{and} \quad \alpha_-U_+^s &= U_+^s.\end{aligned}\right\} \quad (s = 1, 2) \quad (7)$$

Thus α_+ and α_- are given by

$$\alpha_+ = \begin{vmatrix} e, & 0 \\ 0, & -e \end{vmatrix} \quad \text{and} \quad \alpha_- = \begin{vmatrix} 0, & e \\ e, & 0 \end{vmatrix}, \qquad \dots \quad (8)$$

where e is the 2×2 unit matrix.

Next let us consider the operator which interchanges only the spin states with same energy. The matrix Σ, for such an operator, is

$$\Sigma = \begin{vmatrix} \sigma_1, & 0 \\ 0, & \sigma_1 \end{vmatrix} \qquad \dots \quad (9)$$

where σ_1 is the Pauli matrix given above. It clearly commutes with α_+ and α_-. The most general matrix which commutes with both α_+ and α_- is a linear combination of

$$\alpha_j \alpha_k, \quad \alpha_+ \alpha_- \alpha_k, \quad \alpha_+ \alpha_- \alpha_j.$$

Again for symmetry, let us take Σ to be proportional to $\alpha_j \alpha_k$ so that

$$\Sigma = i\alpha_j \alpha_k. \qquad \dots \quad (10)$$

In order to find α_j, α_k separately we note that

$$\alpha_+ \alpha_j U_+^s = -\alpha_j U_+^s \quad \text{and} \quad \alpha_+ \alpha_j U_-^s = +\alpha_j U_-^s.$$

Thus the matrix α_j is of the form

$$\alpha_j = \begin{vmatrix} 0, & \sigma \\ \tilde{\sigma}, & 0 \end{vmatrix}, \qquad \dots \quad (11)$$

where σ is any 2×2 matrix. It is easy to see that σ should satisfy the following conditions

$$\left.\begin{aligned} \sigma \tilde{\sigma} = e, \quad \sigma + \tilde{\sigma} = 0 \\ \text{and} \qquad \sigma \sigma_1 + \sigma_1 \sigma = 0, \end{aligned}\right\} \qquad \dots \quad (12)$$

such that $\alpha_j^2 = 1$, and α_j anti-commutes with α_- and α_+. Hence

$$\sigma = i(\cos\theta\, \sigma_2 + \sin\theta\, \sigma_3), \qquad \dots \quad (13)$$

where θ is any real parameter. Hence from Eq. (10), it follows that

$$\alpha_k = \begin{vmatrix} 0, & \sigma' \\ \tilde{\sigma}', & 0 \end{vmatrix} \qquad \dots \quad (14)$$

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$$\left.\begin{aligned} \text{with} \qquad \sigma' = i(\sin\theta\, \sigma_2 - \cos\theta\, \sigma_3) \\ \text{so that} \qquad \sigma\sigma' + \sigma'\sigma = 0. \end{aligned}\right\} \qquad \dots \quad (15)$$

As mentioned above $\vec{j}$ and $\vec{k}$ are indeterminate but for a rotation in the $(\vec{j} - \vec{k})$ plane, thus we can choose θ, in Eqs. (13) and (15), to be zero so that

$$\alpha_j = i \begin{vmatrix} 0, & \sigma_2 \\ -\sigma_2 & 0 \end{vmatrix}; \qquad \alpha_k = i \begin{vmatrix} 0, & -\sigma_3 \\ \sigma_3, & 0 \end{vmatrix}. \quad \ldots (16)$$

Finally the fifth mutually anti-commuting matrix

$$\alpha_+\alpha_-\alpha_j\alpha_k = -i\alpha_+\alpha_-\Sigma = i \begin{vmatrix} 0, & -\sigma_1 \\ \sigma_1, & 0 \end{vmatrix} \quad \ldots (17)$$

which is a matrix with only non-vanishing anti-diagonal elements.

DISCUSSIONS

The construction of the matrices shows how the Pauli matrices enter in the expression of the Dirac matrices in an obvious manner. The typical block character of all the four α_-, α_j, α_k, $\alpha_+\alpha_-\alpha_j\alpha_k$ is evident from their anti-commuting properties with α_+, which is diagonal. All these four matrices connect positive energy states to the negative energy and vice versa. It is interesting to note that

$$i\alpha_j\alpha_k = \begin{vmatrix} \sigma_1, & 0 \\ 0, & \sigma_1 \end{vmatrix}, \qquad i\alpha_+\alpha_-\alpha_j = \begin{vmatrix} \sigma_2, & 0 \\ 0 & \sigma_2 \end{vmatrix}$$

and

$$-i\alpha_+\alpha_-\alpha_k = \begin{vmatrix} \sigma_3, & 0 \\ 0, & \sigma_3 \end{vmatrix}. \quad \ldots (18)$$

They are isomorphic to the Pauli matrices. It may be pointed out that these are also the only three linearly independent matrices which commute with α_+ and α_-. Thus our choice of $\Sigma = i\alpha_j\alpha_k$, in Eq. (10), is nothing but a choice of a particular representation of Pauli matrices. Any other choice would amount to a 3-dimensional real orthogonal rotation among these three matrices.

The representation constructed here is slightly different from those usually referred (Dirac (1928), Kronig (1938)). In this representation, α_p, β (obtained from α_+ and α_-) and α_j are real; α_k and $\alpha_p\beta\alpha_j\alpha_k$ are purely imaginary. Thus α_k and $\alpha_p\beta\alpha_j\alpha_k$ are anti-symmetric and α_p, β, and α_j are symmetric. The matrix B (Pauli (1936), which induces the transposition, is $\alpha_+\alpha_-\alpha_j$. This representation is one of the second category of representation as noted by the author (1965).

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