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Quantum mechanics of a class of noncentral exponential potentials in two dimensions

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Abstract : We make use of an ansatz for the eigenfunction to obtain an exact analytic solution of the Schrödinger wave equation for a class of noncentral (NC) exponential potentials in two-dimensions on the lines described earlier [*Ann Phys*. **206** 90 (1991)] Several interesting special cases of the derived NC exponential potential of very general nature, are investigated In particular, a Morse-class of NC potentials is obtained

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1. Introduction

Theoretical understanding of several newly discovered phenomena [1] in physics and chemistry now requires a study of noncentral (NC) and anharmonic potentials in both classical and quantum mechanics. Some attempts [2] have already been made in this direction to obtain an exact solution to the Schrödinger wave equation for a certain type of NC potentials. In spite of the fact that the Schrödinger equation for all NC anharmonic potentials remains linear (unlike the corresponding classical equation of motion), a simple analysis has shown [3] that it does not admit the solution for all such systems.

Earlier, using a simple method [4], we have studied [3] the solvability of the Schrödinger equation for a variety of central and NC potentials in two dimensions. For the NC potentials of the type $V(x,y) = \sum_{i,j=0}^{N} b_{ij} x^i y^j (i + j \leq N$ and i, j are not zero simultaneously) with $N = 2$ and 4, we have found that a normalizable solution to the wave

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equation with nonzero eigenvalues is not possible* unless some inverse harmonic terms $(b_1/x^2 + b_2/y^2)$ and/or cross terms of the type $(b_3\ x/y + b_4\ y/x)$,..... are added to $V(x, y)$. Also, for central potential of exponential-type (namely Morse potential [5]) an exact solution to the wave equation is obtained without demanding any additional constraint on the potential parameters, which normally is the case with other potentials.

As far as the study of exponential potentials in quantum mechanics is concerned not many cases are found to be of physical interest. Again, among those which are of physical interest, an exact solution of the wave equation has not been possible for all. More often, the potentials studied are either one-dimensional or three-dimensional with radial symmetry. The Morse potential, in its central form, studied in a variety of problems [6] in physics and chemistry, is sometimes used for testing [7] the elegance of the underlying mathematical technique. On the other hand, if the solution of the wave equation with Morse (or Morse-type) potential in its NC form becomes available, it will naturally add to the domain of applicability of this important potential.

In the present work, we use the eigenfunction-ansatz-method to study a class of NC exponential potentials in two-dimensions. In particular, a NC exponential potential of very general form which admits the solution of the wave equation, is derived in the next section. In Section 3, we discuss some special cases of this generalized form. The Morse-class of NC potentials obtained in Section 3, is studied in detail in Section 4. Finally, the results are discussed and summarized in Section 5. In the Appendix we investigate the classical integrability (in the sense of Whittaker [8]) of the derived NC exponential potential by way of constructing the second invariant for this system.

2. General form of the noncentral exponential potential

We consider the solution to the Schrödinger wave equation

$$\phi_{xx} + \phi_{yy} + [\lambda - v(x,y)]\,\phi(x,y) = 0, \tag{1}$$

where $\lambda = 2\mu E/\hbar^2$, $v(x,y) = 2\,\mu V(x,y)/\hbar$. Here, we slightly depart from our standard method followed earlier [3] in the sense that instead of starting with a known form of the potential in advance, we shall determine the potential itself that can provide a solution to the wave eq. (1). For the eigenfunction $\phi(x,y)$, we make an ansatz [3]

$$\phi(x, y) = \exp(g(x, y)), \tag{2}$$

*It may be mentioned that an inadvertent error has crept in, in Ref. (3). In fact, there in all those NC power potentials containing the terms either with odd powers of x or of y or of both, it should occur $|x|$ and $|y|$ (in place of x and y) and with the same odd powers. This will, however, not affect the results and conclusions of earlier work except for confirming the normalizability of the corresponding eigenfunction. The author wishes to thank Dr. A V Turbiner for bringing this error to his notice.

where $g(x, y)$ is now set in the form

$$g(x,y) = \beta_1 x + \beta_2 y + \beta_3 \cdot \exp(\alpha_1 x) + \beta_4 \cdot \exp(\alpha_2 y) + \beta_5 \cdot \exp(\alpha_3 x + \alpha_4 y), \tag{3}$$

to give

$$\begin{aligned}\phi_{xx} + \phi_{yy} = \Big[&(\beta_1^2 + \beta_2^2) + \beta_3^2 \alpha_1^2 \cdot \exp(2\alpha_1 x) + \beta_4^2 \alpha_2^2 \cdot \exp(2\alpha_2 y) \\ &+ \beta_3 \alpha_1 (2\beta_1 + \alpha_1) \cdot \exp(\alpha_1 x) + \beta_4 \alpha_2 (2\beta_2 + \alpha_2) \cdot \exp(\alpha_2 y) \\ &+ \beta_5^2 (\alpha_3^2 + \alpha_4^2) \cdot \exp(2(\alpha_3 x + \alpha_4 y)) \\ &+ \beta_5 (2\beta_1 \alpha_3 + 2\beta_2 \alpha_4 + \alpha_3^2 + \alpha_4^2) \cdot \exp(\alpha_3 x + \alpha_4 y) \\ &+ 2\beta_3 \beta_5 \alpha_1 \alpha_3 \cdot \exp\{(\alpha_1 + \alpha_3) x + \alpha_4 y\} \\ &+ 2\beta_4 \beta_5 \alpha_2 \alpha_4 \cdot \exp\{\alpha_3 x + (\alpha_2 + \alpha_4) y\}\Big] \; \phi(x,y).\end{aligned} \tag{4}$$

A comparison of eq. (4) with eq. (1) yields an expression

$$\lambda = -(\beta_1^2 + \beta_2^2), \tag{5}$$

for the eigenvalues, and an expression for the potential :

$$\begin{aligned}v(x,y) = &\beta_3^2 \alpha_1^2 \cdot \exp(2\alpha_1 x) + \beta_4^2 \alpha_2^2 \cdot \exp(2\alpha_2 y) \\ &+ \beta_3 \alpha_1 (2\beta_1 + \alpha_1) \cdot \exp(\alpha_1 x) + \beta_4 \alpha_2 (2\beta_2 + \alpha_2) \cdot \exp(\alpha_2 y) \\ &+ \beta_5^2 (\alpha_3^2 + \alpha_4^2) \cdot \exp(2(\alpha_3 x + \alpha_4 y)) \\ &+ \beta_5 (2\beta_1 \alpha_3 + 2\beta_2 \alpha_4 + \alpha_3^2 + \alpha_4^2) \cdot \exp(\alpha_3 x + \alpha_4 y) \\ &+ 2\beta_3 \beta_5 \alpha_1 \alpha_3 \cdot \exp[(\alpha_1 + \alpha_3) x + \alpha_4 y], \\ &+ 2\beta_4 \beta_5 \alpha_2 \alpha_4 \cdot \exp[\alpha_3 x + (\alpha_2 + \alpha_4) y],\end{aligned} \tag{6}$$

which admits the solution to eq. (1). As far as the normalization of the eigenfunction $\phi(x,y)$ is concerned it can be carried out from

$$\int \int |\phi(x,y)|^2 \, dx.dy = 1, \tag{7}$$

by setting β_i's in (3) in such a way that the integral in (7) remains a proper integral. In the next section, we discuss some interesting special cases of potential (6).

3. Some special cases

Here, we discuss two classes of potentials as special cases of (6). One corresponds to the choice when some of the β_i's and/or α_i's become zero (cases (1), (2) and (4) below) and other corresponds to the situation when some of the β_i's and α_i's are mutually related (case (3) below).

Case (1) :

When either $\alpha_3 = \alpha_4 = 0$ or $\beta_5 = 0$, the potential (6) takes the form

$$v(x,y) = \beta_3\alpha_1\left[\beta_3\alpha_1\cdot\exp(\alpha_1 x) + 2\beta_1 + \alpha_1\right]\cdot\exp(\alpha_1 x)$$
$$+ \beta_4\alpha_2\left[\beta_4\alpha_2\cdot\exp(\alpha_2 y) + 2\beta_2 + \alpha_2\right]\cdot\exp(\alpha_2 y), \tag{8}$$

for which the eigenvalue λ is given by (5) and the eigenfunction becomes

$$\phi(x,y) = N.\exp\left[\beta_1 x + \beta_2 y + \beta_3\cdot\exp(\alpha_1 x) + \beta_4\cdot\exp(\alpha_2 y)\right],$$

where the normalization constant, N, can be determined from (7).

Case (2) :

When either $\alpha_1 = \alpha_2 = 0$ or $\beta_3 = \beta_4 = 0$, the potential (6) becomes

$$v(x,y) = \beta_5^2\left(\alpha_3^2 + \alpha_4^2\right)\cdot\exp\left\{2(\alpha_3 x + \alpha_4 y)\right\}$$
$$+ \beta_5\left(2\beta_1\alpha_3 + 2\beta_2\alpha_4 + \alpha_3^2 + \alpha_4^2\right)\cdot\exp(\alpha_3 x + \alpha_4 y), \tag{9}$$

for which the eigenvalue λ is again given by (5) and the eigenfunction now takes the form

$$\phi(x,y) = N\cdot\exp\left[\beta_1 x + \beta_2 y + \beta_5\cdot\exp(\alpha_3 x + \alpha_4 y)\right].$$

Case (3) :

When $\beta_1 = -\alpha_1/2$, $\beta_2 = -\alpha_2/2$; $\alpha_1 = \alpha_3$, $\alpha_2 = \alpha_4$, the potential (6) reduces to the form

$$v(x,y) = \alpha_1^2\left[\beta_3\cdot\exp(\alpha_1 x) + \beta_5\cdot\exp(\alpha_1 x + \alpha_2 y)\right]^2$$
$$+ \alpha_2^2\left[\beta_4\cdot\exp(\alpha_2 y) + \beta_5\cdot\exp(\alpha_1 x + \alpha_2 y)\right]^2, \tag{10}$$

and the corresponding eigenvalue and eigenfunction are given by

$$\lambda = -\left(\alpha_1^2 + \alpha_2^2\right)/4, \tag{10a}$$

$$\phi(x,y) = N\cdot\exp\big[-(1/2)\alpha_1 x - (1/2)\alpha_2 y + \beta_3\cdot\exp(\alpha_1 x) + \beta_4\cdot\exp(\alpha_2 y) + \beta_5\cdot\exp(\alpha_1 x + \alpha_2 y)\big]. \tag{10b}$$

However, for the choice $\alpha_1 = -\alpha_3$, $\alpha_2 = -\alpha_4$, potential (6) takes the form

$$v(x,y) = \alpha_1^2\left[\beta_3\cdot\exp(\alpha_1 x) - \beta_5\cdot\exp(-\alpha_1 x - \alpha_2 y)\right]^2 + \alpha_2^2\left[\beta_4\cdot\exp(\alpha_2 y) - \beta_5\cdot\exp(-\alpha_1 x - \alpha_2 y)\right]^2 + 2\beta_5\left(\alpha_1^2 + \alpha_2^2\right)\cdot\exp(-\alpha_1 x - \alpha_2 y)\,. \tag{11}$$

While the eigenvalue λ is again given by (10a), $\phi(x,y)$ can be obtained from (2) as before. Similarly, the potentials along with corresponding eigenvalues and eigenfunctions can be derived for several other choices of β_1, β_2, β_3 and α_i's. However, these cases are not of much physical interest in the present context.

Case (4) :

When $\beta_1 = \beta_2 = 0$, it can be seen from (5) that the eigenvalue λ turns out to be zero for the potential,

$$v(x,y) = \beta_3\alpha_1^2\exp(\alpha_1 x)\cdot\left[\beta_3\cdot\exp(\alpha_1 x) + 1\right] + \beta_4\alpha_2^2\exp(\alpha_2 y) \times\left[\beta_4\exp(\alpha_2 y) + 1\right] + \beta_5\left(\alpha_3^2 + \alpha_4^2\right)\exp(\alpha_3 x + \alpha_4 y) \times\left[\beta_5\exp(\alpha_3 x + \alpha_4 y) + 1\right], \tag{12}$$

with the eigenfunction

$$\phi(x,y) = N\exp\left[\beta_3\exp(\alpha_1 x) + \beta_4\exp(\alpha_2 y) + \beta_5\exp(\alpha_3 x + \alpha_4 y)\right], \tag{13}$$

representing a zero-energy solution to eq. (1).

Although the basic structure of potentials discussed above (cf. cases (1) – (4)) is fixed by way of obtaining them as special cases of potential (6), yet their generalized character can be noticed in terms of the remaining parameters. While cases (1) and (2) will be analyzed in detail in the next section, it is interesting to note that the potentials obtained in case (3) inspite of having a bound and normalizable state, do not possess a local minimum in the finite xy-domain. This perhaps could be a case of the bound states in the continuum [9]; or else these rather unusual bound states may correspond to some metastable state in the two-dimensional potential which dissociate immediately through the phenomenon of tunnelling [10] along one of the dimensions. Inspite of the fact that Toda potential [11] as such could not be accommodated in the structure (6), three-term Toda-type potentials (cf. case (3)) which admit the solution to eq. (1), can easily be derived.

4. Morse-class of potentials in two-dimensions

From the point of view of understanding much more complex crystalline systems, the studies of one- and two-dimensional models are of substantial pedagogical value. In this context, while one dimensional models would demand much more idealistic situation, the two-dimensional models could indeed be of somewhat more practical use. Inspite of its complicated form, the Morse potential [5] has been in use for a long time not only in explaining the molecular spectra [6] and deuteron problem [12] but also in describing some of the crystalline substances. A Morse pair-wise potential has been used [13] to describe the properties of an infinite array of atoms. It may mentioned that in most of the applications the Morse potential with radial symmetry has been used mainly because of the difficulties in dealing with the noncentral Morse function. The cases (1) and (2), discussed in Section 3, clearly offer examples of Morse-type potentials in two dimensions.

If we define, $X = \exp(\alpha_1 x)$, $Y = \exp(\alpha_2 y)$, then it can be seen that the potential (8) has a minimum at

$$X \equiv X_0 = -(2\beta_1 + \alpha_1)/(2\beta_3\alpha_1); \; Y \equiv Y_0 = -(2\beta_2 + \alpha_2)/(2\beta_4\alpha_2) \quad (14)$$

with the minimum value of $v(x, y)$ as

$$v(x_0, y_0) = -(1/4)\left[(2\beta_1 + \alpha_1)^2 + (2\beta_2 + \alpha_2)^2\right]. \quad (15)$$

On the other hand, it can be noticed that in order to have the minimum point of (8) in the finite xy-domain X_0 and Y_0 in (14) should be positive definite. As a result either β_3 should be negative for positive β_1 and α_1, or else if $\alpha_1 < 0$, then β_3 should be positive such that $2\beta_1 > |\alpha_1|$. Also, either β_4 should be negative for positive β_2 and α_2, or else if $\alpha_2 < 0$, then β_4 should be positive such that $2\beta_2 > |\alpha_2|$.

A similar analysis can be carried out for the case (2) (cf. potential (9)). In this case, however, the possibilities of extremum point exist only with respect to the product XY (note that, here $X = \exp(\alpha_3 x)$, $Y = \exp(\alpha_4 y)$ at the point characterized by

$$XY \equiv X_0 Y_0 = -\frac{\left(2\beta_1\alpha_3 + 2\beta_2\alpha_4 + \alpha_3^2 + \alpha_4^2\right)}{2\beta_5\left(\alpha_3^2 + \alpha_4^2\right)}, \quad (16)$$

with the extremum value of $v(x,y)$ as

$$v(X_0 Y_0) = -\frac{1}{4}\frac{\left(2\beta_1\alpha_3 + 2\beta_2\alpha_4 + \alpha_3^2 + \alpha_4^2\right)^2}{\left(\alpha_3^2 + \alpha_4^2\right)}. \quad (17)$$

Again note that here for the positive definite value of the product X_0Y_0 in (16), β_5 should be negative.

Now, by defining β_i's as $\beta_3 \equiv -b_3 = -\exp(-\alpha_1 x_0)$; $\beta_4 \equiv -b_4 = -\exp(-\alpha_2 y_0)$; $\beta_5 \equiv -b_5 = -\exp(-\alpha_3 x_0 - \alpha_4 y_0)$, it is not difficult to express the potentials (8) and (9) in the forms,

$$v(x,y) = \alpha_1^2 \exp\{2\alpha_1(x-x_o)\} - \alpha_1(2\beta_1+\alpha_1)\exp\{\alpha_1(x-x_0)\}$$
$$+ \alpha_2^2 \exp\{2\alpha_2(y-y_0)\} - \alpha_2(2\beta_2+\alpha_2)\exp\{\alpha_2(y-y_0)\} \quad (18)$$

and
$$v(x,y) = \left(\alpha_3^2 + \alpha_4^2\right)\exp\{2\alpha_3(x-x_0) + 2\alpha_4(y-y_0)\}$$
$$-\left(2\beta_1\alpha_3 + 2\beta_2\alpha_4 + \alpha_3^2 + \alpha_4^2\right)\exp\{\alpha_3(x-x_0)+\alpha_4(y-y_0)\}, \quad (19)$$

respectively. Further, for the case when $\beta_1 = \alpha_1/2$, $\beta_2 = \alpha_2/2$, the form (18) reduces to

$$v_1(x,y) = \alpha_1^2\left[\exp\{2\alpha_1(x-x_0)\} - 2\exp\{\alpha_1(x-x_0)\}\right]$$
$$+ \alpha_2^2\left[\exp\{2\alpha_2(y-y_0)\} - 2\exp\{\alpha_2(y-y_0)\}\right], \quad (20)$$

with the minimum value $v(x_0,y_0) = -\left(\alpha_1^2 + \alpha_2^2\right)$, at the point (x_0,y_0). Similarly, for $\beta_1 = \alpha_3/2$, $\beta_2 = \alpha_4/2$, the potential (19) takes the form

$$v_2(x,y) = \left(\alpha_3^2 + \alpha_4^2\right)\left[\exp\{2\alpha_3(x-x_0) + 2\alpha_4(y-y_0)\}\right.$$
$$\left.-2\exp\{\alpha_3(x-x_0) + \alpha_4(y-y_0)\}\right], \quad (21)$$

with the minimum value $v(x_0,y_0) = -\left(\alpha_3^2 + \alpha_4^2\right)$, at the point characterized by the product $X_0Y_0 = \exp(\alpha_3 x_0 + \alpha_4 y_0)$. The eigenvalue and the eigenfunction corresponding to the potentials (20) and (21) (labelled as 1 and 2) now become

$$\lambda_1 = -(1/4)\left(\alpha_1^2 + \alpha_2^2\right), \quad (22a)$$

$$\psi_1 = N_1 \exp\left[(1/2)\alpha_1 x + (1/2)\alpha_2 y - \exp\{\alpha_1(x-x_0)\}\right.$$
$$\left.- \exp\{\alpha_2(y-y_0)\}\right] \quad (22b)$$

and
$$\lambda_2 = -(1/4)\left(\alpha_3^2 + \alpha_4^2\right), \quad (23a)$$

$$\psi_2 = N_2 \exp\Big[(1/2)\,\alpha_3 x + (1/2)\,\alpha_4 y - \exp\{\alpha_3(x-x_0)\}$$

$$- \exp\{\alpha_4(y-y_0)\}\Big] \qquad (23b)$$

respectively.

It can be seen that the forms (20) and (21) are more akin to the standard Morse potential. For a highly simplified case when

$$\alpha_1 = \alpha_2 = 1,\ x_0 = y_0 = 1 \text{ and } \alpha_3 = \alpha_4 = 1, \qquad (24)$$

the plots of the potentials (20) and (21) are shown in Figures 1 and 2 and the behaviour of the corresponding eigenfunctions is depicted in Figures 3 and 4, respectively. Normalizations of

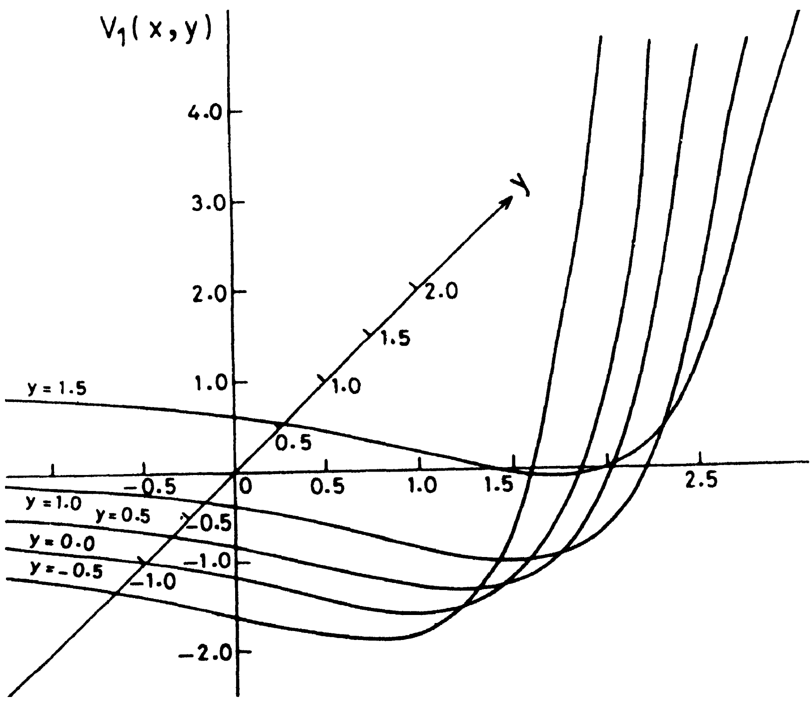

Figure 1. Two-dimensional Morse potential (20) for some typical values of the parameters given in eq. (24)

the eigenfunctions (22b) and (23b) can be carried out [14] using (7). For example, for (22b) N_1 turns out to be

$$N_1 = \left[\alpha_1\sigma_1\,\alpha_2\sigma_2\right]^{1/2},$$

where $\sigma_1 = 2\exp(-\alpha_1 x_0)$, $\sigma_2 = 2\exp(-\alpha_2 y_0)$. However, for the case (24) N_1 reduces to a very simple form $N_1 = 2\,e^{-1}$ with $e = 2.7182$.

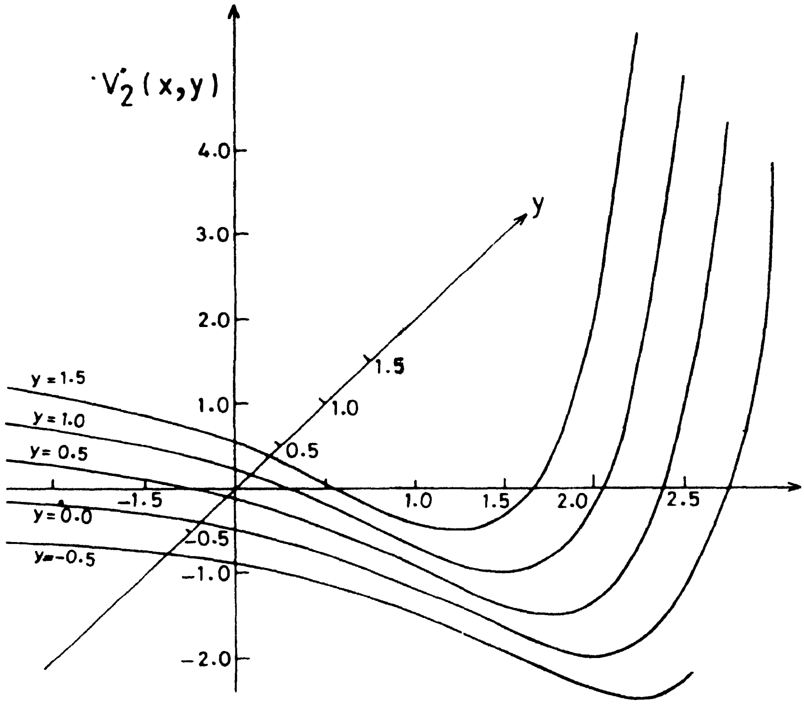

Figure 2. Two-dimensional Morse potential (21) for some typical values of the parameters given in eq. (24).

5. Discussion and summary

Using a simple-minded ansatz for the solution of the Schrödinger wave equation a very general form of the NC exponential potential (cf. eq. (6)) in two dimensions is derived. Interestingly, two wellknown classes of exponential potentials (namely, Morse- and Toda-type potentials) turn out to be special cases of this general form (6). While the Morse-class of potentials is found to admit an ideal quantum-bound-state problem, the Toda-class somehow does not do that. Further, two explicit forms of the Morse potential in two dimensions are investigated in detail.

Markworth [13] studied the properties of an infinite linear array of regularly-spaced atoms using a Morse pair-wise interaction between the nearest neighbours. From the computed equilibrium value of the lattice parameter (a_0) for b.c.c. iron, the information is obtained [15] regarding the stability of the one-dimensional crystal. In an analogous manner the stability of two-dimensional monatomic or one-dimensional diatomic crystals can be studied using the two-dimensional Morse potentials derived in the present work.

Regarding the classical integrability of potential (6) it may be mentioned that only a restricted class of this potential *i.e.* the form (9), admits the second order (in momenta) invariant (cf. Appendix) which can be expressed as

$$I = (1/2)\,\dot{\xi}^2 + \left(\alpha_3^2 + \alpha_4^2\right) v(\xi),$$

where $\xi = \alpha_3 x + \alpha_4 y$, is some preferred direction in the xy-plane. Thus, I, while expressible in the Hamiltonian form, is structurally different from the Hamiltonian itself. Following the method of Holt [16] we have also checked that there does not exist a third order invariant for the system (6). In fact, potential (9) offers one more example of a system which is classically integrable as well as quantum solvable [17].

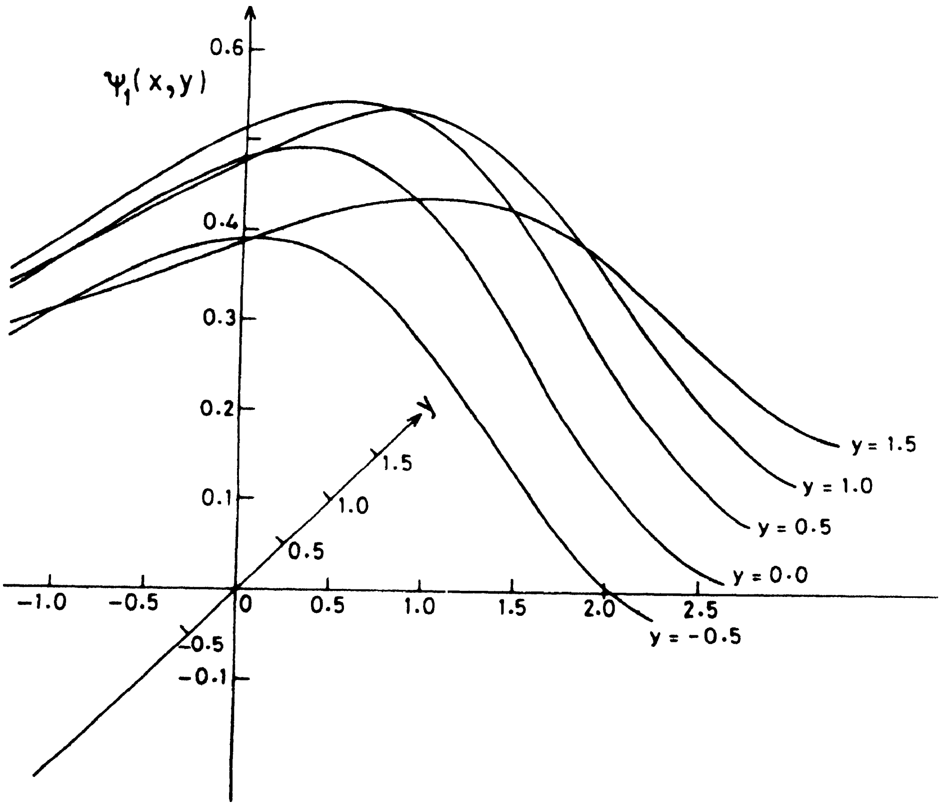

Figure 3. Behaviour of the eigenfunction (not normalized) corresponding to the potential shown in Figure 1

It may be mentioned that Toda-type potentials derived here (cf. case (3) of Section 3) are of somewhat different nature from that of the conventional one [11] in the sense that they are not classically integrable and they exhibit somewhat peculiar behaviour as far as the quantum-bound-state problem is concerned (cf. Section 3). Whereas the standard periodic Toda potential is an integrable system and exhibits [18] several salient features at the quantum level. While the present Toda-type potentials require further investigations, there exists* another Toda-class of potentials which, in fact is classically integrable and admits second order invariants.

*see, Kaushal and Mishra in Ref. [20].

In the present work, while we have restricted to the simple ansatz (2) for the eigenfunction, a much deeper study of the NC exponential potentials is possible by considering (see Taylor and Leach in Ref. [2]) the form $\phi(x,y) = f(x,y)\exp(g(x,y))$, where $f(x,y)$ is a polynomial. However, the exercise as a whole, in this case, turns out to be very complicated particularly for the exponential potentials as compared to their utility. Moreover, we have obtained here only one eigenstate and that too only for some permissible potentials. Even these results can offer a check on the efficiency of numerical algorithms (such as finite difference or perturbation expansions) and provide a complete set of eigenstates for these potentials.

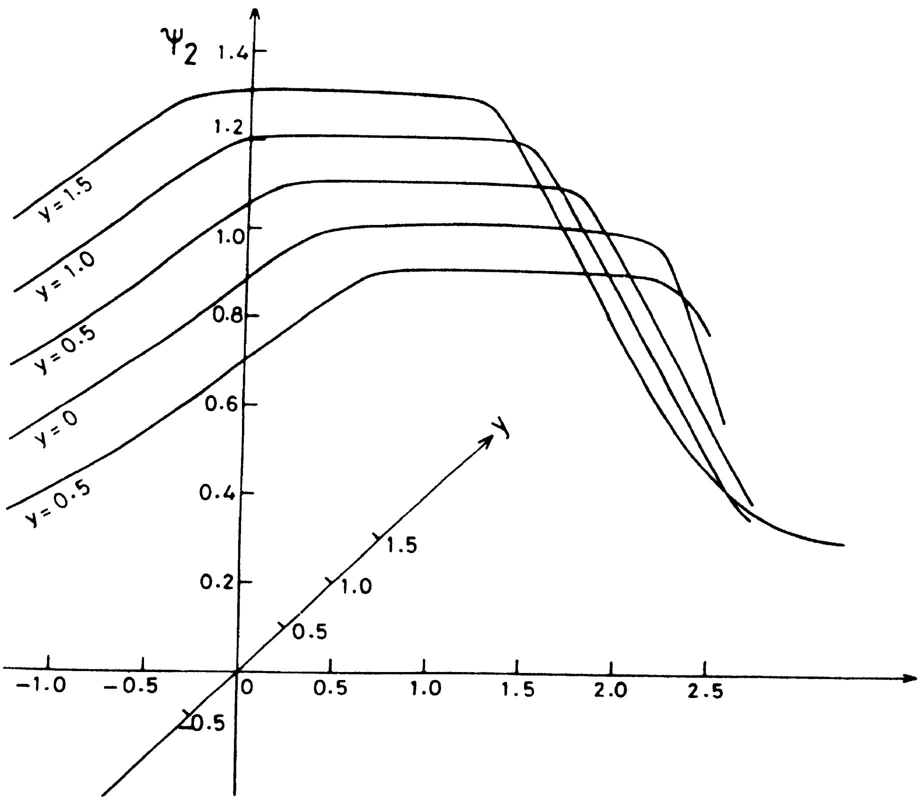

Figure 4. Behaviour of the eigenfunction (not normalized) corresponding to the potential shown in Figure 2.

To summarize, we mention that the classical and quantum mechanics of a class of NC exponential potentials in two dimensions is studied. We have not only obtained an exact normalizable solution to the Schrödinger wave equation for these potentials but also established the classical integrability by way of constructing the second invariant. Quantum solutions are exact in the sense that there are no restrictions on the parameters of the derived potential unlike the cases studied [3] earlier. While some of the cases discussed here could be useful in molecular chemistry and solid state physics, the methodology can be applied in solving exactly the problem [19] of planar and diffused channel waveguides.

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Appendix

Here we investigate the classical integrability of the system (6) by way of constructing the second invariant for it. Let this system admits an invariant of the form

$$I = a_0 + (1/2)\, a_{11}\, \dot{x}^2 + a_{12}\, \dot{x}\, \dot{y} + (1/2)\, a_{22}\, \dot{y}^2, \tag{A1}$$

where a_0 and a_{ij} are the functions of x and y. The potential $v(x,y)$ must satisfy a 'potential' equation [11,16,20]

$$(3/2)\,(c_1 y + c_2)\,(\partial v/\partial x) \; - \; (3/2)\,(c_1 x + c_4)\,(\partial v/\partial y)$$

$$-\;(1/2)\,(c_1 xy \;+\; c_2 x \;+\; c_4 y \;-\; 2c_6)\left(\left(\partial^2 v/\partial x^2\right) - \left(\partial^2 v/\partial x^2\right)\right)$$

$$+\left[(1/2)\left(c_1(y^2 - x^2) \;+\; c_2 y \;-\; c_4 x \;+\; c_3 \;-\; c_5\right)\right]\left(\partial^2 v/\partial x \partial y\right) \;=\; 0, \qquad \text{(A2)}$$

where the coefficient functions a_0 and a_{ij} are expressed in terms of the arbitrary constants c_i's. This equation can be derived readily from the set of six equations obtained from

$$dI/dt \;=\; [I, H]_{\text{PB}} \;=\; 0, \qquad \text{(A3)}$$

using $\ddot{x} \;=\; -\partial v/\partial x,\; \ddot{y} \;=\; -\partial v/\partial y$. Here, H is the Hamiltonian. We use the form (6) in eq. (A2) and as a result of the rationalization of the latter equation one obtains

$$\alpha_1 = \alpha_2 = 0, \;\text{ and also } c_1 = c_2 = c_4 = 0. \qquad \text{(A4)}$$

Subsequently, one also obtains the relation

$$c_6\left(\alpha_4^2 \;-\; \alpha_3^2\right) \;+\; (c_3 \;-\; c_5)\,\alpha_3\alpha_4 \;=\; 0, \qquad \text{(A5)}$$

which fixes the values of the remaining arbitrary constants as

$$c_5 \;=\; \alpha_4^2;\; c_3 \;=\; \alpha_3^2, \;\text{ and } c_6 \;=\; \alpha_3\alpha_4. \qquad \text{(A6)}$$

Thus, the coefficient functions a_{ij}'s in (A1) turn out to be

$$a_{11} \;=\; c_3 \;=\; \alpha_3^2;\; a_{22} \;=\; c_5 \;=\; \alpha_4^2;\; a_{12} \;=\; c_6 \;=\; \alpha_3\alpha_4. \qquad \text{(A7)}$$

A unique expression for a_0 in (A1) can be obtained from the integration of the equations [20]

$$\partial a_0/\partial x \;=\; a_{11}(\partial v/\partial x) \;+\; a_{12}(\partial v/\partial y);\; \partial a_0/\partial y \;=\; a_{12}(\partial v/\partial x) \;+\; a_{22}(\partial v/\partial y)$$

in the form as

$$a_0 \;=\; \beta_5\left(\alpha_3^2 \;+\; \alpha_4^2\right)\left[\beta_5\left(\alpha_3^2 \;+\; \alpha_4^2\right)\cdot\exp\left\{2\left(\alpha_3 x \;+\; \alpha_4 y\right)\right\}\right.$$

$$\left. +\;\left(2\beta_1\alpha_3 \;+\; 2\beta_2\alpha_4 \;+\; \alpha_3^2 \;+\; \alpha_4^2\right)\cdot\exp\left(\alpha_3 x \;+\; \alpha_4 y\right)\right].$$

Finally, for the potential which turns out to be the same as discussed in case (2) above (cf. Section 3, eq. (9)), the invariant (A1) takes the form

$$I = \beta_5\left(\alpha_3^2 + \alpha_4^2\right)\left[\beta_5\left(\alpha_3^2 + \alpha_4^2\right)\cdot\exp\left\{2\left(\alpha_3 x + \alpha_4 y\right)\right\}\right.$$

$$\left. + \left(2\beta_1\alpha_3 + 2\beta_2\alpha_4 + \alpha_3^2 + \alpha_4^2\right)\cdot\exp\left(\alpha_3\dot{x} + \alpha_4\dot{y}\right)\right]$$

$$+ (1/2)\cdot\left(\alpha_3\dot{x} + \alpha_4\dot{y}\right)^2. \tag{A8}$$

As a check it is not difficult to verify that the invariant (A8) is in conformity with (A3) for the potential (9).