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Interior solution for a very slowly rotating star in an isotropic coordinate system

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Abstract : The interior solution in an isotropic coordinate system has been investigated to produce an approximate solution for a very slowly rotating perfect fluid The solution is valid in the case of a very slowly rotating star

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1. Introduction

Almost every object in the sky exhibits some form of rotation. Kerr [1] has derived an axially symmetric solution of the Einstein's vacuum field equations for a rotating star. However, corresponding interior solution for a rotating Kerr metric is a major unsolved problem in general relativity. Hartle [2, and references mentioned therein], studied the equilibrium configurations of a rotating star. In order to study the stability and deformation of the figure resulting from the rotation of neutron stars and other supermassive stars, Hartle restricted himself to slow rotation so as to consider the rotation as a small perturbation on an already known non-rotating configuration. Recently, Rawal *et al* [3] have investigated an approximate Schwarzschild interior solution, for a very slowly rotating star. Out of curiosity, it will be interesting to know, what would be interior solution for a very slowly rotating star of constant angular velocity ω, in isotropic coordinates, if one restricted one self to ω so small that the terms involving ω^2, may be neglected, but the ratio ω^2/ρ, ρ being constant density, has significant value and hence may not be neglected. This consists in obtaining the Wyman's [4] interior solution to Einstein's equations [5]

$$R_{\mu}^{\upsilon} - \frac{1}{2} g_{\mu}^{\upsilon} R = -8\pi T_{\mu}^{\upsilon} \tag{1.1}$$

for a very slowly rotating star.

2. Interior solution for a very slowly rotating star in isotropic coordinates

Schwarzschild solution is obtained in terms of a set of spherical polar coordinates : r, θ, Φ, t. The choice of this particular set of coordinates was motivated by the radial symmetry, time-independence and relative simplicity required for the basic line element having spherical symmetry. However, it is characteristic of general relativity that there are usually many convenient coordinate systems available in which to work and the coordinates r, θ, Φ, t in which the Schwarzschild line element is expressed are not the only coordinates which correspond to our intuitive notions of radial and angular markers. In this paper, we want to consider another convenient set of coordinates and investigate the Schwarzschild line element for slowly rotating star exprcssed in the new coordinates. The main reason for seeking an alternative set of coordinates is that we would like to express ds^2 in a form which is independent of the particular space coordinates used. This sort of line element agrees most closely with our intuitive notion of space which is based mainly on Euclidean geometry. For this reason, such a line element is called a conformal line element. The coordinates which express the line element in such a form are called isotropic coordinates. Isotropic coordinate system has advantage that the coordinate length $\sqrt{dx^2 + dy^2 + dz^2}$ of a small rod which is rigid does not alter when its orientation is altered and is useful when the space is partitioned by rigid scales or by light-triangulations in a small region, for example, in terrestrial measurements, since the ultimate measurements involved in any observation are carried out in a terrestrial laboratory, we ought, strictly speaking always to employ the isotropic system. Which conforms to assumptions made in these measurements. But the terrestrial laboratory is falling freely towards the Sun, and is therefore accelerated relatively to the coordinates (x, y, z, t). But on the Earth the quantity $\frac{m}{r}$ is negligibly small, so that the two systems coalesce with one another and with Euclidean coordinates. Non-Euclidean geometry is required only when m/r is not negligible. As soon as the light-waves have been safely steered into the terrestrial laboratory, the need for non-Euclidean geometry is at an end, and the difference between the isotropic and non-isotropic systems practically disappear. Therefore, isotropic coordinate system is used for the advantage it offers. We consider the metric of the form

$$ds^2 = -e^{\mu}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) + e^{\upsilon}\left(1 - \omega^2 r^2 \sin^2\theta\right) dt^2$$
$$+ 2\omega r^2 \sin^2\theta e^{\mu} d\phi \, dt , \tag{2.1}$$

where $\mu \equiv \mu(r,\theta)$ and $\upsilon \equiv (r,\theta)$, ω is the constant angular velocity of the star, and $G =$ (gravitational constant) $= c$ (velocity of light) $= 1$. We assume that the star is rotating very slowly *i.e.* ω is very small and therefore, terms involving ω^2 can be neglected compared to higher order terms. It can be shown that terms involving ω^2 are smaller by many orders of magnitude in comparison with other higher order terms at the surface of the planets and sun.

As a result, metric (2.1) takes the form :

$$ds^2 = -e^{\mu}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) + e^{\upsilon}dt^2$$
$$+ 2\omega r^2 \sin^2\theta e^{\mu} d\phi\, dt\,. \tag{2.2}$$

Einstein's interior field equations eq. (1.1) are obtained with terms containing $\dot{\mu} = \dfrac{\partial\mu}{\partial\theta}$ and $\dot{\upsilon} = \dfrac{\partial\upsilon}{\partial\theta}$, their products, their squares, and their second order derivatives. As star is rotating very slowly, the star is slightly oblate, as a result $\dot{\mu}$ and $\dot{\upsilon}$ are very small and hence can be neglected. After approximation, field equations take the following form :

$$\frac{\mu'^2}{4} + \frac{\mu'\upsilon'}{2} + \frac{\mu' + \upsilon'}{r} = 8\pi p e^{\mu}\,, \tag{2.3a}$$

$$\frac{\mu'' + \upsilon''}{2} + \frac{\upsilon'^2}{4} + \frac{\mu' + \upsilon'}{2r} = 8\pi p e^{\mu}\,, \tag{2.3b}$$

$$\mu'' + \frac{\mu'^2}{4} + \frac{2\mu'}{r} = -8\pi\rho e^{\mu}\,. \tag{2.3c}$$

Under the assumption made above for $\dot{\mu}$ and $\dot{\upsilon}$, the field eq. (2.3a)–(2.3c) assume the same form as in the case of isotropic static interior solution. The solutions of the above field equations, therefore, have the same form as that in the static isotropic case, but while integrating them, the constants of integration turn out to be functions of θ as well. As a result, one can have an isotropic metric for a very slowly rotating and very slightly oblate star of perfect fluid of constant density ρ rotating with constant angular velocity ω to be

$$ds^2 = -\left[\frac{2R}{e^{c}r^2 + e^{-c}}\right]^2 \left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$
$$+ \left[\frac{Ar^2 + B}{e^{c}r^2 + e^{-c}}\right]^2 dt^2 + 2\omega r^2 \sin^2\theta \left[\frac{2R}{e^{c}r^2 + e^{-c}}\right]^2 d\phi\, dt\,, \tag{2.4}$$

where $A \equiv A(\theta)$, $B \equiv B(\theta)$ and $e^{c} \equiv e^{c}(\theta)$, and $R^2 = \frac{3}{8\pi\rho}$. Now $A(\theta)$, $B(\theta)$, $e^{c}(\theta)$ remain to be determined. Ramsey [6] has given guidelines to find the expression for a Newtonian potential $V(r, \theta)$, for a rotating oblate sphere with

$$V(r,\theta) = \frac{m}{r} + \frac{a^3}{r^3}\left[\frac{\omega^2 a^2}{2} - \frac{\epsilon m}{a}\right]\left(\cos^2\theta - 1/3\right). \tag{2.5}$$

where $\omega^2 = \frac{16\pi\rho\epsilon}{15}$ ρ is constant density of an oblate sphere of equatorial radius a and oblateness ϵ. Using $\rho = \frac{3m}{4\pi a^3}$ the expression for $V(r, \theta)$ in terms of mass m and oblateness ϵ at the equator $r = a$, taking negative sign for potential can be written as,

$$V(r,\theta) = -\frac{m}{a}\left[1 + \frac{\epsilon}{5}\left(1 - 3\cos^2\theta\right)\right]. \tag{2.6}$$

In order to determine $A(\theta)$, $B(\theta)$ and $e^c(\theta)$ we make use of the well known approximation $g_{44} = 1 + 2V(r,\theta)$, where $V(r,\theta)$ is the Newtonian potential of a rotating and slightly oblate sphere at $r = a$, as given by eq. (2.6) and we follow the method of boundary conditions given by Wyman [4] to determine A, B and C. At the equator $r = a$ continuity of e^{ν}, e^{μ} implies

$$Aa^2 + B = \left(e^c a^2 + e^{-c}\right)\beta \tag{2.7}$$

and

$$2R = \left[1 + \frac{m}{2a}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]^2 \left(e^c a^2 + e^{-c}\right), \tag{2.8}$$

where

$$\beta = \frac{\left[1 - \frac{m}{2a}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]}{\left[1 + \frac{m}{2a}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]}.$$

Similarly, pressure $p = 0$ at $r = a$ implies

$$A\left(e^{-2c} - 2a^2\right) + B\left(e^{2c}a^2 - 2\right) = 0. \tag{2.9}$$

Solving (2.7), (2.9) for A, B we obtain

$$A = \beta\left(2 - e^{2c}a^2\right)/\left(e^{-c} - a^2 e^c\right), \tag{2.10}$$

$$B = \beta\left(e^{-2c} - 2a^2\right)/\left(e^{-c} - a^2 e^c\right). \tag{2.11}$$

For positive pressure, $e^{-c} - a^2 e^c > 0$, equation (2.8) can be written in the form

$$e^c a^2 + e^{-c} = 2R/(1 + \alpha)^2, \tag{2.12}$$

where $\alpha = \frac{m}{2a}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}$. Squaring both sides and subtracting $4a^2$ from both sides of (2.12), we get

$$\left(e^{-c} - e^c a^2\right) = 4R^2\left[1 - y(1 + \alpha)^4\right]/(1 + \alpha)^4, \tag{2.13}$$

where $y = a^2/R^2$. Taking positive square root we have,

$$e^{-c} - e^c a^2 = 2R\left[1 - y\,(a + \alpha)^4\right]^{1/2}/(1 + \alpha)^2. \tag{2.14}$$

The continuity of de^{ν}/dr at $r = a$, leads to the equation

$$2a^2\left(Ae^{-c} - Be^{c}\right)/\left(e^{c}a^2 + e^{-c}\right)^2 = \frac{2\alpha}{(1 + \alpha)^2}. \tag{2.15}$$

Using eqs. (2.10), (2.11), (2.12) and (2.14), the above eq. (2.15) will have the form :

$$y/\left[1 - y(1 + \alpha)^4\right]^{1/2} = \frac{4\alpha}{(1 + \alpha)^5(1 - \alpha)}. \tag{2.16}$$

Solving this for y we get

$$y = \frac{4\alpha}{(1 + \alpha)^6}. \tag{2.17}$$

Using eqs. (2.12), (2.14) and (2.17), we find

$$e^{c}(\theta) = \left[\frac{m}{2a^3}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]^{1/2} \tag{2.18}$$

or

$$e^{c}(\theta) = \left[\frac{1}{4R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]^{1/2}. \tag{2.18a}$$

Similarly the values of $A(\theta)$ and $B(\theta)$ are found to be

$$A(\theta\ ) = \left[\frac{1}{4R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]^{1/2}$$

$$\times \frac{\left[4a - \frac{a^3}{2R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]}{\left[2a + \frac{a^3}{2R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - \cos^2\theta\right)\right\}\right]}, \tag{2.19}$$

and

$$B(\theta) = \left[\frac{4R^2}{\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}}\right]^{1/2}$$

$$\times \frac{\left[2a - \frac{a^3}{R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]}{\left[2a + \frac{a^3}{R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]}, \tag{2.20}$$

where $R^2 = \frac{3}{8\pi\rho}$.

As a result, our final solution for gravitationl potentials will have form

$$e^{\mu} = \frac{\left[1 + \frac{a^2}{4R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]^6}{\left[1 + \frac{r^2}{4R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]^2}. \tag{2.21}$$

and

$$e^{\nu} = \frac{\left[1 - \frac{a^2}{4R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]^2}{\left[1 + \frac{r^2}{4R^2}\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\}\right]^2}. \tag{2.22}$$

Further approximation of above potentials take the following form

$$e^{\mu} \cong 1 + \frac{3a^2}{2R^2} - \frac{r^2}{2R^2} \quad 1 + \frac{3a^2}{2R^2} \quad \frac{1}{2R^2}$$

$$\times \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\left| r^2\left(1 + \frac{3a^2}{R^2}\right) - 3a^2 \right. \tag{2.23}$$

and

$$e \cong 1 - \quad 2R^2 \quad 2R^2 \quad 1 - \quad 2R^2 \mid \quad 2R^2$$

$$\times \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right) \; r^2\left(1 - \frac{a^2}{R^2}\right) + a^2 \tag{2.24}$$

Hence, our metric for very slow rotation has the form given by eq. (2.2), with the gravitational potentials e^{μ} and e^{ν} as given by eqs. (2.21) and (2.22) respectively and on approximation these potentials reduce to the form given by eqs. (2.23) and (2.24).

3. Conclusion

Thus, we have obtained zeroeth order approximate solution of Einstein's field equations for a very slowly rotating and slightly oblate star using an isotropic metric. The density of slightly oblate sphere is bounded in mass and size. These bounds in our case are $m\left\{1 + \frac{\varepsilon}{5}\left(1 - 3\cos^2\theta\right)\right\} \leq 0.4a$ and $a^2 \leq 0.27R^2$ respectively.

Our solution has been obtained by a simple method and is valid for 'very small' angular velocity ω, with the condition that the ratio ω^2/ρ is small but still has significant value. For such a case of weak field and low velocity approximation, the solar system is a good example. One can easily see that as the rotation stops ($\omega = 0$, $\varepsilon = o$) the result of our solution reduces to static interior solution for isotropic coordinat.s.

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