CHERENKOV RADIATION IN DIELECTRIC MEDIUM WITH CONDUCTIVITY

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ABSTRACT. Cherenkov radiation through dielecric medium having conductivity has been considered. It has been shown that Cherenkov radiation is obtained at any velocity of the particle through medium though the attenuation will be large for low velocities. Some features of penetration length has been discussed which are unlike those of usual electromagnetic radiation in conducting media. The opening angle also seems to be of interest.

The problem of Cherenkov radiation in nonconducting media has been discussed by many investigators (1953). In practice however materials generally have varried amount of conductivity and thus lead to attenuation. It is therefore interesting to see the relative importance of the conductivity and the particle velocity in the production of Cherenkov radiation.

We start with Maxwell's equations with Fourier components;

$$\left.\begin{aligned}
\text{rot } H_\omega &= \frac{i\omega n^2}{c} E_\omega + \frac{4\pi}{c} j_\omega + \frac{4\pi}{c}\sigma E_\omega \\
\text{rot } E_\omega &= -\frac{i\omega}{c} H\omega \\
\text{div } D_\omega &= 4\pi\rho \\
\text{div } H_\omega &= 0 \\
D_\omega &= n^2 E_\omega
\end{aligned}\right\} \quad \ldots \quad (1)$$

where ω is the frequency, n^2 is the dielectric constant of the medium and σ is the conductivity of the medium. n and σ in general depend on ω.

Introducing scalar and vector potentials ϕ and A as usual we have

$$\left.\begin{aligned}
H_\omega &= \text{rot } A_\omega \\
E_\omega &= -\frac{i\omega}{c} A_\omega - \text{grad}\,\phi_\omega \\
\nabla^2 A_\omega + \frac{1}{c^2}(\omega^2 n^2 - i4\pi\sigma\omega) A_\omega &= -\frac{4\pi}{c} j_\omega \\
\nabla^2 \phi_\omega + \frac{1}{c^2}(\omega^2 n^2 - i4\pi\sigma\omega)\phi_\omega &= -\frac{4\pi}{n^2}\rho
\end{aligned}\right\} \quad \ldots \quad (2)$$

with the modified Lorentz condition

$$\text{div } A_\omega + \frac{1}{c}(4\pi\sigma + i\omega n^2)\,\phi_\omega = 0.$$

Let the particle be moving along the z direction with uniform velocity v. Then in cylindrical co-ordinates the components of vector potential are

$A_\rho = 0 = A_\varphi$ and

$$\frac{\partial^2 A_z}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial A_z}{\partial \rho}+\frac{\partial^2 A_z}{\partial z^2}+\frac{1}{c^2}(\omega^2 n^2 - i4\pi\sigma\omega)A_z = -\frac{e}{\pi c\rho}e^{-i\omega\frac{z}{v}}\delta(\rho) \quad \ldots \quad (3)$$

$$\text{Let } A_z = u(\rho)e^{-i\omega\frac{z}{v}} \quad \ldots \quad (4)$$

From (3) $$\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+s^2 u = -\frac{e}{\pi c\rho}\delta(\rho), \quad \ldots \quad (5)$$

where $s^2 = \frac{w^2}{v^2}\left\{(n^2\beta^2-1)-i\frac{4\pi\sigma\beta^2}{\omega}\right\}$, $\beta = \frac{v}{c}$.

If $s = s_1 + is_2$, then

$$s_1^2 - s_2^2 = \frac{w^2}{v^2}(n^2\beta^2-1) \quad \ldots \quad (6)$$

$$s_1 s_2 = -\frac{2\pi\sigma\omega}{c^2} \quad \ldots \quad (7)$$

From these we have

$$\left.\begin{aligned}
s_1 &= \pm\frac{\omega}{\sqrt{2}v}\left[\left\{(n^2\beta^2-1)^2+\frac{16\pi^2\sigma^2\beta^4}{\omega^2}\right\}^{\frac{1}{2}}+(n^2\beta^2-1)\right]^{\frac{1}{2}},\\
s_2 &= \pm\frac{\omega}{\sqrt{2}v}\left[\left\{(n^2\beta^2-1)^2+\frac{16\pi^2\sigma^2\beta^4}{\omega^2}\right\}^{\frac{1}{2}}-(n^2\beta^2-1)\right\}^{\frac{1}{2}} \text{ when } n^2\beta^2-1 \geqslant 0\\
&\text{and}\\
s_1 &= \pm\frac{\omega}{\sqrt{2}v}\left[\left\{(1-n^2\beta^2)^2+\frac{16\pi^2\sigma^2\beta^4}{\omega^2}\right\}^{\frac{1}{2}}-(1-n^2\beta^2)\right]^{\frac{1}{2}},\\
s_2 &= \pm\frac{\omega}{\sqrt{2}v}\left[\left\{(1-n^2\beta^2)^2+\frac{16\pi^2\sigma^2\beta^4}{\omega^2}\right\}^{\frac{1}{2}}+(1-n^2\beta^2)\right]^{\frac{1}{2}} \text{ when } n^2\beta^2-1 < 0.
\end{aligned}\right\}$$

$$\ldots \quad (8)$$

The right hand side of (7) is independent of v and it represents recetangular hyperbola with v varying. In (8) there is an ambiguity in sign and it may lead to different values of s_1 and s_2. But from (7) s_1 and s_2 have opposite signs, so only two cases are possible.

Case I : $s_1 > 0$ and $s_2 < 0$.

s_1 and s_2 lie on the same branch of the hyperbola (7). For the outgoing wave $u = BH_o^{(2)}(s\rho)$ and $B = -(ie/2c)$ which is fixed by the strength of the singularity at $\rho = 0$.

Here $A_z(\omega) = BH_0^{(2)}(s\rho)\, e^{-\frac{i\omega z}{v}+i\omega t}$

Using the asymptotic form of $H_0^{(2)}(s\rho)$ for $|s\rho| >> 1$,

$$A_z(\omega) = -iB_1(\alpha_1+i\beta_1)\frac{1}{\sqrt{\rho}}e^{s_2\rho}\, e^{i\omega\chi}, \qquad \ldots (9)$$

where $$B_1 = \frac{e}{2c}\sqrt{\frac{2}{\pi(s^2_1+s^2_2)}},$$

$$\alpha_1 = \frac{1}{\sqrt{2}}(\sqrt{s^2_1+s_2^2}+s_1)^{\frac{1}{2}}, \quad \beta_1 = \frac{1}{\sqrt{2}}(\sqrt{s^2_1+s^2_2}-s_1)^{\frac{1}{2}}$$

and $$\chi = t-\left(\frac{z}{v}+\frac{s_1}{\omega}\rho\right)+\frac{\pi}{4\omega} \qquad \ldots (10)$$

Case II. $s_1 < 0$ and $s_2 > 0$.

In this case s_1 and s_2 lie on the opposite branch of the hyperbola (7).

For the outgoing wave $u = \frac{ie}{2c}H_0^{(1)}(s\rho)$

and $$A_z(\omega) = \frac{ie}{2c}H_0^{(1)}(s\rho)\, e^{-i\omega\frac{z}{v}+i\omega t}$$

In the asymptotic form when $(s\rho) >> 1$,

$$A_z(\omega) = -iB_1+(\alpha_1+i\beta_1)\frac{1}{\sqrt{\rho}}e^{-s_2\rho}e^{\omega\left\{t-\left(\frac{z}{v}-\frac{s_1}{\omega}\rho\right)+\frac{3\pi}{4\omega}\right\}} \qquad \ldots (11)$$

The expression of $A_z(\omega)$ in (9) and (11) are identical. There is only a phase difference of π. More over there is no discontinuity of s_1 and s_2 in either case.

By (2) and (9) the field strengths are

$$\left.\begin{aligned}
H_\varphi(\omega) &= -iB_1(\alpha_1+i\beta_1)\left[\frac{1}{2\rho}-s_2+is_1\right]\frac{1}{\sqrt{\rho}}\,e^{s_2\rho}\,e^{i\omega\chi}\\
E_\rho(\omega) &= B_1\,\frac{wc}{v}\,(\alpha_1+i\beta_1)\,\frac{4\pi\sigma-iwn^2}{16\pi^2\sigma^2+\omega^2n^4}\left[\frac{1}{2\rho}-s_2+is_1\right]\frac{1}{\sqrt{\rho}}\,e^{s_2\rho}e^{i\omega\chi}\\
E_z(\omega) &= -iB_1(\alpha_1+i\beta_1)\left[\frac{iw}{c}+\frac{\omega^2c}{v^2}\,\frac{4\pi\sigma-i\omega n^2}{16\pi^2\sigma^2+\omega^2n^4}\right]\frac{1}{\sqrt{\rho}}\,e^{s_2\rho}e^{i\omega\chi}
\end{aligned}\right\}\quad\dots\ (12)$$

$$\left.\begin{aligned}
ReH_\varphi(\omega) &= B_1\,\frac{1}{\sqrt{\rho}}\,e^{s_2\rho}\Big[\Big\{\beta_1\Big(\frac{1}{2\rho}-s_2\Big)+\alpha_1s_1\Big\}\cos\omega\chi\\
&\qquad+\Big\{\alpha_1\Big(\frac{1}{2\rho}-s_2\Big)-\beta_1s_1\Big\}\sin\omega\chi\Big]\\
ReE_\rho(\omega) &= B_1\frac{wc}{v(16\pi^2\sigma^2+w^2n^4)}\,\frac{1}{\sqrt{\rho}}\,e^{s_2\rho}\Big[\Big\{(4\pi\sigma\alpha_1+\omega n^2\beta_1)\Big(\frac{1}{2\rho}-s_2\Big)\\
&\quad-(4\pi\sigma\beta_1-\omega n^2\alpha_1)s_1\Big\}\cos\omega\chi-\Big\{(4\pi\sigma\alpha_1+\omega n^2\beta_1)s_1+(4\pi\sigma\beta_1-\omega n^2\alpha_1)\\
&\qquad\Big(\frac{1}{2\rho}-s_2\Big)\Big\}\sin\omega\chi\Big]\\
ReE_z(\omega) &= -B_1\,\frac{1}{\sqrt{\rho}}\,e^{s_2\rho}\Big[\Big\{\frac{\beta_1}{v^2}\,\frac{4\pi\omega^2c\sigma}{16\pi^2\sigma^2+\omega^2n^4}\\
&\qquad+\alpha_1\Big(\frac{\omega}{c}-\frac{\omega^3cn^2}{v^2}\,\frac{1}{16\pi\sigma^2+w^2n^4}\Big)\Big\}\cos\omega\chi\\
&\quad+\Big\{\frac{\alpha_1}{v^2}\,\frac{4\pi w^2c\sigma}{16\pi^2\sigma^2+w^2n^4}-\beta_1\Big(\frac{\omega}{c}-\frac{w^3cn^2}{v^2}\,\frac{1}{16\pi^2\sigma^2+\omega^2n^4}\Big)\Big\}\sin\omega\chi\Big]
\end{aligned}\right\}\quad\dots\ (13)$$

$$\left.\begin{aligned}
H_\varphi &= \int_0^\infty 2[ReH_\varphi(\omega)]d\omega\\
E_\rho &= \int_0^\infty 2[ReE_\rho(\omega)]d\omega\\
E_z &= \int_0^\infty 2[ReE_z(\omega)]d\omega
\end{aligned}\right\}\quad\dots\ (14)$$

CONCLUSION

(i) The value of A_z in (9) reveals that no restriction on the particle velocity is required for the outgoing wave propagation, though it is damped. Thus Cherenkov radiation takes place at any velocity of the particle in a medium having certain amount of conductivity.

(ii) The energy radiated through the surface of a cylinder per unit length whose axis is the line of motion of the electron is given by

$$\frac{dW}{dt} = - \quad 2 \int_{-\infty}^{\infty} E_z H_\varphi dt$$

$$= \frac{e^2}{c} \int_0^\infty \frac{1}{s_1^2+s_2^2} \Big[\{\alpha_1(s_1-s_2)-\beta_1(s_1+s_2)\} \Big\{ \frac{1}{v^2} (\alpha_1+\beta_1) \frac{4\pi\omega^2 c\sigma}{16\pi^2\sigma^2+\omega^2 n^4}$$

$$+(\alpha_1-\beta_1)\Big(\frac{\omega}{c} - \frac{\omega^3 c b^2}{v^2} \frac{1}{16\pi^2\sigma^2+\omega^2 n^4} \Big) \Big\} \Big] d\omega \quad \dots \quad (15)$$

This expression leads to the correct limiting case $\sigma = 0$ which is give by

$$\frac{dW}{dl} = \frac{e^2}{c^2} \int_0^\infty \Big(1 - \frac{1}{\beta^2 n^2} \Big) \omega d\omega.$$

(iii) From (10) the semi-vertical angle of the cone of radiation $\theta_1 = \tan^{-1}(vs_1/\omega)$. It is different from Chrenkov relation $\theta = \cos^{-1}(1/n\beta)$. Comparing the values of θ_1 with those of θ against v one can observe that the cone of Cherenkov radiation in a conducting medium is generally wider than that of a non-conducting medium. For the values of v between 0 and c/n, θ has no value (i.e. no Cherkenkov radiation takes place) but θ_1 has singificant values which means that the Cherenkov radiation takes place in conducting medium.

(iv) The penetration length

$$\left. \begin{aligned} a &= \frac{\sqrt{2}v}{\omega\Big[\Big\{(n^2\beta^2-1)^2+\frac{16\pi^2\sigma^2\beta^4}{w^2}\Big\}^{\frac{1}{2}} -(n^2\beta^2-1)\Big]^{\frac{1}{2}}} \quad \text{when } n^2\beta^2-1 \geqslant 0, \\ &\text{or} \\ a &= \frac{\sqrt{2}v}{\omega\Big[\Big\{(1-n^2\beta^2)^2+\frac{16\pi^2\sigma^2\beta^4}{\omega^2}\Big\}^{\frac{1}{2}} +(1-n^2\beta^2)\Big]^{\frac{1}{2}}} \quad \text{when } n^2\beta^2-1 < 0. \end{aligned} \right\} \quad \dots \quad (16)$$

The values of a for typical values of v are given below :

(I) $v = \frac{c}{n}$, $a = \frac{c}{\sqrt{2\pi\omega\sigma}}$

(II) $v = c$ and $n = 1$, $a = \frac{c}{\sqrt{2\pi\omega\sigma}}$

(III) when $v = c$, $n > 1$ and σ is very small, $a \simeq \frac{c}{2\pi} \frac{\sqrt{n^2-1}}{\sigma}$ .... (17)

In the usual electromagnetic phenomenon the penetration length $= \frac{cn}{2\pi\sigma}$. It is interesting to compare (17) with this expression.

(v) There is a critical value of v depending on the medium. The penetration length changes rapidly with v until v attains critical value after which it changes very slowly.

(vi) The wave length of the radiation is given by

$$\lambda = \frac{2\sqrt{2\pi}v}{\omega\left[\left\{(n^2\beta^2-1)^2+\frac{16\pi^2\sigma^2\beta^4}{w^2}\right\}^{\frac{1}{2}}+(n^2\beta^2+1)\right]^{\frac{1}{2}}} \text{ when } n^2\beta^2-1 \geqslant 0,$$

and

$$\lambda = \frac{2\sqrt{2\pi}v}{\omega\left[\left\{(1-n^2\beta^2)^2+\frac{16\pi^2\sigma^2\beta^4}{\omega^2}\right\}^{\frac{1}{2}}+(n^2\beta^2+1)\right]^{\frac{1}{2}}} \text{ when } n^2\beta^2-1 < 0. \quad \ldots (18)$$

One sees that a and λ are comparable when $n^2\beta^2-1 < 0$. At the first sight it may be seen that under these circumstances Cherenkov Radiation though produced it becomes unimportant. But that is actually not the case. Since if the conducting material be in the form of a thin wafer, a narrow cylinder or something of this sort then outside the medium, e.g. in vacuum one will have a substantial effect. Calculation stemming from these consideration are in progress and will be communicated elsewhere.

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REFERENCE

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