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Static spheres of charged perfect fluid embedded in a Einstein universe

Abdussattar
Department of Mathematics, Banaras Hindu University, Varanasi-221 005, India

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Abstract : A class of exact, static and spherically symmetric solutions of Einstein's field equations is obtained for a charged perfect fluid distribution representing spheres of charged perfect fluid embedded in a Einstein universe If an electron is modelled as a charged perfect fluid sphere obeying Einstein-Maxwell equations in the background of Einstein universe, it is found that it need not contain negative rest mass density contrary to the result of Bonnor and Cooperstock [1] for an electron in an otherwise empty universe

Keywords : Charged fluid sphers, Einstein universe, static spherically symmetric gravitational fields

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1. Introduction

The Reissner-Nordstrom solution in curvature coordinates,

$$ds^2 = -\left[1 - (2m/r) + \left(e^2/r^2\right)\right]^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$
$$+\left[1 - (2m/r) + \left(e^2/r^2\right)\right]dt^2 \qquad (1)$$

describes the external gravitational field of a charged fluid sphere at rest at the origin in an otherwise empty universe. The arbitrary constants m and e appearing in eq. (1) are identified as the mass and the charge of the fluid sphere respectively. Every internal solution for a sphere of charged fluid in general relativity is continued at the boundary $r = a$ of the sphere with the Nordstrom solution [1]. It is physically implausible to visualize a charged fluid sphere in an otherwise empty universe. Rather, it appears more realistic to consider charged fluid spheres embedded in some cosmological background. In the vicinity of the charged fluid sphere, the Nordstrom field will however, dominate over the cosmological field.

Consequently, the gravitational field just outside the source of the Nordstrom solution, may be described by the Nordstrom solution with a small perturbation exerted by the cosmological field.

With this in view, a static spherically symmetric metric is considered in a suitable form and a class of internal solutions for spheres of charged perfect fluid has been obtained using Einstein-Maxwell's equations. An exact, static and spherically symmetric solution of Einstein's field equations is also obtained for a perfect fluid distribution representing the external gravitational field of a charged fluid sphere embedded in a Einstein universe. The continuity of the two solutions at the boundary $r = a$ of the charged perfect fluid sphere is discussed. Considering a charged perfect fluid sphere as an approximate classical model for an electron embedded in a cosmological background of simple Einstein universe, it is shown that it need not contain negative rest mass density as against the result of Bonnor and Cooperstock for an electron in an otherwise empty universe.

2. The metric and the field equations

The coefficients g_{11} and g_{44} of the Reissner-Nordstrom solution have the interesting property that in curvature coordinates they satisfy the relation $g_{11}g_{44} = -1$. Making use of this relation for perfect fluid spheres with charge, it was shown by Gron [2] and Gantreau [3] that such spheres give rise to gravitational repulsion. Here, a static spherically symmetric metric is considered in the form

$$ds^2 = -e^{-\nu+f}\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + e^{\nu}dt^2, \tag{2}$$

where ν and f are functions of r alone. Einstein's field equations

$$R_i^j - \tfrac{1}{2}Rg_i^j + \Lambda g_i^j = -2\left\{(\rho + p)v_i v^j - pg_i^j + E_i^j\right\} \tag{3}$$

for the metric (2) and with a charged perfect fluid distribution, lead to the following three equations

$$e^{\nu-f}\left[\left(\nu'/r\right) + \left(1/r^2\right)\right] - \left(1/r^2\right) + \Lambda = 2p - 2E_1^1, \tag{4}$$

$$e^{\nu-f}\left[\left(\nu''/2\right) + \left(\nu'^2/2\right) + \left(\nu'/r\right) - \left(f'\nu'/4\right) - \left(f'/2r\right)\right]$$

$$+\Lambda = 2p - 2E_2^2 = 2p - 2E_3^3, \tag{5}$$

and

$$e^{\nu-f}\left[\left\{\left(f' - \nu'\right)/r\right\} - \left(1/r^2\right)\right] + \left(1/r^2\right) - \Lambda = 2\rho + 2E_4^4, \tag{6}$$

where a dash ($'$) denotes differentiation with respect to r. E_i^j is the electromagnetic energy tensor given by

$$E_i^j = (1/4)\,g_i^j F_{rs}F^{rs} - F_{ir}F^{jr}, \tag{7}$$

where F_{ij} is the electromagnetic field tensor satisfying the Maxwell equations

$$F_{ij} = \phi_{i,j} - \phi_{j,i}, \tag{8}$$

and

$$\left(\sqrt{(-g)}F^{ij}\right), j = \sqrt{(-g)}\sigma u^i, \tag{9}$$

where ϕ_i is the electromagnetic four potential vector, σ is the charge density of the distribution and u^i is the fluid flow vector satisfying the condition $g_{ij}u^i u^j = 1$. A comma (,) followed by a suffix denotes partial derivative with respect to the coordinates.

3. A class of internal solutions

We take $\alpha(r)$ to be the electrostatic potential inside a sphere of charged perfect fluid, that is $\phi_i = (0,0,0\,\alpha(r))$ with the only surviving component of the electromagnetic field tensor F_{ij} as $F_{41} = -F_{14} = \alpha'$. The non-vanishing components of the electromagnetic energy tensor E_i^j given by eq. (7) in this case, are obtained as

$$E_4^4 = E_1^1 = -E_2^2 = -E_3^3 = \tfrac{1}{2}(\alpha')^2 e^{-f}. \tag{10}$$

From eqs. (4), (5) and (10) we get

$$\left(\nu''/2\right) + \left(\nu'^2/2\right) - \left(f'\nu'/4\right) - \left(f'/2r\right) - 2(\alpha')^2 e^{-\nu}$$
$$+ \left(1/r^2\right)e^{-\nu+f} - \left(1/r^2\right) = 0. \tag{11}$$

The substitution of $e^{\nu} = F$ reduces eq. (11) to the form

$$-f'e^{-f} + \left[\left(2F''r^2 - 4F - 8(\alpha')^2 r^2\right) / \left(F'r^2 + 2Fr\right)\right]e^{-f}$$
$$+ \left(4/(F'r^2 + 2Fr)\right) = 0. \tag{12}$$

Integrating this equation we get

$$e^{-f} = \exp\left\{-\int \left\{\left(2F''r^2 - 4F - 8(\alpha')^2 r^2\right)/(F'r^2 + 2Fr)\right\}dr\right\}$$
$$\left[\int \left\{\left(-4/(F'r^2 + 2Fr)\right) \exp\left\{\int \left\{\left(2F''r^2 - 4F - 8(\alpha')^2 r^2\right)/\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left(F'r^2 + 2Fr\right)\right\}dr\right\}\right\}dr + K\right], \tag{13}$$

where K is a constant of integration, With f being given by eq. (13) in terms of F and α, the metric (2) inside the sphere of charged perfect fluid may be written as

$$ds^2 = -\left(e^f/F\right)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + Fdt^2. \tag{14}$$

Here, F and α are two arbitrary functions of r satisfying the physical requirements. The metric (14) therefore, represents a class of internal solutions for spheres of charged perfect fluid. Neglecting the cosmological term, the material distribution inside the sphere of charged perfect fluid is given by

$$p = \left(e^{-f}/2\right)\left\{(F'/r) + (F/r^2) + (\alpha')^2\right\} - \left(1/2r^2\right), \tag{15}$$

and

$$\rho = (e^{-f}/2)\left\{(F/r)\left\{\left(2F''r^2 - 4F - 8(\alpha')^2 r^2\right)/\left(F'r^2 + 2Fr\right)\right\}\right.$$
$$\left. - (F'/r) - (F/r^2) - (\alpha')^2\right\} + \left(4F/r(F'r^2 + 2Fr)\right) + (1/r^2). \tag{16}$$

The charge density σ of the distribution is obtained as

$$\sigma = \left(e^{-f}\sqrt{F}/2r^2\right)\left\{f'\alpha' r^2 - 2\alpha'' r^2 - 4\alpha' r\right\}. \tag{17}$$

4. The external solution

In order to determine the external gravitational field of a charged fluid sphere in a cosmological background, we notice that the electrostatic field due to the charge of the spherical body is not confined to the sphere only but it spreads through all space given by the non-vanishing component $F_{41} = -F_{14} = (e/r^2)$ of the electromagnetic field tensor and consequently the electromagnetic energy tensor is also not confined to the sphere only. The gravitational field and the associated perfect fluid distribution external to the charged fluid sphere is determined by substituting in eqs. (4) – (6), $e^{\nu} = [1 - (2m/r) + (e^2/r^2)]$ with the non-vanishing components of the electromagnetic energy tensor E_i^j given by $E_1^1 = -E_2^2 = -E_3^3 = E_4^4 = (1/2)(e^2/r^4)$, as in the case of the Nordstrom solution. Eqs. (4) and (5) now yield a differential equation

$$f'\left\{\left(m/2r^2\right) - (1/2r)\right\} + \left\{\left(2e^2/r^4\right) - \left(1/r^2\right)\right\}\left(1 - e^f\right) = 0. \tag{18}$$

Integrating this equation we get

$$e^{-f} = \left\{1 - K(r-m)^2\left(1 - (m/r)\right)^{-(4e^2/m^2)}\exp\left(-4e^2/mr\right)\right\}, \tag{19}$$

where K is a constant of integration. The metric (2) external to the sphere of charged perfect fluid may now be written as

$$ds^2 = -\left[\left(1 - (2m/r) + (e^2/r^2)\right)\left\{1 - K(r-m)^2\left(1 - (m/r)\right)^{-(4e^2/m^2)}\right.\right.$$
$$\left.\left.\exp\left(-4e^2/mr\right)\right\}\right]^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$
$$+ \left\{1 - (2m/r) + (e^2/r^2)\right\}dt^2. \tag{20}$$

If the charge e of the perfect fluid sphere is put equal to zero, the metric (20) reduces to the line-element obtained by Leibovitz [4] as a model of the perfect fluid distribution representing a point mass in a Einstein universe. For the constant $K = 0$, the metric (20) takes the form of the Nordstrom solution describing the external gravitational field of a charged fluid sphere in an otherwise empty universe. The metric (20) therefore, describes the external gravitational field of a charged fluid sphere embedded in a Einstein universe. It may be noted that the line-element (20) cannot be reduced to the special static case of the solutions obtained by Vaidya and Shah [5,6] for the gravitational field of a charged particle embedded in an expanding and in a homogeneous universe. Retaining the cosmological term, the distribution outside the sphere of the charged perfect fluid is given by

$$p = (K/2)\,(r-m)^2\,\left(1 - (m/r)\right)^{-(4e^2/m^2)}\,\exp\left(-4e^2/mr\right) \times \left\{\left(e^2/r^4\right) - \left(1/r^2\right)\right\} + (\Lambda/2), \tag{21}$$

and

$$\rho = (K/2)\,(r-m)\,\left(1 - (m/r)\right)^{-(4e^2/m^2)}\,\exp\left(-4e^2/mr\right)\left[\left((3r-5m)/r^2\right) + \left(e^2/r^5\right)\left\{(3m-r)3r - 4e^2\right\}\right] - (\Lambda/2). \tag{22}$$

5. Boundary conditions and the case of an electron

At the boundary $r = a$ of the charged fluid sphere, we require the continuity of the metric potentials of the two solutions given by eqs. (14) and (20), the continuity of the F_{14} and the continuity of the pressures of the two distributions inside and outside the charged fluid sphere given by eqs. (15) and (21). These continuity equations will determine in any specific model the radius a of the charged fluid sphere for its given mass m and charge e and the arbitrary constants appearing in the solution (14) for the charged fluid sphere. It may be mentioned that such general relativistic models of charged perfect fluid spheres embedded in a cosmological background have no Newtonian analogue.

We now assume a charged perfect fluid sphere as an approximate classical model for an electron. Considering an electron as a charged fluid in an otherwise empty universe, Bonnor and Cooperstock [1] have concluded that an electron must contain some negative rest mass density. Instead of considering an electron in an otherwise empty universe, we consider it in a cosmological background of a simple Einstein universe given by the metric (20). Neglecting the cosmological term and substituting for E_4^4 from eq. (10), eq. (6) can be written as

$$(d/dr)\left(re^{\nu-f}\right) = 1 - r^2\left(2\rho + (\alpha')^2 e^{-f}\right). \tag{23}$$

This gives

$$\left\{\left(e^{\nu-f}\right)_{r=a}\right\}_{\text{internal}} = 1 - a^{-1}\int^{a}\left(2\rho + (\alpha')^2 e^{-f}\right) r^2 dr, \tag{24}$$

which is the value of the metric potential g^{11} of the internal solution at the boundary $r = a$ of an electron. The metric potential g^{11} for the external solution at the boundary $r = a$ of an electron is given by

$$\left\{(e^{\nu-f})_{r=a}\right\}_{\text{external}} = \left[1 - (2m/a) + \left(e^2/a^2\right)\right]\left\{1 - K(a-m)^2\right.$$

$$\left.\left(1 - (m/a)\right)^{-(4e^2/m^2)} \exp\left(-4e^2/ma\right)\right\}. \tag{25}$$

The continuity of the metric potentials g^{11} of the external and internal solutions at the boundary $r = a$ of an electron, gives

$$\left[1 - (2m/a) + \left(e^2/a^2\right)\right]\left\{1 - K(a-m)^2\left(1 - (m/a)\right)^{-(4e^2/m^2)} \exp\left(-4e^2/ma\right)\right\}$$

$$= 1 - a^{-1}\int_0^a \left(2\rho + (\alpha')^2 e^{-f}\right) r^2 dr. \tag{26}$$

In this case, we find that for a, m and e given for an electron, the arbitrary constant K in the left hand side can always be suitably chosen in such a way that the rest mass density ρ need not be negative anywhere within the electron.

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