Indian J. Phys. **67B** (1), 15-22 (1993)

Exact solutions of Schroedinger equation for a linearly kicked harmonic oscillator

A K Sikri, S C Gupta and M L Narchal
Department of Physics, Punjabi University, Patiala-147 002, India

Received 26 August 1992, acepted 23 October 1992

Abstract : The time-dependent Schroedinger equation for a one dimensional harmonic oscillator subject to a sequence of arbitrarily time spaced linear kicks is solved exactly. The wave function after n kicks is used to calculate the expectation values of observables such position, momentum and energy. Explicit results are presented for an oscillator kicked periodically and quasi-periodically.

Keywords : Chaos, quantum recurrence, quasi-periodic.

PACS No. : 05.45. +b.

1. Introduction

The study of classical and quantum systems subject to time periodic potentials has attracted much attention in recent years in the context of classical and quantum chaos [1-10]. The simplest time periodic potential is a sequence of δ kicks equally spaced in time. A number of research papers have appeared in literature on periodically kicked classical systems. Some general results have also been discussed on the quantum behaviour of systems subject to time periodic potentials. One of the principal results in this connection is the theorem on quantum recurrence given by Hogg and Huberman [3]. This theorem rules out the occurrence of quantum chaos in bounded systems characterised by a point spectrum and subject to time periodic potentials.

The question as to whether recurrent behaviour occurs in quasiperiodically kicked systems or in systems subject to other forms of time dependent potentials is still open and requires further study. Whether or not quantum recurrence occurs can be indicated either by the time behaviour of the autocorrelation of the state vector of the system or by the time behaviour of autocorrelation of the expectation value of position vector. These correlations functions can be evaluated if the exact form of the time dependent wave function of the system is known. This would require the exact solution of the time dependent Schroedinger equation for the system. There are hardly any exact solutions available to even the simplest of systems.

In this paper we present an exact solution of Schroedinger equation for a quantum harmonic oscillator subject to a series of kicks arbitrarily spaced in time. The outline of the paper is as follows : In Section (2) we obtain the general solution for an arbitrarily kicked harmonic oscillator. In Section (3) these solutions have been utilised to calculate the expectation values of position, momentum and Hamiltonian operators. Results are dissused in Section (4).

2. Quantum mapping and the general solution to wave equation

Consider a one dimensional harmonic oscillator of unit mass described by the Hamiltonian

$$H = H_0 + \sum_{m=1}^{n} K(\tau_m , x)\, \delta(t - \tau_m) \tag{1}$$

where H_0 is the usual harmonic oscillator Hamiltonian $H_0 = \hbar\omega a^{+}a$. The second term describes a system of δ kicks at times $\tau_1, \tau_2 \ldots\ldots \tau_n$. $K(\tau_m, x)$ is the kick amplitude for the m-th kick which is a function of kicking time and the instantaneous position of the oscillator. For a system of linear kicks

$$K(\tau_m, x) = K(\tau_m)x. \tag{2}$$

The wave function of the system described by Hamiltonian H is a function of time whose development in between the kicks is governed by the Hamiltonian H_0 and its development during the m-th kick is described by the Hamiltonian

$$H_{\text{kick}} = K(\tau_m)\, x\, \delta(t - \tau_m). \tag{3}$$

The time development of the wave function in between the times immediately after the $(m-1)$-th kick and immediately before m-th kick is given by the operator

$$U(\tau_m^{\;-} + \tau_m^{\;+}) = \exp\left[-\frac{i}{\hbar} H_0 (\tau_m - \tau_{m-1})\right] \tag{4}$$

where the symbols + and – after the time τ_m stand for immediately after and immediately before kick.

The development of the system during the m-th kick operating at time τ_m is given by

$$U(\tau_m) = \exp\left[-\frac{i}{\hbar} K(\tau_m)x \right]. \tag{5}$$

The ordered product of operators given by eq. (4) and eq. (5) gives the total development of the system in between times immediately after the $(m-1)$-th kick to immediately after the m-th kick. If we designate this operator as T_m we have

$$T_m = \exp\left[-\frac{i}{\hbar} K(\tau_m)x \right] \exp\left[-\frac{i}{\hbar} H_0 (\tau_m - \tau_{m-1})\right]. \tag{6}$$

Let $|\psi_0>$ be the initial wave function of the oscillator. Then its wave function after n kicks is given by

$$|\psi_n> = T_n T_{n-1} \ldots\ldots T_1 |\psi_o>. \tag{7}$$

Substituting eq. (6) in eq. (7) we obtain

$$\exp\left[\frac{i}{\hbar}H_o \tau_n\right]|\psi_n> = \prod_{m=n}^{1} \exp\left[\frac{i}{\hbar}H_o \tau_m\right]$$

$$\exp\left[-\frac{i}{\hbar}K(\tau_m)x\right]\exp\left[-\frac{i}{\hbar}H_o \tau_m\right]|\psi_o>$$

$$\exp\left[\frac{i}{\hbar}H_o \tau_n\right]|\psi_n> = \prod_{m=n}^{1} P_m \; |\psi_o> \tag{8}$$

where

$$P_m = \exp\left[\frac{i}{\hbar}H_o \tau_m\right]\exp\left[-\frac{i}{\hbar}K(\tau_m)x\right]\exp\left[-\frac{i}{\hbar}H_o \tau_m\right]. \tag{9}$$

Using the operator theorem

$$\exp[\xi A] B \exp[-\xi A] = B + \xi[A, B] + \frac{\xi^2}{2!}[A, [A, B]] + \ldots\ldots \tag{10}$$

we can write operator P_m as

$$P_m = \exp\left[-\frac{iK(\tau_m)}{\hbar}\left(x\cos(\omega\tau_m) + \frac{p}{\omega}\sin(\omega\tau_m)\right)\right]. \tag{11}$$

Using the operator theorem which says that [11]

If $\quad [A, [A, B]] = [B, [A, B]] = 0$

then

$$\exp[A + B] = \exp(A)\exp(B)\exp\left(-\frac{1}{2}[A, B]\right), \tag{12}$$

the expression for the operator P_m takes the form

$$P_m = \exp\left[-\frac{i}{\hbar}K(\tau_m)\,x\cos(\omega\tau_m)\right] \times$$

$$\exp\left[-\frac{i}{\hbar}\frac{K(\tau_m)}{\omega}p\sin(\omega\tau_m)\right]$$

$$\exp\left[\frac{i}{\hbar}\frac{K^2(\tau_m)}{4\omega}\sin(2\omega\tau_m)\right] \tag{13}$$

Thus the mapping which gives the wave function immediately after the n-th kick in terms of the starting wave function $|\psi_o>$ is given by

$$\exp\left[\frac{i}{\hbar}H_o \tau_n\right]|\psi_n> = \prod_{m=n}^{1}\exp\left(-\frac{i}{\hbar}k(\tau_m)\,x\cos(\omega\tau_m)\right) \times$$

$$\exp\left(-\frac{i}{\hbar}\frac{K(\tau_m)}{\omega}p\sin(\omega\tau_m)\right)\times\exp\left(\frac{i}{\hbar}\frac{K^2(\tau_m)}{4\omega}\sin(2\omega\tau_m)\right)|\psi_0>. \quad (14)$$

The operator product on the right hand side of eq. (14) contains an exponential position operator followed by an exponential momentum operator and so on. We can make repeated use of eq. (12) and collect all the position operators together and all the momentum operators together. Thus eq. (14) simplifies to the form

$$\exp\left[\frac{i}{\hbar}H_0\tau_n\right]|\psi_n> = \exp\left[-\frac{ip}{\hbar\omega}\sum_{m=1}^{n}K(\tau_m)\sin(\omega\tau_m)\right]\times$$
$$\exp\left[-\frac{ix}{\hbar}\sum_{m=1}^{n}K(\tau_m)\cos(\omega\tau_m)\right]\times$$
$$\exp\left[-\frac{i}{\hbar\omega}\sum_{m=1}^{n}\sum_{r=m}^{n}K(\tau_m)K(\tau_r)\sin(\omega\tau_m)\times\right.$$
$$\left.\cos(\omega\tau_r)-\frac{1}{4\omega}\sum_{m=1}^{n}K^2(\tau_m)\sin(2\omega\tau_m)\right]|\psi_0>. \quad (15)$$

This may be written in the short form as

$$\exp\left[\frac{i}{\hbar}H_0\tau_n\right]|\psi_n> = \exp\left(-\frac{i}{\hbar}\alpha_n p\right)\times$$
$$\exp\left(-\frac{i}{\hbar}\beta_n x\right)\times$$
$$\exp\left(-\frac{i}{\hbar}\gamma_n\right)|\psi_0> \quad (16)$$

where

$$\alpha_n = \frac{1}{\omega}\sum_{m=1}^{n}K(\tau_m)\sin(\omega\tau_m), \quad (17)$$

$$\beta_n = \sum_{m=1}K(\tau_m)\cos(\omega\tau_m), \quad (18)$$

$$\gamma_n = \frac{1}{\omega}\sum_{m=1}^{n}\sum_{r=m}^{n}K(\tau_m)K(\tau_r)\sin(\omega\tau_m)\cos(\omega\tau_r)$$
$$-\frac{1}{4\omega}\sum_{m=1}^{n}K^2(\tau_m)\sin(2\omega\tau_m). \quad (19)$$

The operator $\exp\left(-\frac{i}{\hbar}\alpha_n p\right) = \exp\left(-\alpha_n \frac{\partial}{\partial x}\right)$ is the translation operator and we can write eq. (16) as

$$\exp\left[\frac{i}{\hbar}H_0 \tau_n\right]|\psi_n> = \exp\left[-\frac{i}{\hbar}(\gamma_n - \beta_n \alpha_n)\right] \times \exp\left[-\frac{i}{\hbar}\beta_n x\right]|\psi_0 (x - \alpha_n)>. \quad (20)$$

This is an important result of this paper. Eq. (20) essentially gives the wave function after n kicks in terms of starting wave function $|\psi_0>$.

3. Expectation values of observables

If O is an operator then its expectation value after n kicks is given by

$$<O>_n = <\psi_n | O | \psi_n>$$

$$<O>_n = <\psi_n \exp(-\frac{i}{\hbar}H_0 \tau_n) | \exp(\frac{i}{\hbar}H_0 \tau_n) O \exp(-\frac{i}{\hbar}H_0 \tau_n) | \exp(\frac{i}{\hbar}H_0 \tau_n) \psi_n> \quad (21)$$

Thus the expectation value of the operator O is given by the expectation value of the operator $\exp(\frac{i}{\hbar}H_0 \tau_n)$ O $\exp(-\frac{i}{\hbar}H_0 \tau_n)$ in the state $|\psi_n>$ given by $\exp\left[-\frac{i}{\hbar}(\gamma_n - \beta_n \alpha_n)\right] \exp\left(-\frac{i}{\hbar}\beta_n x\right) |\psi_0 (x - \alpha_n)>$. Thus the required expectation value is given by

$$<O>_n = \int_{-\infty}^{\infty} \psi_0^* (x - \alpha_n) \exp(\frac{i}{\hbar}\beta_n x) \times \left[\exp\left(\frac{i}{\hbar}H_0 \tau_n\right) O \exp\left(-\frac{i}{\hbar}H_0 \tau_n\right)\right] \times \exp\left(-\frac{i}{\hbar}\beta_n x\right) \psi_0 (x - \alpha_n) \, dx. \quad (22)$$

The operator in the square bracket can be calculated using eq. (12). For position and momentum operators

$$\exp\left(\frac{i}{\hbar}H_0 \tau_n\right) x \exp\left(-\frac{i}{\hbar}H_0 \tau_n\right) = x \cos(\omega\tau_n) + \frac{p}{\omega}\sin(\omega\tau_n) \quad (23)$$

$$\exp\left(\frac{i}{\hbar}H_0 \tau_n\right) p \exp\left(-\frac{i}{\hbar}H_0 \tau_n\right) = p \cos(\omega\tau_n) - \omega x \sin(\omega\tau_n) \quad (24)$$

Using eq. (23) and eq. (24) we obtain from eq. (22)

$$<x>_n = \frac{1}{\omega} \sum_{m=1}^{n} K(\tau_m) \sin \omega (\tau_m - \tau_n), \quad (25)$$

$$<p>_n = -\sum_{m=1}^{n} K(\tau_m) \cos \omega (\tau_m - \tau_n). \tag{26}$$

The expectation value of H_0 after n kicks is

$$E_n = \frac{1}{2} \sum_{q=1}^{n} \sum_{m=1}^{n} K(\tau_m) K(\tau_q) \cos \omega (\tau_m - \tau_q) + E_0 \tag{27}$$

Eq. (25) to eq. (27) represent general results which are applicable to a sequence of arbitrarily time separated kicks. The two special cases of primary interest are the periodically and quasi-periodically kicked oscillators. For a periodically kicked oscillator we substitue

$$\tau_m = m\tau, \qquad K(\tau_m) = K \tag{28}$$

The expectation values given by eqs. (25), (26) and (27) then take the form

$$<x>_n = \frac{K}{2\omega} \frac{\sin(n\omega\tau) - \sin(n-1)\omega\tau - \sin\omega\tau}{1 - \cos\omega\tau}, \tag{29}$$

$$<p>_n = \frac{K}{2} \frac{\cos(n\omega\tau) + \cos(2n+1)\omega\tau - \cos(2n\omega\tau) - \cos(n+1)\omega\tau}{1 - \cos\omega\tau}, \tag{30}$$

$$E_n = E_0 + \frac{K^2}{2} \frac{\sin^2(n\omega\tau/2)}{\sin^2(\omega\tau/2)}. \tag{31}$$

Eq. (31) agrees with the result given by Hogg and Huberman [3].

It is useful to examine the behaviour of autocorrelation function of position with the number of kicks. The autocorrelation function is defined as

$$c(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} <x>_n <x>_{n+m} \tag{32}$$

Using eq. (29) it takes the form

$$c(m) = \frac{1}{8\omega^2 \sin^2 \frac{\omega\tau}{2}} \left[\cos^2\left(\frac{m\omega\tau}{2}\right) + \cos\frac{(m+1)\omega\tau}{2} \cos\frac{(m-1)\omega\tau}{2} \right]. \tag{33}$$

This correlation function does not decay irrespective of the fact whether $\omega\tau$ is rational or irrational.

If an oscillator is quasiperiodically kicked then we may write [6]

$$K(\tau_m) = K\cos(m\omega'\tau) \tag{34}$$

where ω' is the angular frequency defining the periodic variation in kick amplitude. For quasiperiodic kicking we require ω' to be incommensurate with the kicking frequency i.e. $\omega'\tau$ must be an irrational number. If we define

$$\theta = \frac{(\omega + \omega')\tau}{2}, \qquad \phi = \frac{(\omega - \omega')\tau}{2} \tag{35}$$

We obtain for a quasiperiodically kicked oscillator

$$<x>_n = \frac{K}{8\omega\sin^2\theta}[\sin(n\omega\tau) + \sin\{\omega'\tau - (n-1)\omega\tau\} - \sin\{(n+1)\omega'\tau + \omega\tau\} + \sin(n\omega'\tau)]$$

$$+ \frac{K}{8\omega\sin^2\phi}[\sin(n\omega\tau) - \sin\{\omega'\tau + (n-1)\omega\tau\} + \sin\{(n+1)\omega'\tau - \omega\tau\} - \sin(n\omega'\tau)] \quad (36)$$

$$<p>_n = \frac{K}{8\sin^2\theta}[\cos(n\omega\tau) - \cos\{(n+1)\omega\tau + \omega'\tau\} + \cos\{(2n+1)\omega\tau + (n+1)\omega\tau\}$$

$$- \cos\{2n\omega\tau + n\omega'\tau\}] +$$

$$\frac{K}{8\sin^2\phi}[\cos(n\omega\tau) - \cos\{(n+1)\omega\tau - \omega'\tau\} + \cos\{(2n+1)\omega\tau + (n+1)\omega'\tau\}$$

$$- \cos\{2n\omega\tau - n\omega'\tau\}]$$

$$E_n = \frac{K^2}{8}\left[\frac{\sin^2 n\theta}{\sin^2\theta} + \frac{\sin^2 n\phi}{\sin^2\phi} + \frac{2\sin(n\theta)\sin(n\phi)\cos\{(n+1)(\theta-\phi)\}}{\sin\theta\sin\phi}\right.$$

The correlation function for quasi-periodic kicking turns out to be

$$C(m) = \frac{1}{16\omega^2}\left[\frac{\cos(m\theta)\cos(m\phi)}{\sin^2\theta} + \frac{\cos(m\theta)\cos\phi}{\sin^2\phi}\right.$$

$$\left.\frac{\cos(m\phi-\theta)\cos(m\theta+\phi)}{\sin\theta\sin\phi} + \frac{\cos(m\theta-\phi)\cos(m\phi+\theta)}{\sin\theta\sin\phi}\right] \quad (39)$$

4. Conclusions

The exact wave functions for one dimensional harmonic oscillator subject to arbitrary kicks have been obtained. These wave functions have been used to evaluate the expectation values of various observables such as position, momentum and energy. The expression for energy of periodically kicked oscillator agrees with that given by Hogg and Hubermann [3], who have investigated the response of oscillator to periodic kicking by invoking Ehrenfest's theorem. The phenomenon of quantum recurrence occurs as expected. The system is exactly periodic if $\omega\tau$ is a rational multiple of 2π. For $\omega\tau = 2\pi/q$, the expectation phase plot will consist of just q discrete points. It $\omega\tau$ is an irrational multiple of 2π, the system is not periodic but comes arbitrarily close to its initial state infinitely often. The expectation phase trajectory will be a closed curve and the system is quasiperiodic.

In quasiperiodically kicked oscillator we deal with three frequencies namely (1) the natural frequency of oscillator (ω), (2) the kicking frequency $\omega'' = \frac{1}{\tau}$ and (3) frequency ω' with which the kick amplitude is modulated. If ω' / ω'' is a rational multiple of 2π, the energy of the oscillator would exactly recurr. In case ω' / ω'' is an irrational multiple of 2π, the energy will come arbitrarily close to its initial value.

We also present explicit expression for autocorrelation function of expectation value of position operator. The behaviour of this function is oscillatory for both rational and irrational values of ω' / ω''. Thus it appears that a quasi-periodically kicked oscillator will show quantum recurrence, and the theorem on quantum recurrence is of more general applicability than implied in reference (3). The method described in this paper is sufficiently general and can be applied whenever the exponential operators involved can be handled in terms of well known theorems [11].

References

[1] G Casati, B V Chirikov, J Ford and F M Izrailev 1979 *Stochastic behaviour in classical and quantum Hamiltonian Systems* (Vol. 93 of Lecture Notes in Physics) eds. G Casati and J Ford (Berlin : Springer-Verlag)

[2] S Fishman, D R Grempel and R E Prange 1982 *Phys. Rev. Lett.* **49** 509

[3] T Hogg and B A Hubermann 1982 *Phys Rev* **A28** 22

[4] D R Grempel, R E Prange and S Fishman 1984 *Phys Rev.* **A29** 1639

[5] F M Izrailev 1986 *Phys. Rev. Lett.* **56** 541

[6] P W Milonni, J R Ackerhalt and M E Goggin 1987 *Phys. Rev.* **A35** 1714

[7] R Scharf and M Kus 1989 *Phys Rev.* **A40** 4774

[8] T Geisel 1990 *Phys Rev* **A41** 2989

[9] B Milek and P Seba 1990 *Phys Rev.* **A42** 321

[10] R F Fox and B L Lan 1990 *Phys Rev.* **A41** 2952

[11] W Louisell 1964 *Radiation and Noise in Quantum Electronics* (New York : McGraw Hill)