# CHERENKOV RADIATION IN A MEDIUM OF VARIABLE DIELECTRIC CONSTANT

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**ABSTRACT**. Cherenkov effect in a medium with a variable dielectric constant alters the amplitude of each component of the field variable and the output of radiation compared to the usual Cherenkov radiation as obtained by Frank and Tamm (1937). Even if asymptotically the specific inductive capacity becomes a constant, the radiation still then remains different. Semi-vertical angle of the cone of radiation flactuates with the change of dielectric constant.

## INTRODUCTION

Generally we consider the dielectric constant as a scalar constant for isotropic substance or a tensor for anisotropic medium. Besides these two there are substances where the dielectric constants are scalar variables. So it is important to observe the behaviour of the field variables and Cherenkov radiation for a high energy particle through this type of medium.

From Maxwell's equations of field variables with Fourier transformation

$$\left.\begin{aligned} \text{rot } H &= \frac{i\omega\epsilon}{c} E+\frac{4\pi}{c}j \\ \text{rot } E &= -\frac{i\omega}{c} H \\ \text{div } (\epsilon E) &= 4\pi\rho \\ \text{div } H &= 0, \end{aligned}\right\} \qquad \ldots \quad (1)$$

where $\omega$ is the frequency and $\epsilon$ is the dielectric constant.

Eliminating $H$,

$$ic/\omega \text{ rot rot } E = \frac{i\omega\epsilon}{c} E+\frac{4\pi}{c}j \qquad \ldots \quad (2)$$

Let the electron move along the $z$ axis with the velocity $v$ and in cylindrical co-ordinates $(\rho, \phi, z)$, $E$ is independent of $\phi$ and $E_\varphi = 0$.

From (2) $$-\frac{\partial}{\partial z}\left(\frac{\partial E_\rho}{\partial z}-\frac{\partial E_z}{\partial \rho}\right) = \frac{\omega^2}{c^2}\epsilon E_\rho$$

and $$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\rho\left(\frac{\partial E_\rho}{\partial z}-\frac{\partial E_z}{\partial\rho}\right)\right\}=\frac{\omega^2}{c^2}\epsilon E_z-\frac{ie\omega}{\pi c^2\rho}e^{-\frac{i\omega z}{v}}\delta(\rho).$$

Let $$E_\rho=f_1(\rho)e^{-i\frac{\omega z}{v}},\quad E_z=f_2(\rho)\,e^{-\frac{i\omega z}{z}}$$

Then $$s^2f_1=-\frac{i\omega}{v}\frac{\partial f_2}{\partial\rho} \qquad \dots (3)$$

and $$\frac{\partial^2 f_2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f_2}{\partial\rho}+s^2f_2-\frac{1}{\epsilon(\epsilon\beta^2-1)}\frac{\partial\epsilon}{\partial\rho}\frac{\partial f_2}{\partial\rho}=\frac{ies^2}{\pi\epsilon\omega}\frac{\partial(\rho)}{\rho} \qquad \dots (4)$$

where $$s^2=\frac{\omega^2}{v^2}(\epsilon\beta^2-1),\quad \beta^2=\frac{v^2}{c^2}.$$

If we put $f_2(\rho)=u(\rho)H_0^{(2)}(s\rho)$ in (4) and also we consider as $u=u_0+u_1$, $\epsilon=\epsilon_0+\epsilon_1$, where $u_0$ and $\epsilon_0$ are absolute constants, $u_1$ and $\epsilon_1$ are functions of $\rho$ only but they are small. In this case neglecting second order small quantities we get

$$\frac{\partial^2u_1}{\partial\rho^2}+\left(\frac{1}{\rho}-2s_0\frac{H_1}{H_0}\right)\frac{\partial u_1}{\partial\rho}-\left\{\tfrac{1}{2}\frac{\omega^2u_0}{c^2s_0}\frac{H_1}{H_0}\rho\frac{\partial^2\epsilon_1}{\partial\rho^2}\right.$$
$$\left.+\left(\frac{\omega^2u_0}{c^2}\rho+\tfrac{1}{2}\frac{\omega^2u_0}{c^2s_0}\frac{H_1}{H_0}-\frac{\omega^2}{\epsilon_0}\frac{u_0}{v^2s_0}\frac{H_1}{H_0}\right)\frac{\partial\epsilon_1}{\partial\rho}\right\}=0 \qquad \dots (5)$$

Here $H_1=H_1^{(2)}(s\rho)$, $H_0=H_0^{(2)}(s\rho)$ (Hankel's functions)

From (5) $$u_1=\int\left\{\frac{1}{\rho\chi_1(\rho)}\int\rho\chi_1(\rho)\chi(\rho)d\rho\right\}d\rho+C_1\int\frac{d\rho}{\rho\chi_1(\rho)}+C_2 \qquad \dots (6)$$

where $C_1$ and $C_2$ are constants and

$$\chi_1(\rho)=e^{-2S_0\int\frac{H_1}{H_0}d\rho},$$

$$\chi(\rho)=\frac{1}{2}\frac{\omega^2u_0}{c^2s_0}\frac{H_1}{H_0}\rho\frac{\partial^2\epsilon_1}{\partial\rho^2}+\left(\frac{\omega^2u_0}{c^2}\rho+\frac{1}{2}\frac{\omega^2u_0}{c^2s_0}\frac{H_1}{H_0}-\frac{\omega^2u_0}{\epsilon_0s_0v^2}\frac{H_1}{H_0}\right)\frac{\partial\epsilon_1}{\partial\rho},$$

$$s_0^2=\frac{\omega^2}{v^2}(\epsilon_0\beta^2-1).$$

There is a singularity at $\rho=0$ in (4).
From (4) we have

$$\lim_{\rho\to0}\rho\frac{\partial f_2}{\partial\rho}=\left[\frac{ies^2}{\pi\epsilon\omega}\right]_{\rho=0}$$

or

$$\lim_{\rho\to0}\left[\frac{-2i}{\pi}(u_0+u_1)+\rho\frac{\partial u_1}{\partial\rho}\left(1-\cdot5772i-\frac{i2}{\pi}\log\frac{s_0}{2}\right)\right.$$
$$\left.-\frac{i2}{\pi}\rho\log\rho\frac{\partial u_1}{\partial\rho}\right]=\frac{ie}{\pi\omega\epsilon_0}\left(s_0^2+\frac{\omega^2}{v^2}\cdot\frac{k_0}{\epsilon_0}\right) \qquad \dots (7)$$

where $k_0$ is the value of $\epsilon_1$ at $\rho = 0$.

Here $$u_0 = -\frac{e s_0^2}{2\omega\epsilon_0} \qquad \ldots (8)$$

If $u_1$ is constant then $$u_1 = -\frac{e\omega k_0}{2v^2\epsilon_0^2} \qquad \ldots (9)$$

From (7),

$$\lim_{\rho\to 0}\Big[-\frac{i2}{\pi}\Big(\int\Big\{\frac{1}{\rho\chi_1(\rho)}\int\rho\chi_1(\rho)\chi(\rho)d\rho\Big\}d\rho+C_1\int\frac{d\rho}{\rho\chi_1(\rho)}+C_2$$
$$+\Big(1-\cdot 5772i-\frac{i2}{\pi}\log\frac{s_0}{2}\Big)\Big\{\frac{\int\rho\chi_1(\rho)\chi(\rho)d\rho}{\chi_1(\rho)}+\frac{C_1}{\chi_1(\rho)}\Big\}$$
$$-\frac{i2}{\pi}\log\rho\Big\{\frac{\int\rho\chi_1(\rho)\chi(\rho)d\rho}{\chi_1(\rho)}+\frac{C_1}{\chi_1(\rho)}\Big\}\Big]=\frac{ie\omega k_0}{\pi v^2\epsilon_0^2} \qquad \ldots (10)$$

Again at a large distance $u_1$ and $\epsilon_1$ must be bounded for physical solution. Thus

$$\lim_{\rho\to\infty}\Big[\int\Big\{\frac{1}{\rho\chi_1(\rho)}\int\rho\chi_1(\rho)\chi(\rho)d\rho\Big\}d\rho+C_1\int\frac{d\rho}{\rho\chi_1(\rho)}+C_2\Big]$$
$$= \text{a bounded quantity.} \qquad \ldots (11)$$

From (10) and (11) $C_1$ and $C_2$ can be found out.

The components of the field variables are

$$\left.\begin{aligned}
E_\rho(\omega) &= -\frac{i\omega}{vs^2}\frac{\partial E_z}{\partial\rho}\\
&=\Big\{\frac{i\omega}{vs_0}(u_0+u_1)H_1-\frac{i\omega}{vs_0^2}H_0\frac{\partial u_1}{\partial\rho}-\frac{i}{2}\frac{\omega^3 u_0}{c^2vs_0^3}\epsilon_1 H_1\\
&\qquad+\frac{i}{2}\frac{\omega^3 u_0}{c^2vs_0^3}\rho H_1\frac{\partial\epsilon_1}{\partial\rho}\Big\}e^{i\omega\left(t-\frac{z}{v}\right)}\\
E_z(\omega) &= (u_0+u_1)H_0\, e^{i\omega\left(t-\frac{z}{v}\right)}\\
H_\varphi(\omega) &= -\frac{ic}{\omega}\Big(\frac{\omega^2}{v^2s^2}+1\Big)\frac{\partial E_z}{\partial\rho}\\
&=\Big(\frac{i\omega}{c}\frac{u_0}{s_0}\epsilon_0 H_1+\frac{i\omega}{c}\frac{\epsilon_0}{s_0}u_1H_1-\frac{i}{2}\frac{\omega^3}{c^3s_0^3}\epsilon_0u_0\epsilon_1H_1+\frac{i\omega u_0}{cs_0}\epsilon_1H_1\\
&\qquad-\frac{i\omega\epsilon_0}{cs_0^2}H_0\frac{\partial u_1}{\partial\rho}+\frac{i}{2}\frac{\omega^3u_0}{c^3s_0^3}\epsilon_0\rho H_1\frac{\partial\epsilon_1}{\partial\rho}\Big)e^{i\omega\left(t-\frac{z}{v}\right)}
\end{aligned}\right\} \qquad \ldots (12)$$

For
$$\left.\begin{aligned} \omega > 0,\ ReE_\rho &= 2 \int ReE_\rho(\omega) d\omega \\ ReE_z &= 2 \int ReE_z(\omega) d\omega \\ ReH_\varphi &= 2 \int ReH_\varphi(\omega) d\omega \end{aligned}\right\} \quad \ldots (13)$$

(i) The results of (12) reveal that the components of electromagnetic intensities are different from usual forms of them. The expressions of them are rather unwieldy, but we can compute them numerically upto a certain order.

(ii) At large distance

$$E_z(\omega) = \left[ -\frac{es_0^2}{2w\epsilon_0} + \frac{e}{8} \frac{\omega}{c^2\epsilon_0} k_1 + (a+ib) \right] \sqrt{\frac{2}{\pi s_0 \rho}}\, e^{i\omega\lambda}$$

$$H_\varphi(\omega) = \left[ \frac{es_0}{2c} - \frac{3}{8} \frac{e\omega^2}{c^3 s_0} k_1 + \frac{es_0}{2c\epsilon_0} k_1 - \frac{e\omega^2}{4s_0c^3} k_2 - \frac{\omega\epsilon_0}{cs_0} (a+ib) \right.$$

$$\left. - \frac{i\omega}{c} \frac{\epsilon_0}{s_0^2} (a_1+ib_1) \right] \sqrt{\frac{2}{\pi s_0 \rho}}\, e^{i\omega\lambda},$$

where

$$\lim_{\rho \to \infty} \epsilon_1 = k_1, \quad \lim_{\rho \to \infty} \left( \rho \frac{\partial \epsilon_1}{\partial \rho} \right) = k_2, \quad \lim_{\rho \to \infty} u_1 = a+ib,$$

$$\lim_{\rho \to \infty} \frac{\partial u_1}{\partial \rho} = a_1 + ib_1, \quad \lambda = t - \frac{z}{v} - \frac{s\rho}{\omega} + \frac{\pi}{4\omega}.$$

(iii) Cherenkov radiation $W$ through the surface of a cylinder of length $l$ is

$$W = 2\pi\rho l \int_{-\infty}^{\infty} \frac{c}{4\pi} [EH] dt$$

or
$$\frac{dW}{dl} = -\frac{c\rho}{2} \int_{-\infty}^{\infty} R_e E_z \cdot R_e H_\varphi dt$$

$$= \int_0^{\omega_{max}} \left[ \frac{e^2}{c^2} \left( 1 - \frac{1}{\epsilon_0 \beta^2} \right) \omega - \frac{e^2}{c^2} \frac{k_1}{\epsilon_0} \omega + \frac{e^2 s_0^2}{\omega \epsilon_0^2} k_1 \right.$$

$$\left. - \frac{e^2}{2c^2} \frac{\omega}{\epsilon_0} k_2 - 4e(a-b) + \frac{e}{2s_0} (a_1 + b_1) \right] d\omega$$

If $\epsilon_1$ is constant then $k_1 = k_0 = \epsilon_{01}$ (say), $k_2 = 0$,

$$u_1 = a = -\frac{e\omega\epsilon_{01}}{2v^2\epsilon_0^2}, \quad b = 0, \quad a_1 = 0 = b_1.$$

In this case

$$\frac{dw}{dl} = \frac{e^2}{c^2} \int \left[ \left(1 - \frac{1}{\epsilon_0 \beta^2}\right) + \frac{c^2 \epsilon_{01}}{v^2 \epsilon_0^2} \right] \omega dw$$

(iv) If $\theta$ is the semi-vertical angle of the cone of radiation then

$$\cos\theta = \frac{1}{\sqrt{\epsilon_0}\beta} \left(1 - \frac{\epsilon_1}{2\epsilon_0}\right)$$

If now $v < \frac{c}{\sqrt{\epsilon_0}}$ , $\theta$ may still have a real value which means that radiation can exist even then. As $\epsilon_1$ is a variable quantity, i.e. a function of distance, the semi-vertical angle of the above cone also varies with distance.

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