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Euclidean and Hermitian Clifford analysis on superspace

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A mis abuelos, padres,
hermana y esposa

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Contents

1 Introduction

In this introductory chapter we describe the mathematical framework for this thesis. To that end, we first provide a brief overview of so-called Euclidean and Hermitian Clifford analysis and the underlying abstract radial algebra. Secondly, we explain the meaning of analysis on superspace, comment on some of the approaches used for this study and discuss the extension of Euclidean Clifford analysis to superspace.

Finally, we list our main goals and provide a detailed overview of the contents of the thesis.

1.1 Euclidean and Hermitian Clifford analysis

Clifford analysis nowadays is a well established mathematical discipline constituting a natural refinement of harmonic analysis. In its most simple setting, it focusses on the null solutions of the Dirac operator $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$, where the elements $(e_1, \ldots, e_m)$ form an orthonormal basis for Euclidean space $\mathbb{R}^m$ and underly the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. This setting is known as the Euclidean case (also the term orthogonal Clifford analysis can be found in the literature). The fundamental group leaving the Dirac operator invariant is $\mathrm{Spin}(m)$ which is a double covering of $\mathrm{SO}(m)$. Standard references on Euclidean Clifford analysis are [16, 44, 48].

By taking the dimension even, say $2m$, and introducing a so-called complex structure $J \in \mathrm{SO}(2m)$, the fundamental elements of Hermitian Clifford analysis arise in a natural way from the Euclidean setting. The Hermitian case focusses on h-monogenic functions, h-monogenicity being expressed by means of two mutually adjoint Dirac operators which are invariant under the realization of the unitary group $\mathrm{U}(m)$ in $\mathrm{Spin}(2m)$. Indeed, the action of the projection operators $\frac{1}{2}(1 \pm iJ)$ on the initial orthonormal basis $(e_1, \ldots, e_{2m})$ leads to the Witt basis elements $(\mathfrak{f}_j : j = 1, \ldots, m)$ and $(\mathfrak{f}_j^\dagger : j = 1, \ldots, m)$, producing a direct sum decomposition of $\mathbb{C}^{2m}$ in two components. The elements of $\mathrm{SO}(2m)$ leaving those subspaces invariant generate a subgroup which is doubly covered by a subgroup of $\mathrm{Spin}(2m)$ denoted $\mathrm{Spin}_J(2m)$, and is isomorphic with the unitary group $\mathrm{U}(m)$. The Hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$ are obtained by projection of the gradient on the aforementioned invariant subspaces, whence they are invariant under the action of $\mathrm{Spin}_J(2m)$.

This Hermitian decomposition has been thoroughly studied in several papers, see for example [9, 14]. Results concerning spherical monogenics, invariant differential operators, a Fischer decomposition and integral representation formulae (Bochner-Martinelli, Cauchy) have already been obtained, see [9, 14, 39, 15, 1]. Furthermore, Hermitian Clifford analysis was addressed in [64] where several complex operators $\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger, \partial_{\underline{u}}, \partial_{\underline{u}}^\dagger, \ldots$ were considered giving rise to new syzygy complexes. Those results motivated for the first time the use of the radial algebra in the Hermitian setting, which is independent of the choice of the dimension parameter.

The radial algebra framework is defined through an algebra $R(S)$ generated by a set S of abstract vector variables $x, y, \ldots$, where classical notions of Clifford analysis are reintroduced axiomatically. For example, Dirac operators are axiomatically defined as endomorphisms on $R(S)$, more precisely as a vector derivative denoted by ∂_x, $x \in S$ (see e.g. [66, 70]). The radial algebra possesses some important properties, of which the most powerful most probably is its independence of any particular dimension m, which is now abstractly defined as a complex parameter stemming from the evaluation $\partial_x[x] = m$. In addition, $R(S)$ is independent of the choice of an underlying vector space V to which the vector variables belong and of any chosen quadratic form on V. This gives rise to important applications of the radial algebra setting in the study of the Fischer decomposition and the Dirac complex (see e.g [21, 65, 64]).

In a number of papers [33, 68, 67] other applications have been considered such as the possibility to derive a theory of Dirac operators in superspace from the standard Euclidean one and to give a meaning to spaces with negative integer dimension.

1.2 Clifford analysis on superspace

Superanalysis or analysis on superspace was introduced by Berezin in order to study mathematical problems associated with the physical idea of supersymmetry, see e.g. [6, 7]. The most important feature of superanalysis was developing a formal calculus in a Grassmann algebra where commuting (bosonic) and anti-commuting (fermionic) variables appear on equal footing. Nowadays, superspace and the corresponding supermanifolds play an important rôle in contemporary theoretical physics, e.g. in the particle theory of supersymmetry, supergravity or superstring theories, etc.

There are different mathematical approaches for the study of superspace. A first approach is based on differential geometry where the concept of a point of a graded manifold is primordial. In this case, the bosonic and fermionic variables are represented as co-ordinates in the even $\Lambda_{\overline{0}}$ and odd $\Lambda_{\overline{1}}$ subspaces of some underlying graded commutative Banach superalgebra Λ. This gives rise to considering superspaces of the form $\mathbb{R}^{p|q}(\Lambda) = \Lambda_{\overline{0}}^{p} \times \Lambda_{\overline{1}}^{q}$ and developing an analysis based on the usual topological notions of neighbourhood, continuity, differentiability, etc. Without claiming completeness, we refer the reader to [72, 55, 45, 63] for an overview of this approach.

A different approach to the mathematical foundation of superanalysis is the one developed by Berezin, Kostant and other authors, see e.g. [7, 57, 60]. This approach comes from modern algebraic geometry, where a supermanifold is defined as a differentiable manifold and a set of superfunctions on it as a structural sheaf. This setting allows to arrive at a calculus on superspace which does not depend on possible representations of the variables as co-ordinates on an underlying superalgebra. This calculus collects important constructions such as the *Berezin integral* with respect to anti-commuting variables, and the so-called *Berezinian* which is the analogue of the Jacobian for the change of anti-commuting variables. The two mentioned approaches to superanalysis have been proven to be equivalent in the categorical sense, see e.g. [5].

As mentioned before, harmonic and Euclidean Clifford analysis have been extended to superspace by means of a representation of the radial algebra. This extension uses the definition of a supermanifold as in the approach of Berezin and Kostant (see [7, 57]). This theory introduces some important differential operators (such as Dirac and Laplace operators) on the flat supermanifold $\mathbb{R}^{m|2n}$, and uses them in the study of special functions, orthogonal polynomials, integration, etc. For a nice overview on this development we refer the reader to the Ph.D. theses [29, 22] and the papers [33, 32, 35, 31, 38, 24, 25, 26, 27, 28, 30, 34, 37, 36].

1.3 Objectives

The above-mentioned extension of Clifford analysis to superspace plays a central rôle in our research, which has three main goals. Our first aim is to extend so-called Hermitian Clifford analysis to superspace (Chapters 2, 3). The second goal is to provide a suitable definition for the (super) spin group and studying the underlying group actions in both Euclidean and Hermitian Clifford analysis in superspace (Chapters 4, 5). Finally, the third objective is to further develop integration theory in this setting by introducing and studying integration over general domains and surfaces in superspace depending on bosonic and fermionic variables on equal footing (Chapters 6, 7). The remainder of this introductory chapter is devoted to a detailed analysis of these objectives and their achievement within the contents of this thesis.

In order to introduce a theory of Hermitian Dirac operators in superspace, we first need to set the correct rules that allow such a canonical extension. In Chapter 2, we deal with this task. We first provide some basics of the classical theory of radial algebras as an abstract approach to Euclidean Clifford analysis. The algebra of endomorphisms and the notion of radial algebra representation will be amply discussed. With the introduction of a so-called complex structure we arrive at the Hermitian radial algebra setting, which constitutes an abstract version of the Hermitian monogenic function theory. An important example of a representation of the radial algebra with a complex structure will be presented at the level of endomorphisms.

In Chapter 3, we formulate the basic definitions needed for Hermitian Clifford analysis in superspace. This is done by studying the corresponding representation of the Hermitian radial algebra. To that end, we first recall the main aspects of the extension of Euclidean Clifford analysis to this setting. In particular, the vector multipliers give rise to a natural way of introducing a complex structure on superspace which immediately leads to the corresponding extensions all basic objects such as Hermitian Dirac operators, complex Euler operators, etc. Moreover, it is proven that all defined objects satisfy the abstract relations provided in Chapter 2 for the Hermitian radial algebra.

Chapter 4 is devoted to providing a definition for the spin group in superspace as a set of elements describing every super-rotation through Clifford multiplication. To that end, we consider linear actions on supervector variables using both commuting and anti-commuting coefficients in a Grassmann algebra. This allows to study the invariance of the inner product in superspace through a classical group theoretic approach which contains all information on the underlying symmetry superalgebras obtained in [22, 23]. We first provide some basics on Grassmann algebras, Grassmann envelopes and supermatrices. Next, we further develop the Clifford setting in superspace by introducing the Lie algebra of superbivectors. An extension of this algebra is crucial in the description of the super spin group. While studying the invariance of the bilinear form that extends the Euclidean inner product to superspace, we obtain the so-called group of superrotations SO_0 whose Lie algebra $\mathfrak{so}_0$ turns out to be a Grassmann envelope of $\mathfrak{osp}(m|2n)$. It is also proven that every super-rotation can be uniquely decomposed as the product of three exponentials acting on some special subspaces of $\mathfrak{so}_0$. Finally, we study the problem of defining the spin group in this setting and its differences with the classical case. It is shown that the compositions of even numbers of vector reflections are not sufficient to fully describe SO_0 since they only allow for an orthogonal structure and do not include the symplectic part of SO_0. Next we propose an alternative, by defining the spin group through the exponential of extended superbivectors and showing that they indeed cover the whole set of superrotations. In addition, we explicitly describe a subset Ξ which is a double covering of SO_0 and contains in particular every fractional Fourier transform.

The main goal of Chapter 5 is to study the action of the spin group in superspace on superfunctions. We first study the invariance of the Dirac operator in superspace under the classical H and L actions. In addition, we consider the Hermitian Clifford setting in superspace, where we study the group invariance of the Hermitian inner product of supervectors introduced in Chapter 3. The group of complex supermatrices leaving this inner product invariant constitutes an extension of $\mathrm{U}(m) \times \mathrm{U}(n)$ and is isomorphic to the subset $\mathrm{SO}_0^{\mathbf{J}}$ of SO_0, consisting of those elements which commute with the complex structure $\mathbf{J}$. The realization of $\mathrm{SO}_0^{\mathbf{J}}$ within the spin group is studied simultaneously with the invariance under its actions of the super Hermitian Dirac system. It is interesting to note that the spin element leading to the complex structure can be expressed in terms of the n-dimensional Fourier transform.

Distributions in superspace constitute a very useful tool for establishing an integration theory. In particular, distributions were used in [24] to obtain a suitable extension of the Cauchy formula to superspace and to define integration over the superball and the supersphere through the Heaviside and Dirac distributions, respectively. In Chapter 6, we extend the distributional approach to integration over more general domains and surfaces in superspace. The notions of domain and surface in superspace are defined by smooth bosonic phase functions g. This allows to define domain integrals and oriented (as well as non-oriented) surface integrals in terms of the Heaviside and Dirac distributions of the superfunction g. It will be shown that the presented definition for the integrals does not depend on the choice of the phase function g defining the corresponding domain or surface. In addition, some examples of integration over a super-paraboloid and a super-hyperboloid will be presented. Finally, a new distributional Cauchy-Pompeiu formula will be obtained, which generalizes and unifies the previously known approaches.

In Chapter 7, we address the problem of establishing a Cauchy integral formula in the framework of Hermitian Clifford analysis in superspace. To this end, we use the general distributional approach to integration provided in Chapter 6. This allows to obtain a successful extension of the classical Bochner-Martinelli formula to superspace by means of the corresponding projections on the space of spinor-valued superfunctions. This is inspired by the close relation between the theory of Hermitian monogenic functions and the theory of holomorphic functions of several complex variables in the purely bosonic case, see [11]. The connection between Hermitian monogenicity and holomorphicity in superspace is established by considering a specific class of spinor-valued superfunctions (Section 7.4). As one may have expected, the obtained (super) Hermitian Cauchy integral formula reduces, when considering the correct projections, to a new extension of the Bochner-Martinelli formula for holomorphic functions in superspace.

The results of Chapters 2, 3 and 5 have already been published in three articles, respectively given by references [41], [42] and [43]. The results of Chapters 4, 6 and 7 have been submitted for publication [40, 49, 8].

2

Radial algebras

In this chapter we introduce the notion of radial algebra which describes Clifford analysis in a more abstract setting. This notion will be used later for the introduction of Euclidean and Hermitian Clifford analysis in superspace.

2.1 Motivation and importance of the radial algebra

Many of the special functions that play a fundamental rôle in Clifford analysis are functions of zonal type i.e. functions depending of several Clifford vector variables $\underline{x}, \underline{y}, \ldots$ and their inner products. Fundamental examples of such functions are the reproducing kernel $\frac{1}{k!}\langle \underline{x}, \underline{y}\rangle^k$ and the monogenic part of its Fischer decomposition

$$\frac{1}{k!}\langle \underline{x}, \underline{y}\rangle^k + a_1 \underline{x}\,\underline{y}\langle \underline{x}, \underline{y}\rangle^{k-1} + \cdots + a_k \underline{x}^k \underline{y}^k,$$

where the coefficients $a_1, \ldots, a_k$ only depend on k and the dimension m. It is easy to understand that these zonal functions are in principle the same in every dimension m; the dimension parameter m stems from the repeated action of the Dirac operator $\partial_{\underline{x}}$ on the Fischer decomposition, using the evaluation $\partial_{\underline{x}}[\underline{x}] = -m$.

These observations lead to the idea of defining an algebra $R(S)$ of abstract vector variables and reintroducing the Dirac operators axiomatically as endomorphisms on $R(S)$, i.e. as vector derivatives denoted by ∂_x, $x \in S$. A first account on such an axiomatization can be found in [66] and was inspired by the work on "geometric calculus" presented by Hestenes and Sobczyk ([51]).

Important aspects of this radial algebra approach are:

- $R(S)$ does not depend on a particular dimension m. The vector derivative ∂_x leads to the introduction of the abstract scalar parameter $\partial_x[x] = m$, which can be considered as a continuous parameter in $\mathbb{R}$ or even in $\mathbb{C}$.
- Using the assignment $x \to \underline{x} = \sum_{j=1}^{m} x_j e_j$, $x \in S$, we obtain a representation of $R(S)$ by an algebra $R(\underline{S})$ of Clifford polynomials. This map is injective provided that S is finite and $m \geq \mathrm{Card}(S)$.
- The algebra $R(S)$ is independent of both the choice of an underlying vector space V on which the vector variables are defined, and the choice of a quadratic form on V.

These aspects show the computational strength of the radial algebra since they allow to compute symbolic expressions, depending on vector variables, following a minimal set of rules independently of specific bilinear forms and signatures. This gives rise to special functions in which the dimension m becomes a complex variable to which all methods of holomorphic functions can be applied. A typical example is the Fischer decomposition, which produces coefficients that are rational functions of m and that are valid outside the poles of these coefficients. Also the Fischer decomposition in the case of several vector variables is better studied in the radial algebra setting because its terms always are unique. In the Clifford polynomial setting the relation $\sum_{j=1}^{k} \underline{x}_j \, M_j(\underline{x}_1, \ldots, \underline{x}_k) = 0$ may have non zero monogenic solutions M_j if the dimension $m < 2k - 1$. This gives rise to exceptional syzygies for the Dirac complex (see [21, 65, 64]) that do not appear in the radial algebra setting.

In a number of papers [33, 68, 67] other applications have been considered such as the possibility to derive a theory of Dirac operators in superspace from the standard Euclidean one and to give a meaning to spaces with negative integer dimension. Indeed, to set up a Clifford calculus in superspace it is necessary to work with fundamental objects such as vector variables, directional and vector derivatives that satisfy the main "laws" of Clifford analysis. These "laws" are provided in a natural way by the axiomatic framework of radial algebras.

To address certain typical questions in the framework of algebras such as representation theoretical issues, it is necessary to provide concrete representations of the radial algebra. By a representation we mean an algebra homomorphism from $R(S)$ to an algebra A. In particular, vector variables may be represented through co-ordinate variables defined in a given underlying vector space V endowed with a bilinear form. This leads to different invariance groups depending on the specific representation. For example, in the classical Clifford-polynomial representation we have $V = \mathbb{R}^m$ with the rotation group $SO(m)$. But in superspace the corresponding underlying vector space for the radial algebra representation has the form $V = \mathbb{R}^{m|2n}(\Lambda)$, where Λ denotes some graded commutative Banach superalgebra. With such a representation one obtains a rotation group of supermatrices that contains $SO(m) \times Sp(2n)$ as its real projection. These results can be found in detail in Chapter 4.

In this chapter we first recall the main algebraic properties of the radial algebra and some of its most important endomorphisms. In this way the building blocks of Euclidean Clifford analysis naturally arise from a specific representation of the radial algebra. To that end we provide an overview of the results obtained in [66] together with detailed proofs for a number of results which were not presented there. This approach will be used later on to introduce Euclidean Clifford analysis in superspace.

In addition, we will establish a similar framework for the Hermitian setting. To that end we introduce the so-called Hermitian radial algebra which constitutes an abstract description of the main objects of Hermitian Clifford analysis. This will allow us later on to define the main framework for Hermitian Clifford analysis in superspace.

2.2 Algebraic properties of the radial algebra

Definition 2.1. *Given a set S of symbols $x, y, z, \ldots$ the radial algebra $R(S)$ is defined as the associative algebra over $\mathbb{R}$ freely generated by S and subject to the axiom*

$$\textbf{(A1)} \qquad [\{x, y\}, z] = 0 \quad \textit{for any } x, y, z \in S,$$

where $\{a, b\} = ab + ba$ and $[a, b] = ab - ba$. Elements in S are called abstract vector variables.

Axiom **(A1)** intrinsically means that the anti-commutator of two abstract vector variables is a scalar, i.e. a quantity that commutes with every other element in the algebra. It is clearly inspired by the similar property for Clifford vector variables. In order to formalize the relation between the radial algebra and the Clifford algebra we introduce the notion of a radial algebra representation.

Definition 2.2. *A radial algebra representation is an algebra homomorphism $\Psi : R(S) \to A$ from $R(S)$ into an algebra A. The term representation also refers to the range $\Psi(R(S)) \subset A$ of that mapping. By convenience we denote $\Psi(R(S))$ by $R(\Psi(S))$ where $\Psi(S) := \{\Psi(x) : x \in S\}$.*

The easiest and at the same time most important example of a radial algebra representation is the algebra generated by standard Clifford vector variables.

Example 2.1. *Consider the real Clifford algebra $\mathbb{R}_{0,m}$ generated by the orthonormal basis $\{e_1, \ldots, e_m\}$ of $\mathbb{R}^m$ governed by the multiplication rules*

$$e_j e_k + e_k e_j = -2\delta_{j,k}, \qquad j, k = 1, \ldots, m.$$

The classical Clifford-polynomial representation is established for a finite set S of ℓ abstract vector variables ($\ell \in \mathbb{N}$) by considering the mapping

$$x \to \underline{x} = \sum_{j=1}^{m} x_j e_j, \qquad x \in S, \tag{2.1}$$

where we associate to each $x \in S$ a set of m real variables $\{x_1, \ldots, x_m\}$ or equivalently, a Clifford vector variable $\underline{x}$ defined in $\mathbb{R}^m$. In this way, we obtain the set $\bigcup_{x \in S}\{x_1, \ldots, x_m\}$ of ℓm real variables that, when combined with the Clifford generators e_j ($j = 1 \ldots, m$), generate an algebra of Clifford-valued polynomials denoted by $\mathcal{A}_{m,0}$.

The correspondences (2.1) naturally extend to an algebra homomorphism from $R(S)$ to $\mathcal{A}_{m,0}$ since for any two Clifford vector variables $\underline{x}, \underline{y}$ we have that

$$\{\underline{x}, \underline{y}\} = \underline{x}\,\underline{y} + \underline{y}\,\underline{x} = -2\sum_{j=1}^{m} x_j y_j, \tag{2.2}$$

is a central element in $\mathcal{A}_{m,0}$, i.e. the axiom **(A1)** *is fulfilled.*

The set of ℓ Clifford vector variables established by (2.1) is denoted by $\underline{S} := \{\underline{x} : x \in S\}$ and the corresponding radial algebra representation in $\mathcal{A}_{m,0}$ is denoted by $R(\underline{S})$.

This representation justifies the use of radial algebras in some applications related to Clifford analysis. The number of elements to consider in S normally depends on the application that is going to be treated. In this thesis it always suffices to use a finite number of elements for S.

The simplest case is obtained for $S = \{x\}$ in which case $R(S)$ is mapped into the real algebra of polynomials of the form $\underline{x}^s$, where $\underline{x}^{2s} = (-|\underline{x}|^2)^s$ and $\underline{x}^{2s+1} = \underline{x}(-|\underline{x}|^2)^s$. To have a non-trivial radial algebra for Clifford analysis, the above set of radially symmetric functions $\underline{x}^s$ is too limited. One at least needs objects of the form $S = \{x, u\}$ where the corresponding Clifford vectors are given by $x \to \underline{x} = \sum_{k=1}^{m} x_k e_k$ and $u \to \underline{u} = \sum_{k=1}^{m} u_k e_k$. Here the vector x is considered as the variable vector and u as a parameter vector. The elements of the algebra $R(S)$ clearly have the form $F = A + Bx + Cu + Dx \wedge u$ where A, B, C, D are polynomials of the three variables $x^2, u^2, x \cdot u$ while $x \wedge u = \frac{1}{2}[x, u]$. With these two vector variables one can abstractly produce the so-called zonal monogenic polynomials, see [44]. More in general a typical choice for S would be $S = \{x_1, \ldots, x_s\} \cup \{u_1, \ldots, u_t\}$, where the variables x_j are the vector variables on which functions depend and u_j are additional parameter vectors. This choice for S is used when studying monogenic functions in several vector variables $x_1, \ldots, x_s$, see also [21].

The main difference between the Clifford algebra and the radial algebra lies in the fact that the abstract vector variables $x \in S$ have a merely symbolic nature; they are not vectors belonging to an a priory defined vector space V with some dimension m and some quadratic form on it.

Remark 2.1. *The representation given in Example 2.1 maps elements in S to vector variables defined in the vector space $\mathbb{R}^m$ which is endowed with the bilinear form $\mathcal{B}(e_j, e_k) =$*

$\{e_j, e_k\} = -2\delta_{j,k}$ with signature $(0, m)$. It is possible to obtain similar representations considering $\mathbb{R}^m$ endowed with different bilinear forms. In particular, one can consider the above bilinear form to have signature (p, q) ($p + q = m$, $p > 0$). The Clifford algebra associated to such a radial algebra representation is $\mathbb{R}_{p,q} := Alg_{\mathbb{R}}\{\varepsilon_1, \ldots, \varepsilon_p, e_1, \ldots, e_q\}$ governed by the rules $\{\varepsilon_j, \varepsilon_k\} = 2\delta_{j,k}$, $\{\varepsilon_j, e_k\} = 0$, $\{e_j, e_k\} = -2\delta_{j,k}$.

Based on (2.2), we define for two vector variables $x, y \in S$ the so-called *dot product*

$$x \cdot y = \frac{1}{2}\{x, y\}. \tag{2.3}$$

In the Clifford-polynomial representation, formula (2.2) shows that $\underline{x} \cdot \underline{y} = -\langle \underline{x}, \underline{y} \rangle$ where $\langle \underline{x}, \underline{y} \rangle = \sum_{j=1}^m x_j y_j$ is the Euclidean inner product in $\mathbb{R}^m$.

In the study of the algebraic structure of $R(S)$ also the wedge product of an arbitrary number of vectors will play an important rôle; it is defined by

$$x_1 \wedge \ldots \wedge x_k = \frac{1}{k!} \sum_{\pi} \operatorname{sgn}(\pi)\, x_{\pi(1)} \cdots x_{\pi(k)}, \qquad \forall x_1, \ldots, x_k \in S,$$

where π runs over the group $Sym(k)$ of all permutations of the set $\{1, \ldots, k\}$ and $\operatorname{sgn}(\pi)$ denotes the sign of the permutation π.

As mentioned before, for the case of two vectors, this definition reduces to

$$x \wedge y = \frac{1}{2}[x, y], \qquad \forall x, y \in S. \tag{2.4}$$

It can also be extended in a natural way to elements $x_1, \ldots, x_k \in \langle S \rangle$ where $\langle S \rangle$ is the $\mathbb{R}$-vector space generated by S. The following properties then hold, see [66]; they are the abstract counterparts of the corresponding properties in a Clifford algebra.

Lemma 2.1. *Let $x_1, \ldots, x_k \in \langle S \rangle$ then,*

(i) $x_{\pi(1)} \wedge \ldots \wedge x_{\pi(k)} = sgn(\pi)\, x_1 \wedge \ldots \wedge x_k$;

(ii) $x_1 \wedge \ldots \wedge x_k$ *is multilinear on* $\langle S \rangle$;

(iii) $x_1 \wedge \ldots \wedge x_k = 0$ *if and only if* $x_1, \ldots, x_k$ *are linearly dependent;*

(iv) in the set of vector variables S it holds that $x_1 \wedge \ldots \wedge x_k = 0$ if and only if $x_j = x_\ell$ for some pair $j, \ell \in \{1, \ldots, k\}$.

Lemma 2.2. *Every element $F(x_1, \ldots, x_\ell) \in R(S)$, generated by variables $x_1, \ldots, x_\ell \in S$, can be written as*

$$F(x_1, \ldots, x_\ell) = \sum_{k=0}^{\ell} \sum_{A} F_A\, x_{j_1} \wedge \ldots \wedge x_{j_k}, \tag{2.5}$$

where $A = \{j_1, \ldots, j_k\}$ with $1 \leq j_1 < \ldots < j_k \leq \ell$, the coefficients F_A are linear combinations of products of inner products $x_j \cdot x_r$, and $k = 0$ corresponds to the real term.

Proof.

Since the element $F(x_1,\ldots,x_\ell)$ is, apart from a possible real term, a linear combination of products of vector variables $x_{k_1}\cdots x_{k_s}$, it suffices to prove the lemma for such an individual term. This is easily done by induction. The initial case $s=1$ is trivial. Assuming now that (2.5) is true for every product of at most $s-1$ vector variables, the product of s vectors leads to two different cases. First, if two indices in the above product are equal, it reduces to

$$x_{k_1}\cdots x_{k_s}=\sum \text{scalar l.o.p.},$$

i.e., a linear combination of lower order products, on which the induction hypothesis can be applied, whence the decomposition follows. If, on the other hand, we have s different vector variables, as in $x_1\cdots x_s$, we can directly see from **(A1)** that

$$\text{sgn}(\pi)x_{\pi(1)}\cdots x_{\pi(s)}=x_1\ldots x_s+\sum \text{scalar l.o.p.}$$

whence $x_1\ldots x_s=x_1\wedge\ldots\wedge x_s+\sum$ scalar l.o.p. and the decomposition again follows by induction. □

The above lemma leads to the introduction of the k-vector structure of $R(S)$ as follows. Let $R_0(S)$ be the algebra generated by all scalar objects $x\cdot y$, $x,y\in S$, called the *scalar subalgebra* of $R(S)$, which, by **(A1)**, is contained in the center $C(R(S))$ of $R(S)$. An element $F\in R(S)$ is called a k-vector if F may be written as a sum of elements of the form $b\,x_{j_1}\wedge\ldots\wedge x_{j_k}$ where $b\in R_0(S)$, $x_{j_\ell}\in S$. The space of all k-vectors is denoted by $R_k(S)$.

Remark 2.2. *In the case $k=2$, $R_k(S)$ is the space of the so-called bivectors spanned over $R_0(S)$ by the wedge product of two abstract vector variables in S, see (2.4). The corresponding algebra of bivectors $\mathbb{R}^{(2)}_{0,m}$ in the Clifford algebra $\mathbb{R}_{0,m}$ consists of elements of the form*

$$\sum_{1\le j<k\le m} b_{j,k}\,e_je_k,\qquad b_{j,k}\in\mathbb{R}.$$

The uniqueness of the decomposition (2.5) can be proven using the Clifford polynomial representation for finitely generated radial algebras established in Example 2.1. It just suffices to note that, for some particular choices of the dimension m of the underlying vector space $\mathbb{R}^m$, such a representation is an isomorphism. This characterization of $R\{x_1,\ldots,x_\ell\}$ is given by the following result which was proven in [66].

Theorem 2.1. *The map*

$$\underline{\cdot}\,:R\{x_1,\ldots,x_\ell\}\to Alg\{\underline{x_1},\ldots,\underline{x_\ell}\}$$

defined in (2.1) is an algebra isomorphism if and only if $m\ge\ell$.

Making use of the well-known properties of Clifford algebras we obtain the following direct consequences.

Corollary 2.1. *Every element $F \in R(S)$ may be decomposed in a unique way as a finite sum of the form*

$$F = [F]_0 + [F]_1 + \dots$$

where the operators $[\cdot]_k : R(S) \to R_k(S) : F \to [F]_k$ project any given object into its k-vector part.

Corollary 2.2. *Let $C(R(S))$ denote the center of $R(S)$, then the following holds:*

(i) if S is infinite or has an even number of elements, then $C(R(S)) = R_0(S)$;

(ii) if S has an odd number of elements, say $2\ell+1$, then $C(R(S))$ is generated by $R_0(S)$ and the element $x_1 \wedge \dots \wedge x_{2\ell+1}$.

Corollary 2.3. *The following computation rules for the product of a vector with a k-vector hold:*

(i) $[x_1 \cdots x_k]_k = x_1 \wedge \dots \wedge x_k$;

(ii) $x\, x_1 \wedge \dots \wedge x_k = [x\, x_1 \wedge \dots \wedge x_k]_{k-1} + [x\, x_1 \wedge \dots \wedge x_k]_{k+1}$;

(iii) $[x\, x_1 \wedge \dots \wedge x_k]_{k+1} = x \wedge x_1 \wedge \dots \wedge x_k = \frac{1}{2}\Big(x x_1 \wedge \dots \wedge x_k + (-1)^k x_1 \wedge \dots \wedge x_k x\Big)$;

(iv) $[x\, x_1 \wedge \dots \wedge x_k]_{k-1} = x \cdot (x_1 \wedge \dots \wedge x_k) = \frac{1}{2}\Big(x x_1 \wedge \dots \wedge x_k - (-1)^k x_1 \wedge \dots \wedge x_k x\Big)$,

and similarly for the right multiplication.

The proofs of the above results show that some identities in the radial setting can be obtained in two ways: in a direct axiomatic way and by means of the Clifford-polynomial representation. But this is possible only if the dimension of the chosen representation is sufficiently high compared to the amount of vector variables involved. One example of this issue is given by Theorem 2.1 but this is not the only one. Other conditions on the dimension of the Clifford representation need to be considered in the study of other problems like the Dirac complex, see for example [21, 65].

Through this thesis we mainly work with axiomatic proofs for radial algebra identities, although some of them could be proven using the Clifford-polynomial representation. The main reason for this is to show the computational strength of this method, independent of a specific vector space for the vector variables.

2.3 Endomorphisms on the radial algebra

In this section, we introduce some important elements of the algebra of endomorphisms $\mathrm{End}(R(S))$ defined on the radial algebra $R(S)$. They constitute an important piece of the theory because, through the action of some radial algebra representations, they transform into endomorphisms and operators of interest in classical analysis.

Definition 2.3 (extension of Ψ from $R(S)$ to $\mathrm{End}(R(S))$). *Let $\Psi : R(S) \to R(\Psi(S))$ be a radial algebra representation and let $E \in End(R(S))$ be an endomorphism leaving the kernel of Ψ invariant, i.e.*

$$E(\ker \Psi) \subset \ker \Psi, \quad \textit{with} \quad \ker \Psi := \{F \in R(S) : \Psi(F) = 0\}. \tag{2.6}$$

Then the action $\Psi(E)$ of Ψ on E can be defined as the following endomorphism over $R(\Psi(S))$

$$\Psi(E)[\Psi(F)] := \Psi(E[F]), \qquad F \in R(S).$$

Remark 2.3. *The compatibility condition (2.6) is necessary in order to have the endomorphism $\Psi(E)$ well-defined. Indeed, Definition 2.3 is independent of the choice of $F \in R(S)$ if and only if for every pair $F_1, F_2 \in R(S)$ such that $\Psi(F_1) = \Psi(F_2)$ one has*

$$\Psi(E(F_1)) = \Psi(E(F_2)), \quad \textit{or equivalently,} \quad \Psi(E(F_1 - F_2)) = 0;$$

meaning that $E(\ker \Psi) \subset \ker \Psi$.

Observe that any injective radial algebra representation satisfies the condition (2.6) for every $E \in End(R(S))$. This is the case of the Clifford-polynomial representation defined in (2.1) if $m \geq \ell$, see Theorem 2.1.

Remark 2.4. *It is easily seen that $\Psi : End(R(S)) \to End(R(\Psi(S)))$ is an algebra homomorphism, i.e.*

$$\Psi(E_1 \circ E_2) = \Psi(E_1) \circ \Psi(E_2).$$

2.3.1 Involutions and vector multiplication

Definition 2.4 (main involution and conjugation). *The main involution $\widetilde{\cdot}$ and the conjugation $\overline{\cdot}$ are defined on $R(S)$ by the relations*

$$\overline{FG} = \overline{G}\,\overline{F}, \qquad \widetilde{FG} = \widetilde{F}\,\widetilde{G}, \quad F, G \in R(S),$$

$$\overline{x} = \widetilde{x} = -x, \quad x \in S.$$

By direct computation the following properties are obtained.

Lemma 2.3. *Let $x_1, \ldots, x_k \in S$, $f \in R_0(S)$ and $F \in R(S)$. Then*

(i) $\widetilde{\widetilde{F}} = \overline{\overline{F}} = F$ *and* $\widetilde{\overline{F}} = \overline{\widetilde{F}}$;

(ii) $\widetilde{f} = \overline{f} = f$;

(iii) $x_1 \widetilde{\wedge \ldots \wedge} x_k = (-1)^k x_1 \wedge \ldots \wedge x_k$;

(iv) $\overline{x_1 \wedge \ldots \wedge x_k} = (-1)^{\frac{k(k+1)}{2}} x_1 \wedge \ldots \wedge x_k = (-1)^k x_k \wedge \ldots \wedge x_1$.

It is easily seen that the above defined main involution and conjugation are mapped by the Clifford-polynomial representation (see Definition 2.3 and Example 2.1) into the classical main involution and conjugation; which are defined in $\mathbb{R}_{0,m}$ by

$$\overline{e_{j_1}\cdots e_{j_k}} = (-1)^k e_{j_k}\cdots e_{j_1}, \qquad \widetilde{e_{j_1}\cdots e_{j_k}} = (-1)^k e_{j_1}\cdots e_{j_k}.$$

Definition 2.5 (vector multiplication). *With every $x\in S$ one can associate the basic endomorphisms on $R(S)$:*

$$x : F \to xF, \qquad x| : F \to \widetilde{F}x. \tag{2.7}$$

The set $S\cup S| := \{x, x| : x\in S\}$ generates a subalgebra of $\mathrm{End}(R(S))$ which we denote by $R(S\cup S|)$ and which is subject to the rules

$$\begin{array}{ll} \textbf{(A1|)} & \{x,y|\}=0,\quad \{x,y\}=-\{x|,y|\} \quad \forall x,y\in S, \\ \textbf{(A2|)} & \{x,y\} \text{ is a central element in } R(S\cup S|) \quad \forall x,y\in S. \end{array}$$

In the same way "right" versions of the above endomorphisms may be defined:

$$\cdot x : F \to Fx, \quad \cdot x| : F \to x\widetilde{F}.$$

The algebra $\mathrm{Alg}\{\cdot x, \cdot x| : x\in S\}\subset \mathrm{End}(R(S))$ is isomorphic to $R(S\cup S|)$ and they are connected by the relations:

$$\cdot x = x|\circ \widetilde{\cdot}\ , \qquad \cdot x| = x\circ \widetilde{\cdot}\ .$$

Hence, in order to study the whole subalgebra of endomorphisms of $R(S)$ given by compositions of vector multiplications, it is enough to consider one of both copies (right or left) of $R(S\cup S|)$ in $\mathrm{End}(R(S))$. From now on we will consider only the left one, which is given by (2.7). In this way, it holds that $R(S)\subset R(S\cup S|)\subset \mathrm{End}(R(S))$.

Remark 2.5. *The possibility of seeing $R(S)$ as a subalgebra of End($R(S)$) shows the convenience of using the abuse of notation $\Psi(E)$ for the action of the radial algebra representation Ψ on the endomorphism $E\in$ End($R(S)$). Indeed, Definition 2.3 provides an extension of Ψ from $R(S)$ to End($R(S)$).*

In the language of Clifford algebras it is well known that $\mathrm{End}(\mathbb{R}_{0,m})\cong\mathbb{R}_{m,m}$. This isomorphism is given by the following identification between the generators of both algebras

$$\begin{aligned} e_j &\longleftrightarrow e_j : \ F\mapsto e_jF, \qquad j=1,\ldots,m,\\ \varepsilon_j &\longleftrightarrow e_j| : \ F\mapsto \widetilde{F}e_j, \qquad j=1,\ldots,m. \end{aligned} \tag{2.8}$$

Then $x|$ is mapped, by the Clifford-polynomial representation, to the endomorphism defined by the $\mathbb{R}_{m,0}$ Clifford vector variable $\underline{x}| = \sum_{j=1}^m x_j\varepsilon_j$ through the above correspondence.

2.3.2 Directional derivatives

Definition 2.6 (directional derivatives). *For every pair $x, y \in S$ the operator $D_{y,x} \in End(R(S))$ is defined by*

$$\textbf{(DD1)} \qquad D_{y,x}[FG] = D_{y,x}[F]G + FD_{y,x}[G], \qquad \forall F, G \in R(S),$$

$$\textbf{(DD2)} \qquad D_{y,x}[z] = \delta_{x,z}\, y, \qquad \forall z \in S,$$

where $\delta_{x,z}$ is the Kronecker delta

Remark 2.6. *For $x = y$ the operator $D_{x,x}$ corresponds to the Euler operator E_x, measuring the degree of homogeneity with respect to x, as expressed in the following lemma.*

Lemma 2.4. *In the product $x_1 \cdots x_\ell$ the variable $x \in S$ occurs k times if and only if*

$$E_x[x_1 \cdots x_\ell] = k\, x_1 \cdots x_\ell$$

In particular, $F \in R(S \setminus \{x\})$ if and only if $E_x[F] = 0$.

Proof.

The defining relations **(DD1)-(DD2)** yield

$$E_x[x_1 \cdots x_\ell] = \sum_{j=1}^{\ell} x_1 \cdots E_x[x_j] \cdots x_\ell = \left(\sum_{j=1}^{\ell} \delta_{x,x_j}\right) x_1 \cdots x_\ell$$

where the scalar appearing between brackets clearly counts the number of occurrences of x. □

Lemma 2.5. *The directional derivative $D_{y,x}$, $x, y \in S$, satisfies the following properties:*

(i) *$D_{y,x}$ maps $R_0(S)$ into $R_0(S)$;*

(ii) $\widetilde{D_{y,x}[F]} = D_{y,x}[\widetilde{F}]$ *for all $F \in R(S)$;*

(iii) $\overline{D_{y,x}[F]} = D_{y,x}[\overline{F}]$ *for all $F \in R(S)$.*

Proof.

(i) In view of **(DD1)** it suffices to prove the result for the generators of $R_0(S)$; hence, consider $\{w, z\}$ for an arbitrary pair $w, z \in S$. It is easily checked that

$$D_{y,x}[\{w, z\}] = \delta_{x,z}\{y, w\} + \delta_{x,w}\{y, z\} \in R_0(S).$$

(ii) Every element in $R(S)$ can be written as a sum of elements of the form $f_1F_1 + xf_2F_2$, where $f_1, f_2 \in R_0(S)$ and $F_1, F_2 \in R(S \setminus \{x\})$. Hence, in order to prove the result, it suffices to consider the canonical form[1] $F = f_1F_1 + xf_2F_2$. It then holds that

$$\widetilde{D_{y,x}[F]} = \widetilde{D_{y,x}[f_1]F_1} + \widetilde{D_{y,x}[f_2]xF_2} + \widetilde{f_2yF_2} = \widetilde{D_{y,x}[f_1]}\widetilde{F_1} - \widetilde{D_{y,x}[f_2]}x\widetilde{F_2} - f_2y\widetilde{F_2}.$$

[1]This kind of decomposition is called canonical decomposition of F with respect to x.

On the other hand, $\widetilde{F} = f_1\widetilde{F_1} - f_2x\widetilde{F_2}$, whence

$$D_{y,x}[\widetilde{F}] = D_{y,x}[f_1]\widetilde{F_1} - D_{y,x}[f_2]x\widetilde{F_2} - f_2y\widetilde{F_2}.$$

The equality then follows from (i) and Lemma 2.3.

(iii) As above, we may take $F = f_1F_1 + xf_2F_2$, for which it holds that

$$\overline{D_{y,x}[F]} = \overline{D_{y,x}[f_1]F_1 + D_{y,x}[f_2]xF_2 + f_2yF_2} = \overline{F_1}\,\overline{D_{y,x}[f_1]} - \overline{F_2}x\overline{D_{y,x}[f_2]} - \overline{F_2}yf_2.$$

Here one has $\overline{F} = f_1\overline{F_1} - \overline{F_2}xf_2$, whence

$$D_{y,x}[\overline{F}] = \overline{F_1}D_{y,x}[f_1] - \overline{F_2}yf_2 - \overline{F_2}xD_{y,x}[f_2],$$

and the equality again follows from (i) and Lemma 2.3. □

These properties of the directional derivative $D_{y,x}$ show that it behaves as a scalar first order differential operator, and it is easily seen to be mapped by the Clifford-polynomial representation to the scalar operator

$$D_{\underline{y},\underline{x}} = \sum_{j=1}^{m} y_j \partial_{x_j} \tag{2.9}$$

associated to the Clifford variables $\underline{x}$ and $\underline{y}$. Also, the following commutation relations with the vector multiplication endomorphisms hold.

Lemma 2.6. *Let $x, y, z \in S$. Then*

(i) $[D_{y,x}, z] = \delta_{x,z}\, y$;

(ii) $[D_{y,x}, z|] = \delta_{x,z}\, y|$.

2.3.3 Vector derivative

Definition 2.7 (vector derivative). *Given $x \in S$, the left and right endomorphisms $\partial_x[\cdot]$ and $[\cdot]\partial_x$ on $R(S)$ are defined by the axioms*

(D1) $\partial_x[fF] = \partial_x[f]F + f\partial_x[F]$,
$[fF]\partial_x = F[f]\partial_x + f[F]\partial_x, \qquad \forall f \in R_0(S),\ \forall F \in R(S)$;

(D2) $\partial_x[G] = 0,\ \partial_x[xG] = \partial_x[x]G$,
$[G]\partial_x = 0,\ [Gx]\partial_x = G[x]\partial_x, \qquad \forall G \in R(S \setminus \{x\})$;

(D3) $[\partial_x F]\,\partial_y = \partial_x\,[F\partial_y]\,, \quad \forall x, y \in S$;

(D4) $\partial_x[x^2] = [x^2]\partial_x = 2x, \quad \forall x \in S$,
$\partial_x\{x, y\} = \{x, y\}\partial_x = 2y, \quad \forall y \neq x$.

Remark 2.7. *It is tacitly assumed that, for any $T \subset S$ with $x \in T$, the values of $\partial_x[F]$ and $[F]\partial_x$ for $F \in R(T)$ do not depend on whether $\partial_x[\cdot]$ or $[\cdot]\partial_x$ are considered as elements of End$(R(T))$ or as elements of End$(R(S))$. This assumption is referred to as an "unpronounced axiom" in [66].*

This formal vector derivative ∂_x is mapped by the Clifford-polynomial representation to (minus) the classical Dirac operator (or gradient)

$$-\partial_{\underline{x}} = -\sum_{j=1}^{m} e_j \partial_{x_j}. \tag{2.10}$$

Remark 2.8. *By the canonical decomposition of $F \in R(S)$ it is easily seen that, under the assumption* **(D1)**, **(D2)** *is equivalent to*

$$\partial_x[FG] = \partial_x[F]G, \ \ [GF]\partial_x = G[F]\partial_x, \qquad \forall F \in R(S), \, \forall G \in R(S \setminus \{x\}). \tag{2.11}$$

Theorem 2.2. *The axioms* **(D1)-(D4)** *yield a consistent definition of the endomorphisms $\partial_x[\cdot]$ and $[\cdot]\partial_x$, mapping $R_0(S)$ into $R_1(S)$. Moreover, $\partial_x[x] = [x]\partial_x = \mathfrak{m} \in \mathbb{R}$.*

Proof.

Observe that **(D1)**–**(D3)** are satisfied by every first order Clifford differential operator with constant coefficients; the specific nature of the vector derivative is determined by **(D4)**. Again writing every element of $R(S)$ as a sum of terms of the form $F = f_1F_1 + xf_2F_2$, with $f_j \in R_0(S)$, $F_j \in R(S \setminus \{x\})$, $j = 1, 2$, we have, on account of **(D1)** and **(D2)**, that

$$\partial_x[F] = \partial_x[f_1]F_1 + \partial_x[f_2]xF_2 + f_2\partial_x[x]F_2,$$

whence ∂_x is determined in the whole of $R(S)$ by its action on $R_0(S)$ and on x. For the scalar subalgebra $R_0(S)$ we only need to determine the action of ∂_x on the generators $\{y, z\}$ with $y, z \in S$. By **(D2)** and **(D4)** we have

$$\partial_x\{y, z\} = 2(\delta_{z,x}y + \delta_{y,x}z) \in R_1(S).$$

So we only still need to determine $\partial_x[x]$. Using **(D3)** and **(D4)** we obtain

$$\begin{aligned} 2\partial_x[x] &= \partial_x\left[x^2\partial_x\right] = \left[\partial_x x^2\right]\partial_x = 2[x]\partial_x, \\ 2\partial_x[x] &= \partial_x\left[\{x, y\}\partial_y\right] = \left[\partial_x\{x, y\}\right]\partial_y = 2[y]\partial_y. \end{aligned}$$

Hence, for any pair of vectors $x, y \in S$, it holds that $\partial_x[x] = \partial_y[y]$, implying that $\partial_x[x]$ equals a constant $\mathfrak{m} \in \mathbb{R}$, and similarly for the right action. $\square$

Remark 2.9. *The parameter $\mathfrak{m} = \partial_x[x] = [x]\partial_x$ is called the abstract dimension of R(S). In the $\mathbb{R}_{0,m}$-Clifford polynomial representation $\mathfrak{m}$ gets mapped to $-\partial_{\underline{x}}[\underline{x}] = m$.*

Similar to Lemma 2.5 we can prove the following properties.

Lemma 2.7. *Let $x \in S$ and $F \in R(S)$. Then*

(i) $\widetilde{\partial_x[F]} = -\partial_x[\widetilde{F}]$ *and* $\widetilde{[F]\partial_x} = -[\widetilde{F}]\partial_x$*;*

(ii) $\overline{\partial_x[F]} = -[\overline{F}]\partial_x$ *and* $\overline{[F]\partial_x} = -\partial_x[\overline{F}]$*.*

Remark 2.10. *The property (ii) of the above Lemma shows that, unless explicitly needed, it suffices to study only the left action of the operator ∂_x (i.e. the endomorphism $\partial_x[\cdot]$), since the right action is obtained after conjugation.*

Theorem 2.3. *Let $x, y \in S$. Then*

(i) $\{\partial_x, y\} = 2D_{y,x} + \delta_{x,y}\mathfrak{m}$*;*

(ii) $\{\partial_x, y\} = \cdot\{\partial_x, y\}$*, where* $\cdot\{\partial_x, y\}$ *denotes the right action of the operator* $\{\partial_x, y\}$*, i.e.* $[F]\{\partial_x, y\} = [F]\partial_x y + [Fy]\partial_x$*.*

Proof.

(i) Consider $F \in R(S)$ in its canonical decomposition as before; then

$$\begin{aligned} y\partial_x[F] &= y\partial_x[f_1]F_1 + y\partial_x[f_2]xF_2 + \mathfrak{m}f_2yF_2 \\ \partial_x[yF] &= \partial_x[f_1]yF_1 + \delta_{x,y}f_1\mathfrak{m}F_1 + \partial_x[f_2]yxF_2 + 2f_2yF_2 - \mathfrak{m}f_2yF_2 + \delta_{x,y}\mathfrak{m}f_2xF_2, \end{aligned}$$

whence

$$\{\partial_x, y\}[F] = \{\partial_x, y\}[f_1]F_1 + \{\partial_x, y\}[f_2]xF_2 + 2f_2yF_2.$$

Similarly,

$$(2D_{y,x} + \delta_{x,y}\mathfrak{m})\,[F] = (2D_{y,x} + \delta_{x,y}\mathfrak{m})\,[f_1]F_1 + (2D_{y,x} + \delta_{x,y}\mathfrak{m})\,[f_2]xF_2 + 2f_2yF_2.$$

It thus suffices to prove that $\{\partial_x, y\}$ and $2D_{y,x} + \delta_{x,y}\mathfrak{m}$ have the same action on scalars, or still, on the generators of $R_0(S)$. But for each pair $z, w \in S$ one has

$$\begin{aligned} \{\partial_x, y\}\,[\{z, w\}] &= y\partial_x\{z, w\} + \partial_x\{z, w\}y + \delta_{x,y}\mathfrak{m}\{z, w\} \\ &= 2\delta_{z,x}\{y, w\} + 2\delta_{w,x}\{y, z\} + \delta_{x,y}\mathfrak{m}\{z, w\} \\ &= \ (2D_{y,x} + \delta_{x,y}\mathfrak{m})\,[\{z, w\}]\,. \end{aligned}$$

(ii) The above result shows that $\{\partial_x, y\}$ is a scalar operator, for which, on account of Lemma 2.5, it holds that

$$\overline{\{\partial_x, y\}[F]} = \{\partial_x, y\}[\overline{F}],$$

for all $F \in R(S)$. The proof then follows from Lemma 2.7:

$$\overline{[F]\{\partial_x, y\}} = \overline{[F]\partial_x\, y + [Fy]\partial_x} = y\partial_x[\overline{F}] + \partial_x[y\overline{F}] = \{\partial_x, y\}[\overline{F}] = \overline{\{\partial_x, y\}[F]}.$$

□

The above theorem states that the operator $\{\partial_x, y\}$ behaves as a scalar first order differential operator. In addition, property (ii) allows us to define the directional derivative $D_{y,x}$ from the right by means of $[\cdot]D_{y,x} := D_{y,x}[\cdot]$ as it is expected from an scalar operator. This theorem also provides an alternative approach for defining the vector derivative ∂_x by recursion based on (i).

The following commutation relation with the directional derivatives holds.

Theorem 2.4. *Let $x, y, z \in S$. Then $[D_{y,x}, \partial_z] = -\delta_{y,z}\partial_x$.*

Proof.

Writing elements of $R(S)$ in their canonical decomposition with respect to $x, z \in S$, viz.

$$F = f_1F_1 + f_2xF_2 + f_3zF_3 + f_4xzF_4, \qquad f_j \in R_0(S),\ F_j \in R\left(S \setminus \{x,z\}\right),\ j = 1,\dots,4,$$

we obtain by Lemma 2.5

$$\begin{aligned} D_{y,x}\left[\partial_z[F]\right] = {} & D_{y,x}\left[\partial_z[f_1]\right]F_1 + D_{y,x}\left[\partial_z[f_2]\right]xF_2 + \partial_z[f_2]yF_2 + \delta_{x,z}D_{y,x}[f_2]\mathfrak{m}F_2 \\ & + D_{y,x}\left[\partial_z[f_3]\right]zF_3 + \delta_{x,z}\partial_z[f_3]yF_3 + D_{y,x}[f_3]\mathfrak{m}F_3 \\ & + D_{y,x}\left[\partial_z[f_4]\right]xzF_4 + \partial_z[f_4]yzF_4 + \delta_{x,z}\partial_z[f_4]xyF_4 \\ & + (2-\mathfrak{m}+\delta_{x,z}\mathfrak{m})D_{y,x}[f_4]xF_4 + (2-\mathfrak{m}+\delta_{x,z}\mathfrak{m})f_4yF_4, \end{aligned}$$

while

$$\begin{aligned} \partial_z\left[D_{y,x}[F]\right] = {} & \partial_z\left[D_{y,x}[f_1]\right]F_1 + \partial_z\left[D_{y,x}[f_2]\right]xF_2 + \delta_{x,z}D_{y,x}[f_2]\mathfrak{m}F_2 + \partial_z[f_2]yF_2 + \delta_{z,y}f_2\mathfrak{m}F_2 \\ & + \partial_z\left[D_{y,x}[f_3]\right]zF_3 + D_{y,x}[f_3]\mathfrak{m}F_3 + \delta_{x,z}\partial_z[f_3]yF_3 + \delta_{x,z}\delta_{z,y}f_3\mathfrak{m}F_3 \\ & + \partial_z\left[D_{y,x}[f_4]\right]xzF_4 + (2-\mathfrak{m}+\delta_{x,z}\mathfrak{m})D_{y,x}[f_4]xF_4 + \partial_z[f_4]yzF_4 \\ & + (2-\mathfrak{m}+\delta_{z,y}\mathfrak{m})f_4yF_4 + \delta_{x,z}\partial_z[f_4]xyF_4 + (2\delta_{z,y}+\mathfrak{m}-\delta_{z,y}\mathfrak{m})\delta_{x,z}f_4yF_4. \end{aligned}$$

Hence

$$\begin{aligned} \left[D_{y,x},\partial_z\right][F] = {} & \left[D_{y,x},\partial_z\right][f_1]F_1 + \left[D_{y,x},\partial_z\right][f_2]xF_2 + \left[D_{y,x},\partial_z\right][f_3]zF_3 + \left[D_{y,x},\partial_z\right][f_4]xzF_4 \\ & - \delta_{z,y}\big(f_2\mathfrak{m}F_2 + \delta_{x,z}f_3\mathfrak{m}F_3 + (\mathfrak{m}+2\delta_{z,x}-\delta_{z,x}\mathfrak{m})f_4yF_4\big), \end{aligned}$$

and on the other hand,

$$\begin{aligned} -\delta_{z,y}\partial_x[F] = {} & -\delta_{z,y}\partial_x[f_1]F_1 - \delta_{z,y}\partial_x[f_2]xF_2 - \delta_{z,y}\partial_x[f_3]zF_3 - \delta_{z,y}\partial_x[f_4]xzF_4 \\ & - \delta_{z,y}\big(f_2\mathfrak{m}F_2 + \delta_{x,z}f_3\mathfrak{m}F_3 + (\mathfrak{m}+2\delta_{z,x}-\delta_{z,x}\mathfrak{m})f_4zF_4\big). \end{aligned}$$

It thus suffices to prove the equality $[D_{y,x}, \partial_z] = -\delta_{y,z}\partial_x$ in the scalar subalgebra $R_0(S)$. Direct computation yields

$$\left[D_{y,x},\partial_z\right][fg] = \left[D_{y,x},\partial_z\right][f]g + f\left[D_{y,x},\partial_z\right][g], \qquad f,g \in R_0(S),$$

whence we only have to consider the generators of $R_0(S)$, for which it indeed holds that:

$$\begin{aligned} \left[D_{y,x},\partial_z\right]\{u,w\} &= D_{y,x}\left[2(\delta_{z,w}u+\delta_{z,u}w)\right] - \partial_z\left[\delta_{x,u}\{y,w\}+\delta_{x,w}\{y,u\}\right] \\ &= -2\delta_{z,y}(\delta_{x,u}w+\delta_{x,w}u) \\ &= -\delta_{y,z}\partial_x\{u,w\}. \end{aligned}$$

□

2.3.4 Other endomorphisms

In what follows we use, for $x \in S$, the endomorphisms

$$\partial_x : F \to \partial_x F, \qquad \partial_x| : F \to \widetilde{F}\partial_x. \tag{2.12}$$

The endomorphism $\partial_x|$ is mapped, by the Clifford-polynomial representation, to the differential operator defined by

$$-\partial_{\underline{x}}| = -\sum_{j=1}^{m} \varepsilon_j \partial_{x_j}, \tag{2.13}$$

where we are considering the identifications (2.8) for the generators ε_j.

Theorem 2.5. *Let $x, y, z \in S$. Then the following (anti-)commutation relations hold:*

(i) $\{\partial_x, y\} = -\{\partial_x|, y|\} = 2D_{y,x} + \delta_{x,y}\mathfrak{m}$*;*

(ii) $[\{\partial_x, y\}, z] = 2\delta_{x,z}\, y$*;*

(iii) $[\{\partial_x, y\}, \partial_z] = -2\delta_{z,y}\, \partial_x$*;*

(iv) $\{\partial_x, y|\} = \{\partial_x|, y\} = 0$ *for* $x \neq y$*;*

(v) $\{\partial_x, x|\} = -\{\partial_x|, x\} = B$*, independent of* $x \in S$*.*

Proof.

Properties (i)-(iv) are direct consequences of Lemmas 2.6, 2.7 and Theorems 2.3, 2.4. Now, in order to prove (v), let us first show that $\{\partial_x, x|\} = \{\partial_x|, x\}$. By means of the axioms **(D1)**, **(D3)** and **(D4)** we easily obtain for every $F \in R(S)$ that

$$\widetilde{F}x = \frac{1}{2}\left([x^2\widetilde{F}]\partial_x - x^2[\widetilde{F}]\partial_x\right),$$

whence

$$\begin{aligned}
\{\partial_x, x|\}[F] &= \partial_x[\widetilde{F}x] - \partial_x[\widetilde{F}]x \\
&= \frac{1}{2}\left(\left[\partial_x[x^2\widetilde{F}]\right]\partial_x - \partial_x[x^2][\widetilde{F}]\partial_x - x^2\partial_x\left[[\widetilde{F}]\partial_x\right]\right) - \partial_x[\widetilde{F}]x \\
&= [x\widetilde{F}]\partial_x - x[\widetilde{F}]\partial_x + \frac{1}{2}\left(\partial_x[\widetilde{F}]2x + x^2\left[\partial_x[\widetilde{F}]\right]\partial_x - x^2\partial_x\left[[\widetilde{F}]\partial_x\right]\right) - \partial_x[\widetilde{F}]x \\
&= [x\widetilde{F}]\partial_x - x[\widetilde{F}]\partial_x \\
&= -\{\partial_x|, x\}[F].
\end{aligned}$$

We now still need to prove that this result is independent of x. For a pair $x, y \in S$ with $x \neq y$ we have

$$\widetilde{F}x = \frac{1}{2}\left([\widetilde{F}\{x,y\}]\partial_y - \{x,y\}[\widetilde{F}]\partial_y\right),$$

whence

$$
\begin{aligned}
\{\partial_x, x|\}[F] &= \frac{1}{2}\left(\partial_x\left[[\widetilde{F}\{x,y\}]\partial_y\right] - \partial_x\left[\{x,y\}[\widetilde{F}]\partial_y\right]\right) - \partial_x[\widetilde{F}]x \\
&= \frac{1}{2}\left(\left[2y\widetilde{F} + \{x,y\}\partial_x[\widetilde{F}]\right]\partial_y - 2y[\widetilde{F}]\partial_y - \{x,y\}\partial_x\left[[\widetilde{F}]\partial_y\right]\right) - \partial_x[\widetilde{F}]x \\
&= [y\widetilde{F}]\partial_y - y[\widetilde{F}]\partial_y + \frac{1}{2}\Big(\partial_x[\widetilde{F}]2x + \{x,y\}\left[\partial_x[\widetilde{F}]\right]\partial_y - \{x,y\}\partial_x\left[[\widetilde{F}]\partial_y\right]\Big) - \partial_x[\widetilde{F}]x \\
&= [y\widetilde{F}]\partial_y - y[\widetilde{F}]\partial_y \\
&= -\{\partial_y|, y\}[F],
\end{aligned}
$$

completing the proof. □

Remark 2.11. *The new operator B is mapped by the Clifford-polynomial representation to the $\mathbb{R}_{0,m}$-valued operator*

$$F \to -\sum_{j=1}^{m} e_j \widetilde{F} e_j,$$

the latter being the realization in $End(\mathbb{R}_{0,m}) \cong \mathbb{R}_{m,m}$ of the bivector $\underline{B} = -\sum_{j=1}^{m} e_j\varepsilon_j$ if one takes into account the correspondences (2.8).

Theorem 2.6. *Let $x, y \in S$, $f \in R_0(S)$, $v \in R_1(S)$ and $F \in R(S)$. Then,*

(i) $\widetilde{B[F]} = B[\widetilde{F}]$ *and* $\overline{B[F]} = B[\overline{F}]$*;*

(ii) $B[fF] = fB[F]$*;*

(iii) $[B, x] = -2x|$ *and* $[B, x|] = -2x$*;*

(iv) $B[vF] = vB[F] - 2\widetilde{F}v$ *and* $B[Fv] = B[F]v - 2v\widetilde{F}$*;*

(v) $[D_{y,x}, B] = 0$*;*

(vi) $[B, \partial_x] = -2\partial_x|$ *and* $[B, \partial_x|] = -2\partial_x$*.*

Proof.

(i) By direct calculation we obtain:

$$
\begin{aligned}
\widetilde{B[F]} &= \widetilde{\partial_x[\widetilde{F}x]} - \widetilde{\partial_x[\widetilde{F}]x} = \partial_x[Fx] - \partial_x[F]x = B[\widetilde{F}], \\
\overline{B[F]} &= \overline{\partial_x[\widetilde{F}x]} - \overline{\partial_x[\widetilde{F}]x} = \left[x\overline{\widetilde{F}}\right]\partial_x - x\left[\overline{\widetilde{F}}\right]\partial_x = B[\overline{F}].
\end{aligned}
$$

(ii) Also here, direct calculation yields the desired property:

$$B[fF] = \partial_x[f\widetilde{F}x] - \partial_x[f\widetilde{F}]x = f(\partial_x[\widetilde{F}x] - \partial_x[\widetilde{F}]x) = fB[F].$$

(*iii*) Observe that

$$\begin{aligned} xB[F] &= x[x\widetilde{F}]\partial_x - x^2[\widetilde{F}]\partial_x, \\ B[xF] &= [x\widetilde{xF}]\partial_x - x[\widetilde{xF}]\partial_x = -[x^2\widetilde{F}]\partial_x + x[x\widetilde{F}]\partial_x \\ &= -2\widetilde{F}x - x^2[\widetilde{F}]\partial_x + x[x\widetilde{F}]\partial_x, \end{aligned}$$

whence $[B,x][F] = -2\widetilde{F}x = -2x|[F]$. On the other hand,

$$\begin{aligned} x|B[F] &= \widetilde{B[F]}x = \partial_x[Fx]x - \partial_x[F]x^2, \\ B[x|F] &= B[\widetilde{F}x] = \partial_x[\widetilde{\widetilde{F}x}x] - \partial_x[\widetilde{\widetilde{F}x}]x = -2xF - x^2\partial_x[F] + \partial_x[Fx]x, \end{aligned}$$

from which it follows that $[B,x|][F] = -2xF$.

(*iv*) If $v \in R_1(S)$ then we can write $v = \sum_{j=1}^s f_j x_j$ where $f_j \in R_0(S)$ and $x_j \in S$. Both equalities then directly follow from the linearity of B and properties (*ii*)-(*iii*).

(*v*) By means of Lemma 2.6 and Theorem 2.4 we obtain for $B = \{\partial_z, z|\}$ that

$$\begin{aligned} [D_{y,x}, B] &= [D_{y,x}, \partial_z\, z|] + [D_{y,x}, z|\,\partial_z] \\ &= [D_{y,x}, \partial_z]\, z| + \partial_z[D_{y,x}, z|] + [D_{y,x}, z|]\partial_z + z|\,[D_{y,x}, \partial_z] \\ &= -\delta_{y,z}\partial_x\, z| + \delta_{x,z}\partial_z\, y| + \delta_{x,z} y|\,\partial_z - \delta_{y,z} z|\,\partial_x \\ &= \delta_{x,z}\{\partial_z, y|\} - \delta_{z,y}\{\partial_x, z|\} \end{aligned}$$

which equals zero for any $z \in S$, on account of Theorem 2.5.

(*vi*) Consider F in its canonical decomposition with respect to x. Then by (*ii*) and (*iv*) we have

$$\begin{aligned} B[\partial_x F] &= B[\partial_x[f_1]F_1] + B[\partial_x[f_2]xF_2] + f_2\mathfrak{m}B[F_2] \\ &= \partial_x[f_1]B[F_1] - 2\widetilde{F_1}\partial_x[f_1] \\ &\quad + \partial_x[f_2]xB[F_2] - 2\partial_x[f_2]\widetilde{F_2}x + 2x\widetilde{F_2}\partial_x[f_2] + f_2\mathfrak{m}B[F_2]. \end{aligned}$$

Also, in view of property (*v*) and since $F_j \in R(S \setminus \{x\})$ we get

$$E_x[B[F_j]] = B[E_x[F_j]] = 0,$$

whence $B[F_j] \in R(S \setminus \{x\})$ and

$$\begin{aligned} \partial_x[B[F]] &= \partial_x\left[f_1 B[F_1] + f_2 B[xF_2]\right] = \partial_x\left[f_1 B[F_1] + f_2 xB[F_2] - 2f_2\widetilde{F_2}x\right] \\ &= \partial_x[f_1]B[F_1] + \partial_x[f_2]xB[F_2] + f_2\mathfrak{m}B[F_2] - 2\partial_x[f_2]\widetilde{F_2}x - 2f_2\partial_x[\widetilde{F_2}x]. \end{aligned}$$

It then follows that

$$[B,\partial_x][F] = -2\widetilde{F_1}\partial_x[f_1] + 2x\widetilde{F_2}\partial_x[f_2] + 2f_2\partial_x[\widetilde{F_2}x],$$

while on the other hand,

$$-2\partial_x|[F] = -2[\widetilde{F}]\partial_x = -2\left(\widetilde{F_1}[f_1]\partial_x - x\widetilde{F_2}[f_2]\partial_x - f_2[x\widetilde{F_2}]\partial_x\right).$$

But, since $F_2 \in R(S \setminus \{x\})$ we obtain:

$$\partial_x[\widetilde{F_2}x] = \partial_x[\widetilde{F_2}x] - \partial_x[\widetilde{F_2}]x = B[F_2] = [x\widetilde{F_2}]\partial_x - x[\widetilde{F_2}]\partial_x = [x\widetilde{F_2}]\partial_x,$$

which proves the first equality. The other equality can be proven in a similar way using the (right) canonical decomposition of F with respect to x, i.e. $F = f_1F_1 + f_2F_2x$. □

These properties allow us to conclude that the second order operator $\{\partial_x, \partial_y\}$ behaves, as expected, as a scalar operator. In fact, it is mapped by the Clifford polynomial representation to $-2\sum_{j=1}^m \partial_{x_j}\partial_{y_j}$. The next result illustrates the scalar behavior of $\{\partial_x, \partial_y\}$ in the radial algebra.

Theorem 2.7. *Let $x, y, z \in S$ and $F \in R(S)$. Then*

(i) $[\{\partial_x, \partial_y\}, z] = 2(\delta_{x,z}\partial_y + \delta_{y,z}\partial_x)$;

(ii) $\overline{\{\partial_x, \partial_y\}[F]} = \{\partial_x, \partial_y\}[\overline{F}]$;

(iii) $\{\partial_x, \partial_y\} = \cdot\{\partial_x, \partial_y\}$.

Proof.

(i) On account of Theorems 2.3 and 2.4 we obtain that:

$$\begin{aligned}
\{\partial_x, \partial_y\}z &= \partial_x\partial_y z + \partial_y\partial_x z \\
&= \partial_x(-z\partial_y + 2D_{z,y} + \delta_{y,z}\mathfrak{m}) + \partial_y(-z\partial_x + 2D_{z,x} + \delta_{x,z}\mathfrak{m}) \\
&= -\partial_x z\partial_y - \partial_y z\partial_x + 2\partial_x D_{z,y} + \delta_{y,z}\mathfrak{m}\partial_x + 2\partial_y D_{z,x} + \delta_{x,z}\mathfrak{m}\partial_y \\
&= z\{\partial_x, \partial_y\} - 2[D_{z,x}, \partial_y] - 2[D_{z,y}, \partial_x] \\
&= z\{\partial_x, \partial_y\} + 2(\delta_{x,z}\partial_y + \delta_{y,z}\partial_x).
\end{aligned}$$

(ii) It suffices to prove this property for products of the form $x_{j_1}\cdots x_{j_k}$ with $x_{j_\ell} \in S$; this is done by induction on the number of vectors. The case $k = 1$ is trivial. Now, suppose that (ii) holds for $F \in R(S)$ (F being a product of $k-1$ vectors) then we have to prove that

$$\overline{\{\partial_x, \partial_y\}[zF]} = \{\partial_x, \partial_y\}[\overline{zF}]$$

for every $z \in S$. From (i) it follows that

$$\overline{\{\partial_x, \partial_y\}[zF]} = -\{\partial_x, \partial_y\}[\overline{F}]z - 2\delta_{x,z}[\overline{F}]\partial_y - 2\delta_{y,z}[\overline{F}]\partial_x.$$

Theorem 2.5 now yields that $\{\partial_x, z|\} = \delta_{x,z}B$ which implies

$$\partial_x[Fz] = \partial_x[F]z + \delta_{x,z}B[\widetilde{F}]$$

for every $x, z \in S$, $F \in R(S)$. Then, using Theorem 2.6 (vi), we obtain

$$\begin{aligned}
\{\partial_x, \partial_y\}[\overline{zF}] &= -\left(\partial_x\left[\partial_y[\overline{F}z]\right] + \partial_y\left[\partial_x[\overline{F}z]\right]\right)\\
&= -\partial_x\left[\partial_y[\overline{F}]z + \delta_{y,z}B[\widetilde{\overline{F}}]\right] - \partial_y\left[\partial_x[\overline{F}]z + \delta_{x,z}B[\widetilde{\overline{F}}]\right]\\
&= -\{\partial_x, \partial_y\}[\overline{F}]z + \delta_{x,z}\left[B, \partial_y\right][\widetilde{\overline{F}}] + \delta_{y,z}\left[B, \partial_x\right][\widetilde{\overline{F}}]\\
&= -\{\partial_x, \partial_y\}[\overline{F}]z - 2\delta_{x,z}[\overline{F}]\partial_y - 2\delta_{y,z}[\overline{F}]\partial_x\\
&= \overline{\{\partial_x, \partial_y\}[zF]}.
\end{aligned}$$

(iii) This is a direct consequence of Lemma 2.7 and the previous property. □

Thus, the endomorphisms $\{\partial_x, \partial_x| : x \in S\}$ generate a subalgebra of $\mathrm{End}(R(S))$, subject to the rules

$$\{\partial_x, \partial_y|\} = 0, \qquad \{\partial_x, \partial_y\} = -\{\partial_x|, \partial_y|\}.$$

Crucial in harmonic analysis is the appearance of the Lie algebra $\mathfrak{sl}(2)$ generated by the Laplace operator and the norm of the vector variable. Similarly, in Euclidean Clifford analysis one obtains a representation of the Lie superalgebra $\mathfrak{osp}(1|2)$ with odd generators given by the Dirac operator and the vector variable. Both results can be obtained in the radial algebra level as a consequence of Theorems 2.3, 2.4, 2.7 and Lemma 2.6.

Proposition 2.1. *Let $x \in S$, then the operators $x^2, \partial_x^2 \in \mathrm{End}(R(S))$ generate the Lie algebra $\mathfrak{sl}(2)$. In particular, the representation $\mathfrak{sl}(2) \subset \mathrm{End}(R(S))$ is given by the correspondences*

$$\begin{aligned}
H &= \frac{1}{2}\left(E_x + \frac{\mathfrak{m}}{2}\right),\\
E^+ &= \frac{x^2}{2},\\
E^- &= -\frac{\partial_x^2}{2},
\end{aligned}$$

where

$$\begin{aligned}
[E^+, E^-] &= 2H,\\
[H\ \ , E^{\pm}] &= \pm E^{\pm}.
\end{aligned}$$

Proposition 2.2. *Let $x \in S$, then the operators $x, \partial_x \in \mathrm{End}(R(S))$ are odd generators of the Lie superalgebra $\mathfrak{osp}(1|2)$. In particular, the representation $\mathfrak{osp}(1|2) \subset \mathrm{End}(R(S))$ is given by the correspondences*

$$H = \frac{1}{2}\left(\mathbb{E}_x + \frac{\mathfrak{m}}{2}\right), \qquad F^+ = \frac{x}{2\sqrt{2}},$$
$$E^+ = \frac{x^2}{2}, \qquad F^- = \frac{\partial_x}{2\sqrt{2}},$$
$$E^- = -\frac{\partial_x^2}{2},$$

where

$$[E^+, E^-] = 2H, \qquad [H \ , F^\pm] = \pm\frac{1}{2}F^\pm, \qquad \{F^\pm, F^\pm\} = \pm\frac{1}{2}E^\pm,$$
$$[H \ , E^\pm] = \pm E^\pm, \qquad [E^\pm, F^\mp] = -F^\pm, \qquad \{F^+, F^-\} = \frac{1}{2}H,$$
$$[E^\pm, F^\pm] = 0.$$

2.4 Endomorphisms on $R(S \cup S|, B)$

From the observations above on the endomorphisms of $R(S)$ it follows that

$$R(S) \subset R(S \cup S|) \subset R(S \cup S|, B) \subset \mathrm{End}(R(S)),$$

where $R(S \cup S|, B)$ is the algebra generated by the set of endomorphisms $S \cup S| \cup \{B\}$. Making use of the isomorphism $\mathbb{R}_{m,m} \cong \mathrm{End}(\mathbb{R}_{0,m})$ given by the relations (2.8), one obtains that $R(S \cup S|, B)$ is mapped by the Clifford-polynomial representation to the algebra $R(\underline{S} \cup \underline{S}|, \underline{B})$ generated by the bivector $\underline{B} = -\sum_{j=1}^m e_j \varepsilon_j$ and Clifford vector variables of the form

$$\underline{x} = \sum_{j=1}^m x_j e_j, \qquad \underline{x}| = \sum_{j=1}^m x_j \varepsilon_j.$$

The representation $R(\underline{S} \cup \underline{S}|, \underline{B})$ is included in the algebra of $\mathbb{R}_{m,m}$-valued polynomials and naturally admits the action of the operators $D_{\underline{y},\underline{x}}$, $\partial_{\underline{x}}$, $\partial_{\underline{x}}|$ defined in (2.9), (2.10) and (2.13), respectively.

In this section we establish the same consistent extensions on the radial algebra level. These results will used in relation with a complex structure in Section 2.6 and Section 3.3. We extend the definitions of $D_{y,x}, \partial_x, \partial_x|$ from the initial radial algebra $R(S)$ to $R(S \cup S|, B)$ preserving the meaning of $D_{y,x}$ as a directional derivative and of ∂_x and $\partial_x|$ as vector derivatives. This means, for example, that we have to redefine $\partial_x|$ on $R(S \cup S|, B)$ such that it behaves as the abstract equivalent of $-\sum_{j=1}^m \varepsilon_j \partial_{x_j}$ acting on $\mathbb{R}_{m,m}$-valued functions. Clearly this implies that we cannot simply apply the canonical embedding of $\mathrm{End}(R(S))$ in $\mathrm{End}(\mathrm{End}(R(S)))$ by means of the composition of endomorphisms.

As stated in the previous section, we can embed $R(S \cup S|, B)$ in $\mathrm{End}(R(S))$ by means of left or right representations of the operators $x, x|$. Since both representations are isomorphic, we will continue to consider only the left one.

2.4.1 Extension of the directional derivative

First we will extend the definition of $D_{y,x}$ for $x, y \in S$ to $R(S \cup S|)$ assuming the axiom **(DD1)** valid for every pair $F, G \in R(S \cup S|)$ and extending **(DD2)** as follows

$$\textbf{(DD2)} \quad D_{y,x}[z] = \delta_{x,z}\, y, \quad D_{y,x}[z|] = \delta_{x,z}\, y|, \qquad z \in S.$$

Then it is easy to prove the following result.

Lemma 2.8. *The extension $D_{y,x} : R(S \cup S|) \to \mathrm{End}(R(S))$ satisfies $D_{y,x}[A] = [D_{y,x}, A]$, for all $A \in R(S \cup S|)$.*

Proof.

In view of Lemma 2.6 we have

$$[D_{y,x}, z] = \delta_{x,z}\, y = D_{y,x}[z], \qquad [D_{y,x}, z|] = \delta_{x,z}\, y| = D_{y,x}[z|].$$

The result then follows by induction since for every pair $A, C \in R(S \cup S|)$ it holds that

$$[D_{y,x}, AC] = [D_{y,x}, A]\, C + A\, [D_{y,x}, C]\, .$$

□

This property allows us to extend the notion of directional derivative to the whole space $\mathrm{End}(R(S))$ by $D_{y,x}[A] := [D_{y,x}, A]$ for all $A \in \mathrm{End}(R(S))$. Properties of this extension are:

- $D_{y,x}$ maps $R(S)$ into $R(S)$ and $R(S|)$ into $R(S|)$;
- $D_{y,x}[\partial_z] = [D_{y,x}, \partial_z] = -\delta_{y,z}\partial_x$;
- $D_{y,x}[B] = [D_{y,x}, B] = 0$.

2.4.2 Extension of the vector derivative

In order to define a consistent action of the vector derivatives on $R(S \cup S|)$ we have to keep in mind that every element of $R(S \cup S|)$ acts on $R(S)$ basically as a multiplication.

In Clifford analysis the action of the Dirac operator on a product of Clifford-valued functions is given by

$$\partial_{\underline{x}}[AF] = \partial_{\underline{x}}[A]F + \mathring{\partial}_{\underline{x}}[A\mathring{F}],$$

where we used the so-called Hestenes overdot notation:

$$\mathring{\partial}_{\underline{x}}[A\mathring{F}] = \sum_{j=0}^{m} e_j A\, \partial_{x_j}[F].$$

This inspires us for the abstract definition of ∂_x and $\partial_x|$ on $R(S \cup S|)$.

Definition 2.8. *The operators $\partial_x \cdot, \partial_x| \cdot : R(S \cup S|) \to End(R(S))$ are defined as follows*

$$\partial_x(A)[F] = \partial_x\,[A[F]] - \mathring{\partial}_x[A[\mathring{F}]], \qquad \forall A \in R(S \cup S|),\ \forall F \in R(S),$$

$$\partial_x|(A)[F] = \partial_x|\,[A[F]] - \mathring{\partial}_x|[A[\mathring{F}]], \qquad \forall A \in R(S \cup S|),\ \forall F \in R(S),$$

where the abstract overdot action is defined recursively by

(OD1) $\mathring{\partial}_x[1[\mathring{F}]] = \partial_x[F]$

(OD|1) $\mathring{\partial}_x|[1[\mathring{F}]] = \partial_x|[F]$

(OD2) $\mathring{\partial}_x[yA[\mathring{F}]] = -y\mathring{\partial}_x[A[\mathring{F}]] + 2A[D_{y,x}[F]]$

(OD|2) $\mathring{\partial}_x|[yA[\mathring{F}]] = -y\mathring{\partial}_x|[A[\mathring{F}]]$

(OD3) $\mathring{\partial}_x[y|A[\mathring{F}]] = -y|\mathring{\partial}_x[A[\mathring{F}]]$

(OD|3) $\mathring{\partial}_x|[y|A[\mathring{F}]] = -y|\mathring{\partial}_x|[A[\mathring{F}]] - 2A[D_{y,x}[F]].$

Remark 2.12. *The defining rules* **(OD1)**- **(OD|3)** *for the abstract overdot action are clearly inspired by the corresponding similar actions in the Clifford setting. For instance, the relation* **(OD2)** *corresponds to the following property in the Clifford representation.*

$$\begin{aligned}
-\mathring{\partial}_{\underline{x}}[\underline{y}A\mathring{F}] &= -\sum_{j=1}^{m} e_j \underline{y} A \partial_{x_j}[F] \\
&= -\sum_{j=1}^{m} (-\underline{y}e_j - 2y_j) A \partial_{x_j}[F] \\
&= \underline{y}\sum_{j=1}^{m} e_j A \partial_{x_j}[F] + 2A\sum_{j=1}^{m} y_j \partial_{x_j}[F] \\
&= -\underline{y}\left(-\mathring{\partial}_{\underline{x}}[A\mathring{F}]\right) + 2AD_{\underline{y},\underline{x}}[F].
\end{aligned}$$

It now is easy to derive the recursive formulae for ∂_x and $\partial_x|$.

Theorem 2.8. *Let $A \in R(S \cup S|)$ then*

(i) $\partial_x[yA] = -y\partial_x[A] + 2D_{y,x}[A] + \delta_{x,y}\mathfrak{m}A$;

(ii) $\partial_x[y|A] = -y|\partial_x[A] + \delta_{x,y}BA$;

(iii) $\partial_x|[yA] = -y\partial_x|[A] - \delta_{x,y}BA$;

(iv) $\partial_x|[y|A] = -y|\partial_x|[A] - 2D_{y,x}[A] - \delta_{x,y}\mathfrak{m}A$.

Proof.

By Theorems 2.3 and 2.5 and Lemma 2.8 we obtain for every $F \in R(S)$ that

$$\begin{aligned}\partial_x(yA)[F] &= \partial_x\left[yA[F]\right] - \overset{\circ}{\partial}_x[yA[\overset{\circ}{F}]]\\ &= -y\partial_x[A[F]] + 2D_{y,x}[A[F]] + \delta_{x,y}\mathfrak{m}A[F] + y\overset{\circ}{\partial}_x[A[\overset{\circ}{F}]] - 2A[D_{y,x}[F]]\\ &= -y(\partial_x[A[F]] - \overset{\circ}{\partial}_x[A[\overset{\circ}{F}]]) + 2\left[D_{y,x}, A\right][F] + \delta_{x,y}\mathfrak{m}A[F]\\ &= (-y\partial_x[A] + 2D_{y,x}[A] + \delta_{x,y}\mathfrak{m}A)[F],\end{aligned}$$

and also

$$\begin{aligned}\partial_x(y|A)[F] &= \partial_x\left[y|A[F]\right] - \overset{\circ}{\partial}_x[y|A[\overset{\circ}{F}]]\\ &= -y|\partial_x[A[F]] + \delta_{x,y}B[A[F]] + y|\overset{\circ}{\partial}_x[A[\overset{\circ}{F}]]\\ &= -y|\partial_x(A)[F] + \delta_{x,y}B[A[F]].\end{aligned}$$

Moreover,

$$\begin{aligned}\partial_x|(yA)[F] &= \partial_x|\left[yA[F]\right] - \overset{\circ}{\partial}_x|[yA[\overset{\circ}{F}]]\\ &= -y\partial_x|[A[F]] - \delta_{x,y}B\left[A[F]\right] + y\overset{\circ}{\partial}_x|[A[\overset{\circ}{F}]]\\ &= -y\partial_x|(A)[F] - \delta_{x,y}B[A[F]],\end{aligned}$$

and

$$\begin{aligned}\partial_x|(y|A)[F] &= \partial_x|\left[y|A[F]\right] - \overset{\circ}{\partial}_x|[y|A[\overset{\circ}{F}]]\\ &= -y|\partial_x|[A[F]] - 2D_{y,x}[A[F]] - \delta_{x,y}\mathfrak{m}A[F] + y|\overset{\circ}{\partial}_x|[A[\overset{\circ}{F}]] + 2A[D_{y,x}[F]]\\ &= -y|(\partial_x|[A[F]] - \overset{\circ}{\partial}_x|[A[\overset{\circ}{F}]]) - 2\left[D_{y,x}, A\right][F] - \delta_{x,y}\mathfrak{m}A[F]\\ &= (-y|\partial_x|[A] - 2D_{y,x}[A] - \delta_{x,y}\mathfrak{m}A)[F].\end{aligned}$$

□

These recursive relations thus provide an alternative definition of the left operators ∂_x and $\partial_x|$ on $R(S \cup S|)$. A similar approach is possible for the right operators $\cdot\partial_x$, and $\cdot\partial_x|$, which can be defined by $[1]\partial_x = 0 = [1]\partial_x|$ and

$$\begin{aligned} [Ay]\partial_x &= -[A]\partial_x\, y + 2D_{y,x}[A] + \delta_{x,y}\mathfrak{m}A; \\ [Ay|]\partial_x &= -[A]\partial_x\, y| - \delta_{x,y}AB; \\ [Ay]\partial_x| &= -[A]\partial_x|\, y + \delta_{x,y}AB; \\ [Ay|]\partial_x| &= -[A]\partial_x|\, y| - 2D_{y,x}[A] - \delta_{x,y}\mathfrak{m}A. \end{aligned} \tag{2.14}$$

Remark 2.13. *As for the left actions, the recursive relations (2.14) for the right actions of the operators ∂_x and $\partial_x|$ are inspired by similar properties that hold in the Clifford setting. For example, the last relation in (2.14) corresponds to the following Clifford property:*

$$\begin{aligned} -[A\underline{y}|]\partial_{\underline{x}}| &= -\sum_{j=1}^{m} \partial_{x_j}[A\underline{y}|]\varepsilon_j \\ &= -\sum_{j=1}^{m} \Big(\partial_{x_j}[A]\,\underline{y}|\,\varepsilon_j + \delta_{\underline{x},\underline{y}}A\Big) \\ &= -\sum_{j=1}^{m} \partial_{x_j}[A]\left(-\varepsilon_j\,\underline{y}| + 2y_j\right) - \delta_{\underline{x},\underline{y}}mA \\ &= -\left(-[A]\partial_{\underline{x}}|\right)\underline{y}| - 2D_{\underline{y},\underline{x}}[A] - \delta_{\underline{x},\underline{y}}mA. \end{aligned}$$

Taking into account also the mapping properties of $D_{y,x}$ it can easily be verified that:

- $\partial_x : R(S) \mapsto R(S)$;
- $\partial_x| : R(S|) \mapsto R(S|)$;
- $\partial_x, \partial_x| : R(S \cup S|) \mapsto R(S \cup S|, B)$.

We may thus extend the definition of ∂_x and $\partial_x|$ to $R(S\cup S|, B)$. To this end, we first need to define $\partial_x[B], \partial_x|[B]$ and then to find the corresponding recursion formulae involving B. In order to evaluate the action of the vector derivatives on B we will extend their domain of definition.

Definition 2.9. [operator independent of x] *An operator $A \in End(R(S))$ is called independent of $x \in S$ if*

$$E_x[A] \equiv [E_x, A] = 0.$$

The subalgebra of endomorphisms independent of x is denoted by I_x.

For an $\mathbb{R}_{m,m}$-valued polynomial A, independent of the vector variable $\underline{x}$ in the Clifford-polynomial representation, we have

$$\partial_{\underline{x}}[AF] = \sum_{j=0}^{m} e_j \partial_{x_j}[AF] = \sum_{j=0}^{m} e_j A\, \partial_{x_j}[F] = \overset{\circ}{\partial}_{\underline{x}}[A\overset{\circ}{F}],$$

whence it seems logical to define on the abstract radial algebra level the overdot action of $\partial_x, \partial_x|$ on elements of I_x in the canonical way by means of composition, meaning that

$$\mathring{\partial}_x[A[\mathring{F}]] := \partial_x[A[F]], \qquad \mathring{\partial}_x|[A[\mathring{F}]] := \partial_x|\,[A[F]] \;\text{ for all }\; A \in I_x.$$

As a consequence, Definition 2.8 gives,

$$\partial_x[A] = \partial_x|[A] = 0, \qquad \text{for all } A \in I_x.$$

Since Theorem 2.6 (v) states that $B \in I_x$ for all $x \in S$, it follows that $\partial_x[B] = \partial_x|[B] = 0$.

We now are able to define ∂_x and $\partial_x|$ on $R(S \cup S|, B)$ using Definition 2.8 for every $A \in R(S \cup S|, B)$ and extending the definition of the overdot action to $R(S \cup S|, B)$. Additionally to the axioms **(OD1)-(OD3)**, **(OD|1)-(OD|3)**, two new axioms are needed, prescribing the corresponding recursion relations involving B:

(OD4) $\qquad \mathring{\partial}_x[BA[\mathring{F}]] = B\mathring{\partial}_x[A[\mathring{F}]] + 2\mathring{\partial}_x|[A[\mathring{F}]],$

(OD|4) $\qquad \mathring{\partial}_x|[BA[\mathring{F}]] = B\mathring{\partial}_x|[A[\mathring{F}]] + 2\mathring{\partial}_x[A[\mathring{F}]].$

Then the results of Theorem 2.8 remain valid, together with the new relations

$$\partial_x[BA] = B\partial_x[A] + 2\partial_x|[A], \qquad \partial_x|[BA] = B\partial_x|[A] + 2\partial_x[A]. \tag{2.15}$$

Indeed, in view of Theorem 2.6, one easily obtains, for every $F \in R(S)$, that

$$\begin{aligned}
\partial_x(BA)[F] &= \partial_x[BA[F]] - \mathring{\partial}_x[BA[\mathring{F}]] \\
&= B\partial_x[A[F]] + 2\partial_x|[A[F]] - \left(B\mathring{\partial}_x[A[\mathring{F}]] + 2\mathring{\partial}_x|[A[\mathring{F}]]\right) \\
&= B\left(\partial_x[A[F]] - \mathring{\partial}_x[A[\mathring{F}]]\right) + 2\left(\partial_x|[A[F]] - \mathring{\partial}_x|[A[\mathring{F}]]\right) \\
&= \left(B\partial_x[A] + 2\partial_x|[A]\right)[F],
\end{aligned}$$

and also,

$$\begin{aligned}
\partial_x|(BA)[F] &= \partial_x|[BA[F]] - \mathring{\partial}_x|[BA[\mathring{F}]] \\
&= B\partial_x|[A[F]] + 2\partial_x[A[F]] - \left(B\mathring{\partial}_x|[A[\mathring{F}]] + 2\mathring{\partial}_x[A[\mathring{F}]]\right) \\
&= B\left(\partial_x|[A[F]] - \mathring{\partial}_x|[A[\mathring{F}]]\right) + 2\left(\partial_x[A[F]] - \mathring{\partial}_x[A[\mathring{F}]]\right) \\
&= \left(B\partial_x|[A] + 2\partial_x[A]\right)[F].
\end{aligned}$$

Again we can define the right versions $\cdot\partial_x, \cdot\partial_x|$ on $R(S \cup S|, B)$ in a similar way, where

$$[AB]\partial_x = [A]\partial_x\, B - 2[A]\partial_x| \qquad [AB]\partial_x| = [A]\partial_x|\, B - 2[A]\partial_x.$$

Remark 2.14. *As in the previous definitions, the above relations are inspired by similar properties in the Clifford setting. In particular, one has*

$$-[A\underline{B}]\partial_{\underline{x}} = -\sum_{j=1}^{m} \partial_{x_j}[A]\underline{B}\, e_j = -\sum_{j=1}^{m} \partial_{x_j}[A]\,(e_j\underline{B} - 2\varepsilon_j) = \left(-[A]\partial_{\underline{x}}\right)\underline{B} - 2\left(-[A]\partial_{\underline{x}}|\right).$$

Indeed, it suffices to observe that $\underline{B} = -\sum_{j=1}^{m} e_j\varepsilon_j$ *and in consequence,* $[\underline{B}, e_j] = -2\varepsilon_j$.

Following the previous recursive approach it now is possible to prove similar properties to **(D1)-(D4)**.

Theorem 2.9. *The algebra* $R(S\cup S|, B)$ *is subject to* **(A1|)** *and moreover*

(A2|) $\{x,y\}$ *is a central element in* $R(S\cup S|, B)$, $\quad \forall x,y\in S$.

(A3|) $[B,x] = -2x|, \quad [B,x|] = -2x, \quad \forall x\in S$.

In addition, $\partial_x, \partial_x| \in End(R(S\cup S|, B))$ *are such that*

(D1|) $\partial_x[fF] = \partial_x[f]F + f\partial_x[F], \quad \partial_x|[fF] = \partial_x|[f]F + f\partial_x|[F],$
$[fF]\partial_x = F[f]\partial_x + f[F]\partial_x, \quad [fF]\partial_x| = F[f]\partial_x| + f[F]\partial_x|,$
$f\in R_0(S\cup S|) := \mathrm{Alg}_{\mathbb{R}}\{\{x,y\} : x,y\in S\cup S|\} = R_0(S), \quad F\in R(S\cup S|, B),$

(D2|) $\partial_x[FG] = \partial_x[F]G, \quad [GF]\partial_x = G[F]\partial_x, \quad G\in R\big((S\setminus\{x\})\cup(S|\setminus\{x|\}), B\big),$
$\partial_x|[FG] = \partial_x|[F]G, \quad [GF]\partial_x| = G[F]\partial_x|,$

(D3|) $\partial_x[x] = -\partial_x|[x|] = [x]\partial_x = -[x|]\partial_x| = \mathfrak{m},$
$\partial_x[x|] = -\partial_x|[x] = B = -[x|]\partial_x = [x]\partial_x|,$

(D4|) $\partial_x[x^2] = [x^2]\partial_x = 2x, \quad \partial_x|[x|^2] = [x|^2]\partial_x| = -2x|, \quad x\in S,$
$\partial_x\{x,y\} = \{x,y\}\partial_x = 2y, \quad \partial_x|\{x|,y|\} = \{x|,y|\}\partial_x| = -2y|, \quad x\neq y.$

(D5|) $[B,\partial_x] = -2\partial_x|, \quad [B,\partial_x|] = -2\partial_x.$

The operator ∂_x *thus is the vector derivative in* $R(S)$ *and* $\partial_x|$ *is the vector derivative in* $R(S|)$.

Proof.

Properties **(A1|)** and **(A3|)** were previously obtained, as is the case for the commutation of $\{x,y\}$ with every element of $S\cup S|$, meaning that for **(A2|)** we only still have to check the commutation of $\{x,y\}$ with B:

$$[B,\{x,y\}] = [B,x]y + x[B,y] + [B,y]x + y[B,x] = -2\{x|,y\} - 2\{x,y|\} = 0.$$

Let us now compute the basic evaluations, starting with **(D3|)-(D4|)**:

- $\partial_x[x] = -\partial_x|[x|] = [x]\partial_x = -[x|]\partial_x| = \mathfrak{m}$.

  Observe that by **(OD2)** we have $\mathring{\partial}_x[x\mathring{F}] = -x\partial_x[F] + 2E_x[F]$, whence Theorem 2.3 yields

$$\partial_x(x)[F] = \partial_x[xF] - \mathring{\partial}_x[x\mathring{F}] = -x\partial_x[F] + 2E_x[F] + \mathfrak{m}F - \mathring{\partial}_x[x\mathring{F}] = \mathfrak{m}F,$$

  implying that $\partial_x[x] = \mathfrak{m}$. In a similar way $[x]\partial_x = -\partial_x|[x|] = -[x|]\partial_x| = \mathfrak{m}$ are proven.

- $\partial_x[x^2] = [x^2]\partial_x = 2x, \quad \partial_x|[x|^2] = [x|^2]\partial_x| = -2x|$.

  It is easy to check that $\mathring{\partial}_x[x^2\mathring{F}] = -x\mathring{\partial}_x[x\mathring{F}] + 2xE_x[F] = x^2\partial_x[F]$. Then

$$\partial_x(x^2)[F] = \partial_x[x^2F] - \mathring{\partial}_x[x^2\mathring{F}] = 2xF + x^2\partial_x[F] - x^2\partial_x[F] = 2xF$$

  implying that $\partial_x[x^2] = 2x$, while for the right action

$$[x^2]\partial_x = -[x]\partial_x\, x + 2E_x[x] + \mathfrak{m}x = 2x.$$

  The corresponding properties for $\partial_x|$ are proven in a similar way.

- $\partial_x\{x,y\} = \{x,y\}\partial_x = 2y, \quad \partial_x|\{x|,y|\} = \{x|,y|\}\partial_x| = -2y|, \quad x \neq y$.

  In this case, it holds that

$$\begin{aligned}\mathring{\partial}_x[\{x,y\}\mathring{F}] &= \mathring{\partial}_x[xy\mathring{F}] + \mathring{\partial}_x[yx\mathring{F}]\\ &= -x\mathring{\partial}_x[y\mathring{F}] + 2yE_x[F] - y\mathring{\partial}_x[x\mathring{F}] + 2xD_{y,x}[F]\\ &= x\left(y\partial_x[F] - 2D_{y,x}[F]\right) + 2yE_x[F] + y\left(x\partial_x[F] - 2E_x[F]\right) + 2xD_{y,x}[F]\\ &= \{x,y\}\partial_x[F],\end{aligned}$$

  whence

$$\partial_x(\{x,y\})[F] = \partial_x\left[\{x,y\}F\right] - \mathring{\partial}_x[\{x,y\}\mathring{F}] = 2yF,$$

  implying that $\partial_x\{x,y\} = 2y$. On the other hand,

$$\{x,y\}\partial_x = [xy]\partial_x + [yx]\partial_x = -[x]\partial_x\, y + 2D_{y,x}[x] - [y]\partial_x\, x + 2E_x[y] + \mathfrak{m}y = 2y.$$

  Again, the proof of the corresponding properties for $\partial_x|$ runs along similar lines.

- $\partial_x[x|] = -\partial_x|[x] = B = -[x|]\partial_x = [x]\partial_x|$

  On account of Theorem 2.5 we have

$$\partial_x(x|)[F] = \partial_x[x|F] - \mathring{\partial}_x[x|\mathring{F}] = \partial_x[x|F] + x|\,\partial_x[F] = \{\partial_x, x|\}[F] = B[F],$$

$$\partial_x|(x)[F] = \partial_x|[xF] - \mathring{\partial}_x|[x\mathring{F}] = \partial_x|[xF] + x\,\partial_x|[F] = \{\partial_x|, x\}[F] = -B[F].$$

  On the other hand $[x|]\partial_x = -[1]\partial_x\, x| - B = -B$, and $[x]\partial_x| = -[1]\partial_x|\, x + B = B$.

Next we check **(D1|)**.

In view of **(A1|)** it holds that $R_0(S \cup S|) = R_0(S)$. It now suffices to prove the property for the generators of $R_0(S)$, since, if $f, g \in R_0(S)$ satisfy **(D1|)** for every $F \in R(S \cup S|)$, we have

$$\begin{aligned}
\partial_x[fgF] &= (\partial_x[f]g + f\partial_x[g])\,F + fg\partial_x[F] = \partial_x[fg]F + fg\partial_x[F],\\
[fgF]\partial_x &= F\,(g\,[f]\partial_x + f\,[g]\partial_x) + fg\,[F]\partial_x = F\,[fg]\partial_x + fg\,[F]\partial_x,\\
\partial_x|[fgF] &= (\partial_x|[f]g + f\partial_x|[g])\,F + fg\partial_x|[F] = \partial_x|[fg]F + fg\partial_x|[F],\\
[fgF]\partial_x| &= F\,(g\,[f]\partial_x| + f\,[g]\partial_x|) + fg\,[F]\partial_x| = F\,[fg]\partial_x| + fg\,[F]\partial_x|,
\end{aligned}$$

whence also fg satisfies **(D1|)**.

Now, invoking Theorem 2.8, we obtain for every pair $y, z \in S$ and $F \in R(S \cup S|)$ that:

$$\begin{aligned}
\partial_x[\{y,z\}F] &= \partial_x[yzF] + \partial_x[zyF]\\
&= -y\partial_x[zF] + 2D_{y,x}[zF] + \delta_{x,y}\mathfrak{m}zF - z\partial_x[yF] + 2D_{z,x}[yF] + \delta_{x,z}\mathfrak{m}yF\\
&= -y\,(-z\partial_x[F] + 2D_{z,x}[F] + \delta_{x,z}\mathfrak{m}F) + 2zD_{y,x}[F] + 2\delta_{x,z}yF + \delta_{x,y}\mathfrak{m}zF\\
&\quad - z\,(-y\partial_x[F] + 2D_{y,x}[F] + \delta_{x,y}\mathfrak{m}F) + 2yD_{z,x}[F] + 2\delta_{x,y}zF + \delta_{x,z}\mathfrak{m}yF\\
&= \{y,z\}\partial_x[F] + 2\,(\delta_{x,z}y + \delta_{x,y}z)\,F\\
&= \{y,z\}\partial_x[F] + \partial_x[\{y,z\}]\,F,
\end{aligned}$$

and also, in view of **(A3|)**,

$$\begin{aligned}
\partial_x|[\{y,z\}F] &= \partial_x|[yzF] + \partial_x|[zyF]\\
&= -y\partial_x|[zF] - \delta_{x,y}BzF - z\partial_x|[yF] - \delta_{x,z}ByF\\
&= -y\,(-z\partial_x|[F] - \delta_{x,z}BF) - \delta_{x,y}BzF - z\,(-y\partial_x|[F] - \delta_{x,y}BF) - \delta_{x,z}ByF\\
&= \{y,z\}\partial_x|[F] - \delta_{x,z}[B,y]F - \delta_{x,y}[B,z]F\\
&= \{y,z\}\partial_x|[F] + 2\,(\delta_{x,z}y| + \delta_{x,y}z|)\,F\\
&= \{y,z\}\partial_x|[F] + \partial_x|[\{y,z\}]\,F.
\end{aligned}$$

In the same way, using now (2.14), we get

$$\begin{aligned}
[\{y,z\}F]\partial_x &= [Fyz]\partial_x + [Fzy]\partial_x\\
&= -[Fy]\partial_x\,z + 2D_{z,x}[Fy] + \delta_{z,x}\mathfrak{m}Fy - [Fz]\partial_x\,y + 2D_{y,x}[Fz] + \delta_{y,x}\mathfrak{m}Fz\\
&= -\,(-[F]\partial_x\,y + 2D_{y,x}[F] + \delta_{y,x}\mathfrak{m}F)\,z + 2D_{z,x}[F]y + 2\delta_{x,y}Fz + \delta_{z,x}\mathfrak{m}Fy\\
&\quad -\,(-[F]\partial_x\,z + 2D_{z,x}[F] + \delta_{z,x}\mathfrak{m}F)\,y + 2D_{y,x}[F]z + 2\delta_{x,z}Fy + \delta_{y,x}\mathfrak{m}Fz\\
&= [F]\partial_x\,\{y,z\} + 2F\,(\delta_{x,y}z + \delta_{x,z}y)\\
&= [F]\partial_x\,\{y,z\} + F\,[\{y,z\}]\partial_x,
\end{aligned}$$

and moreover, in view of **(A3|)**,

$$\begin{aligned}
[\{y,z\}F]\partial_x| &= [Fyz]\partial_x| + [Fzy]\partial_x| \\
&= -[Fy]\partial_x|\, z + \delta_{x,z}FyB - [Fz]\partial_x|\, y + \delta_{x,y}FzB \\
&= -\left(-[F]\partial_x|\, y + \delta_{x,y}FB\right) z + \delta_{x,z}FyB - \left(-[F]\partial_x|\, z + \delta_{x,z}FB\right) y + \delta_{x,y}FzB \\
&= [F]\partial_x|\, \{y,z\} - \delta_{x,y}F[B,z] - \delta_{x,z}F[B,y] \\
&= [F]\partial_x|\, \{y,z\} + 2F\left(\delta_{x,z}y| + \delta_{x,y}z|\right) \\
&= [F]\partial_x|\, \{y,z\} + F\ [\{y,z\}]\partial_x|.
\end{aligned}$$

Then the validity of **(D1|)** follows from the previous computations.

In order to prove **(D2|)**, first note that all $x, y \in S$ with $y \neq x$ we have

$$\begin{aligned}
\partial_x[y] &= -y\partial_x[1] + 2D_{y,x}[1] = 0, & \partial_x[y|] &= -y|\partial_x[1] = 0, \\
\partial_x|[y] &= -y\partial_x|[1] = 0, & \partial_x|[y|] &= -y|\partial_x|[1] - 2D_{y,x}[1] = 0, \\
[y]\partial_x| &= -[1]\partial_x|\ y = 0, & [y|]\partial_x| &= -[1]\partial_x|\ y| - 2D_{y,x}[1] = 0, \\
[y]\partial_x &= -[1]\partial_x\ y + 2D_{y,x}[1] = 0, & [y|]\partial_x &= -[1]\partial_x\ y| = 0.
\end{aligned}$$

Hence it can be proven as a consequence of Theorem 2.8 and formulae (2.14) that

$$\partial_x[G] = [G]\partial_x = \partial_x|[G] = [G]\partial_x| = 0,$$

for every $G \in R\big(\,(S \setminus \{x\}) \cup (S| \setminus \{x|\})\,, B\big)$.

The (left) canonical decomposition in $R(S \cup S|, B)$ with respect to $x \in S$ has the form

$$F = f_1F_1 + f_2xF_2 + f_3x|F_3 + f_4xx|F_4,$$

where $f_j \in R_0(S)$ and $F_j \in R\big(\,(S \setminus \{x\}) \cup (S| \setminus \{x|\})\,, B\big)$. Then,

$$\begin{aligned}
\partial_x[FG] = \partial_x[f_1]F_1G + \partial_x[f_2]xF_2G + f_2\partial_x[xF_2G] + \partial_x[f_3]x|F_3G \\
+ f_3\partial_x[x|F_3G] + \partial_x[f_4]xx|F_4G + f_4\partial_x[xx|F_4G].
\end{aligned}$$

However by Theorem 2.8 we have,

$$\begin{aligned}
\partial_x[xF_2G] &= -x\partial_x[F_2G] + 2E_x[F_2G] + \mathfrak{m}F_2G = \mathfrak{m}F_2G = \partial_x[xF_2]G, \\
\partial_x[x|F_3G] &= -x|\partial_x[F_3G] + BF_3G = BF_3G = \partial_x[x|F_3]G, \\
\partial_x[xx|F_4G] &= -x\partial_x[x|F_4G] + 2E_x[x|F_4G] + \mathfrak{m}x|F_4G = -xBF_4G + 2x|F_4G + \mathfrak{m}x|F_4G \\
&= \partial_x[xx|F_4]G,
\end{aligned}$$

whence $\partial_x[FG] = \partial_x[F]G$ is obtained by direct computation. In the same way, we compute

$$\begin{aligned}
\partial_x|[FG] = \partial_x|[f_1]F_1G + \partial_x|[f_2]xF_2G + f_2\partial_x|[xF_2G] + \partial_x|[f_3]x|F_3G \\
+ f_3\partial_x|[x|F_3G] + \partial_x|[f_4]xx|F_4G + f_4\partial_x|[xx|F_4G],
\end{aligned}$$

and using again Theorem 2.8 we obtain,

$$\begin{aligned}
\partial_x|[xF_2G] &= -x\partial_x|[F_2G] - BF_2G = -BF_2G = \partial_x|[xF_2]G,\\
\partial_x|[x|F_3G] &= -x|\partial_x|[F_3G] - 2E_x[F_3G] - \mathfrak{m}F_3G = -\mathfrak{m}F_3G = \partial_x|[x|F_3]G,\\
\partial_x|[xx|F_4G] &= -x\partial_x|[x|F_4G] - Bx|F_4G = \mathfrak{m}xF_4G - Bx|F_4G = \partial_x|[xx|F_4]G.
\end{aligned}$$

Hence, $\partial_x|[FG] = \partial_x|[F]G$ directly follows.

In order to prove the statements regarding to the actions from the right we follow the same order of ideas. The right canonical decomposition for elements of $R(S\cup S|, B)$ with respect to $x\in S$ has the form

$$F = f_1F_1 + f_2F_2x + f_3F_3x| + f_4F_4xx|.$$

We then compute

$$\begin{aligned}
[GF]\partial_x = GF_1\ [f_1]\partial_x &+ GF_2x\ [f_2]\partial_x + f_2\ [GF_2x]\partial_x + GF_3x|\ [f_3]\partial_x\\
&+ f_3\ [GF_3x|]\partial_x + GF_4xx|\ [f_4]\partial_x + f_4\ [GF_4xx|]\partial_x,
\end{aligned}$$

and

$$\begin{aligned}
[GF]\partial_x| = GF_1\ [f_1]\partial_x| &+ GF_2x\ [f_2]\partial_x| + f_2\ [GF_2x]\partial_x| + GF_3x|\ [f_3]\partial_x|\\
&+ f_3\ [GF_3x|]\partial_x| + GF_4xx|\ [f_4]\partial_x| + f_4\ [GF_4xx|]\partial_x|.
\end{aligned}$$

Now, in virtue of (2.14), we obtain

$$\begin{aligned}
[GF_2x]\partial_x &= -[GF_2]\partial_x\ x + 2E_x[GF_2] + \mathfrak{m}GF_2 = \mathfrak{m}GF_2 = G\ [F_2x]\partial_x,\\
[GF_3x|]\partial_x &= -[GF_3]\partial_x\ x| - GF_3B = -GF_3B = G\ [F_3x|]\partial_x,\\
[GF_4xx|]\partial_x &= -[GF_4x]\partial_x\ x| - GF_4xB = -\mathfrak{m}GF_4x| - GF_4xB = G\ [F_4xx|]\partial_x,
\end{aligned}$$

and

$$\begin{aligned}
[GF_2x]\partial_x| &= -[GF_2]\partial_x|\ x + GF_2B = GF_2B = G\ [F_2x]\partial_x|,\\
[GF_3x|]\partial_x| &= -[GF_3]\partial_x|\ x| - 2E_x[GF_3] - \mathfrak{m}GF_3 = -\mathfrak{m}GF_3 = G\ [F_3x|]\partial_x|,\\
[GF_4xx|]\partial_x| &= -[GF_4x]\partial_x|\ x| - 2E_x[GF_4x] - \mathfrak{m}GF_4x\\
&= -GF_4Bx| - 2GF_4x - \mathfrak{m}GF_4x = G\ [F_4xx|]\partial_x|.
\end{aligned}$$

Then both results $[GF]\partial_x = G[F]\partial_x$ and $[GF]\partial_x| = G[F]\partial_x|$ follow by direct computation.

Finally, **(D5|)** is obtained as an application of (2.15) which completes the proof of the theorem. □

In Table 2.1, we summarize the main endomorphisms of $R(S)$ and $R(S\cup S|, B)$ studied until now, together with their Clifford-polynomial representation.

	Radial algebra $R(S)$	Clifford representation $\underline{\cdot}_m : R(S) \to R(\underline{S}) \subset \mathcal{A}_{m,0}$	Clifford representation of $R(S \cup S\|, B)$ using isomorphism $\mathrm{End}(\mathbb{R}_{0,m}) \cong \mathbb{R}_{m,m}$
vector variables/	$x, y, \ldots \in S$	$\underline{x} = \sum_{j=1}^m e_j x_j$	$\underline{x} = \sum_{j=1}^m e_j x_j$
vector multipliers	$x\|, y\|, \ldots \in S\|$	$\sum_{j=1}^m e_j \| x_j$	$\underline{x}\| = \sum_{j=1}^m \varepsilon_j x_j$
vector derivatives	∂_x	$-\partial_{\underline{x}} = -\sum_{j=1}^m e_j \partial_{x_j}$	$-\partial_{\underline{x}} = -\sum_{j=1}^m e_j \partial_{x_j}$
	$\partial_x\|$	$-\sum_{j=1}^m e_j\| \, \partial_{x_j}$	$-\partial_{\underline{x}}\| = -\sum_{j=1}^m \varepsilon_j \partial_{x_j}$
B operator	$B = \{\partial_x, x\|\}$	$-\sum_{j=1}^m e_j e_j\|$	$\underline{B} = -\sum_{j=1}^m e_j \varepsilon_j$
Directional derivative	$D_{y,x}$	$D_{\underline{y},\underline{x}} = \sum_{j=1}^m y_j \partial_{x_j}$	$D_{\underline{y},\underline{x}} = \sum_{j=1}^m y_j \partial_{x_j}$

Table 2.1: Clifford-polynomial representation of the radial algebra

Now, in general, every property of the vector derivatives in $R(S)$ can be generalized to the $R(S\cup S|,B)$ setting. For example, the next result is a generalization of Theorem 2.5 (v).

Lemma 2.9. *Let $x\in S$, then for every $F\in R(S\cup S|,B)$ we have*

$$\partial_x[Fx]-\partial_x[F]x=[xF]\partial_x-x[F]\partial_x:=\mathcal{B}_1[F], \tag{2.16}$$
$$\partial_x|[Fx]-\partial_x|[F]x=([x|F]\partial_x-x|[F]\partial_x):=\mathcal{B}_2[F], \tag{2.17}$$
$$\partial_x|[Fx|]-\partial_x|[F]x|=[x|F]\partial_x|-x|[F]\partial_x|:=\mathcal{B}_3[F], \tag{2.18}$$
$$\partial_x[Fx|]-\partial_x[F]x|=([xF]\partial_x|-x[F]\partial_x|):=\mathcal{B}_4[F]. \tag{2.19}$$

In addition, the operators $\mathcal{B}_j$ do not depend on x, $j=1,\dots,4$.

Remark 2.15. *Note that these operators are mapped by the Clifford-polynomial representation to the Clifford operators*

$$\mathcal{B}_1[F]=-\sum_{j=1}^m e_jFe_j,\qquad \mathcal{B}_2[F]=-\sum_{j=1}^m \varepsilon_jFe_j,$$
$$\mathcal{B}_3[F]=-\sum_{j=1}^m \varepsilon_jF\varepsilon_j,\qquad \mathcal{B}_4[F]=-\sum_{j=1}^m e_jF\varepsilon_j.$$

Proof.

Observe that all operators involved in (2.16)-(2.19) are $R_0(S)$-linear. Then, as every element of $R(S\cup S|,B)$ is a linear combination of terms of the form $F=V_1\cdots V_k$ with $V_j\in S\cup S|\cup\{B\}$, it suffices to prove the lemma for such terms.

We will proceed by induction on k. For $k=0$ we have $F=1$, whence it is easily obtained that

$$\partial_x[x]-\partial_x[1]x=\mathfrak{m}=[x]\partial_x-x[1]\partial_x,$$
$$\partial_x|[x]-\partial_x|[1]x=-B=([x|]\partial_x-x|[1]\partial_x),$$
$$\partial_x|[x|]-\partial_x|[1]x|=-\mathfrak{m}=[x|]\partial_x|-x|[1]\partial_x|,$$
$$\partial_x[x|]-\partial_x[1]x|=B=([x]\partial_x|-x[1]\partial_x|),$$

with $\mathfrak{m}$ and B clearly not depending on x.

Now assuming that all the statements (2.16)-(2.19) hold for every product F of at most k elements from $S\cup S|\cup\{B\}$, it should be proven that (2.16)-(2.19) remain valid for yF, $y|F$ and BF. From (2.15) we obtain the following relations

$$\partial_x[BFx]-\partial_x[BF]x=B\left(\partial_x[Fx]-\partial_x[F]x\right)+2\left(\partial_x|[Fx]-\partial_x|[F]x\right),$$
$$[xBF]\partial_x-x[BF]\partial_x=B([xF]\partial_x-x[F]\partial_x)+2([x|F]\partial_x-x|[F]\partial_x),$$

and

$$\begin{aligned}
\partial_x|[BFx] - \partial_x|[BF]x &= B(\partial_x|[Fx] - \partial_x|[F]x) + 2(\partial_x[Fx] - \partial_x[F]x),\\
[x|BF]\partial_x - x|[BF]\partial_x &= B([x|F]\partial_x - x|[F]\partial_x) + 2([xF]\partial_x - x[F]\partial_x).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_x|[BFx|] - \partial_x|[BF]x| &= B\left(\partial_x|[Fx|] - \partial_x|[F]x|\right) + 2\left(\partial_x[Fx|] - \partial_x[F]x|\right),\\
[x|BF]\partial_x| - x|[BF]\partial_x| &= B([x|F]\partial_x| - x|[F]\partial_x|) + 2([xF]\partial_x| - x[F]\partial_x|),
\end{aligned}$$

and also

$$\begin{aligned}
\partial_x[BFx|] - \partial_x[BF]x| &= B(\partial_x[Fx|] - \partial_x[F]x|) + 2(\partial_x|[Fx|] - \partial_x|[F]x|),\\
[xBF]\partial_x| - x[BF]\partial_x| &= B([xF]\partial_x| - x[F]\partial_x|) + 2([x|F]\partial_x| - x|[F]\partial_x|).
\end{aligned}$$

Combining the above equalities and using the induction hypothesis, it is shown that BF satisfies (2.16)-(2.19).

Let us prove now that yF, $y \in S$, satisfies (2.16)-(2.17). From the recursive formulae in Theorem 2.8 we obtain

$$\begin{aligned}
\partial_x[yFx] - \partial_x[yF]x &= -y\partial_x[Fx] + 2D_{y,x}[Fx] + \delta_{x,y}\mathfrak{m}Fx\\
&\quad + \left(y\partial_x[F] - 2D_{y,x}[F] - \delta_{x,y}\mathfrak{m}F\right)x\\
&= -y\left(\partial_x[Fx] - \partial_x[F]x\right) + 2Fy,
\end{aligned}$$

while for both cases $x = y$ and $x \neq y$ we have

$$[xyF]\partial_x - x[yF]\partial_x = \left[\{x,y\}F - yxF\right]\partial_x - x[yF]\partial_x = 2Fy - y\left([xF]\partial_x - x[F]\partial_x\right).$$

Hence, yF satisfies (2.16). In order to prove (2.17) for yF, it is convenient to use the following property:

$$yF = yV_1 \cdots V_k = \lambda V_1 \cdots V_k\, y + \sum \text{scalar l.o.p.}, \quad \text{where } \lambda \in \mathbb{R}.$$

In view of the induction hypothesis, this means that proving (2.17) for yF is equivalent to proving it for Fy. To this end, observe that

$$\partial_x|[Fyx] - \partial_x|[Fy]x = 2y|F - \left(\partial_x|[Fx] - \partial_x|[F]x\right)y,$$

again for both cases $x = y$ and $x \neq y$. Also from (2.14) we get

$$\begin{aligned}
[x|Fy]\partial_x - x|[Fy]\partial_x &= -[x|F]\partial_x\, y + 2D_{y,x}[x|F] + x|\left([F]\partial_x\, y - 2D_{y,x}[F]\right)\\
&= -\left([x|F]\partial_x - x|[F]\partial_x\right)y + 2y|F,
\end{aligned}$$

from which it easily follows that (2.17) holds for Fy and hence for yF.

The statements (2.18)-(2.19) can be proven for $y|F$ following the same order of ideas. Indeed, from Theorem 2.8 one obtains

$$\begin{aligned}\partial_x|[y|Fx|] - \partial_x|[y|F]x| &= -y|\partial_x|[Fx|] - 2D_{y,x}[Fx|] - \delta_{x,y}\mathfrak{m}Fx| \\ &\quad - (-y|\partial_x|[F] - 2D_{y,x}[F] - \delta_{x,y}\mathfrak{m}F)\, x| \\ &= -y|\,(\partial_x|[Fx|] - \partial_x|[F]x|) - 2Fy|,\end{aligned}$$

while for both cases $x = y$ and $x \neq y$ one has

$$\begin{aligned}[x|y|F]\partial_x| \;-\; x|[y|F]\partial_x| &= \big[\{x|, y|\}F - y|x|F\big]\partial_x| \;-\; x|[y|F]\partial_x| \\ &= -2Fy| \;-\; y|\,([x|F]\partial_x| - x|[F]\partial_x|)\,.\end{aligned}$$

Comparing the previous two formulae, it follows that $y|F$ satisfies (2.18). In order to prove (2.19) for $y|F$, we now use the property

$$y|F = y|V_1 \cdots V_k = \lambda V_1 \cdots V_k\, y| + \sum \text{scalar l.o.p.}, \quad \text{where } \lambda \in \mathbb{R},$$

from which it follows that proving (2.19) for $y|F$ is equivalent to proving it for $Fy|$. Now, observe that

$$\partial_x[Fy|x|] - \partial_x[Fy|]x| = -2yF - (\partial_x[Fx|] - \partial_x[F]x|)\, y|.$$

Moreover, from (2.14) we get

$$\begin{aligned}[xFy|]\partial_x| - x[Fy|]\partial_x| &= -[xF]\partial_x|\, y| - 2D_{y,x}[xF] - x\,(-[F]\partial_x|\, y| - 2D_{y,x}[F]) \\ &= -\,([xF]\partial_x| - x[F]\partial_x|)\, y| - 2yF.\end{aligned}$$

Hence, $y|F$ satisfies (2.19).

The proof that $y|F$ satisfies (2.16)-(2.17) directly follows from the identity $By - yB = -2y|$. Similarly, the proof that yF satisfies (2.18)-(2.19) follows from the identity $By| - y|B = -2y$. In this way, we have proven that (2.16)-(2.19) hold for every $F \in R(S \cup S|, B)$.

Finally, the independence of the operators $\mathcal{B}_j$ of x can be proven very easily. For instance, for $\mathcal{B}_1$, the equality

$$\partial_x[Fx] - \partial_x[F]x = \partial_y[Fy] - \partial_y[F]y, \qquad x, y \in S,$$

is obtained by induction as a direct consequence of the recursion formulae in Theorem 2.8. □

It now is possible to prove by induction some generalizations of Theorem 2.6 (vi).

Lemma 2.10. *Let $x \in S$. Then*

$$\begin{array}{llll}\{\mathcal{B}_1, \partial_x\} = 2(\cdot\partial_x), & \{\mathcal{B}_2, \partial_x\} = 0, & \{\mathcal{B}_3, \partial_x\} = 0, & \{\mathcal{B}_4, \partial_x\} = 2(\cdot\partial_x|), \\ \{\mathcal{B}_1, \cdot\partial_x\} = 2\partial_x, & \{\mathcal{B}_2, \cdot\partial_x\} = 2\partial_x|, & \{\mathcal{B}_3, \cdot\partial_x\} = 0, & \{\mathcal{B}_4, \cdot\partial_x\} = 0, \\ \{\mathcal{B}_1, \partial_x|\} = 0, & \{\mathcal{B}_2, \partial_x|\} = -2(\cdot\partial_x), & \{\mathcal{B}_3, \partial_x|\} = -2(\cdot\partial_x|), & \{\mathcal{B}_4, \partial_x|\} = 0, \\ \{\mathcal{B}_1, \cdot\partial_x|\} = 0, & \{\mathcal{B}_2, \cdot\partial_x|\} = 0, & \{\mathcal{B}_3, \cdot\partial_x|\} = -2\partial_x|, & \{\mathcal{B}_4, \cdot\partial_x|\} = -2\partial_x.\end{array}$$

Finally, as a direct consequence of Lemmas 2.9 and 2.10, we can obtain, applying the same induction procedure, the following important generalization of **(D3)**.

Theorem 2.10. *The operators* $\partial_x, \partial_x|$ *satisfy in* $R(S \cup S|, B)$

$$\begin{aligned} &[\partial_x F]\,\partial_y = \partial_x\,[F\partial_y]\,, &&[\partial_x|F]\,\partial_y = \partial_x|\,[F\partial_y]\,, &&\forall x, y \in S,\\ &[\partial_x F]\,\partial_y| = \partial_x\,[F\partial_y|]\,, &&[\partial_x|F]\,\partial_y| = \partial_x|\,[F\partial_y|]\,. \end{aligned}$$

2.5 Hermitian radial algebra

In this section we introduce the Hermitian radial algebra which constitutes an abstract version of the Hermitian monogenic function theory. In [64] the Hermitian radial algebra was introduced by means of a set of abstract complex variables and the corresponding constraints. In this thesis, it is more convenient to start from a classical radial algebra and to introduce a complex structure on it. This approach induces the one from [64] as a consequence.

2.5.1 Complex structure

Inspired by the classical complex structure in $\mathbb{R}^{2m}$ we give the following definition of a complex structure on the radial algebra $R(S)$.

Definition 2.10 (abstract complex structure). *Let* $J : S \to J(S)$ *be a bijective map which produces a new set* $J(S)$ *of abstract vector variables which is a disjoint copy of* S, *i.e.* $S \cap J(S) = \emptyset$. *Moreover consider the algebra* $R(S \cup J(S), \mathcal{B})$ *generated over the real numbers by the set* $S \cup J(S)$ *and a new symbol* $\mathcal{B}$. *The pair* $(J, \mathcal{B})$ *is called a complex structure if on* $R(S \cup J(S), \mathcal{B})$ *there holds:*

(AH1) $\{x, y\} = \{J(x), J(y)\}, \quad \{J(x), y\} = -\{x, J(y)\} \quad \forall x, y \in S,$

(AH2) $\{x, y\}$ *and* $\{J(x), y\}$ *are central elements,* $\forall x, y \in S,$

(AH3) $[\mathcal{B}, x] = -2J(x), \quad [\mathcal{B}, J(x)] = 2x, \quad \forall x \in S.$

Since the algebraic rôle of $\mathcal{B}$ *is determined by* J *through the axiom* **(AH3)**, *the term "complex structure" will refer only to* J *from now on. The algebra* $R(S \cup J(S), \mathcal{B})$ *is called a radial algebra with complex structure.*

Remark 2.16. *As before, we will use the notation*

$$R_0(S \cup J(S)) := Alg_{\mathbb{R}}\,\{\{x, y\}, \{J(x), y\} : x, y \in S\}$$

for the scalar subalgebra of $R(S \cup J(S), \mathcal{B})$.

Definition 2.11. *A representation of the radial algebra with complex structure* $R(S \cup J(S), \mathcal{B})$ *is an algebra homomorphism* Ψ *from* $R(S \cup J(S), \mathcal{B})$ *into an algebra* A. *The* *term representation also refers to the range* $\Psi(R(S \cup J(S), \mathcal{B})) \subset A$ *of that mapping. In particular, the complex structure* J *defines a map over the set* $\Psi(S) \subset A$ *by*

$$\Psi(x) \to \Psi(J(x)), \qquad x \in S.$$

Such a map is called "a complex structure on the representation $R(\Psi(S))$*" and is denoted by* $\Psi(J)$*.*

Remark 2.17. *The restriction to* $R(S)$ *of an algebra homomorphism* $\Psi : R(S \cup J(S), \mathcal{B}) \to A$ *clearly constitutes a representation for the radial algebra* $R(S)$*, see Definition 2.2. But the inverse statement does not hold, i.e. not every representation for* $R(S)$ *can be extended to a representation of* $R(S \cup J(S), \mathcal{B})$*. This is well illustrated by the Clifford-polynomial representation described in Example 2.1. Indeed, if the representation* $\underline{\cdot}$ *can be extended from* $R(S)$ *to* $R(S \cup J(S), \mathcal{B})$*, then* J *gets mapped to a linear operator* $\underline{J}$ *that satisfies in* $R(\underline{S})$ *the rules* **(AH1)**-**(AH3)***. In particular,* **(AH1)** *implies that* $\underline{J} \in O(m)$ *and that* $\underline{J}^2 = -I_m$*, where* I_m *is the identity matrix. This last condition clearly forces the dimension* m *to be even. Hence, the Clifford-polynomial representation can be extended to a representation of the radial algebra with complex structure only if the dimension* m *is even.*

Taking the dimension in Example 2.1 now to be $2m$ *we map* J *to the complex structure on* $\mathbb{R}^{2m}$ *given by*

$$\underline{J} = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}.$$

In this way, the Clifford-polynomial representation gets extended from $R(S)$ *to* $R(S \cup J(S), \mathcal{B})$ *by considering the mapping*

$$x \to \underline{x} = \sum_{j=1}^{m} (x_j e_j + x_{m+j} e_{m+j}), \qquad J(x) \to \underline{J}(\underline{x}) = \sum_{j=1}^{m} (x_{m+j} e_j - x_j e_{m+j}),$$

while the element $\mathcal{B}$ *is mapped to the bivector*

$$\underline{\mathcal{B}} = \sum_{j=1}^{m} e_j e_{m+j}.$$

It is easily seen that this map constitutes a homomorphism from $R(S \cup J(S), \mathcal{B})$ *to the algebra* $\mathcal{A}_{2m,0}$ *of* $\mathbb{R}_{0,2m}$*-valued polynomials. Indeed, the elements* $\underline{x}$*,* $\underline{J}(\underline{x})$*,* $\underline{\mathcal{B}}$ *generate an algebra, denoted by* $R(\underline{S} \cup \underline{J}(\underline{S}), \underline{\mathcal{B}})$*, which satisfies* **(AH1)**-**(AH3)***. The complex structure* $\underline{J}$ *on* $\mathbb{R}^{2m}$ *is the one used for introducing the so-called Hermitian Clifford analysis, see e.g. [9, 14].*

The map $J : S \to J(S)$ is extended to the whole algebra $R(S \cup J(S), \mathcal{B})$ by linearity, and by the additional rules

$$J(FG) = J(F)J(G), \qquad \forall F, G \in R(S \cup J(S), \mathcal{B}), \tag{2.20}$$

$$J(\mathcal{B}) = \mathcal{B}, \qquad J(J(x)) = -x, \quad \forall x \in S. \tag{2.21}$$

This extension is consistent with (**AH1**)-(**AH3**); in particular, direct computation shows that

$$J\big(\{x,y\}\big)=\{x,y\},\qquad J\left(\{J(x),y\}\right)=\{J(x),y\},\qquad \text{for all } x,y\in S.$$

This means, together with (2.20), that J is the identity map on $R_0(S\cup J(S))$.

Remark 2.18. *Given a representation Ψ of the radial algebra with complex structure $R(S\cup J(S),\mathcal{B})$ (see Definition 2.11), the complex structure $\Psi(J)$ defined on $\Psi(S)$ can be extended to the whole algebra $\Psi(R(S\cup J(S),\mathcal{B}))$ also by means of the rules (2.20)-(2.21).*

In the particular case of the Clifford-polynomial representation, the complex structure $\underline{J}$ gets extended to an algebra automorphism in the algebra $\mathcal{A}_{2m,0}$ of Clifford-valued polynomials.

Proposition 2.3. *J is an algebra automorphism on $R(S\cup J(S),\mathcal{B})$.*

Proof.

Every element of $R(S\cup J(S),\mathcal{B})$ can be written as a combination of terms of the form $f\,x_1\cdots x_\ell\,J(y_1)\cdots J(y_k)\mathcal{B}^s$, with $f\in R_0(S\cup J(S))$ and $x_1,\ldots,x_\ell,y_1,\ldots,y_k\in S$. The result then follows from

$$J^2\big(f\;x_1\cdots x_lJ(y_1)\cdots J(y_k)\mathcal{B}^s\big)=(-1)^{l+k}f\;x_1\cdots x_lJ(y_1)\cdots J(y_k)\mathcal{B}^s.$$

□

The conjugation can be easily redefined in this context by the rules

$$\overline{\mathcal{B}}=-\mathcal{B},\qquad \overline{x}=-x,\quad \overline{J(x)}=-J(x),\quad x\in S,$$
$$\overline{ab}=\overline{b}\,\overline{a},\quad a,b\in R(S\cup J(S),\mathcal{B}).$$

Also the definition of the vector derivatives ∂_x and $\partial_{J(x)}$ is extended from $R(S)$ and $R(J(S))$ to $R(S\cup J(S),\mathcal{B})$ by the following axioms:

(DH1) $\partial_x[fF]=\partial_x[f]F+f\partial_x[F],\quad \partial_{J(x)}[fF]=\partial_{J(x)}[f]F+f\partial_{J(x)}[F],$
$[fF]\partial_x=F[f]\partial_x+f[F]\partial_x,\quad [fF]\partial_{J(x)}=F[f]\partial_{J(x)}+f[F]\partial_{J(x)},$
for $f\in R_0(S\cup J(S))$ and $F\in R(S\cup J(S),\mathcal{B})$,

(DH2) $\partial_x[G]=[G]\partial_x=\partial_{J(x)}[G]=[G]\partial_{J(x)}=0,$

$$\begin{cases}\partial_x[xG]=\partial_x[x]G\\ \partial_x[J(x)G]=\partial_x[J(x)]G\\ \partial_{J(x)}[xG]=\partial_{J(x)}[x]G\\ \partial_{J(x)}[J(x)G]=\partial_{J(x)}[J(x)]G\\ \partial_x[xJ(x)G]=\partial_x[xJ(x)]G\\ \partial_{J(x)}[xJ(x)G]=\partial_{J(x)}[xJ(x)]G\end{cases}\qquad \begin{cases}[Gx]\partial_x=G[x]\partial_x\\ [GJ(x)]\partial_x=G[J(x)]\partial_x\\ [Gx]\partial_{J(x)}=G[x]\partial_{J(x)}\\ [GJ(x)]\partial_{J(x)}=G[J(x)]\partial_{J(x)}\\ [GxJ(x)]\partial_x=G[xJ(x)]\partial_x\\ [GxJ(x)]\partial_{J(x)}=G[xJ(x)]\partial_{J(x)}\end{cases}$$

for $G\in R\big((S\setminus\{x\})\cup J\left(S\setminus\{x\}\right),\mathcal{B}\big)$,

(DH3) $\partial_x[x] = [x]\partial_x = \mathfrak{m} = \partial_{J(x)}[J(x)] = [J(x)]\partial_{J(x)},$
$\partial_x[J(x)] = -[J(x)]\partial_x = 2\mathcal{B} = -\partial_{J(x)}[x] = [x]\partial_{J(x)}.$
where $\mathfrak{m}$ denotes the abstract dimension of $R(S)$,

(DH4)
$$\begin{cases} \partial_x[x^2] = [x^2]\partial_x = 2x, & \partial_{J(x)}[x^2] = [x^2]\partial_{J(x)} = 2J(x), \\ \partial_x[xJ(x)] = (\mathfrak{m}+2)J(x) - 2x\mathcal{B}, & [xJ(x)]\partial_x = -(\mathfrak{m}-2)J(x) - 2x\mathcal{B}, \\ \partial_{J(x)}[xJ(x)] = -(\mathfrak{m}+2)x - 2J(x)\mathcal{B}, & [xJ(x)]\partial_{J(x)} = (\mathfrak{m}-2)x - 2J(x)\mathcal{B}, \end{cases}$$
for $x \in S$, and
$$\begin{cases} \partial_x\{x,y\} = \{x,y\}\partial_x = 2y = \partial_{J(x)}\{J(x),y\} = \{J(x),y\}\partial_{J(x)}, \\ \partial_{J(x)}\{x,y\} = \{x,y\}\partial_{J(x)} = 2J(y) = -\partial_x\{J(x),y\} = -\{J(x),y\}\partial_x, \end{cases}$$
for $x, y \in S, x \neq y$.

Following the same idea as in Definition 2.3, the action of every representation Ψ of $R(S \cup J(S), \mathcal{B})$ on the algebra of endomorphisms on $R(S \cup J(S), \mathcal{B})$ can be defined by

$$\Psi(E)\,[\Psi(F)] := \Psi(E[F]), \quad E \in \mathrm{End}\,(R(S \cup J(S), \mathcal{B}))\,,\, F \in R(S \cup J(S), \mathcal{B}). \tag{2.22}$$

Remark 2.19. *Following (2.22), it is easily seen that the vector derivatives ∂_x, $\partial_{J(x)}$ are mapped by the Clifford polynomial representation to the Dirac operators:*

$$-\partial_{\underline{x}} = -\sum_{j=1}^{m}(e_j\partial_{x_j} + e_{m+j}\partial_{x_{m+j}}), \qquad -\partial_{\underline{J}(\underline{x})} = \underline{J}(-\partial_{\underline{x}}) = -\sum_{j=1}^{m}(e_j\partial_{x_{m+j}} - e_{m+j}\partial_{x_j}),$$

respectively. The abstract dimension $\mathfrak{m}$ gets mapped in this case to $2m$.

Remark 2.20. *Similar to Remark 2.8 we here have that* **(DH2)** *is equivalent to*

$$\begin{cases} \partial_x[FG] = \partial_x[F]G, \\ [GF]\partial_x = G[F]\partial_x, \end{cases} \qquad \begin{cases} \partial_{J(x)}[FG] = \partial_{J(x)}[F]G, \\ [GF]\partial_{J(x)} = G[F]\partial_{J(x)}. \end{cases}$$

where $F \in R(S \cup J(S), \mathcal{B})$ and $G \in R\big((S \setminus \{x\}) \cup J(S \setminus \{x\}), \mathcal{B}\big)$.

Following Theorem 2.2 it can be proven that **(DH1)-(DH4)** lead to a consistent definition of the endomorphisms ∂_x and $\partial_{J(x)}$. Indeed, in this case the canonical decomposition with respect to $x \in S$ is given by

$$F = f_1F_1 + f_2xF_2 + f_3J(x)F_3 + f_4xJ(x)F_4,$$

where $f_j \in R_0(S \cup J(S))$ and $F_j \in R\big((S \setminus \{x\}) \cup J(S \setminus \{x\}), \mathcal{B}\big)$. Consequently, **(DH1)-(DH4)** lead to

$$\begin{aligned} \partial_x[F] &= \partial_x[f_1]F_1 + \partial_x[f_2]xF_2 + f_2\mathfrak{m}F_2 + \partial_x[f_3]J(x)F_3 + 2f_3\mathcal{B}F_3 \\ &\quad + \partial_x[f_4]xJ(x)F_4 + (\mathfrak{m}+2)f_4J(x)F_4 - 2f_4x\mathcal{B}F_4, \\ \partial_{J(x)}[F] &= \partial_{J(x)}[f_1]F_1 + \partial_{J(x)}[f_2]xF_2 - 2f_2\mathcal{B}F_2 + \partial_{J(x)}[f_3]J(x)F_3 + \mathfrak{m}f_3F_3 \\ &\quad + \partial_{J(x)}[f_4]xJ(x)F_4 - (\mathfrak{m}+2)f_4xF_4 - 2f_4J(x)\mathcal{B}F_4. \end{aligned}$$

The introduction of the complex structure J also enables us to extend the notion of directional derivative. For every pair $x \in S$, $y \in S \cup J(S)$, the map $D_{y,x} \in \mathrm{End}(R(S \cup J(S), \mathcal{B}))$ is defined by an adequate extension of **(DD1)** and the following version of **(DD2)**:

$$\textbf{(DDH2)} \qquad D_{y,x}[z] = \delta_{x,z} y, \quad D_{y,x}[J(z)] = \delta_{x,z} J(y), \qquad z \in S,$$
$$D_{y,x}[\mathcal{B}] = 0.$$

This extension allows to consider directional derivatives of the form $D_{J(y),x}$ with $y \in S$. In that case we obtain from the previous axiom that

$$D_{J(y),x}[z] = \delta_{x,z} J(y), \quad D_{J(y),x}[J(z)] = -\delta_{x,z} y.$$

Remark 2.21. *$D_{y,x}$ is mapped by the Clifford polynomial representation to*

$$D_{\underline{y},\underline{x}} = \sum_{j=1}^{m} (y_j \partial_{x_j} + y_{m+j} \partial_{x_{m+j}})$$

while $D_{J(y),x}$ corresponds to the twisted scalar operator

$$D_{\underline{J}(\underline{y}),\underline{x}} = \sum_{j=1}^{m} (y_{m+j} \partial_{x_j} - y_j \partial_{x_{m+j}}).$$

Similar to Lemmas 2.5 and 2.6 we have the following properties.

Lemma 2.11. *Let $x, z \in S$ and $y \in S \cup J(S)$. Then*

(i) $D_{y,x}$ *maps* $R_0(S \cup J(S))$ *into* $R_0(S \cup J(S))$;

(ii) $\overline{D_{y,x}[F]} = D_{y,x}[\overline{F}], \quad \forall F \in R(S \cup J(S), \mathcal{B})$;

(iii) $[D_{y,x}, z] = \delta_{x,z} y, \quad [D_{y,x}, J(z)] = \delta_{x,z} J(y)$;

(iv) $[D_{y,x}, \mathcal{B}] = 0$.

Lemma 2.12. *The complex structure J satisfies the following properties:*

(i) $J(\partial_x[F]) = \partial_{J(x)}[J(F)], \qquad \forall x \in S$;

(ii) $J(\partial_{J(x)}[F]) = -\partial_x[J(F)], \quad \forall x \in S$;

(iii) $J(D_{y,x}[F]) = D_{y,x}[J(F)], \quad \forall x \in S,\ y \in S \cup J(S)$.

Proof.

Using the canonical decomposition of $F \in R(S \cup J(S), \mathcal{B})$, then applying $\partial_{J(x)}$, ∂_x and $D_{y,x}$ to

$$F = f_1 F_1 + f_2 x F_2 + f_3 J(x) F_3 + f_4 x J(x) F_4,$$

and

$$J(F) = f_1 J(F_1) + f_2 J(x) J(F_2) - f_3 x J(F_3) - f_4 J(x) x J(F_4)$$

and comparing the obtained results, it is directly seen that the statements $(i) - (iii)$ only still have to be proven for the generators of $R_0(S \cup J(S))$. For (i) and (ii) we easily obtain

$$\begin{aligned} J\left(\partial_x[\{y,z\}]\right) &= 2\left(\delta_{z,x} J(y) + \delta_{y,x} J(z)\right) = \partial_{J(x)}[\{y,z\}], \\ J\left(\partial_x[\{J(y),z\}]\right) &= 2\left(\delta_{y,x} z - \delta_{z,x} y\right) = \partial_{J(x)}[\{J(y),z\}], \end{aligned}$$

and

$$\begin{aligned} J\left(\partial_{J(x)}[\{y,z\}]\right) &= -2\left(\delta_{z,x} y + \delta_{y,x} z\right) = -\partial_x[\{y,z\}], \\ J\left(\partial_{J(x)}[\{J(y),z\}]\right) &= 2\left(\delta_{y,x} J(z) - \delta_{z,x} J(y)\right) = -\partial_x[\{J(y),z\}]. \end{aligned}$$

The property (iii) directly follows from Lemma 2.11 (i) and from the fact that J is the identity operator when restricted to $R_0(S \cup J(S))$. □

The above lemma provides an alternative definition for $\partial_{J(x)}$ given by the action of J on ∂_x, i.e.

$$\partial_{J(x)}[J(F)] = J(\partial_x[F])$$

or $\partial_{J(x)} = J(\partial_x)$ for short. Now the corresponding recursive formulae for ∂_x and $\partial_{J(x)}$ are proven.

Theorem 2.11. *Let $x, y \in S$ and $F \in R(S \cup J(S), \mathcal{B})$. Then*

$$\begin{aligned} \partial_x[yF] &= -y\partial_x[F] + 2D_{y,x}[F] + \delta_{x,y}\mathfrak{m}F, && (2.23) \\ \partial_x[J(y)F] &= -J(y)\partial_x[F] + 2D_{J(y),x}[F] + 2\delta_{x,y}\mathcal{B}F, && (2.24) \\ \partial_x[\mathcal{B}F] &= \mathcal{B}\partial_x[F] + 2\partial_{J(x)}[F], && (2.25) \\ \partial_{J(x)}[yF] &= -y\partial_{J(x)}[F] - 2D_{J(y),x}[F] - 2\delta_{x,y}\mathcal{B}F, \\ \partial_{J(x)}[J(y)F] &= -J(y)\partial_{J(x)}[F] + 2D_{y,x}[F] + \delta_{x,y}\mathfrak{m}F, \\ \partial_{J(x)}[\mathcal{B}F] &= \mathcal{B}\partial_{J(x)}[F] - 2\partial_x[F]. \end{aligned}$$

Proof.

We restrict ourselves to formulae (2.23)–(2.25), which yield the other three by application of J and the results of Lemma 2.12. Now, (2.23)–(2.25) look very similar to Theorems 2.3 and 2.8, whence their proofs will follow the same order of ideas. However, they are technically much more involved since the canonical decomposition now contains additional terms corresponding to the complex structure. Since (2.23) has already been proven in the radial algebra $R(S)$, we will give here the main ideas of the proofs of (2.24)–(2.25).

From the action of ∂_x on $F = f_1 F_1 + f_2 x F_2 + f_3 J(x) F_3 + f_4 x J(x) F_4$ it follows that

$$\begin{aligned} J(y)\partial_x[F] = {} & J(y)\partial_x[f_1]F_1 + J(y)\partial_x[f_2]xF_2 + J(y)\partial_x[f_3]J(x)F_3 + J(y)\partial_x[f_4]xJ(x)F_4 \\ & + f_2\mathfrak{m}J(y)F_2 + 2f_3 J(y)\mathcal{B}F_3 + (\mathfrak{m}+2)f_4 J(y)J(x)F_4 - 2f_4 J(y)x\mathcal{B}F_4. \end{aligned}$$

We now compute $\partial_x[J(y)F]$ with

$$J(y)F = f_1 J(y)F_1 + f_2 J(y)xF_2 + f_3 J(y)J(x)F_3 + f_4 J(y)xJ(x)F_4,$$

using **(DH3)-(DH4)**. We consecutively find

$$\begin{aligned}
\partial_x[J(y)] &= 2\delta_{x,y}\mathcal{B},\\
\partial_x[J(y)x] &= -(\mathfrak{m}-2)J(y) + 2\delta_{x,y}\mathcal{B}x,\\
\partial_x[J(y)J(x)] &= 2y - 2\mathcal{B}J(y) + 2\delta_{x,y}\mathcal{B}J(x),\\
\partial_x[J(y)xJ(x)] &= 2J(y)J(x) + 2\{x, J(y)\}\mathcal{B} - (\mathfrak{m}+2)\{x,y\} + (\mathfrak{m}+2)J(x)J(y)\\
&\quad - 2xJ(y)\mathcal{B} - 2xy + 2\delta_{x,y}\, xJ(x)\mathcal{B},
\end{aligned}$$

yielding

$$\begin{aligned}
\partial_x[J(y)F] = {}& \partial_x[f_1]J(y)F_1 + \partial_x[f_2]J(y)xF_2 + \partial_x[f_3]J(y)J(x)F_3 + \partial_x[f_4]J(y)xJ(x)F_4\\
&+ (2-\mathfrak{m})f_2 J(y)F_2 + 2f_3 yF_3 - 2f_3\mathcal{B}J(y)F_3 + 2f_4 J(y)J(x)F_4\\
&+ 2f_4\{x, J(y)\}\mathcal{B}F_4 - (\mathfrak{m}+2)f_4\{x,y\}F_4 + (\mathfrak{m}+2)f_4 J(x)J(y)F_4\\
&- 2f_4 xJ(y)\mathcal{B}F_4 - 2f_4 xyF_4 + 2\delta_{x,y}\left(f_1\mathcal{B}F_1 + f_2\mathcal{B}xF_2 + f_3\mathcal{B}J(x)F_3\right)\\
&+ 2\delta_{x,y} f_4\mathcal{B}xJ(x)F_4.
\end{aligned}$$

Eventually we obtain,

$$\begin{aligned}
\{\partial_x, J(y)\}[F] = {}& \{\partial_x, J(y)\}[f_1]F_1 + \{\partial_x, J(y)\}[f_2]xF_2 + \{\partial_x, J(y)\}[f_3]J(x)F_3\\
&+ \{\partial_x, J(y)\}[f_4]xJ(x)F_4 + 2f_2 J(y)F_2 - 2f_3 yF_3 + 2f_4 J(y)J(x)F_4\\
&- 2f_4 xyF_4.
\end{aligned}$$

On the other hand, it is easily seen that

$$\begin{aligned}
\left(2D_{J(y),x} + 2\delta_{x,y}\mathcal{B}\right)[F] = {}& \left(2D_{J(y),x} + 2\delta_{x,y}\mathcal{B}\right)[f_1]F_1 + \left(2D_{J(y),x} + 2\delta_{x,y}\mathcal{B}\right)[f_2]xF_2\\
&+ \left(2D_{J(y),x} + 2\delta_{x,y}\mathcal{B}\right)[f_3]J(x)F_3\\
&+ \left(2D_{J(y),x} + 2\delta_{x,y}\mathcal{B}\right)[f_4]xJ(x)F_4\\
&+ 2f_2 J(y)F_2 - 2f_3 yF_3 + 2f_4 J(y)J(x)F_4 - 2f_4 xyF_4,
\end{aligned}$$

whence, in order to prove (2.24), it now suffices to show that $\{\partial_x, J(y)\}$ and $2D_{J(y),x} + 2\delta_{x,y}\mathcal{B}$ coincide on the generators of $R_0(S \cup J(S))$. Direct computation indeed shows that

$$\begin{aligned}
\{\partial_x, J(y)\}[\{z,w\}] &= 2\big(\delta_{x,z}\{w, J(y)\} + \delta_{x,w}\{z, J(y)\}\big) + 2\delta_{x,y}\mathcal{B}\{z,w\},\\
\{\partial_x, J(y)\}[\{J(z),w\}] &= 2\big(\delta_{x,w}\{z,y\} - \delta_{x,z}\{w,y\}\big) + 2\delta_{x,y}\mathcal{B}\{J(z),w\},
\end{aligned}$$

which coincide with $\left(2D_{J(y),x} + 2\delta_{x,y}\mathcal{B}\right)[\{z,w\}]$ and $\left(2D_{J(y),x} + 2\delta_{x,y}\mathcal{B}\right)[\{J(z),w\}]$ respectively.

In order to prove (2.25) we first observe

$$\begin{aligned}\mathcal{B}\partial_x[F] &= \mathcal{B}\partial_x[f_1]F_1 + \mathcal{B}\partial_x[f_2]xF_2 + \mathcal{B}\partial_x[f_3]J(x)F_3 + \mathcal{B}\partial_x[f_4]xJ(x)F_4\\ &\quad + f_2\mathfrak{m}\mathcal{B}F_2 + 2f_3\mathcal{B}^2F_3 + (\mathfrak{m}+2)f_4\mathcal{B}J(x)F_4 - 2f_4\mathcal{B}x\mathcal{B}F_4.\end{aligned}$$

We now compute $\partial_x[\mathcal{B}F]$ using **(A3|)** and **(DH3)-(DH4)**. We consecutively find

$$\begin{aligned}\partial_x[\mathcal{B}x] &= \partial_x[x\mathcal{B} - 2J(x)] = (\mathfrak{m}-4)\mathcal{B},\\ \partial_x[\mathcal{B}J(x)] &= \partial_x[J(x)\mathcal{B} + 2x] = 2\mathcal{B}^2 + 2\mathfrak{m},\\ \partial_x[\mathcal{B}xJ(x)] &= \partial_x[xJ(x)\mathcal{B}] = (\mathfrak{m}+2)J(x)\mathcal{B} - 2x\mathcal{B}^2\\ &= (\mathfrak{m}+2)\mathcal{B}J(x) - 2(\mathfrak{m}+2)x - 2\mathcal{B}x\mathcal{B} - 4J(x)\mathcal{B},\end{aligned}$$

which yields

$$\begin{aligned}\partial_x[\mathcal{B}F] &= \partial_x[f_1]\mathcal{B}F_1 + \partial_x[f_2]\mathcal{B}xF_2 + \partial_x[f_3]\mathcal{B}J(x)F_3 + \partial_x[f_4]\mathcal{B}xJ(x)F_4\\ &\quad + (\mathfrak{m}-4)f_2\mathcal{B}F_2 + 2f_3\mathcal{B}^2F_3 + 2f_3\mathfrak{m}F_3 + (\mathfrak{m}+2)f_4\mathcal{B}J(x)F_4\\ &\quad - 2(\mathfrak{m}+2)f_4xF_4 - 2f_4\mathcal{B}x\mathcal{B}F_4 - 4f_4J(x)\mathcal{B}F_4.\end{aligned}$$

Then we obtain

$$\begin{aligned}[\partial_x,\mathcal{B}][F] &= [\partial_x,\mathcal{B}][f_1]F_1 + [\partial_x,\mathcal{B}][f_2]xF_2 + [\partial_x,\mathcal{B}][f_3]J(x)F_3 + [\partial_x,\mathcal{B}][f_4]xJ(x)F_4\\ &\quad - 4f_2\mathcal{B}F_2 + 2f_3\mathfrak{m}F_3 - 2(\mathfrak{m}+2)f_4xF_4 - 4f_4J(x)\mathcal{B}F_4.\end{aligned}$$

On the other hand, it is easily seen that

$$\begin{aligned}2\partial_{J(x)}[F] &= 2\partial_{J(x)}[f_1]F_1 + 2\partial_{J(x)}[f_2]xF_2 + 2\partial_{J(x)}[f_3]J(x)F_3 + 2\partial_{J(x)}[f_4]xJ(x)F_4\\ &\quad - 4f_2\mathcal{B}F_2 + 2f_3\mathfrak{m}F_3 - 2(\mathfrak{m}+2)f_4xF_4 - 4f_4J(x)\mathcal{B}F_4,\end{aligned}$$

whence it suffices to show that the operators $[\partial_x,\mathcal{B}]$ and $2\partial_{J(x)}$ coincide on the generators of $R_0(S\cup J(S))$. To that end, observe that

$$\begin{aligned}[\partial_x,\mathcal{B}]\{y,z\} &= \big[\partial_x[\{y,z\}],\mathcal{B}\big] = 2J\big(\partial_x[\{y,z\}]\big) = 2\partial_{J(x)}[\{y,z\}],\\ [\partial_x,\mathcal{B}]\{J(y),z\} &= \big[\partial_x[\{J(y),z\}],\mathcal{B}\big] = 2J\big(\partial_x[\{J(y),z\}]\big) = 2\partial_{J(x)}[\{J(y),z\}].\end{aligned}$$

□

Using the recursion formulae for the vector and directional derivatives given in Lemma 2.11 and Theorem 2.11 one can proof the following extensions of Theorems 2.4 and 2.7 to the Radial algebra with complex structure.

Proposition 2.4. *Let $x,y\in S$. Then in $R(S\cup J(S),\mathcal{B})$ one has:*

i) $\{\partial_x,\partial_y\} = \{\partial_{J(x)},\partial_{J(y)}\}$,

ii) $\{\partial_{J(x)},\partial_y\} = -\{\partial_x,\partial_{J(y)}\}$.

Proposition 2.5. *Let $x, y, z \in S$. Then in $R(S \cup J(S), \mathcal{B})$ one has:*

i) $[D_{y,x}, \partial_z] = -\delta_{y,z}\partial_x$,

ii) $[D_{y,x}, \partial_{J(z)}] = -\delta_{y,z}\partial_{J(x)}$,

iii) $[D_{J(y),x}, \partial_{J(z)}] = -\delta_{y,z}\partial_x$,

iv) $[D_{J(y),x}, \partial_z] = \delta_{y,z}\partial_{J(x)}$.

Proposition 2.6. *Let $x, y, z, w \in S$. Then in $R(S \cup J(S), \mathcal{B})$ one has:*

i) $[D_{y,x}, D_{w,z}] = \delta_{x,w}D_{y,z} - \delta_{y,z}D_{w,x}$,

ii) $[D_{J(y),x}, D_{J(w),z}] = \delta_{y,z}D_{w,x} - \delta_{x,w}D_{y,z}$,

iii) $[D_{J(y),x}, D_{w,z}] = \delta_{x,w}D_{J(y),z} - \delta_{y,z}D_{J(w),x}$.

2.5.2 Hermitian radial algebra

We now introduce the vector variables and operators in the Hermitian setting using the above complex structure J. To this end, let $R_{\mathbb{C}}(S \cup J(S), \mathcal{B})$ be the complexification of $R(S \cup J(S), \mathcal{B})$, i.e.

$$R_{\mathbb{C}}(S \cup J(S), \mathcal{B}) = R(S \cup J(S), \mathcal{B}) \oplus i\, R(S \cup J(S), \mathcal{B}),$$

where i denotes the imaginary unit in $\mathbb{C}$ ($i^2 = -1$) and commutes with every element in $R(S \cup J(S), \mathcal{B})$.

We define in $R_{\mathbb{C}}(S \cup J(S), \mathcal{B})$ the involution $\cdot^\dagger$ known as the *Hermitian conjugation*, by

$$(a + ib)^\dagger = \overline{a} - i\overline{b}, \quad a, b \in R(S \cup J(S), \mathcal{B}). \tag{2.26}$$

The complex vector variables are elements of the (mutually conjugate) sets

$$S_{\mathbb{C}} := \left\{ Z = \frac{1}{2}\big(x + iJ(x)\big) : x \in S \right\},$$
$$S_{\mathbb{C}}^\dagger := \left\{ Z^\dagger = -\frac{1}{2}\big(x - iJ(x)\big) : x \in S \right\}.$$

These complex abstract vector variables generate, together with $\mathcal{B}$, the Hermitian radial algebra:

$$R(S_{\mathbb{C}}, S_{\mathbb{C}}^\dagger, \mathcal{B}) = \mathrm{Alg}_{\mathbb{C}}\{S_{\mathbb{C}} \cup S_{\mathbb{C}}^\dagger \cup \{\mathcal{B}\}\} \subset R_{\mathbb{C}}(S \cup J(S), \mathcal{B}),$$

which submits to the following rules (equivalent to **(AH1)-(AH3)**):

(AH1*) $\{Z, U\} = 0, \quad \{Z^\dagger, U^\dagger\} = 0, \quad \forall Z, U \in S_{\mathbb{C}}$,

(AH2*) $[V, \{Z, U^\dagger\}] = 0, \quad [V^\dagger, \{Z, U^\dagger\}] = 0, \quad \forall Z, U, V \in S_{\mathbb{C}}$,

(AH3*) $[\mathcal{B}, Z] = 2iZ, \quad [\mathcal{B}, Z^\dagger] = -2iZ^\dagger, \quad \forall Z \in S_{\mathbb{C}}$.

Every representation Ψ of the radial algebra with complex structure $R(S \cup J(S), \mathcal{B})$ naturally defines a representation of the Hermitian radial algebra $R(S_{\mathbb{C}}, S_{\mathbb{C}}^{\dagger}, \mathcal{B})$ generated by the elements

$$\Psi(Z) := \frac{1}{2}\left(\Psi(x) + i\Psi(J(x))\right), \qquad \Psi(Z^{\dagger}) := -\frac{1}{2}\left(\Psi(x) - i\Psi(J(x))\right), \qquad x \in S;$$

and $\Psi(\mathcal{B})$.

Remark 2.22. *In the Clifford polynomial representation, the actions of the projection operators* $\pm\frac{1}{2}(\mathbf{1} \pm i\underline{J})$ *on the* $\mathbb{R}_{0,2m}$*-valued vector variables* $\underline{x}$ *generate the maps* $\underline{Z}$ *and* $\underline{Z}^{\dagger}$ *of the complex abstract vector variables* Z *and* $Z^{\dagger}$ *respectively, i.e.*

$$Z \longrightarrow \underline{Z} = \frac{1}{2}(\mathbf{1} + i\underline{J})[\underline{x}] = \frac{1}{2}\sum_{j=1}^{m}(x_j + i x_{m+j})(e_j - i e_{m+j}) = \sum_{j=1}^{m} z_j \mathfrak{f}_j,$$

$$Z^{\dagger} \longrightarrow \underline{Z}^{\dagger} = -\frac{1}{2}(\mathbf{1} - i\underline{J})[\underline{x}] = -\frac{1}{2}\sum_{j=1}^{m}(x_j - i x_{m+j})(e_j + i e_{m+j}) = \sum_{j=1}^{m} z_j^c \mathfrak{f}_j^{\dagger},$$

where $z_j := x_j + i x_{m+j}$, $z_j^c := x_j - i x_{m+j}$ *and*

$$\mathfrak{f}_j = \frac{1}{2}(\mathbf{1} + i\underline{J})[e_j] = \frac{1}{2}(e_j - i e_{m+j}), \qquad j = 1, \dots, m,$$
$$\mathfrak{f}_j^{\dagger} = -\frac{1}{2}(\mathbf{1} - i\underline{J})[e_j] = -\frac{1}{2}(e_j + i e_{m+j}), \qquad j = 1, \dots, m, \tag{2.27}$$

are the well-known Witt basis elements.

Next the Hermitian vector derivatives $\partial_Z, \partial_{Z^{\dagger}} \in \text{End}\Big(R(S_{\mathbb{C}}, S_{\mathbb{C}}^{\dagger}, \mathcal{B})\Big)$ are given by

$$\partial_Z := \frac{1}{4}\left(\partial_x - i\partial_{J(x)}\right), \qquad \partial_{Z^{\dagger}} := -\frac{1}{4}\left(\partial_x + i\partial_{J(x)}\right),$$

where clearly $\partial_x, \partial_{J(x)}$ are assumed to be linear in the complexification of $R(S \cup J(S), \mathcal{B})$. The operators $\partial_Z, \partial_{Z^{\dagger}}$ satisfy the following relations, equivalent to **(DH1)-(DH4)**:

(DH1*) $\partial_Z[fF] = \partial_Z[f]F + f\partial_Z[F], \quad \partial_{Z^{\dagger}}[fF] = \partial_{Z^{\dagger}}[f]F + f\partial_{Z^{\dagger}}[F],$
$[fF]\partial_Z = F[f]\partial_Z + f[F]\partial_Z, \quad [fF]\partial_{Z^{\dagger}} = F[f]\partial_{Z^{\dagger}} + f[F]\partial_{Z^{\dagger}},$
$f \in R_0(S_{\mathbb{C}}, S_{\mathbb{C}}^{\dagger}, \mathcal{B}) := \text{Alg}_{\mathbb{C}}\{\{Z, U\} : Z, U \in S_{\mathbb{C}} \cup S_{\mathbb{C}}^{\dagger}\}, \quad F \in R(S_{\mathbb{C}}, S_{\mathbb{C}}^{\dagger}, \mathcal{B}),$

(DH2*)
$$\begin{cases} \partial_Z[G] = [G]\partial_Z = 0, \\ \partial_Z[ZG] = \partial_Z[Z]G, \\ [GZ]\partial_Z = G[Z]\partial_Z \end{cases} \qquad \forall G \in \text{Alg}_{\mathbb{C}}\big((S_{\mathbb{C}} \setminus \{Z\}) \cup S_{\mathbb{C}}^{\dagger} \cup \{\mathcal{B}\}\big),$$

$$\begin{cases} \partial_{Z^{\dagger}}[G] = [G]\partial_{Z^{\dagger}} = 0, \\ \partial_{Z^{\dagger}}[Z^{\dagger}G] = \partial_{Z^{\dagger}}[Z^{\dagger}]G, \\ [GZ^{\dagger}]\partial_{Z^{\dagger}} = G[Z^{\dagger}]\partial_{Z^{\dagger}} \end{cases} \qquad \forall G \in \text{Alg}_{\mathbb{C}}\Big(S_{\mathbb{C}} \cup \Big(S_{\mathbb{C}}^{\dagger} \setminus \{Z^{\dagger}\}\Big) \cup \{\mathcal{B}\}\Big),$$

(DH3*) $\partial_Z[Z] = \frac{1}{2}(\frac{\mathfrak{m}}{2} + i\mathcal{B}), \quad \partial_{Z^\dagger}[Z^\dagger] = \frac{1}{2}(\frac{\mathfrak{m}}{2} - i\mathcal{B}),$
$[Z]\partial_Z = \frac{1}{2}(\frac{\mathfrak{m}}{2} - i\mathcal{B}), \quad [Z^\dagger]\partial_{Z^\dagger} = \frac{1}{2}(\frac{\mathfrak{m}}{2} + i\mathcal{B}),$

(DH4*) $\partial_Z(U,Z) = U^\dagger = (U,Z)\partial_Z,$
$\partial_{Z^\dagger}(Z,U) = \partial_{Z^\dagger}(U^\dagger, Z^\dagger) = U = (U^\dagger, Z^\dagger)\partial_{Z^\dagger},$
for all $Z, U \in S_{\mathbb{C}}$, and where $(U,Z) := \{U^\dagger, Z\}$.

Remark 2.23. *The vector derivatives ∂_Z and $\partial_{Z^\dagger}$ are mapped by the Clifford-polynomial representation to the Hermitian Dirac operators in the Clifford setting:*

$$\partial_{\underline{Z}} = -\frac{1}{4}\left(\partial_{\underline{x}} - i\partial_{\underline{J(x)}}\right) = -\frac{1}{4}\sum_{j=1}^{m}(e_j + ie_{m+j})(\partial_{x_j} - i\partial_{x_{m+j}}) = \sum_{j=1}^{m}\mathfrak{f}_j^\dagger \partial_{z_j},$$
$$\partial_{\underline{Z}^\dagger} = \frac{1}{4}\left(\partial_{\underline{x}} + i\partial_{\underline{J(x)}}\right) = \frac{1}{4}\sum_{j=1}^{m}(e_j - ie_{m+j})(\partial_{x_j} + i\partial_{x_{m+j}}) = \sum_{j=1}^{m}\mathfrak{f}_j \partial_{z_j^c},$$

where $\partial_{z_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{x_{m+j}})$ and $\partial_{z_j^c} := \frac{1}{2}(\partial_{x_j} + i\partial_{x_{m+j}})$ as usual.

Remark 2.24. *Note that on the polynomial level Z and $Z^\dagger$ behave as independent variables, as illustrated by $\partial_Z[Z^\dagger] = \partial_{Z^\dagger}[Z] = 0$. This is also noticeable in the relations* **(DH4*)** *which are valid for every $Z, U \in S_{\mathbb{C}}$ including $Z = U$.*

Remark 2.25. *As in the previous cases, we can easily prove that* **(DH2*)** *is equivalent to*

$$\begin{cases}\partial_Z[FG] = \partial_Z[F]G, \\ [GF]\partial_Z = G[F]\partial_Z,\end{cases} \qquad \forall F \in R(S_{\mathbb{C}}, S_{\mathbb{C}}^\dagger, \mathcal{B}), \ \forall G \in \mathrm{Alg}_{\mathbb{C}}\big((S_{\mathbb{C}} \setminus \{Z\}) \cup S_{\mathbb{C}}^\dagger \cup \{\mathcal{B}\}\big),$$
$$\begin{cases}\partial_{Z^\dagger}[FG] = \partial_{Z^\dagger}[F]G, \\ [GF]\partial_{Z^\dagger} = G[F]\partial_{Z^\dagger},\end{cases} \qquad \forall F \in R(S_{\mathbb{C}}, S_{\mathbb{C}}^\dagger, \mathcal{B}), \ \forall G \in \mathrm{Alg}_{\mathbb{C}}\Big(S_{\mathbb{C}} \cup \left(S_{\mathbb{C}}^\dagger \setminus \{Z^\dagger\}\right) \cup \{\mathcal{B}\}\Big).$$

In [64], **(AH1*)-(AH3*)** and **(DH1*)-(DH4*)** were used to define the Hermitian radial algebra. Here, we have obtained them by the introduction of a complex structure J subject to **(AH1)-(AH3)** and **(DH1)-(DH4)**. Conversely, the complex structure can also be obtained from the Hermitian radial algebra; indeed, it suffices to consider

$$x = Z - Z^\dagger, \quad J(x) = -i(Z + Z^\dagger), \quad \partial_x = 2\left(\partial_Z - \partial_{Z^\dagger}\right), \quad \partial_{J(x)} = 2i\left(\partial_Z + \partial_{Z^\dagger}\right), \quad \forall Z \in S_{\mathbb{C}}.$$

On account of Theorem 2.11 direct computation yields the following recursion formula for ∂_Z:

$$\partial_Z[UF] = -U\partial_Z[F] + \frac{1}{2}\left(D_{y,x} + iD_{J(y),x}\right)[F] + \frac{1}{4}\delta_{Z,U}(\mathfrak{m} + 2i\mathcal{B})F,$$

where $Z = \frac{1}{2}(x + iJ(x))$, $U = \frac{1}{2}(y + iJ(y))$ and $F = F_1 + iF_2$ with $F_1, F_2 \in R(S \cup J(S), \mathcal{B})$. In this way we obtain the *complex directional derivative* $D_{U,Z}$ and its Hermitian conjugate $D_{U,Z}^\dagger$ given by

$$D_{U,Z} := \frac{1}{2}\left(D_{y,x} + iD_{J(y),x}\right), \quad D_{U,Z}^\dagger := \frac{1}{2}\left(D_{y,x} - iD_{J(y),x}\right). \tag{2.28}$$

Remark 2.26. *The Clifford polynomial representation maps $D_{U,Z}$ and $D_{U,Z}^\dagger$ to the complex differential operators respectively*

$$D_{\underline{U},\underline{Z}} = \frac{1}{2}\left(D_{\underline{y},\underline{x}} + iD_{\underline{J(y)},\underline{x}}\right) = \frac{1}{2}\sum_{j=1}^{m}(y_j + iy_{m+j})(\partial_{x_j} - i\partial_{x_{m+j}}) = \sum_{j=1}^{m} u_j \partial_{z_j},$$
$$D_{\underline{U},\underline{Z}}^\dagger = \frac{1}{2}\left(D_{\underline{y},\underline{x}} - iD_{\underline{J(y)},\underline{x}}\right) = \frac{1}{2}\sum_{j=1}^{m}(y_j - iy_{m+j})(\partial_{x_j} + i\partial_{x_{m+j}}) = \sum_{j=1}^{m} u_j^c \partial_{z_j^c}.$$

From the definitions of $D_{y,x}$ and $D_{J(y),x}$ we can easily obtain the following properties of the complex directional derivatives, which can be used also as defining relations:

$$\begin{aligned} &D_{U,Z}[FG] = D_{U,Z}[F]G + FD_{U,Z}[G], \quad D_{U,Z}[W] = \delta_{Z,W}U, \quad D_{U,Z}[W^\dagger] = 0, \\ &D_{U,Z}^\dagger[FG] = D_{U,Z}^\dagger[F]G + FD_{U,Z}^\dagger[G], \quad D_{U,Z}^\dagger[W^\dagger] = \delta_{Z,W}U^\dagger, \quad D_{U,Z}^\dagger[W] = 0. \end{aligned} \tag{2.29}$$

For $Z = U$ we obtain the Hermitian Euler operators $E_Z := D_{Z,Z}$ and $E_{Z^\dagger} := D_{Z,Z}^\dagger$. As it is expected, these Euler operators yield the respective degrees of the variables Z and $Z^\dagger$ in every product of vector variables; this property immediately follows from (2.29).

Lemma 2.13. *Let $V_j \in S_\mathbb{C} \cup S_\mathbb{C}^\dagger$, $j = 1, \ldots, s$. Then:*

- *Z occurs k times in $V_1 \cdots V_s$ if and only if $E_Z[V_1 \cdots V_s] = kV_1 \cdots V_s$;*
- *$Z^\dagger$ occurs ℓ times in $V_1 \cdots V_s$ if and only if $E_{Z^\dagger}[V_1 \cdots V_s] = \ell V_1 \cdots V_s$.*

Summarizing, we can now rephrase the recursion formulae given in Theorem 2.11 as follows:

$$\begin{cases} \{\partial_Z, U\} = D_{U,Z} + \frac{1}{2}\delta_{Z,U}(\frac{\mathfrak{m}}{2} + i\mathcal{B}), \\ \{\partial_Z, U^\dagger\} = 0, \\ [\partial_Z, \mathcal{B}] = 2i\partial_Z, \end{cases} \qquad \begin{cases} \{\partial_{Z^\dagger}, U^\dagger\} = D_{U,Z}^\dagger + \frac{1}{2}\delta_{Z,U}(\frac{\mathfrak{m}}{2} - i\mathcal{B}), \\ \{\partial_{Z^\dagger}, U\} = 0, \\ \left[\partial_Z^\dagger, \mathcal{B}\right] = -2i\partial_{Z^\dagger}. \end{cases} \tag{2.30}$$

Similar to the Euclidean case, it is known form Hermitian Clifford analysis that the complex vector variables $\underline{Z}, \underline{Z}^\dagger$ and the Hermitian Dirac operators $\partial_{\underline{Z}}, \partial_{\underline{Z}^\dagger}$ generate the Lie superalgebra $\mathfrak{sl}(1|2)$, see [12]. This result also holds in the Hermitian radial algebra and can be proven as a consequence of the commutation relations given in Propositions 2.4, 2.5 and 2.6.

Proposition 2.7. *Let* $Z = \frac{1}{2}\big(x + iJ(x)\big) \in S_{\mathbb{C}}$, $x \in S$. *Then the operators* $Z, Z^\dagger, \partial_Z, \partial_{Z^\dagger} \in \mathrm{End}(R(S_{\mathbb{C}}, S_{\mathbb{C}}^\dagger, \mathcal{B}))$ *are odd generators of the Lie superalgebra* $\mathfrak{sl}(1|2)$. *The representation* $\mathfrak{sl}(1|2) \subset \mathrm{End}(R(S_{\mathbb{C}}, S_{\mathbb{C}}^\dagger, \mathcal{B}))$ *is given by the correspondences*

$$\begin{aligned}
H &= \frac{1}{2}\left(\mathbb{E}_x + \frac{\mathfrak{m}}{2}\right) = \frac{1}{2}\left(\mathbb{E}_Z + \mathbb{E}_{Z^\dagger} + \frac{\mathfrak{m}}{2}\right), & G^+ &= -\frac{Z}{\sqrt{2}},\\
E^+ &= \frac{x^2}{2} = -\frac{1}{2}\{Z, Z^\dagger\}, & G^- &= \sqrt{2}\partial_{Z^\dagger},\\
E^- &= -\frac{\partial_x^2}{2} = 2\{\partial_Z, \partial_{Z^\dagger}\}, & \overline{G^+} &= \frac{Z^\dagger}{\sqrt{2}},\\
L &= -\frac{i}{2}\left(D_{J(x),x} + \mathcal{B}\right) = \frac{1}{2}\left(\mathbb{E}_Z - \mathbb{E}_{Z^\dagger} - i\mathcal{B}\right), & \overline{G^-} &= \sqrt{2}\partial_Z,
\end{aligned}$$

where

$$\begin{aligned}
&[E^+, E^-] = 2H, && [H\ , G^\pm] = \pm\frac{1}{2}G^\pm, && \{G^\pm, G^\pm\} = 0 = \{\overline{G^\pm}, \overline{G^\pm}\},\\
&[H\ , E^\pm] = \pm E^\pm, && [H\ , \overline{G^\pm}] = \pm\frac{1}{2}\overline{G^\pm}, && \{G^\pm, G^\mp\} = 0 = \{\overline{G^\pm}, \overline{G^\mp}\},\\
&[L\ , H\ ] = [L, E^\pm] = 0, && [E^\pm, G^\pm] = [E^\pm, \overline{G^\pm}] = 0, && \{G^\pm, \overline{G^\pm}\} = E^\pm,\\
& && [E^\pm, G^\mp] = -G^\pm, && \{G^\pm, \overline{G^\mp}\} = L \mp H,\\
& && [E^\pm, \overline{G^\mp}] = \overline{G^\pm}, && \\
& && [L\ , G^\pm] = \frac{1}{2}G^\pm, && \\
& && [L\ , \overline{G^\pm}] = -\frac{1}{2}\overline{G^\pm}. &&
\end{aligned}$$

2.6 Relation between $R(S \cup S|, B)$ and $R(S \cup J(S), \mathcal{B})$

In the $\mathbb{R}_{0,2m}$-Clifford representation we have, by the identification $e_{m+j} = i\varepsilon_j$ for $j = 1, \ldots, m$, that the vector variables can be written as

$$\begin{aligned}
\underline{x} &= \sum_{j=1}^{m} x_j e_j + i\sum_{j=1}^{m} x_{m+j}\varepsilon_j = \underline{a} + i\underline{b}|,\\
\underline{J}(\underline{x}) &= \sum_{j=1}^{m} x_{m+j} e_j - i\sum_{j=1}^{m} x_j \varepsilon_j = \underline{b} - i\underline{a}|,
\end{aligned} \tag{2.31}$$

with $\underline{a}, \underline{b} \in \mathbb{R}_{0,m}$ and $\underline{a}|, \underline{b}| \in \mathbb{R}_{m,0}$. An abstract equivalent of this approach can also be developed in the radial algebra setting. In Section 2.3 we obtained from $R(S)$ the superset $R(S \cup S|, B)$ on the level of the endomorphisms, after which, in Section 2.4, we defined the corresponding generalizations of the operators $D_{y,x}, \partial_x, \partial_x|$ on this new algebraic structure. From this setting it is possible to obtain a representation of the radial algebra with complex structure $R(S \cup J(S), \mathcal{B})$.

In order to establish such a representation, we need two different radial algebras $R(S)$ and $R(S_1)$ where $S = \{x, y, \ldots\}$ and $S_1 = \{a, b, \ldots\}$ are disjoint sets of ℓ and 2ℓ abstract vector variables, respectively. We assign to each $x \in S$ an ordered pair $O_x = (a, b)$ of elements of S_1 such that the associated family of subsets $\{C_x = \{a, b\} : x \in S\}$ constitutes a partition of S_1. Let us now introduce the mapping

$$x \to X = a + ib|, \qquad\qquad O_x = (a, b), \;\; x \in S,$$

from S to the set $S_2 = \{a + ib| : (a, b) = O_x, x \in S\}$ composed of *"abstract doubled vector variables"*. It is clearly seen that S_2 is a subset of the complexification of the algebra $R(S_1 \cup S_1|, B_1) \subset End(R(S_1))$, see Section 2.3. Here, B_1 denotes the corresponding B operator on $R(S_1)$, i.e.

$$B_1 = \{\partial_a, a|\}, \qquad \text{for all } a \in S_1.$$

We also denote by $\mathfrak{m}_1$ the abstract dimension corresponding to $R(S_1)$, i.e.

$$\mathfrak{m}_1 = \partial_a[a], \qquad \text{for all } a \in S_1.$$

In view of **(A1)-(A1|)** we obtain for every pair $X = a + ib|$ and $Y = c + id|$ in S_2 that

$$\{X, Y\} = \{a, c\} - \{b|, d|\} = \{a, c\} + \{b, d\}, \tag{2.32}$$

which clearly commutes with every element in $R(S_1 \cup S_1|, B_1)$. Then S_2 satisfies **(A1)** and in consequence it generates a radial algebra representation $R(S_2)$ of $R(S)$.

It is possible to define a complex structure J_2 on $R(S_2)$ by

$$J_2(X) = J_2(a + ib|) = b - ia|, \qquad \forall X \in S_2, \tag{2.33}$$

and the corresponding vector derivatives are given by

$$\partial_X := \partial_a + i\partial_b|, \qquad\qquad \partial_{J_2(X)} := \partial_b - i\partial_a|. \tag{2.34}$$

This map J_2 in fact is a complex structure. Indeed, $J_2(S_2)$ is a disjoint copy of S_2 and it is carefully checked below that **(AH1) -(AH3)** are fulfilled.

We first check **(AH1)-(AH2)**. Using **(A1|)** and (2.32) we obtain

$$\{J_2(X), J_2(Y)\} = \{b - ia|, d - ic|\} = \{b, d\} - \{a|, c|\} = \{a, c\} + \{b, d\} = \{X, Y\},$$

and also

$$\begin{aligned} \{J_2(X), Y\} &= \{b - ia|, c + id|\} = \{b, c\} - \{a, d\}, \\ \{X, J_2(Y)\} &= \{a + ib|, d - ic|\} = \{a, d\} - \{b, c\}. \end{aligned}$$

Consequently, $\{X, Y\}$, $\{J_2(X), Y\}$ are central elements and $\{J_2(X), Y\} = -\{X, J_2(Y)\}$.

Next, in order to check **(AH3)**, we first need to obtain a candidate to represent the element $\mathcal{B}$ in this setting. To this end we compute the action $\partial_X[J_2(X)]$; by Theorem 2.9 **(D3|)** we have

$$\partial_X[J_2(X)] = (\partial_a + i\partial_b|)(b - ia|) = -i(\partial_a[a|] - \partial_b|[b]) = -2iB_1.$$

Then on view of **(A3|)** we get

$$[-iB_1, X] = [-iB_1, a + ib|] = [B_1, b|] - i[B_1, a] = -2b + 2ia| = -2J_2(X),$$
$$[-iB_1, J_2(X)] = [-iB_1, b - ia|] = -[B_1, a|] - i[B_1, b] = 2a + 2ib| = 2X.$$

This way, we have proven that the correspondences

$$x \to X = a + ib|, \qquad J(x) \to J_2(X) = b - ia|, \qquad \mathcal{B} \to -iB_1,$$

define a representation of the radial algebra with complex structure $R(S\cup J(S),\mathcal{B})$ into the complexification of $R(S_1 \cup S_1|, B_1)$. Such a representation will be denoted by

$$\Psi_2 : R(S\cup J(S),\mathcal{B}) \to R(S_2 \cup J_2(S_2), -iB_1).$$

As mentioned before, by this representation the vector derivatives ∂_x, $\partial_{J(x)}$ get mapped into the operators ∂_X, $\partial_{J_2(X)}$ defined in (2.34).

Proposition 2.8. *The operators ∂_X, $\partial_{J_2(X)}$ satisfy* **(DH1)**- **(DH4)**.

Proof.

It suffices to check **(DH3)-(DH4)**, since **(DH1)** and **(DH2)** are direct consequences of (**D1**|) and (**D2**|). For **(DH3)** we obtain, on account of (**D3**|),

$$\partial_X[X] = (\partial_a + i\partial_b|)(a + ib|) = \partial_a[a] - \partial_b|[b|] = 2\mathfrak{m}_1,$$
$$\partial_{J_2(X)}[J_2(X)] = (\partial_b - i\partial_a|)(b - ia|) = \partial_b[b] - \partial_a|[a|] = 2\mathfrak{m}_1,$$
$$\partial_{J_2(X)}[X] = (\partial_b - i\partial_a|)(a + ib|) = i(\partial_b[b|] - \partial_a|[a]) = 2iB_1,$$

The corresponding right actions can be computed in a similar way. From these equalities, we see that the abstract dimension corresponding to the radial algebra $R(S_2)$, generated by the doubled vector variables, is in fact the double of the abstract dimension $\mathfrak{m}_1$ of $R(S_1)$.

To prove **(DH4)**, we first observe that

$$X^2 = (a + ib|)^2 = a^2 - b|^2 + i\{a, b|\} = a^2 + b^2.$$

Then by (**D4**|) we obtain

$$\partial_X[X^2] = (\partial_a + i\partial_b|)(a^2 + b^2) = \partial_a[a^2] + i\partial_b|[b^2] = 2a + i2b| = 2X,$$
$$\partial_{J_2(X)}[X^2] = (\partial_b - i\partial_a|)(a^2 + b^2) = \partial_b[b^2] - i\partial_a|[a^2] = 2b - i2a| = 2J_2(X).$$

The computations for $[X^2]\partial_X$ and $[X^2]\partial_{J_2(X)}$ are similar. For the product $XJ_2(X)$ we have $XJ_2(X) = (a+ib|)(b-ia|) = (ab+b|a|) + i(b|b-aa|)$. Using Theorems 2.8 and 2.9 we obtain

$$\begin{aligned}
\partial_X[XJ_2(X)] &= (\partial_a + i\partial_b|)\big[(ab+b|a|) + i(b|b-aa|)\big] \\
&= \Big(\partial_a[ab+b|a|] - \partial_b|[b|b-aa|]\Big) + i\,\Big(\partial_a[b|b-aa|] + \partial_b|[ab+b|a|]\Big) \\
&= \Big(2\mathfrak{m}_1 b - 2b|B_1 + 2b\Big) + i\,\Big(2aB_1 - 2a| - 2\mathfrak{m}_1 a|\Big) \\
&= (2\mathfrak{m}_1+2)(b-ia|) + 2(a+ib|)iB_1 \\
&= (2\mathfrak{m}_1+2)J_2(X) + 2X\,iB_1,
\end{aligned}$$

$$\begin{aligned}
\partial_{J_2(X)}[XJ_2(X)] &= (\partial_b - i\partial_a|)\big[(ab+b|a|) + i(b|b-aa|)\big] \\
&= \Big(\partial_b[ab+b|a|] + \partial_a|[b|b-aa|]\Big) + i\,\Big(-\partial_a|[ab+b|a|] + \partial_b[b|b-aa|]\Big) \\
&= \Big(2a - 2\mathfrak{m}_1 a + 2B_1 a|\Big) + i\,\Big(2B_1 b + 2b| - 2\mathfrak{m}_1 b|\Big) \\
&= (-2\mathfrak{m}_1+2)(a+ib|) + 2iB_1(b-ia|) \\
&= (2-2\mathfrak{m}_1)X + 2iB_1\,J_2(X) \\
&= -(2+2\mathfrak{m}_1)X + 2J_2(X)\,iB_1.
\end{aligned}$$

and similarly for the corresponding right actions. Finally we have, for $X = a+ib|$, $Y = c+id|$ in $\mathcal{S}$ with $X \neq Y$,

$$\begin{aligned}
\partial_{J_2(X)}\{J_2(X),Y\} &= (\partial_b - i\partial_a|)(\{b,c\} - \{a,d\}) \\
&= \partial_b\{b,c\} + i\partial_a|\{a,d\} = 2c + 2id| = 2Y, \\
\partial_X\{X,Y\} &= (\partial_a + i\partial_b|)(\{a,c\} + \{b,d\}) \\
&= \partial_a\{a,c\} + i\partial_b|\{b,d\} = 2c + 2id| = 2Y, \\
\partial_{J_2(X)}\{X,Y\} &= (\partial_b - i\partial_a|)(\{a,c\} + \{b,d\}) \\
&= \partial_b\{b,d\} - i\partial_a|\{a,c\} = 2d - 2ic| = 2J_2(Y), \\
\partial_X\{J_2(X),Y\} &= (\partial_a + i\partial_b|)(\{b,c\} - \{a,d\}) \\
&= -\partial_a\{a,d\} + i\partial_b|\{b,c\} = -2d + 2ic| = -2J_2(Y).
\end{aligned}$$

and in much the same way $\{J_2(X),Y\}\partial_{J_2(X)}$, $\{X,Y\}\partial_X$, $\{X,Y\}\partial_{J_2(X)}$ and $\{J_2(X),Y\}\partial_X$ can be computed. □

To complete the scheme, let us now finally investigate the form of the directional derivatives in this setting.

Proposition 2.9. *By the representation Ψ_2 the directional derivatives $D_{y,x} \in End(R(S \cup J(S), \mathcal{B}))$, $x \in S$, $y \in S \cup J(S)$, get mapped to the operators*

$$D_{Y,X} = D_{c,a} + D_{d,b}, \qquad D_{J_2(Y),X} = D_{d,a} - D_{c,b},$$

where $X = a+ib|$, $Y = c+id| \in S_2$.

Proof.

Using Theorem 2.8 and taking into account that $a \neq d$ and $b \neq c$ we obtain

$$\begin{aligned} 2D_{Y,X} &= \{\partial_X, Y\} - \delta_{X,Y} 2\mathfrak{m}_1 \\ &= \{\partial_a + i\partial_b|, c + id|\} - 2\delta_{X,Y}\mathfrak{m}_1 \\ &= (\{\partial_a, c\} - \{\partial_b|, d|\}) + i\left(\{\partial_a, d|\} + \{\partial_b|, c\}\right) - 2\delta_{X,Y}\mathfrak{m}_1 \\ &= (2D_{c,a} + \delta_{a,c}\mathfrak{m}_1 + 2D_{d,b} + \delta_{b,d}\mathfrak{m}_1) - 2\delta_{X,Y}\mathfrak{m}_1 \\ &= 2\left(D_{c,a} + D_{d,b}\right). \end{aligned}$$

In a similar way we obtain $D_{J_2(Y),X} = D_{d,a} - D_{c,b}$. □

Given any representation $R(\Psi(S_1))$ of the radial algebra $R(S_1)$, the representation Ψ_2 shows a way of defining a complex structure. Indeed, the composition

$$\Psi \circ \Psi_2 : R(S \cup J(S), \mathcal{B}) \to \Psi(R(S_2 \cup J_2(S_2), -iB_1)) \tag{2.35}$$

is a representation for $R(S \cup J(S), \mathcal{B})$ with vector variables

$$\Psi(X) = \Psi(a) + i\Psi(b|), \quad a, b \in S_1, \tag{2.36}$$

and complex structure

$$\Psi(J_2(X)) = \Psi(b) - i\Psi(a|), \quad a, b \in S_1. \tag{2.37}$$

Formulae (2.31) show that this is the case for the Clifford polynomial representation $\underline{\cdot}_{2m}$ of $R(S \cup J(S), \mathcal{B})$ into $\mathcal{A}_{2m,0}$ given in Remark 2.17. Indeed, the mapping $\underline{\cdot}_{2m}$ is the composition of the representation $\underline{\cdot}_m$ from $R(S_1)$ into $\mathcal{A}_{m,0}$ given in example 2.1 with Ψ_2. In other words, the following diagram is commutative:

$$\begin{array}{ccc} R(S \cup J(S), \mathcal{B}) & \xrightarrow{\Psi_2} & R(S_2 \cup J_2(S_2), -iB_1) \\ & \underline{\cdot}_{2m} \searrow & \downarrow \underline{\cdot}_m \\ & & R(\underline{S} \cup \underline{J}(\underline{S}), \underline{\mathcal{B}}) \end{array}$$

We summarize the main aspects of this representation in Table 2.2.

2.7 Concluding remarks

In this chapter we have carefully developed the radial algebra and the Hermitian radial algebra, as the respective correct abstract frameworks for Euclidean and Hermitian Clifford analysis, starting from results already stated in [66] and inspired by an alternative approach in [64]. In the next chapter the Hermitian framework will be combined with the abstract setting for Clifford calculus on superspace, developed in [33], in order to establish Hermitian Clifford analysis in superspace.

	Radial algebra with a complex structure $R(S \cup J(S), \mathcal{B})$	Clifford-polynomial representation $R(\underline{S} \cup \underline{J}(\underline{S}), \underline{\mathcal{B}}) \subset \mathcal{A}_{2m,0}$	Ψ_2-representation $R(S_2 \cup J_2(S_2), -iB_1)$ $\subset R(S_1 \cup S_1\vert, B_1)$
vector variables	$x, y, \ldots \in S$	$\underline{x} = \sum_{j=1}^m (e_j x_j + e_{m+j} x_{m+j})$ $\underline{y} = \sum_{j=1}^m (e_j y_j + e_{m+j} y_{m+j})$	$X = a + ib\vert\,,\ a, b \in S_1,\ a \neq b$ $Y = c + id\vert\,,\ c, d \in S_1,\ c \neq d$
complex structure	$(J, \mathcal{B})$	$\underline{J}(\underline{x}) = \sum_{j=1}^m (x_{m+j} e_j - x_j e_{m+j})$ $\underline{\mathcal{B}} = \sum_{j=1}^m e_j e_{m+j} = i \sum_{j=1}^m e_j \varepsilon_j = -i\underline{B}$ ($\underline{B}$ is bivector realization of B in $R(\underline{S})$)	$J_2(X) = b - ia\vert$ $-iB_1$ (B_1 is the B operator in $R(S_1)$)
vector derivatives	∂_x $\partial_{J(x)}$	$-\partial_{\underline{x}} = -\sum_{j=1}^m (e_j \partial_{x_j} + e_{m+j} \partial_{x_{m+j}})$ $-\partial_{\underline{J}(\underline{x})} = -\sum_{j=1}^m (e_j \partial_{x_{m+j}} - e_{m+j} \partial_{x_j})$	$\partial_X = \partial_a + i\partial_b$ $\partial_{J_2(X)} = \partial_b - i\partial_a\vert$
abstract dimension	$\mathfrak{m}$	$2m$	$2\mathfrak{m}_1$
directional derivatives	$D_{y,x}$ $D_{J(y),x}$	$D_{\underline{y},\underline{x}} = \sum_{j=1}^m (y_j \partial_{x_j} + y_{m+j} \partial_{x_{m+j}})$ $D_{\underline{J}(\underline{y}),\underline{x}} = \sum_{j=1}^m (y_{m+j} \partial_{x_j} - y_j \partial_{x_{m+j}})$	$D_{Y,X} = D_{c,a} + D_{d,b}$ $D_{J_2(Y),X} = D_{d,a} - D_{c,b}$

Table 2.2: Clifford-polynomial representation of the radial algebra with a complex structure

3

Hermitian Clifford analysis on superspace

In the previous chapter we have introduced the so-called Hermitian radial algebra in order to establish the rules allowing for a canonical extension of Hermitian Clifford analysis to superspace. The objective of the current chapter now is to explicitly introduce the fundamental objects of Hermitian Clifford analysis in the superspace setting. This construction is inspired by the successful extension of orthogonal Clifford analysis to superspace, see [33, 68, 32, 69, 71, 67]. In particular the radial algebra was proven to be an efficient tool for giving a meaning to vector spaces of negative dimension, and defining the fundamental objects of Clifford analysis, such as vector variables and vector derivatives, in such a case.

In this chapter we first provide a brief overview on the superanalysis framework. Then we establish the corresponding representation of the radial algebra in superspace together with the mapping of the main radial algebra endomorphisms into this setting. In addition, we will establish the notion of a complex structure, as well as its realization as a bivector, which then will lead to the Witt basis, the Hermitian vector variables, the Hermitian vector derivatives and the complex Euler operators in superspace. These notions constitute the starting point for the study of this representation of the Hermitian radial algebra. In the next chapters we plan to address some other classical issues such as the underlying group structure, spin representations, invariance of the Dirac operators under spin actions and also Bochner-Martinelli formulae in the Hermitian superspace setting.

3.1 Preliminaries on superanalysis

In this section we introduce the building blocks of analysis in superspace, see e.g. [7, 72, 55]. We first give a short introduction to superspaces and commutative superalgebras. In this setting, we provide a brief overview to differential calculus on a commutative Banach superalgebra following the approach by Vladimirov and Volovich [72]. This approach will not be used it in this thesis, but it illustrates how superanalysis works when one considers co-ordinates defined on an underlying superalgebra. Then we introduce our approach to superanalysis which follows the extension of harmonic and Clifford analysis to superspace (see [32, 33, 35, 31]). It considers commuting and anti-commuting variables in a purely symbolic way, defining an associated supermanifold as in the approach of Berezin, see [7].

3.1.1 Differential calculus on a commutative Banach superalgebra

We recall that a vector space V over the field $\mathbb{K}$ (in this thesis $\mathbb{K}$ will always be $\mathbb{R}$ or $\mathbb{C}$) is $\mathbb{Z}_2$-graded (also called super vector space) if it decomposes as the direct sum of two subspaces

$$V = V_{\overline{0}} \oplus V_{\overline{1}}, \qquad \{\overline{0}, \overline{1}\} \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}.$$

Vectors that are elements of either $V_{\overline{0}}$ or $V_{\overline{1}}$ are said to be homogeneous. The parity of a nonzero homogeneous element is *even* or *odd* according to whether it is in $V_{\overline{0}}$ or in $V_{\overline{1}}$, respectively. This is, $V_{\overline{0}}$ is the subspace of all even elements and $V_{\overline{1}}$ is the subspace of all odd elements.

Example 3.1. *The easiest example of a graded vector space is $V = \mathbb{K}^{p,q}$ with standard basis consisting of the vectors $e_1, \ldots, e_p, \grave{e}_1, \ldots, \grave{e}_q$, where*

$$e_j = (0, \ldots, 1, \ldots, 0)^T, \qquad \text{with 1 on the } j\text{-th position}, \qquad j = 1, \ldots, p,$$
$$\grave{e}_j = (0, \ldots, 1, \ldots, 0)^T, \qquad \text{with 1 on the } (p+j)\text{-th position}, \qquad j = 1, \ldots, q.$$

The elements $e_1, \ldots, e_p$ span the subspace $\mathbb{K}^{p,0}$ of even elements and $\grave{e}_1, \ldots, \grave{e}_q$ span the subspace $\mathbb{K}^{0,q}$ of odd elements. As a vector space $\mathbb{K}^{p,q}$ is clearly isomorphic to $\mathbb{K}^{p+q}$.

A superalgebra Λ over $\mathbb{K}$ is an algebra over $\mathbb{K}$ such that it constitutes a Z_2-graded vector space $\Lambda = \Lambda_{\overline{0}} \oplus \Lambda_{\overline{1}}$, and the multiplication on Λ preserves the gradation, i.e.

$$\Lambda_{\overline{j}} \Lambda_{\overline{k}} \subset \Lambda_{\overline{j+k}}, \quad \text{for } \overline{j}, \overline{k} \in \mathbb{Z}_2.$$

In particular, the subspace $\Lambda_{\overline{0}}$ is a subalgebra of Λ. In the superalgebra Λ we introduce the gradation automorphism $\cdot^* : \Lambda \to \Lambda$ by

$$\begin{cases} v^* = v, & v \in \Lambda_{\overline{0}}, \\ \grave{v}^* = -\grave{v}, & \grave{v} \in \Lambda_{\overline{1}}, \end{cases} \qquad (AB)^* = A^* B^*, \quad A, B \in \Lambda. \tag{3.1}$$

Note that $\cdot^*$ is an involution, i.e. $(A^*)^* = A$. In addition, $A^* = A$ if and only if $A \in \Lambda_{\overline{0}}$.

A superalgebra $\Lambda = \Lambda_{\overline{0}} \oplus \Lambda_{\overline{1}}$ is said to be *(super)commutative* if arbitrary homogeneous elements satisfy

$$vw = wv, \qquad \grave{v}\grave{w} = -\grave{w}\grave{v}, \qquad v\grave{w} = \grave{w}v, \qquad v, w \in \Lambda_{\overline{0}}, \;\; \grave{v}, \grave{w} \in \Lambda_{\overline{1}}.$$

Thus, even elements and elements of different parities commute, while odd elements anti-commute. It then holds that

$$\grave{v}A = A^*\grave{v}, \qquad \grave{v} \in \Lambda_{\overline{1}}, \; A \in \Lambda.$$

One approach to superanalysis consists in defining a differential calculus over a commutative superalgebra Λ, see e.g. [72]. To that end it is convenient to consider an identity element $1 \in \Lambda_{\overline{0}}$. In that way one may see the field $\mathbb{K} \subset \Lambda_{\overline{0}}$ as a subalgebra of Λ. In addition, it is necessary to equip the commutative superalgebra Λ with a Banach space structure, by introducing a norm $\|\cdot\|_\Lambda : \Lambda \to \mathbb{R}$ which satisfies the condition

$$\|ab\|_\Lambda \leq \|a\|_\Lambda \, \|b\|_\Lambda, \quad a, b \in \Lambda, \qquad \|1\|_\Lambda = 1, \tag{3.2}$$

where $\Lambda_{\overline{0}}$ and $\Lambda_{\overline{1}}$ are closed subspaces. Observe that the inequality (3.2) ensures that the multiplication operation is continuous. Such an algebra is said to be a *commutative Banach superalgebra.*

The annihilator of the set of odd elements ($\Lambda_{\overline{1}}$-annihilator) is introduced by

$${}^{\perp}\Lambda_{\overline{1}} := \{a \in \Lambda : \; \grave{v}\, a = a\, \grave{v} = 0, \, \forall \grave{v} \in \Lambda_{\overline{1}}\}.$$

Example 3.2 (Grassmann algebras). *An example of commutative Banach superalgebra is provided by any Grassmann algebra of dimension 2^N generated over $\mathbb{K}$ by the odd canonical generators $f_1 \ldots, f_N$ satisfying the conditions $f_j f_k + f_k f_j = 0$. In case we need to explicitly indicate the system of odd generators, we denote our Grassmann algebra by $\mathfrak{G}(f_1 \ldots, f_N)$ and when only their number is important we write $\mathfrak{G}_N$. When necessary, we make a distinction between the real and the complex Grassmann algebras by using the notation $\mathbb{K}\mathfrak{G}_N$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).*

A basis for $\mathfrak{G}_N$ consists of elements of the form $f_\emptyset = 1$, $f_A = f_{j_1} \cdots f_{j_k}$ for $A = \{j_1, \ldots, j_k\}$ $(1 \leq j_1 < \ldots < j_k \leq N)$. Hence an arbitrary element $a \in \mathfrak{G}_N$ has the form

$$a = \sum_{A \subset \{1,\ldots,N\}} a_A f_A, \quad a_A \in \mathbb{K}. \tag{3.3}$$

Every $a \in \mathfrak{G}_N$ may be written as the sum $a = a_0 + \mathbf{a}$ of a number $a_0 := a_\emptyset \in \mathbb{K}$ and a nilpotent element $\mathbf{a} = \sum_{|A| \leq 1} a_A f_A$ (in particular $\mathbf{a}^{N+1} = 0$). The elements a_0, $\mathbf{a}$ are called the body and the nilpotent part of $a \in \mathfrak{G}_N$, respectively.

In this case we will denote the even and odd subspaces by $\mathfrak{G}_N^{(ev)}$ and $\mathfrak{G}_N^{(odd)}$, respectively. The space $\mathfrak{G}_N^{(ev)}$ (respectively, $\mathfrak{G}_N^{(odd)}$) consists of the elements for which each term in the *expansion (3.3) contains only an even (respectively, odd) number of generators. In this way, the gradation automorphism (3.1) is given by* $a^* = \sum_{A\subset\{1,\dots,N\}}(-1)^{|A|}a_A f_A$.

The structure of a Banach superalgebra is introduced in $\mathfrak{G}_N$ *with the norm* $\|a\|_{\mathfrak{G}} = \sum_A |a_A|$ *for* $a \in \mathfrak{G}_N$ *written in the form (3.3).*

The annihilator ${}^{\perp}\mathfrak{G}_N^{(odd)}$ *is easily seen to be* ${}^{\perp}\mathfrak{G}_N^{(odd)} = \{\lambda f_1 \dots, f_N : \lambda \in \mathbb{K}\}$.

Given the graded vector space $\mathbb{R}^{p,q}$ and a commutative Banach superalgebra $\Lambda = \Lambda_{\overline{0}} \oplus \Lambda_{\overline{1}}$ over $\mathbb{R}$, the superspace $\mathbb{R}^{p,q}(\Lambda)$ of dimension (p,q) over Λ is defined as

$$\mathbb{R}^{p,q}(\Lambda) = \underbrace{\Lambda_{\overline{0}} \times \cdots \times \Lambda_{\overline{0}}}_{p} \times \underbrace{\Lambda_{\overline{1}} \times \cdots \times \Lambda_{\overline{1}}}_{q} = \Lambda_{\overline{0}}^p \times \Lambda_{\overline{1}}^q. \tag{3.4}$$

The superspace $\mathbb{R}^{p,q}(\Lambda)$ becomes a Banach space if, for every supervector

$$\mathbf{x} = (\underline{x}, \underline{\grave{x}}) = (x_1, \dots, x_p, \grave{x}_1, \dots, \grave{x}_q) \in \mathbb{R}^{p,q}(\Lambda),$$

we consider the norm

$$\|\mathbf{x}\|_{p,q}^2 = \sum_{j=1}^{p} \|x_j\|_\Lambda^2 + \sum_{j=1}^{q} \|\grave{x}_j\|_\Lambda^2.$$

Note that the superspace is $\mathbb{R}^{p,q}(\Lambda)$ is a $\Lambda_{\overline{0}}$-module.

In the approach to superanalysis developed by Vladimirov and Volovich (see [72]), the superspace $\mathbb{R}^{p,q}(\Lambda)$ plays the same rôle in superanalysis as the space $\mathbb{R}^p$ in classical analysis. They study functions $F : \mathcal{U} \to \Lambda$ where $\mathcal{U}$ is an open set of $\mathbb{R}^{p,q}(\Lambda)$. Observe that in particular, if $\Lambda_{\overline{0}} = \mathbb{R}$ and $\Lambda_{\overline{1}} = \{0\}$, then $\mathbb{R}^{p,0} = \mathbb{R}^p$.

In the works [55, 72], it has been seen that the $\Lambda_{\overline{1}}$-annihilator plays an important rôle in the study of super-differentiable functions. For example, two different polynomials in anti-commuting variables may define the same Λ-valued function if the $\Lambda_{\overline{1}}$-annihilator is different of $\{0\}$. This is easily seen if one considers $\Lambda = \mathfrak{G}_N$; in this case the polynomial $\grave{x}_1\grave{x}_2 \dots \grave{x}_{N+1}$ is identically zero. This fact has important consequences in the super-differentiation theory such as the non-uniqueness of the odd derivatives if ${}^{\perp}\Lambda_{\overline{1}} \neq \{0\}$, see [72]. On the other hand, for ${}^{\perp}\Lambda_{\overline{1}} = \{0\}$ two useful and convenient properties are fulfilled:

1. two polynomials define the same function if and only if they are identical;

2. the odd derivative is unique according to the definition given in [72].

For the sake of simplicity, one may consider the superalgebra Λ such that ${}^{\perp}\Lambda_{\overline{1}} = \{0\}$. By this restriction one does not lose generality since every graded-commutative superalgebra can be embedded in a superalgebra where the annihilator of the odd subspace equals to $\{0\}$.

Definition 3.1 (super-differentiability [55, 72]). *A function $F : \mathcal{U} \to \Lambda$ is said to be super-differentiable at the point $\mathbf{x} \in \mathcal{U}$ if there exist elements $F_1(\mathbf{x}), \ldots, F_p(\mathbf{x})$ and $\grave{F_1}(\mathbf{x}), \ldots, \grave{F_q}(\mathbf{x})$ in Λ such that*

$$F(\mathbf{x}+\mathbf{y}) = F(\mathbf{x}) + \sum_{j=1}^{p} y_j F_j(\mathbf{x}) + \sum_{j=1}^{q} \grave{y_j} \grave{F_j}(\mathbf{x}) + \tau(\mathbf{x},\mathbf{y}), \tag{3.5}$$

where

$$\lim_{\|\mathbf{y}\|_{p,q}\to 0} \frac{\|\tau(\mathbf{x},\mathbf{y})\|_\Lambda}{\|\mathbf{y}\|_{p,q}} = 0.$$

The elements $F_j(\mathbf{x})$ are called (bosonic or even) partial derivatives of F with respect to x_j, $j = 1, \ldots, p$, while the elements $\grave{F_k}(\mathbf{x})$ are called (fermionic or odd) partial derivatives of F with respect to $\grave{x_k}$, , $k = 1, \ldots, q$. They are respectively denoted by

$$F_j(\mathbf{x}) = \partial_{x_j}[F](\mathbf{x}) = \frac{\partial F}{\partial x_j}(\mathbf{x}), \qquad \grave{F_k}(\mathbf{x}) = \partial_{\grave{x_k}}[F](\mathbf{x}) = \frac{\partial F}{\partial \grave{x_k}}(\mathbf{x}).$$

The even derivatives ∂_{x_j} always are uniquely defined. As mentioned before, the assumption ${}^\perp\Lambda_{\overline{1}} = \{0\}$ ensures that the odd fermionic derivatives $\partial_{\grave{x_j}}$ also are unique, see [72, 55].

Remark 3.1. *The partial derivatives defined in (3.5) are called left derivatives. Right derivatives can be introduced similarly, but then we must replace the expression (3.5) by*

$$F(\mathbf{x}+\mathbf{y}) = F(\mathbf{x}) + \sum_{j=1}^{p} F_j(\mathbf{x}) y_j + \sum_{j=1}^{q} \grave{F_j}(\mathbf{x}) \grave{y_j} + \tau(\mathbf{x},\mathbf{y}).$$

It is readily seen that the right and left derivatives with respect to $x_1 \ldots, x_p$ are identical, but those with respect to $\grave{x_1}, \ldots, \grave{x_q}$ are, in general, different. The left and right odd derivatives of a superfunction F are denoted by $\partial_{\grave{x_k}}[F] = \frac{\partial}{\partial \grave{x_k}} F$ and $[F]\partial_{\grave{x_k}} = F \frac{\partial}{\partial \grave{x_k}}$, respectively.

Example 3.3. *Letting the operator $\partial_{\grave{x_j}}$ act on the product $\grave{x_j}\grave{x_k}$, with $j \neq k$, we obtain*

$$\partial_{\grave{x_j}}[\grave{x_j}\grave{x_k}] = \grave{x_k}$$

while

$$[\grave{x_j}\grave{x_k}]\partial_{\grave{x_j}} = -[\grave{x_k}\grave{x_j}]\partial_{\grave{x_j}} = -\grave{x_k}.$$

The above notion of even partial derivative clearly coincides with the classical partial derivative of real analysis when $\Lambda_{\overline{0}} = \mathbb{R}$. It is easily proven that classical properties such as the Leibniz formula and the chain rule still hold for even differentiation. Similar properties also hold for the odd derivatives, but they require minor modifications related to the Z_2-gradation of Λ, see [55, pp. 14-15] for more details.

3.1.2 Supermanifolds and superanalysis

Next to the approach in the previous section, calculus in superspace can also be developed independently of the underlying Banach superalgebra Λ. Instead of co-ordinates on $\Lambda_{\overline{0}}$, $\Lambda_{\overline{1}}$, we consider in this thesis variables in a purely symbolic way. This is, the set of commuting (bosonic) and anti-commuting (fermionic) variables consist of independent symbols $x_1, \ldots, x_p, \grave{x}_1, \ldots, \grave{x}_q$; which gives rise to the supervector variable

$$\mathbf{x} = (\underline{x}, \underline{\grave{x}}) = (x_1, \ldots, x_p, \grave{x}_1, \ldots, \grave{x}_q)\,. \tag{3.6}$$

The algebra generated by these variables over the field $\mathbb{K}$ is denoted by $\mathbb{K}\mathcal{P}$ and is given by

$$\mathbb{K}\mathcal{P} := \mathrm{Alg}_{\mathbb{K}}(x_1, \ldots, x_p, \grave{x}_1, \ldots, \grave{x}_q) = \mathbb{K}[x_1, \ldots, x_p] \otimes \mathfrak{G}_q,$$

where $\mathfrak{G}_q = \mathfrak{G}(\grave{x}_1, \ldots, \grave{x}_q)$ is the Grassmann algebra generated by the anti-commuting variables $\grave{x}_1, \ldots \grave{x}_q$.

The bosonic and fermionic partial derivatives $\partial_{x_j} = \frac{\partial}{\partial x_j}$, $\partial_{\grave{x}_j} = \frac{\partial}{\partial \grave{x}_j}$ are defined as endomorphisms on $\mathbb{K}\mathcal{P}$ by the relations

$$\begin{cases} \partial_{x_j}[1] = 0, \\ \partial_{x_j} x_k - x_k \partial_{x_j} = \delta_{j,k}, \\ \partial_{x_j} \grave{x}_k = \grave{x}_k \partial_{x_j}, \end{cases} \qquad \begin{cases} \partial_{\grave{x}_j}[1] = 0, \\ \partial_{\grave{x}_j} \grave{x}_k + \grave{x}_k \partial_{\grave{x}_j} = \delta_{j,k}, \\ \partial_{\grave{x}_j} x_k = x_k \partial_{\grave{x}_j}, \end{cases} \tag{3.7}$$

which can be recursively applied for both left and right actions. From this definition it immediately follows that

$$\partial_{x_j}\partial_{x_k} = \partial_{x_k}\partial_{x_j}, \qquad \partial_{\grave{x}_j}\partial_{\grave{x}_k} = -\partial_{\grave{x}_k}\partial_{\grave{x}_j}, \qquad \partial_{x_j}\partial_{\grave{x}_k} = \partial_{\grave{x}_k}\partial_{x_j}.$$

Remark 3.2. *When the variables x_j, $\grave{x}_j$ are represented by co-ordinates with values in $\Lambda_{\overline{0}}$ and $\Lambda_{\overline{1}}$ respectively, the notion of partial derivatives given in (3.7) coincides with Definition 3.1 at the polynomial level.*

For the study of more general functions in superanalysis we consider the definition of a supermanifold as in the approach of Berezin and Kostant, see [7, 57, 22]. A sheaf $\mathcal{O}$ of algebras on a p-dimensional manifold $\mathcal{M}_0$, maps every open subset U in $\mathcal{M}_0$ into an algebra $\mathcal{O}(U)$. This mapping is subject to well-know conditions, see [7, 57]. The standard example is the structure sheaf that maps every open subset $U \in \mathcal{M}_0$ to the commutative algebra $C^\infty(U)$ of complex-valued smooth functions on U; this sheaf is denoted by $C^\infty_{\mathcal{M}_0}$. A supermanifold of dimension $p|q$ then is defined as a ringed space $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$, where $\mathcal{M}_0$ is an underlying smooth p-dimensional manifold and $\mathcal{O}_{\mathcal{M}}$ is the structure sheaf that maps every open subset $U \in \mathcal{M}_0$ into the algebra $C^\infty(U) \otimes \mathfrak{G}_q$ of smooth functions in U with values in the Grassmann algebra $\mathfrak{G}_q$. Such a sheaf $\mathcal{O}_{\mathcal{M}}$ is denoted by $C^\infty_{\mathcal{M}_0} \otimes \mathfrak{G}_q$.

Associated to the variables $x_1, \ldots, x_p, \grave{x}_1, \ldots, \grave{x}_q$ we consider the flat supermanifold

$$\mathbb{R}^{p|q} = (\mathbb{R}^p, \mathcal{O}_{\mathbb{R}^{p|q}}) = (\mathbb{R}^p, C^\infty_{\mathbb{R}^p} \otimes \mathfrak{G}_q)\,;$$

where $\mathfrak{G}_q = \mathfrak{G}(\grave{x}_1, \ldots, \grave{x}_q)$ is the Grassmann algebra generated by the anti-commuting variables $\grave{x}_1, \ldots \grave{x}_q$. The full algebra of functions in this supermanifold is $C^\infty(\mathbb{R}^p) \otimes \mathfrak{G}_q$ which consists of superfunctions of the supervector variable $\mathbf{x} = (\underline{x}, \underline{\grave{x}})$ of the form

$$F(\mathbf{x}) = F(\underline{x}, \underline{\grave{x}}) = \sum_{A\subset\{1,\ldots,q\}} F_A(\underline{x})\, \underline{\grave{x}}_A, \tag{3.8}$$

where $\underline{\grave{x}}_A$ is defined as $\grave{x}_{j_1} \ldots \grave{x}_{j_k}$ with $A = \{j_1, \ldots, j_k\}$ $(1 \leq j_1 < \ldots < j_k \leq q)$, and $F_A \in C^\infty(\mathbb{R}^p)$. Similarly, one may consider other spaces of superfunctions of the form $\mathcal{F} \otimes \mathfrak{G}_q$ where $\mathcal{F} = C^k(\Omega), L_2(\Omega), \ldots,$ with $\Omega \subset \mathbb{R}^p$. In general, the bosonic functions F_A in (3.8) are complex-valued. We say that F is a *real superfunction* when all elements F_A are real-valued.

The bosonic and fermionic partial derivatives defined in (3.7) for $\mathbb{K}\mathcal{P}$ naturally extend to $C^1(\Omega) \otimes \mathfrak{G}_q$ ($\Omega \subset \mathbb{R}^m$ is an open subset). For every $F \in C^1(\Omega) \otimes \mathfrak{G}_q$ one can easily verify that

$$\partial_{\grave{x}_j}[F] = -[F^*]\partial_{\grave{x}_j}, \qquad \left(\partial_{\grave{x}_j}[F]\right)^* = -\partial_{\grave{x}_j}[F^*]. \tag{3.9}$$

where $\cdot^*$ is the gradation automorphism for the Grassmann algebra $\mathfrak{G}_q$ defined in Example 3.2, i.e.

$$F^*(\mathbf{x}) = \sum_{A\subset\{1,\ldots,q\}} (-1)^{|A|} F_A(\underline{x})\underline{\grave{x}}_A.$$

In the same way, the following Leibniz formulae can be easily proven.

Theorem 3.1 (Leibniz formulae [7, p. 75]). *The formulae*

$$\begin{aligned} &\partial_{x_j}[FG] = \partial_{x_j}[F]\, G + F\, \partial_{x_j}[G], && j = 1, \ldots, p, \\ &\partial_{\grave{x}_j}[FG] = \partial_{\grave{x}_j}[F]\, G + F^*\, \partial_{\grave{x}_j}[G] && j = 1, \ldots, q, \\ &[FG]\partial_{\grave{x}_j} = F\left([G]\partial_{\grave{x}_j}\right) + \left([F]\partial_{\grave{x}_j}\right) G^*, && j = 1, \ldots, q, \end{aligned} \tag{3.10}$$

hold for $F, G \in C^1(\Omega) \otimes \mathfrak{G}_q$.

The body F_0 and the nilpotent part $\mathbf{F}$ of a superfunction $F(\mathbf{x})$ of the form (3.8) are obtained by writing $F(\mathbf{x}) = F_0(\underline{x}) + \mathbf{F}(\underline{x}, \underline{\grave{x}})$ where

$$F_0(\underline{x}) = F_\emptyset(\underline{x}), \quad \text{and} \quad \mathbf{F}(\underline{x}, \underline{\grave{x}}) = \sum_{|A|\geq 1} F_A(\underline{x})\, \underline{\grave{x}}_A.$$

Definition 3.2. *Consider a superfunction* $F = \sum_A F_A \underline{\grave{x}}_A \in C^\infty(\mathbb{R}^\ell) \otimes \mathfrak{G}_s$, *even real-valued superfunctions* $y_j(\mathbf{x}) \in C^\infty(\Omega) \otimes \mathfrak{G}_q^{(ev)}$, $j = 1, \ldots, \ell$ *and odd superfunctions* $\grave{y}_k(\mathbf{x}) \in C^\infty(\Omega) \otimes \mathfrak{G}_q^{(odd)}$, $k = 1, \ldots, s$ *with* Ω *an open subset of* $\mathbb{R}^p$. *We expand every* y_j *as the sum of its body and its nilpotent part, i.e.* $y_j(\mathbf{x}) = [y_j]_0(\underline{x}) + \mathbf{y_j}(\mathbf{x})$. *The composed superfunction* $F(\mathbf{y}(\mathbf{x})) = F\left(y_1(\mathbf{x}), \ldots, y_\ell(\mathbf{x}), \grave{y}_1(\mathbf{x}), \ldots, \grave{y}_s(\mathbf{x})\right) \in C^\infty(\Omega) \otimes \mathfrak{G}_q$ *is defined as*

$$F(\mathbf{y}(\mathbf{x})) = \sum_{A\subset\{1,\ldots,s\}} F_A(y_1(\mathbf{x}), \ldots, y_\ell(\mathbf{x}))\, \underline{\grave{y}}_A(\mathbf{x}),$$

where the even superfunctions $F_A(y_1(\mathbf{x}),\ldots,y_\ell(\mathbf{x}))$ *are determined by means of the Taylor expansion as*

$$F_A(y_1(\mathbf{x}),\ldots,y_\ell(\mathbf{x}))=\sum_{k_1,\ldots,k_\ell\geq 0}\frac{F_A^{(k_1,\ldots,k_\ell)}\left([y_1]_0(\underline{x}),\ldots,[y_\ell]_0(\underline{x})\right)}{k_1!\cdots k_\ell!}\mathbf{y_1}(\mathbf{x})^{k_1}\cdots\mathbf{y}_\ell(\mathbf{x})^{k_\ell},$$

and $\underline{\grave{y}}_A(\mathbf{x})=\grave{y}_{j_1}(\mathbf{x})\ldots\grave{y}_{j_k}(\mathbf{x})$ *for* $A=\{j_1,\ldots,j_k\}$ $(1\leq j_1<\ldots<j_k\leq s)$.

Remark 3.3. *Note that the series in the above definition of* F_A *is finite in view of the nilpotency of* $\mathbf{y_j}(\mathbf{x})$*. Moreover, it is clear that Definition 3.2 can be used for functions that are not* C^∞ *as long as all the derivatives appearing in the formula exist.*

Using this definition for composition of superfunctions it is possible to prove the following chain rule in superspace, see e.g. [7, p. 75]

Theorem 3.2 (Chain rule [7, 33]). *Consider the composed superfunction* $F(\mathbf{y}(\mathbf{x}))\in C^\infty(\Omega)\otimes\mathfrak{G}_q$ *under the same conditions of Definition 3.2. Hence,*

$$\begin{aligned}
\frac{\partial F}{\partial x_k}&=\sum_{j=1}^{\ell}\frac{\partial y_j}{\partial x_k}\frac{\partial F}{\partial y_j}+\sum_{j=1}^{s}\frac{\partial \grave{y}_j}{\partial x_k}\left(\frac{\partial}{\partial \grave{y}_j}F\right), && k=1,\ldots,p,\\
\frac{\partial}{\partial \grave{x}_k}F&=\sum_{j=1}^{\ell}\left(\frac{\partial}{\partial \grave{x}_k}y_j\right)\frac{\partial F}{\partial y_j}+\sum_{j=1}^{s}\left(\frac{\partial}{\partial \grave{x}_k}\grave{y}_j\right)\left(\frac{\partial}{\partial \grave{y}_j}F\right), && k=1,\ldots,q,\\
F\frac{\partial}{\partial \grave{x}_k}&=\sum_{j=1}^{\ell}\frac{\partial F}{\partial y_j}\left(y_j\frac{\partial}{\partial \grave{x}_k}\right)+\sum_{j=1}^{s}\left(F\frac{\partial}{\partial \grave{y}_j}\right)\left(\grave{y}_j\frac{\partial}{\partial \grave{x}_k}\right), && k=1,\ldots,q.
\end{aligned}$$

3.2 Radial algebra representation is superspace

In [31, 33, 34, 35, 37] the theory of harmonic and Clifford analysis in superspace has been developed. We now introduce the building blocks of such a theory following the radial algebra representation approach.

In order to establish the Clifford setting in superspace we consider the homogeneous basis $e_1,\ldots,e_m,\grave{e}_1,\ldots,\grave{e}_{2n}$ of $\mathbb{R}^{m,2n}$ endowed with an orthogonal and a symplectic structure. This is done by means of the commutation rules

$$e_je_k+e_ke_j=-2\delta_{j,k},\quad e_j\grave{e}_k+\grave{e}_ke_j=0,\quad \grave{e}_j\grave{e}_k-\grave{e}_k\grave{e}_j=g_{j,k},\tag{3.11}$$

where the symplectic form $g_{j,k}$ is defined by

$$g_{2j,2k}=g_{2j-1,2k-1}=0,\qquad g_{2j-1,2k}=-g_{2k,2j-1}=\delta_{j,k},\qquad j,k=1,\ldots,n.$$

The even dimension of the odd subspace $\mathbb{R}^{0,2n}$ is needed to enable the symplectic structure. Following the above relations, elements in $\mathbb{R}^{m,2n}$ generate an infinite dimensional algebra denoted by $\mathcal{C}_{m,2n}$.

Remark 3.4. *The elements* $\grave{e}_j$*, and the algebra* $\mathcal{C}_{0,2n}$ *generated by them, may be represented by polynomial differential operators in* n *dimensions where we introduce real variables* a_j *and the corresponding derivatives* ∂_{a_j} $(j = 1, \dots, n)$ *and make the assignments :*

$$\grave{e}_{2j-1} \to \partial_{a_j}, \quad \grave{e}_{2j} \to a_j, \quad j = 1, \dots, n, \tag{3.12}$$

see [68]. One indeed has the "Weyl algebra defining relations"

$$\partial_{a_j} a_k - a_k \partial_{a_j} = \delta_{j,k}$$

as operators on polynomials in $a_1, \dots, a_n$*. This approach is entirely consistent with the defining relations of the algebra generated only by the* $\grave{e}_j$*'s. When working with the whole set of Clifford generators* e_j *and* $\grave{e}_j$ *which satisfy the anti-commuting relation* $e_j \grave{e}_k = -\grave{e}_k e_j$*, the identification (3.12) no longer holds. However, if one introduces an extra orthogonal Clifford algebra generator* e_{m+1} *with*

$$e_{m+1}^2 = -1, \qquad \textit{and} \qquad e_{m+1} e_j = -e_j e_{m+1}, \quad (j = 1, \dots, m),$$

one may make the assignment

$$\grave{e}_{2j-1} \to e_{m+1} \partial_{a_j}, \quad \grave{e}_{2j} \to -e_{m+1} a_j, \quad j = 1, \dots, n.$$

In this way, it is proven that there exists a non-trivial algebra generated by the whole set $\{e_1, \dots, e_m, \grave{e}_1 \dots, \grave{e}_{2n}\}$ *subject to the defining relations (3.11).*

The classical representation of the radial algebra $R(S)$ in superspace, where S is a finite set composed by ℓ abstract vector variables ($\ell > 1$), starts with the mapping

$$x \to \mathbf{x} = \underline{x} + \underline{\grave{x}} = \sum_{j=1}^{m} x_j e_j + \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j, \quad x \in S, \tag{3.13}$$

between S and the set of independent supervector variables $\mathbf{S} = \{\mathbf{x} : x \in S\}$. For each $x \in S$ we consider in (3.13) m bosonic variables $x_1, \dots, x_m$ and $2n$ fermionic variables $\grave{x}_1, \dots, \grave{x}_{2n}$. The projections $\underline{x} = \sum_{j=1}^{m} x_j e_j$ and $\underline{\grave{x}} = \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j$ are called the bosonic and fermionic vector variables, respectively. The set of independent supervector variables $\mathbf{S}$, obtained through the correspondence (3.13), generates a radial algebra representation $R(\mathbf{S})$ as we will show next.

Let us define the sets VAR and $\grave{VAR}$ of bosonic and fermionic variables

$$VAR = \bigcup_{\mathbf{x} \in \mathbf{S}} \{x_1, \dots, x_m\}, \qquad \grave{VAR} = \bigcup_{\mathbf{x} \in \mathbf{S}} \{\grave{x}_1, \dots, \grave{x}_{2n}\} \tag{3.14}$$

respectively, where the sets $\{x_1, \dots, x_m\}$ and $\{\grave{x}_1, \dots, \grave{x}_{2n}\}$ correspond to the bosonic and fermionic vector variables associated to each $\mathbf{x} \in \mathbf{S}$ through the correspondences (3.13). In this way, VAR contains $m\ell$ bosonic variables and $\grave{VAR}$ contains $2n\ell$ fermionic variables. They give rise to the algebra of super-polynomials $\mathcal{V} = \mathrm{Alg}_{\mathbb{R}}\{VAR \cup \grave{VAR}\}$ which is extended to the algebra of Clifford-valued super-polynomials

$$\mathcal{A}_{m,2n} = \mathcal{V} \otimes \mathcal{C}_{m,2n},$$

where the elements of $\mathcal{V}$ commute with the elements of $\mathcal{C}_{m,2n}$.

The algebra $\mathcal{V}$ clearly is $\mathbb{Z}_2$-graded. Indeed, $\mathcal{V} = \mathcal{V}_{\overline{0}} \oplus \mathcal{V}_{\overline{1}}$ where $\mathcal{V}_{\overline{0}}$ consists of all commuting super-polynomials and $\mathcal{V}_{\overline{1}}$ consists of all anti-commuting super-polynomials in $\mathcal{V}$.

Note that in the bosonic case, elements in VAR generate an infinite dimensional polynomial algebra, while the elements e_j generate a finite dimensional Clifford algebra. Conversely, in the fermionic case, elements in $\grave{VAR}$ generate a finite dimensional Grassmann algebra, and the elements $\grave{e}_j$ generate an infinite dimensional Weyl algebra. The super-vector variables $\mathbf{x} \in \mathbf{S}$ properly combine these properties, whence the correspondence (3.13) defines a radial algebra representation

$$\Psi_{m,2n} : R(S) \to \mathcal{A}_{m,2n}. \tag{3.15}$$

Indeed, the fundamental axiom **(A1)** is fulfilled in this setting since for every pair $\mathbf{x}, \mathbf{y} \in \mathbf{S}$

$$\{\mathbf{x}, \mathbf{y}\} = -2\sum_{j=1}^{m} x_j y_j + \sum_{j=1}^{n} (\grave{x}_{2j-1}\grave{y}_{2j} - \grave{x}_{2j}\grave{y}_{2j-1}) \tag{3.16}$$

is a central element in $\mathcal{A}_{m,2n}$. Formula (3.16) is normally used to define a generalized inner product in superspace (see Chapter 4).

It is important to note that $R(\mathbf{S})$ is a subalgebra strictly contained in $\mathcal{A}_{m,2n}$. This can be easily seen by noticing that elements of the form $v\grave{e}_j, \grave{v}e_j$ with $v \in VAR, \grave{v} \in \grave{VAR}$ do not belong to $R(\mathbf{S})$. From now on, we will refer to the representation $R(\mathbf{S})$ as the radial algebra embedded in $\mathcal{A}_{m,2n}$. Such a representation allows to develop a nice extension of Clifford analysis to superspace as it was shown in [33, 32].

Remark 3.5. *Consider the variables in VAR and $\grave{VAR}$ represented as co-ordinates with values in $\Lambda_{\overline{0}}$ and $\Lambda_{\overline{1}}$ respectively as in the Vladimirov-Volovich approach, see Section 3.1.1. Then, in the above representation of the radial algebra, the corresponding underlying vector space on which the vector variables $\mathbf{x}$ are defined is given by $V = \mathbb{R}^{m,2n}(\Lambda)$. It is easily seen that the algebra generated by all vectors in $\mathbb{R}^{m,2n}(\Lambda)$ is strictly contained in $A \otimes \mathcal{C}_{m,2n}$. In particular, $\mathbb{R}^{m,2n}(\Lambda)$ does not contain the symplectic Clifford generators $\grave{e}_j$ $(j = 1, \ldots, 2n)$. This makes the notion of vector variable in this supersymmetry setting more restrictive than in the Clifford-polynomial representation. We recall that in that case, the corresponding underlying vector space is $V = \mathbb{R}^m$ which contains all orthogonal Clifford generators e_j $(j = 1, \ldots, m)$.*

The difference is seen by noticing that the orthogonal Clifford generators satisfy the axiom **(A1)** *since $\{e_j, e_k\}$ always is a scalar. But that is not the case for the anti-commutator $\{\grave{e}_j, \grave{e}_k\}$ of symplectic Clifford generators. To obtain a representation of the radial algebra in superspace it is necessary to combine the symplectic generators $\grave{e}_j$ with anti-commuting variables. In that way we get the commuting element $\{\grave{v}\grave{e}_j, \grave{w}\grave{e}_k\} = \grave{v}\grave{w}[\grave{e}_j, \grave{e}_k]$.*

3.2.1 Representation of the main endomorphisms

The endomorphisms defined in Section 2.3 on the radial algebra $R(S)$ can be naturally mapped by $\Psi_{m,2n}$ into the algebra of endomorphisms over $R(\mathbf{S})$, see Definition 2.3. In this section we will describe the extension of some of these endomorphisms from $R(\mathbf{S})$ to $\mathcal{A}_{m,2n}$.

Conjugation: The conjugation admits an extension from the radial algebra representation $R(\mathbf{S})$ to $\mathcal{A}_{m,2n}$. In fact, we can define $\bar{\cdot} \in \mathrm{End}(\mathcal{A}_{m,2n})$ by means of the following rules:

i) $\bar{\cdot}$ is the identity map on $\mathcal{V}$.

ii) $\overline{e_{j_1}\cdots e_{j_k}\grave{e}_{l_1}\cdots\grave{e}_{l_s}} = (-1)^{k+\frac{s(s+1)}{2}}\ \grave{e}_{l_s}\cdots\grave{e}_{l_1}e_{j_k}\cdots e_{j_1}$.

This extension still is an involution on $\mathcal{A}_{m,2n}$ but the anti-automorphism property, i.e. $\overline{FG} = \overline{G}\,\overline{F}$, which is fulfilled in the radial algebra is no longer satisfied in $\mathcal{A}_{m,2n}$. For example, observe that

$$\overline{\grave{e}_j\grave{e}_k} = -\grave{e}_k\grave{e}_j \neq \grave{e}_k\grave{e}_j = \overline{\grave{e}_k}\;\overline{\grave{e}_j},$$

and

$$\overline{\grave{v}e_j\,\grave{w}e_k} = \grave{v}\grave{w}e_ke_j \neq -\grave{v}\grave{w}e_ke_j = \overline{\grave{w}e_k}\;\overline{\grave{v}e_j}.$$

Main involution: The main involution can also be extended from the radial algebra to $\mathcal{A}_{m,2n}$. The algebra homomorphism $\tilde{\cdot}$ can be defined in a natural way by

i) $\tilde{\cdot}$ is the identity map on $\mathcal{V}$.

ii) $\tilde{e}_j = -e_j, \quad \tilde{\grave{e}}_j = -\grave{e}_j$.

iii) $\widetilde{FG} = \tilde{F}\tilde{G}$.

Its restricted actions to the bosonic and the fermionic part respectively are called the *bosonic and fermionic main involutions* and are defined by the following relations:

Bosonic main involution $\tilde{\cdot}^b$

i) $\tilde{\cdot}^b$ is the identity map on $\mathcal{V}$,

ii) $\tilde{e}_j^{\,b} = -e_j, \quad \tilde{\grave{e}}_j^{\,b} = \grave{e}_j$,

iii) $\widetilde{FG}^b = \tilde{F}^b\,\tilde{G}^b$.

Fermionic main involution $\tilde{\cdot}^f$

i) $\tilde{\cdot}^f$ is the identity map on $\mathcal{V}$,

ii) $\tilde{e}_j^{\,f} = e_j, \quad \tilde{\grave{e}}_j^{\,f} = -\grave{e}_j$,

iii) $\widetilde{FG}^f = \tilde{F}^f\,\tilde{G}^f$.

It is easily seen that the main involution $\tilde{\cdot}$ is the composition of its bosonic and fermionic restrictions.

The fermionic main involution is closely related to the $\mathbb{Z}_2$-gradation of the superalgebra $\mathcal{V}$. The gradation automorphism $\cdot^*$ defined for Grassmann algebras in Example 3.2 can be extended to $\mathcal{A}_{m,2n}$ by means of the rules:

i) $v^* = v$ and $\grave{v}^* = -\grave{v}$ for every $v \in VAR, \grave{v} \in V\grave{A}R$.

ii) $\cdot^*$ is the identity map in $\mathcal{C}_{m,2n}$.

iii) $(FG)^* = F^*G^*$.

Observe that the restriction of $\cdot^*$ to the radial algebra $R(\mathbf{S})$ coincides with the fermionic main involution. In fact, for every vector variable $\mathbf{x} \in \mathbf{S}$ we have:

$$\widetilde{\mathbf{x}}^f = \sum_{j=1}^{m} x_j e_j - \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j = \mathbf{x}^*.$$

In addition this grading automorphism is such that

$$\grave{v}F = F^* \, \grave{v}; \qquad F \in \mathcal{A}_{m,2n}, \; \grave{v} \in V\grave{A}R.$$

Vector derivative: The partial derivatives with respect to the variables in $VAR \cup V\grave{A}R$ are defined as endomorphisms in $\mathcal{V}$ by means of the recursive relations (3.7). We recall that in this case we are considering $p = m\ell$ bosonic variables and $q = 2n\ell$ fermionic variables. These derivatives trivially extend to $\mathcal{A}_{m,2n}$ by means of the commuting relations

$$\partial_{x_j} e_j = e_j \partial_{x_j}, \qquad \partial_{x_j} \grave{e}_j = \grave{e}_j \partial_{x_j}, \qquad \partial_{\grave{x}_j} e_j = e_j \partial_{\grave{x}_j}, \qquad \partial_{\grave{x}_j} \grave{e}_j = \grave{e}_j \partial_{\grave{x}_j}. \tag{3.17}$$

The bosonic Dirac operator $\partial_{\underline{x}}$ and the fermionic Dirac operator $\partial_{\underline{\grave{x}}}$ associated to the vector variable $\mathbf{x} \in \mathbf{S}$ are introduced by

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}, \qquad \partial_{\underline{\grave{x}}} = 2 \sum_{j=1}^{n} \left(\grave{e}_{2j} \partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1} \partial_{\grave{x}_{2j}} \right). \tag{3.18}$$

The representation of the vector derivative in $R(\mathbf{S})$ is obtained by means of the left and right super Dirac operators which are defined by

$$\begin{aligned} \partial_{\mathbf{x}} \cdot &= \partial_{\underline{\grave{x}}} \cdot - \partial_{\underline{x}} \cdot \, ; & \qquad F &\to \partial_{\underline{\grave{x}}}[F] - \partial_{\underline{x}}[F] = \partial_{\mathbf{x}}[F], \\ \cdot \partial_{\mathbf{x}} &= - \cdot \partial_{\underline{\grave{x}}} - \cdot \partial_{\underline{x}} \, ; & \qquad F &\to -[F]\partial_{\underline{\grave{x}}} - [F]\partial_{\underline{x}} = [F]\partial_{\mathbf{x}}. \end{aligned}$$

Indeed, in [33] it was proven that the above operators satisfy all axioms **(D1)- (D4)** used in the definition of vector derivative on the level of radial algebra. In this case the abstract dimension is mapped to $\partial_{\mathbf{x}}[\mathbf{x}] = [\mathbf{x}]\partial_{\mathbf{x}} = m - 2n =: M$. This parameter M is called the superdimension.

On the radial algebra level, the left and right actions of the vector derivative are connected by means of the conjugation since $\overline{\partial_{\mathbf{x}}[F]} = -[\overline{F}]\partial_{\mathbf{x}}$ holds for every $F \in R(\mathbf{S})$, see Lemma 2.7. This property, however, is fully dependent on the radial algebra structure, constituting the reason why the above relation is no longer fulfilled in general on $\mathcal{A}_{m,2n}$. For example, for the element $F = \grave{x}_{2j-1}$ we obtain that

$$\overline{\partial_{\mathbf{x}}[F]} = \overline{\partial_{\underline{\grave{x}}}[F]} = 2\overline{\grave{e}_{2j}} = -2\grave{e}_{2j}, \quad \text{while} \quad -[\overline{F}]\partial_{\mathbf{x}} = [F]\partial_{\underline{\grave{x}}} = 2\grave{e}_{2j}.$$

Remark 3.6. *In [23], a slightly different super Dirac operator was considered in order to obtain an* $\mathfrak{osp}(m|2n)$*-symmetry. The supervector variable in that setting was defined as*

$$\mathbf{X} = \sum_{j=0}^{m} X_j E_j + \sum_{j=1}^{2n} X_{m+j} E_{m+j},$$

corresponding to a super Dirac operator of the form

$$\partial_{\mathbf{X}} = \sum_{j=0}^{m} \partial_{X_j} E_j + \sum_{j=1}^{2} \left(\partial_{X_{m+j}} E_{m+n+j} - \partial_{X_{m+n+j}} E_{m+j} \right),$$

where $X_1, \ldots, X_m$ *are bosonic variables,* $X_{m+1}, \ldots, X_{m+2n}$ *are fermionic variables and the elements* E_k*,* $k = 1, \ldots, m+2n$*, generate a super Clifford algebra subject to the rules,*

$$\begin{aligned} E_k E_\ell + E_\ell E_k &= -2\delta_{k,\ell}, & & 1 \leq k, \ell \leq m, \\ E_k E_{m+\ell} + E_{m+\ell} E_k &= 0, & & 1 \leq k \leq m, \quad 1 \leq \ell \leq 2n, \\ E_{m+k} E_{m+\ell} - E_{m+\ell} E_{m+k} &= 0, & & 1 \leq k, \ell \leq n, \\ E_{m+k} E_{m+n+\ell} - E_{m+n+\ell} E_{m+k} &= 2\delta_{k,\ell}, & & 1 \leq k, \ell \leq n, \\ E_{m+n+k} E_{m+n+\ell} - E_{m+n+\ell} E_{m+n+k} &= 0, & & 1 \leq k, \ell \leq n. \end{aligned}$$

The main difference between this setting and the one considered in the underlying thesis is that the fermionic variables anti-commute with the symplectic generators. More precisely,

$$X_{m+j} E_{m+k} = -E_{m+k} X_{m+j}, \qquad 1 \leq j, k \leq 2n, \tag{3.19}$$

while the other commutation rules remain the same, i.e. $X_j E_k = E_k X_j$ *and* $X_k E_j = E_j X_k$ *for* $1 \leq j \leq m$*,* $1 \leq k \leq m + 2n$*. Nevertheless, both approaches are closely related since both of them allow for a radial algebra representation. The explicit connection between both settings is given in terms of the fermionic main involution by means of the following transformations:*

$$\begin{aligned} &X_j = x_j, & &E_j = e_j, & &\partial_{X_j} = \partial_{x_j}, & &j = 1 \ldots, m, \\ &X_{m+j} = \frac{-i}{\sqrt{2}} \grave{x}_{2j-1} \tilde{\cdot}^f, & &E_{m+j} = i\sqrt{2}\, \tilde{\cdot}^f\, \grave{e}_{2j-1}, & &\partial_{X_{m+j}} = i\sqrt{2}\, \partial_{\grave{x}_{2j-1}} \tilde{\cdot}^f, & &j = 1 \ldots, n, \\ &X_{m+n+j} = \frac{-i}{\sqrt{2}} \grave{x}_{2j} \tilde{\cdot}^f, & &E_{m+n+j} = i\sqrt{2}\, \tilde{\cdot}^f\, \grave{e}_{2j}, & &\partial_{X_{m+n+j}} = i\sqrt{2}\, \partial_{\grave{x}_{2j}} \tilde{\cdot}^f, & &j = 1 \ldots, n. \end{aligned}$$

The involution $\tilde{\cdot}^f$ *is very useful in this context since it squares to the identity operator and anti-commutes with the* $\grave{e}_j$*'s. Hence, we obtain*

$$\begin{aligned} X_{m+j} E_{m+j} &= \grave{x}_{2j-1} \grave{e}_{2j-1}, & X_{m+n+j} E_{m+n+j} &= \grave{x}_{2j} \grave{e}_{2j}, & &j = 1 \ldots, n, \\ \partial_{X_{m+j}} E_{m+n+j} &= -2\partial_{\grave{x}_{2j-1}} \grave{e}_{2j}, & \partial_{X_{m+n+j}} E_{m+j} &= -2\partial_{\grave{x}_{2j}} \grave{e}_{2j-1}, & &j = 1 \ldots, n, \end{aligned}$$

yielding $\mathbf{X} = \mathbf{x}$ *and* $\partial_{\mathbf{X}} = -\partial_{\mathbf{x}}$*. As mentioned before, the relations (3.19) allow for an orthosymplectic symmetry of* $\partial_{\mathbf{X}}$*, see [23]. In Chapters 4 and 5, we approach this* *situation from a group theoretical point of view. In particular, we prove that the operator* $\partial_{\mathbf{x}}$ *is spin invariant, which is equivalent to saying that it is invariant under the action of a Grassmann envelope of* $\mathfrak{osp}(m|2n)$.

Vector multipliers: In the Clifford-polynomial representation the vector multipliers x and $x|$ can be easily redefined using the basis multipliers e_j and $e_j|$, see (2.8). In particular, this means that the $\cdot|$ action is linear with respect to the variables x_j. Based on the same idea, we define the following basis multipliers in the superspace representation:

$$
\begin{aligned}
e_j &: F \to e_j F, & e_j| &: F \to \widetilde{F} e_j, \\
\grave{e}_j &: F \to e_j F, & \grave{e}_j| &: F \to \widetilde{F}^b \grave{e}_j.
\end{aligned}
$$

They make it possible to write the $\mathbf{x}|$ operator defined on the radial algebra $R(\mathbf{S})$ as

$$\mathbf{x}| = \underline{x}| + \underline{\grave{x}}| = \sum_{j=1}^{m} x_j e_j| + \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j|.$$

In fact, for every $F \in \mathcal{A}_{m,2n}$ we have that

$$\mathbf{x}|[F] = \sum_{j=1}^{m} x_j \widetilde{F} e_j + \sum_{j=1}^{2n} \grave{x}_j \widetilde{F}^b \grave{e}_j = \sum_{j=1}^{m} \widetilde{F} x_j e_j + \sum_{j=1}^{2n} \left(\widetilde{F}^b\right)^* \grave{x}_j \grave{e}_j.$$

In particular, for $F \in R(\mathbf{S})$ we obtain $\widetilde{F} = \left(\widetilde{F}^b\right)^*$ and in consequence $\mathbf{x}|[F] = \widetilde{F}\mathbf{x}$.

Using the identifications:

$$
\begin{aligned}
e_{m+j} &= i e_j| & j &= 1, \ldots, m, \\
\grave{e}_{2n+j} &= i \grave{e}_j| & j &= 1, \ldots, 2n,
\end{aligned}
\tag{3.20}
$$

it is easily proven that the operators $e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}, \grave{e}_1, \ldots, \grave{e}_{2n}, \grave{e}_{2n+1}, \ldots, \grave{e}_{4n}$ satisfy the commutation relations given in (3.11), i.e. they generate the algebra $\mathcal{C}_{2m,4n}$.

For future computations we will need the following relations that can be easily proven using mathematical induction and the recursion formulae (3.7).

Lemma 3.1. *Let* $\mathbf{x} = \underline{x} + \underline{\grave{x}}$ *be a supervector variable. Then the following identities hold in* $\mathcal{A}_{m,2n}$

$$
\begin{array}{lllll}
\left[\partial_{x_j}, \bar{\cdot}\right] = 0, & \left[\partial_{\grave{x}_j}, \bar{\cdot}\right] = 0, & \left[\partial_{x_j}, e_j|\right] = 0, & \left\{\partial_{\underline{x}}, \widetilde{\cdot}^b\right\} = 0, & \left[\partial_{\underline{\grave{x}}}, \widetilde{\cdot}^b\right] = 0, \\
\left[\partial_{x_j}, \widetilde{\cdot}^b\right] = 0, & \left[\partial_{\grave{x}_j}, \widetilde{\cdot}^b\right] = 0, & \left[\partial_{\grave{x}_j}, e_j|\right] = 0, & \left[\partial_{\underline{x}}, \widetilde{\cdot}^f\right] = 0, & \left\{\partial_{\underline{\grave{x}}}, \widetilde{\cdot}^f\right\} = 0, \\
\left[\partial_{x_j}, \widetilde{\cdot}^f\right] = 0, & \left[\partial_{\grave{x}_j}, \widetilde{\cdot}^f\right] = 0, & \left[\partial_{x_j}, \grave{e}_j|\right] = 0, & \left\{\partial_{\underline{x}}, \widetilde{\cdot}\right\} = 0, & \left\{\partial_{\underline{\grave{x}}}, \widetilde{\cdot}\right\} = 0, \\
\left[\partial_{x_j}, \widetilde{\cdot}\right] = 0, & \left[\partial_{\grave{x}_j}, \widetilde{\cdot}\right] = 0, & \left[\partial_{\grave{x}_j}, \grave{e}_j|\right] = 0, & \left[\partial_{\underline{x}}, \cdot^*\right] = 0, & \left\{\partial_{\underline{\grave{x}}}, \cdot^*\right\} = 0, \\
\left[\partial_{x_j}, \cdot^*\right] = 0, & \left\{\partial_{\grave{x}_j}, \cdot^*\right\} = 0, & & &
\end{array}
$$

In addition, for every $F \in \mathcal{A}_{m,2n}$ *one has that* $\partial_{\grave{v}}[F] = -[F^*]\partial_{\grave{v}}$.

Remark 3.7. *The results in Lemma 3.1 are an extension to $\mathcal{A}_{m,2n}$ of the formulae (3.9).*

Lemma 3.2. *Let $F, G \in \mathcal{A}_{m,2n}$. Then one has*

$$\begin{cases} \partial_{x_j}[FG] = \partial_{x_j}[F]G, \\ [GF]\partial_{x_j} = G[F]\partial_{x_j}, \qquad \text{if } G \in Alg_{\mathbb{R}}\Big(\big(VAR \setminus \{x_j\}\big) \cup VAR\grave{}\Big) \otimes \mathcal{C}_{m,2n}, \\ \partial_{x_j}[G] = 0 = [G]\partial_{x_j}, \end{cases}$$

$$\begin{cases} \partial_{\grave{x}_j}[FG] = \partial_{\grave{x}_j}[F]G, \\ [GF]\partial_{\grave{x}_j} = G[F]\partial_{\grave{x}_j}, \qquad \text{if } G \in Alg_{\mathbb{R}}\Big(VAR \cup \big(VAR\grave{} \setminus \{\grave{x}_j\}\big)\Big) \otimes \mathcal{C}_{m,2n}, \\ \partial_{\grave{x}_j}[G] = 0 = [G]\partial_{\grave{x}_j}. \end{cases}$$

Operator B: Following the radial algebra approach of Theorem 2.5 and using Lemmas 3.1 and 3.2, we compute the representation of the operator $B = \{\partial_x, x|\}$ acting on $F \in \mathcal{A}_{m,2n}$ as follows:

$$\begin{aligned} \{\partial_{\mathbf{x}}, \mathbf{x}|\}[F] &= \partial_{\mathbf{x}}\big[\mathbf{x}|[F]\big] + \mathbf{x}|\big[\partial_{\mathbf{x}}[F]\big] \\ &= (\partial_{\underline{\grave{x}}} - \partial_{\underline{x}})\,(\underline{x}|[F] + \underline{\grave{x}}|[F]) + (\underline{x}| + \underline{\grave{x}}|)\,\big(\partial_{\underline{\grave{x}}}[F] - \partial_{\underline{x}}[F]\big) \\ &= \Big(-\partial_{\underline{x}}\Big[\widetilde{F}\underline{x}\Big] + \partial_{\underline{x}}\Big[\widetilde{F}\Big]\,\underline{x}\Big) + \Big(-\partial_{\underline{x}}\Big[\Big(\widetilde{F}^b\Big)^*\underline{\grave{x}}\Big] + \partial_{\underline{x}}\Big[\Big(\widetilde{F}^b\Big)^*\Big]\,\underline{\grave{x}}\Big) \\ &\quad + \Big(\partial_{\underline{\grave{x}}}\Big[\widetilde{F}\underline{x}\Big] - \partial_{\underline{\grave{x}}}\Big[\widetilde{F}\Big]\,\underline{x}\Big) + \Big(\partial_{\underline{\grave{x}}}\Big[\Big(\widetilde{F}^b\Big)^*\underline{\grave{x}}\Big] - \partial_{\underline{\grave{x}}}\Big[\Big(\widetilde{F}^b\Big)^*\Big]\,\underline{\grave{x}}\Big) \\ &= \Big(-\partial_{\underline{x}}\Big[\widetilde{F}\underline{x}\Big] + \partial_{\underline{x}}\Big[\widetilde{F}\Big]\,\underline{x}\Big) + \Big(\partial_{\underline{\grave{x}}}\Big[\Big(\widetilde{F}^b\Big)^*\underline{\grave{x}}\Big] - \partial_{\underline{\grave{x}}}\Big[\Big(\widetilde{F}^b\Big)^*\Big]\,\underline{\grave{x}}\Big). \end{aligned}$$

However, Remark 2.11 shows that

$$-\partial_{\underline{x}}\Big[\widetilde{F}\underline{x}\Big] + \partial_{\underline{x}}\Big[\widetilde{F}\Big]\,\underline{x} = -\sum_{j=1}^{m} e_j\widetilde{F}e_j.$$

On the other hand,

$$\begin{aligned} \partial_{\underline{\grave{x}}}\Big[\Big(\widetilde{F}^b\Big)^*\underline{\grave{x}}\Big] &= 2\left(\sum_{j=1}^{n}\big(\grave{e}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}}\big)\right)\left(\sum_{k=1}^{2n}\grave{x}_k\widetilde{F}^b\grave{e}_k\right) \\ &= 2\sum_{\substack{1\le j\le n\\ 1\le k\le 2n}}\grave{e}_{2j}\Big(\delta_{2j-1,k}\,\widetilde{F}^b\grave{e}_k - \grave{x}_k\partial_{\grave{x}_{2j-1}}[\widetilde{F}^b]\grave{e}_k\Big) \\ &\quad - 2\sum_{\substack{1\le j\le n\\ 1\le k\le 2n}}\grave{e}_{2j-1}\Big(\delta_{2j,k}\,\widetilde{F}^b\grave{e}_k - \grave{x}_k\partial_{\grave{x}_{2j}}[\widetilde{F}^b]\grave{e}_k\Big) \end{aligned}$$

yielding,

$$\begin{aligned}\partial_{\grave{\underline{x}}}\left[\left(\widetilde{F}^b\right)^*\grave{\underline{x}}\right] &= 2\sum_{\substack{1\le j\le n\\1\le k\le 2n}}\left(\delta_{2j-1,k}\,\grave{e}_{2j}\widetilde{F}^b\grave{e}_k-\delta_{2j,k}\,\grave{e}_{2j-1}\widetilde{F}^b\grave{e}_k\right)+\\ &\quad+2\sum_{\substack{1\le j\le n\\1\le k\le 2n}}\left(\grave{e}_{2j}\partial_{\grave{x}_{2j-1}}[(\widetilde{F}^b)^*]\grave{x}_k\grave{e}_k-\grave{e}_{2j-1}\partial_{\grave{x}_{2j}}[(\widetilde{F}^b)^*]\grave{x}_k\grave{e}_k\right)\\ &= 2\sum_{j=1}^n\left(\grave{e}_{2j}\widetilde{F}^b\grave{e}_{2j-1}-\grave{e}_{2j-1}\widetilde{F}^b\grave{e}_{2j}\right)+\partial_{\grave{\underline{x}}}\left[\left(\widetilde{F}^b\right)^*\right]\grave{\underline{x}}.\end{aligned}$$

We thus get

$$\{\partial_{\mathbf{x}},\mathbf{x}|\}[F]=-\sum_{j=1}^m e_j\widetilde{F}e_j+2\sum_{j=1}^n\left(\grave{e}_{2j}\widetilde{F}^b\grave{e}_{2j-1}-\grave{e}_{2j-1}\widetilde{F}^b\grave{e}_{2j}\right).$$

We can now write the representation of the operator $B=\{\partial_x,x|\}$ in superspace as a special "bivector" $\mathscr{B}$ in $\mathcal{C}_{2m,4n}$:

$$\begin{aligned}\mathscr{B} &= -\sum_{j=1}^m e_je_j|+2\sum_{j=1}^n\left(\grave{e}_{2j}\grave{e}_{2j-1}|-\grave{e}_{2j-1}\grave{e}_{2j}|\right)\\ &= i\left(\sum_{j=1}^m e_je_{m+j}+2\sum_{j=1}^n\left(\grave{e}_{2j-1}\grave{e}_{2n+2j}-\grave{e}_{2j}\grave{e}_{2n+2j-1}\right)\right).\end{aligned}$$

In $\mathcal{A}_{m,2n}$ we can also define the bosonic and fermionic differential operators

$$\partial_{\underline{x}}|=\sum_{j=1}^m e_j|\,\partial_{x_j},\qquad \partial_{\grave{\underline{x}}}|=2\sum_{j=1}^n\left(\grave{e}_{2j}|\,\partial_{\grave{x}_{2j-1}}-\grave{e}_{2j-1}|\,\partial_{\grave{x}_{2j}}\right).$$

They lead to the representation of the endomorphism $\partial_x|$ (see (2.12)) in this setting; which is given by the super differential operator $\partial_{\mathbf{x}}|=\partial_{\grave{\underline{x}}}|-\partial_{\underline{x}}|$ acting from the left. Using Lemma 3.1 we can compute the action of this operator as follows,

$$\begin{aligned}\partial_{\mathbf{x}}|[F] &= \partial_{\grave{\underline{x}}}|[F]-\partial_{\underline{x}}|[F]\\ &= 2\sum_{j=1}^n\left(\grave{e}_{2j}|\,\partial_{\grave{x}_{2j-1}}[F]-\grave{e}_{2j-1}|\,\partial_{\grave{x}_{2j}}[F]\right)-\sum_{j=1}^m e_j|\,\partial_{x_j}[F]\\ &= 2\sum_{j=1}^n\left(\partial_{\grave{x}_{2j-1}}[\widetilde{F}^b]\grave{e}_{2j}-\partial_{\grave{x}_{2j}}[\widetilde{F}^b]\grave{e}_{2j-1}\right)-\sum_{j=1}^m\partial_{x_j}[\widetilde{F}]e_j\\ &= -2\sum_{j=1}^n\left([(\widetilde{F}^b)^*]\partial_{\grave{x}_{2j-1}}\grave{e}_{2j}-[(\widetilde{F}^b)^*]\partial_{\grave{x}_{2j}}\grave{e}_{2j-1}\right)-\sum_{j=1}^m[\widetilde{F}]\partial_{x_j}e_j\\ &= -[(\widetilde{F}^b)^*]\partial_{\grave{\underline{x}}}-[\widetilde{F}]\partial_{\underline{x}}.\end{aligned}$$

Then, for $F \in R(\mathbf{S})$ we have $\partial_{\mathbf{x}}|[F] = [\widetilde{F}]\partial_{\mathbf{x}}$ (see (2.12)). As it was proven in Theorem 2.5, the equality $-\{\partial_{\mathbf{x}}|, \mathbf{x}\} = \mathscr{B}$ holds on the radial algebra $R(\mathbf{S})$. By straightforward computation we can check that it remains valid in $\mathcal{A}_{m,2n}$.

3.3 Complex structures

Following the approach given in (2.35), we obtain a representation of the radial algebra with complex structure $R(S \cup J(S), \mathcal{B})$ in superspace by means of the composition

$$\Psi_{m,2n} \circ \Psi_2 : R(S \cup J(S), \mathcal{B}) \to \mathcal{A}_{2m,4n}.$$

Here, the doubled vector variables $\Psi_{m,2n}(X)$, see (2.36), have the form

$$\mathbf{x} = \sum_{j=1}^{m} x_j e_j + \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j + i\left(\sum_{j=1}^{m} x_{m+j} e_j| + \sum_{j=1}^{2n} \grave{x}_{2n+j} \grave{e}_j|\right) = \sum_{j=1}^{2m} x_j e_j + \sum_{j=1}^{4n} \grave{x}_j \grave{e}_j = \underline{x} + \underline{\grave{x}},$$

while the action of the complex structure $\mathcal{J} := \Psi_{m,2n}(J_2)$, see (2.37), is given by

$$\mathcal{J}(\mathbf{x}) = \mathcal{J}(\underline{x}) + \mathcal{J}(\underline{\grave{x}}) = \sum_{j=1}^{m} (x_{m+j} e_j - x_j e_{m+j}) + \sum_{j=1}^{2n} (\grave{x}_{2n+j} \grave{e}_j - \grave{x}_j \grave{e}_{2n+j}).$$

In other words, the representation $\Psi_{2n,4n}$ defined in (3.15) extends to the homomorphism $\Psi_{m,2n} \circ \Psi_2$ from $R(S \cup J(S), \mathcal{B})$ into $R(\mathbf{S} \cup \mathcal{J}(\mathbf{S}), -i\mathscr{B}) \subset \mathcal{A}_{2m,4n}$ by means of the commutative diagram

$$\begin{array}{ccc} R(S \cup J(S), \mathcal{B}) & \xrightarrow{\Psi_2} & R(S_2 \cup J_2(S_2), -iB_1) \\ & \searrow^{\Psi_{2m,4n}} & \downarrow \Psi_{m,2n} \\ & & R(\mathbf{S} \cup \mathcal{J}(\mathbf{S}), -i\mathscr{B}) \end{array} \tag{3.21}$$

For the sake of simplicity, we have abused of the notations $\mathbf{x}$ and $\mathbf{S}$ used in the previous sections for the representation $\Psi_{m,2n}$. In case we explicitly need to indicate the dimensions $p, 2q$ of the representation $\Psi_{p,2q}$ we use the notations $\mathbf{x} \in \mathcal{A}_{p,2q}$ and $\mathbf{S} \subset \mathcal{A}_{p,2q}$.

The complex structure $\mathcal{J}$ can be extended from $R(\mathbf{S} \cup \mathcal{J}(\mathbf{S}), -i\mathscr{B})$ to an algebra automorphism on $\mathcal{A}_{2m,4n}$ by means of the following relations:

i) $\mathcal{J}$ is the identity map on $\mathcal{V}$.

ii) $\mathcal{J}(e_j) = -e_{m+j}, \quad \mathcal{J}(e_{m+j}) = e_j, \quad j = 1, \ldots, m,$
$\mathcal{J}(\grave{e}_j) = -\grave{e}_{2n+j}, \quad \mathcal{J}(\grave{e}_{2n+j}) = \grave{e}_j, \quad j = 1, \ldots, 2n.$

iii) $\mathcal{J}(FG) = \mathcal{J}(F)\mathcal{J}(G), \quad F, G \in \mathcal{A}_{2m,4n}.$

The restriction of $\mathcal{J}$ to $\mathcal{C}_{2m,0} = \mathbb{R}_{0,2m}$ exactly yields the complex structure $\underline{J}$ used in classical Hermitian Clifford analysis, see [14, 9] and Remark 2.17. On the other hand, the restriction to $\mathcal{C}_{0,4n} = \text{Alg}_{\mathbb{R}}(\grave{e}_1, \ldots, \grave{e}_{4n})$ brings new insights to this study since it acts on objects of a different nature.

In order to study the action of $\mathcal{J}$ on $\mathcal{C}_{0,4n}$ let us consider the representation (3.12) of $\mathcal{C}_{0,4n}$ in terms of polynomial differential operators. In this particular case, one needs $2n$ real variables a_j, b_j $(j = 1, \ldots, n)$ and the assignments

$$\grave{e}_{2j-1} \to \partial_{a_j}, \quad \grave{e}_{2j} \to a_j, \quad \grave{e}_{2n+2j-1} \to \partial_{b_j}, \quad \grave{e}_{2n+2j} \to b_j. \tag{3.22}$$

In the closure of the set of polynomials in the variables a_j, b_j we may consider the element

$$e^{i\langle a,b\rangle} = \exp\left(i \sum_{j=1}^{n} a_j b_j\right) = \prod_{j=1}^{n} e^{i a_j b_j}, \quad \text{where} \quad e^{i a_j b_j} = \sum_{k=0}^{\infty} \frac{(i a_j b_j)^k}{k!}.$$

This function works as a "projection wall" under the action of the above Weyl generators, i.e.

$$\partial_{a_j} e^{i\langle a,b\rangle} = i\, b_j\, e^{i\langle a,b\rangle}, \quad \partial_{b_j} e^{i\langle a,b\rangle} = i\, a_j\, e^{i\langle a,b\rangle}.$$

These relations yield a projection $\mathbf{J}$ of the complex structure $\mathcal{J}$. Indeed, by letting $\mathcal{J}(\grave{e}_{2j-1})$ and $\mathcal{J}(\grave{e}_{2j})$ act on $e^{i\langle a,b\rangle}$ we obtain, using the substitutions (3.22), that

$$\begin{aligned} \mathcal{J}(\grave{e}_{2j-1}) \left[e^{i\langle a,b\rangle}\right] &= -\partial_{b_j} e^{i\langle a,b\rangle} = -a_j\, i e^{i\langle a,b\rangle} = -\grave{e}_{2j} \left[i e^{i\langle a,b\rangle}\right], \\ \mathcal{J}(\grave{e}_{2j}) \left[e^{i\langle a,b\rangle}\right] &= -b_j e^{i\langle a,b\rangle} = \partial_{a_j}\, i e^{i\langle a,b\rangle} = \grave{e}_{2j-1} \left[i e^{i\langle a,b\rangle}\right]. \end{aligned}$$

In this way, we define the projection morphism $\mathbf{J} : \mathcal{C}_{0,4n} \to \mathcal{C}_{0,2n}$ by means of the relation

$$\mathcal{J}(\grave{e}_j) \left[e^{i\langle a,b\rangle}\right] = i^{1-\left\lfloor \frac{j}{2n+1} \right\rfloor} \mathbf{J}(\grave{e}_j) \left[e^{i\langle a,b\rangle}\right], \qquad j = 1, \ldots, 4n.$$

This operator $\mathbf{J}$ projects the whole action of $\mathcal{J}$ onto the algebra $\mathcal{C}_{0,2n} = \text{Alg}_{\mathbb{R}}(\grave{e}_1, \ldots, \grave{e}_{2n})$. In particular,

$$\mathbf{J}(\grave{e}_{2j-1}) = -\mathbf{J}(\grave{e}_{2n+2j}) = -\grave{e}_{2j}, \quad \mathbf{J}[\grave{e}_{2j}] = \mathbf{J}(\grave{e}_{2n+2j-1}) = \grave{e}_{2j-1}.$$

This avoids the "redundancy" caused by doubling the already doubled fermionic part. The above relations allow to extend the radial algebra representation $\Psi_{2m,2n} : R(S) \to \mathcal{A}_{2m,2n}$ to a representation of $R(S \cup J(S), \mathcal{B})$, where only the bosonic dimension has been doubled. Indeed, the restriction of $\mathbf{J}$ to $\mathcal{C}_{0,2n}$ can be extended to an algebra automorphism of $\mathcal{A}_{2m,2n}$ as follows

i) $\mathbf{J}$ is the identity in $\mathcal{V}$.

ii) $\mathbf{J}(e_j) = -e_{m+j}, \quad \mathbf{J}(e_{m+j}) = e_j, \quad j = 1, \ldots, m,$
$\mathbf{J}(\grave{e}_{2j-1}) = -\grave{e}_{2j}, \quad \mathbf{J}(\grave{e}_{2j}) = \grave{e}_{2j-1}, \quad j = 1, \ldots, n.$

iii) $\mathbf{J}(FG) = \mathbf{J}(F)\mathbf{J}(G), \quad F, G \in \mathcal{A}_{2m,2n}.$

In the next section we will show that, indeed, $\mathbf{J}$ satisfies the complex structure axioms **(AH1)-(AH3)**.

3.3.1 Verification of the complex structure axioms

Here we verify that the action of $\mathbf{J}$ on the radial algebra $R(\mathbf{S})$ embedded in $\mathcal{A}_{2m,2n}$ satisfies the complex structure axioms, i.e. that $\mathbf{J}$ is a complex structure on $R(\mathbf{S}) \subset \mathcal{A}_{2m,2n}$.

We recall that, in this setting, the supervector variables take the form

$$\mathbf{x} = \underline{x} + \underline{\grave{x}} = \sum_{j=1}^{m} (x_j e_j + x_{m+j} e_{m+j}) + \sum_{j=1}^{n} (\grave{x}_{2j-1} \grave{e}_{2j-1} + \grave{x}_{2j} \grave{e}_{2j}),$$

and the action of $\mathbf{J}$ is given by,

$$\mathbf{J}(\mathbf{x}) = \mathbf{J}(\underline{x}) + \mathbf{J}(\underline{\grave{x}}) = \sum_{j=1}^{m} (x_{m+j} e_j - x_j e_{m+j}) + \sum_{j=1}^{n} (\grave{x}_{2j} \grave{e}_{2j-1} - \grave{x}_{2j-1} \grave{e}_{2j}). \tag{3.23}$$

Checking (AH1)-(AH2):

$$\{\mathbf{x}, \mathbf{y}\} = \{\mathbf{J}(\mathbf{x}), \mathbf{J}(\mathbf{y})\}, \quad \{\mathbf{J}(\mathbf{x}), \mathbf{y}\} = -\{\mathbf{x}, \mathbf{J}(\mathbf{y})\} \quad \mathbf{x}, \mathbf{y} \in \mathbf{S}.$$

We clearly have

$$\begin{aligned}
\{\underline{x}, \underline{y}\} &= -2 \sum_{j=1}^{m} (x_j y_j + x_{m+j} y_{m+j}) = \{\mathbf{J}(\underline{x}), \mathbf{J}(\underline{y})\}, \\
\{\underline{x}, \underline{\grave{y}}\} &= \sum_{j,k} x_j \grave{y}_k \{e_j, \grave{e}_k\} = 0 = \{\mathbf{J}(\underline{x}), \mathbf{J}(\underline{\grave{y}})\}, \\
\{\underline{\grave{x}}, \underline{\grave{y}}\} &= \sum_{j,k} \grave{x}_j \grave{y}_k [\grave{e}_j, \grave{e}_k] = \sum_{j=1}^{n} \grave{x}_{2j-1} \grave{y}_{2j} - \grave{x}_{2j} \grave{y}_{2j-1}.
\end{aligned}$$

and also

$$\begin{aligned}
\{\mathbf{J}(\underline{\grave{x}}), \mathbf{J}(\underline{\grave{y}})\} &= \sum_{j,k=1}^{n} \{\grave{x}_{2j} \grave{e}_{2j-1} - \grave{x}_{2j-1} \grave{e}_{2j},\ \grave{y}_{2k} \grave{e}_{2k-1} - \grave{y}_{2k-1} \grave{e}_{2k}\} \\
&= \sum_{j,k=1}^{n} -\grave{x}_{2j} \grave{y}_{2k-1} [\grave{e}_{2j-1}, \grave{e}_{2k}] - \grave{x}_{2j-1} \grave{y}_{2k} [\grave{e}_{2j}, \grave{e}_{2k-1}] \\
&= \sum_{j=1}^{n} \grave{x}_{2j-1} \grave{y}_{2j} - \grave{x}_{2j} \grave{y}_{2j-1}.
\end{aligned}$$

Hence we conclude

$$\{\mathbf{x}, \mathbf{y}\} = -2 \sum_{j=1}^{m} (x_j y_j + x_{m+j} y_{m+j}) + \sum_{j=1}^{n} \grave{x}_{2j-1} \grave{y}_{2j} - \grave{x}_{2j} \grave{y}_{2j-1} = \{\mathbf{J}(\mathbf{x}), \mathbf{J}(\mathbf{y})\},$$

which clearly is a central element in $\mathcal{A}_{2m,2n}$. On the other hand, we have

$$
\begin{aligned}
\{\mathbf{J}(\underline{x}), \underline{y}\} &= -2\sum_{j=1}^{m} (x_{m+j}y_j - x_j y_{m+j}) = -\{\underline{x}, \mathbf{J}(\underline{y})\},\\
\{\mathbf{J}(\underline{x}), \underline{\grave{y}}\} &= \sum_{j,k} \{x_{m+j}e_j - x_j e_{m+j}, \grave{y}_k \grave{e}_k\} = 0 = \{\underline{x}, \mathbf{J}(\underline{\grave{y}})\},\\
\{\mathbf{J}(\underline{\grave{x}}), \underline{\grave{y}}\} &= \sum_{j,k=1}^{n} \{\grave{x}_{2j}\grave{e}_{2j-1} - \grave{x}_{2j-1}\grave{e}_{2j},\ \grave{y}_{2k-1}\grave{e}_{2k-1} + \grave{y}_{2k}\grave{e}_{2k}\}\\
&= \sum_{j,k=1}^{n} \grave{x}_{2j}\grave{y}_{2k}[\grave{e}_{2j-1}, \grave{e}_{2k}] - \grave{x}_{2j-1}\grave{y}_{2k-1}[\grave{e}_{2j}, \grave{e}_{2k-1}]\\
&= \sum_{j=1}^{n} \grave{x}_{2j-1}\grave{y}_{2j-1} + \grave{x}_{2j}\grave{y}_{2j}
\end{aligned}
$$

and

$$
\begin{aligned}
\{\underline{\grave{x}}, \mathbf{J}(\underline{\grave{y}})\} &= \sum_{j,k=1}^{n} \{\grave{x}_{2j-1}\grave{e}_{2j-1} + \grave{x}_{2j}\grave{e}_{2j},\ \grave{y}_{2k}\grave{e}_{2k-1} - \grave{y}_{2k-1}\grave{e}_{2k}\}\\
&= \sum_{j,k=1}^{n} -\grave{x}_{2j-1}\grave{y}_{2k-1}[\grave{e}_{2j-1}, \grave{e}_{2k}] + \grave{x}_{2j}\grave{y}_{2k}[\grave{e}_{2j}, \grave{e}_{2k-1}]\\
&= -\sum_{j=1}^{n} \grave{x}_{2j-1}\grave{y}_{2j-1} + \grave{x}_{2j}\grave{y}_{2j}.
\end{aligned}
$$

Thus we obtain

$$
\{\mathbf{J}(\mathbf{x}), \mathbf{y}\} = -2\sum_{j=1}^{m} (x_{m+j}y_j - x_j y_{m+j}) + \sum_{j=1}^{n} (\grave{x}_{2j-1}\grave{y}_{2j-1} + \grave{x}_{2j}\grave{y}_{2j}) = -\{\mathbf{x}, \mathbf{J}(\mathbf{y})\},
$$

which also is a central element in $\mathcal{A}_{2m,2n}$. □

Checking the axiom **(AH3)** requires the introduction of a suitable element $\mathbf{B} \in \mathcal{A}_{2m,2n}$. In accordance with axiom **(DH3)** we have that such an element $\mathbf{B}$ is determined by the action of the vector derivative $\partial_{\mathbf{x}}$ on $\mathbf{J}(\mathbf{x})$. Thus we define,

$$
\begin{aligned}
\mathbf{B} := \frac{1}{2}\partial_{\mathbf{x}}[\mathbf{J}(\mathbf{x})] &= \frac{1}{2}(\partial_{\underline{\grave{x}}} - \partial_{\underline{x}})(\mathbf{J}(\underline{x}) + \mathbf{J}(\underline{\grave{x}})) = \frac{1}{2}\left(\partial_{\underline{\grave{x}}}[\mathbf{J}(\underline{\grave{x}})] - \partial_{\underline{x}}[\mathbf{J}(\underline{x})]\right)\\
&= \sum_{j=1}^{m} e_j e_{m+j} - \sum_{j=1}^{2n} \grave{e}_j^{\,2}.
\end{aligned}
\tag{3.24}
$$

This definition is consistent with (2.21) since $\mathbf{J}(\mathbf{B}) = \mathbf{B}$.

Checking (AH3):

$$
[\mathbf{B}, \mathbf{x}] = -2\mathbf{J}(\mathbf{x}), \quad [\mathbf{B}, \mathbf{J}(\mathbf{x})] = 2\mathbf{x}, \qquad \mathbf{x} \in \mathbf{S}.
$$

Let us write $\mathbf{B} := \mathbf{B}_b - \mathbf{B}_f$ with $\mathbf{B}_b = \sum_{j=1}^{m} e_j e_{m+j}$ and $\mathbf{B}_f = \sum_{j=1}^{n} \left(\grave{e}_{2j-1}^{2} + \grave{e}_{2j}^{2}\right)$. We immediately obtain,

$$
\begin{aligned}
[\mathbf{B}_b, \underline{x}] &= \sum_{j,k} x_k [e_j e_{m+j}, e_k] = -2 \sum_{j=1}^{m} (x_{m+j} e_j - x_j e_{m+j}) = -2\mathbf{J}(\underline{x}),\\
[\mathbf{B}_b, \underline{\grave{x}}] &= \sum_{j,k} \grave{x}_k [e_j e_{m+j}, \grave{e}_k] = 0 = \sum_{j,k} x_k [\grave{e}_j^{2}, e_k] = [\mathbf{B}_f, \underline{x}],\\
[\mathbf{B}_f, \underline{\grave{x}}] &= \sum_{j,k=1}^{2n} \left[\grave{e}_j^{2}, \grave{x}_k \grave{e}_k\right] = \sum_{j,k=1}^{n} \grave{x}_{2k} \left[\grave{e}_{2j-1}^{2}, \grave{e}_{2k}\right] + \grave{x}_{2k-1} \left[\grave{e}_{2j}^{2}, \grave{e}_{2k-1}\right]\\
&= 2 \sum_{j=1}^{n} (\grave{x}_{2j} \grave{e}_{2j-1} - \grave{x}_{2j-1} \grave{e}_{2j}) = 2\mathbf{J}(\underline{\grave{x}}).
\end{aligned}
$$

Then we conclude that

$$
[\mathbf{B}, \mathbf{x}] = [\mathbf{B}_b, \underline{x}] - [\mathbf{B}_f, \underline{\grave{x}}] = -2\mathbf{J}(\underline{x}) - 2\mathbf{J}(\underline{\grave{x}}) = -2\mathbf{J}(\mathbf{x}).
$$

The other statement in **(AH3)** is obtained by applying $\mathbf{J}$ to the above relation. □

In this way we have proven that the mapping $\Psi_{2m,2n} : R(S \cup J(S), \mathcal{B}) \to \mathcal{A}_{2m,2n}$ given by

$$
x \to \mathbf{x}, \qquad J(x) \to \mathbf{J}(\mathbf{x}), \qquad \mathcal{B} \to \mathbf{B}, \tag{3.25}
$$

constitutes a representation of the radial algebra with complex structure $R(S \cup J(S), \mathcal{B})$ in $\mathcal{A}_{2m,2n}$. The range $\Psi_{2m,2n}\left(R(S \cup J(S), \mathcal{B})\right)$ will be denoted by $R(\mathbf{S} \cup \mathbf{J}(\mathbf{S}), \mathbf{B})$.

We now possess of all elements to complete the commutative diagram (3.21) as follows,

$$
\begin{array}{ccc}
R(S \cup J(S), \mathcal{B}) & \xrightarrow{\Psi_2} & R(S_2 \cup J_2(S_2), -iB_1)\\
\Psi_{2m,2n}\downarrow & \searrow^{\Psi_{2m,4n}} & \downarrow \Psi_{m,2n}\\
R(\mathbf{S} \cup \mathbf{J}(\mathbf{S}), \mathbf{B}) & \xleftarrow[P_{\mathcal{J},\mathbf{J}}]{} & R(\mathbf{S} \cup \mathcal{J}(\mathbf{S}), -i\mathscr{B})
\end{array}
$$

where the algebra homomorphism

$$
P_{\mathcal{J},\mathbf{J}} : R(\mathbf{S} \cup \mathcal{J}(\mathbf{S}), -i\mathscr{B}) \subset \mathcal{A}_{2m,4n} \to R(\mathbf{S} \cup \mathbf{J}(\mathbf{S}), \mathbf{B}) \subset \mathcal{A}_{2m,2n}
$$

is defined by

$$
\begin{aligned}
\mathbf{x} \in \mathcal{A}_{2m,4n} &\to \mathbf{x} \in \mathcal{A}_{2m,2n},\\
\mathcal{J}(\mathbf{x}) \in \mathcal{A}_{2m,4n} &\to \mathbf{J}(\mathbf{x}) \in \mathcal{A}_{2m,2n},\\
-i\mathscr{B} \in \mathcal{A}_{2m,4n} &\to \mathbf{B} \in \mathcal{A}_{2m,2n}.
\end{aligned}
$$

3.3.2 Vector derivatives $\partial_{\mathbf{x}}$ and $\partial_{\mathbf{J(x)}}$

In Section 2.5.1, the vector derivatives ∂_x, $\partial_{J(x)}$ were extended from $R(\mathcal{S})$ and $R(J(\mathcal{S}))$, respectively, to endomorphisms in $R(\mathcal{S}\cup J(\mathcal{S}),\mathbf{B})$. The goal of this section is to study the corresponding vector derivatives $\partial_{\mathbf{x}}$, $\partial_{\mathbf{J(x)}}$ (see (2.22)) in the above described radial algebra representation $R(\mathbf{S}\cup\mathbf{J}(\mathbf{S}),\mathbf{B})\subset\mathcal{A}_{2m,2n}$.

We first observe that the partial derivatives $\partial_{x_j}, \partial_{\grave{x}_j}$ always commute with the complex structure $\mathbf{J}$. Then, the corresponding action of $\mathbf{J}$ on the super Dirac operator $\partial_{\mathbf{x}}$ can be easily seen by means of the following bosonic and fermionic twisted Dirac operators on $\mathcal{A}_{2m,2n}$:

$$\begin{aligned}\partial_{\mathbf{J}(\underline{x})} &:= \mathbf{J}(\partial_{\underline{x}}) = \sum_{j=1}^{m}\left(e_j\partial_{x_{m+j}} - e_{m+j}\partial_{x_j}\right),\\ \partial_{\mathbf{J}(\underline{\grave{x}})} &:= \mathbf{J}(\partial_{\underline{\grave{x}}}) = 2\sum_{j=1}^{n}\left(\grave{e}_{2j-1}\partial_{\grave{x}_{2j-1}} + \grave{e}_{2j}\partial_{\grave{x}_{2j}}\right),\end{aligned}$$

where $\partial_{\underline{x}}, \partial_{\underline{\grave{x}}}$ are defined as in (3.18), i.e.

$$\begin{aligned}\partial_{\underline{x}} &= \sum_{j=1}^{m}\left(e_j\partial_{x_j} + e_{m+j}\partial_{x_{m+j}}\right),\\ \partial_{\underline{\grave{x}}} &= 2\sum_{j=1}^{n}\left(\grave{e}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}}\right).\end{aligned}$$

The twisted super Dirac operator $\partial_{\mathbf{J(x)}}$ then is defined by

$$\begin{cases}\partial_{\mathbf{J(x)}}\cdot := \mathbf{J}(\partial_{\mathbf{x}}\cdot) = \partial_{\mathbf{J}(\underline{\grave{x}})}\cdot - \partial_{\mathbf{J}(\underline{x})}\cdot\,,\\ \cdot\partial_{\mathbf{J(x)}} = \mathbf{J}(\cdot\partial_{\mathbf{x}}) = -\cdot\,\partial_{\mathbf{J}(\underline{\grave{x}})} - \cdot\partial_{\mathbf{J}(\underline{x})},\end{cases} \quad \text{where} \quad \begin{cases}\partial_{\mathbf{x}}\cdot = \partial_{\underline{\grave{x}}}\cdot - \partial_{\underline{x}}\cdot\,,\\ \cdot\partial_{\mathbf{x}} = -\cdot\,\partial_{\underline{\grave{x}}} - \cdot\partial_{\underline{x}}.\end{cases} \tag{3.26}$$

This means that the above actions are subject to the relations:

$$\begin{cases}\mathbf{J}(\partial_{\mathbf{x}}[F]) = \partial_{\mathbf{J(x)}}[\mathbf{J}(F)],\\ \mathbf{J}([F]\partial_{\mathbf{x}}) = [\mathbf{J}(F)]\partial_{\mathbf{J(x)}},\end{cases} \qquad \begin{cases}\mathbf{J}(\partial_{\mathbf{J(x)}}[F]) = -\partial_{\mathbf{x}}[\mathbf{J}(F)],\\ \mathbf{J}([F]\partial_{\mathbf{J(x)}}) = -[\mathbf{J}(F)]\partial_{\mathbf{x}},\end{cases} \tag{3.27}$$

for every $F\in\mathcal{A}_{2m,2n}$.

Similar properties to (3.27) are satisfied by the vector derivatives ∂_x, $\partial_{J(x)}$ at the radial algebra level, see Lemma 2.12. In fact, the operators $\partial_{\mathbf{x}}$ and $\partial_{\mathbf{J(x)}}$ are the corresponding representations of ∂_x and $\partial_{J(x)}$ in superspace. In order to show that this is true, we now verify that the definitions (3.26) are in agreement with the axioms **(DH1)-(DH4)** established in the radial algebra setting. We first check the basic evaluations.

Checking (DH3)

$$\begin{aligned}\partial_{\mathbf{x}}[\mathbf{x}] = [\mathbf{x}]\partial_{\mathbf{x}} = M = \partial_{\mathbf{J(x)}}[\mathbf{J}(\mathbf{x})] = [\mathbf{J}(\mathbf{x})]\partial_{\mathbf{J(x)}},\\ \partial_{\mathbf{x}}[\mathbf{J}(\mathbf{x})] = -[\mathbf{J}(\mathbf{x})]\partial_{\mathbf{x}} = 2\mathbf{B} = -\partial_{\mathbf{J(x)}}[\mathbf{x}] = [\mathbf{x}]\partial_{\mathbf{J(x)}}.\end{aligned}$$

It is known from [33] that $\partial_{\mathbf{x}}[\mathbf{x}] = [\mathbf{x}]\partial_{\mathbf{x}} = 2m-2n =: M$, since $\partial_{\mathbf{x}}$ represents the original vector derivative in $\mathcal{A}_{2m,2n}$. Then applying $\mathbf{J}$ and using (3.27) we obtain $\partial_{\mathbf{J}(\mathbf{x})}[\mathbf{J}(\mathbf{x})] = [\mathbf{J}(\mathbf{x})]\partial_{\mathbf{J}(\mathbf{x})} = 2m-2n$.

The relation $\partial_{\mathbf{x}}[\mathbf{J}(\mathbf{x})] = 2\mathbf{B}$ was the one used in (3.24). Furthermore,

$$\begin{aligned}
-[\mathbf{J}(\mathbf{x})]\partial_{\mathbf{x}} &= \left(\mathbf{J}(\underline{x}) + \mathbf{J}(\underline{\grave{x}})\right)\left(\partial_{\underline{\grave{x}}} + \partial_{\underline{x}}\right) = [\mathbf{J}(\underline{x})]\partial_{\underline{x}} + [\mathbf{J}(\underline{\grave{x}})]\partial_{\underline{\grave{x}}} \\
&= 2\sum_{j=1}^{n}(-\grave{e}_{2j-1}^{2} - \grave{e}_{2j}^{2}) + \sum_{j=1}^{m}(-e_{m+j}e_j + e_j e_{m+j}) \\
&= 2\mathbf{B}.
\end{aligned}$$

Applying $\mathbf{J}$ on the above equalities and using again (3.27) we conclude that $-\partial_{\mathbf{J}(\mathbf{x})}[\mathbf{x}] = 2\mathbf{B} = [\mathbf{x}]\partial_{\mathbf{J}(\mathbf{x})}$. □

Checking (DH4)

$$\begin{cases}
\partial_{\mathbf{x}}[\mathbf{x}^2] = [\mathbf{x}^2]\partial_{\mathbf{x}} = 2\mathbf{x}, & \partial_{\mathbf{J}(\mathbf{x})}[\mathbf{x}^2] = [\mathbf{x}^2]\partial_{\mathbf{J}(\mathbf{x})} = 2\mathbf{J}(\mathbf{x}), \\
\partial_{\mathbf{x}}[\mathbf{x}\mathbf{J}(\mathbf{x})] = (M+2)\mathbf{J}(\mathbf{x}) - 2\mathbf{x}\mathbf{B}, & [\mathbf{x}\mathbf{J}(\mathbf{x})]\partial_{\mathbf{x}} = -(M-2)\mathbf{J}(\mathbf{x}) - 2\mathbf{x}\mathbf{B}, \\
\partial_{\mathbf{J}(\mathbf{x})}[\mathbf{x}\mathbf{J}(\mathbf{x})] = -(M+2)\mathbf{x} - 2\mathbf{J}(\mathbf{x})\mathbf{B}, & [\mathbf{x}\mathbf{J}(\mathbf{x})]\partial_{\mathbf{J}(\mathbf{x})} = (M-2)\mathbf{x} - 2\mathbf{J}(\mathbf{x})\mathbf{B},
\end{cases}$$

for $\mathbf{x} \in \mathbf{S}$, and

$$\begin{cases}
\partial_{\mathbf{x}}\{\mathbf{x},\mathbf{y}\} = \{\mathbf{x},\mathbf{y}\}\partial_{\mathbf{x}} = 2\mathbf{y} = \partial_{\mathbf{J}(\mathbf{x})}\{\mathbf{J}(\mathbf{x}),\mathbf{y}\} = \{\mathbf{J}(\mathbf{x}),\mathbf{y}\}\partial_{\mathbf{J}(\mathbf{x})}, \\
\partial_{\mathbf{J}(\mathbf{x})}\{\mathbf{x},\mathbf{y}\} = \{\mathbf{x},\mathbf{y}\}\partial_{\mathbf{J}(\mathbf{x})} = 2\mathbf{J}(\mathbf{y}) = -\partial_{\mathbf{x}}\{\mathbf{J}(\mathbf{x}),\mathbf{y}\} = -\{\mathbf{J}(\mathbf{x}),\mathbf{y}\}\partial_{\mathbf{x}},
\end{cases}$$

for $\mathbf{x},\mathbf{y} \in \mathbf{S}, \mathbf{x} \neq \mathbf{y}$.

The equalities $\partial_{\mathbf{x}}[\mathbf{x}^2] = [\mathbf{x}^2]\partial_{\mathbf{x}} = 2\mathbf{x}$ and $\partial_{\mathbf{x}}\{\mathbf{x},\mathbf{y}\} = \{\mathbf{x},\mathbf{y}\}\partial_{\mathbf{x}} = 2\mathbf{y}$ $(\mathbf{x} \neq \mathbf{y})$ were obtained in [33]. Letting act $\mathbf{J}$ on each of the previous relations we get,

$$\partial_{\mathbf{J}(\mathbf{x})}[\mathbf{x}^2] = [\mathbf{x}^2]\partial_{\mathbf{J}(\mathbf{x})} = 2\mathbf{J}(\mathbf{x}), \quad \text{and} \quad \partial_{\mathbf{J}(\mathbf{x})}\{\mathbf{x},\mathbf{y}\} = \{\mathbf{x},\mathbf{y}\}\partial_{\mathbf{J}(\mathbf{x})} = 2\mathbf{J}(\mathbf{y}).$$

We also find

$$\begin{aligned}
\partial_{\mathbf{x}}[\mathbf{x}\mathbf{J}(\mathbf{x})] &= \left(\partial_{\underline{\grave{x}}} - \partial_{\underline{x}}\right)\left(\underline{\grave{x}} + \underline{x}\right)\left(\mathbf{J}(\underline{x}) + \mathbf{J}(\underline{\grave{x}})\right) \\
&= \left(\partial_{\underline{\grave{x}}} - \partial_{\underline{x}}\right)\left[\underline{x}\mathbf{J}(\underline{x}) + \underline{x}\mathbf{J}(\underline{\grave{x}}) + \underline{\grave{x}}\mathbf{J}(\underline{x}) + \underline{\grave{x}}\mathbf{J}(\underline{\grave{x}})\right] \\
&= -\partial_{\underline{\grave{x}}}\left[\mathbf{J}(\underline{\grave{x}})\underline{x}\right] + \partial_{\underline{\grave{x}}}\left[\underline{\grave{x}}\mathbf{J}(\underline{x})\right] + \partial_{\underline{\grave{x}}}\left[\underline{\grave{x}}\mathbf{J}(\underline{\grave{x}})\right] - \partial_{\underline{x}}\left[\underline{x}\mathbf{J}(\underline{x})\right] - \partial_{\underline{x}}\left[\underline{x}\mathbf{J}(\underline{\grave{x}})\right] + \partial_{\underline{x}}\left[\mathbf{J}(\underline{x})\underline{\grave{x}}\right].
\end{aligned}$$

However

$$\begin{aligned}
&\partial_{\underline{\grave{x}}}\left[\mathbf{J}(\underline{\grave{x}})\underline{x}\right] = \partial_{\underline{\grave{x}}}\left[\mathbf{J}(\underline{\grave{x}})\right]\underline{x} = -2\mathbf{B}_f\,\underline{x}, && \partial_{\underline{\grave{x}}}\left[\underline{\grave{x}}\mathbf{J}(\underline{x})\right] = \partial_{\underline{\grave{x}}}\left[\underline{\grave{x}}\right]\mathbf{J}(\underline{x}) = -2n\,\mathbf{J}(\underline{x}), \\
&\partial_{\underline{x}}\left[\underline{x}\mathbf{J}(\underline{\grave{x}})\right] = \partial_{\underline{x}}\left[\underline{x}\right]\mathbf{J}(\underline{\grave{x}}) = -2m\,\mathbf{J}(\underline{\grave{x}}), && \partial_{\underline{x}}\left[\mathbf{J}(\underline{x})\underline{\grave{x}}\right] = \partial_{\underline{x}}\left[\mathbf{J}(\underline{x})\right]\underline{\grave{x}} = -2\mathbf{B}_b\,\underline{\grave{x}},
\end{aligned}$$

$$
\begin{aligned}
\partial_{\underline{\grave{x}}}\left[\underline{\grave{x}}\mathbf{J}(\underline{\grave{x}})\right] &= 2\sum_{j=1}^{n} \grave{e}_{2j}\partial_{\grave{x}_{2j-1}}\left[\underline{\grave{x}}\mathbf{J}(\underline{\grave{x}})\right] - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}}\left[\underline{\grave{x}}\mathbf{J}(\underline{\grave{x}})\right] \\
&= 2\sum_{j=1}^{n} \grave{e}_{2j}\left(\grave{e}_{2j-1}\mathbf{J}(\underline{\grave{x}}) + \underline{\grave{x}}\grave{e}_{2j}\right) - \grave{e}_{2j-1}\left(\grave{e}_{2j}\mathbf{J}(\underline{\grave{x}}) - \underline{\grave{x}}\grave{e}_{2j-1}\right) \\
&= 2\left(\sum_{j=1}^{n} \grave{e}_{2j}\grave{e}_{2j-1} - \grave{e}_{2j-1}\grave{e}_{2j}\right)\mathbf{J}(\underline{\grave{x}}) + 2\sum_{j=1}^{n}\left(\grave{e}_{2j-1}\underline{\grave{x}}\grave{e}_{2j-1} + \grave{e}_{2j}\underline{\grave{x}}\grave{e}_{2j}\right) \\
&= -2n\mathbf{J}(\underline{\grave{x}}) + 2\sum_{j=1}^{n}\left(\grave{e}_{2j-1}^{2}\,\underline{\grave{x}} - \grave{e}_{2j-1}\grave{x}_{2j} + \grave{e}_{2j}^{2}\,\underline{\grave{x}} + \grave{e}_{2j}\grave{x}_{2j-1}\right) \\
&= -(2n+2)\mathbf{J}(\underline{\grave{x}}) + 2\mathbf{B}_f\,\underline{\grave{x}}
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_{\underline{x}}\left[\underline{x}\mathbf{J}(\underline{x})\right] &= \sum_{j=1}^{m} e_j\left(e_j\mathbf{J}(\underline{x}) - \underline{x}e_{m+j}\right) + e_{m+j}\left(e_{m+j}\mathbf{J}(\underline{x}) + \underline{x}e_j\right) \\
&= -2m\mathbf{J}(\underline{x}) + \sum_{j=1}^{m}\left(e_{m+j}\,\underline{x}\,e_j - e_j\,\underline{x}\,e_{m+j}\right) \\
&= -2m\mathbf{J}(\underline{x}) + \sum_{j=1}^{m}\left(2e_je_{m+j}\,\underline{x} + 2x_{m+j}e_j - 2x_je_{m+j}\right) \\
&= (-2m+2)\mathbf{J}(\underline{x}) + 2\mathbf{B}_b\,\underline{x}.
\end{aligned}
$$

Then we conclude that

$$
\begin{aligned}
\partial_{\mathbf{x}}[\mathbf{x}\mathbf{J}(\mathbf{x})] &= 2\mathbf{B}_f\,\underline{x} - 2n\,\mathbf{J}(\underline{x}) - (2n+2)\mathbf{J}(\underline{\grave{x}}) + 2\mathbf{B}_f\,\underline{\grave{x}} \\
&\quad - (-2m+2)\mathbf{J}(\underline{x}) - 2\mathbf{B}_b\,\underline{x} + 2m\,\mathbf{J}(\underline{\grave{x}}) - 2\mathbf{B}_b\,\underline{\grave{x}} \\
&= (2\mathbf{B}_f - 2\mathbf{B}_b)\underline{x} + (2\mathbf{B}_f - 2\mathbf{B}_b)\underline{\grave{x}} \\
&\quad + (2m-2n-2)\mathbf{J}(\underline{x}) + (2m-2n-2)\mathbf{J}(\underline{\grave{x}}) \\
&= (2m-2n-2)\mathbf{J}(\mathbf{x}) - 2\mathbf{B}\mathbf{x} \\
&= (2m-2n+2)\mathbf{J}(\mathbf{x}) - 2\mathbf{x}\mathbf{B}.
\end{aligned} \tag{3.28}
$$

Letting act $\mathbf{J}$ on (3.28) we obtain:

$$
\partial_{\mathbf{J}(\mathbf{x})}[\mathbf{x}\mathbf{J}(\mathbf{x})] = -(M+2)\mathbf{x} - 2\mathbf{J}(\mathbf{x})\mathbf{B}, \tag{3.29}
$$

and conjugating both (3.28) and (3.29), we get

$$
[\mathbf{x}\mathbf{J}(\mathbf{x})]\partial_{\mathbf{x}} = -(M-2)\mathbf{J}(\mathbf{x}) - 2\mathbf{x}\mathbf{B}, \quad \text{and} \quad [\mathbf{x}\mathbf{J}(\mathbf{x})]\partial_{\mathbf{J}(\mathbf{x})} = (M-2)\mathbf{x} - 2\mathbf{J}(\mathbf{x})\mathbf{B}.
$$

Furthermore we have

$$
\begin{aligned}
\partial_{\mathbf{x}}\{\mathbf{J}(\mathbf{x}),\mathbf{y}\} &= \left(\partial_{\underline{\grave{x}}} - \partial_{\underline{x}}\right)\left(-2\sum_{j=1}^{m}(x_{m+j}y_j - x_j y_{m+j}) + \sum_{j=1}^{n}(\grave{x}_{2j-1}\grave{y}_{2j-1} + \grave{x}_{2j}\grave{y}_{2j})\right)\\
&= 2\sum_{j=1}^{m}(e_{m+j}y_j - e_j y_{m+j}) + 2\sum_{j=1}^{n}(\grave{e}_{2j}\grave{y}_{2j-1} - \grave{e}_{2j-1}\grave{y}_{2j})\\
&= -2\mathbf{J}(\underline{y}) - 2\mathbf{J}(\underline{\grave{y}}) = -2\mathbf{J}(\mathbf{y})\\
&= \left(-2\sum_{j=1}^{m}(x_{m+j}y_j - x_j y_{m+j}) + \sum_{j=1}^{n}(\grave{x}_{2j-1}\grave{y}_{2j-1} + \grave{x}_{2j}\grave{y}_{2j})\right)\left(-\partial_{\underline{\grave{x}}} - \partial_{\underline{x}}\right)\\
&= \{\mathbf{J}(\mathbf{x}),\mathbf{y}\}\partial_{\mathbf{x}}.
\end{aligned}
$$

Finally, from the action of $\mathbf{J}$ on the above relations we obtain

$$\partial_{\mathbf{J}(\mathbf{x})}\{\mathbf{J}(\mathbf{x}),\mathbf{y}\} = 2\mathbf{y} = \{\mathbf{J}(\mathbf{x}),\mathbf{y}\}\,\partial_{\mathbf{J}(\mathbf{x})}.$$

□

Checking (DH1)

$$
\begin{aligned}
&\partial_{\mathbf{x}}[fF] = \partial_{\mathbf{x}}[f]F + f\partial_{\mathbf{x}}[F], \quad \partial_{\mathbf{J}(\mathbf{x})}[fF] = \partial_{\mathbf{J}(\mathbf{x})}[f]F + f\partial_{\mathbf{J}(\mathbf{x})}[F],\\
&[fF]\partial_{\mathbf{x}} = F[f]\partial_{\mathbf{x}} + f[F]\partial_{\mathbf{x}}, \quad [fF]\partial_{\mathbf{J}(\mathbf{x})} = F[f]\partial_{\mathbf{J}(\mathbf{x})} + f[F]\partial_{\mathbf{J}(\mathbf{x})},\\
&f\in R_0(\mathbf{S}\cup\mathbf{J}(\mathbf{S})), \quad F\in R(\mathbf{S}\cup\mathbf{J}(\mathbf{S}),\mathbf{B}).
\end{aligned}
$$

It suffices to prove **(DH1)** just for those f generating $R_0(\mathbf{S}\cup\mathbf{J}(\mathbf{S}))$, which are given by the anti-commutators $\{\mathbf{x},\mathbf{y}\}$, $\mathbf{x}^2$, $\{\mathbf{J}(\mathbf{x}),\mathbf{y}\}$ with $\mathbf{x},\mathbf{y}\in\mathbf{S}$. By straightforward computation we obtain for every $F\in\mathcal{A}_{2m,2n}$ that

$$
\begin{cases}
\partial_{\mathbf{x}}[\{\mathbf{x},\mathbf{y}\}F] = 2\mathbf{y}F + \{\mathbf{x},\mathbf{y}\}\partial_{\mathbf{x}}[F],\\
\partial_{\mathbf{x}}[\mathbf{x}^2F] = 2\mathbf{x}F + \mathbf{x}^2\partial_{\mathbf{x}}[F],\\
\partial_{\mathbf{x}}[\{\mathbf{J}(\mathbf{x}),\mathbf{y}\}F] = -2\mathbf{J}(\mathbf{y})F + \{\mathbf{J}(\mathbf{x}),\mathbf{y}\}\partial_{\mathbf{x}}[F],
\end{cases}
$$

$$
\begin{cases}
[\{\mathbf{x},\mathbf{y}\}F]\partial_{\mathbf{x}} = 2F\mathbf{y} + \{\mathbf{x},\mathbf{y}\}[F]\partial_{\mathbf{x}},\\
[\mathbf{x}^2F]\partial_{\mathbf{x}} = 2F\mathbf{x} + \mathbf{x}^2[F]\partial_{\mathbf{x}},\\
[\{\mathbf{J}(\mathbf{x}),\mathbf{y}\}F]\partial_{\mathbf{x}} = -2F\mathbf{J}(\mathbf{y}) + \{\mathbf{J}(\mathbf{x}),\mathbf{y}\}[F]\partial_{\mathbf{x}}.
\end{cases}
$$

Finally, the other relations can be obtained by means of the action of $\mathbf{J}$ on the above equalities. □

The statement in **(DH2)** being a trivial consequence of Lemma 3.2, we omit its proof.

3.3.3 Directional derivatives

We now are able to obtain explicit expressions for the directional derivatives $D_{\mathbf{y},\mathbf{x}}$ and $D_{\mathbf{J}(\mathbf{y}),\mathbf{x}}$ in the representation $R(\mathbf{S}\cup\mathbf{J}(\mathbf{S}),\mathbf{B})$. From the radial algebra framework (see Theorem 2.11) it is known that

$$\{\partial_{\mathbf{x}},\mathbf{y}\} = 2D_{\mathbf{y},\mathbf{x}} + \delta_{\mathbf{x},\mathbf{y}}M, \qquad \{\partial_{\mathbf{x}},\mathbf{J}(\mathbf{y})\} = 2D_{\mathbf{J}(\mathbf{y}),\mathbf{x}} + 2\delta_{\mathbf{x},\mathbf{y}}\mathbf{B}.$$

For the operator $\{\partial_{\mathbf{x}},\mathbf{y}\}$ we first obtain

$$\begin{aligned}
\{\partial_{\mathbf{x}},\mathbf{y}\} &= \left\{-\sum_{j=1}^{2m} e_j\partial_{x_j} + 2\sum_{j=1}^{n}\left(\grave{e}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}}\right), \sum_{k=1}^{2m} y_k e_k + \sum_{k=1}^{2n}\grave{y}_k\grave{e}_k\right\}\\
&= -\sum_{j,k=1}^{2m}\left\{e_j\partial_{x_j}, y_k e_k\right\} - \sum_{\substack{1\le j\le 2m\\ 1\le k\le 2n}}\left\{e_j\partial_{x_j}, \grave{y}_k\grave{e}_k\right\}\\
&\quad + 2\sum_{\substack{1\le j\le n\\ 1\le k\le 2m}}\Big(\left\{\grave{e}_{2j}\partial_{\grave{x}_{2j-1}}, y_k e_k\right\} - \left\{\grave{e}_{2j-1}\partial_{\grave{x}_{2j}}, y_k e_k\right\}\Big)\\
&\quad + 2\sum_{\substack{1\le j\le n\\ 1\le k\le 2n}}\Big(\left\{\grave{e}_{2j}\partial_{\grave{x}_{2j-1}}, \grave{y}_k\grave{e}_k\right\} - \left\{\grave{e}_{2j-1}\partial_{\grave{x}_{2j}}, \grave{y}_k\grave{e}_k\right\}\Big).
\end{aligned}$$

However, from (3.11) and (3.7), we obtain the following relations for every pair $v,w\in VAR$ and every pair $\grave{v},\grave{w}\in V\grave{A}R$ (see (3.14)):

$$\begin{aligned}
\{e_j\partial_v, e_k w\} &= -\delta_{v,w}e_k e_j - 2\delta_{j,k}\delta_{v,w} - 2\delta_{j,k}w\partial_v,\\
\{e_j\partial_v, \grave{e}_k\grave{v}\} &= 0 = \{\grave{e}_j\partial_{\grave{v}}, e_k v\},\\
\{\grave{e}_{2j}\partial_{\grave{v}}, \grave{e}_k\grave{w}\} &= \delta_{\grave{v},\grave{w}}\,\grave{e}_k\grave{e}_{2j} - \delta_{2j-1,k}\,\delta_{\grave{v},\grave{w}} + \delta_{2j-1,k}\,\grave{w}\,\partial_{\grave{v}},\\
\{\grave{e}_{2j-1}\partial_{\grave{v}}, \grave{e}_k\grave{w}\} &= \delta_{\grave{v},\grave{w}}\,\grave{e}_k\grave{e}_{2j-1} + \delta_{2j,k}\,\delta_{\grave{v},\grave{w}} - \delta_{2j,k}\,\grave{w}\,\partial_{\grave{v}},
\end{aligned}$$

whence,

$$\begin{aligned}
\{\partial_{\mathbf{x}},\mathbf{y}\} = &-\sum_{j,k=1}^{2m}\left(-\delta_{\mathbf{x},\mathbf{y}}\delta_{j,k}e_k e_j - 2\delta_{j,k}\delta_{\mathbf{x},\mathbf{y}} - 2\delta_{j,k}y_k\partial_{x_j}\right)\\
&+ 2\sum_{\substack{1\le j\le n\\ 1\le k\le 2n}}\left(\delta_{\mathbf{x},\mathbf{y}}\delta_{2j-1,k}\,\grave{e}_k\grave{e}_{2j} - \delta_{2j-1,k}\,\delta_{\mathbf{x},\mathbf{y}} + \delta_{2j-1,k}\,\grave{y}_k\,\partial_{\grave{x}_{2j-1}}\right)\\
&- 2\sum_{\substack{1\le j\le n\\ 1\le k\le 2n}}\left(\delta_{\mathbf{x},\mathbf{y}}\delta_{2j,k}\,\grave{e}_k\grave{e}_{2j-1} + \delta_{2j,k}\,\delta_{\mathbf{x},\mathbf{y}} - \delta_{2j,k}\,\grave{y}_k\,\partial_{\grave{x}_{2j}}\right),
\end{aligned}$$

and consequently,

$$
\begin{aligned}
\{\partial_{\mathbf{x}}, \mathbf{y}\} &= \sum_{j=1}^{2m} \left(\delta_{\mathbf{x},\mathbf{y}} + 2y_j \partial_{x_j}\right) + 2\sum_{j=1}^{n} \delta_{\mathbf{x},\mathbf{y}} (\grave{e}_{2j-1}\grave{e}_{2j} - \grave{e}_{2j}\grave{e}_{2j-1} - 2) \\
&\quad + 2\sum_{j=1}^{n} \grave{y}_{2j-1}\partial_{\grave{x}_{2j-1}} + \grave{y}_{2j}\partial_{\grave{x}_{2j}} \\
&= \delta_{\mathbf{x},\mathbf{y}}(2m-2n) + 2\left(\sum_{j=1}^{2m} y_j \partial_{x_j} + \sum_{j=1}^{2n} \grave{y}_j \partial_{\grave{x}_j}\right).
\end{aligned}
$$

Similarly, for the operator $\{\partial_{\mathbf{x}}, \mathbf{J}(\mathbf{y})\}$, we obtain

$$
\begin{aligned}
\{\partial_{\mathbf{x}}, \mathbf{J}(\mathbf{y})\} &= \left\{\partial_{\underline{\grave{x}}} - \partial_{\underline{x}}, \sum_{k=1}^{2m}(y_{m+k}e_k - y_k e_{m+k}) + \sum_{k=1}^{2n}(\grave{y}_{2k}\grave{e}_{2k-1} - \grave{y}_{2k-1}\grave{e}_{2k})\right\} \\
&= -\sum_{j,k=1}^{m} \left\{e_j\partial_{x_j}, e_k y_{m+k}\right\} - \left\{e_j\partial_{x_j}, e_{m+k}y_k\right\} \\
&\quad - \sum_{j,k=1}^{m} \left\{e_{m+j}\partial_{x_{m+j}}, e_k y_{m+k}\right\} - \left\{e_{m+j}\partial_{x_{m+j}}, e_{m+k}y_k\right\} \\
&\quad + 2\sum_{j,k=1}^{n} \left\{\grave{e}_{2j}\partial_{\grave{x}_{2j-1}}, \grave{e}_{2k-1}\grave{y}_{2k}\right\} - \left\{\grave{e}_{2j}\partial_{\grave{x}_{2j-1}}, \grave{e}_{2k}\grave{y}_{2k-1}\right\} \\
&\quad + 2\sum_{j,k=1}^{n} -\left\{\grave{e}_{2j-1}\partial_{\grave{x}_{2j}}, \grave{e}_{2k-1}\grave{y}_{2k}\right\} + \left\{\grave{e}_{2j-1}\partial_{\grave{x}_{2j}}, \grave{e}_{2k}\grave{y}_{2k-1}\right\},
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\{\partial_{\mathbf{x}}, \mathbf{J}(\mathbf{y})\} &= -\sum_{j,k=1}^{m} \left(-2\delta_{j,k}\, y_{m+k}\partial_{x_j} + \delta_{\mathbf{x},\mathbf{y}}\, \delta_{j,k}\, e_{m+k}e_j - \delta_{\mathbf{x},\mathbf{y}}\, \delta_{j,k}\, e_k e_{m+j} + 2\delta_{j,k}\, y_k \partial_{x_{m+j}}\right) \\
&\quad + 2\sum_{j,k=1}^{n} \left(\delta_{j,k}\, \grave{y}_{2k}\partial_{\grave{x}_{2j-1}} - \delta_{\mathbf{x},\mathbf{y}}\, \delta_{j,k}\, \grave{e}_{2k}\grave{e}_{2j} - \delta_{\mathbf{x},\mathbf{y}}\, \delta_{j,k}\, \grave{e}_{2k-1}\grave{e}_{2j-1} - \delta_{j,k}\, \grave{y}_{2k-1}\partial_{\grave{x}_{2j}}\right) \\
&= 2\sum_{j=1}^{m} \left(\delta_{\mathbf{x},\mathbf{y}} e_j e_{m+j} + y_{m+j}\partial_{x_j} - y_j \partial_{m+j}\right) \\
&\quad + 2\sum_{j=1}^{n} \Big(-\delta_{\mathbf{x},\mathbf{y}}(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2) + \grave{y}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{y}_{2j-1}\partial_{\grave{x}_{2j}}\Big) \\
&= 2\delta_{\mathbf{x},\mathbf{y}}\mathbf{B} + 2\left(\sum_{j=1}^{m}(y_{m+j}\partial_{x_j} - y_j\partial_{x_{m+j}}) + \sum_{j=1}^{n}(\grave{y}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{y}_{2j-1}\partial_{\grave{x}_{2j}})\right).
\end{aligned}
$$

Then, we have found the following expressions for the directional derivatives

$$D_{\mathbf{y},\mathbf{x}} = D_{\underline{y},\underline{x}} + D_{\underline{\grave{y}},\underline{\grave{x}}} = \sum_{j=1}^{2m} y_j \partial_{x_j} + \sum_{j=1}^{2n} \grave{y}_j \partial_{\grave{x}_j},$$
$$D_{\mathbf{J}(\mathbf{y}),\mathbf{x}} = D_{\mathbf{J}(\underline{y}),\underline{x}} + D_{\mathbf{J}(\underline{\grave{y}}),\underline{\grave{x}}} = \sum_{j=1}^{m}(y_{m+j}\partial_{x_j} - y_j \partial_{x_{m+j}}) + \sum_{j=1}^{n}(\grave{y}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{y}_{2j-1}\partial_{\grave{x}_{2j}}),$$

which lead to the Euler operator

$$\mathbb{E}_{\mathbf{x}} := D_{\mathbf{x},\mathbf{x}} = \sum_{j=1}^{2m} x_j \partial_{x_j} + \sum_{j=1}^{2n} \grave{x}_j \partial_{\grave{x}_j}.$$

As it is known, $\mathbb{E}_{\mathbf{x}}$ measures the degree of homogeneity in the supervector variable $\mathbf{x}$ of every element of the radial algebra embedded in $\mathcal{A}_{2m,2n}$. This situation can be generalized to the polynomial setting.

In the algebra $\mathbb{R}\mathcal{P} := \mathbb{R}[x_1, \ldots, x_{2m}] \otimes \mathfrak{G}(\grave{x}_1, \ldots, \grave{x}_{2n})$, we say that a polynomial $R_k(\mathbf{x}) = R_k(x_1, \ldots, x_{2m}, \grave{x}_1, \ldots, \grave{x}_{2n})$ is homogeneous of degree $k \in \mathbb{N}\cup\{0\}$ if, for every $\lambda \in \mathbb{R}\backslash\{0\}$, it holds that

$$R_k(\lambda \mathbf{x}) = \lambda^k R_k(\mathbf{x}).$$

The vector space of homogeneous polynomials of degree k in $\mathbb{R}\mathcal{P}$ is denoted by $\mathbb{R}\mathcal{P}_k$. A basis for $\mathbb{R}\mathcal{P}_k$ consist of elements of the form $x_1^{\alpha_1} \cdots x_{2m}^{\alpha_{2m}} \grave{x}_1^{\beta_1} \cdots \grave{x}_{2n}^{\beta_{2n}}$ where $\alpha_j \in \mathbb{N}$, $\beta_j \in \{0,1\}$ and with $\sum_{j=1}^{2m} \alpha_j + \sum_{j=1}^{2n} \beta_j = k$. It is moreover easily seen that $\mathbb{R}\mathcal{P}_k$ is a finite dimensional vector space with dimension

$$\dim \mathbb{R}\mathcal{P}_k = \sum_{j=0}^{\min(k,2n)} \binom{2n}{j}\binom{k-j+2m-1}{2m-1}.$$

It can be directly verified that $\mathbb{R}\mathcal{P}_k$ is an eigenspace of $\mathbb{E}_{\mathbf{x}}$ with eigenvalue k. The same conclusion holds for $\mathbb{R}\mathcal{P}_k \otimes \mathcal{C}_{2m,2n}$.

3.4 Hermitian setting in superspace

In the complexification $\mathbb{C}\mathcal{A}_{2m,2n}$ of $\mathcal{A}_{2m,2n}$ we define the involution $\cdot^\dagger$ as

$$(a + ib)^\dagger = \overline{a} - i\overline{b}, \qquad a, b \in \mathcal{A}_{2m,2n},$$

which is an extension to $\mathbb{C}\mathcal{A}_{2m,2n}$ of the Hermitian conjugation over the complexification of the radial algebra representation with complex structure $R(\mathbf{S} \cup \mathbf{J}(\mathbf{S}), \mathbf{B})$, see (2.26).

The corresponding representation on superspace of the Hermitian radial algebra introduced in Section 2.5.2 will be denoted by $R(\mathbf{S}_\mathbb{C}, \mathbf{S}_\mathbb{C}^\dagger, \mathbf{B})$. It is generated over the complex numbers by the sets of Hermitian vector variables

$$\mathbf{S}_{\mathbb{C}} = \left\{\mathbf{Z} = \frac{1}{2}\Big(\mathbf{x} + i\mathbf{J}(\mathbf{x})\Big) : \mathbf{x} \in \mathbf{S}\right\}, \qquad \mathbf{S}_{\mathbb{C}}^{\dagger} = \left\{\mathbf{Z}^{\dagger} = -\frac{1}{2}\Big(\mathbf{x} - i\mathbf{J}(\mathbf{x})\Big) : \mathbf{x} \in \mathbf{S}\right\},$$

and the element $\mathbf{B}$ defined in (3.24). The complex vector variables $\mathbf{Z}$, $\mathbf{Z}^{\dagger}$ in superspace can be written as

$$\begin{aligned} \mathbf{Z} = {} & \frac{1}{2}\Big(\mathbf{x} + i\mathbf{J}(\mathbf{x})\Big) = \frac{1}{2}\Big(\underline{x} + i\mathbf{J}(\underline{x})\Big) + \frac{1}{2}\Big(\underline{\grave{x}} + i\mathbf{J}(\underline{\grave{x}})\Big) =: \underline{Z} + \underline{\grave{Z}}, \\ \mathbf{Z}^{\dagger} = {} & -\frac{1}{2}\Big(\mathbf{x} - i\mathbf{J}(\mathbf{x})\Big) = -\frac{1}{2}\Big(\underline{x} - i\mathbf{J}(\underline{x})\Big) - \frac{1}{2}\Big(\underline{\grave{x}} - i\mathbf{J}(\underline{\grave{x}})\Big) =: \underline{Z}^{\dagger} + \underline{\grave{Z}}^{\dagger}, \end{aligned}$$

where the bosonic Hermitian vector variables $\underline{Z}$ and $\underline{Z}^{\dagger}$ are defined as in Remark 2.22, i.e.

$$\underline{Z} = \frac{1}{2}\Big(\underline{x} + i\mathbf{J}(\underline{x})\Big) = \frac{1}{2}\sum_{j=1}^{m}(x_j + ix_{m+j})(e_j - ie_{m+j}) = \sum_{j=1}^{m} z_j \mathfrak{f}_j, \tag{3.30}$$

$$\underline{Z}^{\dagger} = -\frac{1}{2}\Big(\underline{x} - i\mathbf{J}(\underline{x})\Big) = -\frac{1}{2}\sum_{j=1}^{m}(x_j - ix_{m+j})(e_j + ie_{m+j}) = \sum_{j=1}^{m} z_j^c \mathfrak{f}_j^{\dagger}, \tag{3.31}$$

while the fermionic Hermitian vector variables $\underline{\grave{Z}}$ and $\underline{\grave{Z}}^{\dagger}$ are given by

$$\underline{\grave{Z}} = \frac{1}{2}\Big(\underline{\grave{x}} + i\mathbf{J}(\underline{\grave{x}})\Big) = \frac{1}{2}\sum_{j=1}^{n}(\grave{x}_{2j-1} + i\grave{x}_{2j})(\grave{e}_{2j-1} - i\grave{e}_{2j}) = \sum_{j=1}^{n} \grave{z}_j \grave{\mathfrak{f}}_j, \tag{3.32}$$

$$\underline{\grave{Z}}^{\dagger} = -\frac{1}{2}\Big(\underline{\grave{x}} - i\mathbf{J}(\underline{\grave{x}})\Big) = -\frac{1}{2}\sum_{j=1}^{n}(\grave{x}_{2j-1} - i\grave{x}_{2j})(\grave{e}_{2j-1} + i\grave{e}_{2j}) = \sum_{j=1}^{n} \grave{z}_j^c \grave{\mathfrak{f}}_j^{\dagger}. \tag{3.33}$$

Together with the commuting complex variables $z_j = x_j + ix_{m+j}$ and $z_j^c = x_j - ix_{m+j}$, we consider in this representation the anti-commuting variables $\grave{z}_j = \grave{x}_{2j-1} + i\grave{x}_{2j}$ and $\grave{z}_j^c = \grave{x}_{2j-1} - i\grave{x}_{2j}$. Moreover, the Witt basis introduced in (2.27) gets extended to

$$\begin{cases} \mathfrak{f}_j = \frac{1}{2}(e_j - ie_{m+j}), \\ \mathfrak{f}_j^{\dagger} = -\frac{1}{2}(e_j + ie_{m+j}), \end{cases} \qquad \begin{cases} \grave{\mathfrak{f}}_j = \frac{1}{2}(\grave{e}_{2j-1} - i\grave{e}_{2j}), \\ \grave{\mathfrak{f}}_j^{\dagger} = -\frac{1}{2}(\grave{e}_{2j-1} + i\grave{e}_{2j}). \end{cases}$$

It is easily seen that these Witt basis elements generate the complexification of $\mathcal{C}_{2m,2n}$ and are subject to the following commutation rules

$$\begin{cases} \mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = 0, \\ \mathfrak{f}_j^{\dagger}\mathfrak{f}_k^{\dagger} + \mathfrak{f}_k^{\dagger}\mathfrak{f}_j^{\dagger} = 0, \\ \mathfrak{f}_j\mathfrak{f}_k^{\dagger} + \mathfrak{f}_k^{\dagger}\mathfrak{f}_j = \delta_{j,k}, \end{cases} \qquad \begin{cases} \grave{\mathfrak{f}}_j\grave{\mathfrak{f}}_k - \grave{\mathfrak{f}}_k\grave{\mathfrak{f}}_j = 0, \\ \grave{\mathfrak{f}}_j^{\dagger}\grave{\mathfrak{f}}_k^{\dagger} - \grave{\mathfrak{f}}_k^{\dagger}\grave{\mathfrak{f}}_j^{\dagger} = 0, \\ \grave{\mathfrak{f}}_j\grave{\mathfrak{f}}_k^{\dagger} - \grave{\mathfrak{f}}_k^{\dagger}\grave{\mathfrak{f}}_j = -\frac{i}{2}\delta_{j,k}, \end{cases}$$

$$\begin{cases} \mathfrak{f}_j\grave{\mathfrak{f}}_k + \grave{\mathfrak{f}}_k\mathfrak{f}_j = 0, \\ \mathfrak{f}_j\grave{\mathfrak{f}}_k^{\dagger} + \grave{\mathfrak{f}}_k^{\dagger}\mathfrak{f}_j = 0, \end{cases} \qquad \begin{cases} \mathfrak{f}_j^{\dagger}\grave{\mathfrak{f}}_k + \grave{\mathfrak{f}}_k\mathfrak{f}_j^{\dagger} = 0, \\ \mathfrak{f}_j^{\dagger}\grave{\mathfrak{f}}_k^{\dagger} + \grave{\mathfrak{f}}_k^{\dagger}\mathfrak{f}_j^{\dagger} = 0. \end{cases}$$

As mentioned in Section 2.5.2, the Hermitian radial algebra representation $R(\mathbf{S}_{\mathbb{C}}, \mathbf{S}_{\mathbb{C}}^{\dagger}, \mathbf{B})$ submits to the rules **(AH1*)-(AH3*)**. In particular, $\{\mathbf{Z}, \mathbf{U}^{\dagger}\}$ is a commuting object in the whole algebra $\mathbb{C}\mathcal{A}_{2m,2n}$ and has the form

$$\{\mathbf{Z}, \mathbf{U}^{\dagger}\} = \sum_{j=1}^{m} z_j u_j^c - \frac{i}{2} \sum_{j=1}^{n} \grave{z}_j \grave{u}_j^c. \tag{3.34}$$

The complex conjugation $\cdot^c$ acts on the complexification of $\mathcal{V}$ as the corresponding restriction of $\cdot^{\dagger}$, whence it is clear that

$$\{\mathbf{Z}, \mathbf{U}^{\dagger}\}^c = \sum_{j=1}^{m} z_j^c u_j + \frac{i}{2} \sum_{j=1}^{n} \grave{z}_j^c \grave{u}_j = \sum_{j=1}^{m} u_j z_j^c - \frac{i}{2} \sum_{j=1}^{n} \grave{u}_j \grave{z}_j^c = \{\mathbf{U}, \mathbf{Z}^{\dagger}\},$$

meaning that formula (3.34) can be used as a generalized Hermitian inner product.

We also introduce the left and right actions of the Hermitian vector derivatives in this setting using radial algebra notions. These are

$$\partial_{\mathbf{Z}} := \frac{1}{4}\left(\partial_{\mathbf{x}} - i\partial_{\mathbf{J}(\mathbf{x})}\right), \qquad \partial_{\mathbf{Z}^{\dagger}} := -\frac{1}{4}\left(\partial_{\mathbf{x}} + i\partial_{\mathbf{J}(\mathbf{x})}\right), \tag{3.35}$$

which are valid for both left and right actions of the operators $\partial_{\mathbf{Z}}$, $\partial_{\mathbf{Z}^{\dagger}}$. These actions can be re-written as

$$\begin{aligned}
\partial_{\mathbf{Z}} \cdot &= \frac{1}{4}\big(\partial_{\mathbf{x}} \cdot - i\partial_{\mathbf{J}(\mathbf{x})} \cdot\big) = \frac{1}{4}\big(\partial_{\underline{\grave{x}}} \cdot - i\partial_{\mathbf{J}(\underline{\grave{x}})} \cdot\big) - \frac{1}{4}\big(\partial_{\underline{x}} \cdot - i\partial_{\mathbf{J}(\underline{x})} \cdot\big) = \partial_{\underline{\grave{Z}}} \cdot + \partial_{\underline{Z}} \cdot,\\
\partial_{\mathbf{Z}^{\dagger}} \cdot &= -\frac{1}{4}\big(\partial_{\mathbf{x}} \cdot + i\partial_{\mathbf{J}(\mathbf{x})} \cdot\big) = -\frac{1}{4}\big(\partial_{\underline{\grave{x}}} \cdot + i\partial_{\mathbf{J}(\underline{\grave{x}})} \cdot\big) + \frac{1}{4}\big(\partial_{\underline{x}} \cdot + i\partial_{\mathbf{J}(\underline{x})} \cdot\big) = \partial_{\underline{\grave{Z}}^{\dagger}} \cdot + \partial_{\underline{Z}^{\dagger}} \cdot,\\
\cdot \partial_{\mathbf{Z}} &= \frac{1}{4}\big(\cdot\, \partial_{\mathbf{x}} - i \,\cdot\, \partial_{\mathbf{J}(\mathbf{x})}\big) = \frac{1}{4}\big(-\cdot \partial_{\underline{\grave{x}}} + i \,\cdot\, \partial_{\mathbf{J}(\underline{\grave{x}})}\big) - \frac{1}{4}\big(\cdot\, \partial_{\underline{x}} - i \,\cdot\, \partial_{\mathbf{J}(\underline{x})}\big) = -\cdot\, \partial_{\underline{\grave{Z}}} + \cdot \partial_{\underline{Z}},\\
\cdot \partial_{\mathbf{Z}^{\dagger}} &= -\frac{1}{4}\big(\cdot\, \partial_{\mathbf{x}} + i \,\cdot\, \partial_{\mathbf{J}(\mathbf{x})}\big) = \frac{1}{4}\big(\cdot\, \partial_{\underline{\grave{x}}} + i \,\cdot\, \partial_{\mathbf{J}(\underline{\grave{x}})}\big) + \frac{1}{4}\big(\cdot\, \partial_{\underline{x}} + i \,\cdot\, \partial_{\mathbf{J}(\underline{x})}\big) = -\cdot\, \partial_{\underline{\grave{Z}}^{\dagger}} + \cdot \partial_{\underline{Z}^{\dagger}}.
\end{aligned}$$

The operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^{\dagger}}$ are the bosonic Hermitian Dirac operators defined in Remark 2.23, i.e.

$$\partial_{\underline{Z}} = -\frac{1}{4}\big(\partial_{\underline{x}} - i\partial_{\mathbf{J}(\underline{x})}\big) = \sum_{j=1}^{m} \mathfrak{f}_j^{\dagger}\, \partial_{z_j}, \qquad \partial_{\underline{Z}^{\dagger}} = \frac{1}{4}\big(\partial_{\underline{x}} + i\partial_{\mathbf{J}(\underline{x})}\big) = \sum_{j=1}^{m} \mathfrak{f}_j\, \partial_{z_j^c},$$

while the fermionic Hermitian Dirac operators $\partial_{\underline{\grave{Z}}}$ and $\partial_{\underline{\grave{Z}}^{\dagger}}$ are given by

$$\partial_{\underline{\grave{Z}}} = \frac{1}{4}\big(\partial_{\underline{\grave{x}}} - i\partial_{\mathbf{J}(\underline{\grave{x}})}\big) = 2i \sum_{j=1}^{n} \grave{\mathfrak{f}}_j^{\dagger}\, \partial_{\grave{z}_j}, \qquad \partial_{\underline{\grave{Z}}^{\dagger}} = -\frac{1}{4}\big(\partial_{\underline{\grave{x}}} + i\partial_{\mathbf{J}(\underline{\grave{x}})}\big) = -2i \sum_{j=1}^{n} \grave{\mathfrak{f}}_j\, \partial_{\grave{z}_j^c}.$$

where

$$\begin{aligned}
\partial_{z_j} &= \frac{1}{2}(\partial_{x_j} - i\partial_{x_{m+j}}), & \partial_{z_j^c} &= \frac{1}{2}(\partial_{x_j} + i\partial_{x_{m+j}}),\\
\partial_{\grave{z}_j} &= \frac{1}{2}(\partial_{\grave{x}_{2j-1}} - i\partial_{\grave{x}_{2j}}), & \partial_{\grave{z}_j^c} &= \frac{1}{2}(\partial_{\grave{x}_{2j-1}} + i\partial_{\grave{x}_{2j}}),
\end{aligned}$$

are the classical Cauchy-Riemann operators and their conjugates with respect to the variables z_j and $\grave{z}_j$.

As it was shown in Section 2.5.2, the operators $\partial_{\mathbf{Z}}$ and $\partial_{\mathbf{Z}^\dagger}$ satisfy the relations **(DH1*)-(DH4*)**. These relations can also be checked using the above explicit expressions, e.g.:

$$\partial_{\mathbf{Z}}[\mathbf{Z}] = \partial_{\underline{\grave{Z}}}[\underline{\grave{Z}}] + \partial_{\underline{Z}}[\underline{Z}] = 2i\sum_{j=1}^{n} \grave{\mathfrak{f}}_j^{\dagger}\, \grave{\mathfrak{f}}_j + \sum_{j=1}^{m} \mathfrak{f}_j^{\dagger}\, \mathfrak{f}_j = \frac{1}{2}\left[(m-n) + i\mathbf{B}\right].$$

We also obtain explicit formulae for the complex directional derivatives in superspace following the radial algebra approach. In fact, for every pair of Hermitian vector variables $\mathbf{Z} = \frac{1}{2}\left(\mathbf{x} + i\mathbf{J}(\mathbf{x})\right)$ and $\mathbf{U} = \frac{1}{2}\left(\mathbf{y} + i\mathbf{J}(\mathbf{y})\right)$ we have (see (2.28))

$$\begin{aligned} D_{\mathbf{U},\mathbf{Z}} &= \frac{1}{2}\left(D_{\mathbf{y},\mathbf{x}} + iD_{\mathbf{J}(\mathbf{y}),\mathbf{x}}\right) = D_{\underline{U},\underline{Z}} + D_{\underline{\grave{U}},\underline{\grave{Z}}},\\ D^{\dagger}_{\mathbf{U},\mathbf{Z}} &= \frac{1}{2}\left(D_{\mathbf{y},\mathbf{x}} - iD_{\mathbf{J}(\mathbf{y}),\mathbf{x}}\right) = D^{\dagger}_{\underline{U},\underline{Z}} + D^{\dagger}_{\underline{\grave{U}},\underline{\grave{Z}}}. \end{aligned}$$

where the bosonic and fermionic directional derivatives and their Hermitian conjugates are given by

$$\begin{cases} D_{\underline{U},\underline{Z}} = \frac{1}{2}\left(D_{\underline{y},\underline{x}} + iD_{\mathbf{J}(\underline{y}),\underline{x}}\right) = \sum_{j=1}^{m} u_j\partial_{z_j}, \\ D_{\underline{\grave{U}},\underline{\grave{Z}}} = \frac{1}{2}\left(D_{\underline{\grave{y}},\underline{\grave{x}}} + iD_{\mathbf{J}(\underline{\grave{y}}),\underline{\grave{x}}}\right) = \sum_{j=1}^{n} \grave{u}_j\partial_{\grave{z}_j}, \end{cases}$$

$$\begin{cases} D^{\dagger}_{\underline{U},\underline{Z}} = \frac{1}{2}\left(D_{\underline{y},\underline{x}} - iD_{\mathbf{J}(\underline{y}),\underline{x}}\right) = \sum_{j=1}^{m} u_j^c\, \partial_{z_j^c}, \\ D^{\dagger}_{\underline{\grave{U}},\underline{\grave{Z}}} = \frac{1}{2}\left(D_{\underline{\grave{y}},\underline{\grave{x}}} - iD_{\mathbf{J}(\underline{\grave{y}}),\underline{\grave{x}}}\right) = \sum_{j=1}^{n} \grave{u}_j^c\, \partial_{\grave{z}_j^c}. \end{cases}$$

By the formulae (2.30) obtained for the Hermitian radial algebra we have that

$$\{\partial_{\mathbf{Z}}, \mathbf{U}\} = D_{\mathbf{U},\mathbf{Z}} + \frac{1}{2}\delta_{\mathbf{Z},\mathbf{U}}\big((m-n) + i\mathbf{B}\big), \quad \{\partial_{\mathbf{Z}^\dagger}, \mathbf{U}^\dagger\} = D^{\dagger}_{\mathbf{U},\mathbf{Z}} + \frac{1}{2}\delta_{\mathbf{Z},\mathbf{U}}\big((m-n) - i\mathbf{B}\big),$$

hold in the representation $R(\mathbf{S}_{\mathbb{C}}, \mathbf{S}_{\mathbb{C}}^{\dagger}, \mathbf{B})$. Moreover, it can be easily checked that the above relations remain valid in $\mathbb{C}\mathcal{A}_{2m,2n}$.

In the case where $\mathbf{Z} = \mathbf{U}$ we obtain the Hermitian Euler operators

$$\mathbb{E}_{\mathbf{Z}} = D_{\mathbf{Z},\mathbf{Z}} = \sum_{j=1}^{m} z_j\partial_{z_j} + \sum_{j=1}^{n} \grave{z}_j\partial_{\grave{z}_j} \quad \text{and} \quad \mathbb{E}_{\mathbf{Z}^\dagger} = D^{\dagger}_{\mathbf{Z},\mathbf{Z}} = \sum_{j=1}^{m} z_j^c\, \partial_{z_j}^c + \sum_{j=1}^{n} \grave{z}_j^c\, \partial_{\grave{z}_j^c},$$

which split the Euler operator $\mathbb{E}_{\mathbf{x}}$ as $\mathbb{E}_{\mathbf{x}} = \mathbb{E}_{\mathbf{Z}} + \mathbb{E}_{\mathbf{Z}^\dagger}$.

It follows from Lemma 2.13 that $\mathbb{E}_{\mathbf{Z}}$ and $\mathbb{E}_{\mathbf{Z}}^{\dagger}$ measure the degree of homogeneity of the vector variables $\mathbf{Z}$ and $\mathbf{Z}^\dagger$, respectively, on every element of the Hermitian radial algebra representation $R(\mathbf{S}_{\mathbb{C}}, \mathbf{S}_{\mathbb{C}}^{\dagger}, \mathbf{B})$. As expected, also this property generalizes to

$$\mathbb{C}\mathcal{P} = \mathbb{C}[x_1, \ldots, x_{2m}] \otimes \mathfrak{G}(\grave{x}_1, \ldots, \grave{x}_{2n}).$$

To this end, we refine the notion of a k-homogeneous polynomial to a (bi-)homogeneous polynomial of degree (p, q) with $p + q = k$. We first note that every polynomial in the variables $(x_1, \ldots, x_{2m}, \grave{x}_1, \ldots, \grave{x}_{2n})$ may be written as a polynomial in the variables $(z_1, \ldots, z_m, z_1^c, \ldots, z_m^c, \grave{z}_1, \ldots, \grave{z}_n, \grave{z}_1^c, \ldots, \grave{z}_n^c)$. Hence, a polynomial

$$R_{p,q}(\mathbf{Z}, \mathbf{Z}^\dagger) = R_{p,q}(z_1, \ldots, z_m, z_1^c, \ldots, z_m^c, \grave{z}_1, \ldots, \grave{z}_n, \grave{z}_1^c, \ldots, \grave{z}_n^c) \in \mathbb{C}\mathcal{P}$$

is said to be homogeneous of degree (p, q), $p, q \in \mathbb{N}^2 \cup \{0\}$, if for all $\lambda \in \mathbb{C} \setminus \{0\}$ it holds that

$$R_{p,q}(\lambda \mathbf{Z}, \lambda^c \mathbf{Z}^\dagger) = \lambda^p \, (\lambda^c)^q \, R_{p,q}(\mathbf{Z}, \mathbf{Z}^\dagger).$$

The space of all homogenous polynomials of degree (p, q) in $\mathbb{C}\mathcal{P}$ is denoted by $\mathbb{C}\mathcal{P}_{p,q}$. A basis for $\mathbb{C}\mathcal{P}_{p,q}$ consists of the elements

$$z_1^{\alpha_1} \cdots z_m^{\alpha_m} \, \grave{z}_1^{\beta_1} \cdots \grave{z}_n^{\beta_n} \, (z_1^c)^{\gamma_1} \cdots (z_m^c)^{\gamma_m} \, (\grave{z}_1^c)^{\delta_1} \cdots (\grave{z}_n^c)^{\delta_n}$$

where $\alpha_j, \gamma_j \in \mathbb{N}$, $\beta_j, \delta_j \in \{0, 1\}$ and with $\sum_{j=1}^m \alpha_j + \sum_{j=1}^n \beta_j = p$, $\sum_{j=1}^m \gamma_j + \sum_{j=1}^n \delta_j = q$. It easily follows that $\mathbb{C}\mathcal{P}_{p,q}$ is a finite dimensional complex vector space with dimension

$$\dim \mathbb{C}\mathcal{P}_{p,q} = \left[\sum_{j=0}^{\min(n,p)} \binom{n}{j} \binom{p - j + m - 1}{m - 1} \right] \left[\sum_{j=0}^{\min(n,q)} \binom{n}{j} \binom{q - j + m - 1}{m - 1} \right].$$

In addition the following decomposition holds:

$$\mathbb{C}\mathcal{P}_k = \bigoplus_{j=0}^{k} \mathbb{C}\mathcal{P}_{j,k-j}.$$

In the previous section, it was mentioned that $\mathbb{C}\mathcal{P}_k$ is the eigenspace of $\mathbb{E}_{\mathbf{x}}$ corresponding to the eigenvalue k. A similar property can be proven in the Hermitian context.

Lemma 3.3. *If $R_{p,q}(\mathbf{Z}, \mathbf{Z}^\dagger)$ is a homogeneous polynomial of degree (p, q) then*

$$\mathbb{E}_{\mathbf{Z}}[R_{p,q}] = pR_{p,q}, \quad \text{and} \quad \mathbb{E}_{\mathbf{Z}^\dagger}[R_{p,q}] = qR_{p,q}.$$

Proof.

Differentiating with respect to the complex variable λ and applying the chain rule (see Theorem 3.2), we have on the one hand

$$\begin{aligned} \partial_\lambda R_{p,q}(\lambda \mathbf{Z}, \lambda^c \mathbf{Z}^\dagger) &= \sum_{j=1}^m z_j \partial_{z_j} R_{p,q}(\lambda \mathbf{Z}, \lambda^c \mathbf{Z}^\dagger) + \sum_{j=1}^n \grave{z}_j \partial_{\grave{z}_j} R_{p,q}(\lambda \mathbf{Z}, \lambda^c \mathbf{Z}^\dagger) \\ &= \mathbb{E}_{\mathbf{Z}} R_{p,q}(\lambda \mathbf{Z}, \lambda^c \mathbf{Z}^\dagger), \end{aligned}$$

and on the other hand,

$$\partial_\lambda R_{p,q}(\lambda \mathbf{Z}, \lambda^c \mathbf{Z}^\dagger) = \partial_\lambda \left[\lambda^p \, (\lambda^c)^q \, R_{p,q}(\mathbf{Z}, \mathbf{Z}^\dagger) \right] = p\lambda^{p-1} \, (\lambda^c)^q \, R_{p,q}(\mathbf{Z}, \mathbf{Z}^\dagger),$$

whence,

$$\mathbb{E}_{\mathbf{Z}} R_{p,q}(\lambda \mathbf{Z}, \lambda^c \mathbf{Z}^\dagger) = p\lambda^{p-1} \, (\lambda^c)^q \, R_{p,q}(\mathbf{Z}, \mathbf{Z}^\dagger).$$

In particular, for $\lambda = 1$ we have $\mathbb{E}_{\mathbf{Z}}[R_{p,q}] = pR_{p,q}$. The proof of the other relation is similar. □

3.5 Concluding remarks

We have carefully introduced the defining objects for Hermitian Clifford analysis in superspace through the rules determined by the Hermitian radial algebra. These rules provide a straightforward way of defining a suitable complex structure in this setting, giving rise to the introduction of all basic elements in the Hermitian Clifford calculus. This complex structure can be seen either as a special automorphism on $\mathcal{A}_{2m,2n}$ or as the action of the special bivector $\mathbf{B}$ through its commutator with vector variables. This action of $\mathbf{B}$ allows to interpret the complex structure as a special element of the set of superrotations. In Chapters 4 and 5 this theory will be further developed. This will include a deep study of the group realization of rotations in superspace and the invariance of the super Dirac operators under the action of these groups.

In Table 3.1, we summarize the main aspects of the Clifford-polynomial representation and the representation in superspace of the radial algebra $R(S)$.

Radial algebra setting	Clifford-polynomial representation	Representation in superspace
$R(S)$	$R(\underline{S}) \subset \mathcal{A}_{m,0}$	$R(\mathbf{S}) \subset \mathcal{A}_{m,2n}$
x	$\underline{x} = \sum_{j=1}^{m} e_j x_j$	$\mathbf{x} = \underline{x} + \underline{\grave{x}} = \sum_{j=1}^{m} x_j e_j + \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j$
∂_x	$-\partial_{\underline{x}} = -\sum_{j=1}^{m} e_j \partial_{x_j}$	$\partial_{\mathbf{x}} = \partial_{\underline{\grave{x}}} - \partial_{\underline{x}} = 2\sum_{j=1}^{n} \left(\grave{e}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}}\right) - \sum_{j=1}^{m} e_j \partial_{x_j}$
$B = \{\partial_x, x\|\}$	$-\sum_{j=1}^{m} e_j e_j\|$	$-\sum_{j=1}^{m} e_j e_j\| + 2\sum_{j=1}^{n} (\grave{e}_{2j}\grave{e}_{2j-1}\| - \grave{e}_{2j-1}\grave{e}_{2j}\|)$
$R(S \cup J(S), \mathcal{B})$	$R(\underline{S} \cup \underline{J}(\underline{S}), \underline{\mathcal{B}}) \subset \mathcal{A}_{2m,0}$	$R(\mathbf{S} \cup \mathbf{J}(\mathbf{S}), \mathbf{B}) \subset \mathcal{A}_{2m,2n}$
$J(x) = -\frac{1}{2}[\mathcal{B}, x]$	$\underline{J}(\underline{x}) = -\frac{1}{2}[\underline{\mathcal{B}}, \underline{x}]$ (Remark 2.17)	$\mathbf{J}(\mathbf{x}) = -\frac{1}{2}[\mathbf{B}, \mathbf{x}]$ (see (3.23))
$\mathcal{B}$	$\underline{\mathcal{B}} = \sum_{j=1}^{m} e_j e_{m+j}$	$\mathbf{B} = \sum_{j=1}^{m} e_j e_{m+j} - \sum_{j=1}^{2n} \grave{e}_j{}^2$
$\partial_{J(x)} = J(\partial_x)$	$-\partial_{\underline{J}(\underline{x})} = \underline{J}(-\partial_{\underline{x}})$ (Remark 2.19)	$\partial_{\mathbf{J}(\mathbf{x})} = \mathbf{J}(\partial_{\mathbf{x}}) = \partial_{\mathbf{J}(\underline{\grave{x}})} - \partial_{\mathbf{J}(\underline{x})}$ (see (3.26))
$R(S_{\mathbb{C}}, S_{\mathbb{C}}^{\dagger}, \mathcal{B})$	$R(\underline{S}_{\mathbb{C}}, \underline{S}_{\mathbb{C}}^{\dagger}, \underline{\mathcal{B}}) \subset \mathbb{C}\mathcal{A}_{2m,0}$	$R(\mathbf{S}_{\mathbb{C}}, \mathbf{S}_{\mathbb{C}}^{\dagger}, \mathbf{B}) \subset \mathbb{C}\mathcal{A}_{2m,2n}$
$Z = \frac{1}{2}\big(x + iJ(x)\big)$	$\underline{Z} = \sum_{j=1}^{m} z_j \mathfrak{f}_j$	$\mathbf{Z} = \underline{Z} + \underline{\grave{Z}} = \sum_{j=1}^{m} z_j \mathfrak{f}_j + \sum_{j=1}^{n} \grave{z}_j \grave{\mathfrak{f}}_j$
$Z^{\dagger} = -\frac{1}{2}\big(x - iJ(x)\big)$	$\underline{Z}^{\dagger} = \sum_{j=1}^{m} z_j^c \mathfrak{f}_j^{\dagger}$	$\mathbf{Z}^{\dagger} = \underline{Z}^{\dagger} + \underline{\grave{Z}}^{\dagger} = \sum_{j=1}^{m} z_j^c \mathfrak{f}_j^{\dagger} + \sum_{j=1}^{n} \grave{z}_j^c \grave{\mathfrak{f}}_j{}^{\dagger}$
$\partial_Z = \frac{1}{4}\left(\partial_x - i\partial_{J(x)}\right)$	$\partial_{\underline{Z}} = \sum_{j=1}^{m} \mathfrak{f}_j^{\dagger}\partial_{z_j}$	$\partial_{\mathbf{Z}} = \partial_{\underline{\grave{Z}}} + \partial_{\underline{Z}} = 2i\sum_{j=1}^{n} \grave{\mathfrak{f}}_j{}^{\dagger}\,\partial_{\grave{z}_j} + \sum_{j=1}^{m} \mathfrak{f}_j^{\dagger}\partial_{z_j}$
$\partial_{Z^{\dagger}} := -\frac{1}{4}\left(\partial_x + i\partial_{J(x)}\right)$	$\partial_{\underline{Z}^{\dagger}} = \sum_{j=1}^{m} \mathfrak{f}_j \partial_{z_j^c}$	$\partial_{\mathbf{Z}^{\dagger}} = \partial_{\underline{\grave{Z}}^{\dagger}} + \partial_{\underline{Z}^{\dagger}} = -2i\sum_{j=1}^{n} \grave{\mathfrak{f}}_j \partial_{\grave{z}_j^c} + \sum_{j=1}^{m} \mathfrak{f}_j \partial_{z_j^c}$

Table 3.1: Clifford-polynomial representation and the superspace representation of the radial algebra $R(S)$.

4

The Spin group in superspace

The notion of inner product in the radial algebra $R(S)$ can be abstractly defined as

$$\langle x, y\rangle = -x \cdot y = -\frac{1}{2}\{x, y\}$$

for $x, y \in S$, see (2.3). In the Clifford-polynomial representation this formula clearly coincides with the Euclidean inner product in $\mathbb{R}^m$. The most important invariance group in this case is the set of rotations $\mathrm{SO}(m)$ which is doubly covered by the spin group

$$\mathrm{Spin}(m) := \left\{ \prod_{j=1}^{2k} \underline{w}_j : k \in \mathbb{N}, \underline{w}_j \in \mathbb{S}^{m-1} \right\},$$

where $\mathbb{S}^{m-1} = \{\underline{w} \in \mathbb{R}^m : \underline{w}^2 = -1\}$ denotes the unit sphere in $\mathbb{R}^m$. The relation between $\mathrm{Spin}(m)$ and $\mathrm{SO}(m)$ is easily seen through the Lie group representation $h : \mathrm{Spin}(m) \to \mathrm{SO}(m)$

$$h(s)[\underline{x}] = s\underline{x}\overline{s}, \qquad s \in \mathrm{Spin}(m),\ \underline{x} \in \mathbb{R}^m,$$

which describes the action of every element of $\mathrm{SO}(m)$ in terms of Clifford multiplication. Such a description of the spin group follows from the Cartan-Dieudonné theorem which states that every orthogonal transformation in an m-dimensional symmetric bilinear space can be written as the composition of at most m reflections. Basic references for this setting are [16, 44, 48].

In this chapter we study the similar situation in the radial algebra representation $R(\mathbf{S}) \subset \mathcal{A}_{m,2n}$ in superspace, where the Cartan-Dieudonné theorem is no longer valid. The main goal is to properly define the spin group in superspace as a set of elements describing every super-rotation through Clifford multiplication. To that end, we consider linear actions on supervector variables using both commuting and anti-commuting coefficients in a Grassmann algebra $\mathfrak{G}(f_1, \ldots, f_N)$. This makes it possible to study the group of supermatrices leaving the inner product invariant and to define in a proper way the spin group in superspace. It is worth noticing that the superstructures are absorbed by the Grassmann algebras leading to classical Lie groups and Lie algebras instead of supergroups or superalgebras.

We start with some preliminaries on Grassmann algebras, Grassmann envelopes and supermatrices in Section 4.1. In particular, we carefully recall the notion of an exponential map for Grassmann numbers and supermatrices as elements of finite dimensional associative algebras. In Section 4.2 we further develop the Clifford setting in superspace by introducing the Lie algebra of superbivectors. An extension of this algebra plays an important rôle in the description of the super spin group. The use of the exponential map in such an extension (which takes us out of the radial algebra) necessitates the introduction of the corresponding tensor algebra. Section 4.3 is devoted to the study of the invariance of the "inner product" in superspace. There, we study several groups of supermatrices and in particular, the group of superrotations SO_0 and its Lie algebra $\mathfrak{so}_0$, which combine both orthogonal and symplectic structures. We prove that every superrotation can be uniquely decomposed as the product of three exponentials acting on some special subspaces of $\mathfrak{so}_0$. Finally, in Section 4.4 we study the problem of defining the spin group in this setting and its differences with the classical case. We show that the compositions of even numbers of vector reflections do not suffice to fully describe SO_0 since they only show an orthogonal structure and don't include the symplectic part of SO_0. Next we propose an alternative, by defining the spin group through the exponential of extended superbivectors and we show that they indeed cover the whole set of superrotations. In particular, we explicitly describe a subset Ξ which is a double covering of SO_0 and contains in particular every fractional Fourier transform.

4.1 Linear algebra in $\mathbb{K}^{p,q}(\mathfrak{G}_N)$

In this section we provide some preliminaries on the Grassmann algebras of coefficients, Grassmann envelopes and the algebra of supermatrices. The distinction between real and complex Grassmann coefficients will be necessary throughout the entire study of linear actions on supervector variables. For that reason, the notation $\mathbb{K}\mathfrak{G}_N$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) will be used for the Grassmann algebra of coefficients in Chapters 4 and 5.

4.1.1 Grassmann algebras and Grassmann envelopes

The Grassmann algebra $\mathbb{K}\mathfrak{G}_N = \mathbb{K}\mathfrak{G}_N^{(ev)} \oplus \mathbb{K}\mathfrak{G}_N^{(odd)}$ was introduced as a Banach commutative superalgebra in Example 3.2. Following the general form (3.3) of the elements in

$\mathbb{K}\mathfrak{G}_N$, we define the space of homogeneous elements of degree k by

$$\mathbb{K}\mathfrak{G}_N^{(k)} = \operatorname{span}_{\mathbb{K}}\{f_A : \; |A| = k\},$$

where in particular $\mathbb{K}\mathfrak{G}_N^{(k)} = \{0\}$ for $k > N$. It then easily follows that

$$\mathbb{K}\mathfrak{G}_N = \bigoplus_{k=0}^{N} \mathbb{K}\mathfrak{G}_N^{(k)}, \qquad \mathbb{K}\mathfrak{G}_N^{(k)}\mathbb{K}\mathfrak{G}_N^{(\ell)} \subset \mathbb{K}\mathfrak{G}_N^{(k+\ell)}.$$

In addition, the even and odd subspaces can be written as

$$\mathbb{K}\mathfrak{G}_N^{(ev)} = \bigoplus_{k\geq 0} \mathbb{K}\mathfrak{G}_N^{(2k)}, \qquad \mathbb{K}\mathfrak{G}_N^{(odd)} = \bigoplus_{k\geq 0} \mathbb{K}\mathfrak{G}_N^{(2k+1)}.$$

The projection of $\mathbb{K}\mathfrak{G}_N$ on its k-homogeneous part is denoted by $[\cdot]_k : \mathbb{K}\mathfrak{G}_N \to \mathbb{K}\mathfrak{G}_N^{(k)}$, i.e. $[a]_k = \sum_{|A|=k} a_A f_A$. As in Example 3.2, we denote the body of $a \in \mathbb{K}\mathfrak{G}_N$ by $[a]_0 = a_\emptyset =: a_0$ and its nilpotent part by $\mathbf{a} \in \mathbb{K}\mathfrak{G}_N^+ := \bigoplus_{k=1}^N \mathbb{K}\mathfrak{G}_N^{(k)}$. It is easily seen that the projection $[\cdot]_0 : \mathbb{K}\mathfrak{G}_N \to \mathbb{K}$ is an algebra homomorphism, i.e.

$$[ab]_0 = a_0 b_0, \qquad a, b \in \mathbb{K}\mathfrak{G}_N.$$

Lemma 4.1. *Let $a \in \mathbb{K}\mathfrak{G}_N$ such that $a^2 \in \mathbb{K} \setminus \{0\}$. Then $a \in \mathbb{K}$.*

Proof.

Let us write

$$a = \sum_{j=0}^{N} [a]_j,$$

with $[a]_j \in \mathbb{K}\mathfrak{G}_N^{(j)}$. If $a \notin \mathbb{K}$, let us consider $k \in \{1, \ldots, N\}$ to be the smallest integer such that $[a]_k \neq 0$. Then,

$$a^2 = a_0^2 + 2a_0\,[a]_k + b \in \mathbb{K},$$

where $b \in \bigoplus_{j>k} \mathbb{K}\mathfrak{G}_N^{(j)}$. This implies that $[a]_k = 0$, which is a contradiction. Then, $a \in \mathbb{K}$. □

The exponential of $a \in \mathbb{K}\mathfrak{G}_N$, denoted by e^a or $\exp(a)$, is defined by the power series

$$e^a = \sum_{j=0}^{\infty} \frac{a^j}{j!}. \tag{4.1}$$

Proposition 4.1. *The series (4.1) converges for every $a \in \mathbb{K}\mathfrak{G}_N$ and e^a is a continuous function in $\mathbb{K}\mathfrak{G}_N$.*

Proof.

We recall that $\mathbb{K}\mathfrak{G}_N$ is a Banach space with the norm $\|\cdot\|_{\mathfrak{G}}:\mathbb{K}\mathfrak{G}_N\rightarrow\mathbb{R}$ defined by

$$\|a\|_{\mathfrak{G}}=\sum_{A\subset\{1,\ldots,N\}}|a_A|, \qquad a\in\mathbb{K}\mathfrak{G}_N.$$

In particular, this norm satisfies the inequality

$$\|ab\|_{\mathfrak{G}}\leq\|a\|_{\mathfrak{G}}\,\|b\|_{\mathfrak{G}}, \qquad a,b\in\mathbb{K}\mathfrak{G}_N.$$

Then, it follows that

$$\sum_{j=0}^{\infty}\frac{\|a^j\|_{\mathfrak{G}}}{j!}\leq\sum_{j=0}^{\infty}\frac{\|a\|_{\mathfrak{G}}^j}{j!}=e^{\|a\|_{\mathfrak{G}}}$$

whence (4.1) (absolutely) converges in $\mathbb{K}\mathfrak{G}_N$. Now consider the ball $B_R:=\{a\in\mathbb{K}\mathfrak{G}_N:\|a\|_{\mathfrak{G}}\leq R\}$ for some $R>0$, where it holds that

$$\frac{\|a^j\|_{\mathfrak{G}}}{j!}\leq\frac{\|a\|_{\mathfrak{G}}^j}{j!}\leq\frac{R^j}{j!}.$$

Since the series $\sum_{j=0}^{\infty}\frac{R^j}{j!}$ converges, we have by the Weierstrass M-criterium that $\sum_{j=0}^{\infty}\frac{a^j}{j!}$ uniformly converges in B_R and e^a thus is continuous in B_R. Then, e^a is continuous in $\mathbb{K}\mathfrak{G}_N$. □

Now consider a $\mathbb{Z}_2$-graded vector space $\mathbb{K}^{p,q}=\mathbb{K}^{p,0}\oplus\mathbb{K}^{0,q}$ with standard homogeneous basis $e_1,\ldots,e_p,\grave{e}_1,\ldots,\grave{e}_q$, as introduced in Example 3.1. In [7, p. 91], the *Grassmann envelope* $\mathbb{K}^{p,q}(\mathfrak{G}_N)$ was defined as the set of Grassmann supervectors[1]

$$\mathbf{w}=\underline{w}+\underline{\grave{w}}=\sum_{j=1}^{p}w_je_j+\sum_{j=1}^{q}\grave{w}_j\grave{e}_j, \quad \text{where } w_j\in\mathbb{K}\mathfrak{G}_N^{(ev)}, \quad \grave{w}_j\in\mathbb{K}\mathfrak{G}_N^{(odd)}. \tag{4.2}$$

Remark 4.1. *The Grassmann envelope of a general graded $\mathbb{K}$-vector space $V=V_{\overline{0}}\oplus V_{\overline{1}}$ is similarly defined as*

$$V(\mathfrak{G}_N)=\left(\mathbb{K}\mathfrak{G}_N^{(ev)}\otimes V_{\overline{0}}\right)\oplus\left(\mathbb{K}\mathfrak{G}_N^{(odd)}\otimes V_{\overline{1}}\right).$$

The set $\mathbb{K}^{p,q}(\mathfrak{G}_N)$ is a $\mathbb{K}$-vector space of dimension $2^{N-1}(p+q)$, inheriting the $\mathbb{Z}_2$-grading of $\mathbb{K}^{p,q}$, i.e.

$$\mathbb{K}^{p,q}(\mathfrak{G}_N)=\mathbb{K}^{p,0}(\mathfrak{G}_N)\oplus\mathbb{K}^{0,q}(\mathfrak{G}_N),$$

where $\mathbb{K}^{p,0}(\mathfrak{G}_N)$ denotes the subspace of vectors of the form (4.2) with $\grave{w}_j=0$, and $\mathbb{K}^{0,q}(\mathfrak{G}_N)$ denotes the subspace of vectors of the form (4.2) with $w_j=0$. The subspaces $\mathbb{K}^{p,0}(\mathfrak{G}_N)$ and $\mathbb{K}^{0,q}(\mathfrak{G}_N)$ are called the Grassmann envelopes of $\mathbb{K}^{p,0}$ and $\mathbb{K}^{0,q}$, respectively.

[1]Observe that $\mathbb{K}^{p,q}(\mathfrak{G}_N)$ is the superspace of dimension (p,q) over $\mathbb{K}\mathfrak{G}_N$ defined in (3.4).

In $\mathbb{K}^{p,q}(\mathfrak{G}_N)$, there exists a subspace which is naturally isomorphic to $\mathbb{K}^{p,0}$. It consists of vectors (4.2) of the form $\mathbf{w} = \sum_{j=1}^{m} w_j e_j$ with $w_j \in \mathbb{K}$. The map $[\cdot]_0 : \mathbb{K}^{p,q}(\mathfrak{G}_N) \to \mathbb{K}^{p,0}$ defined by $[\mathbf{w}]_0 = \sum_{j=1}^{p} [w_j]_0 \, e_j$ will be useful.

We recall that the standard basis of $\mathbb{K}^{p,q}$ is represented by column vectors, see Example 3.1. In this basis, elements of $\mathbb{K}^{p,q}(\mathfrak{G}_N)$ take the form $\mathbf{w} = (w_1, \ldots, w_p, \grave{w}_1, \ldots, \grave{w}_q)^T$.

4.1.2 Supermatrices

The $\mathbb{Z}_2$-grading of $\mathbb{K}^{p,q}$ yields the $\mathbb{Z}_2$-grading of the space $\mathrm{End}\,(\mathbb{K}^{p,q})$ of endomorphisms on $\mathbb{K}^{p,q}$. This space is isomorphic to the space $\mathrm{Mat}(p|q)$ of block matrices of the form

$$M = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & \grave{B} \\ \grave{C} & 0 \end{pmatrix} \tag{4.3}$$

where[2] $A \in \mathbb{K}^{p\times p}$, $\grave{B} \in \mathbb{K}^{p\times q}$, $\grave{C} \in \mathbb{K}^{q\times p}$ and $D \in \mathbb{K}^{q\times q}$. The first term in (4.3) is the even part of M and the second term is the odd one.

Remark 4.2. *The super vector space* $\mathrm{Mat}(p|q)$ *is a Lie superalgebra with the Lie superbracket given by the graded commutator,*

$$[X, Y] = XY - (-1)^{|X||Y|} YX$$

for homogeneous elements $X, Y \in \mathrm{Mat}(p|q)$. *Here the grading function* $|X|$ *is defined as* 0 *if* X *is even and* 1 *if* X *is odd. When seen as a Lie superalgebra,* $\mathrm{Mat}(p|q)$ *is denoted by* $\mathfrak{gl}(p|q)(\mathbb{K})$. *It is easily seen that the Grassmann envelope of any Lie subsuperalgebra of* $\mathfrak{gl}(p|q)(\mathbb{K})$ *is a classical Lie algebra.*

The Grassmann envelope of $\mathrm{Mat}(p|q)$ is denoted by $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$ and consists of matrices of the form (4.3), however with entries in $\mathbb{K}\mathfrak{G}_N$ (namely, even entries for A, D and odd entries for $\grave{B}$, $\grave{C}$). Elements in $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$ are called *supermatrices.*

The $\mathbb{Z}_2$-grading of $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$, inherited from $\mathrm{Mat}(p|q)$, together with the usual matrix multiplication, provides a superalgebra structure on this Grassmann envelope. More precisely, for any $k \in \mathbb{N}$, let $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(k)})$ be the space of all homogeneous supermatrices of degree k. This is, $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(2k)})$ consists of all diagonal block matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad A \in \left(\mathbb{K}\mathfrak{G}_N^{(2k)}\right)^{p\times p}, \;\; D \in \left(\mathbb{K}\mathfrak{G}_N^{(2k)}\right)^{q\times q},$$

while $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(2k+1)})$ consists of all off-diagonal block matrices

$$\begin{pmatrix} 0 & \grave{B} \\ \grave{C} & 0 \end{pmatrix}, \qquad \grave{B} \in \left(\mathbb{K}\mathfrak{G}_N^{(2k+1)}\right)^{p\times q}, \;\; \grave{C} \in \left(\mathbb{K}\mathfrak{G}_N^{(2k+1)}\right)^{q\times p}.$$

[2]Given a set $\mathbf{S}$, we use the notation $\mathbf{S}^{p\times q}$ to refer to the set of matrices of order $p \times q$ with entries in $\mathbf{S}$.

These subspaces define a grading in $\text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$ by

$$\text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N) = \bigoplus_{k=0}^{N} \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(k)}),$$

$$\text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(k)}) \ \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(\ell)}) \subset \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(k+\ell)}).$$

Then, clearly, every supermatrix M can be written as the sum of a *body* matrix $M_0 \in \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(0)})$ and a nilpotent element $\mathbf{M} \in \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{+}) := \bigoplus_{k=1}^{N} \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(k)})$. In accordance with the general ideas valid for Grassmann algebras and Grassmann envelopes we define the algebra homomorphism

$$[\cdot]_0 : \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N) \to \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(0)})$$

as the projection:

$$M = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \longrightarrow \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix} = M_0 = [M]_0$$

where A_0 and D_0 are the *body* projections of A and D on $\mathbb{K}^{p\times p}$ and $\mathbb{K}^{q\times q}$ respectively. We recall that $\text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^{(0)})$ is equal to the even subalgebra of $\text{Mat}(p|q)$. Given a set of supermatrices $\mathbf{S}$ we define $[\mathbf{S}]_0 = \{[M]_0 : M \in \mathbf{S}\}$.

Every supermatrix M defines a linear operator on $\mathbb{K}^{p,q}(\mathfrak{G}_N)$ which acts on a Grassmann supervector $\mathbf{w} = \underline{w} + \underline{\grave{w}}$ by left multiplication with its column representation:

$$M\mathbf{w} = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \begin{pmatrix} \underline{w} \\ \underline{\grave{w}} \end{pmatrix} = \begin{pmatrix} A\underline{w} + \grave{B}\underline{\grave{w}} \\ \grave{C}\underline{w} + D\underline{\grave{w}} \end{pmatrix} \in \mathbb{K}^{p,q}(\mathfrak{G}_N).$$

In order to study some group structures in $\text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$ we start from the Lie group $\text{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$ of all invertible elements of $\text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$. The following theorem states a well-known characterization of this group, see [7].

Theorem 4.1. *Let* $M = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \in \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$. *Then the following statements are equivalent.*

(i) $M \in \text{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$.

(ii) A, D *are invertible.*

(iii) A_0, D_0 *are invertible.*

In addition, for every $M \in \text{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$ *its inverse is given by*

$$M^{-1} = \begin{pmatrix} \left(A - \grave{B}D^{-1}\grave{C}\right)^{-1} & -A^{-1}\grave{B}\left(D - \grave{C}A^{-1}\grave{B}\right)^{-1} \\ -D^{-1}\grave{C}\left(A - \grave{B}D^{-1}\grave{C}\right)^{-1} & \left(D - \grave{C}A^{-1}\grave{B}\right)^{-1} \end{pmatrix}.$$

Remark 4.3. *The group* $\mathrm{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$ *is an extension of the general linear group* $\mathrm{GL}(p)$ *of invertible matrices in* $\mathbb{R}^{p\times p}$.

The usual definitions of transpose, trace and determinant of a matrix are not appropriate in the graded case. For example, although the classical transpose

$$M^T = \begin{pmatrix} A^T & \grave{C}^T \\ \grave{B}^T & D^T \end{pmatrix}$$

of a supermatrix M is a well defined element of $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$, we have in general that

$$(ML)^T \neq L^T M^T$$

unlike the classical property. This problem is fixed by introducing the *supertranspose* by

$$M^{ST} = \begin{pmatrix} A^T & \grave{C}^T \\ -\grave{B}^T & D^T \end{pmatrix}.$$

The transpose and supertranspose operations satisfy the following relations, see [7].

Proposition 4.2. *Let* $M, L \in \mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$, $x \in \mathbb{K}^{p,q}(\mathfrak{G}_N)$, $\grave{B} \in \left(\mathbb{K}\mathfrak{G}_N^{(odd)}\right)^{p\times q}$ *and* $\grave{C} \in \left(\mathbb{K}\mathfrak{G}_N^{(odd)}\right)^{q\times p}$. *Then,*

(i) $\left(\grave{B}\grave{C}\right)^T = -\grave{C}^T\grave{B}^T$,

(ii) $(ML)^{ST} = L^{ST}M^{ST}$,

(iii) $(Mx)^T = x^T M^{ST}$,

(iv) $\left(M^{ST}\right)^{ST} = \begin{pmatrix} A & -\grave{B} \\ -\grave{C} & D \end{pmatrix} = SMS$, *where*[3] $S = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$,

(v) $\left(M^{-1}\right)^{ST} = \left(M^{ST}\right)^{-1}$ *for every* $M \in \mathrm{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$.

The situation for the trace is similar. The usual trace $\mathrm{tr}(M)$ of an element $M \in \mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$ is well defined, but in general one has that

$$\mathrm{tr}(ML) \neq \mathrm{tr}(LM)$$

for $M, L \in \mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$. The notion of supertrace provides a solution to this problem; it is defined as the map $\mathrm{str} : \mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N) \to \mathbb{K}\mathfrak{G}_N^{(ev)}$ given by

$$\mathrm{str}\begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D).$$

The following properties easily follow from the above definition, see [7].

[3] I_k denotes the identity matrix in $\mathbb{R}^{k\times k}$.

Proposition 4.3. *Let* $M, L \in \mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$*,* $\grave{B} \in \left(\mathbb{K}\mathfrak{G}_N^{(odd)}\right)^{p\times q}$ *and* $\grave{C} \in \left(\mathbb{K}\mathfrak{G}_N^{(odd)}\right)^{q\times p}$*. Then*

(i) $\mathrm{tr}\left(\grave{B}\,\grave{C}\right) = -\,\mathrm{tr}\left(\grave{C}\,\grave{B}\right)$*,*

(ii) $\mathrm{str}(ML) = \mathrm{str}(LM)$*,*

(iii) $\mathrm{str}\left(M^{ST}\right) = \mathrm{str}(M)$*.*

The *superdeterminant* or *Berezinian* is a function from $\mathrm{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$ to $\mathbb{K}\mathfrak{G}_N^{(ev)}$ defined by:

$$\mathrm{sdet}(M) = \frac{\det(A - \grave{B}D^{-1}\grave{C})}{\det(D)} = \frac{\det(A)}{\det(D - \grave{C}A^{-1}\grave{B})}.$$

Some of its basic properties are given in the following proposition, see [7].

Proposition 4.4. *Let* $M, L \in \mathrm{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$*, then*

(i) $\mathrm{sdet}(ML) = \mathrm{sdet}(M)\,\mathrm{sdet}(L)$*,*

(ii) $\mathrm{sdet}\left(M^{ST}\right) = \mathrm{sdet}(M)$*.*

In the vector space $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$ we introduce the norm

$$\|M\| = \sum_{j,k=1}^{p+q} \|m_{j,k}\|_{\mathfrak{G}},$$

where $m_{j,k} \in \mathbb{K}\mathfrak{G}_N$ $(j,k = 1,\ldots,p+q)$ are the entries of $M \in \mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$. As was the case in $\mathbb{K}\mathfrak{G}_N$, also this norm satisfies the inequality $\|ML\| \leq \|M\|\|L\|$ for every pair $M, L \in \mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$, leading to the absolute convergence of the series

$$\exp(M) = \sum_{j=0}^{\infty} \frac{M^j}{j!}$$

and hence, the continuity of the exponential map in $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$. It is easily seen that also the supertranspose, the supertrace and the superdeterminant are continuous maps. Some properties of the exponential are gathered in the following proposition.

Proposition 4.5. *Let* $M, L \in \mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$*. Then*

(i) $e^0 = I_{p+q}$*;*

(ii) $\left(e^M\right)^{ST} = e^{M^{ST}}$*;*

(iii) $e^{M+L} = e^M\, e^L$ *whenever* $ML = LM$*;*

(iv) $e^M \in \mathrm{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$ *and* $\left(e^M\right)^{-1} = e^{-M}$*;*

(v) $e^{(a+b)M} = e^{aM}e^{bM}$ *for every pair* $a, b \in \mathbb{K}\mathfrak{G}_N^{(ev)}$*;*

(vi) $e^{CMC^{-1}} = Ce^M C^{-1}$ *for every* $C \in \mathrm{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$*;*

(vii) e^{tM} *($t \in \mathbb{R}$) is a smooth curve in* $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$*, with*

$$\frac{d}{dt}e^{tM} = Me^{tM} = e^{tM}M, \qquad \text{and} \qquad \left.\frac{d}{dt}e^{tM}\right|_{t=0} = M.$$

(viii) $\mathrm{sdet}\left(e^M\right) = e^{\mathrm{str}(M)}$.

Remark 4.4. *The proofs of (i)-(vii) are straightforward computations. A detailed proof for (viii) can be found in [7, pp. 101-103]. Similar properties to (i) and (iii)-(vii) can be obtained for the exponential map in* $\mathbb{K}\mathfrak{G}_N$*.*

We also can define the notion of logarithm for a supermatrix $M \in \mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$ by

$$\ln(M) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{(M - I_{p+q})^j}{j}, \tag{4.4}$$

wherever it converges.

Proposition 4.6.

(i) The series (4.4) converges and yields a continuous function near I_{p+q}*.*

(ii) In $\mathrm{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N)$*, let* U *be a neighbourhood of* I_{p+q} *on which* $\ln$ *is defined and let* V *be a neighbourhood of* 0 *such that* $\exp(V) := \{e^M : M \in V\} \subset U$*. Then*

$$e^{\ln(M)} = M, \quad \forall M \in U, \qquad\qquad \ln(e^L) = L, \quad \forall L \in V$$

Proof.

(i) Observe that

$$\sum_{j=1}^{\infty}\frac{\left\|(M - I_{p+q})^j\right\|}{j} \le \sum_{j=1}^{\infty}\frac{\|M - I_{p+q}\|^j}{j},$$

whence, since the radius of convergence of the last series is 1, (4.4) absolutely converges and defines a continuous function in the ball $\|M - I_{p+q}\| < 1$.

(ii) The statement immediately follows from the absolute convergence of the series for exp and ln, and from the identities $e^{\ln x} = \ln(e^x) = x$ in formal power series in the indeterminate x. □

It is worth mentioning that the same procedure can be repeated in $\mathbb{K}\mathfrak{G}_N$. With the above definitions of the exponential and logarithmic maps, it is possible to obtain all classical results known for Lie groups and Lie algebras of real and complex matrices .

The exponential of a nilpotent matrix $\mathbf{M} \in \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^+)$ reduces to a finite sum, yielding the bijective mapping

$$\exp : \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^+) \to I_{p+q} + \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^+)$$

with inverse

$$\ln : I_{p+q} + \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^+) \to \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^+),$$

since also the second expansion only has a finite number of non-zero terms, whence problems of convergence do not arise.

We recall that a supermatrix M belongs to $\text{GL}(p|q)(\mathbb{K}\mathfrak{G}_N)$ if and only if its body M_0 has an inverse. Then

$$M = M_0(I_{p+q} + M_0^{-1}\mathbf{M}) = M_0 \exp(\mathbf{L}),$$

for some unique $\mathbf{L} \in \text{Mat}(p|q)(\mathbb{K}\mathfrak{G}_N^+)$.

4.2 The algebra $\mathcal{A}_{m,2n} \otimes \mathbb{R}\mathfrak{G}_N$

Let us consider the radial algebra representation $R(\mathbf{S}) \subset \mathcal{A}_{m,2n}$ in superspace introduced in Section 3.2. As mentioned before, one of the goals of this chapter is to study the invariance under linear transformations of the inner product of supervector variables $\mathbf{x}, \mathbf{y} \in \mathbf{S}$ given by

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}} := -\frac{1}{2}\{\mathbf{x}, \mathbf{y}\} = \sum_{j=1}^{m} x_j y_j - \frac{1}{2}\sum_{j=1}^{n}(\grave{x}_{2j-1}\grave{y}_{2j} - \grave{x}_{2j}\grave{y}_{2j-1}) \in \mathcal{V}_{\overline{0}}. \tag{4.5}$$

In order to study linear actions on the algebra $\mathcal{A}_{m,2n} = \mathcal{V} \otimes \mathcal{C}_{m,2n}$ we must consider a suitable set of coefficients. Observe that the field of numbers $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is too limited for that aim since it does not lead to any interaction between even and odd elements. For instance, multiplication by real or complex numbers leaves the decomposition $\mathcal{V} = \mathcal{V}_{\overline{0}} \oplus \mathcal{V}_{\overline{1}}$ of the algebra of super-polynomials generated by the set of variables $VAR \cup \grave{VAR}$ invariant.

The study of linear actions on $\mathcal{A}_{m,2n}$ requires of a set including both commuting and anti-commuting elements. In this thesis we consider the most simple set of such coefficients, i.e. the Grassmann algebra $\mathbb{R}\mathfrak{G}_N$ generated by odd independent elements $f_1 \ldots, f_N$. This leads to the $\mathbb{Z}_2$-graded algebra of super-polynomials with Grassmann coefficients

$$\mathcal{V} \otimes \mathbb{R}\mathfrak{G}_N,$$

generated over $\mathbb{R}$ by the set of $m\ell$ commuting variables VAR and the set of independent $2n\ell+N$ anti-commuting symbols $V\grave{A}R \cup \{f_1, \ldots, f_N\}$. In general we consider the algebra

$$\mathcal{A}_{m,2n} \otimes \mathbb{R}\mathfrak{G}_N = \mathcal{V} \otimes \mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n},$$

of super-polynomials with coefficients in $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$. Here elements of $\mathcal{V} \otimes \mathbb{R}\mathfrak{G}_N$ commute with elements in $\mathcal{C}_{m,2n}$.

In the set of coefficients $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ one has a radial algebra representation by considering supervectors $\mathbf{w} \in \mathbb{R}^{m,2n}(\mathfrak{G}_N)$, i.e.

$$\mathbf{w} = \underline{w} + \underline{\grave{w}} = \sum_{j=1}^{m} w_j e_j + \sum_{j=1}^{2n} \grave{w}_j \grave{e}_j, \qquad w_j \in \mathbb{R}\mathfrak{G}_N^{(ev)}, \quad \grave{w}_j \in \mathbb{R}\mathfrak{G}_N^{(odd)},$$

where clearly the basis elements $e_1, \ldots, e_m, \grave{e}_1, \ldots, \grave{e}_{2n}$ of $\mathbb{R}^{m,2n}$ have to submit to the multiplication rules (3.11). Indeed, the anti-commutator of two constant supervectors $\mathbf{w}, \mathbf{v} \in \mathbb{R}^{m,2n}(\mathfrak{G}_N)$ clearly is a central element in $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$, i.e.

$$\{\mathbf{w}, \mathbf{v}\} = 2\sum_{j=1}^{m} w_j v_j + \sum_{j=1}^{n} (\grave{w}_{2j-1}\grave{v}_{2j} - \grave{w}_{2j}\grave{v}_{2j-1}) \in \mathbb{R}\mathfrak{G}_N^{(ev)}.$$

The subalgebra generated by the Grassmann envelope $\mathbb{R}^{m|2n}(\mathfrak{G}_N)$ of constant supervectors is called the *radial algebra embedded in* $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$. This algebra is denoted by $\mathbb{R}_{m|2n}(\mathfrak{G}_N)$. Observe that $\mathbb{R}_{m|2n}(\mathfrak{G}_N)$ is a finite dimensional vector space since it is generated by the union of the sets

$$\{f_A e_j \mid A \subset \{1, \ldots, N\}, |A| \text{ even}\ , j = 1, \ldots, m\},$$
$$\{f_A \grave{e}_j \mid A \subset \{1, \ldots, N\}, |A| \text{ odd}\ , j = 1, \ldots, 2n\},$$

and there is a finite number of possible products amongst these generators.

Every element in $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ can be written as a finite sum of terms of the form $a e_{j_1} \cdots e_{j_k} \grave{e}_1^{\alpha_1} \cdots \grave{e}_{2n}^{\alpha_{2n}}$ where $a \in \mathbb{R}\mathfrak{G}_N$, $1 \leq j_1 \leq \ldots \leq j_k \leq m$ and $(\alpha_1, \ldots, \alpha_{2n}) \in (\mathbb{N} \cup \{0\})^{2n}$ is a multi-index. In this algebra we consider the corresponding generalization of the projection $[\cdot]_0$ which now goes from $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ to $\mathcal{C}_{m,2n}$ and is defined by

$$[a e_{j_1} \cdots e_{j_k} \grave{e}_1^{\alpha_1} \cdots \grave{e}_{2n}^{\alpha_{2n}}]_0 = [a]_0 e_{j_1} \cdots e_{j_k} \grave{e}_1^{\alpha_1} \cdots \grave{e}_{2n}^{\alpha_{2n}}.$$

We now can define linear actions on supervector variables $\mathbf{x} \in \mathbf{S}$ by means of supermatrices $M \in \text{Mat}(m|2n)(\mathfrak{G}_N)$. We recall that the basis elements $e_1, \ldots, e_m, \grave{e}_1, \ldots, \grave{e}_{2n}$ can be written as column vectors, see Example 3.1. Then, by writing the $\mathbf{x} = \underline{x} + \underline{\grave{x}} \in \mathbf{S}$ in its column representation we obtain,

$$M\mathbf{x} = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{\grave{x}} \end{pmatrix} = \begin{pmatrix} A\underline{x} + \grave{B}\underline{\grave{x}} \\ \grave{C}\underline{x} + D\underline{\grave{x}} \end{pmatrix}. \tag{4.6}$$

This action produces a new supervector variable $M\mathbf{x} = (y_1, \ldots, y_m, \grave{y}_1, \ldots, \grave{y}_{2n})^T$ where the y_j are even elements of $\mathcal{V} \otimes \mathbb{R}\mathfrak{G}_N$ while the $\grave{y}_j$ are odd ones. It is obvious that the result in Proposition 4.2 (*iii*) also applies to these actions, i.e.

$$(M\mathbf{x})^T = \mathbf{x}^T M^{ST}.$$

4.2.1 Superbivectors

Superbivectors in $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ play a very important rôle when studying the invariance of the inner product (4.5). Following the radial algebra approach (see (2.4)), the space of bivectors is generated by the wedge product of supervectors of $\mathbb{R}^{m|2n}(\mathfrak{G}_N)$, i.e.

$$\begin{aligned} \mathbf{w} \wedge \mathbf{v} &= \frac{1}{2}[\mathbf{w}, \mathbf{v}] \\ &= \sum_{1 \leq j < k \leq m} (w_j v_k - w_k v_j) e_j e_k + \sum_{\substack{1 \leq j \leq m \\ 1 \leq k \leq 2n}} (w_j \grave{v}_k - \grave{w}_k v_j) e_j \grave{e}_k + \sum_{1 \leq j \leq k \leq 2n} (\grave{w}_j \grave{v}_k + \grave{w}_k \grave{v}_j)\, \grave{e}_j \odot \grave{e}_k, \end{aligned}$$

where $\grave{e}_j \odot \grave{e}_k = \frac{1}{2}\{\grave{e}_j, \grave{e}_k\}$. Hence, the space $\mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$ of superbivectors consists of elements of the form

$$B = \sum_{1 \leq j < k \leq m} b_{j,k}\, e_j e_k + \sum_{\substack{1 \leq j \leq m \\ 1 \leq k \leq 2n}} \grave{b}_{j,k}\, e_j \grave{e}_k + \sum_{1 \leq j \leq k \leq 2n} B_{j,k}\, \grave{e}_j \odot \grave{e}_k, \tag{4.7}$$

where $b_{j,k} \in \mathbb{R}\mathfrak{G}_N^{(ev)}$, $\grave{b}_{j,k} \in \mathbb{R}\mathfrak{G}_N^{(odd)}$ and $B_{j,k} \in \mathbb{R}\mathfrak{G}_N^{(ev)} \cap \mathbb{R}\mathfrak{G}_N^{+}$. Observe that the coefficients $B_{j,k}$ are commuting but nilpotent since they are generated by elements of the form $\grave{w}_j \grave{v}_k + \grave{w}_k \grave{v}_j$ that belong to $\mathbb{R}\mathfrak{G}_N^{+}$. This constitutes an important limitation for the space of superbivectors because it means that $\mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$ does not allow for any other structure than the orthogonal one. In fact, the real projection $[B]_0$ of every superbivector B is just the classical Clifford bivector:

$$[B]_0 = \sum_{1 \leq j < k \leq m} [b_{j,k}]_0\; e_j e_k \in \mathbb{R}^{(2)}_{0,m}.$$

Hence it is necessary to introduce an extension $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ of $\mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$ containing elements B of the form (4.7) but with $B_{j,k} \in \mathbb{R}\mathfrak{G}_N^{(ev)}$. This extension enables us to consider two different structures in the same element B: the orthogonal and the symplectic one. In fact, in this case we have

$$[B]_0 = \sum_{1 \leq j < k \leq m} [b_{j,k}]_0\; e_j e_k + \sum_{1 \leq j \leq k \leq 2n} [B_{j,k}]_0\; \grave{e}_j \odot \grave{e}_k.$$

Remark 4.5. *Observe that* $\mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$ *and* $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ *are finite dimensional real vector subspaces of* $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ *with*

$$\begin{aligned}\dim \mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N) &= 2^{N-1}\frac{m(m-1)}{2} + 2^{N-1}\, 2mn + \left(2^{N-1}-1\right)\, n(2n+1),\\ \dim \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N) &= 2^{N-1}\frac{m(m-1)}{2} + 2^{N-1}\, 2mn + 2^{N-1}\, n(2n+1).\end{aligned}$$

The extension $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ of the superbivector space clearly lies outside the radial algebra $\mathbb{R}_{m|2n}(\mathfrak{G}_N)$, and its elements generate an infinite dimensional algebra. Elements in $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ are called *extended superbivectors.* Both superbivectors and extended superbivectors preserve several properties of classical Clifford bivectors.

Proposition 4.7. *The space* $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ *is a Lie algebra. In addition,* $\mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$ *is a Lie subalgebra of* $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$.

Proof.

We only need to check that the Lie bracket defined by the commutator in the associative algebra $\mathcal{A}_{m,2n} \otimes \mathbb{R}\mathfrak{G}_N$ is an internal binary operation in $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ and $\mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$. Direct computation shows that for $a, b \in \mathbb{R}\mathfrak{G}_N^{(ev)}$ and $\grave{a}, \grave{b} \in \mathbb{R}\mathfrak{G}_N^{(odd)}$ we get:

$$\begin{aligned}[a e_j e_k,\, b e_r e_s] &= ab\,(2\delta_{j,s}\, e_r e_k - 2\delta_{s,k}\, e_r e_j + 2\delta_{r,j}\, e_k e_s - 2\delta_{r,k}\, e_j e_s),\\ [a e_j e_k,\, \grave{b} e_r \grave{e}_s] &= a\grave{b}\,(2\delta_{r,j}\, e_k \grave{e}_s - 2\delta_{r,k}\, e_j \grave{e}_s),\\ [a e_j e_k,\, b \grave{e}_r \odot \grave{e}_s] &= 0,\\ [\grave{a} e_j \grave{e}_k,\, \grave{b} e_r \grave{e}_s] &= \grave{a}\grave{b}\,(2\delta_{r,j}\, \grave{e}_k \odot \grave{e}_s + (1-\delta_{j,r}) g_{s,k}\, e_j e_r),\\ [\grave{a} e_j \grave{e}_k,\, b \grave{e}_r \odot \grave{e}_s] &= \grave{a} b\,(g_{k,s}\, e_j \grave{e}_r + g_{k,r}\, e_j \grave{e}_s),\\ [a \grave{e}_j \odot \grave{e}_k,\, b \grave{e}_r \odot \grave{e}_s] &= ab\,(g_{j,s}\, \grave{e}_r \odot \grave{e}_k + g_{k,s}\, \grave{e}_r \odot \grave{e}_j + g_{j,r}\, \grave{e}_k \odot \grave{e}_s + g_{k,r}\, \grave{e}_j \odot \grave{e}_s).\end{aligned}$$

□

It is well known from the radial algebra framework that the commutator of a bivector with a vector always yields a linear combination of vectors with coefficients in the scalar subalgebra. Indeed, for the abstract vector variables $x, y, z \in S$ we obtain

$$[x \wedge y, z] = \frac{1}{2}\,[[x,y],z] = \frac{1}{2}[2xy - \{x,y\}, z] = [xy, z] = \{y,z\}x - \{x,z\}y.$$

This property can be easily generalized to $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ by straightforward computation. In particular, the following results hold.

Proposition 4.8. *Let* $\mathbf{x} \in \mathbf{S}$ *be a supervector variable, let* $\{b_1, \ldots, b_{2^{N-1}}\}$ *be a basis for* $\mathbb{R}\mathfrak{G}_N^{(ev)}$ *and let* $\{\grave{b}_1, \ldots, \grave{b}_{2^{N-1}}\}$ *be a basis for* $\mathbb{R}\mathfrak{G}_N^{(odd)}$. *Then,*

$$\begin{aligned}[b_r\, e_j e_k,\, \mathbf{x}] &= 2b_r\,(x_j e_k - x_k e_j), & [b_r\, \grave{e}_{2j} \odot \grave{e}_{2k},\, \mathbf{x}] &= -b_r\,(\grave{x}_{2j-1}\grave{e}_{2k} + \grave{x}_{2k-1}\grave{e}_{2j}),\\ [\grave{b}_r\, e_j \grave{e}_{2k-1},\, \mathbf{x}] &= \grave{b}_r\,(2x_j \grave{e}_{2k-1} + \grave{x}_{2k} e_j), & [b_r\, \grave{e}_{2j-1} \odot \grave{e}_{2k-1},\, \mathbf{x}] &= b_r\,(\grave{x}_{2j}\grave{e}_{2k-1} + \grave{x}_{2k}\grave{e}_{2j-1}),\\ [\grave{b}_r\, e_j \grave{e}_{2k},\, \mathbf{x}] &= \grave{b}_r\,(2x_j \grave{e}_{2k} - \grave{x}_{2k-1} e_j), & [b_r\, \grave{e}_{2j-1} \odot \grave{e}_{2k},\, \mathbf{x}] &= b_r\,(\grave{x}_{2j}\grave{e}_{2k} - \grave{x}_{2k-1}\grave{e}_{2j-1}).\end{aligned}$$

Clearly, the above computations remain valid when replacing $\mathbf{x}$ by a fixed supervector $\mathbf{w} \in \mathbb{R}^{m|2n}(\mathfrak{G}_N)$.

4.2.2 Tensor algebra and exponential map

Since $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ is infinite dimensional, the definition of the exponential map by means of a power series is not as straightforward as it was for the algebras $\mathbb{R}\mathfrak{G}_N$ or $\mathrm{Mat}(p|q)(\mathbb{R}\mathfrak{G}_N)$. A correct definition of the exponential map in $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ requires the introduction of the tensor algebra. More details about the general theory of tensor algebras can be found in several basic references, see e.g. [73, 56, 46].

Let $T(V)$ be the tensor algebra of the vector space V spanned by the basis $B_V = \{f_1, \ldots, f_N, e_1, \ldots, e_m, \grave{e}_1, \ldots, \grave{e}_{2n}\}$, i.e.

$$T(V) = \bigoplus_{j=0}^{\infty} T^j(V)$$

where $T^j(V) = \mathrm{span}_{\mathbb{R}}\{v_1 \otimes \cdots \otimes v_j : v_\ell \in B_V\}$ is the j-fold tensor product of V with itself. Then $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ can be seen as a subalgebra of $T(V)/I$ where $I \subset T(V)$ is the two-sided ideal generated by the elements:

$$\begin{aligned} &f_j \otimes f_k + f_k \otimes f_j, && e_j \otimes e_k + e_k \otimes e_j + 2\delta_{j,k},\\ &f_j \otimes e_k - e_k \otimes f_j, && e_j \otimes \grave{e}_k + \grave{e}_k \otimes e_j,\\ &f_j \otimes \grave{e}_k - \grave{e}_k \otimes f_j, && \grave{e}_j \otimes \grave{e}_k - \grave{e}_k \otimes \grave{e}_j - g_{j,k}. \end{aligned}$$

Indeed, $T(V)/I$ is isomorphic to the extension of $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ which also contains **infinite** sums of arbitrary terms of the form $a e_{j_1} \cdots e_{j_k} \grave{e}_1^{\alpha_1} \cdots \grave{e}_{2n}^{\alpha_{2n}}$ where $a \in \mathbb{R}\mathfrak{G}_N$, $1 \leq j_1 \leq \ldots \leq j_k \leq m$ and $(\alpha_1, \ldots, \alpha_{2n}) \in (\mathbb{N} \cup \{0\})^{2n}$ is a multi-index.

The exponential map

$$\exp(x) = e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

is known to be well defined in the tensor algebra $T(V)$, see e.g. [46], whence it also is well defined in $T(V)/I$. It has the following mapping properties:

$$\exp : \mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n} \to T(V)/I, \qquad \exp : \mathbb{R}_{m|2n}(\mathfrak{G}_N) \to \mathbb{R}_{m|2n}(\mathfrak{G}_N).$$

The first statement directly follows from the definition of $T(V)/I$, while the second one can be obtained following the standard procedure established for $\mathbb{R}\mathfrak{G}_N$ and $\mathrm{Mat}(p|q)(\mathbb{R}\mathfrak{G}_N)$, since the radial algebra $\mathbb{R}_{m|2n}(\mathfrak{G}_N) \subset \mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ is finite dimensional.

4.3 The orthosymplectic structure in $\mathbb{R}^{m|2n}(\mathfrak{G}_N)$

4.3.1 Invariance of the inner product

The inner product (4.5) can be easily written as

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}} = \mathbf{x}^T \mathbf{Q} \mathbf{y}$$

in terms of the supermatrix $\mathbf{Q} = \begin{pmatrix} I_m & 0 \\ 0 & -\frac{1}{2}\Omega_{2n} \end{pmatrix}$, where

$$\Omega_{2n} = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}.$$

In order to find all supermatrices $M \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ leaving the inner product $\langle \cdot, \cdot \rangle$ invariant, we observe that

$$\langle M\mathbf{x}, M\mathbf{y} \rangle_{\mathbb{R}} = \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}} \iff (M\mathbf{x})^T \mathbf{Q} M \mathbf{y} = \mathbf{x}^T \mathbf{Q} \mathbf{y} \iff \mathbf{x}^T \left(M^{ST} \mathbf{Q} M - \mathbf{Q} \right) \mathbf{y} = 0,$$

whence the desired set is given by

$$\mathrm{O}_0 = \mathrm{O}_0(m|2n)(\mathbb{R}\mathfrak{G}_N) = \left\{ M \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N) : M^{ST} \mathbf{Q} M - \mathbf{Q} = 0 \right\},$$

Remark 4.6. *It is clear that elements in the above set of supermatrices also leave the same bilinear form in $\mathbb{R}^{m,2n}(\mathfrak{G}_N)$ invariant, i.e.*

$$-\frac{1}{2}\{M\mathbf{w}, M\mathbf{v}\} = -\frac{1}{2}\{\mathbf{w}, \mathbf{v}\}, \quad M \in \mathrm{O}_0, \ \mathbf{w}, \mathbf{v} \in \mathbb{R}^{m,2n}(\mathfrak{G}_N).$$

In general, every property that holds for supermatrix actions on supervector variables $\mathbf{x} \in \mathbf{S}$ also holds for the same actions on fixed supervectors $\mathbf{w} \in \mathbb{R}^{m,2n}(\mathfrak{G}_N)$.

We now study the algebraic structure of $\mathrm{O}_0(m|2n)(\mathbb{R}\mathfrak{G}_N)$.

Theorem 4.2. *The following statements hold:*

(i) $\mathrm{O}_0(m|2n)(\mathbb{R}\mathfrak{G}_N) \subset \mathrm{GL}(m|2n)(\mathbb{R}\mathfrak{G}_N)$.

(ii) $\mathrm{O}_0(m|2n)(\mathbb{R}\mathfrak{G}_N)$ *is a group under the usual matrix multiplication.*

(iii) $\mathrm{O}_0(m|2n)(\mathbb{R}\mathfrak{G}_N)$ *is a closed subgroup of* $\mathrm{GL}(m|2n)(\mathbb{R}\mathfrak{G}_N)$.

Summarizing, $\mathrm{O}_0(m|2n)(\mathbb{R}\mathfrak{G}_N)$ *is a Lie group.*

Proof.

(*i*) For every $M \in \mathrm{O}_0(m|2n)(\mathbb{R}\mathfrak{G}_N)$ we have

$$\left[M^{ST}\right]_0 \mathbf{Q}[M]_0 - \mathbf{Q} = [M]_0^T \mathbf{Q}[M]_0 - \mathbf{Q} = 0,$$

where

$$[M]_0 = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}.$$

This can be rewritten in terms of the real blocks A_0 and D_0 as

$$\begin{cases} A_0^T A_0 = I_m, \\ D_0^T \Omega_{2n} D_0 = \Omega_{2n}, \end{cases}$$

implying that A_0 and D_0 are invertible matrices. On account of Theorem 4.1, M thus is invertible.

(*ii*) It suffices to prove that matrix inversion and matrix multiplication are internal operations in $\mathrm{O}_0(m|2n)(\mathbb{R}\mathfrak{G}_N)$. Both properties follow by straightforward computation.

(*iii*) Let $\{M_j\}_{j\in\mathbb{N}} \subset \mathrm{O}_0(m|2n)(\mathbb{R}\mathfrak{G}_N)$ be a sequence that converges to a supermatrix $M \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$. Since algebraic operations are continuous in the space $\mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ we have that

$$M^{ST}\mathbf{Q}M - \mathbf{Q} = \lim_{j\to\infty} M_j^{ST}\mathbf{Q}M_j - \mathbf{Q} = 0.$$

□

Proposition 4.9. *The following statements hold:*

(i) A supermatrix $M = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ *belongs to* O_0 *if and only if*

$$\begin{cases} A^T A - \frac{1}{2}\grave{C}^T \Omega_{2n} \grave{C} = I_m, \\ A^T \grave{B} - \frac{1}{2}\grave{C}^T \Omega_{2n} D = 0, \\ \grave{B}^T \grave{B} + \frac{1}{2} D^T \Omega_{2n} D = \frac{1}{2}\Omega_{2n}. \end{cases} \tag{4.8}$$

(ii) $\mathrm{sdet}(M) = \pm 1$ *for every* $M \in \mathrm{O}_0$.

(iii) $[\mathrm{O}_0]_0 = \mathrm{O}(m) \times \mathrm{Sp}_\Omega(2n)$.

Remark 4.7. *As usual,* $\mathrm{O}(m)$ *is the classical orthogonal group in dimension* m *and* $\mathrm{Sp}_\Omega(2n)$ *is the symplectic group defined through the antisymmetric matrix* Ω_{2n}*, i.e.*

$$\mathrm{Sp}_\Omega(2n) = \{D_0 \in \mathbb{R}^{2n\times 2n} : D_0^T \Omega_{2n} D_0 = \Omega_{2n}\}.$$

Proof.

(i) The relation $M^{ST}\mathbf{Q}M = \mathbf{Q}$ can be written in terms of $A, \grave{B}, \grave{C}, D$ as:

$$\begin{pmatrix} A^TA - \frac{1}{2}\grave{C}^T\Omega_{2n}\grave{C} & A^T\grave{B} - \frac{1}{2}\grave{C}^T\Omega_{2n}D \\ -\grave{B}^TA - \frac{1}{2}D^T\Omega_{2n}\grave{C} & -\grave{B}^T\grave{B} - \frac{1}{2}D^T\Omega_{2n}D \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & -\frac{1}{2}\Omega_{2n} \end{pmatrix}.$$

(ii) On account of Proposition 4.4, $M^{ST}\mathbf{Q}M = \mathbf{Q}$ implies that $\text{sdet}(M)^2\,\text{sdet}(\mathbf{Q}) = \text{sdet}(\mathbf{Q})$, whence $\text{sdet}(M)^2 = 1$. The statement then follows from Lemma 4.1.

(iii) See the proof of Theorem 4.2 (i). □

4.3.2 Group of superrotations SO_0.

As in the classical way, we now can introduce the set of *superrotations* by

$$\text{SO}_0 = \text{SO}_0(m|2n)(\mathbb{R}\mathfrak{G}_N) = \{M \in \text{O}_0 : \text{sdet}(M) = 1\}.$$

This is easily seen to be a Lie subgroup of O_0 with real projection equal to $\text{SO}(m) \times \text{Sp}_\Omega(2n)$, where $\text{SO}(m) \subset \text{O}(m)$ is the special orthogonal group in dimension m. In fact, the conditions

$$M^{ST}\mathbf{Q}M = \mathbf{Q}, \qquad \text{and} \qquad \text{sdet}(M) = 1,$$

imply that

$$M_0^T\mathbf{Q}M_0 = \mathbf{Q}, \qquad \text{and} \qquad \text{sdet}(M_0) = 1,$$

whence

$$M_0 = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$$

with $A_0^TA_0 = I_m$, $D_0^T\Omega_{2n}D_0 = \Omega_{2n}$ and $\det(A_0) = \det(D_0)$. But $D_0 \in \text{Sp}_\Omega(2n)$ implies $\det(D_0) = 1$, yielding $\det(A_0) = 1$ and $A_0 \in \text{SO}(m)$.

The following proposition states that, as in the classical case, SO_0 is connected and in consequence, it is the identity component of O_0.

Proposition 4.10. SO_0 *is a connected Lie group.*

Proof.

Since the real projection $\text{SO}(m) \times \text{Sp}_\Omega(2n)$ of SO_0 is a connected group, it suffices to prove that for every $M \in \text{SO}_0$ there exist a continuous path inside SO_0 connecting M with its real projection M_0. To that end, let us write

$$M = \sum_{j=0}^{N}[M]_j,$$

where $[M]_j$ is the projection of M on $\mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N^{(j)})$ for each $j = 0, 1, \ldots, N$. Then, observe that

$$\begin{aligned} M^{ST}\mathbf{Q}M = \mathbf{Q} &\iff \left(\sum_{j=0}^{N}\left[M^{ST}\right]_j\right)\mathbf{Q}\left(\sum_{j=0}^{N}[M]_j\right) = \mathbf{Q} \\ &\iff \sum_{k=0}^{N}\left(\sum_{j=0}^{k}\left[M^{ST}\right]_j\mathbf{Q}[M]_{k-j}\right) = \mathbf{Q} \\ &\iff M_0^T\mathbf{Q}M_0 = \mathbf{Q}, \quad \text{and} \quad \sum_{j=0}^{k}\left[M^{ST}\right]_j\mathbf{Q}[M]_{k-j} = 0, \quad k = 1, \ldots, N. \end{aligned}$$

Let us now take the path

$$M(t) = \sum_{j=0}^{N} t^j [M]_j.$$

For $t \in [0,1]$ this is a continuous path with $M(0) = M_0$ and $M(1) = M$. In addition,

$$M(t)_0^T\mathbf{Q}M(t)_0 = M_0^T\mathbf{Q}M_0 = \mathbf{Q},$$

and for every $k = 1, \ldots, N$ we have,

$$\sum_{j=0}^{k}\left[M(t)^{ST}\right]_j\mathbf{Q}[M(t)]_{k-j} = t^k\sum_{j=0}^{k}\left[M^{ST}\right]_j\mathbf{Q}[M]_{k-j} = 0.$$

Hence, $M(t)^{ST}\mathbf{Q}M(t) = \mathbf{Q}$, $t \in [0,1]$. Finally, observe that $\mathrm{sdet}(M(t)) = 1$ for every $t \in [0,1]$, since $\mathrm{sdet}(M(t)_0) = \mathrm{sdet}(M_0) = 1$. $\square$

We will now investigate the corresponding Lie algebras of O_0 and SO_0.

Theorem 4.3.

(i) The Lie algebra $\mathfrak{so}_0 = \mathfrak{so}_0(m|2n)(\mathbb{R}\mathfrak{G}_N)$ *of* O_0 *coincides with the Lie algebra of* SO_0 *and is given by the space of all "super anti-symmetric" supermatrices*

$$\mathfrak{so}_0 = \{X \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N) : X^{ST}\mathbf{Q} + \mathbf{Q}X = 0\}.$$

(ii) A supermatrix $X = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ *belongs to* $\mathfrak{so}_0$ *if and only if*

$$\begin{cases} A^T + A = 0, \\ \grave{B} - \frac{1}{2}\grave{C}^T\Omega_{2n} = 0, \\ D^T\Omega_{2n} + \Omega_{2n}D = 0. \end{cases} \tag{4.9}$$

(iii) $[\mathfrak{so}_0]_0 = \mathfrak{so}(m) \oplus \mathfrak{sp}_\Omega(2n)$.

Remark 4.8. *As usual,* $\mathfrak{so}(m) = \{A_0 \in \mathbb{R}^{m\times m} : A_0^T + A_0 = 0\}$ *is the special orthogonal Lie algebra in dimension* m *and* $\mathfrak{sp}_\Omega(2n) = \{D_0 \in \mathbb{R}^{2n\times 2n} : D_0^T\Omega_{2n} + \Omega_{2n}D_0 = 0\}$ *is the symplectic Lie algebra defined through the antisymmetric matrix* Ω_{2n}.

Proof.

(*i*) If $X \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ is in the Lie algebra of O_0 then $e^{tX} \in \mathrm{O}_0$ for every $t \in \mathbb{R}$, i.e. $e^{tX^{ST}}\mathbf{Q}e^{tX} - \mathbf{Q} = 0$. Differentiating at $t = 0$ we obtain $X^{ST}\mathbf{Q} + \mathbf{Q}X = 0$. On the other hand, if $X \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ satisfies $X^{ST}\mathbf{Q} + \mathbf{Q}X = 0$, then $X^{ST} = -\mathbf{Q}X\mathbf{Q}^{-1}$. Computing the exponential of tX^{ST} we obtain

$$e^{tX^{ST}} = \sum_{j=0}^{\infty} \frac{\left(\mathbf{Q}(-tX)\mathbf{Q}^{-1}\right)^j}{j!} = \mathbf{Q}e^{-tX}\mathbf{Q}^{-1},$$

which implies that

$$e^{tX^{ST}}\mathbf{Q}e^{tX} - \mathbf{Q} = 0,$$

i.e. $e^{tX} \in \mathrm{O}_0$. As a consequence, $\mathfrak{so}_0$ is the Lie algebra of O_0.

From Proposition 4.5 it easily follows that the Lie algebra of SO_0 is given by

$$\{X \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N) : X^{ST}\mathbf{Q} + \mathbf{Q}X = 0,\ \mathrm{str}(X) = 0\}.$$

But $X^{ST}\mathbf{Q}+\mathbf{Q}X = 0$ implies $\mathrm{str}(X) = 0$. In fact, the condition $X^{ST} = -\mathbf{Q}X\mathbf{Q}^{-1}$ implies that

$$\mathrm{str}(X^{ST}) = -\,\mathrm{str}(\mathbf{Q}X\mathbf{Q}^{-1}) = -\,\mathrm{str}(X),$$

yielding $\mathrm{str}(X) = \mathrm{str}(X^{ST}) = -\,\mathrm{str}(X)$ and $\mathrm{str}(X) = 0$. Hence, the Lie algebra of SO_0 is $\mathfrak{so}_0$.

(*ii*) Observe that the relation $X^{ST}\mathbf{Q} + \mathbf{Q}X = 0$ can be written in terms of $A, \grave{B}, \grave{C}, D$ as follows:

$$\begin{pmatrix} A^T + A & -\frac{1}{2}\grave{C}^T\Omega_{2n} + \grave{B} \\ -\grave{B}^T - \frac{1}{2}\Omega_{2n}\grave{C} & -\frac{1}{2}D^T\Omega_{2n} - \frac{1}{2}\Omega_{2n}D \end{pmatrix} = 0.$$

(*iii*) Let $X = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \in \mathfrak{so}_0$, then

$$X_0 = [X]_0 = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$$

satisfies

$$X_0^{ST}\mathbf{Q} + \mathbf{Q}X_0 = 0.$$

Using (*ii*) we obtain $A_0^T + A_0 = 0$ and $D_0^T\Omega_{2n} + \Omega_{2n}D_0 = 0$ which implies that $A_0 \in \mathfrak{so}(m)$ and $D_0 \in \mathfrak{sp}_\Omega(2n)$. □

Remark 4.9. *The Lie algebra* $\mathfrak{so}_0$ *constitutes a Grassmann envelope of the orthosymplectic Lie superalgebra* $\mathfrak{osp}(m|2n)$*. Here we define* $\mathfrak{osp}(m|2n)$*, in accordance with [23], as the subsuperalgebra of* $\mathfrak{gl}(m|2n)(\mathbb{R})$ *given by,*

$$\mathfrak{osp}(m|2n) := \{X \in \mathfrak{gl}(m|2n)(\mathbb{R}) : X^{ST}\mathbf{G} + \mathbf{G}X = 0\}, \qquad \mathbf{G} = \begin{pmatrix} I_m & 0 \\ 0 & J_{2n} \end{pmatrix},$$

where $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

It suffices to note that $\mathfrak{so}_0$ *is the Grassmann envelope of*

$$\mathfrak{so}_0(m|2n)(\mathbb{R}) := \{X \in \mathfrak{gl}(m|2n)(\mathbb{R}) : X^{ST}\mathbf{Q} + \mathbf{Q}X = 0\},$$

which is isomorphic to $\mathfrak{osp}(m|2n)$*. In order to explicitly find this isomorphism we first need the matrix*

$$R = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{array}\right) \in \mathrm{O}(2n),$$

which satisfies $R^T J_{2n} R = \Omega_{2n}$*. Then the mapping* $\Phi : \mathfrak{so}_0(m|2n)(\mathbb{R}) \to \mathfrak{osp}(m|2n)$*, given by*

$$\Phi(X) = \mathbf{R}^{-1}X\mathbf{R}, \qquad \textit{with} \qquad \mathbf{R} = \begin{pmatrix} I_m & 0 \\ 0 & i\sqrt{2}R^T \end{pmatrix},$$

is easily seen to be a Lie superalgebra isomorphism. Indeed, the matrix $\mathbf{R}$ *is such that* $\mathbf{R}^{ST}\mathbf{Q}\mathbf{R} = \mathbf{G}$*. As a consequence, for every* $X \in \mathfrak{so}_0(m|2n)(\mathbb{R})$ *one has that*

$$\begin{aligned} \Phi(X)^{ST}\mathbf{G} + \mathbf{G}\Phi(X) &= \mathbf{R}^{ST}X^{ST}\left(\mathbf{R}^{-1}\right)^{ST}\mathbf{G} + \mathbf{G}\mathbf{R}^{-1}X\mathbf{R} \\ &= \mathbf{R}^{ST}\left(X^{ST}\mathbf{Q} + \mathbf{Q}X\right)\mathbf{R} \\ &= 0. \end{aligned}$$

The use of Grassmann envelopes allows to study particular aspects of the theory of Lie superalgebras in terms of classical Lie algebras and Lie groups. The $\mathfrak{osp}(m|2n)$*-invariance of the Dirac operator* $\partial_{\mathbf{X}}$ *used in [23] (see Remark 3.6) will be obtained in the next chapter in terms of the invariance of* $\partial_{\mathbf{x}}$ *under the action of the Grassmann envelope* $\mathfrak{so}_0$ *(or equivalently, under the action of the group* SO_0*).*

The connectedness of SO_0 allows to write any of its elements as a finite product of exponentials of supermatrices in $\mathfrak{so}_0$, see [50, p. 71]. In the classical case, a single exponential suffices for such a description since $\mathrm{SO}(m)$ is compact and in consequence

$\exp : \mathfrak{so}(m) \to \mathrm{SO}(m)$ is surjective, see Corollary 11.10 [50, p. 314]. This property, however, does not hold in the group of superrotations SO_0, since the exponential map from $\mathfrak{sp}_\Omega(2n)$ to the non-compact Lie group $\mathrm{Sp}_\Omega(2n) \cong \{I_m\} \times \mathrm{Sp}_\Omega(2n) \subset \mathrm{SO}_0$ is not surjective, whence not every element in SO_0 can be written as a single exponential of a supermatrix in $\mathfrak{so}_0$. Nevertheless, it is possible to find a decomposition for elements of SO_0 in terms of a fixed number of exponentials of $\mathfrak{so}_0$ elements.

Every supermatrix $M \in \mathrm{SO}_0$ has a unique decomposition $M = M_0 + \mathbf{M} = M_0(I_{m+2n} + \mathbf{L})$ where M_0 is its real projection, $\mathbf{M} \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N^+)$ its nilpotent projection and $\mathbf{L} = M_0^{-1}\mathbf{M}$. We will now separately study the decompositions for $M_0 \in \mathrm{SO}(m) \times \mathrm{Sp}_\Omega(2n)$ and $I_{m+2n} + \mathbf{L} \in \mathrm{SO}_0$.

First consider $M_0 \in \mathrm{SO}(m) \times \mathrm{Sp}_\Omega(2n)$. We already mentioned that $\exp : \mathfrak{so}(m) \to \mathrm{SO}(m)$ is surjective, while $\exp : \mathfrak{sp}_\Omega(2n) \to \mathrm{Sp}_\Omega(2n)$ is not. However, it can be proven that

$$\mathrm{Sp}_\Omega(2n) = \exp(\mathfrak{sp}_\Omega(2n)) \cdot \exp(\mathfrak{sp}_\Omega(2n)),$$

invoking the following polar decomposition for real algebraic Lie groups, see Proposition 4.3.3 in [52, p. 74].

Proposition 4.11. *Let $G \subset \mathrm{GL}(p)$ be an algebraic Lie group such that $G = G^T$ and let $\mathfrak{g}$ be its Lie algebra. Then every $A \in G$ can be uniquely written as $A = Re^X$, $R \in G \cap \mathrm{O}(p)$, $X \in \mathfrak{g} \cap \mathrm{Sym}(p)$, where $\mathrm{Sym}(p)$ is the subspace of all symmetric matrices in $\mathbb{R}^{p\times p}$.*

Remark 4.10. *A subgroup $G \subset \mathrm{GL}(p)$ is called algebraic if there exists a family $\{p_j\}_{j\in\Upsilon}$ of real polynomials*

$$p_j(M) = p_j(m_{11}, m_{12}, \ldots, m_{pp}) \in \mathbb{R}[m_{11}, \ldots, m_{pp}]$$

in the entries of the matrix $M \in \mathbb{R}^{p\times p}$ such that

$$G = \{M \in \mathrm{GL}(p) : p_j(M) = 0,\ \forall j \in \Upsilon\}.$$

See [52, p. 73] for more details. Obviously, the groups $\mathrm{O}(m)$, $\mathrm{SO}(m)$, $\mathrm{Sp}_\Omega(2n)$ are algebraic Lie groups.

Taking $p = 2n$ and $G = \mathrm{Sp}_\Omega(2n)$ in the above proposition we get that every symplectic matrix D_0 can be uniquely written as $D_0 = R_0 e^{Z_0}$ with $R_0 \in \mathrm{Sp}_\Omega(2n) \cap \mathrm{O}(2n)$ and $Z_0 \in \mathfrak{sp}_\Omega(2n) \cap \mathrm{Sym}(2n)$. But the group $\mathrm{Sp}_\Omega(2n) \cap \mathrm{O}(2n)$ is isomorphic to the unitary group $\mathrm{U}(n) = \left\{L_0 \in \mathbb{C}^{n\times n} : \left(L_0^T\right)^c L_0 = I_n\right\}$ which is connected and compact. Then the exponential map from the Lie algebra $\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n) \cong \mathfrak{u}(n)$ is surjective on $\mathrm{Sp}_\Omega(2n) \cap \mathrm{O}(2n)$ where $\mathfrak{u}(n) = \left\{L_0 \in \mathbb{C}^{n\times n} : \left(L_0^T\right)^c + L_0 = 0\right\}$ is the unitary Lie algebra in dimension n. This means that $D_0 \in \mathrm{Sp}_\Omega(2n)$ can be written as

$$D_0 = e^{Y_0} e^{Z_0}, \quad Y_0 \in \mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n), \quad Z_0 \in \mathfrak{sp}_\Omega(2n) \cap \mathrm{Sym}(2n).$$

Hence, the supermatrix $M_0 \in \mathrm{SO}(m) \times \mathrm{Sp}_\Omega(2n)$ can be decomposed as

$$M_0 = \begin{pmatrix} e^{X_0} & 0 \\ 0 & e^{Y_0}e^{Z_0} \end{pmatrix} = \begin{pmatrix} e^{X_0} & 0 \\ 0 & e^{Y_0} \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & e^{Z_0} \end{pmatrix} = e^X e^Y,$$

where

$$X = \begin{pmatrix} X_0 & 0 \\ 0 & Y_0 \end{pmatrix} \in \mathfrak{so}(m) \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)]$$

and

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & Z_0 \end{pmatrix} \in \{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathrm{Sym}(2n)].$$

Now consider the element $I_{m+2n} + \mathbf{L} \in \mathrm{SO}_0$. As shown at the end of Section 5.1, the function $\exp : \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N^+) \to I_{m+2n} + \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N^+)$ is a bijection with the logarithmic function defined in (4.4) as its inverse. Then the supermatrix

$$\mathbf{Z} = \ln(I_{m+2n} + \mathbf{L})$$

satisfies

$$e^{\mathbf{Z}} = I_{m+2n} + \mathbf{L}$$

and is nilpotent. Those properties suffice for proving that $\mathbf{Z} \in \mathfrak{so}_0$. From now on we will denote the set $\mathfrak{so}_0 \cap \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N^+)$ of nilpotent elements of $\mathfrak{so}_0$ by $\mathfrak{so}_0(m|2n)(\mathbb{R}\mathfrak{G}_N^+)$.

Proposition 4.12. *Let $\mathbf{Z} \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N^+)$ such that $e^{\mathbf{Z}} \in \mathrm{SO}_0$. Then $\mathbf{Z} \in \mathfrak{so}_0$.*

Proof.

Since $e^{\mathbf{Z}} \in \mathrm{SO}_0$, it is clear that $e^{t\mathbf{Z}} \in \mathrm{SO}_0$ for every $t \in \mathbb{Z}$. Let us prove that the same property holds for every $t \in \mathbb{R}$. The expression

$$e^{t\mathbf{Z}^{ST}} \mathbf{Q} e^{t\mathbf{Z}} - \mathbf{Q}$$

can be written as the following polynomial in the real variable t.

$$\begin{aligned} P(t) = e^{t\mathbf{Z}^{ST}} \mathbf{Q} e^{t\mathbf{Z}} - \mathbf{Q} &= \left[\sum_{j=0}^{N} \frac{t^j (\mathbf{Z}^{ST})^j}{j!}\right] \mathbf{Q} \left[\sum_{k=0}^{N} \frac{t^k \mathbf{Z}^k}{k!}\right] - \mathbf{Q} \\ &= \sum_{k=1}^{N} \sum_{j=0}^{k} \frac{t^j (\mathbf{Z}^{ST})^j}{j!} \, \mathbf{Q} \, \frac{t^{k-j} \mathbf{Z}^{k-j}}{(k-j)!} \\ &= \sum_{k=1}^{N} \frac{t^k}{k!} \left[\sum_{j=0}^{k} \binom{k}{j} (\mathbf{Z}^{ST})^j \mathbf{Q} \mathbf{Z}^{k-j}\right] \\ &= \sum_{k=1}^{N} \frac{t^k}{k!} P_k(\mathbf{Z}), \end{aligned}$$

where $P_k(\mathbf{Z}) = \sum_{j=0}^{k} \binom{k}{j} (\mathbf{Z}^{ST})^j \mathbf{Q} \mathbf{Z}^{k-j}$. If $P(t)$ is not identically zero, i.e. not all the $P_k(\mathbf{Z})$ are 0, we can take $k_0 \in \{1, 2, \ldots, N\}$ to be the largest subindex for which $P_{k_0}(\mathbf{Z}) \neq 0$. Then,

$$\lim_{t \to \infty} \frac{1}{t^{k_0}} P(t) = \frac{P_{k_0}(\mathbf{Z})}{k_0!} \neq 0,$$

contradicting that $P(\mathbb{Z}) = \{0\}$. So $P(t)$ identically vanishes, yielding $e^{t\mathbf{Z}} \in \mathrm{SO}_0$ for every $t \in \mathbb{R}$. $\square$

In this way, we have proven the following result.

Theorem 4.4. *Every supermatrix in* SO_0 *can be written as*

$$M = e^X e^Y e^{\mathbf{Z}}, \quad \textit{with} \quad \begin{cases} X \in \mathfrak{so}(m) \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)], \\ Y \in \{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathrm{Sym}(2n)], \\ \mathbf{Z} \in \mathfrak{so}_0(m|2n)(\mathbb{R}\mathfrak{G}_N^+). \end{cases}$$

Moreover, the elements Y and $\mathbf{Z}$ are unique.

4.3.3 Relation with superbivectors.

Theorem 4.3 allows to compute the dimension of $\mathfrak{so}_0$ as a real vector space.

Corollary 4.1. *The dimension of the real Lie algebra $\mathfrak{so}_0$ is*

$$\dim \mathfrak{so}_0 = 2^{N-1} \left(\frac{m(m-1)}{2} + 2mn + n(2n+1) \right).$$

Proof.

Since $\mathfrak{so}_0$ is the direct sum of the corresponding subspaces of block components $A, \grave{B}, \grave{C}$ and D respectively, it suffices to compute the dimension of each one of them. According to Theorem 4.3 (iii) we have:

$$V_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} : A^T = -A,\ A \in \left(\mathbb{R}\mathfrak{G}_N^{(ev)}\right)^{m\times m} \right\} \cong \mathbb{R}\mathfrak{G}_N^{(ev)} \otimes \mathfrak{so}(m),$$

$$V_2 = \left\{ \begin{pmatrix} 0 & \frac{1}{2}\grave{C}^T\Omega_{2n} \\ \grave{C} & 0 \end{pmatrix} : \grave{C} \in \left(\mathbb{R}\mathfrak{G}_N^{(odd)}\right)^{2n\times m} \right\} \cong \mathbb{R}\mathfrak{G}_N^{(odd)} \otimes \mathbb{R}^{2n\times m},$$

$$V_3 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} : D^T\Omega_{2n} + \Omega_{2n}D = 0,\ D \in \left(\mathbb{R}\mathfrak{G}_N^{(ev)}\right)^{2n\times 2n} \right\} \cong \mathbb{R}\mathfrak{G}_N^{(ev)} \otimes \mathfrak{sp}_\Omega(2n).$$

This leads to,

$$\dim V_1 = 2^{N-1}\frac{m(m-1)}{2}, \qquad \dim V_2 = 2^{N-1}m2n, \qquad \dim V_3 = 2^{N-1}n(2n+1).$$

$\square$

Comparing this result with the one in Remark 4.5 we obtain that $\dim \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N) = \dim \mathfrak{so}_0$. This means that both vector spaces are isomorphic. This isomorphism also holds on the Lie algebra level. Following the classical Clifford approach, the commutator

$$[B, \mathbf{x}] \qquad\qquad B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N),\ \mathbf{x} \in \mathbf{S}, \tag{4.10}$$

should be the key for the Lie algebra isomorphism. Proposition 4.8 shows that for every $B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ the commutator (4.10) defines a linear action on the supervector variable $\mathbf{x} \in \mathbf{S}$ that can be represented by a supermatrix in $\mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$, see (4.6).

Lemma 4.2. *The map* $\phi : \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N) \to \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ *defined by*

$$\phi(B)\mathbf{x} = [B, \mathbf{x}] \qquad\qquad B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N),\ \mathbf{x} \in \mathbf{S}, \tag{4.11}$$

takes values in $\mathfrak{so}_0$. *In particular, if we consider* $\{b_1, \ldots, b_{2^{N-1}}\}$ *and* $\{\grave{b}_1, \ldots, \grave{b}_{2^{N-1}}\}$ *to be the canonical basis of* $\mathbb{R}\mathfrak{G}_N^{(ev)}$ *and* $\mathbb{R}\mathfrak{G}_N^{(odd)}$ *respectively, we obtain the following basis for* $\mathfrak{so}_0$.

$$\begin{aligned}
\phi(b_r e_j e_k) &= 2b_r \begin{pmatrix} E_{k,j} - E_{j,k} & 0 \\ 0 & 0 \end{pmatrix}, && 1 \leq j < k \leq m, \\
\phi(\grave{b}_r\, e_j \grave{e}_{2k-1}) &= \grave{b}_r \begin{pmatrix} 0 & E_{j,2k} \\ 2E_{2k-1,j} & 0 \end{pmatrix}, && 1 \leq j \leq m,\ \ 1 \leq k \leq n, \\
\phi(\grave{b}_r\, e_j \grave{e}_{2k}) &= \grave{b}_r \begin{pmatrix} 0 & -E_{j,2k-1} \\ 2E_{2k,j} & 0 \end{pmatrix}, && 1 \leq j \leq m,\ \ 1 \leq k \leq n, \\
\phi(b_r\, \grave{e}_{2j} \odot \grave{e}_{2k}) &= -b_r \begin{pmatrix} 0 & 0 \\ 0 & E_{2j,2k-1} + E_{2k,2j-1} \end{pmatrix}, && 1 \leq j \leq k \leq n, \\
\phi(b_r\, \grave{e}_{2j-1} \odot \grave{e}_{2k-1}) &= b_r \begin{pmatrix} 0 & 0 \\ 0 & E_{2j-1,2k} + E_{2k-1,2j} \end{pmatrix}, && 1 \leq j \leq k \leq n, \\
\phi(b_r\, \grave{e}_{2j-1} \odot \grave{e}_{2k}) &= b_r \begin{pmatrix} 0 & 0 \\ 0 & E_{2k,2j} - E_{2j-1,2k-1} \end{pmatrix}, && 1 \leq j \leq k \leq n, \\
\phi(b_r\, \grave{e}_{2j} \odot \grave{e}_{2k-1}) &= b_r \begin{pmatrix} 0 & 0 \\ 0 & E_{2j,2k} - E_{2k-1,2j-1} \end{pmatrix}, && 1 \leq j < k \leq n,
\end{aligned}$$

where $1 \leq r \leq 2^{N-1}$, $E_{j,k}$ *denotes the matrix in which only the element on the crossing of the* j*-th row and the* k*-th column equals 1 and all the other entries are zero. The order of* $E_{j,k}$ *should be deduced from the context.*

Proof.

The above equalities can be directly obtained from Proposition 4.8, whence we should only check that all supermatrices obtained above form a basis for $\mathfrak{so}_0$. The matrices $E_{j,k}$ satisfy the relations

$$\begin{aligned}
E_{j,k}^T &= E_{k,j}, \\
E_{j,2k-1}\Omega_{2n} &= E_{j,2k}, \\
E_{j,2k}\Omega_{2n} &= -E_{j,2k-1}, \\
\Omega_{2n}E_{2j,k} &= E_{2j-1,k}, \\
\Omega_{2n}E_{2j-1,k} &= -E_{2j,k}.
\end{aligned}$$

Then

- for $\phi(b_r e_j e_k)$ we have $A = 2b_r\,(E_{k,j} - E_{j,k})$, $\grave{B} = 0$, $\grave{C} = 0$ and $D = 0$, whence
$$A^T = 2b_r\,(E_{j,k} - E_{k,j}) = -A;$$

- for $\phi(\grave{b}_r\, e_j \grave{e}_{2k-1})$ we have $A = 0$, $\grave{B} = \grave{b}_r E_{j,2k}$, $\grave{C} = 2\grave{b}_r E_{2k-1,j}$ and $D = 0$, whence
$$\frac{1}{2}\grave{C}^T \Omega_{2n} = \grave{b}_r E_{j,2k-1}\Omega_{2n} = \grave{b}_r E_{j,2k} = \grave{B};$$

- for $\phi(\grave{b}_r\, e_j \grave{e}_{2k})$ we have $A = 0$, $\grave{B} = -\grave{b}_r E_{j,2k-1}$, $\grave{C} = 2\grave{b}_r E_{2k,j}$ and $D = 0$, whence
$$\frac{1}{2}\grave{C}^T \Omega_{2n} = \grave{b}_r E_{j,2k}\Omega_{2n} = -\grave{b}_r E_{j,2k-1} = \grave{B};$$

- for $\phi(b_r\, \grave{e}_{2j} \odot \grave{e}_{2k})$ we have $A = 0$, $\grave{B} = 0$, $\grave{C} = 0$ and $D = -b_r\,(E_{2j,2k-1} + E_{2k,2j-1})$, whence
$$\begin{aligned} D^T\Omega_{2n} + \Omega_{2n}D &= -b_r\,(E_{2k-1,2j}\Omega_{2n} + E_{2j-1,2k}\Omega_{2n} + \Omega_{2n}E_{2j,2k-1} + \Omega_{2n}E_{2k,2j-1}) \\ &= -b_r\,(-E_{2k-1,2j-1} - E_{2j-1,2k-1} + E_{2j-1,2k-1} + E_{2k-1,2j-1}) = 0; \end{aligned}$$

- for $\phi(b_r\, \grave{e}_{2j-1} \odot \grave{e}_{2k-1})$ we have $A = 0$, $\grave{B} = 0$, $\grave{C} = 0$ and $D = b_r\,(E_{2j-1,2k} + E_{2k-1,2j})$, whence
$$\begin{aligned} D^T\Omega_{2n} + \Omega_{2n}D &= b_r\,(E_{2k,2j-1}\Omega_{2n} + E_{2j,2k-1}\Omega_{2n} + \Omega_{2n}E_{2j-1,2k} + \Omega_{2n}E_{2k-1,2j}) \\ &= b_r\,(E_{2k,2j} + E_{2j,2k} - E_{2j,2k} - E_{2k,2j}) = 0; \end{aligned}$$

- for $\phi(b_r\, \grave{e}_{2j-1} \odot \grave{e}_{2k})$ we have $A = 0$, $\grave{B} = 0$, $\grave{C} = 0$ and $D = b_r\,(E_{2k,2j} - E_{2j-1,2k-1})$, whence
$$\begin{aligned} D^T\Omega_{2n} + \Omega_{2n}D &= b_r\,(E_{2j,2k}\Omega_{2n} - E_{2k-1,2j-1}\Omega_{2n} + \Omega_{2n}E_{2k,2j} - \Omega_{2n}E_{2j-1,2k-1}) \\ &= b_r\,(-E_{2j,2k-1} - E_{2k-1,2j} + E_{2k-1,2j} + E_{2j,2k-1}) = 0. \end{aligned}$$

The above computations show that all supermatrices obtained belong to $\mathfrak{so}_0$. Direct verification shows that they form a set of $2^{N-1}\frac{m(m-1)}{2} + 2^{N-1}\,2mn + 2^{N-1}\,n(2n+1)$ linearly independent elements, i.e. a basis of $\mathfrak{so}_0$. □

Theorem 4.5. *The map $\phi : \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N) \to \mathfrak{so}_0$ defined in (4.11) is a Lie algebra isomorphism.*

Proof.

From Lemma 4.2 it follows that ϕ is a vector space isomorphism. In addition, due to the Jacobi identity in the associative algebra $\mathcal{A}_{m,2n} \otimes \mathbb{R}\mathfrak{G}_N$ we have for all $B_1, B_2 \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ and $\mathbf{x} \in \mathbf{S}$ that

$$\begin{aligned}[\phi(B_1), \phi(B_2)]\,\mathbf{x} &= \phi(B_1)\phi(B_2)\mathbf{x} - \phi(B_2)\phi(B_1)\mathbf{x}\\ &= [B_1, [B_2, \mathbf{x}]] + [B_2, [\mathbf{x}, B_1]]\\ &= \ [[B_1, B_2]\,, \mathbf{x}] = \ \phi\left([B_1, B_2]\right)\mathbf{x}.\end{aligned}$$

implying, that $[\phi(B_1), \phi(B_2)] = \phi\left([B_1, B_2]\right)$, i.e., ϕ is a Lie algebra isomorphism. □

Remark 4.11. *By virtue of Remark 4.9, the algebra of extended superbivectors $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ is a Grassmann envelope of $\mathfrak{osp}(m|2n)$.*

4.4 The Spin group in superspace

So far we have seen that the Lie algebra $\mathfrak{so}_0$ of the Lie group of superrotations SO_0 has a realization in $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ as the Lie algebra of extended superbivectors. In this section, we discuss the proper way of defining the corresponding realization of SO_0 in $T(V)/I$, i.e., the analogue of the Spin group in the Clifford superspace framework.

4.4.1 Supervector reflections

The group of linear transformations generated by the supervector reflections was briefly introduced in [69] using the notion of the unit sphere in $\mathbb{R}^{m,2n}(\mathfrak{G}_N)$ defined as

$$\mathbb{S}(m|2n)(\mathbb{R}\mathfrak{G}_N) = \{\mathbf{w} \in \mathbb{R}^{m,2n}(\mathfrak{G}_N) : \mathbf{w}^2 = -1\}.$$

The *reflection* associated to the supervector $\mathbf{w} \in \mathbb{S}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ is defined by the linear action on supervector variables

$$\psi(\mathbf{w})[\mathbf{x}] = \mathbf{w}\mathbf{x}\mathbf{w}, \qquad\qquad \mathbf{x} \in \mathbf{S}. \tag{4.12}$$

It is known from the radial algebra setting that $\psi(\mathbf{w})[\mathbf{x}]$ yields a new supervector variable. Indeed, for $x, y \in S$ one has

$$yxy = \{x, y\}y - y^2x = \{x, y\}y + x = x - 2\langle x, y\rangle y.$$

Every supervector reflection can be represented by a supermatrix in $\mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$.

Lemma 4.3. *Let $\mathbf{w} = \underline{w} + \underline{\grave{w}} = \sum_{j=1}^m w_j e_j + \sum_{j=1}^{2n} \grave{w}_j \grave{e}_j \in \mathbb{S}(m|2n)(\mathbb{R}\mathfrak{G}_N)$. Then, the linear transformation (4.12) can be represented by a supermatrix*

$$\psi(\mathbf{w}) = \begin{pmatrix} A(\mathbf{w}) & \grave{B}(\mathbf{w}) \\ \grave{C}(\mathbf{w}) & D(\mathbf{w}) \end{pmatrix} \in \mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$$

with

$$
\begin{aligned}
A(\mathbf{w}) &= -2D_{\underline{w}}E_{m\times m}D_{\underline{w}} + I_m,\\
\grave{B}(\mathbf{w}) &= D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\,\Omega_{2n},\\
\grave{C}(\mathbf{w}) &= -2D_{\underline{\grave{w}}}E_{2n\times m}D_{\underline{w}},\\
D(\mathbf{w}) &= D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\,\Omega_{2n} + I_{2n},
\end{aligned}
$$

where

$$
D_{\underline{w}} = \begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_m \end{pmatrix}, \quad D_{\underline{\grave{w}}} = \begin{pmatrix} \grave{w}_1 & & \\ & \ddots & \\ & & \grave{w}_{2n} \end{pmatrix}, \quad E_{p\times q} = \begin{pmatrix} 1 & 1 & \dots & 1\\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{p\times q}.
$$

Proof.

Observe that $\psi(\mathbf{w})[\mathbf{x}] = \mathbf{w}\mathbf{x}\mathbf{w} = \{\mathbf{x},\mathbf{w}\}\mathbf{w} + \mathbf{x} = \sum_{k=1}^{m} y_k e_k + \sum_{k=1}^{2n} \grave{y}_k \grave{e}_k$, where

$$
\begin{aligned}
y_k &= -2\left(\sum_{j=1}^{m} w_j w_k x_j\right) + x_k + \sum_{j=1}^{n} \grave{w}_{2j-1} w_k \grave{x}_{2j} - \grave{w}_{2j} w_k \grave{x}_{2j-1},\\
\grave{y}_k &= -2\left(\sum_{j=1}^{m} w_j \grave{w}_k x_j\right) + \grave{x}_k + \sum_{j=1}^{n} -\grave{w}_{2j-1} \grave{w}_k \grave{x}_{2j} + \grave{w}_{2j} \grave{w}_k \grave{x}_{2j-1}.
\end{aligned}
$$

Then,

$$
\psi(\mathbf{w})\mathbf{x} = \begin{pmatrix} A(\mathbf{w}) & \grave{B}(\mathbf{w}) \\ \grave{C}(\mathbf{w}) & D(\mathbf{w}) \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{\grave{x}} \end{pmatrix}
$$

where,

$$
A(\mathbf{w}) = -2\begin{pmatrix} w_1^2 & w_2w_1 & \dots & w_mw_1\\ w_1w_2 & w_2^2 & \dots & w_mw_2\\ \vdots & \vdots & \ddots & \vdots\\ w_1w_m & w_2w_m & \dots & w_m^2 \end{pmatrix} + I_m = -2D_{\underline{w}}E_{m\times m}D_{\underline{w}} + I_m,
$$

$$
\grave{B}(\mathbf{w}) = \begin{pmatrix} -\grave{w}_2w_1 & \grave{w}_1w_1 & \dots & -\grave{w}_{2n}w_1 & \grave{w}_{2n-1}w_1\\ -\grave{w}_2w_2 & \grave{w}_1w_2 & \dots & -\grave{w}_{2n}w_2 & \grave{w}_{2n-1}w_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -\grave{w}_2w_m & \grave{w}_1w_m & \dots & -\grave{w}_{2n}w_m & \grave{w}_{2n-1}w_m \end{pmatrix} = D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\,\Omega_{2n},
$$

$$
\grave{C}(\mathbf{w}) = -2\begin{pmatrix} w_1\grave{w}_1 & w_2\grave{w}_1 & \dots & w_m\grave{w}_1\\ w_1\grave{w}_2 & w_2\grave{w}_2 & \dots & w_m\grave{w}_2\\ \vdots & \vdots & \ddots & \vdots\\ w_1\grave{w}_{2n} & w_2\grave{w}_{2n} & \dots & w_m\grave{w}_{2n} \end{pmatrix} = -2D_{\underline{\grave{w}}}E_{2n\times m}D_{\underline{w}},
$$

$$D(\mathbf{w}) = \begin{pmatrix} \grave{w}_2\grave{w}_1 & -\grave{w}_1\grave{w}_1 & \dots & \grave{w}_{2n}\grave{w}_1 & -\grave{w}_{2n-1}\grave{w}_1 \\ \grave{w}_2\grave{w}_2 & -\grave{w}_1\grave{w}_2 & \dots & \grave{w}_{2n}\grave{w}_2 & -\grave{w}_{2n-1}\grave{w}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \grave{w}_2\grave{w}_{2n} & -\grave{w}_1\grave{w}_{2n} & \dots & \grave{w}_{2n}\grave{w}_{2n} & -\grave{w}_{2n-1}\grave{w}_{2n} \end{pmatrix} + I_{2n}$$

$$= D_{\underline{\grave{w}}} E_{2n\times 2n} D_{\underline{\grave{w}}}\,\Omega_{2n} + I_{2n}.$$

□

Remark 4.12. *Algebraic operations with the matrices $A(\mathbf{w}), \grave{B}(\mathbf{w}), \grave{C}(\mathbf{w}), D(\mathbf{w})$ are easy since*

$$E_{p\times m} D_{\underline{w}}^2 E_{m\times q} = \left(\sum_{j=1}^{m} w_j^2\right) E_{p\times q}, \tag{4.13}$$

$$E_{p\times 2n} D_{\underline{\grave{w}}} \Omega_{2n} D_{\underline{\grave{w}}} E_{2n\times q} = 2\left(\sum_{j=1}^{n} \grave{w}_{2j-1}\grave{w}_{2j}\right) E_{p\times q}. \tag{4.14}$$

Proposition 4.13. *Let $\mathbf{w} \in \mathbb{S}(m|2n)(\mathbb{R}\mathfrak{G}_N)$. Then $\psi(\mathbf{w}) \in \mathrm{O}_0$ and* $\operatorname{sdet}(\psi(\mathbf{w})) = -1$.

Proof.

In order to prove that $\psi(\mathbf{w}) \in \mathrm{O}_0$ it suffices to prove that $A(\mathbf{w}), \grave{B}(\mathbf{w}), \grave{C}(\mathbf{w}), D(\mathbf{w})$ satisfy (4.8). This can be easily done using (4.13)-(4.14) and the identity

$$-1 = \mathbf{w}^2 = -\sum_{j=1}^{m} w_j^2 + \sum_{j=1}^{n} \grave{w}_{2j-1}\grave{w}_{2j}.$$

In fact, we have

$$\begin{aligned} A(\mathbf{w})^T A(\mathbf{w}) &= 4D_{\underline{w}} E_{m\times m} D_{\underline{w}}^2 E_{m\times m} D_{\underline{w}} - 4D_{\underline{w}} E_{m\times m} D_{\underline{w}} + I_m \\ &= 4\left(\sum_{j=1}^{m} w_j^2\right) D_{\underline{w}} E_{m\times m} D_{\underline{w}} - 4D_{\underline{w}} E_{m\times m} D_{\underline{w}} + I_m, \end{aligned}$$

and

$$\grave{C}(\mathbf{w})^T \Omega_{2n} \grave{C}(\mathbf{w}) = 4D_{\underline{w}} E_{m\times 2n} D_{\underline{\grave{w}}} \Omega_{2n} D_{\underline{\grave{w}}} E_{2n\times m} D_{\underline{w}} = 8\left(\sum_{j=1}^{n} \grave{w}_{2j-1}\grave{w}_{2j}\right) D_{\underline{w}} E_{m\times m} D_{\underline{w}}.$$

Then,

$$A(\mathbf{w})^T A(\mathbf{w}) - \frac{1}{2}\grave{C}(\mathbf{w})^T \Omega_{2n} \grave{C}(\mathbf{w}) = 4\left[\sum_{j=1}^{m} w_j^2 - 1 - \sum_{j=1}^{n} \grave{w}_{2j-1}\grave{w}_{2j}\right] D_{\underline{w}} E_{m\times m} D_{\underline{w}} + I_m = I_m.$$

Also,

$$\begin{aligned}A(\mathbf{w})^T \grave{B}(\mathbf{w}) &= -2D_{\underline{w}}E_{m\times m}D^2_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} + D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\,\Omega_{2n}\\ &= -2\left(\sum_{j=1}^{m} w_j^2\right) D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} + D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\,\Omega_{2n},\end{aligned}$$

and

$$\begin{aligned}\grave{C}(\mathbf{w})^T\Omega_{2n}D(\mathbf{w}) &= -2D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\Omega_{2n}D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} - 2D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\Omega_{2n}\\ &= -4\left(\sum_{j=1}^{n} \grave{w}_{2j-1}\grave{w}_{2j}\right) D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} - 2D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\Omega_{2n}.\end{aligned}$$

Hence,

$$A(\mathbf{w})^T\grave{B}(\mathbf{w}) - \frac{1}{2}\grave{C}(\mathbf{w})^T\Omega_{2n}D(\mathbf{w}) = 2\left[-\sum_{j=1}^{m} w_j^2 + 1 + \sum_{j=1}^{n}\grave{w}_{2j-1}\grave{w}_{2j}\right] D_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} = 0.$$

In the same way we have

$$\grave{B}(\mathbf{w})^T\grave{B}(\mathbf{w}) = -\Omega_{2n}D_{\underline{\grave{w}}}E_{2n\times m}D^2_{\underline{w}}E_{m\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} = -\left(\sum_{j=1}^{m} w_j^2\right)\Omega_{2n}D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n},$$

and

$$\begin{aligned}D(\mathbf{w})^T\Omega_{2n}D(\mathbf{w}) &= \Omega_{2n}D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n}D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} + 2\Omega_{2n}D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} + \Omega_{2n}\\ &= 2\left(\sum_{j=1}^{n}\grave{w}_{2j-1}\grave{w}_{2j} + 1\right)\Omega_{2n}D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} + \Omega_{2n},\end{aligned}$$

whence

$$\begin{aligned}\grave{B}(\mathbf{w})^T\grave{B}(\mathbf{w}) + \frac{1}{2}D^T(\mathbf{w})\Omega_{2n}D(\mathbf{w}) &= \left[-\sum_{j=1}^{m} w_j^2 + 1 + \sum_{j=1}^{n}\grave{w}_{2j-1}\grave{w}_{2j}\right]\Omega_{2n}D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n} + \frac{1}{2}\Omega_{2n}\\ &= \frac{1}{2}\Omega_{2n}.\end{aligned}$$

Then, $A(\mathbf{w}), \grave{B}(\mathbf{w}), \grave{C}(\mathbf{w}), D(\mathbf{w})$ satisfy (4.8) and in consequence, $\psi(\mathbf{w}) \in \mathrm{O}_0$. To prove that $\operatorname{sdet}(\psi(\mathbf{w})) = -1$, first observe that $\psi(\mathbf{w}) = \psi(\mathbf{w})^{-1}$ since

$$\psi(\mathbf{w}) \circ \psi(\mathbf{w})[\mathbf{x}] = \mathbf{w}\mathbf{w}\mathbf{x}\mathbf{w}\mathbf{w} = \mathbf{x}.$$

Hence, due to Theorem 4.1 we obtain

$$A(\mathbf{w}) = \left(A(\mathbf{w}) - \grave{B}(\mathbf{w})D(\mathbf{w})^{-1}\grave{C}(\mathbf{w})\right)^{-1},$$

yielding

$$\operatorname{sdet}(\psi(\mathbf{w})) = \frac{\det\left[A(\mathbf{w}) - \grave{B}(\mathbf{w})D(\mathbf{w})^{-1}\grave{C}(\mathbf{w})\right]}{\det[D(\mathbf{w})]} = \frac{1}{\det[A(\mathbf{w})]\det[D(\mathbf{w})]}.$$

We will compute $\det[D(\mathbf{w})]$ using the formula $\det[D(\mathbf{w})] = \exp(\operatorname{tr}\ln D(\mathbf{w}))$ and the fact that $D(\mathbf{w}) - I_{2n}$ is a nilpotent matrix. Observe that

$$\ln D(\mathbf{w}) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{(D(\mathbf{w}) - I_{2n})^j}{j} = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{\left(D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n}\right)^j}{j}.$$

It follows from (4.14) that

$$\left(D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n}\right)^j = 2^{j-1}\left(\sum_{j=1}^{n}\grave{w}_{2j-1}\grave{w}_{2j}\right)^{j-1} D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n}.$$

Then,

$$\ln D(\mathbf{w}) = \left[\sum_{j=1}^{\infty}(-1)^{j+1}\frac{2^{j-1}\left(\sum_{j=1}^{n}\grave{w}_{2j-1}\grave{w}_{2j}\right)^{j-1}}{j}\right] D_{\underline{\grave{w}}}E_{2n\times 2n}D_{\underline{\grave{w}}}\Omega_{2n}$$

and in consequence,

$$\operatorname{tr}\ln D(\mathbf{w}) = -\sum_{j=1}^{\infty}(-1)^{j+1}\frac{2^{j}\left(\sum_{j=1}^{n}\grave{w}_{2j-1}\grave{w}_{2j}\right)^{j}}{j} = -\ln\left(1 + 2\sum_{j=1}^{n}\grave{w}_{2j-1}\grave{w}_{2j}\right).$$

Hence

$$\det(D(\mathbf{w})) = \frac{1}{1 + 2\sum\limits_{j=1}^{n}\grave{w}_{2j-1}\grave{w}_{2j}}.$$

Similar computations yield

$$\det(A(\mathbf{w})) = 1 - 2\sum_{j=1}^{m} w_j^2$$

which shows that

$$\operatorname{sdet}(\psi(\mathbf{w})) = \frac{1 + 2\sum_{j=1}^{n}\grave{w}_{2j-1}\grave{w}_{2j}}{1 - 2\sum_{j=1}^{m} w_j^2} = -1.$$

□

We can now define the *bosonic Pin group* in superspace as

$$\operatorname{Pin}_b(m|2n)(\mathfrak{G}_N) = \{\mathbf{w}_1\cdots\mathbf{w}_k \,:\, \mathbf{w}_j \in \mathbb{S}(m|2n)(\mathbb{R}\mathfrak{G}_N), k\in\mathbb{N}\},$$

and extend the map ψ to a Lie group homomorphism $\psi : \mathrm{Pin}_b(m|2n)(\mathfrak{G}_N) \to \mathrm{O}_0$ by

$$\psi(\mathbf{w}_1 \cdots \mathbf{w}_k)[\mathbf{x}] = \mathbf{w}_1 \cdots \mathbf{w}_k \, \mathbf{x} \, \mathbf{w}_k \cdots \mathbf{w}_1 = \psi(\mathbf{w}_1) \circ \cdots \circ \psi(\mathbf{w}_k)[\mathbf{x}].$$

It is clearly seen that the restriction of ψ to the *bosonic spin group*, defined as

$$\mathrm{Spin}_b(m|2n)(\mathfrak{G}_N) = \{\mathbf{w}_1 \cdots \mathbf{w}_{2k} : \mathbf{w}_j \in \mathbb{S}(m|2n)(\mathbb{R}\mathfrak{G}_N), k \in \mathbb{N}\},$$

takes values in the subgroup $\mathrm{SO}_0 \subset \mathrm{O}_0$.

In the classical case, the Pin group and the Spin group are double coverings of the groups $\mathrm{O}(m)$ and $\mathrm{SO}(m)$ respectively. A natural question in this setting is whether $\mathrm{Pin}_b(m|2n)(\mathfrak{G}_N)$ and $\mathrm{Spin}_b(m|2n)(\mathfrak{G}_N)$ cover the groups O_0 and SO_0. The answer to this question is negative and the main reason for this is that the real projection of every vector $\mathbf{w} \in \mathbb{S}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ is in the unitary sphere $\mathbb{S}^{m-1}$ of $\mathbb{R}^m$, i.e.,

$$[\mathbf{w}]_0 = \sum_{j=1}^{m} [w_j]_0 e_j \qquad \text{and} \qquad [\mathbf{w}]_0^2 = -1.$$

Then, the real projection of $\psi\left(\mathrm{Pin}_b(m|2n)(\mathfrak{G}_N)\right)$ is just $\mathrm{O}(m)$, while $[\mathrm{O}_0]_0 = \mathrm{O}(m) \times \mathrm{Sp}_\Omega(2n)$. This means that these bosonic versions of Pin and Spin do not describe the symplectic parts of O_0 and SO_0. This phenomenon is due to the natural structure of supervectors: their real projections belong to a space with an orthogonal structure while the symplectic structure plays no rôle. Up to a nilpotent vector, they are classical Clifford vectors, whence it is impossible to generate by this approach the real symplectic geometry that is also present in the structure of O_0 and SO_0. That is why we have chosen the name of "bosonic" Pin and "bosonic" Spin groups. This also explains why we had to extend the space of superbivectors in section 4.2.1. The ordinary superbivectors in $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ are generated over $\mathbb{R}\mathfrak{G}_N^{(ev)}$ by the wedge product of supervectors. Then, they can only describe $\mathfrak{so}(m)$ and not $\mathfrak{sp}_\Omega(2n)$ and in consequence, they do not cover $\mathfrak{so}_0$.

As in the classical setting (see [47]), it is possible to obtain the following result that shows, from another point of view, that $\mathrm{Pin}_b(m|2n)(\mathfrak{G}_N)$ cannot completely describe O_0.

Proposition 4.14. *The Lie algebra of* $\mathrm{Pin}_b(m|2n)(\mathfrak{G}_N)$ *is included in* $\mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$.

Proof.

Let $\gamma(t) = \mathbf{w}_1(t) \cdots \mathbf{w}_k(t)$ be a path in $\mathrm{Pin}_b(m|2n)(\mathfrak{G}_N)$ with $\mathbf{w}_j(t) \in \mathbb{S}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ for every $t \in \mathbb{R}$ and $\gamma(0) = 1$. The tangent to γ at $t = 0$ is

$$\left.\frac{d\gamma}{dt}\right|_{t=0} = \sum_{j=1}^{k} \mathbf{w}_1(0) \cdots \mathbf{w}_j'(0) \cdots \mathbf{w}_k(0).$$

We will show that each summand of $\left.\frac{d\gamma}{dt}\right|_{t=0}$ belongs to $\mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$.

For $j = 1$ we have $\mathbf{w}_1'(0)\mathbf{w}_2(0)\cdots\mathbf{w}_k(0) = -\mathbf{w}_1'(0)\mathbf{w}_1(0)$. But $\mathbf{w}_1(t)\mathbf{w}_1(t) \equiv -1$ implies

$$\{\mathbf{w}_1'(0), \mathbf{w}_1(0)\} = \mathbf{w}_1'(0)\mathbf{w}_1(0) + \mathbf{w}_1(0)\mathbf{w}_1'(0) = 0.$$

Then $\mathbf{w}_1'(0)\mathbf{w}_1(0) = \frac{1}{2}\{\mathbf{w}_1'(0), \mathbf{w}_1(0)\} + \mathbf{w}_1'(0) \wedge \mathbf{w}_1(0) = \mathbf{w}_1'(0) \wedge \mathbf{w}_1(0) \in \mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$. For $j = 2$,

$$\begin{aligned}\mathbf{w}_1(0)\mathbf{w}_2'(0)\cdots\mathbf{w}_k(0) &= \mathbf{w}_1(0)\mathbf{w}_2'(0)\mathbf{w}_2(0)\mathbf{w}_1(0)\\ &= -\left[\mathbf{w}_1(0)\mathbf{w}_2'(0)\mathbf{w}_1(0)\right]\left[\mathbf{w}_1(0)\mathbf{w}_2(0)\mathbf{w}_1(0)\right]\\ &= -\psi(\mathbf{w}_1(0))[\mathbf{w}_2'(0)]\ \psi(\mathbf{w}_1(0))[\mathbf{w}_2(0)].\end{aligned}$$

But $\psi(\mathbf{w}_1(0)) \in \mathrm{O}_0$ preserves the inner product (see remark 4.6), so

$$\mathbf{w}_1(0)\mathbf{w}_2'(0)\cdots\mathbf{w}_k(0) = \psi(\mathbf{w}_1(0))[\mathbf{w}_2'(0)] \wedge \psi(\mathbf{w}_1(0))[\mathbf{w}_2(0)] \in \mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N).$$

We can proceed similarly for every $j = 3, \ldots, k$. □

4.4.2 A proper definition for the group $\mathrm{Spin}(m|2n)(\mathfrak{G}_N)$

The above approach shows that the radial algebra setting does not contain a suitable realization of SO_0 in the Clifford superspace framework. Observe that the Clifford representation of $\mathfrak{so}_0$ given by $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ lies outside of the radial algebra $\mathbb{R}_{m|2n}(\mathfrak{G}_N)$, which suggests that something similar should happen with the corresponding Lie group SO_0. In this case, a proper definition for the Spin group would be generated by the exponentials (in general contained in $T(V)/I$) of all elements in $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$, i.e.

$$\mathrm{Spin}(m|2n)(\mathfrak{G}_N) := \left\{e^{B_1}\cdots e^{B_k} : B_1, \ldots, B_k \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N), k \in \mathbb{N}\right\},$$

and the action of this group on supervector variables $\mathbf{x} \in \mathbf{S}$ is given by the group homomorphism $h : \mathrm{Spin}(m|2n)(\mathfrak{G}_N) \to \mathrm{SO}_0$ defined by

$$h(e^B)[\mathbf{x}] = e^B\mathbf{x}e^{-B}, \qquad B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N),\ \mathbf{x} \in \mathbf{S}. \tag{4.15}$$

In fact, for every extended superbivector B, $h(e^B)$ maps supervector variables into new supervector variables and admits a supermatrix representation in $\mathrm{Mat}(m|2n)(\mathbb{R}\mathfrak{G}_N)$ belonging to SO_0. This is summarized below.

Proposition 4.15. *Let $B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$. Then,*

$$h(e^B)[\mathbf{x}] = e^{\phi(B)}\mathbf{x}, \qquad \forall \mathbf{x} \in \mathbf{S}.$$

Proof.

In the associative algebra $\mathcal{A}_{m,2n} \otimes \mathbb{R}\mathfrak{G}_N$, the identity

$$\underbrace{[B,[B\ldots[B,\mathbf{x}]\ldots]]}_{k} = \sum_{j=0}^{k} \binom{k}{j} B^j \mathbf{x}(-B)^{k-j}$$

holds. Then,

$$\begin{aligned}
h(e^B)[\mathbf{x}] = e^B\mathbf{x}e^{-B} &= \left(\sum_{k=0}^{\infty}\frac{B^k}{k!}\right)\mathbf{x}\left(\sum_{k=0}^{\infty}\frac{(-B)^k}{k!}\right)\\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{B^j}{j!}\,\mathbf{x}\,\frac{(-B)^{k-j}}{(k-j)!}\right)\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{j=0}^{k}\binom{k}{j}B^j\mathbf{x}(-B)^{k-j}\right)\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\underbrace{[B,[B\ldots[B,\mathbf{x}]\ldots]]}_{k}\\
&= \sum_{k=0}^{\infty}\frac{\phi(B)^k\mathbf{x}}{k!} = e^{\phi(B)}\mathbf{x}.
\end{aligned}$$

□

Remark 4.13. *Proposition 4.15 means that the Lie algebra isomorphism* $\phi : \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N) \rightarrow \mathfrak{so}_0$ *is the derivative at the origin (or infinitesimal representation) of the Lie group homomorphism* $h : \mathrm{Spin}(m|2n)(\mathfrak{G}_N) \rightarrow \mathrm{SO}_0$*, i.e.,*

$$e^{t\phi(B)} = h(e^{tB}) \qquad \forall t \in \mathbb{R},\ B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N). \tag{4.16}$$

On account of the connectedness of SO_0 it can be shown that the group $\mathrm{Spin}(m|2n)(\mathfrak{G}_N)$ is a realization of SO_0 in $T(V)/I$ through the representation h.

Theorem 4.6. *For every* $M \in \mathrm{SO}_0$ *there exists an element* $s \in \mathrm{Spin}(m|2n)(\mathfrak{G}_N)$ *such that* $h(s) = M$.

Proof.

Since SO_0 is a connected Lie group (Proposition 4.10), for every supermatrix $M \in \mathrm{SO}_0$ there exist $X_1, \ldots, X_k \in \mathfrak{so}_0$ such that $e^{X_1}\cdots e^{X_k} = M$, see Corollary 3.47 in [50]. Taking $B_1, \ldots, B_k \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ such that $\phi(B_j) = X_j$, $j = 1, \ldots, k$, we obtain

$$M\mathbf{x} = e^{X_1}\cdots e^{X_k}\,\mathbf{x} = e^{\phi(B_1)}\cdots e^{\phi(B_k)}\,\mathbf{x} = h(e^{B_1})\circ\cdots\circ h(e^{B_k})[\mathbf{x}] = h(e^{B_1}\cdots e^{B_k})[\mathbf{x}].$$

Then, $s = e^{B_1}\cdots e^{B_k} \in \mathrm{Spin}(m|2n)(\mathfrak{G}_N)$ satisfies $h(s) = M$. □

Table 4.1 below provides a comparative overview concerning the Spin group in both settings: Euclidean Clifford analysis and its extension to superspace.

	Euclidean Clifford analysis	Clifford analysis in superspace
Bilinear form	$\langle \underline{x}, \underline{y} \rangle_{\mathbb{R}} = \sum_{j=1}^{m} x_j y_j$	$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}} = \sum_{j=1}^{m} x_j y_j - \frac{1}{2} \sum_{j=1}^{n} (\grave{x}_{2j-1} \grave{y}_{2j} - \grave{x}_{2j} \grave{y}_{2j-1})$
Invariance	$\mathrm{SO}(m)$	$\mathrm{SO}_0(m\|2n)(\mathbb{R}\mathfrak{G}_N)$
Body	$\mathrm{SO}(m)$	$\mathrm{SO}(m) \times \mathrm{Sp}(2n)$
Lie algebra	$\mathfrak{so}(m)$	$\mathfrak{so}_0(m\|2n)(\mathbb{R}\mathfrak{G}_N)$
Real dimension	$\frac{m(m-1)}{2}$	$2^{N-1} \left(\frac{m(m-1)}{2} + 2mn + n(2n+1) \right)$
Bivectors	$\sum_{1 \leq j < k \leq m} b_{j,k}\, e_j e_k$	$\sum_{1 \leq j < k \leq m} b_{j,k}\, e_j e_k + \sum_{\substack{1 \leq j \leq m \\ 1 \leq k \leq 2n}} \grave{b}_{j,k}\, e_j \grave{e}_k + \sum_{1 \leq j \leq k \leq 2n} B_{j,k}\, \grave{e}_j \odot \grave{e}_k$
Iwasawa decomposition	$M_0 = e^X$, $X \in \mathfrak{so}(m)$	$M = e^X e^Y e^{\mathbf{Z}}$, $X \in \mathfrak{so}(m) \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)]$, $Y \in \{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathrm{Sym}(2n)]$, $\mathbf{Z} \in \mathfrak{so}_0(m\|2n)(\mathbb{R}\mathfrak{G}_N^+)$
Spin group/ elements	$\mathrm{Spin}(m)$ $\underline{v}_1 \cdots \underline{v}_{2k}$, $v_j \in \mathbb{S}^{m-1}$	$\mathrm{Spin}(m\|2n)(\mathfrak{G}_N)$ $e^{B_1} \cdots e^{B_k}$, $B_j \in \mathbb{R}^{(2)E}_{m\|2n}(\mathfrak{G}_N)$

Table 4.1: Comparative overview of the Spin realization of the rotation group.

The decomposition of SO_0 given in Theorem 4.4 provides the exact number of exponentials of extended superbivectors to be considered in $\mathrm{Spin}(m|2n)(\mathfrak{G}_N)$ in order to cover the whole group SO_0. If we consider the subspaces Ξ_1, Ξ_2, Ξ_3 of $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ given by

$$\begin{aligned} \Xi_1 &= \phi^{-1}\big(\mathfrak{so}(m) \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)]\big), & \dim \Xi_1 &= \frac{m(m-1)}{2} + n^2, \\ \Xi_2 &= \phi^{-1}\big(\{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathrm{Sym}(2n)]\big), & \dim \Xi_2 &= n^2 + n, \\ \Xi_3 &= \phi^{-1}\big(\mathfrak{so}_0(m|2n)(\mathbb{R}\mathfrak{G}_N^+)\big) & \dim \Xi_3 &= \dim \mathfrak{so}_0 - \frac{m(m-1)}{2} - n(2n+1), \end{aligned} \tag{4.17}$$

we get the decomposition $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N) = \Xi_1 \oplus \Xi_2 \oplus \Xi_3$, leading to the subset

$$\Xi = \exp(\Xi_1) \exp(\Xi_2) \exp(\Xi_3) \subset \mathrm{Spin}(m|2n)(\mathfrak{G}_N),$$

which suffices for describing SO_0. Indeed, from Theorem 4.4 it follows that the restriction $h : \Xi \to \mathrm{SO}_0$ is surjective. We now investigate the explicit form of the superbivectors in each of the subspaces Ξ_1, Ξ_2 and Ξ_3.

Proposition 4.16. *The following statements hold.*

(i) A basis for Ξ_1 *is* $\begin{cases} e_j e_k, & 1 \leq j < k \leq m, \\ \grave{e}_{2j-1} \odot \grave{e}_{2k-1} + \grave{e}_{2j} \odot \grave{e}_{2k}, & 1 \leq j \leq k \leq n, \\ \grave{e}_{2j-1} \odot \grave{e}_{2k} - \grave{e}_{2j} \odot \grave{e}_{2k-1}, & 1 \leq j < k \leq n. \end{cases}$

(ii) A basis for Ξ_2 *is:* $\begin{cases} \grave{e}_{2j-1} \odot \grave{e}_{2j}, & 1 \leq j \leq n, \\ \grave{e}_{2j-1} \odot \grave{e}_{2k-1} - \grave{e}_{2j} \odot \grave{e}_{2k}, & 1 \leq j \leq k \leq n, \\ \grave{e}_{2j-1} \odot \grave{e}_{2k} + \grave{e}_{2j} \odot \grave{e}_{2k-1}, & 1 \leq j < k \leq n. \end{cases}$

(iii) Ξ_3 *consists of all elements of the form (4.7) with* $b_{j,k}, B_{j,k} \in \mathbb{R}\mathfrak{G}_N^{(ev)} \cap \mathbb{R}\mathfrak{G}_N^{+}$ *and* $\grave{b}_{j,k} \in \mathbb{R}\mathfrak{G}_N^{(odd)}$.

Proof.

We first recall that a basis for the Lie algebra $\mathfrak{sp}_\Omega(2n)$ is given by the elements

$$\begin{aligned} A_{j,k} &:= E_{2j,2k-1} + E_{2k,2j-1}, && 1 \leq j \leq k \leq n, \\ B_{j,k} &:= E_{2j-1,2k} + E_{2k-1,2j}, && 1 \leq j \leq k \leq n, \\ C_{j,k} &:= E_{2k,2j} - E_{2j-1,2k-1}, && 1 \leq j \leq k \leq n, \\ D_{j,k} &:= E_{2j,2k} - E_{2k-1,2j-1}, && 1 \leq j < k \leq n, \end{aligned}$$

where the matrices $E_{j,k} \in \mathbb{R}^{n\times n}$ are defined as in Lemma 4.2. It holds that

$$\begin{aligned} A_{j,k}^T &= B_{j,k}, && 1 \leq j \leq k \leq n, \\ C_{j,k}^T &= D_{j,k}, && 1 \leq j < k \leq n, \\ C_{j,j}^T &= C_{j,j}, && 1 \leq j \leq n. \end{aligned}$$

Hence, for every matrix $D_0 \in \mathfrak{sp}_\Omega(2n)$ we have

$$\begin{aligned} D_0 &= \sum_{1\leq j\leq k\leq n} (a_{j,k}A_{j,k} + b_{j,k}B_{j,k} + c_{j,k}C_{j,k}) + \sum_{1\leq j<k\leq n} d_{j,k}D_{j,k}, \\ D_0^T &= \sum_{1\leq j\leq k\leq n} (a_{j,k}B_{j,k} + b_{j,k}A_{j,k}) + \sum_{1\leq j<k\leq n} (c_{j,k}D_{j,k} + d_{j,k}C_{j,k}) + \sum_{j=1}^{n} c_{j,j}C_{j,j}, \end{aligned}$$

where $a_{j,k}, b_{j,k}, c_{j,k}, d_{j,k} \in \mathbb{R}$.

(i) From the previous equalities we get that $D_0^T = -D_0$ if and only if

$$D_0 = \sum_{1\leq j\leq k\leq n} a_{j,k} \left(A_{j,k} - B_{j,k}\right) + \sum_{1\leq j<k\leq n} c_{j,k} \left(C_{j,k} - D_{j,k}\right).$$

Then, a basis for $\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)$ is

$$\{A_{j,k} - B_{j,k} : 1 \leq j \leq k \leq n\} \cup \{C_{j,k} - D_{j,k} : 1 \leq j < k \leq n\}.$$

The remainder of the proof directly follows from Lemma 4.2.

(ii) In this case we have that $D_0^T = D_0$ if and only if

$$D_0 = \sum_{1\leq j\leq k\leq n} a_{j,k}\left(A_{j,k}+B_{j,k}\right) + \sum_{1\leq j<k\leq n} c_{j,k}\left(C_{j,k}+D_{j,k}\right) + \sum_{j=1}^{n} c_{j,j}C_{j,j},$$

whence a basis for $\mathfrak{sp}_\Omega(2n)\cap \mathrm{Sym}(2n)$ is

$\{A_{j,k}+B_{j,k} : 1\leq j\leq k\leq n\}\cup\{C_{j,j} : 1\leq j\leq n\}\cup\{C_{j,k}+D_{j,k} : 1\leq j<k\leq n\}.$

The remainder of the proof directly follows from Lemma 4.2.

$iii)$ This trivially follows from Lemma 4.2. □

4.4.3 Spin covering of the group SO_0

It is a natural question in this setting whether the spin group still is a double covering of the group of rotations, as it is in classical Clifford analysis. In other words, we will investigate how many times $\Xi\subset \mathrm{Spin}(m|2n)(\mathfrak{G}_N)$ covers SO_0, or more precisely, we will determine the cardinality of the set $\{s\in\Xi : h(s)=M\}$ given a certain fixed element $M\in \mathrm{SO}_0$.

From Proposition 4.15 we have that the representation h of an element

$$s=e^{B_1}e^{B_2}e^{B_3}\in\Xi, \qquad\qquad B_j\in\Xi_j,$$

has the form

$$h(s)=e^{\phi(B_1)}e^{\phi(B_2)}e^{\phi(B_3)}.$$

Following the decomposition

$$M=e^Xe^Ye^{\mathbf{Z}}$$

given in Theorem 4.4 for $M\in\mathrm{SO}_0$, we get that $h(s)=M$ if and only if

$$e^{\phi(B_1)}=e^X, \qquad B_2=\phi^{-1}(Y), \qquad B_3=\phi^{-1}(\mathbf{Z}).$$

Then, the cardinality of $\{s\in\Xi : h(s)=M\}$ only depends on the number of extended superbivectors $B_1\in\Xi_1$ that satisfy $e^{\phi(B_1)}=e^X$. It reduces our analysis to finding the kernel of the restriction

$$h|_{\exp(\Xi_1)} : \exp(\Xi_1)\to \mathrm{SO}(m)\times[\mathrm{Sp}_\Omega(2n)\cap\mathrm{SO}(2n)]$$

of the Lie group homomorphism h to $\exp(\Xi_1)$. This kernel is given by

$$\ker h|_{\exp(\Xi_1)}=\{e^B \;:\; e^{\phi(B)}=I_{m+2n},\, B\in\Xi_1\}.$$

We recall, from Proposition 4.16, that $B\in\Xi_1$ may be written as $B=B_o+B_s$ where $B_o\in\mathbb{R}^{(2)}_{0,m}$ is a classical real bivector and

$$B_s\in\phi^{-1}\left(\{0_m\}\times[\mathfrak{sp}_\Omega(2n)\cap\mathfrak{so}(2n)]\right).$$

The components B_o, B_s commute and in consequence, $e^B = e^{B_o}e^{B_s}$. Consider the projections ϕ_o and ϕ_s of ϕ over the algebra of classical bivectors $\mathbb{R}_{0,m}^{(2)}$ and over the algebra $\phi^{-1}\left(\{0_m\}\times[\mathfrak{sp}_\Omega(2n)\cap\mathfrak{so}(2n)]\right)$ respectively, i.e.

$$\phi_o : \mathbb{R}_{0,m}^{(2)} \to \mathfrak{so}(m), \qquad \phi_s : \phi^{-1}\left(\{0_m\}\times[\mathfrak{sp}_\Omega(2n)\cap\mathfrak{so}(2n)]\right) \to \mathfrak{sp}_\Omega(2n)\cap\mathfrak{so}(2n),$$

where

$$\phi(B) = \begin{pmatrix} \phi_o(B_o) & 0 \\ 0 & \phi_s(B_s) \end{pmatrix}, \qquad\qquad B \in \Xi_1.$$

Or equivalently:

$$\begin{cases} \phi_o(B_o)[\underline{x}] = [B_o, \underline{x}], \\ \phi_s(B_s)[\underline{x}\grave{}] = [B_s, \underline{x}\grave{}\,], \end{cases} \qquad\qquad \mathbf{x} = \underline{x} + \underline{x}\grave{} \in \mathbf{S}.$$

Hence $e^{\phi(B)} = I_{m+2n}$ if and only if $e^{\phi_o(B_o)} = I_m$ and $e^{\phi_s(B_s)} = I_{2n}$. For the first condition, we know from classical Clifford analysis that $\mathrm{Spin}(m) = \{e^B : B \in \mathbb{R}_{0,m}^{(2)}\}$ is a double covering of $\mathrm{SO}(m)$ and in consequence,

$$e^{\phi_o(B_o)} = I_m,$$

implies

$$e^{B_0} = \pm 1.$$

Let us now compute all possible values for e^{B_s} for which $e^{\phi_s(B_s)} = I_{2n}$. To that end, we need the following linear algebra result.

Proposition 4.17. *Every matrix $D_0 \in \mathfrak{so}(2n)\cap\mathfrak{sp}_\Omega(2n)$ can be written in the form $D_0 = R\Sigma R^T$ where $R \in \mathrm{SO}(2n)\cap\mathrm{Sp}_\Omega(2n)$ and*

$$\Sigma = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix}, \qquad \theta_j \in \mathbb{R}, \quad j = 1, \ldots, n. \tag{4.18}$$

Proof.

The map $\Psi(D_0) = \frac{1}{2}QD_0\left(Q^T\right)^c$, where

$$Q = \begin{pmatrix} 1 & i & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & i & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & i \end{pmatrix} \in \mathbb{C}^{n\times 2n},$$

is a Lie group isomorphism between $\mathrm{SO}(2n)\cap\mathrm{Sp}_\Omega(2n)$ and $\mathrm{U}(n)$. It is easily proven that Ψ is its own infinitesimal representation on the Lie algebra level, and in consequence, a Lie algebra isomorphism between $\mathfrak{so}(2n)\cap\mathfrak{sp}_\Omega(2n)$ and $\mathfrak{u}(n)$. The inverse of Ψ is given by

$$\Psi^{-1}(L) = \frac{1}{2}\left[\left(Q^T\right)^c\, L\, Q + Q^T\, L^c\, Q^c\right].$$

For every $D_0 \in \mathfrak{so}(2n)\cap\mathfrak{sp}_\Omega(2n)$, let us consider the skew-Hermitian matrix $L = \Psi(D_0) \in \mathfrak{u}(n)$. It is known that every skew-Hermitian matrix is unitarily diagonalizable and all its eigenvalues are purely imaginary, see [54]. Hence, $L = \Psi(D_0)$ can be written as

$$L = U\mathfrak{D}\left(U^T\right)^c$$

where $U \in \mathrm{U}(n)$ and

$$\mathfrak{D} = \mathrm{diag}(-i\theta_1, \ldots, -i\theta_n), \qquad\qquad \theta_j \in \mathbb{R}.$$

Then,

$$D_0 = \Psi^{-1}(L) = R\Sigma R^T$$

where $R = \Psi^{-1}(U) \in \mathrm{SO}(2n)\cap\mathrm{Sp}_\Omega(2n)$ and $\Sigma = \Psi^{-1}(\mathfrak{D})$ has the form (4.18). $\square$

Since $\phi_s(B_s) \in \mathfrak{so}(2n)\cap\mathfrak{sp}_\Omega(2n)$, we have $\phi_s(B_s) = R\Sigma R^T$ as in the previous proposition. Hence, $e^{\phi_s(B_s)} = Re^\Sigma R^T$ where e^Σ is the block-diagonal matrix

$$e^\Sigma = \mathrm{diag}(e^{\theta_1\Omega_2}, \ldots, e^{\theta_n\Omega_2}) \quad \text{with} \quad e^{\theta_j\Omega_2} = \cos\theta_j I_2 + \sin\theta_j\Omega_2.$$

Hence $e^{\phi_s(B_s)} = I_{2n}$ if and only if $e^\Sigma = I_{2n}$, which is seen to be equivalent to $\theta_j = 2k_j\pi$, $k_j \in \mathbb{Z}$ $(j = 1, \ldots, n)$, or to

$$\Sigma = \sum_{j=1}^{n} 2k_j\pi\left(E_{2j-1,2j} - E_{2j,2j-1}\right), \quad k_j \in \mathbb{Z}\ (j = 1, \ldots, n).$$

Now, $\mathrm{SO}(2n)\cap\mathrm{Sp}_\Omega(2n)$ being connected and compact, there exists $B_R \in \phi^{-1}(\mathfrak{so}(2n)\cap\mathfrak{sp}_\Omega(2n))$ such that $R = e^{\phi(B_R)}$. We recall that the h-action leaves any multivector structure invariant, in particular,

$$h[e^B]\left(\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)\right) \subset \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$$

for every $B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$. Then, using the fact that ϕ is the derivative at the origin of h, we get that the extended superbivector

$$h(e^{B_R})[\phi^{-1}(\Sigma)] = e^{B_R}\phi^{-1}(\Sigma)e^{-B_R}$$

is such that

$$\phi\left(e^{B_R}\phi^{-1}(\Sigma)e^{-B_R}\right) = e^{\phi(B_R)}\Sigma e^{-\phi(B_R)} = R\Sigma R^T = \phi(B_s),$$

implying that $B_s = e^{B_R}\phi^{-1}(\Sigma)e^{-B_R}$. Then, in order to compute $e^{B_s} = e^{B_R}\, e^{\phi^{-1}(\Sigma)}\, e^{-B_R}$, we first have to compute $e^{\phi^{-1}(\Sigma)}$. Following the correspondences given in Lemma 4.2 we get

$$\phi^{-1}(\Sigma) = \sum_{j=1}^{n} 2k_j\pi\, \phi^{-1}\left(E_{2j-1,2j} - E_{2j,2j-1}\right) = \sum_{j=1}^{n} k_j\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right).$$

and, in consequence

$$e^{\phi^{-1}(\Sigma)} = \exp\left(\sum_{j=1}^{n} k_j\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right)\right) = \prod_{j=1}^{n} \exp\left[k_j\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right)\right]. \tag{4.19}$$

Let us compute $\exp\left[\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right)\right]$, $j \in \{1, \ldots, n\}$. Consider

$$\mathbf{a} = 2\mathfrak{f}_j = \grave{e}_{2j-1} - i\grave{e}_{2j}, \qquad \mathbf{b} = -2\grave{\mathfrak{f}_j} = \grave{e}_{2j-1} + i\grave{e}_{2j};$$

where i is the usual imaginary unit in $\mathbb{C}$. It is clear that

$$\mathbf{ab} = \grave{e}_{2j-1}^2 + \grave{e}_{2j}^2 + i(\grave{e}_{2j-1}\grave{e}_{2j} - \grave{e}_{2j}\grave{e}_{2j-1}) = \grave{e}_{2j-1}^2 + \grave{e}_{2j}^2 + i,$$

and $[\mathbf{a}, \mathbf{b}] = 2i$ which is a commuting element. Then,

$$\exp\left[\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right)\right] = \exp\left(\pi\,\mathbf{ab} - i\pi\right) = -\exp\left(\pi\,\mathbf{ab}\right).$$

In order to compute $\exp(\pi\,\mathbf{ab})$ we first prove the following results.

Lemma 4.4. *For every $k \in \mathbb{N}$ the following relations hold.*

$$(i)\quad \left[\mathbf{b}^k, \mathbf{a}\right] = -2ik\,\mathbf{b}^{k-1}, \qquad (ii)\quad \mathbf{a}^k\mathbf{b}^k\mathbf{ab} = \mathbf{a}^{k+1}\mathbf{b}^{k+1} - 2ik\,\mathbf{a}^k\mathbf{b}^k.$$

Proof.

(*i*) We proceed by induction. For $k = 1$ we get $[\mathbf{b}, \mathbf{a}] = -2i$ which obviously is true. Now assume that (*i*) is true for $k \geq 1$, then for $k + 1$ we get

$$\mathbf{b}^{k+1}\mathbf{a} = \mathbf{b}\left(\mathbf{b}^k\mathbf{a}\right) = \mathbf{bab}^k - 2ik\,\mathbf{b}^k = (\mathbf{ab} - 2i)\mathbf{b}^k - 2ik\,\mathbf{b}^k = \mathbf{ab}^{k+1} - 2i(k+1)\mathbf{b}^k.$$

(*ii*) From (*i*) we get $\mathbf{a}^k\mathbf{b}^k\mathbf{ab} = \mathbf{a}^k\left(\mathbf{ab}^k - 2ik\,\mathbf{b}^{k-1}\right)\mathbf{b} = \mathbf{a}^{k+1}\mathbf{b}^{k+1} - 2ik\,\mathbf{a}^k\mathbf{b}^k$. □

Lemma 4.5. *For every $k \in \mathbb{N}$ it holds that*

$$(\mathbf{ab})^k = \sum_{j=1}^{k} (2i)^{k-j}\, S(k, j)\, \mathbf{a}^j\mathbf{b}^j,$$

where $S(n, j)$ is the Stirling number of the second kind corresponding to k and j.

Remark 4.14. *The Stirling number of the second kind $S(k,j)$ is the number of ways of partitioning a set of k elements into j non empty subsets. Amongst the properties of Stirling numbers we recall the following ones:*

- $S(k,1) = S(k,k) = 1.$
- $S(k+1,j+1) = S(k,j) + (j+1)S(k,j+1).$
- $\displaystyle\sum_{k=j}^{\infty} S(k,j)\frac{x^k}{k!} = \frac{(e^x-1)^j}{j!}$

Proof of Lemma 4.5.
We proceed by induction. For $k=1$ the statement clearly is true. Now assume it to be true for $k \geq 1$. Using Lemma 4.4, we have for $k+1$ that

$$\begin{aligned}
(\mathbf{ab})^{k+1} &= \sum_{j=1}^{k}(-2i)^{k-j}\, S(k,j)\,\mathbf{a}^j\mathbf{b}^j\mathbf{ab}\\
&= \sum_{j=1}^{k}(-2i)^{k-j}\, S(k,j)\mathbf{a}^{j+1}\mathbf{b}^{j+1} + (-2i)^{k+1-j} j\, S(k,j)\,\mathbf{a}^j\mathbf{b}^j\\
&= (-2i)^k\mathbf{ab} + \left(\sum_{j=1}^{k-1}(-2i)^{k-j}\left[S(k,j)+(j+1)S(k,j+1)\right]\mathbf{a}^{j+1}\mathbf{b}^{j+1}\right) + \mathbf{a}^{k+1}\mathbf{b}^{k+1}\\
&= \sum_{j=1}^{k+1}(-2i)^{k+1-j}\, S(k+1,j)\,\mathbf{a}^j\mathbf{b}^j.
\end{aligned}$$

□

Then we obtain

$$\begin{aligned}
e^{\pi\mathbf{ab}} &= \sum_{k=0}^{\infty}\frac{\pi^k}{k!}(\mathbf{ab})^k = 1 + \sum_{k=1}^{\infty}\sum_{j=1}^{k}\frac{\pi^k}{k!}(-2i)^{k-j}\, S(k,j)\,\mathbf{a}^j\mathbf{b}^j\\
&= 1 + \sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\frac{\pi^k}{k!}(-2i)^{k-j}\, S(k,j)\,\mathbf{a}^j\mathbf{b}^j\\
&= 1 + \sum_{j=1}^{\infty}(-2i)^{-j}\left[\sum_{k=j}^{\infty}\frac{(-2\pi i)^k}{k!}S(k,j)\right]\mathbf{a}^j\mathbf{b}^j\\
&= 1 + \sum_{j=1}^{\infty}(-2i)^{-j}\frac{\left(e^{-2\pi i}-1\right)^j}{j!}\mathbf{a}^j\mathbf{b}^j \;=\; 1,
\end{aligned}$$

from which we conclude that $\exp\left[\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right)\right] = -\exp\left(\pi\,\mathbf{ab}\right) = -1.$

Remark 4.15. *Within the algebra* $\mathcal{C}_{0,2n} = Alg_{\mathbb{R}}\{\grave{e}_1, \ldots, \grave{e}_{2n}\}$ *the elements* $\grave{e}_{2j-1}, \grave{e}_{2j}$ *may be identified with the operators* $e^{\frac{\pi}{4}i}\partial_{a_j}, e^{-\frac{\pi}{4}i}a_j$ *respectively, the* a_j*'s being real variables. Indeed, these identifications immediately lead to the Weyl algebra defining relations*

$$e^{\frac{\pi}{4}i}\partial_{a_j}\, e^{-\frac{\pi}{4}i}a_k - e^{-\frac{\pi}{4}i}a_k\, e^{\frac{\pi}{4}i}\partial_{a_j} = \partial_{a_j}a_k - a_k\partial_{a_j} = \delta_{j,k}. \tag{4.20}$$

Hence $\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2$ *may be identified with the harmonic oscillator* $i\left(\partial_{a_j}^2 - a_j^2\right)$ *and in consequence, the element* $\exp\left[\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right)\right]$ *corresponds to* $\exp\left[\pi i\left(\partial_{a_j}^2 - a_j^2\right)\right]$. *We recall that the classical Fourier transform in one variable can be written as an operator exponential*

$$\mathcal{F}[f] = \exp\left(\frac{\pi}{4}i\right)\exp\left(\frac{\pi}{4}i\left(\partial_{a_j}^2 - a_j^2\right)\right)[f].$$

Hence, $\exp\left[\pi i\left(\partial_{a_j}^2 - a_j^2\right)\right] = -\mathcal{F}^4 = -id$*, where id denotes the identity operator.*

Observe that the representation (4.20) for the $\grave{e}_j$*'s is equivalent to the one introduced in (3.12). The only difference between them is given by the use of the constants* $e^{\pm\frac{\pi}{4}i}$.

Going back to (4.19) we have

$$e^{\phi^{-1}(\Sigma)} = \prod_{j=1}^{n}\exp\left[\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right)\right]^{k_j} = (-1)^{\sum k_j},$$

whence $e^{B_s} = \pm 1$. Then, for $B = B_o + B_s \in \Xi_1$ such that $e^{\phi(B)} = I_{m+2n}$, we have

$$e^B = e^{B_o}e^{B_s} = \pm 1,$$

i.e. $\ker h|_{\exp(\Xi_1)} = \{-1, 1\}$. This way, we have proven the following result.

Theorem 4.7. *The set* $\Xi = \exp(\Xi_1)\exp(\Xi_2)\exp(\Xi_3)$ *is a double covering of* SO_0.

Remark 4.16. *As shown before, every extended superbivector of the form*

$$B = \sum_{j=1}^{n}\frac{\theta_j}{2}\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right), \quad \theta_j \in \mathbb{R},$$

belongs to Ξ_1*. Then, though the identifications (4.20) we can see all operators*

$$\exp\left[\sum_{j=1}^{n}\frac{\theta_j}{2}\pi i(\partial_{a_j}^2 - a_j^2)\right] = \prod_{j=1}^{n}\exp\left[\theta_j\frac{\pi}{2}i(\partial_{a_j}^2 - a_j^2)\right] = \prod_{j=1}^{n}\exp\left(-\theta_j\frac{\pi}{2}i\right)\mathcal{F}_{a_j}^{2\theta_j},$$

as elements of the Spin group in superspace. Here, $\mathcal{F}_{a_j}^{2\theta_j}$ *denotes the one-dimensional fractional Fourier transform of order* $2\theta_j$ *in the variable* a_j.

4.5 Concluding remarks

In this chapter we have shown that vector reflections in superspace do not suffice to describe the set of linear transformations leaving the inner product invariant. This constitutes a very important difference with the classical case in which the algebra of bivectors $\underline{x} \wedge \underline{y}$ is isomorphic to the special orthogonal algebra $\mathfrak{so}(m)$. Such a property is no longer fulfilled in this setting. The real projection of the algebra of superbivectors $\mathbb{R}^{(2)}_{m|2n}(\mathfrak{G}_N)$ does not include the symplectic algebra structure which is present in the Lie algebra of supermatrices $\mathfrak{so}_0$, corresponding to the group of super rotations.

That fact has an major impact on the definition of the Spin group in this setting. The set of elements defined through the multiplication of an even number of unit vectors in $\mathbb{R}^{m,2n}(\mathfrak{G}_N)$ does not suffice for describing $\text{Spin}(m|2n)(\mathfrak{G}_N)$. A suitable alternative, in this case, is to define the (super) spin elements as products of exponentials of **extended** superbivectors. Such an extension of the Lie algebra of superbivectors contains, through the corresponding identifications, harmonic oscillators. In this way, we obtain the Spin group as a cover of the set of superrotations SO_0 through the usual representation h. In addition, every fractional Fourier transform can be identified with a spin element.

5

Spin actions in Euclidean and Hermitian Clifford analysis in superspace

In this chapter, we study the operator actions on superfunctions, associated to the h-representation of the spin group in superspace, see (4.15). These operators are given by

$$H(s)[F(\mathbf{x})] = sF(\bar{s}\mathbf{x}s)\bar{s}, \qquad L(s)[F(\mathbf{x})] = sF(\bar{s}\mathbf{x}s), \qquad s \in \mathrm{Spin}(m|2n)(\mathfrak{G}_N). \tag{5.1}$$

As in the classical case, the super Dirac operator $\partial_{\mathbf{x}}$ is invariant under those actions. Explicitly this invariance is expressed through the commutation of $\partial_{\mathbf{x}}$ with the infinitesimal representation

$$dL(B) = \frac{d}{dt} L(e^{tB})\big|_{t=0}, \qquad\qquad B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N).$$

The action of the representation dL on the basis elements of $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ gives rise to a class of super angular momentum operators.

In addition, we also study these spin actions within the Hermitian Clifford system in superspace. The basics of Hermitian Clifford analysis in superspace were introduced in Chapter 3 following the notion of an abstract complex structure in the Hermitian radial algebra. We first study the group of complex supermatrices $\mathrm{U}_0(m|n)(\mathbb{C}\mathfrak{G}_N)$ leaving the Hermitian inner product $\{\mathbf{Z}, \mathbf{U}^\dagger\}$ of complex supervector variables invariant, see (3.34). The real representation $\mathrm{SO}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ of this group is composed of all $\mathrm{SO}_0(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ supermatrices which commute with the complex structure

$\mathbf{J} \in \mathrm{SO}_0(2m|2n)(\mathbb{R}\mathfrak{G}_N)$. The group $\mathrm{U}_0(m|n)(\mathbb{C}\mathfrak{G}_N)$ contains $\mathrm{U}(m)\times\mathrm{U}(n)$ as its complex projection and preserves the connectedness from its classical antecedent $\mathrm{U}(m)$. But, as is the case for SO_0, also $\mathrm{U}_0(m|n)(\mathbb{C}\mathfrak{G}_N)$ does not preserve the compactness and cannot be described by a single action of the exponential map on its Lie algebra. Nevertheless, its isomorphic copy $\mathrm{SO}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ may be decomposed as the product of only two exponentials, respectively acting on the elements of $\mathfrak{so}(2m) \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)]$ and $\mathfrak{so}_0(2m|2n)(\mathbb{R}\mathfrak{G}_N^+)$ which commute with $\mathbf{J}$.

The subgroup $\mathrm{SO}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ is covered by a subgroup of $\mathrm{Spin}(2m|2n)(\mathfrak{G}_N)$, which is denoted by $\mathrm{Spin}_{\mathbf{J}}(2m|2n)(\mathfrak{G}_N)$ and generated by the exponentials of extended superbivectors that are invariant under the action of the complex structure. Using the above-mentioned decomposition for $\mathrm{SO}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$, we construct a subset $\Xi_{\mathbf{J}}$ of $\Xi \cap \mathrm{Spin}_{\mathbf{J}}(m|2n)(\mathfrak{G}_N)$ which constitutes a double covering of $\mathrm{SO}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$. Those properties allow to prove the invariance of the twisted super Dirac operator $\partial_{\mathbf{J}(\mathbf{x})}$ under the $\mathrm{Spin}_{\mathbf{J}}(2m|2n)(\mathfrak{G}_N)$-actions.

5.1 Spin invariance of the super Dirac operator

In this section we study the invariance of $\partial_{\mathbf{x}}$ under the spin actions defined in (5.1). To that end it suffices to consider the H and L actions on super-polynomials depending on a supervector variable $\mathbf{x} \in \mathbf{S}$, i.e. elements of the space $\mathbb{R}\mathcal{P} = \mathbb{R}[x_1, \ldots, x_m] \otimes \mathbb{R}\mathfrak{G}(\grave{x}_1, \ldots, \grave{x}_{2n})$, but with coefficients in $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$. This is, the subalgebra $\mathcal{P}_{m,2n}^N \subset \mathcal{A}_{m,2n} \otimes \mathbb{R}\mathfrak{G}_N$ defined as

$$\mathcal{P}_{m,2n}^N = \mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n} \otimes \mathbb{R}\mathcal{P} = \mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n} \otimes \mathbb{R}[x_1, \ldots, x_m] \otimes \mathfrak{G}(\grave{x}_1, \ldots, \grave{x}_{2n}).$$

We recall that the partial derivatives ∂_{x_j}, $\partial_{\grave{x}_j}$ are defined in $\mathcal{V} \otimes \mathbb{R}\mathfrak{G}_N$ as in (3.7) where we are considering $p = m\ell$ commuting variables in VAR and $q = 2n\ell + N$ anti-commuting variables in $\grave{VAR} \cup \{f_1 \ldots, f_N\}$. The extension to $\mathcal{A}_{m,2n} \otimes \mathbb{R}\mathfrak{G}_N$ is trivially defined by means of the commuting relations (3.17) between the the derivatives ∂_{x_j}, $\partial_{\grave{x}_j}$ and the Clifford generators e_j's and $\grave{e}_j$'s.

Following the radial algebra approach it is possible to extend some important involutions from $\mathcal{A}_{m,2n}$ to $\mathcal{A}_{m,2n} \otimes \mathbb{R}\mathfrak{G}_N$. In particular, the conjugation can be defined on $\mathcal{A}_{m,2n} \otimes \mathbb{R}\mathfrak{G}_N$ as the linear map satisfying

$$a\, e_{j_1} \ldots e_{j_k} \grave{e}_{\ell_1} \ldots \grave{e}_{\ell_s} = a\, \overline{e_{j_1} \ldots e_{j_k} \grave{e}_{\ell_1} \ldots \grave{e}_{\ell_s}}, \quad a \in \mathcal{V} \otimes \mathbb{R}\mathfrak{G}_N;$$

where (see Section 3.2.1)

$$\overline{e_{j_1} \ldots e_{j_k} \grave{e}_{\ell_1} \ldots \grave{e}_{\ell_s}} = (-1)^{k + \frac{s(s+1)}{2}} \grave{e}_{\ell_s} \ldots \grave{e}_{\ell_1} e_{j_k} \ldots e_{j_1}.$$

This conjugation map can be continuously extended from the algebra of coefficients $\mathbb{R}\mathfrak{G}_N \otimes \mathcal{C}_{m,2n}$ to $T(V)/I$. This leads, amongst others, to relations of the type $\overline{e^F} = e^{\overline{F}}$.

The extension of the set of Clifford-Grassmann coefficients to $T(V)/I$ allows to consider polynomials in the space

$$\mathcal{TP}^N_{m,2n} = T(V)/I \otimes \mathbb{R}[x_1, \ldots, x_m] \otimes \mathfrak{G}(x\grave{}_1, \ldots, x\grave{}_{2n}),$$

i.e. super-polynomials with coefficients in $T(V)/I$. Every linear action on $\mathcal{P}^N_{m|2n}$ can be extended by continuity to $\mathcal{TP}^N_{m|2n}$.

It is easily seen that the H and L spin actions defined in (5.1) are are Lie group homomorphisms from the spin group to the group of automorphisms on $\mathcal{TP}^N_{m|2n}$, i.e. H and L are Lie group representations of $\mathrm{Spin}(m|2n)(\mathfrak{G}_N)$. It is our aim to show that both representations commute with the super Dirac operator $\partial_{\mathbf{x}}$, whence $\partial_{\mathbf{x}}$ can be called an invariant operator under the action of the spin group. The proof can easily be reduced to showing the invariance under the L action, i.e. to showing that

$$[\partial_{\mathbf{x}}, L(s)] = 0, \qquad s \in \mathrm{Spin}(m|2n)(\mathfrak{G}_N).$$

Let $\mathrm{End}(\mathcal{TP}^N_{m|2n})$ be the space of endomorphisms on $\mathcal{TP}^N_{m|2n}$ and

$$dL : \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N) \rightarrow \mathrm{End}(\mathcal{TP}^N_{m|2n})$$

the infinitesimal representation of L, defined, as above, by

$$L(e^B) = e^{dL(B)}, \qquad B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N).$$

It then suffices to prove that $\partial_{\mathbf{x}}$ commutes with $dL(B)$ for every basis element of $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$. To this end, we need the following result.

Proposition 5.1. *Let* $B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$. *Then*

$$dL(B) = B - \sum_{j=1}^{m} ([B, \mathbf{x}])_j \partial_{x_j} - \sum_{j=1}^{2n} ([B, \mathbf{x}])_{m+j} \partial_{x\grave{}_j},$$

where $([B, \mathbf{x}])_j$ *denotes the* j*-th co-ordinate of the supervector variable* $[B, \mathbf{x}]$*, see (4.11).*

Proof.

For every superbivector B we have by definition that

$$dL(B) = \frac{d}{dt} L(e^{tB}) \bigg|_{t=0}.$$

Then

$$dL(B)[F(\mathbf{x})] = \frac{d}{dt} \left[e^{tB} F(e^{-tB} \mathbf{x} e^{tB}) \right] \bigg|_{t=0} = BF(\mathbf{x}) + \frac{d}{dt} \left[F(e^{-tB} \mathbf{x} e^{tB}) \right] \bigg|_{t=0}.$$

Writing $\mathbf{y} = e^{-tB}\mathbf{x}e^{tB} = e^{-t\phi(B)}\mathbf{x}$, we have that

$$\left.\frac{d\mathbf{y}}{dt}\right|_{t=0} = -\phi(B)\mathbf{x} = -[B,\mathbf{x}].$$

Hence, the chain rule in superanalysis (see Theorem 3.2) yields

$$\begin{aligned} dL(B)[F(\mathbf{x})] &= BF(\mathbf{x}) + \left.\left(\sum_{j=1}^{m} \frac{dy_j}{dt}\,\frac{\partial F}{\partial y_j}(e^{-tB}\mathbf{x}e^{tB}) + \sum_{j=1}^{2n} \frac{d\grave{y}_j}{dt}\,\frac{\partial F}{\partial \grave{y}_j}(e^{-tB}\mathbf{x}e^{tB})\right)\right|_{t=0} \\ &= BF(\mathbf{x}) - \sum_{j=1}^{m}\Big([B,\mathbf{x}]\Big)_j \partial_{x_j}[F](\mathbf{x}) - \sum_{j=1}^{2n}\Big([B,\mathbf{x}]\Big)_{m+j} \partial_{\grave{x}_j}[F](\mathbf{x}). \end{aligned}$$

□

Using Proposition 4.8, the following results are now easily obtained for the basis elements of $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$,

$$\begin{aligned} dL(b\,e_je_k) &= -2b\,(x_j\partial_{x_k} - x_k\partial_{x_j} - \frac{1}{2}e_je_k), && 1\leq j<k\leq m,\\ dL(\grave{b}\,e_j\grave{e}_{2k-1}) &= -\grave{b}\,(2x_j\partial_{\grave{x}_{2k-1}} + \grave{x}_{2k}\partial_{x_j} - e_j\grave{e}_{2k-1}), && 1\leq j\leq m,\ 1\leq k\leq n,\\ dL(\grave{b}\,e_j\grave{e}_{2k}) &= -\grave{b}\,(2x_j\partial_{\grave{x}_{2k}} - \grave{x}_{2k-1}\partial_{x_j} - e_j\grave{e}_{2k}), && 1\leq j\leq m,\ 1\leq k\leq n,\\ dL(b\,\grave{e}_{2j-1}\odot\grave{e}_{2k-1}) &= -b(\grave{x}_{2j}\partial_{\grave{x}_{2k-1}} + \grave{x}_{2k}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\odot\grave{e}_{2k-1}), && 1\leq j\leq k\leq n,\\ dL(b\,\grave{e}_{2j}\odot\grave{e}_{2k}) &= b(\grave{x}_{2j-1}\partial_{\grave{x}_{2k}} + \grave{x}_{2k-1}\partial_{\grave{x}_{2j}} + \grave{e}_{2j}\odot\grave{e}_{2k}), && 1\leq j\leq k\leq n,\\ dL(b\,\grave{e}_{2j-1}\odot\grave{e}_{2k}) &= -b(\grave{x}_{2j}\partial_{\grave{x}_{2k}} - \grave{x}_{2k-1}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\odot\grave{e}_{2k}), && 1\leq j\leq k\leq n,\\ dL(b\,\grave{e}_{2j}\odot\grave{e}_{2k-1}) &= -b(-\grave{x}_{2j-1}\partial_{\grave{x}_{2k-1}} + \grave{x}_{2k}\partial_{\grave{x}_{2j}} - \grave{e}_{2j}\odot\grave{e}_{2k-1}), && 1\leq j<k\leq n, \end{aligned}$$

where b, $\grave{b}$ are arbitrary basis elements of $\mathbb{R}\mathfrak{G}_N^{(ev)}$ and $\mathbb{R}\mathfrak{G}_N^{(odd)}$ respectively. These operators explicitly define the $\mathfrak{so}_0$-action on $\mathcal{P}^N_{m|2n}$. We now are in the condition of proving the desired property.

Proposition 5.2. *For every $B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ it holds that*

$$[\partial_{\mathbf{x}}, dL(B)] = 0.$$

Proof.

It suffices to prove the commutation relation of $\partial_{\mathbf{x}}$ with $dL(B)$ for every basis element B of $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$. As an example, take $B = \grave{b}\,e_j\grave{e}_{2k-1}$; it then is easily obtained that:

$$\begin{aligned} [\grave{e}_{2\ell}\,\partial_{\grave{x}_{2\ell-1}}, \grave{b}\,x_j\partial_{\grave{x}_{2k-1}}] &= [\grave{e}_{2\ell}\,\partial_{\grave{x}_{2\ell-1}}, \grave{b}\,\grave{x}_{2k}\partial_{x_j}] = 0,\\ [\grave{e}_{2\ell}\,\partial_{\grave{x}_{2\ell-1}}, \grave{b}\,e_j\grave{e}_{2k-1}] &= -\delta_{k,\ell}\,\grave{b}\,e_j\partial_{\grave{x}_{2\ell-1}},\\ [\grave{e}_{2\ell-1}\,\partial_{\grave{x}_{2\ell}}, \grave{b}\,x_j\partial_{\grave{x}_{2k-1}}] &= [\grave{e}_{2\ell-1}\,\partial_{\grave{x}_{2\ell}}, \grave{b}\,e_j\grave{e}_{2k-1}] = 0,\\ [\grave{e}_{2\ell-1}\,\partial_{\grave{x}_{2\ell}}, \grave{b}\,\grave{x}_{2k}\partial_{x_j}] &= -\delta_{k,\ell}\,\grave{b}\,\grave{e}_{2\ell-1}\partial_{x_j}, \end{aligned}$$

whence

$$[\grave{e}_{2\ell}\,\partial_{\grave{x}_{2\ell-1}}, dL(\mathfrak{b}\,e_j\grave{e}_{2k-1})] = -\delta_{k,\ell}\,\mathfrak{b}\,e_j\partial_{\grave{x}_{2\ell-1}},$$
$$[\grave{e}_{2\ell-1}\,\partial_{\grave{x}_{2\ell}}, dL(\mathfrak{b}\,e_j\grave{e}_{2k-1})] = \delta_{k,\ell}\,\mathfrak{b}\,\grave{e}_{2\ell-1}\partial_{x_j},$$

and yet

$$[\partial_{\underline{\grave{x}}}, dL(\mathfrak{b}\,e_j\grave{e}_{2k-1})] = -2\sum_{\ell=1}^{n}\delta_{k,\ell}\,\mathfrak{b}\left(e_j\partial_{\grave{x}_{2\ell-1}} + \grave{e}_{2\ell-1}\partial_{x_j}\right) = -2\mathfrak{b}\,\left(e_j\partial_{\grave{x}_{2k-1}} + \grave{e}_{2k-1}\partial_{x_j}\right).$$

On the other hand,

$$[e_\ell\,\partial_{x_\ell}, \mathfrak{b}\,x_j\partial_{\grave{x}_{2k-1}}] = \delta_{\ell,j}\,\mathfrak{b}\,e_\ell\partial_{\grave{x}_{2k-1}},$$
$$[e_\ell\,\partial_{x_\ell}, \mathfrak{b}\,\grave{x}_{2k}\partial_{x_j}] = 0,$$
$$[e_\ell\,\partial_{x_\ell}, \mathfrak{b}\,e_j\grave{e}_{2k-1}] = -2\delta_{j,\ell}\,\mathfrak{b}\,\grave{e}_{2k-1}\partial_{x_\ell},$$

whence

$$[\partial_{\underline{x}}, dL(\mathfrak{b}\,e_j\grave{e}_{2k-1})] = -2\sum_{\ell=1}^{m}\delta_{j,\ell}\,\mathfrak{b}\left(e_\ell\partial_{\grave{x}_{2k-1}} + \grave{e}_{2k-1}\partial_{x_\ell}\right) = -2\mathfrak{b}\,\left(e_j\partial_{\grave{x}_{2k-1}} + \grave{e}_{2k-1}\partial_{x_j}\right).$$

Finally, we obtain that

$$[\partial_{\mathbf{x}}, dL(\mathfrak{b}\,e_j\grave{e}_{2k-1})] = [\partial_{\underline{\grave{x}}}, dL(\mathfrak{b}\,e_j\grave{e}_{2k-1})] - [\partial_{\underline{x}}, dL(\mathfrak{b}\,e_j\grave{e}_{2k-1})] = 0.$$

The proof proceeds in a similar way for all other basis elements of $\mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$. □

The above result implies the $\mathfrak{so}_0$-invariance of the super Dirac operator, and as a consequence, its invariance under the spin actions H and L.

Corollary 5.1. *For every $s \in \mathrm{Spin}(m|2n)(\mathfrak{G}_N)$ it holds that*

$$[\partial_{\mathbf{x}}, L(s)] = 0 = [\partial_{\mathbf{x}}, H(s)].$$

Remark 5.1. *As it was announced in Remarks 3.6 and 4.9, here we approached the $\mathfrak{osp}(m|2n)$-invariance of the super Dirac operator from a classical group theoretical point of view, with the help of the Grassmann algebra structure.*

The above-given operators $dL(B)$, with $B \in \mathbb{R}^{(2)E}_{m|2n}(\mathfrak{G}_N)$ being a basis element, exactly coincide with the ones provided in [23] when one gets rid of the Grassmann coefficients *and applies the transformations given in Remark 3.6. These operators are,*

$$
\begin{aligned}
&K_{j,k} = X_j\partial_{X_k} - X_k\partial_{X_j} - \frac{1}{2}E_jE_k, && 1 \leq j,k \leq m,\\
&K_{j,m+k} = -X_j\partial_{X_{m+n+k}} - X_{m+k}\partial_{X_j} + \frac{1}{2}E_jE_{m+n+k}, && 1 \leq j \leq m, \quad 1 \leq k \leq n,\\
&K_{j,m+n+k} = X_j\partial_{X_{m+k}} - X_{m+n+k}\partial_{X_j} - \frac{1}{2}E_jE_{m+k}, && 1 \leq j \leq m, \quad 1 \leq k \leq n,\\
&K_{m+j,m+k} = -X_{m+j}\partial_{X_{m+n+k}} - X_{m+k}\partial_{X_{m+n+j}} - \frac{1}{2}E_{m+n+j}E_{m+n+k}, && 1 \leq j,k \leq n,\\
&K_{m+j,m+n+k} = X_{m+j}\partial_{X_{m+k}} - X_{m+n+k}\partial_{X_{m+n+j}} + \frac{1}{4}\left\{E_{m+n+j},E_{m+k}\right\}, && 1 \leq j,k \leq n,\\
&K_{m+n+j,m+n+k} = X_{m+n+j}\partial_{X_{m+k}} + X_{m+n+k}\partial_{X_{m+j}} - \frac{1}{2}E_{m+j}E_{m+k}, && 1 \leq j,k \leq n.
\end{aligned}
$$

5.2 Linear transformations on Hermitian supervectors

In this section we study some fundamental aspects of Hermitian Clifford analysis in superspace. In particular, we are interested in the group of supermatrices leaving the Hermitian inner product (3.34) invariant. This leads to a restriction of the spin group, depending on the so-called complex structure $\mathbf{J}$ defined in Section 3.3.

5.2.1 Commutation with the complex structure J

The fundaments of Hermitian Clifford analysis in superspace were introduced in Chapter 3 through the representation of the radial algebra with complex structure $R(\mathbf{S} \cup \mathbf{J}(\mathbf{S}), \mathbf{B}) \subset \mathcal{A}_{2m,2n}$. The complex structure $\mathbf{J}$ was introduced in Section 3.3 as an algebra automorphism over $\mathcal{A}_{2m,2n}$. It is easily seen that $\mathbf{J}$ can be trivially extended to $\mathcal{A}_{2m,2n} \otimes \mathbb{R}\mathfrak{G}_N$ by

- *(i)* $\mathbf{J}$ is the identity on $\mathcal{V} \otimes \mathbb{R}\mathfrak{G}_N$;
- *(ii)* $\mathbf{J}(e_j) = -e_{m+j}$, $\mathbf{J}(e_{m+j}) = e_j$, $j = 1,\ldots,m$;
  $\mathbf{J}(\grave{e}_{2j-1}) = -\grave{e}_{2j}$, $\mathbf{J}(\grave{e}_{2j}) = \grave{e}_{2j-1}$, $j = 1,\ldots,n$;
- *(iii)* $\mathbf{J}(FG) = \mathbf{J}(F)\mathbf{J}(G)$ for all $F, G \in \mathcal{A}_{2m,2n} \otimes \mathbb{R}\mathfrak{G}_N$.

The action (3.23) of $\mathbf{J}$ on supervector variables can be written in a supermatrix form as

$$\mathbf{J} = \begin{pmatrix} J_{2m} & 0 \\ 0 & \Omega_{2n} \end{pmatrix},$$

where

$$J_{2m} = \underline{J} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

It is easily seen that $\mathbf{J}$ is an element of both the Lie group $\mathrm{SO}_0(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ and the Lie algebra $\mathfrak{so}_o(2m|2n)(\mathbb{R}\mathfrak{G}_N)$. In addition, $\mathbf{J}^2 = -I_{2m+2n}$.

Remark 5.2. *We recall that for $m = n$, the antisymmetric matrices J_{2n} and Ω_{2n} are used to define two different copies of the symplectic group, i.e.*

$$\mathrm{Sp}_J(2n) = \{A_0 \in \mathbb{R}^{2n\times 2n} : A_0^T J_{2n} A_0 = J_{2n}\},$$
$$\mathrm{Sp}_\Omega(2n) = \{D_0 \in \mathbb{R}^{2n\times 2n} : D_0^T \Omega_{2n} D_0 = \Omega_{2n}\}.$$

The matrices J_{2n} and Ω_{2n} only differ by a permutation of the basis vectors. The relation between them is seen by means of the orthogonal matrix R satisfying $R^T J_{2n} R = \Omega_{2n}$, see Remark 4.9. Hence,

$$D_0^T \Omega_{2n} D_0 = \Omega_{2n} \iff D_0^T R^T J_{2n} R D_0 = R^T J_{2n} R \iff (R D_0 R^T)^T J_{2n}\, R D_0 R^T = J_{2n},$$

meaning that

$$D_0 \in \mathrm{Sp}_\Omega(2n) \iff R D_0 R^T \in \mathrm{Sp}_J(2n)$$

or equivalently,

$$R\,\mathrm{Sp}_\Omega(2n)\, R^T = \mathrm{Sp}_J(2n).$$

The map $\gamma(D_0) = R D_0 R^T$ thus constitutes a Lie group isomorphism between $\mathrm{Sp}_\Omega(2n)$ and $\mathrm{Sp}_J(2n)$, whence also the corresponding Lie algebras

$$\mathfrak{so}_\Omega(2n) = \{D_0 \in \mathbb{R}^{2n\times 2n} : D_0^T \Omega_{2n} + \Omega_{2n} D_0 = 0\},$$
$$\mathfrak{so}_J(2n) = \{A_0 \in \mathbb{R}^{2n\times 2n} : A_0^T J_{2n} + J_{2n} A_0 = 0\},$$

are isomorphic. These observations allow us to speak of "the" symplectic structure independently of Ω_{2n} or J_{2n}.

In accordance with the radial algebra property **(DH3)**, the actions of the Dirac operators $\partial_{\mathbf{x}}$, $\partial_{\mathbf{J(x)}}$ on the supervector variables $\mathbf{x}$ and $\mathbf{J}(\mathbf{x})$ give two important defining elements: the superdimension M and the fundamental bivector $\mathbf{B} \in \mathbb{R}^{(2)E}_{2m|2n}(\mathfrak{G}_N)$. Indeed, in Section 3.3.2 we proved that

$$\partial_{\mathbf{x}}[\mathbf{x}] = \partial_{\mathbf{J(x)}}[\mathbf{J}(\mathbf{x})] = M = 2m - 2n, \qquad \partial_{\mathbf{x}}[\mathbf{J}(\mathbf{x})] = -\partial_{\mathbf{J(x)}}[\mathbf{x}] = 2\mathbf{B},$$

where

$$\mathbf{B} = \sum_{j=1}^{m} e_j e_{m+j} - \sum_{j=1}^{2n} \grave{e}_j{}^2.$$

Since $\mathbf{J} \in \mathfrak{so}_0(2m|2n)(\mathbb{R}\mathfrak{G}_N)$, another important characterization of $\mathbf{B}$ is that

$$[\mathbf{B}, \mathbf{x}] = -2\mathbf{J}(\mathbf{x}) \quad \text{or equivalently} \quad \phi(\mathbf{B}) = -2\mathbf{J}. \tag{5.2}$$

In Section 3.4 we introduced the representation $R(\mathbf{S}_{\mathbb{C}}, \mathbf{S}_{\mathbb{C}}^\dagger, \mathbf{B})$ of the Hermitian radial algebra in superspace. This algebra is easily seen to be a subalgebra of the complexification $\mathbb{C}\mathcal{A}_{2m,2n} \otimes \mathbb{C}\mathfrak{G}_N$ of $\mathcal{A}_{2m,2n} \otimes \mathbb{R}\mathfrak{G}_N$, i.e.

$$\mathbb{C}\mathcal{A}_{2m,2n} \otimes \mathbb{C}\mathfrak{G}_N = (\mathcal{A}_{2m,2n} \otimes \mathbb{R}\mathfrak{G}_N) \oplus i\,(\mathcal{A}_{2m,2n} \otimes \mathbb{R}\mathfrak{G}_N) = \mathcal{V} \otimes \mathcal{C}_{2m,2n} \otimes \mathbb{C}\mathfrak{G}_N,$$

where the imaginary unit i commutes with every element of $\mathcal{A}_{2m,2n}\otimes\mathbb{R}\mathfrak{G}_N$. Similarly the complexification of $T(V)/I$ is $T(V_{\mathbb{C}})/I$. The Hermitian and the complex conjugations are trivially extended from $\mathbb{C}\mathcal{A}_{2m,2n}$ to $\mathbb{C}\mathcal{A}_{2m,2n}\otimes\mathbb{C}\mathfrak{G}_N$ by the rules

$$(a+ib)^\dagger = \overline{a} - i\overline{b}, \qquad\qquad (a+ib)^c = a - ib,$$

where $a, b \in \mathcal{A}_{2m,2n}\otimes\mathbb{R}\mathfrak{G}_N$.

We recall that the projection operators $\frac{1}{2}\left(I_{2m+2n} \pm i\mathbf{J}\right)$ acting on the supervector variables $\mathbf{x}\in\mathbf{S}$ produce the Hermitian supervector variables

$$\mathbf{Z} = \frac{1}{2}\Big(\mathbf{x} + i\mathbf{J}(\mathbf{x})\Big) \qquad\qquad \mathbf{Z}^\dagger = -\frac{1}{2}\Big(\mathbf{x} - i\mathbf{J}(\mathbf{x})\Big).$$

The actions of every supermatrix $M\in \mathrm{Mat}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ on $\mathbf{Z}$, $\mathbf{Z}^\dagger$ are given by

$$\mathbf{Z} \;=\; \frac{1}{2}\left[\mathbf{x} + i\mathbf{J}(\mathbf{x})\right] \;\longmapsto\; \frac{1}{2}\left[M\mathbf{x} + iM\mathbf{J}(\mathbf{x})\right], \tag{5.3}$$

$$\mathbf{Z}^\dagger = -\frac{1}{2}\left[\mathbf{x} - i\mathbf{J}(\mathbf{x})\right] \;\longmapsto\; -\frac{1}{2}\left[M\mathbf{x} - iM\mathbf{J}(\mathbf{x})\right]. \tag{5.4}$$

In particular, we are interested in supermatrices preserving the Hermitian structure, i.e. commuting with the projection operators $\frac{1}{2}\left(I_{2m+2n}\pm i\mathbf{J}\right)$. The subspace

$$\mathrm{Mat}_{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N) \subset \mathrm{Mat}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$$

of such supermatrices clearly is given by

$$\mathrm{Mat}_{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N) = \{M\in \mathrm{Mat}(2m|2n)(\mathbb{R}\mathfrak{G}_N) : M\mathbf{J} = \mathbf{J}M\}.$$

In this way, for every $M\in \mathrm{Mat}_{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ the linear transformations (5.3)-(5.4) yield two new Hermitian supervector variables $\mathbf{U}$, $\mathbf{U}^\dagger$ depending on $\mathbf{y} = M\mathbf{x}$, i.e.

$$\mathbf{U} \;=\; \frac{1}{2}\left[\mathbf{y} + i\mathbf{J}(\mathbf{y})\right], \qquad\qquad \mathbf{U}^\dagger = -\frac{1}{2}\left[\mathbf{y} - i\mathbf{J}(\mathbf{y})\right].$$

Proposition 5.3. *Let* $M = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D\end{pmatrix} \in \mathrm{Mat}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$. *Then the following statements are equivalent:*

(i) $M\mathbf{J} = \mathbf{J}M$;

(ii) the matrices, $A, \grave{B}, \grave{C}, D$ *satisfy*

$$AJ_{2m} = J_{2m}A, \quad \grave{B}\Omega_{2n} = J_{2m}\grave{B}, \quad \grave{C}J_{2m} = \Omega_{2n}\grave{C}, \quad D\Omega_{2n} = \Omega_{2n}D.$$

(iii) the matrices, $A, \grave{B}, \grave{C}, D$ *have the form*

$$A = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1\end{pmatrix}, \qquad \grave{B} = \begin{pmatrix} \grave{B}_1 \\ \grave{B}_1\Omega_{2m}\end{pmatrix}, \qquad \grave{C} = \left(\grave{C}_1 \middle| -\Omega_{2n}\grave{C}_1\right),$$

and

$$D = \left\{D_{jk}\right\}_{j,k=1,\ldots,n} \quad \textit{with} \quad D_{jk} = \begin{pmatrix} a_{jk} & b_{jk} \\ -b_{jk} & a_{jk} \end{pmatrix},$$

where $A_1, A_2 \in \left(\mathbb{R}\mathfrak{G}_N^{(ev)}\right)^{m\times m}$, $\grave{B}_1 \in \left(\mathbb{R}\mathfrak{G}_N^{(odd)}\right)^{m\times 2n}$, $\grave{C}_1 \in \left(\mathbb{R}\mathfrak{G}_N^{(odd)}\right)^{2n\times m}$ *and* $a_{jk}, b_{jk} \in \mathbb{R}\mathfrak{G}_N^{(ev)}$.

Proof.

The equivalence between (i) and (ii) easily follows from the block structure of the supermatrices M and $\mathbf{J}$. To prove the equivalence between (ii) and (iii) we first write

$$\begin{aligned} A &= \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, & A_j &\in \left(\mathbb{R}\mathfrak{G}_N^{(ev)}\right)^{m\times m}, \\ \grave{B} &= \begin{pmatrix} \grave{B}_1 \\ \grave{B}_2 \end{pmatrix}, & \grave{B}_j &\in \left(\mathbb{R}\mathfrak{G}_N^{(odd)}\right)^{m\times 2n}, \\ \grave{C} &= \left(\grave{C}_1 \middle| \grave{C}_2\right), & \grave{C}_j &\in \left(\mathbb{R}\mathfrak{G}_N^{(odd)}\right)^{2n\times m}, \\ D &= \left\{D_{jk}\right\}_{j,k=1,\ldots,n}, & D_{jk} &\in \left(\mathbb{R}\mathfrak{G}_N^{(ev)}\right)^{2\times 2}. \end{aligned}$$

Then, direct computations show that

$$\begin{aligned} AJ_{2m} = J_{2m}A &\iff \begin{pmatrix} -A_2 & A_1 \\ -A_4 & A_3 \end{pmatrix} = \begin{pmatrix} A_3 & A_4 \\ -A_1 & -A_2 \end{pmatrix} \iff A = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}, \\ \grave{B}\Omega_{2n} = J_{2m}\grave{B} &\iff \begin{pmatrix} \grave{B}_1\Omega_{2n} \\ \grave{B}_2\Omega_{2n} \end{pmatrix} = \begin{pmatrix} \grave{B}_2 \\ -\grave{B}_1 \end{pmatrix} \iff \grave{B} = \begin{pmatrix} \grave{B}_1 \\ \grave{B}_1\Omega_{2m} \end{pmatrix}, \\ \grave{C}J_{2m} = \Omega_{2n}\grave{C} &\iff \left(-\grave{C}_2 \middle| \grave{C}_1\right) = \left(\Omega_{2n}\grave{C}_1 \middle| \Omega_{2n}\grave{C}_2\right) \iff \left(\grave{C}_1 \middle| -\Omega_{2n}\grave{C}_1\right), \\ D\Omega_{2n} = \Omega_{2n}D &\iff D_{jk}\Omega_2 = \Omega_2 D_{jk} \iff D_{jk} = \begin{pmatrix} a_{jk} & b_{jk} \\ -b_{jk} & a_{jk} \end{pmatrix}. \end{aligned}$$

□

We recall that every complex supervector variable $\mathbf{Z}$ and its Hermitian conjugate $\mathbf{Z}^\dagger$ have the following form, see (3.30)-(3.33),

$$\mathbf{Z} = \sum_{j=1}^{m} z_j\mathfrak{f}_j + \sum_{j=1}^{n} \grave{z}_j\grave{\mathfrak{f}}_j, \qquad \mathbf{Z}^\dagger = \sum_{j=1}^{m} z_j^c\mathfrak{f}_j^\dagger + \sum_{j=1}^{n} \grave{z}_j^c\,\grave{\mathfrak{f}}_j^{\,\dagger},$$

whence they can be written as the column vectors

$$\begin{aligned} \mathbf{Z} &= \begin{pmatrix} \underline{Z} \\ \underline{\grave{Z}} \end{pmatrix} = \left(z_1, \ldots, z_m, \grave{z}_1, \ldots, \grave{z}_n\right)^T, \\ \mathbf{Z}^c &= \begin{pmatrix} \underline{Z}^c \\ \underline{\grave{Z}}^c \end{pmatrix} = \left(z_1^c, \ldots, z_m^c, \grave{z}_1^c, \ldots, \grave{z}_n^c\right)^T. \end{aligned}$$

In this way, the equalities $\mathbf{Z} = \frac{1}{2}[\mathbf{x} + i\mathbf{J}(\mathbf{x})]$ and $\mathbf{Z}^\dagger = -\frac{1}{2}[\mathbf{x} - i\mathbf{J}(\mathbf{x})]$ can be rewritten in the matrix[1] form as:

$$\mathbf{Z} = \mathbf{P}\mathbf{x}, \qquad \mathbf{Z}^c = \mathbf{P}^c\mathbf{x}, \qquad \text{where} \quad \mathbf{P} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

and with

$$P = \left(\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & i & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & i \end{array}\right) = \left(I_m \,\middle|\, iI_m\right) \in \mathbb{C}^{m\times 2m},$$

$$Q = \begin{pmatrix} 1 & i & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & i \end{pmatrix} \in \mathbb{C}^{n\times 2n}.$$

These complex matrices play a rôle in every computation involving the supervectors $\mathbf{Z}$ and $\mathbf{x}$. In particular, they satisfy the following relations.

$$\mathbf{P}\left(\mathbf{P}^T\right)^c = 2I_{m+n}, \qquad \left(\mathbf{P}^T\right)^c \mathbf{P} = I_{2m+2n} + i\mathbf{J}, \qquad \mathbf{P}\mathbf{J} = -i\mathbf{P}. \tag{5.5}$$

Every linear transformation of supervector variables in $\mathbf{S}$ defined by a supermatrix $M \in \text{Mat}_{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ is associated to a transformation of Hermitian supervectors in $\mathbf{S}_\mathbb{C}$ determined by

$$\mathbf{U} = \frac{1}{2}\left[M\mathbf{x} + i\mathbf{J}(M\mathbf{x})\right], \quad \mathbf{x} \in \mathbf{S}.$$

This transformation can be written in terms of a supermatrix $\psi(M) \in \text{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N)$ as

$$\mathbf{U} = \psi(M)\mathbf{Z} \qquad \text{with} \quad \mathbf{Z} = \frac{1}{2}(\mathbf{x} + i\mathbf{J}(\mathbf{x})).$$

Indeed, the above relation can be rewritten as $\mathbf{P}M\mathbf{x} = \psi(M)\mathbf{P}\mathbf{x}$, meaning that $\psi(M)\mathbf{P} = \mathbf{P}M$, or equivalently,

$$\psi(M) = \frac{1}{2}\mathbf{P}M\left(\mathbf{P}^T\right)^c. \tag{5.6}$$

Using Proposition 5.3 we easily get

$$\psi\left(\begin{array}{cc|c} A_1 & A_2 & \grave{B}_1 \\ -A_2 & A_1 & \grave{B}_1\Omega_{2n} \\ \hline \grave{C}_1 & -\Omega_{2n}\grave{C}_1 & D \end{array}\right) = \begin{pmatrix} A_1 - iA_2 & \grave{B}_1\left(Q^T\right)^c \\ Q\grave{C}_1 & \frac{1}{2}QD\left(Q^T\right)^c \end{pmatrix}. \tag{5.7}$$

Remark 5.3. *It is known from Proposition 5.3 that the matrix D is composed of 2×2 blocks D_{jk}. It then easily follows that $\frac{1}{2}QD\left(Q^T\right)^c = \{a_{jk} - ib_{jk}\}_{j,k=1,\dots,n}$.*

[1] The complex conjugate M^c of a supermatrix $M \in \text{Mat}(p|q)(\mathbb{C}\mathfrak{G}_N)$ is defined componentwise.

Proposition 5.4. *The map* $\psi : \mathrm{Mat}_{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N) \to \mathrm{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N)$ *is a real algebra isomorphism.*

Proof.

Using the properties of $\mathbf{P}$ given in (5.5) it easily follows that ψ is invertible and its inverse is given by

$$\psi^{-1}(L) = \frac{1}{2}\left[\left(\mathbf{P}^T\right)^c L\,\mathbf{P} + \mathbf{P}^T L^c\,\mathbf{P}^c\right], \qquad L \in \mathrm{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N).$$

That ψ is an algebra isomorphism follows from its real-linearity and from

$$\begin{aligned}
\psi(M_1)\psi(M_2) &= \frac{1}{4}\mathbf{P}M_1\left(\mathbf{P}^T\right)^c\,\mathbf{P}M_2\left(\mathbf{P}^T\right)^c \\
&= \frac{1}{4}\mathbf{P}M_1\left(I_{2m+2n} + i\mathbf{J}\right)M_2\left(\mathbf{P}^T\right)^c \\
&= \frac{1}{4}\left[\mathbf{P}M_1M_2\left(\mathbf{P}^T\right)^c + i\mathbf{P}\mathbf{J}M_1M_2\left(\mathbf{P}^T\right)^c\right] \\
&= \frac{1}{2}\mathbf{P}M_1M_2\left(\mathbf{P}^T\right)^c = \psi(M_1M_2).
\end{aligned}$$

□

5.2.2 Invariance of the Hermitian inner product. The group $\mathrm{SO}_0^{\mathbf{J}}$.

We are now interested in the invariance, under linear actions, of the inner product (3.34) between Hermitian supervectors, i.e.

$$\langle \mathbf{Z}, \mathbf{U}\rangle_{\mathbb{C}} = \{\mathbf{Z}, \mathbf{U}^\dagger\} = \sum_{j=1}^{m} z_j u_j^c - \frac{i}{2}\sum_{j=1}^{n} \grave{z}_j \grave{u}_j^c, \qquad \mathbf{Z}, \mathbf{U} \in \mathbf{S}_{\mathbb{C}}.$$

In particular, we want to describe the set of supermatrices $M \in \mathrm{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N)$ satisfying

$$\langle M\mathbf{Z}, M\mathbf{U}\rangle_{\mathbb{C}} = \langle \mathbf{Z}, \mathbf{U}\rangle_{\mathbb{C}}, \qquad \mathbf{Z}, \mathbf{U} \in \mathbf{S}_{\mathbb{C}}. \tag{5.8}$$

Observe that $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ can be written in a matrix form as

$$\langle \mathbf{Z}, \mathbf{U}\rangle_{\mathbb{C}} = \mathbf{Z}^T\mathbf{H}\mathbf{U}^c, \quad \text{where} \quad \mathbf{H} = \begin{pmatrix} I_m & 0 \\ 0 & -\frac{i}{2}I_n \end{pmatrix}.$$

Then a super matrix $M \in \mathrm{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N)$ satisfies the condition (5.8) if and only if

$$\mathbf{Z}^T\left(M^{ST}\mathbf{H}M^c - \mathbf{H}\right)\mathbf{U}^c = 0, \qquad \text{or equivalently} \quad M^{ST}\mathbf{H}M^c - \mathbf{H} = 0.$$

Hence the set of supermatrices leaving the inner product (3.34) invariant is given by

$$\mathrm{U}_0 = \mathrm{U}_0(m|n)(\mathbb{C}\mathfrak{G}_N) = \left\{M \in \mathrm{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N) : \left(M^{ST}\right)^c\mathbf{H}^cM - \mathbf{H}^c = 0\right\},$$

which is a closed subgroup of $\mathrm{GL}(m|n)(\mathbb{C}\mathfrak{G}_N)$ and in consequence a Lie group.

Proposition 5.5. *The following statements hold:*

(i) a supermatrix $M = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \in \mathrm{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N)$ *belongs to* U_0 *if and only if*

$$\begin{cases} \left(A^T\right)^c A + \frac{i}{2}\left(\grave{C}^T\right)^c \grave{C} = I_m, \\ \left(A^T\right)^c \grave{B} + \frac{i}{2}\left(\grave{C}^T\right)^c D = 0, \\ -\left(\grave{B}^T\right)^c \grave{B} + \frac{i}{2}\left(D^T\right)^c D = \frac{i}{2} I_n; \end{cases}$$

(ii) $(\mathrm{sdet}\, M)^c\, \mathrm{sdet}\, M = 1$ *for every* $M \in \mathrm{U}_0$*;*

(iii) $[\mathrm{U}_0]_0 = \mathrm{U}(m) \times \mathrm{U}(n)$*, where* $\mathrm{U}(k)$ *denotes the unitary group of order* k*.*

Proof.

(*i*) The relation $\left(M^{ST}\right)^c \mathbf{H}^c M = \mathbf{H}^c$ can be written in terms of $A, \grave{B}, \grave{C}, D$ as

$$\begin{pmatrix} \left(A^T\right)^c A + \frac{i}{2}\left(\grave{C}^T\right)^c \grave{C} & \left(A^T\right)^c \grave{B} + \frac{i}{2}\left(\grave{C}^T\right)^c D \\ -\left(\grave{B}^T\right)^c A + \frac{i}{2}\left(D^T\right)^c \grave{C} & -\left(\grave{B}^T\right)^c \grave{B} + \frac{i}{2}\left(D^T\right)^c D \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & \frac{i}{2} I_n \end{pmatrix}.$$

(*ii*) This is easily obtained from $\left(M^{ST}\right)^c \mathbf{H}^c M = \mathbf{H}^c$.

(*iii*) Applying the homomorphism $[\cdot]_0$ to each of the relations in (*i*) we get for

$$[M]_0 = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$$

that $\left(A_0^T\right)^c A_0 = I_m$, $\left(D_0^T\right)^c D_0 = I_n$ or equivalently $A_0 \in \mathrm{U}(m)$, $D_0 \in \mathrm{U}(n)$. □

As it was done in the previous chapter for SO_0, it can be proven that this generalization of the unitary groups $\mathrm{U}(m)$ and $\mathrm{U}(n)$ is connected.

Proposition 5.6. U_0 *is a connected Lie group.*

Proof.

This proof is similar to the one of Proposition 4.10 showing the connectedness of SO_0. The strategy is to find for every $M \in \mathrm{U}_0$ a continuous path $M(t)$ $(0 \leq t \leq 1)$ connecting, inside U_0, M with its complex projection M_0, which belongs to the connected group $\mathrm{U}(m) \times \mathrm{U}(n)$. This continuous path is given by

$$M(t) = \sum_{j=0}^{N} t^j [M]_j,$$

where $[M]_j$ is the projection of M on $\mathrm{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N^{(j)})$. □

As in the classical case, the Lie group $\mathrm{U}_0(m|n)(\mathbb{C}\mathfrak{G}_N)$ is related to $\mathrm{SO}_0(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ through the complex structure $\mathbf{J}$. Indeed, it is clear that supermatrices in the group

$$\mathrm{SO}_0^{\mathbf{J}} = \mathrm{SO}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N) = \mathrm{SO}_0(2m|2n)(\mathbb{R}\mathfrak{G}_N) \cap \mathrm{Mat}_{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N),$$

have the double property of keeping the bilinear forms $\{\mathbf{x},\mathbf{y}\}$ and $\{\mathbf{x},\mathbf{J}(\mathbf{y})\}$ for $\mathbf{x},\mathbf{y}\in\mathbf{S}$ invariant. But for the Hermitian inner product we have

$$\langle \mathbf{Z},\mathbf{U}\rangle_{\mathbb{C}} = \{\mathbf{Z},\mathbf{U}^\dagger\} = \left\{\frac{1}{2}(\mathbf{x}+i\mathbf{J}(\mathbf{x})), -\frac{1}{2}(\mathbf{y}-i\mathbf{J}(\mathbf{y}))\right\} = -\frac{1}{2}\big[\{\mathbf{x},\mathbf{y}\} - i\{\mathbf{x},\mathbf{J}(\mathbf{y})\}\big],$$

whence the action of every supermatrix in $\mathrm{SO}_0^{\mathbf{J}}$ leaves the Hermitian inner product invariant as well. This property is summarized below.

Proposition 5.7. *The map ψ defined in (5.6) is a Lie group isomorphism between $\mathrm{SO}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N)$ and $\mathrm{U}_0(m|n)(\mathbb{C}\mathfrak{G}_N)$.*

Proof.

It is clear that ψ defines a smooth map and in addition, it was proven in Proposition 5.4 that ψ is a group homomorphism. It thus only remains to show that $\psi(\mathrm{SO}_0^{\mathbf{J}}) = \mathrm{U}_0$. To that end, first observe that $\psi(\mathbf{Q}) = \mathbf{H}^c$ and

$$\left(\psi(M)^{ST}\right)^c = \frac{1}{2}\mathbf{P}M^{ST}\left(\mathbf{P}^T\right)^c = \psi(M^{ST}).$$

Using Proposition 5.4 we get for every $M\in\mathrm{SO}_0^{\mathbf{J}}$ that

$$\left(\psi(M)^{ST}\right)^c\mathbf{H}^c\psi(M) - \mathbf{H}^c = \psi(M^{ST})\psi(\mathbf{Q})\psi(M) - \psi(\mathbf{Q}) = \psi\left(M^{ST}\mathbf{Q}M - \mathbf{Q}\right) = 0,$$

meaning that $\psi(M)\in\mathrm{U}_0$, and consequently $\psi(\mathrm{SO}_0^{\mathbf{J}})\subset\mathrm{U}_0$. On the other hand, for $L\in\mathrm{U}_0$ we get

$$\psi^{-1}(L)^{ST} = \frac{1}{2}\left[\mathbf{P}^T L^{ST}\mathbf{P}^c + \left(\mathbf{P}^T\right)^c\left(L^{ST}\right)^c\mathbf{P}\right] = \psi^{-1}\left(\left(L^{ST}\right)^c\right).$$

Hence,

$$\begin{aligned}\psi^{-1}(L)^{ST}\mathbf{Q}\psi^{-1}(L) - \mathbf{Q} &= \psi^{-1}(L)^{ST}\psi^{-1}(\mathbf{H}^c)\psi^{-1}(L) - \psi^{-1}(\mathbf{H}^c)\\ &= \psi^{-1}\left(\left(L^{ST}\right)^c\mathbf{H}^cL - \mathbf{H}^c\right)\\ &= 0.\end{aligned}$$

This shows that the supermatrix $M=\psi^{-1}(L)$ belongs to O_0, but we still have to prove that $\mathrm{sdet}(M)=1$. To that end it suffices to compute $\mathrm{sdet}(M_0)$, since $\mathrm{sdet}(M) = \mathrm{sdet}(M_0)$. From Proposition 4.9 we know that

$$M_0 = \begin{pmatrix} A_0 & 0\\ 0 & D_0\end{pmatrix}$$

with $A_0 \in \mathrm{O}(2m)$ and $D_0 \in \mathrm{Sp}_\Omega(2n)$. In addition, $M_0\mathbf{J} = \mathbf{J}M_0$ which implies in particular that $A_0 J_{2m} = J_{2m} A_0$. Combining this with $A_0^T A_0 = I_{2m}$, straightforward computations yield $A_0^T J_{2m} A_0 = J_{2m}$, or still $A_0 \in \mathrm{Sp}_J(2m)$. Hence, $\det(A_0) = \det(D_0) = 1$ since the determinant of a symplectic matrix always equals 1, implying that $\mathrm{sdet}(M_0) = 1$. Whence $M = \psi^{-1}(L) \in \mathrm{SO}_0^{\mathbf{J}}$ and in consequence $\psi^{-1}(\mathrm{U}_0) \subset \mathrm{SO}_0^{\mathbf{J}}$. □

The above proposition leads to the following result on the Lie algebra level.

Proposition 5.8. *The Lie algebras of* $\mathrm{SO}_0^{\mathbf{J}}$ *and* U_0 *are given by*

$$\begin{aligned} \mathfrak{so}_0^{\mathbf{J}} &= \mathfrak{so}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N) = \{X \in \mathfrak{so}_0(2m|2n)(\mathbb{R}\mathfrak{G}_N) : X\mathbf{J} = \mathbf{J}X\}, \\ \mathfrak{u}_0 &= \mathfrak{u}_0(m|n)(\mathbb{C}\mathfrak{G}_N) = \{X \in \mathrm{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N) : \left(X^{ST}\right)^c \mathbf{H}^c + \mathbf{H}^c X = 0\}, \end{aligned} \tag{5.9}$$

respectively. In addition, $\psi : \mathfrak{so}_0^{\mathbf{J}} \to \mathfrak{u}_0$ *is a Lie algebra isomorphism.*

Proof.

If $X \in \mathrm{Mat}(m|n)(\mathbb{C}\mathfrak{G}_N)$ belongs to the Lie algebra of U_0, it satisfies $e^{tX} \in \mathrm{U}_0$ for every $t \in \mathbb{R}$. Differentiating both sides of the equality

$$e^{t(X^{ST})^c}\mathbf{H}^c e^{tX} = \mathbf{H}^c$$

and evaluating at $t = 0$, we get

$$\left(X^{ST}\right)^c \mathbf{H}^c + \mathbf{H}^c X = 0.$$

On the other hand, it is easily seen that every supermatrix satisfying the above condition is such that $e^{tX} \in \mathrm{U}_0$ for every $t \in \mathbb{R}$. Then, the Lie algebra of U_0 is the one given in (5.9).

The Lie algebra of $\mathrm{SO}_0^{\mathbf{J}}$ is similarly obtained: differentiating both sides of $e^{tM}\mathbf{J} = \mathbf{J}e^{tM}$, we get $M\mathbf{J} = \mathbf{J}M$. Vice versa, if $M\mathbf{J} = \mathbf{J}M$ then clearly $e^{tM}\mathbf{J} = \mathbf{J}e^{tM}$.

Since $\psi : \mathrm{SO}_0^{\mathbf{J}} \to \mathrm{U}_0$ is a Lie group isomorphism, its infinitesimal representation $d\psi : \mathfrak{so}_0^{\mathbf{J}} \to \mathfrak{u}_0$ turns out to be a Lie algebra isomorphism, see [50]. The map $d\psi$ is obtained from ψ through the relation $e^{t\,d\psi(X)} = \psi(e^{tX})$, $t \in \mathbb{R}$, $X \in \mathfrak{so}_0^{\mathbf{J}}$. Differentiating at $t = 0$ we get

$$d\psi(X) = \frac{d}{dt}\psi(e^{tX})\Big|_{t=0} = \frac{d}{dt}\left[\frac{1}{2}\mathbf{P}e^{tX}\left(\mathbf{P}^T\right)^c\right]\Big|_{t=0} = \frac{1}{2}\mathbf{P}X\left(\mathbf{P}^T\right)^c = \psi(X).$$

Hence, ψ is its own infinitesimal representation. □

Remark 5.4. *Straightforward computations show that the body projections of* $\mathrm{SO}_0^{\mathbf{J}}$, U_0, $\mathfrak{so}_0^{\mathbf{J}}$ *and* $\mathfrak{u}_0$ *are given by the sets*

$$\begin{aligned} \left[\mathrm{SO}_0^{\mathbf{J}}\right]_0 &= \left[\mathrm{SO}(2m) \cap \mathrm{Sp}_J(2m)\right] \times \left[\mathrm{SO}(2n) \cap \mathrm{Sp}_\Omega(2n)\right], & [\mathrm{U}_0]_0 &= \mathrm{U}(m) \times \mathrm{U}(n), \\ \left[\mathfrak{so}_0^{\mathbf{J}}\right]_0 &= \left[\mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)\right] \times \left[\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)\right], & [\mathfrak{u}_0]_0 &= \mathfrak{u}(m) \times \mathfrak{u}(n), \end{aligned}$$

where

$$\mathfrak{u}(k) = \{A_0 \in \mathbb{C}^{k\times k} : \left(A_0^T\right)^c + A_0 = 0\}$$

is the classical unitary Lie algebra in dimension k. Here, ψ is a Lie group isomorphism between $[\mathrm{SO}_0^{\mathbf{J}}]_0$ and $[\mathrm{U}_0]_0$ (respectively a Lie algebra isomorphism between $[\mathfrak{so}_0^{\mathbf{J}}]_0$ and $[\mathfrak{u}_0]_0$). The projections $\psi_{J_{2m}}$ and $\psi_{\Omega_{2n}}$ of ψ over $[\mathrm{SO}(2m)\cap \mathrm{Sp}_J(2m)]$ and $[\mathrm{SO}(2n)\cap \mathrm{Sp}_\Omega(2n)]$ ($[\mathfrak{so}(2m)\cap\mathfrak{sp}_J(2m)]$ and $[\mathfrak{so}(2n)\cap \mathfrak{sp}_\Omega(2n)]$), respectively, are given by

$$\psi_{J_{2m}}(A_0) = \frac{1}{2}PA_0\left(P^T\right)^c \qquad \psi_{\Omega_{2n}}(D_0) = \frac{1}{2}QD_0\left(Q^T\right)^c.$$

They define the Lie group (Lie algebra) isomorphisms

$$\begin{aligned} &\psi_{J_{2m}} : [\mathrm{SO}(2m)\cap \mathrm{Sp}_J(2m)] \to \mathrm{U}(m), && \psi_{J_{2m}} : [\mathfrak{so}(2m)\cap \mathfrak{sp}_J(2m)] \to \mathfrak{u}(m),\\ &\psi_{\Omega_{2n}} : [\mathrm{SO}(2n)\cap \mathrm{Sp}_\Omega(2n)] \to \mathrm{U}(n), && \psi_{\Omega_{2n}} : [\mathfrak{so}(2n)\cap \mathfrak{sp}_\Omega(2n)] \to \mathfrak{u}(n). \end{aligned}$$

In the Euclidean Clifford setting in superspace, it has been shown that the fundamental symmetry group SO_0 is connected but non-compact, the non-compactness being due to the realization of $\mathrm{Sp}_\Omega(2n)$ in the real projection of SO_0. The introduction of the complex structure, and the consequent refinement of the symmetry group, causes the corresponding body projection $[\mathrm{U}_0]_0 = \mathrm{U}(m)\times \mathrm{U}(n)$ to be compact, while U_0 remains non-compact. Indeed, an example of an unbounded sequence $\{M(k)\}_{k\in\mathbb{N}}$ in U_0 is

$$M(k) = \begin{pmatrix} I_m & -\frac{i}{2}k\,\mathfrak{b}\,E_{m\times n} \\ k\,\mathfrak{b}\,E_{n\times m} & I_n \end{pmatrix},$$

where $\mathfrak{b}\in \mathbb{C}\mathfrak{G}_N^{(odd)}$ and $E_{p\times q}$ is defined as in Lemma 4.3. As a consequence the map $\exp : \mathfrak{so}_0^{\mathbf{J}} \to \mathrm{SO}_0^{\mathbf{J}}$ may not be surjective. However, $\mathrm{SO}_0^{\mathbf{J}}$ can be fully described by products of exponentials acting in some special subalgebras of $\mathfrak{so}_0^{\mathbf{J}}$. Indeed, write $M\in \mathrm{SO}_0^{\mathbf{J}}$ as

$$M = M_0 + \mathbf{M} = M_0(I_{2m+2n} + \mathbf{L}),$$

where $M_0 \in [\mathrm{SO}_0^{\mathbf{J}}]_0$ is the real projection of M, $\mathbf{M}\in \mathrm{Mat}(2m|2n)(\mathbb{R}\mathfrak{G}_N^+)$ is its nilpotent projection and $\mathbf{L} = M_0^{-1}\mathbf{M}$. Since $[\mathrm{SO}_0^{\mathbf{J}}]_0 \cong \mathrm{U}(m)\times\mathrm{U}(n)$ is connected and compact, the exponential map is surjective on this group, see Corollary 11.10 in [50], whence we can write

$$M_0 = e^X, \qquad\qquad X \in [\mathfrak{so}(2m)\cap \mathfrak{sp}_J(2m)] \times [\mathfrak{so}(2n)\cap\mathfrak{sp}_\Omega(2n)].$$

As explained in Section 4.1.2, there is only one matrix in $\mathrm{Mat}(2m|2n)(\mathbb{R}\mathfrak{G}_N^+)$ whose exponential equals $I_{2m+2n}+\mathbf{L}$, viz $\mathbf{Z} = \ln(I_{2m+2n}+\mathbf{L})$. Following the decomposition of SO_0 in Theorem 4.4, we get that $\mathbf{Z}\in \mathfrak{so}_0(2m|2n)(\mathbb{R}\mathfrak{G}_N^+)$. In addition, we recall that $I_{2m+2n}+\mathbf{L}$ commutes with $\mathbf{J}$, meaning $\mathbf{JL} = \mathbf{LJ}$. Thence

$$\mathbf{Z} = \ln(I_{2m+2n}+\mathbf{L}) = \sum_{j=1}^{N}(-1)^{j+1}\frac{\mathbf{L}^j}{j}$$

commutes with $\mathbf{J}$ as well. In this way, we have obtained the following refinement of Theorem 4.4.

Theorem 5.1. *Every supermatrix* $M \in \mathrm{SO}_0^{\mathbf{J}}$ *can be written as*

$$M = e^X e^{\mathbf{Z}} \quad \textit{where} \quad \begin{cases} X \in [\mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)] \times [\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)], \\ \mathbf{Z} \in \mathfrak{so}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N^+), \end{cases}$$

with $\mathfrak{so}_0^{\mathbf{J}}(2m|2n)(\mathbb{R}\mathfrak{G}_N^+) := \mathfrak{so}_0^{\mathbf{J}} \cap \mathrm{Mat}(2m|2n)(\mathbb{R}\mathfrak{G}_N^+)$. *In addition, the element* $\mathbf{Z}$ *is unique.*

5.3 Spin realization of $\mathrm{SO}_0^{\mathbf{J}}$

In this section we aim at finding the spin realization of the group $\mathrm{SO}_0^{\mathbf{J}}$, i.e. the subgroup $\mathrm{Spin}_{\mathbf{J}}(2m|2n)(\mathfrak{G}_N)$ of $\mathrm{Spin}(2m|2n)(\mathfrak{G}_N)$ containing all spin elements which correspond to elements of $\mathrm{SO}_0^{\mathbf{J}}$ through the h-representation (4.15). To that end we must first find the representation of the Lie subalgebra $\mathfrak{so}_0^{\mathbf{J}} \subset \mathfrak{so}_0$ in the algebra of extended superbivectors, i.e. $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}}) \subset \mathbb{R}_{2m|2n}^{(2)E}(\mathfrak{G}_N)$. The exponentials of these bivectors yield all elements of $\mathrm{Spin}_{\mathbf{J}}(2m|2n)(\mathfrak{G}_N)$, leaving the super Dirac operators $\partial_{\mathbf{x}}$ and $\partial_{\mathbf{J(x)}}$ invariant.

5.3.1 The Lie algebras $\mathfrak{so}_0^{\mathbf{J}}$ and $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$.

Propositions 5.3 and 5.8 show that $\mathfrak{so}_0^{\mathbf{J}}$ can be described as the set of supermatrices of the form

$$M = \left(\begin{array}{cc|c} A_1 & A_2 & \grave{B}_1 \\ -A_2 & A_1 & \grave{B}_1\Omega_{2n} \\ \hline \grave{C}_1 & -\Omega_{2n}\grave{C}_1 & D \end{array}\right) \qquad \text{with} \qquad \begin{cases} A_1^T + A_1 = 0, \\ A_2^T - A_2 = 0, \\ \grave{B}_1 - \grave{C}_1^{\,T}\Omega_{2n} = 0, \\ D^T + D = 0, \end{cases} \tag{5.10}$$

since the conditions for being an element of $\mathfrak{so}_0$ given in (4.9) can rewritten in this case as

$$\begin{aligned} A^T + A &= \begin{pmatrix} A_1^T + A_1 & -A_2^T + A_2 \\ A_2^T - A_2 & A_1^T + A_1 \end{pmatrix} = 0, \\ \grave{B} - \grave{C}^T\Omega_{2n} &= \begin{pmatrix} \grave{B}_1 \\ \grave{B}_1\Omega_{2n} \end{pmatrix} - \begin{pmatrix} \grave{C}_1^{\,T} \\ \grave{C}_1^{\,T}\Omega_{2n} \end{pmatrix}\Omega_{2n} = 0, \\ D^T\Omega_{2n} + \Omega_{2n}D &= (D^T + D)\Omega_{2n} = 0. \end{aligned}$$

The relations in (5.10) provide an easy way of computing the dimension of $\mathfrak{so}_0^{\mathbf{J}}$. Indeed, we can write $\mathfrak{so}_0^{\mathbf{J}}$ as the direct sum of the real subspaces W_1, W_2, W_3, W_4 where

$$W_1 = \left\{ \left(\begin{array}{cc|c} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) : A_1^T + A_1 = 0,\ A_1 \in \left(\mathbb{R}\mathfrak{G}_N^{(ev)}\right)^{m\times m} \right\},$$

$$W_2 = \left\{ \left(\begin{array}{cc|c} 0 & A_2 & 0 \\ -A_2 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) : A_2^T - A_2 = 0,\ A_2 \in \left(\mathbb{R}\mathfrak{G}_N^{(ev)}\right)^{m\times m} \right\},$$

$$W_3 = \left\{ \left(\begin{array}{cc|c} 0 & 0 & \grave{C_1}^T \Omega_{2n} \\ 0 & 0 & -\grave{C_1}^T \\ \hline \grave{C_1} & -\Omega_{2n}\grave{C_1} & 0 \end{array}\right) : \grave{C_1} \in \left(\mathbb{R}\mathfrak{G}_N^{(odd)}\right)^{2n\times m} \right\},$$

$$W_4 = \left\{ \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & D \end{array}\right) : D\Omega_{2n} = \Omega_{2n} D,\ D^T + D = 0,\ D \in \left(\mathbb{R}\mathfrak{G}_N^{(ev)}\right)^{2n\times 2n} \right\},$$

This leads to

$$\dim W_1 = 2^{N-1}\frac{m(m-1)}{2}, \qquad \dim W_2 = 2^{N-1}\frac{m(m+1)}{2},$$
$$\dim W_3 = 2^{N-1}2mn, \qquad \dim W_4 = 2^{N-1}n^2,$$

whence

$$\dim \mathfrak{so}_0^{\mathbf{J}} = 2^{N-1}\left[\frac{m(m-1)}{2} + \frac{m(m+1)}{2} + 2mn + n^2\right] = 2^{N-1}(m+n)^2.$$

We now look for the representation of $\mathfrak{so}_0^{\mathbf{J}}$ in the Lie algebra of extended bivectors. To that end, consider $M \in \mathfrak{so}_0$, $B = \phi^{-1}(M) \in \mathbb{R}^{(2)E}_{2m|2n}(\mathfrak{G}_N)$ and $\mathbf{x} \in \mathbf{S}$. Then

$$\begin{aligned} M\mathbf{J} = \mathbf{J}M &\iff M\mathbf{J}\mathbf{x} = \mathbf{J}M\mathbf{x}, \\ &\iff \phi(B)(\mathbf{J}(\mathbf{x})) = \mathbf{J}(\phi(B)(\mathbf{x})), \\ &\iff [B, \mathbf{J}(\mathbf{x})] = \mathbf{J}\left([B,\mathbf{x}]\right) = [\mathbf{J}(B), \mathbf{J}(\mathbf{x})], \\ &\iff B = \mathbf{J}(B). \end{aligned}$$

We recall that $\mathbf{J}$ is an automorphism on $\mathcal{A}_{2m,2n} \otimes \mathbb{R}\mathfrak{G}_N$.

In this way, we have obtained that

$$\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}}) = \{B \in \mathbb{R}^{(2)E}_{2m|2n}(\mathfrak{G}_N) : B = \mathbf{J}(B)\}.$$

In order to find the explicit form of the elements in $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$ we need the following computations:

$$\begin{cases} \mathbf{J}(e_j e_k) = e_{m+j}e_{m+k} & 1 \le j < k \le m, \\ \mathbf{J}(e_j e_{m+k}) = e_k e_{m+j} & 1 \le j,k \le m, \\ \mathbf{J}(e_{m+j}e_{m+k}) = e_j e_k & 1 \le j < k \le m, \end{cases}$$

$$\begin{cases} \mathbf{J}(e_j\grave{e}_{2k-1}) = e_{m+j}\grave{e}_{2k} & 1\leq j\leq m,\ 1\leq k\leq n,\\ \mathbf{J}(e_j\grave{e}_{2k}) = -e_{m+j}\grave{e}_{2k-1} & 1\leq j\leq m,\ 1\leq k\leq n,\\ \mathbf{J}(e_{m+j}\grave{e}_{2k-1}) = -e_j\grave{e}_{2k} & 1\leq j\leq m,\ 1\leq k\leq n,\\ \mathbf{J}(e_{m+j}\grave{e}_{2k}) = e_j\grave{e}_{2k-1} & 1\leq j\leq m,\ 1\leq k\leq n, \end{cases}$$

$$\begin{cases} \mathbf{J}(\grave{e}_{2j-1}\odot\grave{e}_{2k-1}) = \grave{e}_{2j}\odot\grave{e}_{2k} & 1\leq j\leq k\leq n,\\ \mathbf{J}(\grave{e}_{2j-1}\odot\grave{e}_{2k}) = -\grave{e}_{2j}\odot\grave{e}_{2k-1} & 1\leq j\leq k\leq n,\\ \mathbf{J}(\grave{e}_{2j}\odot\grave{e}_{2k-1}) = -\grave{e}_{2j-1}\odot\grave{e}_{2k} & 1\leq j< k\leq n,\\ \mathbf{J}(\grave{e}_{2j}\odot\grave{e}_{2k}) = \grave{e}_{2j-1}\odot\grave{e}_{2k-1} & 1\leq j\leq k\leq n. \end{cases}$$

Applying $\mathbf{J}$ to both sides of the equality

$$B = \sum_{1\leq j<k\leq 2m} b_{j,k}\, e_je_k + \sum_{\substack{1\leq j\leq 2m\\ 1\leq k\leq 2n}} \grave{b}_{j,k}\, e_j\grave{e}_k + \sum_{1\leq j\leq k\leq 2n} B_{j,k}\, \grave{e}_j\odot\grave{e}_k,$$

we obtain that $B=\mathbf{J}(B)$ is equivalent to

$$\begin{cases} b_{j,k} = b_{m+j,m+k} & 1\leq j<k\leq m,\\ b_{j,m+k} = b_{k,m+j} & 1\leq j,k\leq m, \end{cases}$$

$$\begin{cases} \grave{b}_{j,2k-1} = \grave{b}_{m+j,2k} & 1\leq j\leq m,\ 1\leq k\leq n,\\ \grave{b}_{j,2k} = -\grave{b}_{m+j,2k-1} & 1\leq j\leq m,\ 1\leq k\leq n, \end{cases}$$

$$\begin{cases} B_{2j-1,2k-1} = B_{2j,2k} & 1\leq j\leq k\leq n,\\ B_{2j-1,2k} = -B_{2j,2k-1} & 1\leq j<k\leq n,\\ B_{2j-1,2j} = 0 & 1\leq j\leq n. \end{cases}$$

Hence, $B\in\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$ if and only if $B=B_1+B_2+B_3$ where B_1,B_2,B_3 are of the form

$$\begin{aligned} B_1 =& \sum_{1\leq j<k\leq m} b_{j,k}(e_je_k + e_{m+j}e_{m+k}) + \sum_{j=1}^m b_{j,m+j}\, e_je_{m+j}\\ &+ \sum_{1\leq j<k\leq m} b_{j,m+k}(e_je_{m+k} + e_ke_{m+j}),\\ B_2 =& \sum_{\substack{1\leq j\leq m\\ 1\leq k\leq n}} \grave{b}_{j,2k-1}(e_j\grave{e}_{2k-1} + e_{m+j}\grave{e}_{2k}) + \grave{b}_{j,2k}(e_j\grave{e}_{2k} - e_{m+j}\grave{e}_{2k-1}),\\ B_3 =& \sum_{1\leq j\leq k\leq n} B_{2j-1,2k-1}(\grave{e}_{2j-1}\odot\grave{e}_{2k-1} + \grave{e}_{2j}\odot\grave{e}_{2k})\\ &+ \sum_{1\leq j<k\leq n} B_{2j-1,2k}(\grave{e}_{2j-1}\odot\grave{e}_{2k} - \grave{e}_{2j}\odot\grave{e}_{2k-1}). \end{aligned}$$

Summarizing, we have obtained the following result.

Proposition 5.9. *Let $\{b_1, \dots, b_{2^{N-1}}\}$ and $\{\grave{b}_1, \dots, \grave{b}_{2^{N-1}}\}$ be the canonical basis of $\mathbb{R}\mathfrak{G}_N^{(ev)}$ and $\mathbb{R}\mathfrak{G}_N^{(odd)}$ respectively. Then, a basis for $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$ is given by the elements*

$$\begin{aligned}
b_r(e_j e_k + e_{m+j} e_{m+k}), &\qquad 1 \le r \le 2^{N-1},\ 1 \le j < k \le m,\\
b_r(e_j e_{m+k} + e_k e_{m+j}), &\qquad 1 \le r \le 2^{N-1},\ 1 \le j \le k \le m,\\
\grave{b}_r(e_j \grave{e}_{2k-1} + e_{m+j} \grave{e}_{2k}), &\qquad 1 \le r \le 2^{N-1},\ 1 \le j \le m,\ 1 \le k \le n,\\
\grave{b}_r(e_j \grave{e}_{2k} - e_{m+j} \grave{e}_{2k-1}), &\qquad 1 \le r \le 2^{N-1},\ 1 \le j \le m,\ 1 \le k \le n,\\
b_r(\grave{e}_{2j-1} \odot \grave{e}_{2k-1} + \grave{e}_{2j} \odot \grave{e}_{2k}), &\qquad 1 \le r \le 2^{N-1},\ 1 \le j \le k \le n,\\
b_r(\grave{e}_{2j-1} \odot \grave{e}_{2k} - \grave{e}_{2j} \odot \grave{e}_{2k-1}), &\qquad 1 \le r \le 2^{N-1},\ 1 \le j < k \le n.
\end{aligned}$$

Remark 5.5. *Obviously the algebras $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$ and $\mathfrak{so}_0^{\mathbf{J}}$ are isomorphic. This fact can be double checked through the previous result from which it follows that* $\dim \phi^{-1}(\mathfrak{so}_0^{\mathbf{J}}) = 2^{N-1}(m+n)^2$.

5.3.2 The group $\mathrm{Spin}_{\mathbf{J}}(2m|2n)(\mathfrak{G}_N)$

We may now introduce the group

$$\mathrm{Spin}_{\mathbf{J}} \equiv \mathrm{Spin}_{\mathbf{J}}(2m|2n)(\mathfrak{G}_N) := \{e^{B_1} \cdots e^{B_k} : B_1, \dots, B_k \in \phi^{-1}(\mathfrak{so}_0^{\mathbf{J}}),\ k \in \mathbb{N}\}.$$

This is a Lie subgroup of $\mathrm{Spin}(2m|2n)(\mathfrak{G}_N)$ which completely describes $\mathrm{SO}_0^{\mathbf{J}}$ through the h-representation, as shown in the next result.

Proposition 5.10. *The group $\mathrm{Spin}_{\mathbf{J}}$ covers $\mathrm{SO}_0^{\mathbf{J}}$.*

Proof.

The Lie group isomorphism $\mathrm{SO}_0^{\mathbf{J}} \cong \mathrm{U}_0$ shows, in view of Proposition 5.6, that $\mathrm{SO}_0^{\mathbf{J}}$ is connected. Hence, for every $M \in \mathrm{SO}_0^{\mathbf{J}}$ there exist $X_1, \dots, X_k \in \mathfrak{so}_0^{\mathbf{J}}$ such that $e^{X_1} \cdots e^{X_k} = M$, see Corollary 3.47 in [50]. Taking $B_j = \phi^{-1}(X_j)$, $j = 1, \dots, k$, and using the relations (4.15) and (4.16), we get for $\mathbf{x} \in \mathbf{S}$ that

$$\begin{aligned}
M\mathbf{x} &= e^{X_1} \cdots e^{X_k}\mathbf{x} = e^{\phi(B_1)} \cdots e^{\phi(B_k)}\mathbf{x}\\
&= h(e^{B_1}) \circ \cdots \circ h(e^{B_k})[\mathbf{x}]\\
&= e^{B_1} \cdots e^{B_k}\, \mathbf{x}\, e^{-B_k} \cdots e^{-B_1} = h(e^{B_1} \cdots e^{B_k})[\mathbf{x}].
\end{aligned}$$

whence $M = h(s)$ with $s = e^{B_1} \cdots e^{B_k} \in \mathrm{Spin}_{\mathbf{J}}$. □

The decomposition for $\mathrm{SO}_0^{\mathbf{J}}$ given in Theorem 5.1 allows to describe the $\mathrm{Spin}_{\mathbf{J}}$-covering of $\mathrm{SO}_0^{\mathbf{J}}$ more precisely. Indeed, following the decomposition given in (4.17) we obtain the Lie subalgebras of $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$

$$\begin{aligned}
\Xi_1^{\mathbf{J}} &= \Xi_1 \cap \phi^{-1}(\mathfrak{so}_0^{\mathbf{J}}) = \phi^{-1}\big([\mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)] \times [\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)]\big),\\
\Xi_3^{\mathbf{J}} &= \Xi_3 \cap \phi^{-1}(\mathfrak{so}_0^{\mathbf{J}}) = \phi^{-1}(\mathfrak{so}_0^{\mathbf{J}}(2m|2n)(\mathfrak{G}_N^{+})),
\end{aligned}$$

yielding the decomposition $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}}) = \Xi_1^{\mathbf{J}} \oplus \Xi_3^{\mathbf{J}}$ (observe that $\Xi_2 \cap \phi^{-1}(\mathfrak{so}_0^{\mathbf{J}}) = \{0\}$). This leads to the subset $\Xi_{\mathbf{J}} = \exp(\Xi_1^{\mathbf{J}}) \exp(\Xi_3^{\mathbf{J}}) \subset \Xi \cap \mathrm{Spin}_{\mathbf{J}}$, which can be seen to constitute a double covering of $\mathrm{SO}_0^{\mathbf{J}}$.

It is easily seen that $\Xi_1^{\mathbf{J}}$ is composed of all elements in $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$ with real coefficients, i.e. $\Xi_1^{\mathbf{J}} = \left[\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})\right]_0$, while $\Xi_3^{\mathbf{J}}$ contains all nilpotent elements of $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$. In this way, we obtain from Proposition 5.9 that a basis for $\Xi_1^{\mathbf{J}}$ is given by

$$\begin{array}{llll} e_j e_k + e_{m+j} e_{m+k}, & 1 \leq j < k \leq m, & \grave{e}_{2j-1} \odot \grave{e}_{2k-1} + \grave{e}_{2j} \odot \grave{e}_{2k}, & 1 \leq j \leq k \leq n, \\ e_j e_{m+k} + e_k e_{m+j}, & 1 \leq j \leq k \leq m, & \grave{e}_{2j-1} \odot \grave{e}_{2k} - \grave{e}_{2j} \odot \grave{e}_{2k-1}, & 1 \leq j < k \leq n. \end{array}$$

We recall that an important element of $\mathrm{SO}_0^{\mathbf{J}}$ is $\mathbf{J}$ itself. In order to find the spin element that represents $\mathbf{J}$, or equivalently, an element $B_{\mathbf{J}} \in \Xi_1^{\mathbf{J}}$ such that $e^{\phi(B_{\mathbf{J}})} = \mathbf{J}$, we first compute

$$\ln(\mathbf{J}) = \begin{pmatrix} \ln J_{2m} & 0 \\ 0 & \ln \Omega_{2n} \end{pmatrix}.$$

Observe that both J_{2m} and Ω_{2n} have eigenvalues $i, -i$, with multiplicity m and n, respectively. It easily follows that

$$\ln J_{2m} = \frac{\pi}{2} J_{2m}, \qquad \ln \Omega_{2m} = \frac{\pi}{2} \Omega_{2m}.$$

Using the relations (5.2) we get $\ln \mathbf{J} = \frac{\pi}{2}\mathbf{J} = -\frac{\pi}{4}\phi(\mathbf{B})$, or equivalently $B_{\mathbf{J}} = -\frac{\pi}{4}\mathbf{B}$. The spin element $s_{\mathbf{J}}$ associated to $\mathbf{J}$ thus is given by

$$\begin{aligned} s_{\mathbf{J}} = \exp(-\frac{\pi}{4}\mathbf{B}) &= \prod_{j=1}^{m} \exp\left(-\frac{\pi}{4} e_j e_{m+j}\right) \prod_{j=1}^{n} \exp\left(\frac{\pi}{4}(\grave{e}_{2j-1}{}^2 + \grave{e}_{2j}{}^2)\right) \\ &= \frac{1}{2^{m/2}} \prod_{j=1}^{m} (1 - e_j e_{m+j}) \prod_{j=1}^{n} \exp\left(\frac{\pi}{4}(\grave{e}_{2j-1}{}^2 + \grave{e}_{2j}{}^2)\right). \end{aligned}$$

Remark 5.6. *In the purely fermionic case, i.e. $m = 0$, the element $s_{\mathbf{J}}$ may be identified with the operator*

$$\exp\left[\frac{\pi}{4} i \sum_{j=1}^{n} \left(\partial_{a_j}^2 - a_j^2\right)\right] = \exp\left(-n\frac{\pi}{4} i\right) \mathcal{F},$$

see remarks 4.15 and 4.16. Here, $\mathcal{F}$ denotes the n-dimensional Fourier transform.

The fundamental extended bivector $\mathbf{B}$ provides other characterizations for $\phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$ and $\mathrm{Spin}_{\mathbf{J}}$.

Proposition 5.11. *Let $B \in \mathbb{R}_{2m|2n}^{(2)E}(\mathfrak{G}_N)$. Then $\phi(B) \in \mathfrak{so}_0^{\mathbf{J}}$ if and only if $B\mathbf{B} = \mathbf{B}B$.*

Proof.

It suffices to observe that

$$\phi(B)\mathbf{J} = \mathbf{J}\phi(B) \iff [\phi(B), \phi(\mathbf{B})] = 0 \iff \phi([B, \mathbf{B}]) = 0 \iff [B, \mathbf{B}] = 0.$$

□

Corollary 5.2. *Let $s \in \mathrm{Spin}_{\mathbf{J}}$. Then $s s_{\mathbf{J}} = s_{\mathbf{J}} s$.*

The following table summarizes the main aspects concerning the spin realization of the unitary group in both Hermitian Clifford analysis and its extension to superspace.

	Hermitian Clifford analysis	Hermitian Clifford analysis in superspace
Bilinear form	$\langle \underline{Z}, \underline{U}\rangle_{\mathbb{C}} = \sum_{j=1}^{m} z_j u_j^c$	$\langle \mathbf{Z}, \mathbf{U}\rangle_{\mathbb{C}} = \sum_{j=1}^{m} z_j u_j^c - \frac{i}{2}\sum_{j=1}^{n} \grave{z}_j \grave{u}_j^c$
Invariance	$\mathrm{U}(m) \cong \mathrm{SO}(2m) \cap \mathrm{Sp}_J(2m)$	$\mathrm{U}_0(m\|n)(\mathbb{C}\mathfrak{G}_N) \cong \mathrm{SO}_0^{\mathbf{J}}(2m\|2n)(\mathbb{R}\mathfrak{G}_N)$
Body	$\mathrm{U}(m)$	$\mathrm{U}(m) \times \mathrm{U}(n)$
Lie algebra	$\mathfrak{u}(m) \cong \mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)$	$\mathfrak{u}_0(m\|n)(\mathbb{C}\mathfrak{G}_N) \cong \mathfrak{so}_0^{\mathbf{J}}(2m\|2n)(\mathbb{R}\mathfrak{G}_N)$
Real dimension	m^2	$2^{N-1}(m+n)^2$
Bivectors	$B \in \mathbb{R}_{0,2m}^{(2)} : B = \underline{J}(B)$	$B \in \mathbb{R}_{2m\|2n}^{(2)E}(\mathfrak{G}_N) : B = \mathbf{J}(B)$
Iwasawa decomposition	$M_0 = e^X$, $X \in \mathfrak{u}(m)$	$M = e^X e^{\mathbf{Z}}$, $M \in \mathrm{SO}_0^{\mathbf{J}}$ $X \in [\mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)] \times [\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)]$, $\mathbf{Z} \in \mathfrak{so}_0(2m\|2n)(\mathbb{R}\mathfrak{G}_N^+)$
Spin group/ elements	$\mathrm{Spin}_{\underline{J}}(2m)$ $s \in \mathrm{Spin}(2m) : [h(s), \underline{J}] = 0$	$\mathrm{Spin}_{\mathbf{J}}(2m\|2n)(\mathfrak{G}_N)$ $e^{B_1} \cdots e^{B_k} : B_j \in \mathbb{R}_{m\|2n}^{(2)E}(\mathfrak{G}_N), \mathbf{J}(B_j) = B_j$

Table 5.1: Comparative overview of the Spin realization of the unitary group.

5.3.3 $\mathrm{Spin}_{\mathbf{J}}$-invariance of $\partial_{\mathbf{J}(\mathbf{x})}$

Our final goal is to show the invariance of the twisted Dirac operator $\partial_{\mathbf{J}(\mathbf{x})}$ under the H and L actions of the group $\mathrm{Spin}_{\mathbf{J}}$, i.e.

$$[\partial_{\mathbf{J}(\mathbf{x})}, L(s)] = 0 = [\partial_{\mathbf{J}(\mathbf{x})}, H(s)], \qquad \forall s \in \mathrm{Spin}_{\mathbf{J}}. \tag{5.11}$$

Following the same reasoning as in Section 5.1, it suffices to prove that $\partial_{\mathbf{J}(\mathbf{x})}$ commutes with the infinitesimal representation $dL(B)$ of $L(e^B)$ for every $B \in \phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$. Using

Proposition 5.1, we obtain for $B = \mathbf{J}(B)$ that

$$\begin{aligned}
\mathbf{J}\left(dL(B)[F]\right) &= \mathbf{J}\left(BF - \sum_{j=1}^{m} \Big([B,x]\Big)_j \partial_{x_j}[F] - \sum_{j=1}^{2n} \Big([B,x]\Big)_{m+j} \partial_{\grave{x}_j}[F]\right) \\
&= B\mathbf{J}(F) - \sum_{j=1}^{m} \Big([B,x]\Big)_j \partial_{x_j}[\mathbf{J}(F)] - \sum_{j=1}^{2n} \Big([B,x]\Big)_{m+j} \partial_{\grave{x}_j}[\mathbf{J}(F)] \\
&= \; dL(B)[\mathbf{J}(F)].
\end{aligned}$$

Corollary 5.1 then yields that

$$\partial_{\mathbf{x}}\left[dL(B)[G]\right] = dL(B)[\partial_{\mathbf{x}}[G]],$$

whence, applying $\mathbf{J}$ to both sides and writing $F = \mathbf{J}(G)$, we get for every $B \in \phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$ that

$$\partial_{\mathbf{J}(\mathbf{x})}\left[dL(B)[F]\right] = dL(B)[\partial_{\mathbf{J}(\mathbf{x})}[F]].$$

In this way, we have proved that $[\partial_{\mathbf{J}(\mathbf{x})}, dL(B)] = 0$ for every $B \in \phi^{-1}(\mathfrak{so}_0^{\mathbf{J}})$ and in consequence (5.11) holds.

6

Distributions and integration in superspace

Integration on superspace is based on the notion of the Berezin integral given by

$$\int_B = \pi^{-n}\, \partial_{\grave{x}_{2n}} \cdots \partial_{\grave{x}_1},$$

see [7]. This functional plays the same rôle in the Grassmann algebra $\mathfrak{G}_{2n}$ as the general real integral

$$\int_{\mathbb{R}^m} dV_{\underline{x}}$$

in classical analysis. Traditionally, the Berezin integral is combined with the classical real integration in order to integrate superfunctions over real domains, i.e. the integral of a superfunction F over $\Omega \subset \mathbb{R}^m$ is given by

$$\int_\Omega \int_B F(\underline{x}, \underline{\grave{x}})\, dV_{\underline{x}}. \tag{6.1}$$

Some important classical results, such as a Stokes and a Cauchy-Pompeiu formula have been established for the super Dirac operator, see [38]. Yet, these extensions have important limitations since they only consist of real integration combined with the Berezin integral, instead of considering general integration over domains and surfaces defined in terms of both commuting and anti-commuting co-ordinates in superspace.

The study of spherical harmonics (and monogenics) has lead to an important development of integration theory in superspace. For example, in [35, 31] the Berezin integral was related to more familiar types of integration like Pizzetti's formula, see [61]. In this way, the integral of a polynomial P over the supersphere was introduced as follows:

$$\int_{SS} P = \sum_{j=0}^{\infty} (-1)^j \frac{2\pi^{M/2}}{2^{2j}\, j!\, \Gamma(j+M/2)} (\Delta^j_{m|2n} P)(0), \tag{6.2}$$

where $\Delta_{m|2n} := -\partial_{\mathbf{x}}^2$ is the super Laplace operator and $M = m - 2n$ the corresponding superdimension. In the later work [24], formula (6.2) was extended to more general superfunctions on the supersphere by considering the integral

$$\int_{SS} F = 2 \int_{\mathbb{R}^m} \int_B \delta(\mathbf{x}^2 + 1) F \, dV_{\underline{x}}, \tag{6.3}$$

where $\delta(\mathbf{x}^2+1)$ denotes the Dirac distribution on the unit supersphere. Following this last distributional approach, some important problems were solved. In particular, closed formulas for the Pizzetti integral and a Cauchy-Pompeiu formula for the supersphere were obtained, see [24].

Nevertheless, the approaches given in (6.1) and (6.3) are still limited. They only refer to the particular cases of integration of superfunctions over real domains or over the supersphere. The main goal of this chapter is to extend and unify both approaches by defining integration over general domains and surfaces in superspace. The principal idea of this extension comes from a distributional approach to classical real integration. Indeed, suppose that $\Omega \subset \mathbb{R}^m$ is a domain (m dimensional manifold) determined by some inequality $g_0(x_1, \ldots, x_m) < 0$ and let $\partial\Omega$ be its boundary ($m-1$ dimensional manifold in $\mathbb{R}^m$) determined by the equation $g_0(x_1, \ldots, x_m) = 0$. Then the integrals over Ω and $\partial\Omega$ can be rewritten as[1]

$$\int_\Omega (\cdot)\, dV_{\underline{x}} = \int_{\mathbb{R}^m} H(-g_0(\underline{x}))(\cdot)\, dV_{\underline{x}}, \quad \text{and} \quad \int_{\partial\Omega} (\cdot)\, dS_{\underline{w}} = \int_{\mathbb{R}^m} \delta(g_0(\underline{x}))\, |\partial_{\underline{x}}[g_0](\underline{x})|\, (\cdot)\, dV_{\underline{x}},$$

respectively; where H is the Heaviside distribution and δ the Dirac distribution. In this way, one may see the integrals $\int_\Omega$ and $\int_{\partial\Omega}$ not as functionals depending on geometrical sets of points Ω and $\partial\Omega$; but as functionals depending on the action of the Heaviside or Dirac distributions on a fixed phase function g_0.

As will be shown in this chapter, this last approach is the more suitable one to extend domain and surface integrals to superspace. In particular, we will illustrate it by integrating over a super-paraboloid and super-hyperboloid. Moreover, this approach will be used to obtain a Cauchy-Pompeiu formula, valid not only for real domains and for the superball, but for every domain with smooth boundary in superspace. This allows to follow a completely analytical method which uses the Cauchy kernel as a true distribution rather than as a smooth function with a singularity at the origin. This distributional Cauchy formula will play an essential rôle in obtaining a Bochner-Martinelli formula for holomorphic functions in superspace, see Chapter 7.

[1]These formulas will be discussed in detail in section 6.3.

6.1 Superfunctions

From now on, we work with superfunctions $F(\mathbf{x})$ of the supervector variable $\mathbf{x}$, see (3.8). In particular, we will consider these functions as elements of the spaces $\mathcal{F} \otimes \mathfrak{G}_{2n}$ where $\mathcal{F} = C^k(\Omega), C^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^m$. Following the classical approach, the (real) support *supp* F of a superfunction F is defined as the closure of the set of all points in $\mathbb{R}^m$ for which the function $F(\cdot, \underline{x\grave{}}) : \mathbb{R}^m \to \mathfrak{G}_{2n}$ is not zero, see (3.8). From this definition, it immediately follows that

$$supp\ F = \bigcup_{A\subset\{1,\dots,2n\}} supp\ F_A.$$

Definition 3.2 shows a nice way of producing interesting even superfunctions out of known functions from real analysis. Let us consider in particular a smooth function $F \in C^\infty(\mathbb{R})$ and an even real superfunction[2] $a = a_0 + \mathbf{a} \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ where a_0 and $\mathbf{a}$ are the body and nilpotent part of a, respectively. The superfunction $F(a(\mathbf{x})) \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ then is defined by

$$F(a) = F(a_0 + \mathbf{a}) = \sum_{j=0}^{n} \frac{\mathbf{a}^j}{j!}\, F^{(j)}(a_0). \tag{6.4}$$

Straightforward calculations show that the above expression is independent of the splitting of the even superfunction a if the function F is analytic in $\mathbb{R}$.

Proposition 6.1. *Let $a, b \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ be real superfunctions such that $a = a_0 + \mathbf{a}$, $b = b_0 + \mathbf{b}$, where a_0, b_0 are the bodies of a, b respectively and $\mathbf{a}$, $\mathbf{b}$ are the corresponding nilpotent parts. Then, for every analytic function $F \in C^\infty(\mathbb{R})$ the following statements hold.*

(i) $F(a + b) = \sum_{j=0}^{n} \frac{\mathbf{b}^j}{j!}\, F^{(j)}(a + b_0)$,

(ii) $F(a + b) = \sum_{j=0}^{\infty} \frac{b_0^j}{j!}\, F^{(j)}(a + \mathbf{b})$,

(iii) $F(a + b) = \sum_{j=0}^{\infty} \frac{b^j}{j!}\, F^{(j)}(a)$.

The easiest application of the composition (6.4) is obtained when defining arbitrary real powers of even superfunctions. Let $a = a_0 + \mathbf{a} \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ be a real superfunction and $p \in \mathbb{R}$, then for $a_0 > 0$ we define

$$a^p = \sum_{j=0}^{n} \frac{\mathbf{a}^j}{j!}\, (-1)^j\, (-p)_j\, a_0^{p-j}, \quad \text{where} \quad (q)_j = \begin{cases} 1, & j = 0, \\ q(q+1)\cdots(q+j-1), & j > 0, \end{cases} \tag{6.5}$$

[2]We recall that a being a real superfunction means that $a = \sum_A a_A \underline{x\grave{}}_A$ where all the elements a_A are real-valued functions.

is the rising Pochhammer symbol. Observe that if the numbers q and $q+j$ are in the set $\mathbb{R}\setminus\{0,-1,-2,\ldots\}$ we can write $(q)_j = \dfrac{\Gamma(q+j)}{\Gamma(q)}$. Making use of this definition of power function in superspace, we can easily prove that its basic properties still hold in this setting.

Lemma 6.1. *Let $a = a_0 + \mathbf{a}$ and $b = b_0 + \mathbf{b}$ be real superfunctions in $C^\infty(\mathbb{R}^m)\otimes\mathfrak{G}_{2n}^{(ev)}$ such that $a_0, b_0 > 0$. Then, for every pair $p, q \in \mathbb{R}$ we have*

$$(i)\ a^p a^q = a^{p+q}, \qquad (ii)\ (ab)^p = a^p b^p, \qquad (iii)\ (a^p)^q = a^{pq}.$$

Proof.

For $a_0, b_0 > 0$, the equalities

$$\begin{aligned}(a_0+X)^p(a_0+X)^q &= (a_0+X)^{p+q},\\ [(a_0+X)(b_0+Y)]^p &= (a_0+X)^p(b_0+Y)^p,\\ ((a_0+X)^p)^q &= (a_0+X)^{pq},\end{aligned}$$

are identities in formal power series in the indeterminates X and Y. Then, making the substitutions $X = \mathbf{a}$ and $Y = \mathbf{b}$ we obtain i), ii) and iii). Observe that the nilpotency of $\mathbf{a}$ and $\mathbf{b}$ avoids every possible convergence issue. □

The *absolute value* function can be defined for real superfunctions in $C^\infty(\mathbb{R}^m)\otimes\mathfrak{G}_{2n}^{(ev)}$ by

$$|a| = (a^2)^{1/2} = \begin{cases} a & \text{if } a_0 \geq 0,\\ -a & \text{if } a_0 < 0.\end{cases}$$

As in the classical case, the absolute value function can be extended to the supervector variable $\mathbf{x}$, since its square is an even super-polynomial, i.e.

$$\mathbf{x}^2 = -\sum_{j=1}^m x_j^2 + \sum_{j=1}^n \grave{x}_{2j-1}\grave{x}_{2j} \in C^\infty(\mathbb{R}^m)\otimes\mathfrak{G}_{2n}^{(ev)}.$$

It is clear that $-\mathbf{x}^2$ has non-negative body. Hence, the element $(-\mathbf{x}^2)^{1/2}$ is well defined. In this way, we define the absolute value of a supervector by

$$|\mathbf{x}| = (-\mathbf{x}^2)^{1/2} = \left(|\underline{x}|^2 - \underline{\grave{x}}^2\right)^{1/2} = \sum_{j=0}^n \frac{(-1)^j\underline{\grave{x}}^{2j}}{j!}\,\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}-j)}|\underline{x}|^{1-2j},$$

where $|\underline{x}| = \left(\sum_{j=1}^m x_j^2\right)^{\frac{1}{2}}$ as usual.

Proposition 6.2. *Let $\mathbf{x}, \mathbf{y}$ be supervector variables and $a \in C^\infty(\mathbb{R}^m)\otimes\mathfrak{G}_{2n}^{(ev)}$ be a real superfunction. Then,* *(i)* $|a\mathbf{x}| = |a||\mathbf{x}|$,

(ii) $|\mathbf{x} + a\mathbf{y}| = |\mathbf{x}| + aF(\mathbf{x}, \mathbf{y}, a)$ *where* $F(\mathbf{x}, \mathbf{y}, a)$ *is a superfunction depending on* $\mathbf{x}, \mathbf{y}$ *and* a*.*

Proof.

(i) By Lemma 6.1 we get

$$|a\mathbf{x}| = \left(-(a\mathbf{x})^2\right)^{\frac{1}{2}} = \left(a^2(-\mathbf{x}^2)\right)^{\frac{1}{2}} = \left(a^2\right)^{\frac{1}{2}} \left(-\mathbf{x}^2\right)^{\frac{1}{2}} = |a||\mathbf{x}|.$$

(ii) We first write

$$(\mathbf{x} + a\mathbf{y})^2 = \mathbf{x}^2 + a\{\mathbf{x}, \mathbf{y}\} + a^2\mathbf{y}^2 = \mathbf{x}^2 - av,$$

where $v = -\{\mathbf{x}, \mathbf{y}\} - a\mathbf{y}^2$ is an even element. Then, using Proposition 6.1 iii), we get

$$\begin{aligned} |\mathbf{x} + a\mathbf{y}| &= \left(-(\mathbf{x} + a\mathbf{y})^2\right)^{\frac{1}{2}} = \left(-\mathbf{x}^2 + av\right)^{\frac{1}{2}} \\ &= \sum_{j=0}^{\infty} \frac{(av)^j}{j!} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} - j)} (-\mathbf{x}^2)^{\frac{1}{2}-j} \\ &= |\mathbf{x}| + aF(\mathbf{x}, \mathbf{y}, a), \end{aligned}$$

where

$$F(\mathbf{x}, \mathbf{y}, a) = \sum_{j=1}^{\infty} \frac{a^{j-1}v^j}{j!} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} - j)} (-\mathbf{x}^2)^{\frac{1}{2}-j}.$$

□

As usual, we say that a function $F \in C^1(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ (Ω being an open subset of $\mathbb{R}^m$) is (left) *super-monogenic* if $\partial_{\mathbf{x}}[F] = 0$. As the super Dirac operator factorizes the super Laplace operator:

$$\Delta_{m|2n} = -\partial_{\mathbf{x}}^2 = \sum_{j=1}^{m} \partial_{x_j}^2 - 4 \sum_{j=1}^{n} \partial_{\grave{x}_{2j-1}} \partial_{\grave{x}_{2j}},$$

monogenicity also constitutes a refinement of harmonicity in superanalysis. More details on the theory of super-monogenic and super-harmonic functions can be found for instance in [35, 31, 37, 36, 23].

6.2 Distributions in superanalysis

In this section we study some properties of distributions in superanalysis. We pay particular attention to the extensions of the Heaviside and Dirac distributions to this setting. They play an important rôle in the definition of domain and surface integrals in superspace, as it will be shown in the next section.

6.2.1 Superdistributions

Let $\mathcal{D}'$ be the space of Schwartz distributions, i.e. the space of generalized functions on the space $C_0^\infty(\mathbb{R}^m)$ of complex-valued C^∞-functions with compact support. As usual, the notation

$$\int_{\mathbb{R}^m} \alpha f \, dV_{\underline{x}} = \langle \alpha, f \rangle, \tag{6.6}$$

where $dV_{\underline{x}} = dx_1 \cdots dx_m$ is the classical m-volume element, is used for the evaluation of the distribution $\alpha \in \mathcal{D}'$ on the test function $f \in C_0^\infty(\mathbb{R}^m)$.

Let $\mathcal{E}'$ be the space of generalized functions on the space $C^\infty(\mathbb{R}^m)$ of C^∞-functions in $\mathbb{R}^m$ (with arbitrary support). We recall that $\mathcal{E}'$ is exactly the subspace of all compactly supported distributions in $\mathcal{D}'$. Indeed, every distribution in $\mathcal{E}' \subset \mathcal{D}'$ has compact support and vice-versa, every distribution in $\mathcal{D}'$ with compact support can be uniquely extended to a distribution in $\mathcal{E}'$, see [19] for more details. This means that, for every $\alpha \in \mathcal{E}'$, evaluations of the form (6.6) extend to $C^\infty(\mathbb{R}^m)$ (instead of $C_0^\infty(\mathbb{R}^m)$).

The space of superdistributions $\mathcal{D}' \otimes \mathfrak{G}_{2n}$ then is defined by all elements of the form

$$\alpha = \sum_{A \subset \{1,\ldots,2n\}} \alpha_A \underline{\grave{x}}_A, \qquad \alpha_A \in \mathcal{D}'. \tag{6.7}$$

Similarly, the subspace $\mathcal{E}' \otimes \mathfrak{G}_{2n}$ is composed by all elements of the form (6.7) but with $\alpha_A \in \mathcal{E}'$.

The analogue of the integral $\int_{\mathbb{R}^m} dV_{\underline{x}}$ in superspace is given by

$$\int_{R^{m|2n}} = \int_{\mathbb{R}^m} dV_{\underline{x}} \int_B = \int_B \int_{\mathbb{R}^m} dV_{\underline{x}},$$

where the bosonic integration is the usual real integration and the integral over fermionic variables is given by the so-called Berezin integral (see [7]), defined by

$$\int_B = \pi^{-n} \, \partial_{\grave{x}_{2n}} \cdots \partial_{\grave{x}_1} = \frac{(-1)^n \pi^{-n}}{4^n n!} \partial_{\underline{\grave{x}}}^{2n}.$$

This enables us to define the action of a superdistribution $\alpha \in \mathcal{D}' \otimes \mathfrak{G}_{2n}$ (resp. $\alpha \in \mathcal{E}' \otimes \mathfrak{G}_{2n}$) on a test superfunction $F \in C_0^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}$ (resp. $F \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}$) by

$$\int_{\mathbb{R}^{m|2n}} \alpha F := \sum_{A,B \subset \{1,\ldots,2n\}} \langle \alpha_A, f_B \rangle \int_B \underline{\grave{x}}_A \, \underline{\grave{x}}_B.$$

As in the classical case, we say that the superdistribution $\alpha \in \mathcal{D}' \otimes \mathfrak{G}_{2n}$ *vanishes in the open set* $\Omega \subset \mathbb{R}^m$ if

$$\int_{\mathbb{R}^{m|2n}} \alpha F = 0$$

for every $F \in C_0^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}$ whose real support is contained in Ω. In the same way, the *support* $supp\ \alpha$ *of* $\alpha \in \mathcal{D}' \otimes \mathfrak{G}_{2n}$ is defined as the complement of the largest open subset of $\mathbb{R}^m$ on which α vanishes. Hence, it can be easily seen that

$$supp\ \alpha = \bigcup_{A \subset \{1,\ldots,2n\}} supp\ \alpha_A.$$

This means that $\mathcal{E}' \otimes \mathfrak{G}_{2n}$ is the subspace of all compactly supported superdistributions in $\mathcal{D}' \otimes \mathfrak{G}_{2n}$.

6.2.2 Multiplication of distributions

We now define the multiplication of distributions with disjoint singular supports. We first recall that the singular support $sing\ supp\,\alpha$ of the distribution $\alpha \in \mathcal{D}'$ is defined by the statement that $\underline{x} \notin sing\ supp\,\alpha$ if and only if there exists a neighbourhood $U_{\underline{x}}$ of $\underline{x} \in \mathbb{R}^m$ such that the restriction of α to $U_{\underline{x}}$ is a smooth function. It is readily seen that

$$sing\ supp\,\alpha \subset supp\,\alpha.$$

Definition 6.1 (multiplication of distributions, [20, 53]). *Consider two distributions* $\alpha, \beta \in \mathcal{D}'$ *such that* $sing\ supp\,\alpha \cap sing\ supp\,\beta = \emptyset$*. The product of distributions* $\alpha\beta$ *is well defined by the formula*

$$\langle \alpha\beta, \phi \rangle = \langle \alpha, \beta\chi\phi \rangle + \langle \beta, \alpha(1-\chi)\phi \rangle, \qquad \phi \in C^\infty(\mathbb{R}^m), \tag{6.8}$$

where $\chi \in C^\infty(\mathbb{R}^m)$ *is equal to zero in a neighbourhood of* $sing\ supp\,\beta$ *and equal to one in a neighbourhood of* $sing\ supp\,\alpha$*.*

Remark 6.1. *For our purposes Definition 6.1 is sufficient, however the product of two distributions in* $\mathcal{D}'$ *can also be defined under more general conditions, see [53, p. 267] and [20] for more details.*

It is easily seen that if $\alpha, \beta \in \mathcal{D}'$ vanish in $\Omega \subset \mathbb{R}^m$ then the product $\alpha\beta$ vanishes in Ω as well. Hence

$$supp\,\alpha\beta \subset supp\,\alpha \cup supp\,\beta.$$

As a consequence, if α and β have compact supports (i.e. $\alpha, \beta \in \mathcal{E}'$) then $\alpha\beta$ also has compact support (i.e. $\alpha\beta \in \mathcal{E}'$). The product (6.8) is associative, commutative and satisfies the Leibniz rule, see [20, 53].

The notion of singular support can be extended to distributions $\alpha \in \mathcal{D}' \otimes \mathfrak{G}_{2n}$ by the statement that $\underline{x} \notin sing\ supp\,\alpha$ if and only if there exists a neighbourhood $U_{\underline{x}}$ of $\underline{x} \in \mathbb{R}^m$ such that the restriction of α to $U_{\underline{x}}$ belongs to $C^\infty(U_{\underline{x}}) \otimes \mathfrak{G}_{2n}$. In this way we obtain for every $\alpha \in \mathcal{D}' \otimes \mathfrak{G}_{2n}$ of the form (6.7) that

$$sing\ supp\,\alpha = \bigcup_{A \subset \{1,\ldots,n\}} sing\ supp\,\alpha_A.$$

In the same way, we define the product of superdistributions $\alpha, \beta \in \mathcal{D}' \otimes \mathfrak{G}_{2n}$ with *sing supp* $\alpha \cap$ *sing supp* $\beta = \emptyset$ by

$$\alpha\beta = \sum_{A,B \subset \{1,\ldots,n\}} \alpha_A \beta_B \, \grave{\underline{x}}_A \, \grave{\underline{x}}_B, \tag{6.9}$$

where the distribution $\alpha_A\beta_B$ is to be understood in the sense of (6.8).

6.2.3 Properties of the δ-distribution in real calculus.

We now list some important properties of the δ-distribution in real calculus. This is necessary to introduce and study the main properties of the Heaviside distribution and all its derivatives in superspace.

Proposition 6.3. *Let $j, k \in \mathbb{N} \cup \{0\}$. Then*

$$\delta^{(j)}(x)\, x^k = \begin{cases} 0, & j < k, \\ (-1)^k\, k! \binom{j}{k}\, \delta^{(j-k)}(x), & k \leq j. \end{cases}$$

Proof.

For every complex-valued test function $f \in C^\infty(\mathbb{R})$ we have

$$\begin{aligned}
\int_{\mathbb{R}} \delta^{(j)}(x) x^k f(x)\, dx &= (-1)^j \left[x^k f(x)\right]^{(j)} \Big|_{x=0} \\
&= (-1)^j \left(\sum_{\ell=0}^{j} \binom{j}{\ell} \left(x^k\right)^{(\ell)} f^{(j-\ell)}(x) \right) \Bigg|_{x=0} \\
&= (-1)^j \left(\sum_{\ell=0}^{\min(j,k)} \binom{j}{\ell} \frac{k!}{(k-l)!} x^{k-\ell} f^{(j-\ell)}(x) \right) \Bigg|_{x=0}.
\end{aligned}$$

Clearly, if $j < k$ the above expression equals 0. For $0 \leq k \leq j$ we obtain,

$$\begin{aligned}
\int_{\mathbb{R}} \delta^{(j)}(x) x^k f(x)\, dx &= (-1)^j \binom{j}{k} k! f^{(j-k)}(0) \\
&= (-1)^j \binom{j}{k} k! \int_{\mathbb{R}} (-1)^{j-k} \delta^{(j-k)}(x) f(x)\, dx \\
&= \int_{\mathbb{R}} (-1)^k k! \binom{j}{k} \delta^{(j-k)}(x) f(x)\, dx.
\end{aligned}$$

□

In order to study the composition of the real δ-distribution with real-valued functions in $\mathbb{R}^m$, we first need the following result.

Proposition 6.4. *Let $g_0 \in C^\infty(\mathbb{R}^m)$ be a real-valued function such that $\partial_{\underline{x}}[g_0] \neq 0$ on the surface $g_0^{-1}(0) := \{\underline{w} \in \mathbb{R}^m : g_0(\underline{w}) = 0\}$. Then for every $j \in \mathbb{N} \cup \{0\}$, it holds that*

$$\partial_{\underline{x}} \left[\delta^{(j)}(g_0(\underline{x}))\right] = \partial_{\underline{x}}[g_0](\underline{x})\, \delta^{(j+1)}(g_0(\underline{x})).$$

Proof.

From the chain rule for partial derivatives acting on the composition $\delta^{(j)}(g_0(\underline{x}))$ it immediately follows that

$$\partial_{x_j}[\delta(g_0^{(j)}(\underline{x}))] = \delta^{(j+1)}(g_0(\underline{x}))\, \partial_{x_j}[g_0](\underline{x}), \tag{6.10}$$

see (6.1.2) in [53, p. 135]. □

Proposition 6.5. *Let $g_0, h_0 \in C^\infty(\mathbb{R}^m)$ be real-valued functions such that $h_0 > 0$ and $\partial_{\underline{x}}[g_0] \neq 0$ on the surface $g_0^{-1}(0)$. Then, for $j \in \mathbb{N} \cup \{0\}$ it holds that*

$$\delta^{(j)}\left(h_0(\underline{x})g_0(\underline{x})\right) = \frac{\delta^{(j)}(g_0(\underline{x}))}{h_0(\underline{x})^{j+1}}. \tag{6.11}$$

Proof.

We proceed by induction on $j \in \mathbb{N} \cup \{0\}$. In order to prove the statement for $j = 0$ we first observe that the following simple layer integral identity holds (see Theorem 6.1.5 in [53, p. 136]):

$$\int_{\mathbb{R}^m} \delta(g_0(\underline{x}))f(\underline{x})\, dV_{\underline{x}} = \int_{g_0^{-1}(0)} \frac{f(\underline{w})}{|\partial_{\underline{x}}[g_0](\underline{w})|}\, dS_{\underline{w}}, \tag{6.12}$$

where $dV_{\underline{x}} = dx_1 \cdots dx_m$ is the classical m-dimensional volume element and $dS_{\underline{w}}$ is the Lebesgue surface measure on the surface $g_0^{-1}(0)$. We also have

$$\partial_{\underline{x}}[h_0 g_0] = \partial_{\underline{x}}[h_0]g_0 + h_0\partial_{\underline{x}}[g_0],$$

which implies

$$\partial_{\underline{x}}[h_0 g_0](\underline{w}) = h_0(\underline{w})\partial_{\underline{x}}[g_0](\underline{w})$$

if $g_0(\underline{w}) = 0$. Thus applying (6.12) to $g_0 h_0$, instead of g_0, we get

$$\int_{\mathbb{R}^m} \delta(h_0(\underline{x})g_0(\underline{x}))f(\underline{x})\, dV_{\underline{x}} = \int_{g_0^{-1}(0)} \frac{f(\underline{w})}{h_0(\underline{w})\, |\partial_{\underline{x}}[g_0](\underline{w})|}\, dS_{\underline{w}} = \int_{\mathbb{R}^m} \frac{\delta(g_0(\underline{x}))}{h_0(\underline{x})} f(\underline{x})\, dV_{\underline{x}},$$

for every complex-valued test function f. Hence,

$$\delta(h_0(\underline{x})g_0(\underline{x})) = \frac{\delta(g_0(\underline{x}))}{h_0(\underline{x})}.$$

Now assume (6.11) to be true for $j \geq 1$. We then prove it for $j+1$. Letting $\partial_{\underline{x}}$ act on both sides of (6.11) we get

$$\partial_{\underline{x}}[h_0]g_0\, \delta^{(j+1)}(h_0 g_0) + h_0 \partial_{\underline{x}}[g_0]\, \delta^{(j+1)}(h_0 g_0) = \frac{\partial_{\underline{x}}[g_0]\, \delta^{(j+1)}(g_0)}{h_0^{j+1}} - \frac{j+1}{h_0^{j+2}}\, \partial_{\underline{x}}[h_0]\, \delta^{(j)}(g_0). \quad (6.13)$$

By Proposition 6.3 and the induction hypothesis we get

$$h_0 g_0\, \delta^{(j+1)}(h_0 g_0) = -(j+1)\delta^{(j)}(h_0 g_0) = -(j+1)\frac{\delta^{(j)}(g_0)}{h_0^{j+1}},$$

which implies,

$$g_0 \delta^{(j+1)}(h_0 g_0) = -(j+1)\frac{\delta^{(j)}(g_0)}{h_0^{j+2}}.$$

Substituting this in (6.13) we easily obtain

$$h_0 \partial_{\underline{x}}[g_0]\, \delta^{(j+1)}(h_0 g_0) = \frac{\partial_{\underline{x}}[g_0]\, \delta^{(j+1)}(g_0)}{h_0^{j+1}},$$

which is equivalent to

$$\delta^{(j+1)}(h_0 g_0) = \frac{\delta^{(j+1)}(g_0)}{h_0^{j+2}}.$$

Observe that the factor $\partial_{\underline{x}}[g_0]$ can be cancelled since $\partial_x[g_0] \neq 0$ in $g_0^{-1}(0)$. □

6.2.4 δ-Distribution in superspace

In this section we introduce the δ-distribution in superspace together with all its derivatives. As usual, the Heaviside distribution will be introduced as the corresponding antiderivative of the Dirac distribution. In [24], these distributions were introduced for some particular cases corresponding to the supersphere.

Consider an even real superfunction $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ such that $\partial_{\underline{x}}[g_0] \neq 0$ on the surface $g_0^{-1}(0)$. The distribution $\delta^{(k)}(g)$ is defined as the Taylor series

$$\delta^{(k)}(g) = \sum_{j=0}^{n} \frac{\mathbf{g}^j}{j!}\, \delta^{(k+j)}(g_0), \qquad k \in \mathbb{N} - 2 := \{-1, 0, 1, 2 \ldots\}.^3$$

The particular case $k = -1$ provides the expression for the antiderivative of δ, i.e. the Heaviside distribution $H = \delta^{(-1)}$ given by

$$H(g) = H(g_0) + \sum_{j=1}^{n} \frac{\mathbf{g}^j}{j!}\, \delta^{(j-1)}(g_0), \qquad \text{where} \qquad H(g_0) = \begin{cases} 1, & g_0 \geq 0, \\ 0, & g_0 < 0. \end{cases} \quad (6.14)$$

[3] For $p, q \in \mathbb{Z}$ we denote $p\mathbb{N}+q := \{pk+q : k \in \mathbb{N}\}$ where $\mathbb{N} := \{1, 2, \ldots\}$ is the set of natural numbers.

These are suitable extensions of the considered distributions to superspace, as we will show throughout this chapter. It is easy to check that a property similar to Proposition 6.1 *i*) holds for the above definitions. i.e.

$$\delta^{(k)}(a+b) = \sum_{j=0}^{n} \frac{\mathbf{b}^j}{j!}\, \delta^{(k+j)}(a+b_0), \qquad k \in \mathbb{N}-2, \tag{6.15}$$

where $a = a_0 + \mathbf{a}$, $b = b_0 + \mathbf{b}$ are real superfunctions in $C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$.

Let us prove now some important properties of the δ-distribution in superspace.

Proposition 6.6. *Let $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ be a real superfunction such that $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$. Then, for $j \in \mathbb{N} \cup \{0\}$ it holds that*

i) $g^j \delta^{(j)}(g) = (-1)^j j!\, \delta(g)$,

ii) $g^{j+1}\delta^{(j)}(g) = 0$.

Proof.

Using Proposition 6.3 we get,

$$\begin{aligned}
g^j \delta^{(j)}(g) &= (g_0+\mathbf{g})^j\, \delta^{(j)}(g_0+\mathbf{g}) = \left[\sum_{k=0}^{j} \binom{j}{k} g_0^{j-k}\mathbf{g}^k\right] \left[\sum_{\ell=0}^{n} \frac{\mathbf{g}^\ell}{\ell!}\, \delta^{(j+\ell)}(g_0)\right] \\
&= \sum_{p=0}^{n} \sum_{k=0}^{\min(j,p)} \binom{j}{k} \frac{\mathbf{g}^p}{(p-k)!}\, g_0^{j-k}\, \delta^{(j+p-k)}(g_0) \\
&= \sum_{p=0}^{n} \mathbf{g}^p \sum_{k=0}^{\min(j,p)} (-1)^{j-k} \binom{j}{k} \frac{(j-k)!}{(p-k)!} \binom{j+p-k}{j-k} \delta^{(p)}(g_0) \\
&= (-1)^j \sum_{p=0}^{n} \frac{\mathbf{g}^p}{p!} \delta^{(p)}(g_0) \left[\sum_{k=0}^{\min(j,p)} (-1)^k \binom{j}{k} \frac{(j+p-k)!}{(p-k)!}\right].
\end{aligned}$$

Writing

$$(j+p-k)! = \frac{(j+p)!}{(j+p)(j+p-1)\cdots(j+p-k+1)} = (-1)^k \frac{(j+p)!}{(-j-p)_k},$$

and

$$(p-k)! = \frac{p!}{(p)(p-1)\cdots(p-k+1)} = (-1)^k \frac{(p)!}{(-p)_k},$$

we get

$$\begin{aligned}
\sum_{k=0}^{\min(j,p)} (-1)^k \binom{j}{k} \frac{(j+p-k)!}{(p-k)!} &= \frac{(j+p)!}{p!} \sum_{k=0}^{\min(j,p)} (-1)^k \binom{j}{k} \frac{(-p)_k}{(-j-p)_k} \\
&= \frac{(j+p)!}{p!}\, {}_2F_1(-j,-p,-p-j,1),
\end{aligned}$$

where ${}_2F_1(a,b,c,z)$ denotes the hypergeometric function with parameters a,b,c in the variable z, see [2, p. 64]. Using the Chu-Vandermonde identity (which is an special case of the Gauss Theorem, see [2, p. 67]) we obtain

$$ {}_2F_1(-j,-p,-p-j,1) = \frac{(-j)_j}{(-j-p)_j} = \frac{(-j)(-j+1)\cdots(-1)}{(-j-p)(-j-p+1)\cdots(-p-1)} = \frac{j!p!}{(j+p)!}. $$

Whence,

$$ \sum_{k=0}^{\min(j,p)} (-1)^k \binom{j}{k} \frac{(j+p-k)!}{(p-k)!} = j!, $$

and as a consequence,

$$ g^j \delta^{(j)}(g) = (-1)^j j! \sum_{p=0}^{n} \frac{\mathbf{g}^p}{p!} \delta^{(p)}(g_0) = (-1)^j j!\, \delta(g). $$

For the proof of (ii) it follows from Proposition 6.3 that

$$ \begin{aligned} g\delta(g) &= (g_0 + \mathbf{g}) \sum_{j=0}^{n} \frac{\mathbf{g}^j}{j!} \delta^{(j)}(g_0) \\ &= \sum_{j=0}^{n} \frac{\mathbf{g}^j}{j!} g_0 \delta^{(j)}(g_0) + \sum_{j=0}^{n-1} \frac{\mathbf{g}^{j+1}}{j!} \delta^{(j)}(g_0) \\ &= -\sum_{j=1}^{n} \frac{\mathbf{g}^j}{(j-1)!} \delta^{(j-1)}(g_0) + \sum_{j=0}^{n-1} \frac{\mathbf{g}^{j+1}}{j!} \delta^{(j)}(g_0) \\ &= 0. \end{aligned} $$

Hence, using (i) we have that $g^{j+1}\delta^{(j)}(g) = (-1)^j j!\, g\delta(g) = 0$. □

Proposition 6.7. *Let $g = g_0 + \mathbf{g}$ and $h = h_0 + \mathbf{h}$ be real superfunctions in $C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ where g_0 and h_0 are their bodies and $\mathbf{g}$ resp. $\mathbf{h}$ their nilpotent parts. Let us assume that $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$ and $h_0 > 0$ in $\mathbb{R}^m$. Then,*

$$ \delta(hg) = \frac{\delta(g)}{h}. $$

Proof.

We first prove the result for $h = h_0 > 0$. From Proposition 6.5 we obtain

$$ \delta(h_0 g) = \delta(h_0 g_0 + h_0 \mathbf{g}) = \sum_{j=0}^{n} \frac{h_0^j \mathbf{g}^j}{j!} \delta^{(j)}(h_0 g_0) = \sum_{j=0}^{n} \frac{\mathbf{g}^j}{j!} \frac{\delta^{(j)}(g_0)}{h_0} = \frac{\delta(g)}{h_0}. $$

Writing $\frac{\mathbf{h}}{h_0} = \mathbf{K}$, we get for $h = h_0 + \mathbf{h}$ that

$$ \delta(hg) = \delta(h_0 g + \mathbf{h} g) = \delta\left(h_0(g + \mathbf{K}g)\right) = \frac{\delta\left((g + \mathbf{K}g)\right)}{h_0}. $$

It thus suffices to prove that for $\mathbf{K}$ nilpotent the equality

$$\delta\left((g+\mathbf{K}g)\right) = \frac{\delta(g)}{1+\mathbf{K}}$$

holds. Finally, using (6.15) and Proposition 6.6 (i) we get

$$\delta(g+\mathbf{K}g) = \sum_{j=0}^{n} \frac{\mathbf{K}^j g^j}{j!}\,\delta^{(j)}(g) = \left(\sum_{j=0}^{n}(-1)^j j!\frac{\mathbf{K}^j}{j!}\right)\delta(g) = \delta(g)\sum_{j=0}^{n}(-1)^j\mathbf{K}^j = \frac{\delta(g)}{1+\mathbf{K}}.$$

□

Proposition 6.8. *Let* $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^m)\otimes\mathfrak{G}_{2n}^{(ev)}$ *be a real superfunction such that* $\partial_{\underline{x}}[g_0] \neq 0$ *on* $g_0^{-1}(0)$. *Hence* $\delta^{(j)}(-g) = (-1)^j\delta^{(j)}(g)$ *for every* $j \in \mathbb{N}\cup\{0\}$.

Proof.

This directly follows from the corresponding property in real analysis; $\delta^{(j)}(-x) = (-1)^j\delta^{(j)}(x)$. Indeed,

$$\delta^{(j)}(-g) = \sum_{k=0}^{n}\frac{(-\mathbf{g})^k}{k!}\delta^{(j+k)}(-g_0) = (-1)^j\sum_{k=0}^{n}\frac{(\mathbf{g})^k}{k!}\delta^{(j+k)}(g_0) = (-1)^j\delta^{(j)}(g).$$

□

6.3 Integration in superspace

In this section we define general domain and surface integration in superspace using the above definitions for the Heaviside and Dirac distributions. This form of integration turns out to be an easy and powerful formalism which has as natural antecedent in the real case.

Indeed, let $\Omega \subset \mathbb{R}^m$ be a domain defined by means of a real-valued function $g_0 \in C(\mathbb{R}^m)$ as $\Omega = \{\underline{x} \in \mathbb{R}^m : g_0(x) < 0\}$. The characteristic function of Ω is given by $H(-g_0)$; this easily leads to the following expression for the integration over Ω

$$\int_\Omega f(\underline{x})\,dV_{\underline{x}} = \int_{\mathbb{R}^m} H(-g_0(\underline{x}))f(\underline{x})\,dV_{\underline{x}}. \qquad (6.16)$$

The function g_0 is called the defining *phase function* of the domain Ω.

Let us now consider $g_0 \in C^\infty(\mathbb{R}^m)$ such that $\partial_{\underline{x}}[g_0] \neq 0$ on the $(m-1)$-surface

$$\Gamma := g_0^{-1}(0) = \{\underline{w} \in \mathbb{R}^m : g_0(\underline{w}) = 0\}.$$

Then the non-oriented integral of a function f over Γ can be written as the simple layer integral

$$\int_{\mathbb{R}^m} \delta(g_0(\underline{x}))\, |\partial_{\underline{x}}[g_0](x)|\, f(\underline{x})\, dV_{\underline{x}} = \int_{\mathbb{R}} \delta(t) \left(\int_{g_0^{-1}(t)} f(\underline{w})\, dS_{\underline{w}} \right) dt = \int_{\Gamma} f(\underline{w})\, dS_{\underline{w}}, \tag{6.17}$$

where $dS_{\underline{w}}$ is the corresponding Lebesgue measure; see Theorem 6.1.5 [53, p. 136]. Observe also that the exterior normal vector to Γ in a point $\underline{w} \in \Gamma$ is given by

$$n(\underline{w}) = \frac{\partial_{\underline{x}}[g_0](\underline{w})}{|\partial_{\underline{x}}[g_0](\underline{w})|}.$$

This leads to the following formula for the oriented surface integral

$$\int_{\Gamma} n(\underline{w}) f(\underline{w})\, dS_{\underline{w}} = \int_{\mathbb{R}^m} \delta(g_0(\underline{x}))\, \partial_{\underline{x}}[g_0](x)\, f(\underline{x})\, dV_{\underline{x}}. \tag{6.18}$$

Formulas (6.16), (6.17) and (6.18) share a very important characteristic: they describe integrals over specific domains and surfaces as integrals over the whole space $\mathbb{R}^m$ depending only on the defining function g_0. In other words, they show the transition of the concept of integral as a functional depending on a set of points of $\mathbb{R}^m$ to a functional depending on a fixed phase function g_0.

This last approach will be used to define domain and surface integration in superspace using formulas which are similar to (6.16),(6.17),(6.18). Given an even real superfunction $g \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$, one may consider the superdistributions $H(-g)$ and $\delta(g)$ as the formal "characteristic functions" for the domain and surface associated to g respectively. As in the classical case, the superfunction g defining a domain in superspace is called a phase function.

Remark 6.2. *Following the above approach we arrive at a calculus in superspace that is independent of the representation of the variables x_j, $\grave{x}_j$ as co-ordinates with values in some commutative Banach superalgebra Λ, see Section 3.1.1. Observe that defining domains and surfaces as sets of points in $\mathbb{R}^{m,2n}(\Lambda)$, see (3.4), may not be convenient since it strongly depends on the topological properties of Λ.*

Example 6.1. *The supersphere $R\mathbb{S}^{2m-1|2n}$ and the corresponding superball $R\mathbb{B}^{2m|2n}$ of radius $R > 0$ are associated to the superfunction*

$$-g(\mathbf{x}) = \mathbf{x}^2 + R^2 = R^2 - |\underline{x}|^2 + \underline{\grave{x}}^2 = R^2 - \sum_{j=1}^{m} x_j^2 + \sum_{j=1}^{2n} \grave{x}_{2j-1}\grave{x}_{2j}.$$

The Dirac distribution corresponding to the supersphere $R\mathbb{S}^{2m-1|2n}$ is

$$\delta(\mathbf{x}^2 + R^2) = \sum_{j=0}^{n} \frac{\underline{\grave{x}}^{2j}}{j!}\, \delta^{(j)}(R^2 - |\underline{x}|^2).$$

The Heaviside distribution corresponding to the superball $R\mathbb{B}^{2m|2n}$ *is*

$$H(\mathbf{x}^2 + R^2) = H(R^2 - |\underline{x}|^2) + \sum_{j=1}^{n} \frac{\underline{\grave{x}}^{2j}}{j!} \delta^{(j-1)}(R^2 - |\underline{x}|^2).$$

6.3.1 Domain integrals in superspace

As mentioned before, our approach considers domains in superspace $\Omega_{m|2n}$ given by characteristic functions of the form $H(-g)$ where $g(\mathbf{x}) = g_0(\underline{x}) + \mathbf{g}(\underline{x}, \underline{\grave{x}}) \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ is a phase function. In this sense, $\Omega_{m|2n}$ plays the same rôle in superanalysis as its associated real domain $\Omega_{m|0} := \{\underline{x} \in \mathbb{R}^m : g_0(\underline{x}) < 0\}$ in classical analysis.

Definition 6.2. *Let* $\Omega_{m|2n}$ *be a domain in superspace (defined as before) satisfying the following two conditions:*

- *the associated real domain* $\Omega_{m|0}$ *has compact closure;*
- *the body* g_0 *of the defining phase function* g *is such that* $\partial_{\underline{x}}[g_0] \neq 0$ *on* $g_0^{-1}(0)$*.*

The integral over $\Omega_{m|2n}$ *then is defined as the functional* $\int_{\Omega_{m|2n}} : C^{n-1}\left(\overline{\Omega_{m|0}}\right) \otimes \mathfrak{G}_{2n} \to \mathbb{C}$ *given by*

$$\int_{\Omega_{m|2n}} F = \int_{\mathbb{R}^{m|2n}} H(-g)F, \qquad\qquad F \in C^{n-1}\left(\overline{\Omega_{m|0}}\right) \otimes \mathfrak{G}_{2n}. \tag{6.19}$$

The evaluation of the expression (6.19) requires the integration of smooth functions on the real domain $\Omega_{m|0}$ to be possible. This is guaranteed by the first condition imposed on the super domain $\Omega_{m|2n}$. On the other hand, if $\mathbf{g}$ is not identically zero, the above definition also involves the action of the Dirac distribution on g_0, see (6.14). For that reason, we restrict our analysis to the case where $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$, in order to ensure that this action is well defined.

The most simple examples for illustrating the use of Definition 6.2 correspond to the cases $g = g_0$ or $g = -\mathbf{x}^2 - R^2$; i.e. integration over real domains or over the superball respectively. The integral (6.19) then is given by

$$\int_{\Omega} \int_{B} F dV_{\underline{x}},$$

and

$$\int_{\mathbb{R}^{m|2n}} H(\mathbf{x}^2 + R^2)F,$$

respectively. These are the two particular cases that have been treated in the literature, see [24, 38]. The superball (and the supersphere) will also be used in this thesis as an illustrative example. In Section 6.4 we work with integrals over other super domains such as a super-paraboloid and a super-hyperboloid.

Proposition 6.9. *The volume of the superball* $R\,\mathbb{B}^{m|2n}$ *of radius* $R > 0$ *is given by*

$$vol(R\,\mathbb{B}^{m|2n}) = \begin{cases} \dfrac{\pi^{M/2}}{\Gamma(\frac{M}{2}+1)} R^M, & M \notin -2\mathbb{N}, \\ 0, & M \in -2\mathbb{N}, \end{cases}$$

where $M = m - 2n$ $(m \neq 0)$ *is the corresponding superdimension.*

Proof.

The volume of the $R\,\mathbb{B}^{m|2n}$ is obtained by integrating the function $F \equiv 1$ over $R\,\mathbb{B}^{m|2n}$ following Definition 6.2; i.e.

$$vol(R\,\mathbb{B}^{m|2n}) = \int_{R\,\mathbb{B}^{m|2n}} 1 = \int_{\mathbb{R}^m} \int_B H(R^2 - |\underline{x}|^2 + \underline{\grave{x}}^2)\, dV_{\underline{x}},$$

where

$$H(R^2 - |\underline{x}|^2 + \underline{\grave{x}}^2) = H(R^2 - |\underline{x}|^2) + \sum_{j=1}^{n} \frac{\underline{\grave{x}}^{2j}}{j!}\, \delta^{(j-1)}(R^2 - |\underline{x}|^2).$$

Since $\underline{\grave{x}}^{2n} = n!\grave{x}_1 \cdots \grave{x}_{2n}$, we obtain

$$\int_B H(R^2 - |\underline{x}|^2 + \underline{\grave{x}}^2) = \pi^{-n}\, \delta^{(n-1)}(R^2 - |\underline{x}|^2),$$

and consequently,

$$vol(R\,\mathbb{B}^{m|2n}) = \pi^{-n} \int_{\mathbb{R}^m} \delta^{(n-1)}(R^2 - |\underline{x}|^2)\, dV_{\underline{x}}. \tag{6.20}$$

Effectuating the change of variables $\underline{x} = r\underline{w}$, where $0 < r < \infty$ and $\underline{w} \in \mathbb{S}^{m-1} := \{\underline{x} \in \mathbb{R}^m : \underline{x}^2 = -1\}$, we get $dV_{\underline{x}} = r^{m-1} dr\, dS_{\underline{w}}$. Hence,

$$\begin{aligned} vol(R\,\mathbb{B}^{m|2n}) &= \pi^{-n} \int_{\mathbb{S}^{m-1}} \left(\int_0^\infty \delta^{(n-1)}(R^2 - r^2)\, r^{m-1}\, dr \right) dS_{\underline{w}} \\ &= \pi^{-n} A_m \int_0^\infty \delta^{(n-1)}(R^2 - r^2)\, r^{m-1}\, dr, \end{aligned}$$

where

$$A_m = \int_{\mathbb{S}^{m-1}} dS_{\underline{w}} = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})}$$

is the area of the unit sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$. Changing variables again, now through

$t = R^2 - r^2$, we obtain $r = \left(R^2 - t\right)^{1/2}$ and $dr = -\frac{1}{2}\left(R^2 - t\right)^{-1/2} dt$. Then, it holds that

$$\begin{aligned} vol(R\,\mathbb{B}^{m|2n}) &= \frac{\pi^{-n}\,A_m}{2} \int_{-\infty}^{R^2} \delta^{(n-1)}(t)\left(R^2 - t\right)^{\frac{m}{2}-1}\,dt \\ &= (-1)^{n-1}\frac{\pi^{-n}\,A_m}{2}\,\frac{d^{n-1}}{dt^{n-1}}\left[\left(R^2 - t\right)^{\frac{m}{2}-1}\right]\Bigg|_{t=0} \\ &= \frac{\pi^{-n}\,A_m}{2}\prod_{j=1}^{n-1}\left(\frac{m}{2} - j\right)R^{m-2n}, \end{aligned} \tag{6.21}$$

where for the special cases $n = 0, 1$ we are considering the expressions $\prod_{j=1}^{-1}\left(\frac{m}{2} - j\right) := \frac{2}{m}$ and $\prod_{j=1}^{0}\left(\frac{m}{2} - j\right) := 1$. Since we consider $m \neq 0$, it is easily seen that

$$\prod_{j=1}^{n-1}\left(\frac{m}{2} - j\right) = \begin{cases} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}-n+1)} & \frac{M}{2} \notin -\mathbb{N}, \\ 0 & \frac{M}{2} \in -\mathbb{N}, \end{cases}$$

whence substitution in (6.21) completes the proof. □

Remark 6.3. *Proposition 6.9 provides a suitable extension to superspace for the volume of the ball* $R\mathbb{B}^m := \{\underline{x} \in \mathbb{R}^m : |\underline{x}| < R\}$ *($R > 0$). We recall that in the case $n = 0$ the volume of $R\mathbb{B}^m$ equals* $\frac{\pi^{m/2}}{\Gamma(\frac{m}{2}+1)}R^m$.

In real analysis the choice of the phase function g_0 defining a certain domain $\Omega_{m|0} \subset \mathbb{R}^m$ is not unique. Indeed, for every real-valued function $h_0 \in C(\mathbb{R}^m)$ with $h_0 > 0$, the function $h_0 g_0$ defines the same domain as g_0, i.e.

$$\Omega_{m|0} = \{\underline{x} \in \mathbb{R}^m : g_0(\underline{x}) < 0\} = \{\underline{x} \in \mathbb{R}^m : h_0(\underline{x})g_0(\underline{x}) < 0\}.$$

A simple example is the unitary ball $\mathbb{B} \subset \mathbb{R}^m$ which can be described by

$$|\underline{x}| - 1 < 0, \quad \text{or} \quad -\underline{x}^2 - 1 < 0,$$

since

$$-\underline{x}^2 - 1 = (|\underline{x}| + 1)(|\underline{x}| - 1).$$

However, integration over $\Omega_{m|0}$ remains independent of the choice of the function g_0 that defines $\Omega_{m|0}$. Indeed, in real analysis it is easily seen that $H(-g_0) = H(-h_0 g_0)$ for $h_0 > 0$. This property remains valid in superspace.

Proposition 6.10. *Let $g = g_0 + \mathbf{g}$ and $h = h_0 + \mathbf{h}$ be real superfunctions in $C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ where g_0 and h_0 are their respective bodies. Let us assume that $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$ and $h_0 > 0$ in $\mathbb{R}^m$. Then*

$$H(hg) = H(g).$$

Proof.

We first prove that $H(h_0 g) = H(g)$. By Proposition 6.5 we obtain $(h_0\mathbf{g})^j \delta^{(j-1)}(h_0 g_0) = \mathbf{g}^j \delta^{(j-1)}(g_0)$ for $j \in \mathbb{N}$. Hence,

$$H(h_0 g) = H(h_0 g_0) + \sum_{j=1}^{n} \frac{(h_0\mathbf{g})^j}{j!} \delta^{(j-1)}(h_0 g_0) = H(g_0) + \sum_{j=1}^{n} \frac{\mathbf{g}^j}{j!} \delta^{(j-1)}(g_0) = H(g).$$

If we write $\mathbf{K} = \frac{\mathbf{h}}{h_0}$, we get for $h = h_0 + \mathbf{h}$ that

$$H(hg) = H(h_0 g + \mathbf{h} g) = H(g + \mathbf{K} g),$$

whence it suffices to prove that $H(g + \mathbf{K}g) = H(g)$. Using (6.15) and Proposition 6.6 (ii) we obtain

$$H(g + \mathbf{K}g) = H(g) + \sum_{j=1}^{n} \frac{\mathbf{K}^j}{j!}\, g^j \delta^{(j-1)}(g) = H(g).$$

□

This allows us to define the notion of a pair of phase functions that define the same domain (or surface) in superspace.

Definition 6.3. *Let $g, \varphi \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ be two phase functions such that $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$ and $\partial_{\underline{x}}[\varphi_0] \neq 0$ on $\varphi_0^{-1}(0)$. They are said to define the same domain (or surface) in superspace if there exists a real superfunction $h \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ with $h_0 > 0$ in $\mathbb{R}^m$ such that $\varphi = hg$.*

Remark 6.4. *Proposition 6.10 shows that the integral over $\Omega_{m|2n}$ defined in (6.19) does not depend on the choice of the superfunction g defining $\Omega_{m|2n}$. The example of the superball $R\,\mathbb{B}^{m|2n}$ of radius $R > 0$ illustrates very well this property. Indeed, the domain $R\,\mathbb{B}^{m|2n}$ can be defined by means of any of the two superfunctions*

$$|\mathbf{x}| - R \qquad \textit{or} \qquad -\mathbf{x}^2 - R^2.$$

Both definitions for $R\,\mathbb{B}^{m|2n}$ can be used in (6.19) without changing the result of the integration since $-\mathbf{x}^2 - R^2 = (|\mathbf{x}| + R)(|\mathbf{x}| - R)$ where

$$h(\mathbf{x}) = |\mathbf{x}| + R = R + \sum_{j=0}^{n} \frac{(-1)^j \underline{\grave{x}}^{2j}}{j!} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} - j)} |\underline{x}|^{1-2j}, \quad \textit{and} \quad h_0(\underline{x}) = R + |\underline{x}| > 0.$$

6.3.2 Surface integrals in superspace

Similarly to the case of super domains, we define a surface $\Gamma_{m-1|2n}$ in superspace by means of $\delta(g)$ where $g(\mathbf{x}) = g_0(\underline{x}) + \mathbf{g}(\underline{x}, \underline{\grave{x}}) \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ is a given phase function. If $\Omega_{m|2n}$ is the super domain associated to g as in Definition 6.2, then we say that $\Gamma_{m-1|2n}$

is the boundary of $\Omega_{m|2n}$ and denote it by $\Gamma_{m-1|2n} := \partial\Omega_{m|2n}$. This way, $\Gamma_{m-1|2n}$ plays the same rôle in superspace as its real surface

$$\Gamma_{m-1|0} := \partial\Omega_{m|0} = \{\underline{x} \in \mathbb{R}^m : g_0(\underline{x}) = 0\}$$

in classical analysis.

Based on the formulae (6.17) and (6.18) concerning the real case we now define the non-oriented and oriented surface integrals in superspace.

Definition 6.4. *Let $\Gamma_{m-1|2n}$ be a surface in superspace (defined as before) satisfying the following two conditions:*

- *the associated real surface $\Gamma_{m-1|0} \subset \mathbb{R}^m$ is a compact set;*
- *the body g_0 of the defining phase function g is such that $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$.*

The non-oriented and oriented surface integrals over $\Gamma_{m-1|2n}$ are then defined as the following functionals on $C^n\left(\Gamma_{m-1|0}\right) \otimes \mathfrak{G}_{2n}$

$$\int_{\Gamma_{m-1|2n}} F = \int_{\mathbb{R}^{m|2n}} \delta(g)\left|\partial_{\mathbf{x}}[g]\right| F, \qquad \int_{\Gamma_{m-1|2n}} \sigma_{\mathbf{x}}\, F = -\int_{\mathbb{R}^{m|2n}} \delta(g)\,\partial_{\mathbf{x}}[g]\, F, \quad (6.22)$$

respectively.

Remark 6.5. *Proposition 6.8 assures that the sign of the superfunction g does not play a rôle in the non-oriented case.*

When $g = g_0$, the integrals (6.22) reduce to the product of the classical real surface integration and the Berezin integral, i.e.,

$$\int_{\Gamma_{m-1|0}} \int_B F(\underline{w}, \grave{x})\, dS_{\underline{w}}, \qquad \int_{\Gamma_{m-1|0}} \int_B n(\underline{w}) F(\underline{w}, \grave{x})\, dS_{\underline{w}}, \quad (6.23)$$

see formulae (6.17)-(6.18).We will now verify that, when restricted to the supersphere $R\mathbb{S}^{2m-1|2n}$, the definition of the non-oriented surface integral still coincides with the one given in [24].

Proposition 6.11. *The non-oriented integral over the supersphere $R\mathbb{S}^{2m-1|2n}$ ($R > 0$) can be written as*

$$\int_{R\mathbb{S}^{m-1,2n}} F = 2\int_{\mathbb{R}^m}\int_B \delta(R^2 + \mathbf{x}^2)\,|\mathbf{x}|\,F(\mathbf{x})\, dV_{\underline{x}} = 2R\int_{\mathbb{R}^m}\int_B \delta(\mathbf{x}^2 + R^2)\, F(\mathbf{x})\, dV_{\underline{x}}. \quad (6.24)$$

Proof.

We first observe that $\partial_{\mathbf{x}}[R^2+\mathbf{x}^2] = 2\mathbf{x}$. Following (6.22), the non-oriented surface integral of F over the supersphere $R\mathbb{S}^{m-1,2n}$ is given by

$$\int_{R\mathbb{S}^{m-1,2n}} F = 2\int_{\mathbb{R}^m}\int_B \delta(R^2+\mathbf{x}^2)\,|\mathbf{x}|\,F(\mathbf{x})\,dV_{\underline{x}}.$$

Let us prove now that, as in the classical case, one can substitute $|\mathbf{x}| = R$ in the above expression. Consider the real distribution $G(t) = \delta(R^2-t)t^{\frac{1}{2}}$, $t \geq 0$. Then, we can write in superspace

$$\delta(R^2+\mathbf{x}^2)\,|\mathbf{x}| = \delta(R^2+\mathbf{x}^2)\,(-\mathbf{x}^2)^{\frac{1}{2}} = G(-\mathbf{x}^2) = G(|\underline{x}|^2 - \grave{\underline{x}}^2) = \sum_{j=0}^{n}\frac{\left(-\grave{\underline{x}}^2\right)^j}{j!}\,G^{(j)}(|\underline{x}|^2).$$

However, in the distributional sense, it is easily seen that

$$G(t) = \delta(R^2-t)t^{\frac{1}{2}} = R\,\delta(t-R^2),$$

whence also

$$G^{(j)}(t) = R\delta^{(j)}(t-R^2), \qquad\qquad j = 0,\ldots,n.$$

Substituting this in the above formula we obtain,

$$\delta(R^2+\mathbf{x}^2)\,|\mathbf{x}| = R\sum_{j=0}^{n}\frac{\left(-\grave{\underline{x}}^2\right)^j}{j!}\delta^{(j)}(|\underline{x}|^2-R^2) = R\,\delta(R^2+\mathbf{x}^2),$$

which completes the proof. □

The non-oriented integration over the supersphere (6.24), and the integration of superfunctions over real surfaces (6.23), have been studied in the literature, see e.g. [24, 38]. In particular, (6.24) was proven to be an extension of Pizzetti's formula for polynomials. The simplest example for application of Pizzetti's formula is obtained when integrating the function $F \equiv 1$; which leads to the surface area of the supersphere $R\mathbb{S}^{m-1,2n}$.

Proposition 6.12. *The surface area of the supersphere $R\mathbb{S}^{m-1,2n}$ of radius $R > 0$ is given by*

$$area(R\,\mathbb{S}^{m-1|2n}) = \begin{cases} \dfrac{2\pi^{M/2}}{\Gamma(\frac{M}{2})}R^{M-1}, & M \notin -2\mathbb{N}+2, \\ 0 & M \in -2\mathbb{N}+2, \end{cases}$$

where $M = m-2n$ $(m \neq 0)$ is the corresponding superdimension.

Proof.

Using (6.24), the area of the surface $R\mathbb{S}^{m-1,2n}$ is given by

$$area(R\mathbb{S}^{m-1,2n}) = \int_{R\mathbb{S}^{m-1,2n}} 1 = 2R\int_{\mathbb{R}^m}\int_B \delta(\mathbf{x}^2+R^2)\,dV_{\underline{x}},$$

where

$$\delta(\mathbf{x}^2 + R^2) = \sum_{j=0}^{n} \frac{\grave{\underline{x}}^{2j}}{j!} \delta^{(j)}(R^2 - |\underline{x}|^2)$$

implies

$$\int_B \delta(\mathbf{x}^2 + R^2) = \pi^{-n}\, \delta^{(n)}(R^2 - |\underline{x}|^2).$$

By the co-ordinate changes $\underline{x} = r\underline{w}$, $\underline{w} \in \mathbb{S}^{m-1}$ and $r = t^{\frac{1}{2}}$ we get

$$\begin{aligned}
area(R\mathbb{S}^{m-1,2n}) &= 2R\pi^{-n} \int_{\mathbb{R}^m} \delta^{(n)}(R^2 - |\underline{x}|^2)\, dV_{\underline{x}} \\
&= 2R\pi^{-n} \int_{\mathbb{S}^{m-1}} \left(\int_0^\infty \delta^{(n)}(R^2 - r^2)\, r^{m-1}\, dr \right) dS_{\underline{w}} \\
&= \pi^{-n}\, A_m\, R \int_0^\infty \delta^{(n)}(R^2 - t) t^{\frac{m}{2}-1}\, dt \\
&= \pi^{-n}\, A_m\, R\, \frac{d^n}{dt^n} \left[t^{\frac{m}{2}-1}\right] \Big|_{t=R^2} \\
&= \pi^{-n}\, A_m \prod_{j=1}^{n} \left(\frac{m}{2} - j\right) R^{m-2n-1}, \qquad (6.25)
\end{aligned}$$

where for the special case $n = 0$ we put by convention $\prod_{j=1}^{0} \left(\frac{m}{2} - j\right) := 1$. Since we consider $m \neq 0$, it is easily seen that

$$\prod_{j=1}^{n} \left(\frac{m}{2} - j\right) = \begin{cases} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}-n)} & \frac{M}{2} \notin -\mathbb{N} + 1, \\ 0 & \frac{M}{2} \in -\mathbb{N} + 1, \end{cases}$$

whence substitution in (6.21) completes the proof. □

As for domain integrals, we can prove that Definition 6.4 does not depend on the choice of the defining superfunction g for the surface $\Gamma_{m-1|2n}$.

Proposition 6.13. *Let $g = g_0 + \mathbf{g}$ and $h = h_0 + \mathbf{h}$ be real superfunctions in $C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ having g_0 and h_0 as their respective bodies. Let us assume that $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$ and $h_0 > 0$ in $\mathbb{R}^m$. Then,*

(i) $\delta(hg)\partial_{\mathbf{x}}[hg] = \delta(g)\partial_{\mathbf{x}}[g]$, *(ii)* $\delta(hg)|\partial_{\mathbf{x}}[hg]| = \delta(g)|\partial_{\mathbf{x}}[g]|$.

Proof.

Using Propositions 6.6 and 6.7 we get

$$\delta(hg)\partial_{\mathbf{x}}[hg] = \frac{\delta(g)}{h} \left(\partial_{\mathbf{x}}[h]g + h\partial_{\mathbf{x}}[g]\right) = \delta(g)\partial_{\mathbf{x}}[g],$$

which proves (i). In addition, by Propositions 6.2 and 6.7 we get

$$\begin{aligned}\delta(hg)|\partial_{\mathbf{x}}[hg]| &= \delta(hg)\big|\partial_{\mathbf{x}}[h]g + h\partial_{\mathbf{x}}[g]\big| \\ &= \frac{\delta(g)}{h}\Big[h|\partial_{\mathbf{x}}[g]| + gF\big(h\partial_{\mathbf{x}}[g], \partial_{\mathbf{x}}[h], g\big)\Big] = \delta(g)|\partial_{\mathbf{x}}[g]|,\end{aligned}$$

which proves (ii). □

6.4 Other examples and applications

In this section we study some more examples of integration in superspace over domains and surfaces. To that end, let us observe first that, as in the classical case, it is possible to define intersections amongst domains as well as between a surface and several domains in superspace.

Indeed, let Ω_j $(j = 1, \ldots, k)$ be domains in superspace defined by means of the phase functions $g_j(\mathbf{x}) = g_{j,0}(\underline{x}) + \mathbf{g}_j(\underline{x}, \underline{x}\grave{}) \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$. The intersection $\Omega_{m|2n} = \cap_{j=1}^k \Omega_k$ is naturally defined as the domain with characteristic function

$$H(-g_1) \ldots H(-g_k).$$

If we also consider a surface $\Gamma_{m-1|2n}$ defined by $\delta(g)$ with $g(\mathbf{x}) = g_0(\underline{x}) + \mathbf{g}(\underline{x}, \underline{x}\grave{}) \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ being a phase function, the intersection $\Gamma_{m-1|2n} \cap \Omega_{m|2n}$ is defined by the characteristic function

$$\delta(g) H(-g_1) \ldots H(-g_k).$$

The domain $\Omega_{m|2n} = \cap_{j=1}^k \Omega_k$ plays the same rôle in superspace as its associated region

$$\Omega_{m|0} := \bigcap_{j=1}^{k} \{\underline{x} \in \mathbb{R}^m : g_{j,0}(\underline{x}) < 0\}$$

in $\mathbb{R}^m$. In the same way, $\Gamma_{m-1|2n} \cap \Omega_{m|2n}$ is the super-analogue of the intersection of the $(m-1)$-surface $\Gamma_{m-1|0} = g_0^{-1}(0)$ with the region $\Omega_{m|0}$.

Now assume that $\Omega_{m|0}$ has a compact closure and the body functions $g_0, g_{1,0}, \ldots, g_{k,0}$ have non vanishing gradients on the respective surfaces $g_0^{-1}(0), g_{1,0}^{-1}(0), \ldots, g_{k,0}^{-1}(0)$. Hence, Definitions 6.2 and 6.4 extend to $\Omega_{m|2n} = \cap_{j=1}^k \Omega_k$ and $\Gamma_{m-1|2n} \cap \Omega_{m|2n}$ respectively by putting

$$\int_{\Omega_{m|2n}} F = \int_{\mathbb{R}^{m|2n}} H(-g_1) \ldots H(-g_k) F,$$

$$\int_{\Gamma_{m-1|2n} \cap \Omega_{m|2n}} F = \int_{\mathbb{R}^{m|2n}} \delta(g) \left|\partial_{\mathbf{x}}[g]\right| H(-g_1) \ldots H(-g_k) F,$$

$$\int_{\Gamma_{m-1|2n} \cap \Omega_{m|2n}} \sigma_{\mathbf{x}} F = -\int_{\mathbb{R}^{m|2n}} \delta(g)\, \partial_{\mathbf{x}}[g]\, H(-g_1) \ldots H(-g_k) F.$$

6.4.1 Super-paraboloid of revolution

Let us consider the paraboloid of revolution in 3 real dimensions defined by

$$x_3 = x_1^2 + x_2^2 = -\underline{\hat{x}}^2 \qquad \text{where} \qquad \underline{\hat{x}} = x_1 e_1 + x_2 e_2.$$

We define its extension to superspace by means of the superfunction

$$g(\mathbf{x}) = -\hat{\mathbf{x}}^2 - x_m = \sum_{j=1}^{m-1} x_j^2 - \sum_{j=1}^{n} \grave{x}_{2j-1}\grave{x}_{2j} - x_m = |\underline{\hat{x}}|^2 - x_m - \underline{\grave{x}}^2 \qquad (m \geq 2),$$

where, in this case, $\underline{\hat{x}} = \underline{x} - x_m e_m$ and $\hat{\mathbf{x}} = \underline{\hat{x}} + \underline{\grave{x}} = \mathbf{x} - x_m e_m$.

The set

$$\{\underline{x} \in \mathbb{R}^m : |\underline{\hat{x}}|^2 - x_m \leq 0\}$$

clearly is non compact. However, its intersection with the region

$$\{\underline{x} \in \mathbb{R}^m : x_m \in [0, h]\} \qquad\qquad (h > 0)$$

gives a compact subset composed by the interior and boundary of the paraboloid $x_m = |\underline{\hat{x}}|^2$ with height $h > 0$. This means that we can integrate over the domain (and surface) defined by the superfunction g in superspace with the restriction $x_m < h$. This object will be called the *super-paraboloid of revolution of height* $h > 0$ and is denoted by $SP_h^{m|2n}$. More precisely, the domain and surface associated to $SP_h^{m|2n}$ are given by the characteristic functions

$$H(-g)H(h - x_m)$$

and

$$\delta(g)H(h - x_m),$$

respectively.

Proposition 6.14. *The volume of* $SP_h^{m|2n}$ *is given by*

$$vol(SP_h^{m|2n}) = \begin{cases} \dfrac{\pi^{\frac{M-1}{2}}}{\Gamma(\frac{M+3}{2})} h^{\frac{M+1}{2}}, & M \notin -2\mathbb{N} + 1, \\ 0 & M \in -2\mathbb{N} + 1. \end{cases} \qquad (6.26)$$

Proof.

Observe that

$$vol(SP_h^{m|2n}) = \int_{\mathbb{R}^{m|2n}} H(-g)H(h - x_m) = \int_0^h \left(\int_{\mathbb{R}^{m-1}} \int_B H(-g)\, dV_{\underline{\hat{x}}} \right) dx_m.$$

Because

$$H(-g) = H(x_m - |\underline{\hat{x}}|^2 + \underline{\grave{x}}^2) = \sum_{j=0}^{n} \frac{\underline{\grave{x}}^{2j}}{j!} \delta^{(j-1)}(x_m - |\underline{\hat{x}}|^2)$$

we have

$$\int_B H(-g) = \pi^{-n}\, \delta^{(n-1)}(x_m - |\underline{\hat{x}}|^2).$$

Hence, by formula (6.20) we get

$$\int_{\mathbb{R}^{m-1}} \int_B H(-g)\, dV_{\underline{\hat{x}}} = \pi^{-n} \int_{\mathbb{R}^{m-1}} \delta^{(n-1)}(x_m - |\underline{\hat{x}}|^2)\, dV_{\underline{\hat{x}}} = vol\left(x_m^{\frac{1}{2}}\, \mathbb{B}^{m-1|2n}\right),$$

which leads to

$$\int_{\mathbb{R}^{m-1}} \int_B H(-g)\, dV_{\underline{\hat{x}}} = \begin{cases} \dfrac{\pi^{\frac{M-1}{2}}}{\Gamma(\frac{M+1}{2})} x_m^{\frac{M-1}{2}}, & M \notin -2\mathbb{N}+1, \\ 0 & M \in -2\mathbb{N}+1, \end{cases}$$

whence for $M \notin -2\mathbb{N}+1$ we obtain

$$vol(SP_h^{m|2n}) = \frac{\pi^{\frac{M-1}{2}}}{\Gamma(\frac{M+1}{2})} \int_0^h x_m^{\frac{M-1}{2}}\, dx_m = \frac{\pi^{\frac{M-1}{2}}}{\Gamma(\frac{M+1}{2})} \frac{h^{\frac{M+1}{2}}}{\frac{M+1}{2}} = \frac{\pi^{\frac{M-1}{2}}}{\Gamma(\frac{M+3}{2})} h^{\frac{M+1}{2}};$$

and $vol(SP_h^{m|2n}) = 0$ for $M \in -2\mathbb{N}+1$. □

Remark 6.6. *The above result is an extension of the volume formulae for the corresponding paraboloids in the known classical cases $m = 2$, $n = 0$ and $m = 3$, $n = 0$.*

- *In the case $m = 2$, $n = 0$, the parabola $SP_h^{2|0} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 \leq x_2 \leq h\}$ is known to have the volume (i.e. area in $\mathbb{R}^2$)*

$$2h^{3/2} - \int_{-h^{\frac{1}{2}}}^{h^{\frac{1}{2}}} x_1^2\, dx_1 = \frac{4}{3} h^{\frac{3}{2}}.$$

This is exactly the result obtained when substituting $M = 2$ in (6.26).

- *In the case $m = 3$, $n = 0$, $SP_h^{3|0} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq x_3 \leq h\}$ is the body generated by the rotation of the curve $x_3 = x_1^2$ around the x_3-axis from $x_1 = 0$ to $x_1 = h^{\frac{1}{2}}$. The volume of $SP_h^{3|0}$ is known to be*

$$2\pi \int_0^{h^{\frac{1}{2}}} x_1(h - x_1^2)\, dx_1 = \frac{\pi}{2} h^2,$$

coinciding with the result obtained when evaluating formula (6.26) for $M = 3$.

Proposition 6.15. *The surface area of $SP_h^{m|2n}$ for $M > 1$ is given by*

$$area(SP_h^{m|2n}) = \frac{\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M+1}{2}\right)} h^{\frac{M-1}{2}}\, {}_2F_1\left(-\frac{1}{2}, \frac{M-1}{2}; \frac{M+1}{2}; -4h\right), \tag{6.27}$$

where ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function, see [2, p. 64].

Proof.

In this case,

$$area(SP_h^{m|2n}) = \int_{\mathbb{R}^{m|2n}} \delta(g) \left|\partial_{\mathbf{x}}[g]\right| H(h-x_m) = \int_0^h \left(\int_{\mathbb{R}^{m-1}} \int_B \delta(g) \left|\partial_{\mathbf{x}}[g]\right| dV_{\underline{\hat{x}}} \right) dx_m.$$

We recall that $\partial_{\mathbf{x}} = \partial_{\hat{\mathbf{x}}} - \partial_{x_m} e_m$. Then

$$\partial_{\mathbf{x}}[g] = (\partial_{\hat{\mathbf{x}}} - \partial_{x_m} e_m)(-\hat{\mathbf{x}}^2 - x_m) = -\partial_{\hat{\mathbf{x}}}[\hat{\mathbf{x}}^2] + \partial_{x_m}[x_m]e_m = -2\hat{\mathbf{x}} + e_m,$$

which leads to $\left|\partial_{\mathbf{x}}[g]\right| = \left(-4\hat{\mathbf{x}}^2+1\right)^{\frac{1}{2}}$.

Let us now consider the distribution $D(t) = \delta(t-x_m)(4t+1)^{\frac{1}{2}}$, $(t>0)$. The evaluation of D in $-\hat{\mathbf{x}}^2 = |\underline{\hat{x}}|^2 - \underline{\grave{x}}^2$ equals $\delta(g)\left|\partial_{\mathbf{x}}[g]\right|$ and is given by

$$D(-\hat{\mathbf{x}}^2) = \sum_{j=0}^{n} \frac{(-1)^j \underline{\grave{x}}^{2j}}{j!} D^{(j)}\left(|\underline{\hat{x}}|^2\right),$$

implying

$$\int_B D(-\hat{\mathbf{x}}^2) = (-1)^n \pi^{-n} D^{(n)}\left(|\underline{\hat{x}}|^2\right).$$

Then,

$$area(SP_h^{m|2n}) = (-1)^n\pi^{-n} \int_0^h \left(\int_{\mathbb{R}^{m-1}} D^{(n)}\left(|\underline{\hat{x}}|^2\right) dV_{\underline{\hat{x}}} \right) dx_m.$$

By the change of variables $\underline{\hat{x}} = r\underline{w}$, $r>0$, $\underline{w}\in\mathbb{S}^{m-2}$ we get $dV_{\underline{\hat{x}}} = r^{m-2}dr\, dS_{\underline{w}}$ and

$$\begin{aligned}
(-1)^n\pi^{-n} \int_{\mathbb{R}^{m-1}} D^{(n)}\left(|\underline{\hat{x}}|^2\right) dV_{\underline{\hat{x}}} &= (-1)^n\pi^{-n} \int_{\mathbb{S}^{m-2}} \left(\int_0^{+\infty} D^{(n)}\left(r^2\right) r^{m-2}\, dr \right) dS_{\underline{w}} \\
&= \frac{(-1)^n\pi^{-n}A_{m-1}}{2} \int_0^{+\infty} D^{(n)}(t)\, t^{\frac{m-3}{2}}\, dt \\
&= \frac{\pi^{-n}A_{m-1}}{2} \int_0^{+\infty} D(t) \frac{d^n}{dt^n}\left[t^{\frac{m-3}{2}}\right] dt \\
&= \frac{\pi^{-n}A_{m-1}\Gamma\left(\frac{m-1}{2}\right)}{2\,\Gamma\left(\frac{m-1}{2}-n\right)} \int_0^{+\infty} \delta(t-x_m)(4t+1)^{\frac{1}{2}} t^{\frac{m-3}{2}-n}\, dt \\
&= \frac{\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M-1}{2}\right)} (4x_m+1)^{\frac{1}{2}}\, x_m^{\frac{M-3}{2}},
\end{aligned}$$

whence,

$$\begin{aligned}
area(SP_h^{m|2n}) &= \frac{\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M-1}{2}\right)} \int_0^h (4x_m+1)^{\frac{1}{2}}\, x_m^{\frac{M-3}{2}}\, dx_m \\
&= \frac{\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M-1}{2}\right)} h^{\frac{M-1}{2}} \int_0^1 (4th+1)^{\frac{1}{2}}\, (t)^{\frac{M-3}{2}}\, dt,
\end{aligned}$$

where the last equality has been obtained from the change of variable $x_m = ht$.

Let us now compute the last integral which only converges when $M > 1$. To that end, we first recall Euler's integral representation formula for hypergeometric functions, see Theorem 2.2.1 [2, p. 65]. For $a, b, c \in \mathbb{R}$ such that $c > b > 0$, the hypergeometric function ${}_2F_1(a, b; c; z)$ can be written as

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-zt)^{-a}\, t^{b-1}\, (1-t)^{c-b-1}\, dt. \tag{6.28} $$

Writing $a = -\frac{1}{2}$, $b = \frac{M-1}{2}$, $c = b+1 = \frac{M+1}{2}$ and $z = -4h$ we obtain for $M > 1$ that

$$ \int_0^1 (4th+1)^{\frac{1}{2}} (t)^{\frac{M-3}{2}}\, dt = \frac{2}{M-1}\, {}_2F_1\left(-\frac{1}{2}, \frac{M-1}{2}; \frac{M+1}{2}; -4h\right), $$

from which the result follows □

Remark 6.7. *Similar to the volume (see Remark 6.6), by (6.27) we obtain an extension of the surface area of the corresponding paraboloids for the known cases $m = 2$, $n = 0$ and $m = 3$, $n = 0$.*

- *In the case $m = 2$, $n = 0$, $SP_h^{2|0}$ is known to have the surface area (i.e. length in this case)*

$$ \int_{-h^{\frac{1}{2}}}^{h^{\frac{1}{2}}} \left(1+4x_1^2\right)^{\frac{1}{2}}\, dx_1 = h^{\frac{1}{2}} (1+4h)^{\frac{1}{2}} + \frac{\sinh^{-1}(2h^{\frac{1}{2}})}{2}. $$

  Substituting $M = 2$ in (6.27) we get the same result:

$$ area(SP_h^{2|0}) = 2h^{\frac{1}{2}}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -4h\right) = h^{\frac{1}{2}} (1+4h)^{\frac{1}{2}} + \frac{\sinh^{-1}(2h^{\frac{1}{2}})}{2}. $$

- *In the case $m = 3$, $n = 0$, the paraboloid $SP_h^{3|0}$ is known to have the surface area*

$$ 2\pi \int_0^h x_3^{\frac{1}{2}} \left(1 + \frac{1}{4x_3}\right)^{\frac{1}{2}} dx_3 = \frac{\pi}{6}\left[(4h+1)^{\frac{3}{2}} - 1\right]. $$

  Substituting $M = 3$ in (6.27) again we get the same result:

$$ area(SP_h^{3|0}) = \pi h\, {}_2F_1\left(-\frac{1}{2}, 1; 2; -4h\right) = \frac{\pi}{6}\left[(4h+1)^{\frac{3}{2}} - 1\right]. $$

6.4.2 Super-hyperboloid of revolution

In 3 real dimensions, we consider the one-sheeted hyperboloid of revolution obtained by rotating the hyperbola $x_1^2 - x_3^2 = 1$ around the x_3-axis. The Cartesian equation of this hyperboloid is

$$ x_1^2 + x_2^2 - x_3^2 = 1. $$

We define its extension to superspace by means of the superfunction

$$g(\mathbf{x}) = \sum_{j=1}^{m-1} x_j^2 - x_m^2 - \sum_{j=1}^{n} \grave{x}_{2j-1}\grave{x}_{2j} - 1 = -\hat{\mathbf{x}}^2 - 1 - x_m^2. \quad (m \geq 2).$$

Observe that the set

$$\{\underline{x} \in \mathbb{R}^m : |\underline{\hat{x}}|^2 - x_m^2 - 1 \leq 0\}$$

is non compact. However its intersection with the region

$$\{\underline{x} \in \mathbb{R}^m : x_m \in [-h, h]\} \qquad (h > 0)$$

gives a compact set (symmetric with respect to the plane $x_m = 0$) that is composed of the interior and the boundary of the hyperboloid $|\underline{\hat{x}}|^2 - x_m^2 = 1$ in $\mathbb{R}^m$ with half height h. This means that we can integrate over the domain (and surface) defined by the superfunction g with the restrictions $-h \leq x_m \leq h$. This object will be called the *super-hyperboloid of revolution of half height* $h > 0$ and is denoted by $SH_h^{m|2n}$. The domain and surface associated to $SH_h^{m|2n}$ are given by the characteristic functions

$$H(-g)H(h - x_m)H(h + x_m)$$

and

$$\delta(g)H(h - x_m)H(h + x_m),$$

respectively.

Proposition 6.16. *The volume of* $SH_h^{m|2n}$ *is given by*

$$vol(SH_h^{m|2n}) = \begin{cases} \frac{2h\pi^{\frac{M-1}{2}}}{\Gamma(\frac{M+1}{2})} \, {}_2F_1\left(\frac{1-M}{2}, \frac{1}{2}; \frac{3}{2}; -h^2\right), & M \notin -2\mathbb{N} + 1, \\ 0, & M \in -2\mathbb{N} + 1. \end{cases} \tag{6.29}$$

Proof.

Observe that

$$\begin{aligned} vol(SH_h^{m|2n}) &= \int_{\mathbb{R}^{m|2n}} H(-g)H(h - x_m)H(h + x_m) \\ &= \int_{-h}^{h} \left(\int_{\mathbb{R}^{m-1}} \int_B H(\hat{\mathbf{x}}^2 + x_m^2 + 1) \, dV_{\underline{\hat{x}}} \right) dx_m, \end{aligned}$$

where

$$\begin{aligned} \int_B H(\hat{\mathbf{x}}^2 + x_m^2 + 1) &= \int_B H(x_m^2 + 1 - |\underline{\hat{x}}|^2 + \grave{\underline{x}}^2) \\ &= \sum_{j=0}^{n} \int_B \frac{\grave{\underline{x}}^{2j}}{j!} \delta^{(j-1)}(x_m^2 + 1 - |\underline{\hat{x}}|^2) \\ &= \pi^{-n} \, \delta^{(n-1)}(x_m^2 + 1 - |\underline{\hat{x}}|^2). \end{aligned}$$

Using (6.20) we obtain,

$$\int_{\mathbb{R}^{m-1}} \int_B H(\hat{\mathbf{x}}^2+x_m^2+1)\, dV_{\hat{\underline{x}}} = \pi^{-n}\int_{\mathbb{R}^{m-1}} \delta^{(n-1)}(x_m^2+1-|\hat{\underline{x}}|^2)\, dV_{\hat{\underline{x}}} = vol\left((x_m^2+1)^{\frac{1}{2}}\mathbb{B}^{m-1|2n}\right).$$

Hence,

$$\int_{\mathbb{R}^{m-1}} \int_B H(\hat{\mathbf{x}}^2 + x_m^2 + 1)\, dV_{\hat{\underline{x}}} = \begin{cases} \frac{\pi^{\frac{M-1}{2}}}{\Gamma(\frac{M+1}{2})}(x_m^2+1)^{\frac{M-1}{2}}, & M \notin -2\mathbb{N}+1, \\ 0, & M \in -2\mathbb{N}+1. \end{cases}$$

This implies, for $M \in -2\mathbb{N}+1$, that $vol(SH_h^{m|2n}) = 0$. But for $M \notin -2\mathbb{N}+1$ we have

$$vol(SH_h^{m|2n}) = \frac{2\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M+1}{2}\right)} \int_0^h (x_m^2+1)^{\frac{M-1}{2}}\, dx_m = \frac{h\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M+1}{2}\right)} \int_0^1 (1+h^2 t)^{\frac{M-1}{2}} t^{-\frac{1}{2}}\, dt,$$

where the last integral has been obtained from the change of variable $t = \frac{x_m^2}{h^2}$. Using Euler's integral representation formula (6.28) for hypergeometric functions we get for $a = \frac{1-M}{2}$, $b = \frac{1}{2}$, $c = b+1 = \frac{3}{2}$ and $z = -h^2$ that

$$\frac{1}{2}\int_0^1 (1+h^2t)^{\frac{M-1}{2}} t^{-\frac{1}{2}}\, dt = {}_2F_1\left(\frac{1-M}{2}, \frac{1}{2}; \frac{3}{2}; -h^2\right),$$

which completes the proof. □

Remark 6.8. *Formula (6.29) constitutes an extension of the volume formulas for the corresponding hyperboloids in the classical cases $m = 2$, $n = 0$ and $m = 3$, $n = 0$.*

- *In the case $m = 2$, $n = 0$, the hyperbola*

$$SH_h^{2|0} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 \leq 1,\ -h \leq x_2 \leq h\}$$

is known to have the volume (i.e. area in $\mathbb{R}^2$)

$$2\int_{-h}^{h} (1+x_2^2)^{\frac{1}{2}}\, dx_2 = 2\left[h(h^2+1)^{\frac{1}{2}} + \sinh^{-1}(h)\right],$$

while evaluating (6.29) for $M = 2$ gives

$$vol(SH_h^{2|0}) = 4h\, {}_2F_1\left(\frac{-1}{2}, \frac{1}{2}; \frac{3}{2}; -h^2\right) = 2\left[h(h^2+1)^{\frac{1}{2}} + \sinh^{-1}(h)\right].$$

- *In the case $m = 3$, $n = 0$,*

$$SH_h^{3|0} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3 \leq 1,\ -h \leq x_3 \leq h\}$$

is known to have the volume

$$\pi \int_{-h}^{h} (x_3^2 + 1)\, dx_3 = 2\pi h \left(1 + \frac{h^2}{3}\right),$$

while substituting $M = 3$ in (6.29) yields

$$vol(SH_h^{3|0}) = 2h\pi\ {}_2F_1\left(-1, \frac{1}{2}; \frac{3}{2}; -h^2\right) = 2\pi h\left(1 + \frac{h^2}{3}\right).$$

Proposition 6.17. *The surface area of $SH_h^{m|2n}$ is given by*

$$area(SH_h^{m|2n}) = \begin{cases} \dfrac{4h\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M-1}{2}\right)} F_1\left(\dfrac{1}{2}; -\dfrac{1}{2}, \dfrac{3-M}{2}; \dfrac{3}{2}; -2h^2, -h^2\right), & M \notin -2\mathbb{N}+3, \\ 0, & M \in -2\mathbb{N}+3, \end{cases} \tag{6.30}$$

where $F_1(a; b_1, b_2; c; z_1, z_2)$ denotes Appell's hypergeometric function, see [4, p. 73].

Proof.

Observe that

$$\begin{aligned} area(SH_h^{m|2n}) &= \int_{\mathbb{R}^{m|2n}} \delta(g)\, |\partial_{\mathbf{x}}[g]|\, H(h - x_m) H(h + x_m) \\ &= \int_{-h}^{h} \left(\int_{\mathbb{R}^{m-1}} \int_B \delta(g)\, |\partial_{\mathbf{x}}[g]|\, dV_{\underline{\hat{x}}}\right) dx_m, \end{aligned}$$

where

$$\partial_{\mathbf{x}}[g] = -(\partial_{\hat{\mathbf{x}}} - \partial_{x_m} e_m)(\hat{\mathbf{x}}^2 + x_m^2 + 1) = -\partial_{\hat{\mathbf{x}}}[\hat{\mathbf{x}}^2] + \partial_{x_m}[x_m^2] e_m = -2\hat{\mathbf{x}} + 2x_m e_m.$$

Then,

$$|\partial_{\mathbf{x}}[g]| = 2|\hat{\mathbf{x}} - x_m e_m| = 2\left(-\hat{\mathbf{x}}^2 + x_m^2\right)^{\frac{1}{2}}.$$

Using Proposition 6.8, we write

$$\delta(g)\, |\partial_{\mathbf{x}}[g]| = 2\delta\left(\hat{\mathbf{x}}^2 + x_m^2 + 1\right)\left(-\hat{\mathbf{x}}^2 + x_m^2\right)^{\frac{1}{2}} = K(-\hat{\mathbf{x}}^2),$$

where K denotes the distribution

$$K(t) = 2\delta\left(x_m^2 + 1 - t\right)\left(t + x_m^2\right)^{\frac{1}{2}}.$$

Moreover

$$K(-\hat{\mathbf{x}}^2) = K(|\underline{\hat{x}}|^2 - \underline{\grave{x}}^2) = \sum_{j=0}^{n} \frac{(-1)^n \underline{\grave{x}}^{2j}}{j!} K^{(j)}(|\underline{\hat{x}}|^2),$$

whence

$$\int_B \delta(g)\,|\partial_{\mathbf{x}}[g]| = \int_B K(-\hat{\mathbf{x}}^2) = (-1)^n \pi^{-n} K^{(n)}(|\underline{\hat{x}}|^2),$$

and finally

$$area(SH_h^{m|2n}) = \int_{-h}^{h} \left(\int_{\mathbb{R}^{m-1}} (-1)^n \pi^{-n} K^{(n)}(|\underline{\hat{x}}|^2)\, dV_{\underline{\hat{x}}} \right) dx_m.$$

Direct computations yield

$$\begin{aligned}
(-1)^n\pi^{-n} \int_{\mathbb{R}^{m-1}} K^{(n)}(|\underline{\hat{x}}|^2)\, dV_{\underline{\hat{x}}} &= (-1)^n\pi^{-n} \int_{\mathbb{S}^{m-2}} \left(\int_0^\infty K^{(n)}(r^2) r^{m-2}\, dr \right) dS_{\underline{w}} \\
&= \frac{(-1)^n \pi^{-n} A_{m-1}}{2} \int_0^\infty K^{(n)}(t) t^{\frac{m-3}{2}}\, dt \\
&= \frac{\pi^{-n} A_{m-1}}{2} \int_0^\infty K(t) \frac{d^n}{dt^n} \left[t^{\frac{m-3}{2}} \right] dt \\
&= \frac{\pi^{-n} A_{m-1}}{2} \prod_{j=1}^{n} \left(\frac{m-1}{2} - j \right) \int_0^\infty K(t) t^{\frac{M-3}{2}}\, dt,
\end{aligned}$$

where for the special case $n = 0$ we put by convention $\prod_{j=1}^{0} \left(\frac{m-1}{2} - j\right) := 1$. For $\frac{M-3}{2} \in -\mathbb{N}$, it immediately follows that $\prod_{j=1}^{n} \left(\frac{m-1}{2} - j\right) = 0$ and in consequence $area(SH_h^{m|2n}) = 0$. But for $\frac{M-3}{2} \notin -\mathbb{N}$ we have,

$$\begin{aligned}
(-1)^n\pi^{-n} \int_{\mathbb{R}^{m-1}} K^{(n)}(|\underline{\hat{x}}|^2)\, dV_{\underline{\hat{x}}} &= \frac{\pi^{-n} A_{m-1} \Gamma\left(\frac{m-1}{2}\right)}{2\,\Gamma\left(\frac{m-1}{2} - n\right)} \int_0^\infty 2\delta\left(x_m^2 + 1 - t\right) \left(t + x_m^2\right)^{\frac{1}{2}} t^{\frac{M-3}{2}}\, dt \\
&= \frac{2\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M-1}{2}\right)} \left(2x_m^2 + 1\right)^{\frac{1}{2}} \left(x_m^2 + 1\right)^{\frac{M-3}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
area(SH_h^{m|2n}) &= \frac{4\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M-1}{2}\right)} \int_0^h \left(2x_m^2 + 1\right)^{\frac{1}{2}} \left(x_m^2 + 1\right)^{\frac{M-3}{2}} dx_m \\
&= \frac{2h\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M-1}{2}\right)} \int_0^1 (2h^2 t + 1)^{\frac{1}{2}} (h^2 t + 1)^{\frac{M-3}{2}} t^{-\frac{1}{2}}\, dt, \qquad (6.31)
\end{aligned}$$

where the last equality has been obtained by the change of variable $t = \frac{x_m^2}{h^2}$.

The last integral can be written in terms of the so-called Appell's hypergeometric function of the first kind. Such a function constitutes an extension of the hypergeometric function of two variables and it is defined by

$$F_1(a; b_1, b_2; c; z_1, z_2) = \sum_{j,k=0}^{\infty} \frac{(a)_{j+k}\,(b_1)_j\,(b_2)_k}{(c)_{j+k}\, j!\, k!} z_1^j z_2^k,$$

see [4, Chapter IX] for more details. We now recall the integral representation of $F_1(a, b_1, b_2; c; z_1, z_2)$ for $c > a > 0$, see [4, p. 77]:

$$F_1(a; b_1, b_2; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-z_1 t)^{-b_1}(1-z_2 t)^{-b_2}\, dt.$$

Taking $a = \frac{1}{2}$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{3-M}{2}$, $c = a + 1 = \frac{3}{2}$, $z_1 = -2h^2$ and $z_2 = -h^2$ we obtain,

$$\int_0^1 (2h^2 t + 1)^{\frac{1}{2}} (h^2 t + 1)^{\frac{M-3}{2}} t^{-\frac{1}{2}}\, dt = 2\, F_1\left(\frac{1}{2}; -\frac{1}{2}, \frac{3-M}{2}; \frac{3}{2}; -2h^2, -h^2\right). \tag{6.32}$$

Finally, substitution of (6.32) into (6.31) yields (6.30). □

Remark 6.9. *Similarly to the previous results, (6.30) extends the known formulae for the surface area of the corresponding hyperboloids in the classical cases $m = 2$, $n = 0$ and $m = 3$, $n = 0$.*

- *The hyperbola $SH_h^{2|0}$ (see Remark 6.8) is known to have the surface area (i.e. length in this case)*

$$S = 4 \int_1^{(1+h^2)^{\frac{1}{2}}} (2x_1^2 - 1)^{\frac{1}{2}} (x_1^2 - 1)^{-\frac{1}{2}}\, dx_1 = 2h \int_0^1 (2h^2 t + 1)^{\frac{1}{2}} t^{-\frac{1}{2}} (h^2 t + 1)^{-\frac{1}{2}}\, dt,$$

  where we have used the change of variable $x_1 = (th^2 + 1)^{\frac{1}{2}}$, $0 \leq t \leq 1$. Then (6.32) immediately shows that

$$S = 4h\, F_1\left(\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -2h^2, -h^2\right)$$

  which is the same result obtained when substituting $M = 2$ in (6.30).

- *The hyperboloid $SH_h^{3|0}$ is known to have the surface area*

$$2\pi \int_{-h}^{h} (2x_3^2 + 1)^{\frac{1}{2}}\, dx_3 = \pi \left[2h(2h^2 + 1)^{\frac{1}{2}} + 2^{\frac{1}{2}} \sinh^{-1}(2^{\frac{1}{2}} h)\right].$$

  On the other hand, substituting $M = 3$ in (6.30) we obtain

$$\begin{aligned} area(SH_h^{3|0}) &= 4h\pi\, F_1\left(\frac{1}{2}; -\frac{1}{2}, 0; \frac{3}{2}; -2h^2, -h^2\right) \\ &= \pi \left[2h(2h^2 + 1)^{\frac{1}{2}} + 2^{\frac{1}{2}} \sinh^{-1}(2^{\frac{1}{2}} h)\right]. \end{aligned}$$

A summary of all previously computed volumes and surface areas is provided in Table 6.1.

	Superball and supersphere of radius $R > 0$	Super-paraboloid of height $h > 0$	Super-hyperboloid of half height $h > 0$
phase function	$g(\mathbf{x}) = -\mathbf{x}^2 - R^2$	$g(\mathbf{x}) = -\hat{\mathbf{x}}^2 - x_m$ $x_m \in [0, h]$, $m \geq 2$	$g(\mathbf{x}) = -\hat{\mathbf{x}}^2 - 1 - x_m^2$ $x_m \in [-h, h]$, $m \geq 2$
volume	$\dfrac{\pi^{M/2}}{\Gamma(\frac{M}{2}+1)} R^M$, $M \notin -2\mathbb{N}$	$\frac{\pi^{\frac{M-1}{2}}}{\Gamma(\frac{M+3}{2})}\, h^{\frac{M+1}{2}}$ $M \notin -2\mathbb{N}+1$	$\frac{2h\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M+1}{2}\right)}\ {}_2F_1\left(\frac{1-M}{2}, \frac{1}{2}; \frac{3}{2}; -h^2\right)$, $M \notin -2\mathbb{N}+1$
area	$\frac{2\pi^{M/2}}{\Gamma(\frac{M}{2})} R^{M-1}$, $M \notin -2\mathbb{N}+2$	$\frac{(\pi h)^{\frac{M-1}{2}}}{\Gamma\left(\frac{M+1}{2}\right)}\ {}_2F_1\left(-\frac{1}{2}, \frac{M-1}{2}; \frac{M+1}{2}; -4h\right)$, $M > 1$	$\frac{4h\pi^{\frac{M-1}{2}}}{\Gamma\left(\frac{M-1}{2}\right)}\ F_1\left(\frac{1}{2}; -\frac{1}{2}, \frac{3-M}{2}; \frac{3}{2}; -2h^2, -h^2\right)$, $M \notin -2\mathbb{N}+3$

Table 6.1: Volumes and surface areas of some domains and their boundaries in superdimension $M = m - 2n$.

6.5 Distributional Cauchy-Pompeiu formula in superspace

The integration over general domains and surfaces in superspace introduced in Section 6.3 allows to extend and unify the respective known versions of the Cauchy-Pompeiu formulae in superspace. In [24, 38], the following Cauchy-Pompeiu formula was proven for bounded domains $\Omega \subset \mathbb{R}^m$:

$$\int_{\partial\Omega}\int_B \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})n(\underline{x})G(\mathbf{x})\,dS_{\underline{x}} + \int_{\Omega}\int_B \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})\left(\partial_{\mathbf{x}}G(\mathbf{x})\right)\,dV_{\underline{x}} = \begin{cases} -G(\mathbf{y}), & \underline{y}\in\Omega, \\ 0, & \underline{y}\notin\overline{\Omega}, \end{cases} \tag{6.33}$$

where $\nu_1^{m|2n}$ is the fundamental solution of the super Dirac operator $\partial_{\mathbf{x}}$, see [37]. The proof of this formula runs along similar lines as the traditional approach (see e.g. [44, p. 147]). It does not require the use of distributions since it only considers integration over real sets composed with the Berezin integral. In [24], another version of the Cauchy-Pompeiu formula was obtained for the unit supersphere $\mathbb{S}^{m-1,2n}$. Its proof is based on (6.33) and uses the distributional approach on the supersphere described in Example 6.1:

$$\int_{\mathbb{R}^{m|2n}} \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})2\underline{x}\delta(\mathbf{x}^2+1)G(\mathbf{x}) + \int_{\mathbb{R}^{m|2n}} \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})H(\mathbf{x}^2+1)\left(\partial_{\mathbf{x}}G(\mathbf{x})\right) = \begin{cases} -G(\mathbf{y}), & \underline{y}\in\mathbb{B}^m, \\ 0, & \underline{y}\notin\overline{\mathbb{B}^m}. \end{cases} \tag{6.34}$$

Both formulae generalize, in a certain way, the classical Cauchy-Pompeiu theorem in $\mathbb{R}^m$; see e.g. [44, p. 147]. Yet this extension is not complete. Indeed, formulas (6.33) and (6.34) only allow for integration over real domains (and surfaces) and the superball (and supersphere) respectively. In this section we show how our general integration approach allows to obtain a distributional Cauchy-Pompeiu formula in superspace for which (6.33)-(6.34) are obtained as particular cases.

We start with the following Stokes theorem in superspace.

Theorem 6.1 (Distributional Stokes Theorem, [24]). *Let $F, G \in C^\infty(\Omega)\otimes\mathfrak{G}_{2n}\otimes\mathcal{C}_{m,2n}$ and $\alpha\in\mathcal{E}'\otimes\mathfrak{G}_{2n}^{(ev)}$ a distribution with compact support such that $\operatorname{supp}\alpha\subset\Omega\subset\mathbb{R}^m$. Then,*

$$\int_{\mathbb{R}^{m|2n}} \left(F\partial_{\mathbf{x}}\right)\alpha G + F\alpha\left(\partial_{\mathbf{x}}G\right) = -\int_{\mathbb{R}^{m|2n}} F\left(\partial_{\mathbf{x}}\alpha\right)G. \tag{6.35}$$

Proof.

The proof is based in two fundamental observations: the support of α is compact and the operator $\int_B \partial_{\grave{x}_j}$ is identically zero. Hence, for $F, G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n}$ we have

$$\int_{\mathbb{R}^m} \partial_{x_j} [F\alpha G] = 0, \qquad \int_B \partial_{\grave{x}_j} [F\alpha G] = 0.$$

But

$$\partial_{\underline{x}}[F\alpha G] = (\partial_{\underline{x}} F)\alpha G + F\alpha(\partial_{\underline{x}} G) + F(\partial_{\underline{x}}\alpha)G,$$

and by (3.10) we get

$$\partial_{\grave{\underline{x}}}[F^*\alpha G] = (\partial_{\grave{\underline{x}}} F^*)\alpha G + F(\partial_{\grave{\underline{x}}}\, \alpha G) = -(F\partial_{\grave{\underline{x}}})\alpha G + F\alpha(\partial_{\grave{\underline{x}}} G) + F(\partial_{\grave{\underline{x}}}\, \alpha)G,$$

whence,

$$\int_{\mathbb{R}^{m|2n}} (\partial_{\underline{x}} F)\alpha G + F\alpha(\partial_{\underline{x}} G) = -\int_{\mathbb{R}^{m|2n}} F(\partial_{\underline{x}}\alpha)G, \tag{6.36}$$

$$\int_{\mathbb{R}^{m|2n}} -(F\partial_{\grave{\underline{x}}})\alpha G + F\alpha(\partial_{\grave{\underline{x}}} G) = -\int_{\mathbb{R}^{m|2n}} F(\partial_{\grave{\underline{x}}}\, \alpha)G. \tag{6.37}$$

Subtracting (6.36) from (6.37) we get (6.35) for $F, G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n}$. The extension to Clifford valued functions in $C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ is easily done by multiplying from the left and from the right with the corresponding Clifford generators $e_j, e_{\grave{j}}$. □

In particular, if we take $\alpha = H(-g)$ with $g \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ being a phase function such that $\{g_0 \leq 0\}$ is compact, we obtain a Stokes formula in superspace compatible with the notions of domain and surface integrals that we have introduced in Section 6.3.

Corollary 6.1. *Let $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ be a phase function such that $\{g_0 \leq 0\}$ is compact and $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$. Then, for $F, G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ such that $\{g_0 \leq 0\} \subset \Omega$ one has*

$$\int_{\mathbb{R}^{m|2n}} H(-g)\left[(F\partial_{\mathbf{x}})\, G + F\, (\partial_{\mathbf{x}} G)\right] = \int_{\mathbb{R}^{m|2n}} F\delta(g)\partial_{\mathbf{x}}[g]G. \tag{6.38}$$

Proof.

The support of $H(-g)$ clearly is $\{g_0 \leq 0\}$, which is compact. Then (6.38) is the result of substituting $\alpha = H(-g)$ in (6.35) and proving that $\partial_{\mathbf{x}}[H(-g)] = -\partial_{\mathbf{x}}[g]\delta(g)$. Indeed, by (6.10) we get

$$
\begin{aligned}
\partial_{x_k}[H(-g)] &= \partial_{x_k}[H(-g_0)] + \sum_{j=1}^{n} \partial_{x_k}\left[\frac{(-\mathbf{g})^j}{j!}\delta^{(j-1)}(-g_0)\right] \\
&= -\delta(g_0)\partial_{x_k}[g_0] - \sum_{j=1}^{n}\left[\frac{(-\mathbf{g})^{j-1}}{(j-1)!}\partial_{x_k}[\mathbf{g}]\delta^{(j-1)}(-g_0) + \frac{(-\mathbf{g})^j}{j!}\delta^{(j)}(-g_0)\partial_{x_k}[g_0]\right] \\
&= -\partial_{x_k}[g_0]\sum_{j=0}^{n}\frac{(-\mathbf{g})^j}{j!}\delta^{(j)}(-g_0) - \partial_{x_k}[\mathbf{g}]\sum_{j=0}^{n-1}\frac{(-\mathbf{g})^j}{j!}\delta^{(j)}(-g_0) \\
&= -\partial_{x_k}[g_0]\delta(g) - \partial_{x_k}[\mathbf{g}]\left(\delta(g) - \frac{\mathbf{g}^n}{n!}\delta^{(n)}(g_0)\right) \\
&= -\partial_{x_k}[g]\delta(g),
\end{aligned}
$$

the last equality being obtained on account of the nilpotency of the element $\partial_{x_k}[\mathbf{g}]$ which implies $\partial_{x_k}[\mathbf{g}]\mathbf{g}^n = 0$.

Using formulae (3.10) we easily get by induction that $\partial_{\grave{x}_k}[\mathbf{g}^j] = j\mathbf{g}^{j-1}\partial_{\grave{x}_k}[\mathbf{g}]$. Then,

$$
\begin{aligned}
\partial_{\grave{x}_k}[H(-g)] &= \sum_{j=1}^{n}\partial_{\grave{x}_k}\left[\frac{(-\mathbf{g})^j}{j!}\delta^{(j-1)}(-g_0)\right] = -\sum_{j=1}^{n}\frac{(-\mathbf{g})^{j-1}}{(j-1)!}\partial_{\grave{x}_k}[\mathbf{g}]\delta^{(j-1)}(-g_0) \\
&= -\partial_{\grave{x}_k}[\mathbf{g}]\left(\delta(g) - \frac{\mathbf{g}^n}{n!}\delta^{(n)}(g_0)\right) \\
&= -\partial_{\grave{x}_k}[\mathbf{g}]\delta(g).
\end{aligned}
$$

Now, one easily concludes that $\partial_{\mathbf{x}}[H(-g)] = -\partial_{\mathbf{x}}[g]\delta(g)$. □

As an immediate consequence one obtains the following Cauchy theorem for super monogenic functions.

Corollary 6.2 (Clifford-Cauchy theorem in superspace). *Let $g = g_0+\mathbf{g} \in C^\infty(\mathbb{R}^m)\otimes \mathfrak{G}_{2n}^{(ev)}$ be a phase function such that $\{g_0 \le 0\}$ is compact and $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$. Then, for every super-monogenic function $G \in C^\infty(\Omega)\otimes \mathfrak{G}_{2n}\otimes \mathcal{C}_{m,2n}$ such that $\{g_0 \le 0\} \subset \Omega$ one has*

$$\int_{\mathbb{R}^{m|2n}} \delta(g)\partial_{\mathbf{x}}[g]G = 0.$$

In order to prove the distributional Cauchy-Pompeiu formula we need a distributional version of the Stokes formula (6.35). We recall that two distributions in $\mathcal{E}'\otimes\mathfrak{G}_{2n}$ can be multiplied if they have disjoint singular supports, see Definition 6.1 and (6.9). Going back to the proof of the Stokes formula (6.35), we have that

$$\int_{\mathbb{R}^m} \partial_{x_j}[\beta\alpha G]\, dV_{\underline{x}} = 0,$$

and

$$\int_B \partial_{\grave{x}_j}[\beta\alpha G] = 0.$$

where $\alpha \in \mathcal{E}' \otimes \mathfrak{G}_{2n}^{(ev)}$ and $\beta \in \mathcal{E}' \otimes \mathfrak{G}_{2n}$ are such that *sing supp* $\alpha \cap$ *sing supp* $\beta = \emptyset$. Hence, we can repeat the proof of Theorem 6.1 to obtain (6.35) but with $F \in \mathcal{E}' \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ satisfying *sing supp* $\alpha \cap$ *sing supp* $F = \emptyset$. Applying this reasoning to (6.38), for which we are considering $\alpha = H(-g)$, we immediately obtain the following consequence.

Corollary 6.3. *Let $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ be a phase function such that $\{g_0 \leq 0\}$ is compact and $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$. Moreover let $\beta \in \mathcal{E}' \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ such that*

$$\textit{sing supp}\, \beta \cap g^{-1}(0) = \emptyset.$$

Then, for $G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ such that $\{g_0 \leq 0\} \subset \Omega$ one has

$$\int_{\mathbb{R}^{m|2n}} H(-g)\left[(\beta\partial_{\mathbf{x}})\, G + \beta\,(\partial_{\mathbf{x}} G)\right] = \int_{\mathbb{R}^{m|2n}} \beta\delta(g)\partial_{\mathbf{x}}[g]G, \tag{6.39}$$

where the distributional products $H(-g)(\beta\partial_{\mathbf{x}})$, $H(-g)\beta$, $\beta\delta(g)$ are understood in the sense of (6.9)-(6.8).

Proof.

It suffices to note that *sing supp* $H(-g) = g_0^{-1}(0)$. □

In [37], the fundamental solution of the super Dirac operator $\partial_{\mathbf{x}}$ was calculated to be,

$$\nu_1^{m|2n} = \pi^n \sum_{k=0}^{n-1} \frac{2^{2k+1}k!}{(n-k-1)!}\, \varphi_{2k+2}^{m|0} \underline{\grave{x}}^{2n-2k-1} - \pi^n \sum_{k=0}^{n} \frac{2^{2k}k!}{(n-k)!}\, \varphi_{2k+1}^{m|0} \underline{\grave{x}}^{2n-2k}, \tag{6.40}$$

where $\varphi_j^{m|0}$ is the fundamental solution of $\partial_{\underline{x}}^j$. Observe that

$$\nu_1^{m|0}(\underline{x}) = -\varphi_1^{m|0}(\underline{x}) = \frac{1}{|\mathbb{S}^{m-1}|}\frac{\underline{x}}{|\underline{x}|^m}.$$

The superdistribution $\nu_1^{m|2n}$ satisfies

$$\partial_{\mathbf{x}}\nu_1^{m|2n}(\mathbf{x}) = \delta(\underline{x})\frac{\pi^n}{n!}\underline{\grave{x}}^{2n} = \delta(\mathbf{x}) = \nu_1^{m|2n}(\mathbf{x})\partial_{\mathbf{x}},$$

where

$$\delta(\mathbf{x}) = \delta(\underline{x})\frac{\pi^n}{n!}\underline{\grave{x}}^{2n}$$

defines the Dirac distribution on the supervector variable $\mathbf{x}$ and $\delta(\underline{x}) = \delta(x_1)\cdots\delta(x_m)$ is the m-dimensional real Dirac distribution. It is easily seen that

$$\langle \delta(\mathbf{x}-\mathbf{y}), G(\mathbf{x})\rangle = \int_{\mathbb{R}^{m|2n}} \delta(\mathbf{x}-\mathbf{y})G(\mathbf{x}) = G(\mathbf{y}), \tag{6.41}$$

or equivalently,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^m} \delta(\underline{x}-\underline{y}) G_A(\underline{x})\, dV_{\underline{x}} = G_A(\underline{y}), \\ \displaystyle\frac{\pi^n}{n!} \int_B (\underline{x}\grave{} - \underline{y}\grave{})^{2n}\, \underline{x}\grave{}_A = \underline{y}\grave{}_A, \end{cases}$$

where $\mathbf{y} = \underline{y} + \underline{y}\grave{}$ and $G \in C^\infty(U_{\underline{y}}) \otimes \mathfrak{G}_{2n}$ with $U_{\underline{y}} \subset \mathbb{R}^m$ being a neighbourhood of $\underline{y}$.

In (6.39) we can effectuate the substitution $\beta = \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})$ with $\mathbf{y} = \underline{y} + \underline{y}\grave{}$ such that $g_0(\underline{y}) \neq 0$. Indeed, it is easily seen that *sing supp* $\nu_1^{m|2n}(\mathbf{x}-\mathbf{y}) = \{\underline{y}\}$. In this way we get

$$\begin{aligned} &\int_{\mathbb{R}^{m|2n}} \delta(\mathbf{x}-\mathbf{y}) H(-g(\mathbf{x})) G(\mathbf{x}) \\ &= \int_{\mathbb{R}^{m|2n}} \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})\, \delta(g(\mathbf{x}))\, (\partial_{\mathbf{x}} g(\mathbf{x}))\, G(\mathbf{x}) - \int_{\mathbb{R}^{m|2n}} \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})\, H(-g(\mathbf{x}))\, (\partial_{\mathbf{x}} G(\mathbf{x}))\,. \end{aligned} \tag{6.42}$$

Let us now examine the distributional product

$$\delta(\mathbf{x}-\mathbf{y}) H(-g(\mathbf{x})) = \frac{\pi^n}{n!} (\underline{x}\grave{} - \underline{y}\grave{})^{2n} \sum_{j=0}^{n} \frac{(-\mathbf{g}(\mathbf{x}))^j}{j!}\, \delta(\underline{x}-\underline{y}) \delta^{(j-1)}(-g_0(\underline{x})).$$

It is clearly seen that *sing supp* $\delta(\underline{x}-\underline{y}) = \{\underline{y}\}$ and *sing supp* $\delta^{(j-1)}(-g_0(\underline{x})) = g_0^{-1}(0)$, whence, (6.8) immediately shows for $g_0(\underline{y}) \neq 0$ that

$$\delta(\underline{x}-\underline{y}) \delta^{(j-1)}(-g_0(\underline{x})) = 0, \qquad j = 1, \ldots, n.$$

Thus $\delta(\mathbf{x}-\mathbf{y}) H(-g(\mathbf{x})) = \delta(\mathbf{x}-\mathbf{y}) H(-g_0(\underline{x}))$ and then, (6.41) yields

$$\int_{\mathbb{R}^{m|2n}} \delta(\mathbf{x}-\mathbf{y}) H(-g(\mathbf{x})) G(\mathbf{x}) = \int_{\mathbb{R}^{m|2n}} \delta(\mathbf{x}-\mathbf{y}) H(-g_0(\underline{x})) G(\mathbf{x}) = H(-g_0(\underline{y})) G(\mathbf{y}). \tag{6.43}$$

Substituting (6.43) into (6.42) we obtain the following distributional Cauchy-Pompeiu formula in superspace.

Theorem 6.2 (Cauchy-Pompeiu theorem in superspace). *Let* $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ *be a phase function such that* $\{g_0 \leq 0\}$ *is compact and* $\partial_{\underline{x}}[g_0] \neq 0$ *on* $g_0^{-1}(0)$*. Then, for* $G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ *such that* $\{g_0 \leq 0\} \subset \Omega$ *one has*

$$\begin{aligned} \int_{\mathbb{R}^{m|2n}} \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})\, \delta(g(\mathbf{x}))\, (\partial_{\mathbf{x}} g(\mathbf{x}))\, G(\mathbf{x}) - \int_{\mathbb{R}^{m|2n}} \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})\, H(-g(\mathbf{x}))\, (\partial_{\mathbf{x}} G(\mathbf{x})) \\ = \begin{cases} G(\mathbf{y}), & g_0(\underline{y}) < 0, \\ 0, & g_0(\underline{y}) > 0. \end{cases} \end{aligned} \tag{6.44}$$

As mentioned before, Theorem 6.2 extends and unifies the known Cauchy-Pompeiu formulae (6.33) and (6.34) (see Theorems 7 and 11 in [24]). Indeed, in the particular case $g = g_0$, formula (6.33) immediately follows from (6.44). On the other hand, for $g(\mathbf{x}) = -\mathbf{x}^2 - 1$, formula (6.44) yields the supersphere case (6.34).

The Cauchy integral formula for super-monogenic functions then reads as follows.

Theorem 6.3 (Cauchy integral formula in superspace). *Let $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ be a phase function such that $\{g_0 \leq 0\}$ is compact and $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$. Then, for every super-monogenic function $G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ such that $\{g_0 \leq 0\} \subset \Omega$ one has*

$$\int_{\mathbb{R}^{m|2n}} \nu_1^{m|2n}(\mathbf{x}-\mathbf{y})\, \delta(g(\mathbf{x}))\, (\partial_{\mathbf{x}} g(\mathbf{x}))\, G(\mathbf{x}) = \begin{cases} G(\mathbf{y}), & g_0(\underline{y}) < 0, \\ 0, & g_0(\underline{y}) > 0. \end{cases} \tag{6.45}$$

7

Bochner-Martinelli formula in superspace

The Bochner-Martinelli integral representation constitutes a classical generalization, to the case of several complex variables, of the Cauchy integral formula for holomorphic functions in the complex plane. This representation reads for every holomorphic complex function f on some bounded domain $\Omega \subset \mathbb{C}^m$, with smooth boundary $\partial\Omega$, as

$$f(\underline{U}) = \int_{\partial\Omega} f(\underline{Z})\, \mathcal{K}(\underline{Z}, \underline{U}), \qquad \underline{U} \in \Omega, \tag{7.1}$$

where $\mathcal{K}(\underline{Z}, \underline{U})$ is the exterior differential form of type $(m, m-1)$ given by

$$\mathcal{K}(\underline{Z}, \underline{U}) = \frac{(m-1)!}{(2\pi i)^m} \sum_{j=1}^{m} (-1)^{j-1} \frac{z_j^c - u_j^c}{|\underline{Z} - \underline{U}|^{2m}}\, \widetilde{\widehat{dz_j^c}},$$

with $\widetilde{\widehat{dz_j^c}} = dz_1^c \wedge \cdots \wedge dz_{j-1}^c \wedge dz_{j+1}^c \wedge \cdots \wedge dz_m^c \wedge dz_1 \wedge \cdots \wedge dz_m$ and $\cdot^c$ denoting the complex conjugation. The form $\mathcal{K}(\underline{Z}, \underline{U})$ is the so-called Bochner-Martinelli kernel. When $m = 1$, this kernel reduces to the Cauchy kernel $(2\pi i)^{-1}(z-u)^{-1}\, dz$, whence formula (7.1) reduces to the traditional Cauchy integral formula in one complex variable. For $m > 1$, $\mathcal{K}(\underline{Z}, \underline{U})$ fails to be holomorphic but it still remains harmonic, see e.g. [59]. Formula (7.1) was obtained independently and through different methods by Martinelli and Bochner, see e.g. [58] for a detailed description. The interest for proving different generalizations of the classical Bochner-Martinelli formula has emerged as a successful research topic.

A second important generalization of the Cauchy integral formula is offered by Euclidean Clifford analysis. In this framework the Clifford-Cauchy integral formula for monogenic functions reads

$$f(\underline{y}) = \int_{\partial\Omega} \varphi_1^{2m|0}(\underline{x} - \underline{y})\, n(\underline{x})\, f(\underline{x})\, dS_{\underline{x}}, \qquad \underline{y} \in \Omega.$$

Observe that this representation corresponds to the pure bosonic version, i.e. $n = 0$, of formula (6.45). This integral formula has been a cornerstone in the development of the Euclidean monogenic function theory.

Both integral representations above were proven to be related when one considers Hermitian Clifford analysis. Indeed, in [11] a Cauchy integral formula for Hermitian monogenic functions was obtained in the purely bosonic case by passing to the framework of circulant (2×2) matrix functions. This Hermitian Cauchy integral representation was proven to reduce to the traditional Bochner-Martinelli formula (7.1) when considering the special case of functions taking values in the zero-homogeneous part of complex spinor space. This means that the theory of Hermitian monogenic functions not only refines Euclidean Clifford analysis (and thus harmonic analysis as well), but also has strong connections with the theory of functions of several complex variables, even encompassing some of its results.

The main goal of this chapter is to extend the Bochner-Martinelli formula (7.1) to superspace by exploiting the above described relation with Clifford analysis. We first address the problem of establishing a Cauchy integral formula in the framework of Hermitian Clifford analysis in superspace (the building blocks of which were introduced in Chapter 3). To this end, we use the general distributional approach to integration in superspace provided in the previous chapter. Finally, we establish the connection between Hermitian monogenicity and holomorphicity in superspace by considering an specific class of spinor valued superfunctions (Section 7.4). As one may have expected, the obtained (super) Hermitian Cauchy integral formula (7.29) reduces, when considering the correct projections, to a new extension of the Bochner-Martinelli formula for holomorphic functions in superspace.

7.1 Hermitian-Stokes and Hermitian-Cauchy theorems

The aim of this section is to translate the distributional Stokes theorem 6.1 and the Clifford-Cauchy theorem 6.2 into the Hermitian Clifford analysis framework in superspace.

In the remainder of this chapter, we will denote by Ω some open region in $\mathbb{R}^{2m}$. As was the case with $\partial_{\mathbf{x}}$, the notion of super-monogenicity may be naturally associated to $\partial_{\mathbf{J(x)}}$ as well. Then a function $F \in C^1(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{2m,2n}$ is called a (left) Hermitian super-monogenic (or hs-monogenic) function if it satisfies the system

$$\partial_{\mathbf{x}}[F] = 0 = \partial_{\mathbf{J(x)}}[F]$$

or equivalently, the system

$$\partial_{\mathbf{Z}}[F] = 0 = \partial_{\mathbf{Z}^\dagger}[F].$$

For further use we recall that

$$(\mathbf{Z})^2 = \left(\mathbf{Z}^\dagger\right)^2 = 0, \qquad\qquad (\partial_{\mathbf{Z}})^2 = (\partial_{\mathbf{Z}^\dagger})^2 = 0,$$

and moreover,

$$\Delta_{2m|2n} = -\partial_{\mathbf{x}}^2 = -\partial_{\mathbf{J}(\mathbf{x})}^2 = 4\left\{\partial_{\mathbf{Z}}, \partial_{\mathbf{Z}^\dagger}\right\}, \qquad\qquad \{\partial_{\mathbf{x}}, \partial_{\mathbf{J}(\mathbf{x})}\} = 0,$$

where

$$\Delta_{2m|2n} = \sum_{j=1}^{2m} \partial_{x_j}^2 - 4\sum_{j=1}^{n} \partial_{\grave{x}_{2j-1}}\partial_{\grave{x}_{2j}}$$

is the corresponding super Laplace operator. Moreover, if one defines

$$|\mathbf{Z}|^2 = |\mathbf{Z}^\dagger|^2 := \left\{\mathbf{Z}, \mathbf{Z}^\dagger\right\},$$

one immediately has

$$|\mathbf{Z}|^2 = |\mathbf{Z}^\dagger|^2 = \sum_{j=1}^{m} z_j z_j^c - \frac{i}{2}\sum_{j=1}^{n} \grave{z}_j \grave{z}_j^c = \sum_{j=1}^{m} x_j^2 - \sum_{j=1}^{n} \grave{x}_{2j-1}\grave{x}_{2j} = |\mathbf{x}|^2 = |\mathbf{J}(\mathbf{x})|^2.$$

In Section 6.5, the theorems of Stokes and Cauchy were already formulated in terms of the super Dirac operator $\partial_{\mathbf{x}}$. Clearly, the action of the complex structure $\mathbf{J}$ allows to restate both theorems for $\partial_{\mathbf{J}(\mathbf{x})}$, leading to their "twisted" formulations below.

Theorem 7.1 (Distributional Stokes Theorems). *Let $F, G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ and $\alpha \in \mathcal{E}' \otimes \mathfrak{G}_{2n}^{(ev)}$ a distribution with compact support such that $supp\,\alpha \subset \Omega$. Then,*

$$\int_{\mathbb{R}^{2m|2n}} (F\partial_{\mathbf{x}})\,\alpha G + F\alpha\,(\partial_{\mathbf{x}}G) = -\int_{\mathbb{R}^{2m|2n}} F\,(\partial_{\mathbf{x}}\alpha)\,G,$$

$$\int_{\mathbb{R}^{2m|2n}} \left(F\partial_{\mathbf{J}(\mathbf{x})}\right)\alpha G + F\alpha\left(\partial_{\mathbf{J}(\mathbf{x})}G\right) = -\int_{\mathbb{R}^{2m|2n}} F\left(\partial_{\mathbf{J}(\mathbf{x})}\alpha\right)G.$$

Corollary 7.1. *Let $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^{2m}) \otimes \mathfrak{G}_{2n}^{(ev)}$ be a phase function such that $\{g_0 \leq 0\}$ is compact and $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$. Then, for $F, G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ such that $\{g_0 \leq 0\} \subset \Omega$ one has*

$$\int_{\mathbb{R}^{2m|2n}} H(-g)\left[(F\partial_{\mathbf{x}})\,G + F\,(\partial_{\mathbf{x}}G)\right] = \int_{\mathbb{R}^{2m|2n}} F\delta(g)\partial_{\mathbf{x}}[g]G,$$

$$\int_{\mathbb{R}^{2m|2n}} H(-g)\left[\left(F\partial_{\mathbf{J}(\mathbf{x})}\right)G + F\left(\partial_{\mathbf{J}(\mathbf{x})}G\right)\right] = \int_{\mathbb{R}^{2m|2n}} F\delta(g)\partial_{\mathbf{J}(\mathbf{x})}[g]G.$$

Corollary 7.2 (Clifford-Cauchy theorems). *Let $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^m) \otimes \mathfrak{G}_{2n}^{(ev)}$ be a phase function such that $\{g_0 \leq 0\}$ is compact and $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$; and consider $G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{m,2n}$ with Ω such that $\{g_0 \leq 0\} \subset \Omega$. Then,* *(i) if G is $\partial_{\mathbf{x}}$-super-monogenic, one has*

$$\int_{\mathbb{R}^{m|2n}} \delta(g)\partial_{\mathbf{x}}[g]G = 0.$$

(ii) if G is $\partial_{\mathbf{J}(\mathbf{x})}$-super-monogenic, one has

$$\int_{\mathbb{R}^{m|2n}} \delta(g)\partial_{\mathbf{J}(\mathbf{x})}[g]G = 0.$$

Using (3.35) we easily obtain the following equivalent results in terms of the Hermitian Dirac operators $\partial_{\mathbf{Z}}$ and $\partial_{\mathbf{Z}^\dagger}$.

Corollary 7.3 (Hermitian Clifford-Stokes theorems). *The following formulae hold under the same conditions as in Theorem 7.1*

$$\int_{\mathbb{R}^{2m|2n}} (F\partial_{\mathbf{Z}})\,\alpha G + F\alpha\,(\partial_{\mathbf{Z}}G) = -\int_{\mathbb{R}^{2m|2n}} F\,(\partial_{\mathbf{Z}}\alpha)\,G, \tag{7.2}$$

$$\int_{\mathbb{R}^{2m|2n}} (F\partial_{\mathbf{Z}^\dagger})\,\alpha G + F\alpha\,(\partial_{\mathbf{Z}^\dagger}G) = -\int_{\mathbb{R}^{2m|2n}} F\,(\partial_{\mathbf{Z}^\dagger}\alpha)\,G. \tag{7.3}$$

Corollary 7.4. *The following formulae hold under the same conditions as in Corollary 7.1*

$$\int_{\mathbb{R}^{2m|2n}} H(-g)\left[(F\partial_{\mathbf{Z}})\,G + F\,(\partial_{\mathbf{Z}}G)\right] = \int_{\mathbb{R}^{2m|2n}} F\delta(g)\partial_{\mathbf{Z}}[g]G,$$

$$\int_{\mathbb{R}^{2m|2n}} H(-g)\left[(F\partial_{\mathbf{Z}^\dagger})\,G + F\,(\partial_{\mathbf{Z}^\dagger}G)\right] = \int_{\mathbb{R}^{2m|2n}} F\delta(g)\partial_{\mathbf{Z}^\dagger}[g]G.$$

Corollary 7.5 (Hermitian Clifford-Cauchy theorems). *Let $g = g_0+\mathbf{g} \in C^\infty(\mathbb{R}^m)\otimes \mathfrak{G}_{2n}^{(ev)}$ be a phase function such that $\{g_0 \leq 0\}$ is compact and $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$. Then for every sh-monogenic function $G \in C^\infty(\Omega)\otimes\mathfrak{G}_{2n}\otimes\mathcal{C}_{m,2n}$ such that $\{g_0 \leq 0\} \subset \Omega$ one has*

$$\int_{\mathbb{R}^{2m|2n}} \delta(g)\partial_{\mathbf{Z}}[g]G = 0, \qquad \int_{\mathbb{R}^{2m|2n}} \delta(g)\partial_{\mathbf{Z}^\dagger}[g]G = 0.$$

7.2 Fundamental solutions for $\partial_{\mathbf{x}}$ and $\partial_{\mathbf{J}(\mathbf{x})}$

In this section we provide explicit expressions for the fundamental solutions of the super Dirac operators $\partial_{\mathbf{x}}$ and $\partial_{\mathbf{J}(\mathbf{x})}$, and perform some important computations with them. First we need the following result, for which we consider a general bosonic dimension $m \in \mathbb{N}$ (not necessarily even).

Theorem 7.2. *If $M = m - 2n \notin -2\mathbb{N}+2$, the fundamental solution $\nu_1^{m|2n}$ given in (6.40) has the form*

$$\nu_1^{m|2n}(\mathbf{x}) = \frac{1}{|\mathbb{S}^{m-1|2n}|}\frac{\mathbf{x}}{|\mathbf{x}|^M},$$

where $|\mathbb{S}^{m-1|2n}| = \frac{2\pi^{M/2}}{\Gamma(\frac{M}{2})}$ *is the surface area of the unit supersphere, see Proposition 6.12.*

Proof.

We first recall that the fundamental solution of Δ_m^k, where $\Delta_m = \sum_{j=1}^m \partial_{x_j}^2$ is the Laplace operator in m bosonic dimensions, is given by

$$\nu_{2k}^{m|0}(\underline{x}) = \frac{|\underline{x}|^{2k-n}}{|\mathbb{S}^{m-1}|\, 2^{k-1}\,(k-1)!\,\prod_{\ell=1}^{k}(2\ell-m)} = \frac{(-1)^k\Gamma\left(\frac{m}{2}-k\right)}{2^{2k}\,\pi^{m/2}\,\Gamma(k)}\,\frac{|\underline{x}|^{2k}}{|\underline{x}|^m}, \tag{7.4}$$

if $m-2k \notin -2\mathbb{N}+2$, see [3]. Then the above formula can be used for every $k \leq n$, since the condition $m-2n \notin -2\mathbb{N}+2$ directly implies $m-2k \notin -2\mathbb{N}+2$. Indeed, it suffices to observe that $m-2k = m-2n+2(n-k)$ with $n-k \geq 0$.

Since $\Delta_m^{k+1} = (-1)^{k+1}\partial_{\underline{x}}^{2k+2}$ we can write

$$\varphi_{2k+2}^{m|0} = (-1)^{k+1}\nu_{2k+2}^{m|0} = \frac{\Gamma\left(\frac{m}{2}-k-1\right)}{2^{2k+2}\,\pi^{m/2}\,\Gamma(k+1)}\,\frac{|\underline{x}|^{2k+2}}{|\underline{x}|^m}, \qquad k=0,1,\ldots,n-1, \tag{7.5}$$

$$\varphi_{2k+1}^{m|0} = \partial_{\underline{x}}\left[\varphi_{2k+2}^{m|0}\right] = -\frac{\Gamma\left(\frac{m}{2}-k\right)}{2^{2k+1}\,\pi^{m/2}\,\Gamma(k+1)}\,\frac{\underline{x}\,|\underline{x}|^{2k}}{|\underline{x}|^m}, \qquad k=0,1,\ldots,n-1. \tag{7.6}$$

It is easily seen that (7.6) also holds for $k=n$. Indeed, writing

$$\varphi_{2n+1}^{m|0} = -\frac{\Gamma\left(\frac{m}{2}-n\right)}{2^{2n+1}\,\pi^{m/2}\,\Gamma(n+1)}\,\frac{\underline{x}\,|\underline{x}|^{2n}}{|\underline{x}|^m},$$

we immediately obtain

$$\partial_{\underline{x}}\left[\varphi_{2n+1}^{m|0}\right] = \frac{\Gamma\left(\frac{m}{2}-n\right)}{2^{2n}\,\pi^{m/2}\,\Gamma(n)}\,\frac{|\underline{x}|^{2n}}{|\underline{x}|^m} = \varphi_{2n}^{m|0}.$$

This means that the above expression for $\varphi_{2n+1}^{m|0}$ constitutes a fundamental solution for $\partial_{\underline{x}}^{2n+1}$.

Now, substituting (7.5)-(7.6) into (6.40) we get

$$\begin{aligned}
\nu_1^{m|2n} &= \frac{\pi^n}{2\pi^{\frac{m}{2}}}\left[\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{m}{2}-k-1\right)}{\Gamma(n-k)}\frac{|\underline{x}|^{2(k+1)}}{|\underline{x}|^m}\grave{\underline{x}}^{2n-2k-1} + \sum_{k=0}^{n}\frac{\Gamma\left(\frac{m}{2}-k\right)}{\Gamma(n-k+1)}\frac{\underline{x}\,|\underline{x}|^{2k}}{|\underline{x}|^m}\grave{\underline{x}}^{2n-2k}\right]\\
&= \frac{1}{2\pi^{\frac{M}{2}}}\left[\sum_{k=1}^{n}\frac{\Gamma\left(\frac{m}{2}-k\right)}{\Gamma(n-k+1)}\frac{|\underline{x}|^{2k}}{|\underline{x}|^m}\grave{\underline{x}}^{2n-2k+1} + \sum_{k=0}^{n}\frac{\Gamma\left(\frac{m}{2}-k\right)}{\Gamma(n-k+1)}\frac{\underline{x}\,|\underline{x}|^{2k}}{|\underline{x}|^m}\grave{\underline{x}}^{2n-2k}\right]\\
&= \frac{1}{2\pi^{\frac{M}{2}}}\,\mathbf{x}\sum_{k=1}^{n}\frac{\Gamma\left(\frac{m}{2}-k\right)}{\Gamma(n-k+1)}|\underline{x}|^{2k-m}\,\grave{\underline{x}}^{2n-2k} + \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{M}{2}}\,\Gamma(n+1)}\,\frac{\underline{x}}{|\underline{x}|^m}\,\grave{\underline{x}}^{2n}.
\end{aligned} \tag{7.7}$$

Recall (see (6.5)) that

$$\frac{1}{|\mathbf{x}|^M} = \left(|\underline{x}|^2 - \underline{x\grave{}}^2\right)^{\frac{-M}{2}} = \sum_{j=0}^{n} \frac{\underline{x\grave{}}^{2j}}{j!} \frac{\Gamma\left(\frac{m}{2} - m + j\right)}{\Gamma\left(\frac{m}{2} - n\right)} |\underline{x}|^{-m+2n-2j}$$
$$= \frac{1}{\Gamma\left(\frac{m}{2} - n\right)} \sum_{k=0}^{n} \frac{\Gamma\left(\frac{m}{2} - k\right)}{\Gamma(n-k+1)} |\underline{x}|^{2k-m}\, \underline{x\grave{}}^{2n-2k}.$$

Substituting the later into (7.7) we obtain

$$\nu_1^{m|2n} = \frac{1}{2\pi^{M/2}} \mathbf{x} \left(\frac{\Gamma\left(\frac{M}{2}\right)}{|\mathbf{x}|^M} - \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma(n+1)} \frac{\underline{x\grave{}}^{2n}}{|\underline{x}|^m} \right) + \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{M/2}\,\Gamma(n+1)} \frac{\underline{x}}{|\underline{x}|^m}\, \underline{x\grave{}}^{2n}$$
$$= \frac{1}{|\mathbb{S}^{m-1|2n}|} \frac{\mathbf{x}}{|\mathbf{x}|^M},$$

which completes the proof. □

If we now consider an even bosonic dimension, i.e. $M = 2m - 2n$, we easily obtain the fundamental solution of $\partial_{\mathbf{J}(\mathbf{x})}$ as shown in the next result.

Corollary 7.6. *For $M = 2m - 2n \notin -2\mathbb{N} + 2$, i.e. $m > n$, the fundamental solutions of $\partial_{\mathbf{x}}$ and $\partial_{\mathbf{J}(\mathbf{x})}$ are given by,*

$$\nu_1^{2m|2n} = \frac{1}{|\mathbb{S}^{2m-1|2n}|} \frac{\mathbf{x}}{|\mathbf{x}|^M},$$

and

$$\mathbf{J}(\nu_1^{2m|2n}) = \frac{1}{|\mathbb{S}^{2m-1|2n}|} \frac{\mathbf{J}(\mathbf{x})}{|\mathbf{x}|^M},$$

respectively.

7.2.1 Finite part distribution and spherical means

One of the fundamental tools used in the distributional calculus with $\nu_1^{2m|2n}$ and $\mathbf{J}(\nu_1^{2m|2n})$ is the distribution *"finite part"* $\mathrm{Fp}\, t_+^\mu$ on the real line. For a better understanding, we give its definition and list some of its main properties.

Let μ be a real parameter, t a real variable and consider the function

$$t_+^\mu = \begin{cases} t^\mu, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

For $\mu > -1$ the function t_+^μ is locally integrable and hence constitutes a regular distribution, i.e.

$$\langle t_+^\mu, \phi \rangle = \int_0^{+\infty} t^\mu \phi(t)\, dt, \qquad \phi \in C_0^\infty(\mathbb{R}). \tag{7.8}$$

The finite part distribution $\operatorname{Fp} t_+^\mu$ is an extension of the regular distribution t_+^μ to every value $\mu \in \mathbb{R}$. The idea of this extension is to consider only the finite part of the integral (7.8). In this way, for $-1 < \mu$, one easily defines

$$\langle \operatorname{Fp} t_+^\mu, \phi \rangle := \langle t_+^\mu, \phi \rangle,$$

while for values of the parameter μ in the strip $-(k+1) < \mu < -k$, $k \in \mathbb{N}$, one puts

$$\langle \operatorname{Fp} t_+^\mu, \phi \rangle := \lim_{\varepsilon \to 0^+} \left(\int_\varepsilon^{+\infty} t^\mu \phi(t)\, dt + \sum_{j=1}^{k} \frac{\phi^{(j-1)}(0)}{(j-1)!} \frac{\varepsilon^{\mu+j}}{(\mu+j)} \right),$$

and finally, for negative entire exponents $\mu = -k$, $k \in \mathbb{N}$,

$$\langle \operatorname{Fp} t_+^\mu, \phi \rangle := \lim_{\varepsilon \to 0^+} \left(\int_\varepsilon^{+\infty} t^{-k} \phi(t)\, dt + \left(\sum_{j=1}^{k-1} \frac{\phi^{(j-1)}(0)}{(j-1)!} \frac{\varepsilon^{-k+j}}{(-k+j)} \right) + \frac{\phi^{(k-1)}(0)}{(k-1)!} \ln(\varepsilon) \right).$$

The notation

$$\operatorname{Fp} \int_0^{+\infty} t^\mu \phi(t)\, dt$$

is often used for $\langle \operatorname{Fp} t_+^\mu, \phi \rangle$.

Proposition 7.1. *The following properties hold for* $\operatorname{Fp} t_+^\mu$.

(i) $t \operatorname{Fp} t_+^\mu = \operatorname{Fp} t_+^{\mu+1}$, $\mu \in \mathbb{R}$,

(ii) $$\frac{d}{dt} \operatorname{Fp} t_+^\mu = \begin{cases} \mu \operatorname{Fp} t_+^{\mu-1}, & \mu \notin -\mathbb{N}+1, \\ (-k) \operatorname{Fp} t_+^{-k-1} + (-1)^k \dfrac{1}{k!} \delta^{(k)}(t), & \mu = -k,\ k \in \mathbb{N}-1. \end{cases}$$

In order to compute finite part distributions in $\mathbb{R}^m$ we need the so-called *generalized spherical means*, see e.g. [18, 17]. Let $\phi \in C_0^\infty(\mathbb{R}^m)$; putting $\underline{x} = r\underline{w}$, $r = |\underline{x}|$, we define the generalized spherical means

$$\Sigma^{(0)}[\phi](r) = \frac{1}{|\mathbb{S}^{m-1}|} \int_{\mathbb{S}^{m-1}} \phi(r\underline{w})\, dS_{\underline{w}},$$
$$\Sigma^{(1)}[\phi](r) = \Sigma^{(0)}[\underline{w}\phi](r) = \frac{1}{|\mathbb{S}^{m-1}|} \int_{\mathbb{S}^{m-1}} \underline{w}\phi(r\underline{w})\, dS_{\underline{w}}.$$

It is clear that

$$\Sigma^{(0)}[\phi] : \mathbb{R}_+ \to \mathbb{C}, \qquad \Sigma^{(1)}[\phi] : \mathbb{R}_+ \to \mathbb{C}^m,$$

are C^∞ functions with singular support. We now list some important properties of these spherical means. The proofs of these results can be found in [18].

Proposition 7.2. *For a test function $\phi \in C_0^\infty(\mathbb{R}^m)$ one has*

(i) $\Sigma^{(0)}[\underline{x}\,\phi] = r\,\Sigma^{(1)}[\phi]$,

(ii) $\Sigma^{(1)}[\underline{x}\,\phi] = -r\,\Sigma^{(0)}[\phi]$,

(iii) $\Sigma^{(0)}[\partial_{\underline{x}}\,\phi] = \left(\partial_r + \frac{m-1}{r}\right)\Sigma^{(1)}[\phi]$,

(iv) $\Sigma^{(1)}[\partial_{\underline{x}}\,\phi] = -\partial_r\,\Sigma^{(0)}[\phi]$,

(v) $\left\langle \delta(r), \Sigma^{(0)}[\phi]\right\rangle = \langle \delta(\underline{x}), \phi\rangle$,

(vi) $\left\langle \delta(r), \Sigma^{(1)}[\phi]\right\rangle = 0$,

(vii) $\left\langle \delta'(r), \Sigma^{(1)}[\phi]\right\rangle = \frac{1}{m}\left\langle \partial_{\underline{x}}\delta(\underline{x}), \phi\right\rangle$.

We now have introduced all elements needed for computing the action of the distribution $\mathrm{Fp}\,|\underline{x}|_+^\lambda$ on a test function $\phi \in C_0^\infty(\mathbb{R}^m)$, i.e.

$$\begin{aligned}\left\langle \mathrm{Fp}\,|\underline{x}|_+^\lambda, \phi\right\rangle &:= \mathrm{Fp}\int_{\mathbb{R}^m} |\underline{x}|^\lambda\,\phi(\underline{x})\,dV_{\underline{x}} = \mathrm{Fp}\int_0^\infty \int_{\mathbb{S}^{m-1}} r^\lambda\,\phi(r\underline{w}) r^{m-1}\,dr\,dS_{\underline{w}}\\ &= \mathrm{Fp}\int_0^\infty r^{\lambda+m-1}\left(\int_{\mathbb{S}^{m-1}}\phi(r\underline{w})\,dS_{\underline{w}}\right)dr\\ &= |\mathbb{S}^{m-1}|\;\mathrm{Fp}\int_0^\infty r^{\lambda+m-1}\Sigma^{(0)}[\phi](r)\,dr\\ &= |\mathbb{S}^{m-1}|\left\langle \mathrm{Fp}\,r_+^{\lambda+m-1}, \Sigma^{(0)}[\phi]\right\rangle.\end{aligned} \tag{7.9}$$

This motivates the introduction of the following distributions (see [18, 17]):

$$\begin{aligned}\langle T_\lambda, \phi\rangle &= |\mathbb{S}^{m-1}|\left\langle \mathrm{Fp}\,r_+^\mu\,,\,\Sigma^{(0)}[\phi]\right\rangle,\\ \langle U_\lambda, \phi\rangle &= |\mathbb{S}^{m-1}|\left\langle \mathrm{Fp}\,r_+^\mu\,,\,\Sigma^{(1)}[\phi]\right\rangle,\end{aligned}$$

where $\mu = \lambda + m - 1$. In this way, one has in $\mathbb{R}^m$ that

$$\mathrm{Fp}\,\frac{|\underline{x}|^{\mu+1}}{|\underline{x}|^m} = T_\lambda, \qquad\qquad \mathrm{Fp}\,\frac{\underline{x}\,|\underline{x}|^\mu}{|\underline{x}|^m} = U_\lambda. \tag{7.10}$$

Indeed, the first equality directly follows from (7.9) while for the second one it suffices to note that

$$\begin{aligned}\left\langle \mathrm{Fp}\,\frac{\underline{x}\,|\underline{x}|^\mu}{|\underline{x}|^m}, \phi\right\rangle &= \mathrm{Fp}\int_{\mathbb{R}^m}\frac{\underline{x}\,|\underline{x}|^\mu}{|\underline{x}|^m}\,\phi(\underline{x})\,dV_{\underline{x}}\\ &= \mathrm{Fp}\int_0^\infty\int_{\mathbb{S}^{m-1}} r^{\mu+1-m}\,\underline{w}\,\phi(r\underline{w}) r^{m-1}\,dr\,dS_{\underline{w}}\\ &= \mathrm{Fp}\int_0^\infty r^\mu\left(\int_{\mathbb{S}^{m-1}}\underline{w}\,\phi(r\underline{w})\,dS_{\underline{w}}\right)dr\\ &= |\mathbb{S}^{m-1}|\left\langle \mathrm{Fp}\,r_+^\mu, \Sigma^{(1)}[\phi]\right\rangle = \langle U_\lambda, \phi\rangle\,.\end{aligned}$$

7.2.2 Distributional calculus for $\nu_1^{2m|2n}$ and $\mathbf{J}(\nu_1^{2m|2n})$

We are now able to compute $\partial_{\mathbf{x}}[\mathbf{J}(\nu_1^{2m|2n})]$ and $\partial_{\mathbf{J}(\mathbf{x})}[\nu_1^{2m|2n}]$ with the help of the above defined distributions.

Proposition 7.3. *For $m > n$ (i.e. $M = 2(m-n) \notin -2\mathbb{N}+2$) it holds that*

$$\partial_{\mathbf{x}}[\mathbf{J}(\nu_1^{2m|2n})] = \frac{2\mathbf{B}}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M} + \frac{M}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{\mathbf{x}\mathbf{J}(\mathbf{x})}{|\mathbf{x}|^{M+2}} + \frac{\mathbf{B}_b}{m}\delta(\mathbf{x}), \tag{7.11}$$

$$\partial_{\mathbf{J}(\mathbf{x})}[\nu_1^{2m|2n}] = \frac{-2\mathbf{B}}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M} + \frac{M}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{\mathbf{J}(\mathbf{x})\mathbf{x}}{|\mathbf{x}|^{M+2}} - \frac{\mathbf{B}_b}{m}\delta(\mathbf{x}), \tag{7.12}$$

where the distribution $\mathrm{Fp}\,\dfrac{1}{|\mathbf{x}|^{M+\alpha}}$*, $\alpha \geq 0$, is defined in superspace as*

$$\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^{M+\alpha}} = \frac{1}{\Gamma(m-n+\frac{\alpha}{2})}\sum_{k=0}^{n}\frac{\Gamma(m-k+\frac{\alpha}{2})}{\Gamma(n-k+1)}\,\mathrm{Fp}\,\frac{|\underline{x}|^{2k}}{|\underline{x}|^{2m+\alpha}}\,\underline{\grave{x}}^{2n-2k}. \tag{7.13}$$

Remark 7.1. *For n=0, formulae (7.11)-(7.12) coincide with the corresponding expressions in the purely bosonic case computed in [11].*

Proof.

From Corollary 7.6 we have in distributional sense that

$$\mathbf{J}(\nu_1^{2m|2n}) = \frac{1}{|\mathbb{S}^{2m-1|2n}|}\mathbf{J}(\mathbf{x})\,\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M}.$$

Hence,

$$\begin{aligned}\partial_{\mathbf{x}}[\mathbf{J}(\nu_1^{2m|2n})] &= \frac{1}{|\mathbb{S}^{2m-1|2n}|}\left(\partial_{\mathbf{x}}\left[\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M}\right]\mathbf{J}(\mathbf{x}) + \mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M}\partial_{\mathbf{x}}\,[\mathbf{J}(\mathbf{x})]\right)\\ &= \frac{2\mathbf{B}}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M} + \frac{1}{|\mathbb{S}^{2m-1|2n}|}\,\partial_{\mathbf{x}}\left[\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M}\right]\mathbf{J}(\mathbf{x}).\end{aligned} \tag{7.14}$$

Using (7.10) and (7.13) we obtain,

$$\partial_{\mathbf{x}}\left[\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M}\right] = \frac{1}{\Gamma(m-n)}\sum_{k=0}^{n}\frac{\Gamma(m-k)}{\Gamma(n-k+1)}\,\partial_{\mathbf{x}}\left[T_{2k-2m}\,\underline{\grave{x}}^{2n-2k}\right], \tag{7.15}$$

where

$$\partial_{\mathbf{x}}\left[T_{2k-2m}\,\underline{\grave{x}}^{2n-2k}\right] = -\partial_{\underline{x}}\,[T_{2k-2m}]\,\underline{\grave{x}}^{2n-2k} + T_{2k-2m}\,\partial_{\underline{\grave{x}}}\left[\underline{\grave{x}}^{2n-2k}\right].$$

By Propositions 7.1 and 7.2 we now get,

$$\begin{aligned}
\langle \partial_{\underline{x}} [T_{2k-2m}] , \phi\rangle &= -\langle T_{2k-2m}, \partial_{\underline{x}} [\phi]\rangle = -|\mathbb{S}^{2m-1}| \left\langle \mathrm{Fp}\, r_+^{2k-1} , \Sigma^{(0)} \left[\partial_{\underline{x}}\phi\right] \right\rangle \\
&= -|\mathbb{S}^{2m-1}| \left\langle \mathrm{Fp}\, r_+^{2k-1} , \left(\partial_r + \frac{2m-1}{r}\right) \Sigma^{(1)}[\phi] \right\rangle \\
&= |\mathbb{S}^{2m-1}| \left\langle \frac{d}{dr}\, \mathrm{Fp}\, r_+^{2k-1} , \Sigma^{(1)}[\phi] \right\rangle - |\mathbb{S}^{2m-1}| \left\langle (2m-1)\, \mathrm{Fp}\, r_+^{2k-2} , \Sigma^{(1)}[\phi] \right\rangle \\
&= |\mathbb{S}^{2m-1}| \left\langle (2k-1)\, \mathrm{Fp}\, r_+^{2k-2} - \delta_{k,0}\delta'(r) - (2m-1)\, \mathrm{Fp}\, r_+^{2k-2} , \Sigma^{(1)}[\phi] \right\rangle \\
&= (2k-2m)|\mathbb{S}^{2m-1}| \left\langle \mathrm{Fp}\, r_+^{2k-2} , \Sigma^{(1)}[\phi] \right\rangle - \delta_{k,0}|\mathbb{S}^{2m-1}| \left\langle \delta'(r) , \Sigma^{(1)}[\phi] \right\rangle \\
&= (2k-2m) \left\langle U_{2k-2m-1} , \phi\right\rangle - \frac{\delta_{k,0}|\mathbb{S}^{2m-1}|}{2m} \left\langle \partial_{\underline{x}}\delta(\underline{x}) , \phi\right\rangle \\
&= \left\langle (2k-2m)U_{2k-2m-1} - \frac{\delta_{k,0}|\mathbb{S}^{2m-1}|}{2m} \partial_{\underline{x}}\delta(\underline{x}) , \phi \right\rangle ,
\end{aligned}$$

or equivalently,

$$\partial_{\underline{x}} [T_{2k-2m}] = (2k-2m)U_{2k-2m-1} - \frac{\delta_{k,0}|\mathbb{S}^{2m-1}|}{2m} \partial_{\underline{x}}\delta(\underline{x}).$$

Moreover,

$$\partial_{\underline{x}\grave{}} \left[\underline{x}\grave{}^{\,2n-2k}\right] = 2(n-k)\underline{x}\grave{}^{\,2n-2k-1}, \qquad k = 0, 1, \ldots , n,$$

where we are formally[1] defining $0\underline{x}\grave{}^{\,-1} := 0$ in the case $k = n$. Hence we conclude that

$$\begin{aligned}
&\partial_{\mathbf{x}} \left[T_{2k-2m}\, \underline{x}\grave{}^{\,2n-2k}\right] \\
&= 2(m-k)U_{2k-2m-1}\, \underline{x}\grave{}^{\,2n-2k} + 2(n-k)\, T_{2k-2m}\, \underline{x}\grave{}^{\,2n-2k-1} + \frac{\delta_{k,0}|\mathbb{S}^{2m-1}|}{2m} \partial_{\underline{x}}\delta(\underline{x})\, \underline{x}\grave{}^{\,2n}.
\end{aligned} \tag{7.16}$$

Substituting this into (7.15) we obtain,

$$\begin{aligned}
&\partial_{\mathbf{x}} \left[\mathrm{Fp}\, \frac{1}{|\mathbf{x}|^M}\right] \\
&= \frac{1}{\Gamma(m-n)} \sum_{k=0}^{n} \frac{2\Gamma(m-k)}{\Gamma(n-k+1)} \left[(m-k)U_{2k-2m-1}\, \underline{x}\grave{}^{\,2n-2k} + (n-k)\, T_{2k-2m}\, \underline{x}\grave{}^{\,2n-2k-1}\right] \\
&\quad + \frac{\Gamma(m)}{\Gamma(m-n)\Gamma(n+1)} \frac{|\mathbb{S}^{2m-1}|}{2m} \partial_{\underline{x}}\delta(\underline{x})\, \underline{x}\grave{}^{\,2n}
\end{aligned}$$

[1] We recall that the element $\underline{x}\grave{}^{\,-1}$ does not exist due to the nilpotency of $\underline{x}\grave{}$.

which yields

$$
\begin{aligned}
&\partial_{\mathbf{x}}\left[\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^{M}}\right]\\
&= \frac{2}{\Gamma(m-n)}\left[\sum_{k=0}^{n}\frac{\Gamma(m-k+1)}{\Gamma(n-k+1)}\,U_{2k-2m-1}\,\grave{\underline{x}}^{2n-2k} + \sum_{k=0}^{n-1}\frac{\Gamma(m-k)}{\Gamma(n-k)}\,T_{2k-2m}\,\grave{\underline{x}}^{2n-2k-1}\right]\\
&\quad + \frac{\pi^{m}}{m\Gamma(m-n)\Gamma(n+1)}\,\partial_{\underline{x}}\delta(\underline{x})\,\grave{\underline{x}}^{2n}\\
&= \frac{2}{\Gamma(m-n)}\left[\sum_{k=0}^{n}\frac{\Gamma(m-k+1)}{\Gamma(n-k+1)}U_{2k-2m-1}\,\grave{\underline{x}}^{2n-2k}+\sum_{k=1}^{n}\frac{\Gamma(m-k+1)}{\Gamma(n-k+1)}T_{2k-2-2m}\,\grave{\underline{x}}^{2n-2k+1}\right]\\
&\quad + \frac{\pi^{m}}{m\Gamma(m-n)\Gamma(n+1)}\,\partial_{\underline{x}}\delta(\underline{x})\,\grave{\underline{x}}^{2n}\\
&= \frac{2}{\Gamma(m-n)}\sum_{k=1}^{n}\frac{\Gamma(m-k+1)}{\Gamma(n-k+1)}\left(\mathrm{Fp}\,\frac{\underline{x}|\underline{x}|^{2k-2}}{|\underline{x}|^{2m}} + \mathrm{Fp}\,\frac{\grave{\underline{x}}|\underline{x}|^{2k-2}}{|\underline{x}|^{2m}}\right)\grave{\underline{x}}^{2n-2k}\\
&\quad + \frac{2\Gamma(m+1)}{\Gamma(m-n)\Gamma(n+1)}U_{-2m-1}\,\grave{\underline{x}}^{2n} + \frac{\pi^{m}}{m\Gamma(m-n)\Gamma(n+1)}\,\partial_{\underline{x}}\delta(\underline{x})\,\grave{\underline{x}}^{2n}\\
&= \frac{2\mathbf{x}}{\Gamma(m-n)}\sum_{k=1}^{n}\left(\frac{\Gamma(m-k+1)}{\Gamma(n-k+1)}\,\mathrm{Fp}\,\frac{|\underline{x}|^{2k}}{|\underline{x}|^{2m+2}}\,\grave{\underline{x}}^{2n-2k}\right) + \frac{2\Gamma(m+1)}{\Gamma(m-n)\Gamma(n+1)}U_{-2m-1}\,\grave{\underline{x}}^{2n}\\
&\quad + \frac{\pi^{m}}{m\Gamma(m-n)\Gamma(n+1)}\,\partial_{\underline{x}}\delta(\underline{x})\,\grave{\underline{x}}^{2n}.
\end{aligned}
$$

Then from (7.13) we get,

$$
\begin{aligned}
\partial_{\mathbf{x}}\left[\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^{M}}\right] &= \frac{2\mathbf{x}}{\Gamma(m-n)}\left[\Gamma(m-n+1)\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^{M+2}} - \frac{\Gamma(m+1)}{\Gamma(n+1)}\,\mathrm{Fp}\,\frac{1}{|\underline{x}|^{2m+2}}\,\grave{\underline{x}}^{2n}\right]\\
&\quad + \frac{2\Gamma(m+1)}{\Gamma(m-n)\Gamma(n+1)}U_{-2m-1}\,\grave{\underline{x}}^{2n} + \frac{\pi^{m}}{m\Gamma(m-n)\Gamma(n+1)}\,\partial_{\underline{x}}\delta(\underline{x})\,\grave{\underline{x}}^{2n}\\
&= 2(m-n)\,\mathrm{Fp}\,\frac{\mathbf{x}}{|\mathbf{x}|^{M+2}} + \frac{\pi^{m}}{m\Gamma(m-n)\Gamma(n+1)}\,\partial_{\underline{x}}\delta(\underline{x})\,\grave{\underline{x}}^{2n} \qquad (7.17)
\end{aligned}
$$

Substitution of (7.17) into (7.14) yields

$$
\begin{aligned}
&\partial_{\mathbf{x}}[\mathbf{J}(\nu_1^{2m|2n})]\\
&= \frac{1}{|\mathbb{S}^{2m-1|2n}|}\left(2\mathbf{B}\,\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^{M}} + M\,\mathrm{Fp}\,\frac{\mathbf{x}\mathbf{J}(\mathbf{x})}{|\mathbf{x}|^{M+2}} + \frac{\pi^{m}}{m\Gamma(m-n)\Gamma(n+1)}\,\partial_{\underline{x}}\delta(\underline{x})\,\mathbf{J}(\underline{x})\,\grave{\underline{x}}^{2n}\right). \qquad (7.18)
\end{aligned}
$$

Observe now that, for any complex valued test function ϕ, one has

$$
\begin{aligned}
\left\langle \partial_{\underline{x}}\delta(\underline{x})\,\mathbf{J}(\underline{x}),\phi\right\rangle &= -\left\langle \delta(\underline{x})\,,\partial_{\underline{x}}\left[\mathbf{J}(\underline{x})\phi\right]\right\rangle\\
&= -\left\langle \delta(\underline{x})\,,-2\mathbf{B}_b\phi + \partial_{\underline{x}}\left[\phi\right]\mathbf{J}(\underline{x})\right\rangle\\
&= \left\langle 2\mathbf{B}_b\delta(\underline{x})\,,\phi\right\rangle,
\end{aligned}
$$

which implies that

$$\partial_{\underline{x}}\delta(\underline{x})\,\mathbf{J}(\underline{x}) = 2\mathbf{B}_b\delta(\underline{x}).$$

Hence, since

$$\delta(\mathbf{x}) = \delta(\underline{x})\frac{\pi^n}{n!}\underline{x}\grave{}^{2n},$$

we obtain,

$$\frac{\pi^m}{m\Gamma(m-n)\Gamma(n+1)}\,\partial_{\underline{x}}\delta(\underline{x})\,\mathbf{J}(\underline{x})\,\underline{x}\grave{}^{2n} = \frac{2\pi^m}{m\Gamma(\frac{M}{2})n!}\,\mathbf{B}_b\delta(\underline{x})\underline{x}\grave{}^{2n} = \frac{|\mathbb{S}^{2m-1|2n}|}{m}\mathbf{B}_b\delta(\mathbf{x}).$$

Finally, substituting the later into (7.18) we obtain (7.11). Formula (7.12) easily follows from applying $\mathbf{J}$ to both sides of (7.11) and using the properties $\mathbf{J}^2[\nu_1^{2m|2n}] = -\nu_1^{2m|2n}$ and $\mathbf{J}(\mathbf{B}) = \mathbf{B}$. □

7.2.3 Hermitian counterparts of $\nu_1^{2m|2n}$ and $\mathbf{J}(\nu_1^{2m|2n})$

Similarly as above, we introduce the following Hermitian counterparts to the pair of fundamental solutions $(\nu_1^{2m|2n}, \mathbf{J}(\nu_1^{2m|2n}))$, for $m > n$:

$$\Psi_1^{m|n} = \nu_1^{2m|2n} + i\mathbf{J}(\nu_1^{2m|2n}), \qquad \Psi_1^{m|n\dagger} = -\left(\nu_1^{2m|2n} - i\mathbf{J}(\nu_1^{2m|2n})\right),$$

or equivalently,

$$\Psi_1^{m|n}(\mathbf{Z}) = \frac{2}{|\mathbb{S}^{2m-1|2n}|}\frac{\mathbf{Z}}{|\mathbf{Z}|^M}, \qquad \Psi_1^{m|n\dagger}(\mathbf{Z}) = \frac{2}{|\mathbb{S}^{2m-1|2n}|}\frac{\mathbf{Z}^\dagger}{|\mathbf{Z}|^M},$$

where we recall that $|\mathbf{Z}| = |\mathbf{x}|$. As in the purely bosonic case, see e.g. [11], $\Psi_1^{m|n}$ and $\Psi_1^{m|n\dagger}$ are not the fundamental solutions of the Hermitian super Dirac operators $\partial_{\mathbf{Z}}$ and $\partial_{\mathbf{Z}^\dagger}$. Indeed, from (7.11)-(7.12) one obtains the following results.

Proposition 7.4.

$$\partial_{\mathbf{Z}}\Psi_1^{m|n} = \frac{m+i\mathbf{B}_b}{2m}\delta(\mathbf{x}) + \frac{\frac{M}{2}+i\mathbf{B}}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M} - \frac{M}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{\mathbf{Z}^\dagger\mathbf{Z}}{|\mathbf{x}|^{M+2}} = \Psi_1^{m|n\dagger}\partial_{\mathbf{Z}^\dagger}, \tag{7.19}$$

$$\partial_{\mathbf{Z}^\dagger}\Psi_1^{m|n} = 0 = \Psi_1^{m|n\dagger}\partial_{\mathbf{Z}}, \tag{7.20}$$

$$\partial_{\mathbf{Z}}\Psi_1^{m|n\dagger} = 0 = \Psi_1^{m|n}\partial_{\mathbf{Z}^\dagger},$$

$$\partial_{\mathbf{Z}^\dagger}\Psi_1^{m|n\dagger} = \frac{m-i\mathbf{B}_b}{2m}\delta(\mathbf{x}) - \frac{\frac{M}{2}+i\mathbf{B}}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M} + \frac{M}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{\mathbf{Z}^\dagger\mathbf{Z}}{|\mathbf{x}|^{M+2}} = \Psi_1^{m|n}\partial_{\mathbf{Z}}.$$

Proof.

We will only prove the left equalities in (7.19) and (7.20) since the remaining ones can be proven along similar lines. We first observe that

$$\begin{aligned}
\partial_{\mathbf{Z}} \Psi_1^{m|n} &= \frac{1}{4}\left(\partial_{\mathbf{x}} - i\partial_{\mathbf{J}(\mathbf{x})}\right)\left(\nu_1^{2m|2n} + i\mathbf{J}(\nu_1^{2m|2n})\right)\\
&= \frac{1}{4}\left[\left(\partial_{\mathbf{x}}\nu_1^{2m|2n} + \partial_{\mathbf{J}(\mathbf{x})}\mathbf{J}(\nu_1^{2m|2n})\right) + i\left(\partial_{\mathbf{x}}\mathbf{J}(\nu_1^{2m|2n}) - \partial_{\mathbf{J}(\mathbf{x})}\nu_1^{2m|2n}\right)\right]\\
&= \frac{1}{4}\left[2\delta(\mathbf{x}) + i\left(\frac{4\mathbf{B}}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M} + \frac{2M}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{\mathbf{x}\mathbf{J}(\mathbf{x})}{|\mathbf{x}|^{M+2}} + \frac{2\mathbf{B}_b}{m}\delta(\mathbf{x})\right)\right]\\
&= \frac{m+i\mathbf{B}_b}{2m}\delta(\mathbf{x}) + \frac{i\mathbf{B}}{|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{1}{|\mathbf{x}|^M} + \frac{iM}{2|\mathbb{S}^{2m-1|2n}|}\,\mathrm{Fp}\,\frac{\mathbf{x}\mathbf{J}(\mathbf{x})}{|\mathbf{x}|^{M+2}}.
\end{aligned}$$

On the other hand it can be easily proven that

$$\mathbf{x}\mathbf{J}(\mathbf{x}) = -i|\mathbf{x}|^2 + 2i\mathbf{Z}^\dagger\mathbf{Z}.$$

Substituting this result into the above formula, we get (7.19). For (7.20) it suffices to note that

$$\begin{aligned}
\partial_{\mathbf{Z}^\dagger} \Psi_1^{m|n} &= -\frac{1}{4}\left(\partial_{\mathbf{x}} + i\partial_{\mathbf{J}(\mathbf{x})}\right)\left(\nu_1^{2m|2n} + i\mathbf{J}(\nu_1^{2m|2n})\right)\\
&= -\frac{1}{4}\left[\left(\partial_{\mathbf{x}}\nu_1^{2m|2n} - \partial_{\mathbf{J}(\mathbf{x})}\mathbf{J}(\nu_1^{2m|2n})\right) + i\left(\partial_{\mathbf{x}}\mathbf{J}(\nu_1^{2m|2n}) + \partial_{\mathbf{J}(\mathbf{x})}\nu_1^{2m|2n}\right)\right]\\
&= 0,
\end{aligned}$$

which completes the proof. □

Proposition 7.4 shows that the functions $\Psi_1^{m|n}$ and $\Psi_1^{m|n\,\dagger}$ are not hs-monogenic. Nevertheless, they can be combined in a (2×2) circulant matrix in order to obtain the Hermitian Cauchy formulae in superspace. This approach is inspired by the one used in the purely bosonic case, see e.g. [11, 62].

Theorem 7.3. *Introducing the particular circulant* (2×2) *matrices* [2]

$$\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)} = \begin{pmatrix} \partial_{\mathbf{Z}} & \partial_{\mathbf{Z}^\dagger} \\ \partial_{\mathbf{Z}^\dagger} & \partial_{\mathbf{Z}} \end{pmatrix}, \quad \boldsymbol{\Psi}_{2\times 2}^{m|n} = \begin{pmatrix} \Psi_1^{m|n} & \Psi_1^{m|n\,\dagger} \\ \Psi_1^{m|n\,\dagger} & \Psi_1^{m|n} \end{pmatrix}, \quad \boldsymbol{\delta} := \delta I_2 = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix},$$

one obtains that

$$\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\, \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}) = \boldsymbol{\delta}(\mathbf{x}) = \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z})\, \boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}.$$

7.3 Hermitian Cauchy-Pompeiu formula in superspace

Theorem 7.3 means that $\boldsymbol{\Psi}_{2\times 2}^{m|n}$ may be considered as a fundamental solution of $\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}$ in the above-introduced matrix context. This observation is crucial for the matrix approach

[2] I_2 denotes the identity matrix of order (2×2).

used in Hermitian Clifford analysis to arrive at a Cauchy integral formula. Moreover, it is remarkable that the Dirac matrix $\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}$ factorizes in some sense the Laplacian, i.e.

$$\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\ \left(\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\right)^\dagger = \frac{1}{4}\left(\begin{array}{cc} \Delta_{2m|2n} & 0 \\ 0 & \Delta_{2m|2n} \end{array}\right),$$

with

$$\left(\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\right)^\dagger = \left(\begin{array}{cc} \partial_{\mathbf{Z}^\dagger} & \partial_{\mathbf{Z}} \\ \partial_{\mathbf{Z}} & \partial_{\mathbf{Z}^\dagger} \end{array}\right).$$

Thus, in the same setting, we associate, with every pair of Clifford-valued superfunctions $G_1, G_2 \in C^\infty(\Omega)\otimes\mathfrak{G}_{2n}\otimes\mathcal{C}_{2m,2n}$, the matrix function

$$\boldsymbol{G}_2^1 = \left(\begin{array}{cc} G_1 & G_2 \\ G_2 & G_1 \end{array}\right). \tag{7.21}$$

Definition 7.1. *The matrix function $\boldsymbol{G}_2^1$ is said to be (left) $\boldsymbol{SH}$-monogenic if*

$$\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\boldsymbol{G}_2^1 = \mathbf{0},$$

where $\mathbf{0}$ *denotes the matrix with zero entries.*

The above definition for $\boldsymbol{SH}$-monogenicity explicitly reads

$$\begin{cases} \partial_{\mathbf{Z}}[G_1] + \partial_{\mathbf{Z}^\dagger}[G_2] = 0, \\ \partial_{\mathbf{Z}}[G_2] + \partial_{\mathbf{Z}^\dagger}[G_1] = 0. \end{cases}$$

When considering in particular $G_1 = G$ and $G_2 = G^\dagger$, the $\boldsymbol{SH}$-monogenicity of the corresponding matrix function

$$\boldsymbol{G} = \left(\begin{array}{cc} G & G^\dagger \\ G^\dagger & G \end{array}\right)$$

does not imply, in general, the sh-monogenicity of G and vice versa. As a clear example to illustrate this consider the matrix $\boldsymbol{G} = \boldsymbol{\Psi}_{2\times 2}^{m|n}$, i.e. $G = \Psi_1^{m|n}$. An important exception to this general remark occurs in the case of Grassmann-valued functions. Indeed, if $G \in C^\infty(\Omega)\otimes\mathfrak{G}_{2n}$ one has

$$\partial_{\mathbf{Z}}[G] + \partial_{\mathbf{Z}^\dagger}[G^\dagger] = \sum_{j=1}^m \left(\mathfrak{f}_j^\dagger\,\partial_{z_j}[G] + \mathfrak{f}_j\,\partial_{z_j^c}[G^c]\right) + 2i\sum_{j=1}^n \left(\mathfrak{f}_j^{\grave{}\dagger}\,\partial_{\grave{z}_j}[G] - \mathfrak{f}_j^{\grave{}}\,\partial_{\grave{z}_j^c}[G^c]\right),$$

$$\partial_{\mathbf{Z}}[G^\dagger] + \partial_{\mathbf{Z}^\dagger}[G] = \sum_{j=1}^m \left(\mathfrak{f}_j^\dagger\,\partial_{z_j}[G^c] + \mathfrak{f}_j\,\partial_{z_j^c}[G]\right) + 2i\sum_{j=1}^n \left(\mathfrak{f}_j^{\grave{}\dagger}\,\partial_{\grave{z}_j}[G^c] - \mathfrak{f}_j^{\grave{}}\,\partial_{\grave{z}_j^c}[G]\right).$$

Hence, in this case the $\boldsymbol{SH}$-monogenicity of $\boldsymbol{G}$ is equivalent to

$$\begin{cases} \partial_{z_j}[G] = \partial_{z_j^c}[G] = 0, & j = 1,\dots,m, \\ \partial_{\grave{z}_j}[G] = \partial_{\grave{z}_j^c}[G] = 0, & j = 1,\dots,n, \end{cases}$$

which can be re-written as

$$\partial_{\mathbf{Z}}[G] = 0 = \partial_{\mathbf{Z}^\dagger}[G].$$

Another important case occurs when considering the matrix function

$$\boldsymbol{G}_0 = GI_2 = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}$$

with $G \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{2m,2n}$. Also in this case the $\boldsymbol{SH}$-monogenicity of $\boldsymbol{G}_0$ is equivalent to the sh-monogenicity of G. It suffices to note that

$$\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\boldsymbol{G}_0 = \begin{pmatrix} \partial_{\mathbf{Z}}[G] & \partial_{\mathbf{Z}^\dagger}[G] \\ \partial_{\mathbf{Z}^\dagger}[G] & \partial_{\mathbf{Z}}[G] \end{pmatrix}.$$

We are now in condition to reformulate the Hermitian-Stokes theorem given in Corollary 7.3, in a matrix form. The proof easily follows by taking deliberate combinations of the formulae (7.2)-(7.3).

Theorem 7.4. *Let $\boldsymbol{F}_2^1$ and $\boldsymbol{G}_2^1$ be a pair of matrix functions of the form (7.21) with entries in $C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{2m,2n}$. Let moreover α and β be distributions in $\mathcal{E}' \otimes \mathfrak{G}_{2n}^{(ev)}$ and consider the circulant matrix distribution*

$$\boldsymbol{\Sigma} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

such that $\operatorname{supp} \Sigma := \operatorname{supp} \alpha \cup \operatorname{supp} \beta$ is a subset of Ω. It then holds that

$$\int_{\mathbb{R}^{2m|2n}} \big(\boldsymbol{F}_2^1 \boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\big)\, \boldsymbol{\Sigma}\, \boldsymbol{G}_2^1 \,+\, \boldsymbol{F}_2^1\, \boldsymbol{\Sigma}\, \big(\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)} \boldsymbol{G}_2^1\big) = -\int_{\mathbb{R}^{2m|2n}} \boldsymbol{F}_2^1 \,\big(\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)} \boldsymbol{\Sigma}\big)\, \boldsymbol{G}_2^1. \quad (7.22)$$

Considering now a phase function $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^{2m}) \otimes \mathfrak{G}_{2n}^{(ev)}$ such that the real set $\{g_0 \leq 0\} \subset \Omega$ is compact and $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$, the corresponding matrix distribution

$$\boldsymbol{\Sigma} = H(-g) I_2 = \begin{pmatrix} H(-g) & 0 \\ 0 & H(-g) \end{pmatrix} \quad (7.23)$$

is a commuting matrix satisfying

$$\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)} \boldsymbol{\Sigma} = -\delta(g) \begin{pmatrix} \partial_{\mathbf{Z}}[g] & \partial_{\mathbf{Z}^\dagger}[g] \\ \partial_{\mathbf{Z}^\dagger}[g] & \partial_{\mathbf{Z}}[g] \end{pmatrix} = -\delta(g) \boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g].$$

Hence formula (7.22) takes the following form, when substituting (7.23):

$$\int_{\mathbb{R}^{2m|2n}} H(-g) \big[\big(\boldsymbol{F}_2^1 \boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\big)\, \boldsymbol{G}_2^1 + \boldsymbol{F}_2^1 \,\big(\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)} \boldsymbol{G}_2^1\big)\big] = \int_{\mathbb{R}^{2m|2n}} \boldsymbol{F}_2^1 \,\delta(g) \boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g]\, \boldsymbol{G}_2^1. \quad (7.24)$$

In order to prove the Hermitian Cauchy-Pompeiu formula in superspace we proceed as in the previous chapter. We first observe that in (7.22) the matrix function $\boldsymbol{F}_2^1$ can be replaced by any matrix distribution

$$\boldsymbol{\Upsilon} = \begin{pmatrix} \gamma & \sigma \\ \sigma & \gamma \end{pmatrix}, \qquad \gamma, \sigma \in \mathcal{E}' \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{2m,2n}, \quad (7.25)$$

such that the sets

$$sing\ supp\,\mathbf{\Upsilon} := sing\ supp\,\gamma \cup sing\ supp\,\sigma,$$

and

$$sing\ supp\,\mathbf{\Sigma} := sing\ supp\,\alpha \cup sing\ supp\,\beta,$$

are disjoint. Under these conditions, (7.22) can be proven for $\boldsymbol{F}_2^1 = \mathbf{\Upsilon}$ by taking deliberate combinations of (7.2)-(7.3) with $F = \gamma$ and $F = \sigma$. We recall that these substitutions are possible since distributions with disjoint singular supports can be multiplied and the Leibniz rule remains valid for such a product, see (6.8) and (6.9). Applying this reasoning to (7.24), for which we are taking $\Sigma = H(-g)I_2$, we immediately obtain the following Hermitian analogue of Corollary 6.3.

Corollary 7.7. *Let $\boldsymbol{G}_2^1$ be a matrix function of the form (7.21) with entries in $C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{2m,2n}$. Let $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^{2m}) \otimes \mathfrak{G}_{2n}^{(ev)}$ be a phase function such that $\{g_0 \leq 0\} \subset \Omega$ is compact and $\partial_{\underline{x}}[g_0] \neq 0$ on $g_0^{-1}(0)$, and let $\mathbf{\Upsilon}$ be a matrix distribution of the form (7.25) such that*

$$sing\ supp\,\mathbf{\Upsilon} \cap g^{-1}(0) = \emptyset.$$

It then holds that

$$\int_{\mathbb{R}^{2m|2n}} H(-g)\left[\left(\mathbf{\Upsilon}\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\right)\boldsymbol{G}_2^1 + \mathbf{\Upsilon}\left(\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\boldsymbol{G}_2^1\right)\right] = \int_{\mathbb{R}^{2m|2n}} \mathbf{\Upsilon}\,\delta(g)\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g]\,\boldsymbol{G}_2^1, \quad (7.26)$$

where the distributional products $H(-g)\left(\mathbf{\Upsilon}\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\right)$, $H(-g)\mathbf{\Upsilon}$ and $\mathbf{\Upsilon}\delta(g)$ are to be understood in the sense of (6.8) and (6.9).

Proof.

It suffices to note that $sing\ supp\,H(-g) = g^{-1}(0)$. □

Let us now consider the supervector $\mathbf{y} = \underline{y} + \underline{y\grave{}}$, its Hermitian counterparts

$$\mathbf{U} = \frac{1}{2}(\mathbf{y} + i\mathbf{J}(\mathbf{y})), \qquad \mathbf{U}^\dagger = -\frac{1}{2}(\mathbf{y} - i\mathbf{J}(\mathbf{y})),$$

and the matrix distribution

$$\mathbf{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U}) = \begin{pmatrix} \Psi_1^{m|n}(\mathbf{Z}-\mathbf{U}) & {\Psi_1^{m|n}}^\dagger(\mathbf{Z}-\mathbf{U}) \\ {\Psi_1^{m|n}}^\dagger(\mathbf{Z}-\mathbf{U}) & \Psi_1^{m|n}(\mathbf{Z}-\mathbf{U}) \end{pmatrix},$$

where we recall that

$$\Psi_1^{m|n}(\mathbf{Z}-\mathbf{U}) = \frac{2}{|\mathbb{S}^{2m-1|2n}|}\frac{\mathbf{Z}-\mathbf{U}}{|\mathbf{Z}-\mathbf{U}|^M}, \qquad {\Psi_1^{m|n}}^\dagger(\mathbf{Z}-\mathbf{U}) = \frac{2}{|\mathbb{S}^{2m-1|2n}|}\frac{\mathbf{Z}^\dagger-\mathbf{U}^\dagger}{|\mathbf{Z}-\mathbf{U}|^M}.$$

The following Hermitian Cauchy-Pompeiu formula in superspace then is established.

Theorem 7.5 (Hermitian Cauchy-Pompeiu formula in superspace). *Let* $\boldsymbol{G}_2^1$ *be a matrix function of the form (7.21) with entries in* $C^\infty(\Omega) \otimes \mathfrak{G}_{2n} \otimes \mathcal{C}_{2m,2n}$*, and let* $g = g_0 + \mathbf{g} \in C^\infty(\mathbb{R}^{2m}) \otimes \mathfrak{G}_{2n}^{(ev)}$ *be a phase function such that* $\{g_0 \leq 0\} \subset \Omega$ *is compact and* $\partial_{\underline{x}}[g_0] \neq 0$ *on* $g_0^{-1}(0)$*. It then holds that*

$$\int_{\mathbb{R}^{2m|2n}} \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})\,\delta(g(\mathbf{x}))\,\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g(\mathbf{x})]\,\boldsymbol{G}_2^1(\mathbf{x}) - \int_{\mathbb{R}^{2m|2n}} H(-g(\mathbf{x}))\,\boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})\left(\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\boldsymbol{G}_2^1(\mathbf{x})\right) = \begin{cases} \boldsymbol{G}_2^1(\mathbf{y}), & g_0(\underline{y}) < 0, \\ 0, & g_0(\underline{y}) > 0. \end{cases} \tag{7.27}$$

Proof.

The outline of this proof is very similar to the one of Theorem 6.2. It is easily seen that $\boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})$ is a matrix distribution in the variable $\mathbf{Z}$ with singular support $\{\underline{y}\}$. Hence, for $g_0(\underline{y}) \neq 0$ we have that

$$\textit{sing supp}\, H(-g(\mathbf{x})) \cap \textit{sing supp}\, \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U}) = \emptyset.$$

This means that one can take $\boldsymbol{\Upsilon} = \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})$ in (7.26). From Theorem 7.3 it follows that

$$\boldsymbol{\delta}(\mathbf{x}-\mathbf{y}) = \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})\,\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)},$$

leading to

$$\int_{\mathbb{R}^{2m|2n}} H(-g(\mathbf{x}))\,\delta(\mathbf{x}-\mathbf{y})\boldsymbol{G}_2^1(\mathbf{x}) = \int_{\mathbb{R}^{2m|2n}} \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})\,\delta(g(\mathbf{x}))\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g(\mathbf{x})]\,\boldsymbol{G}_2^1(\mathbf{x}) - \int_{\mathbb{R}^{2m|2n}} H(-g(\mathbf{x}))\,\boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})\,\left(\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}\boldsymbol{G}_2^1(\mathbf{x})\right). \tag{7.28}$$

In Section 6.5, it was proven that

$$\delta(\mathbf{x}-\mathbf{y})H(-g(\mathbf{x})) = \delta(\mathbf{x}-\mathbf{y})H(-g_0(\underline{x})).$$

Then formula (6.41) yields

$$\int_{\mathbb{R}^{2m|2n}} \delta(\mathbf{x}-\mathbf{y})H(-g(\mathbf{x}))\boldsymbol{G}_2^1(\mathbf{x}) = \int_{\mathbb{R}^{2m|2n}} \delta(\mathbf{x}-\mathbf{y})H(-g_0(\underline{x}))\boldsymbol{G}_2^1(\mathbf{x}) = H(-g_0(\underline{y}))\boldsymbol{G}_2^1(\mathbf{y}).$$

Substitution of the later in (7.28) gives the desired result (7.27). □

This theorem now leads to the following Hermitian Cauchy integral formulae in superspace.

Corollary 7.8. *If the matrix function* $\boldsymbol{G}_2^1$ *is* **SH***-monogenic then,*

$$\int_{\mathbb{R}^{m|2n}} \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})\,\delta(g(\mathbf{x}))\,\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g(\mathbf{x})]\,\boldsymbol{G}_2^1(\mathbf{x}) = \begin{cases} \boldsymbol{G}_2^1(\mathbf{y}), & g_0(\underline{y})<0,\\ 0, & g_0(\underline{y})>0.\end{cases}$$

Corollary 7.9. *If the function* G *is sh-monogenic then,*

$$\int_{\mathbb{R}^{m|2n}} \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})\,\delta(g(\mathbf{x}))\,\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g(\mathbf{x})]\,\boldsymbol{G}_0(\mathbf{x}) = \begin{cases} \boldsymbol{G}_0(\mathbf{y}), & g_0(\underline{y})<0,\\ 0, & g_0(\underline{y})>0.\end{cases} \tag{7.29}$$

The above result may be considered as a Hermitian Cauchy integral theorem for the sh-monogenic function G. For $n=0$ the above result becomes the purely bosonic Hermitian Cauchy integral representation. The study of its boundary limits leads to Hermitian Clifford-Hardy spaces and to a Hermitian Hilbert transform, see e.g. [10].

Remark 7.2. *The second summand at the left hand side of formula (7.27) is the well-known Téodorescu transform, which is denoted by*

$$\boldsymbol{T}_g G_2^1(\mathbf{y}) = -\int_{\mathbb{R}^{2m|2n}} H(-g(\mathbf{x}))\,\boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})\,\boldsymbol{G}_2^1(\mathbf{x}).$$

This operator constitutes a right inverse to the Dirac operator. Indeed, using Theorem 7.3 and (6.41) one easily obtains

$$\boldsymbol{\mathcal{D}}_{(\mathbf{U},\mathbf{U}^\dagger)}\boldsymbol{T}_g G_2^1(\mathbf{y}) = \int_{\mathbb{R}^{2m|2n}} H(-g(\mathbf{x}))\delta(\mathbf{x}-\mathbf{y})\boldsymbol{G}_2^1(\mathbf{x}) = \begin{cases} \boldsymbol{G}_2^1(\mathbf{y}), & g_0(\underline{y})<0,\\ 0, & g_0(\underline{y})>0.\end{cases}$$

The combination of the above inversion formula with the Hermitian Cauchy-Pompeiu Theorem 7.5, yields a Hermitian Koppelman formula in superspace, see e.g. [13]. This formula reads as follows for $g_0(\underline{y})<0$*:*

$$\int_{\mathbb{R}^{m|2n}} \boldsymbol{\Psi}_{2\times 2}^{m|n}(\mathbf{Z}-\mathbf{U})\,\delta(g(\mathbf{x}))\,\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g(\mathbf{x})]\,\boldsymbol{G}_2^1(\mathbf{x}) + \boldsymbol{T}_g\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}G_2^1(\mathbf{y}) + \boldsymbol{\mathcal{D}}_{(\mathbf{U},\mathbf{U}^\dagger)}\boldsymbol{T}_g G_2^1(\mathbf{y}) = 2\boldsymbol{G}_2^1(\mathbf{y}).$$

7.4 Integral formulae for holomorphic functions in superspace

7.4.1 Holomorphicity in superspace and sh-monogenicity

Every superfunction $F(\mathbf{x})\in C^\infty(\Omega)\otimes\mathfrak{G}_{2n}$, $\Omega\subset\mathbb{R}^{2m}$, can be written in terms of the Hermitian supervector variables $\mathbf{Z}$ and $\mathbf{Z}^\dagger$ as

$$F(\mathbf{Z},\mathbf{Z}^\dagger) = \sum_{A,B\subset\{1,\ldots,n\}} F_{A,B}(\underline{Z},\underline{Z}^\dagger)\,\grave{\underline{Z}}_A\,\grave{\underline{Z}}_B^c, \qquad F_{A,B}\in C^\infty(\Omega), \tag{7.30}$$

where

$$\grave{\underline{Z}}_A = \grave{z}_{j_1} \cdots \grave{z}_{j_k}, \qquad A = \{j_1, \ldots, j_k\}, \quad 1 \leq j_1 < \ldots < j_k \leq n,$$
$$\grave{\underline{Z}}_B^c = \grave{z}_{\ell_1}^c \cdots \grave{z}_{\ell_s}^c, \qquad B = \{\ell_1, \ldots, \ell_s\}, \quad 1 \leq \ell_1 < \ldots < \ell_s \leq n.$$

The notion of a holomorphic function in the purely bosonic case then naturally extends to superfunctions.

Definition 7.2. *A superfunction $F(\mathbf{Z}, \mathbf{Z}^\dagger)$ of the form (7.30) is said to be holomorphic in the bosonic and fermionic complex variables $z_1, \ldots, z_m, \grave{z}_1, \ldots, \grave{z}_n$ if*

$$\partial_{z_j^c}[F] = \partial_{\grave{z}_k^c}[F] = 0, \qquad j = 1, \ldots, m, \quad k = 1 \ldots, n.$$

This holomorphicity condition is equivalent to saying that the function F does not depend on the conjugate variables $z_1^c, \ldots, z_m^c, \grave{z}_1^c, \ldots, \grave{z}_n^c$, i.e.

$$F(\mathbf{Z}, \mathbf{Z}^\dagger) = F(\mathbf{Z}) = \sum_{A \subset \{1,\ldots,n\}} F_A(\underline{Z}) \, \grave{\underline{Z}}_A.$$

Let us now connect this holomorphicity notion in superspace with sh-monogenicity. To that end we start by introducing the classical primitive idempotent

$$I_b = \mathfrak{f}_1^\dagger \mathfrak{f}_1 \cdots \mathfrak{f}_m^\dagger \mathfrak{f}_m,$$

where

$$\mathfrak{f}_j^\dagger \mathfrak{f}_j = \frac{1}{2}(1 + i e_j e_{m+j}), \qquad j = 1, \ldots, m.$$

This idempotent clearly satisfies $e_j I_b = -i e_{m+j} I_b$ or equivalently $\mathfrak{f}_j^\dagger I_b = 0$.

A similar element to I_b can be constructed in terms of the symplectic generators $\grave{e}_j$'s. We first recall that the elements $\grave{e}_{2j-1}$, $\grave{e}_{2j}$ can be identified with the following operators when acting on the corresponding spinor space:

$$\grave{e}_{2j-1} \to e_{2m+1} \, \partial_{a_j}, \qquad \grave{e}_{2j} \to -e_{2m+1} \, a_j,$$

where $a_1, \ldots, a_n$ are commuting variables and e_{2m+1} is an additional orthogonal Clifford generator, see e.g. [68, 23] and Remark 3.4. In the remainder of this chapter, we will consider the above-given identification. We then need to find an element I_f similar to I_b in the spinor space that consists of all smooth functions in the variables $a_1, \ldots, a_n$. This element I_f has to satisfy the key property

$$\grave{e}_{2j-1} I_f = -i \grave{e}_{2j} I_f, \qquad \text{or equivalently,} \qquad \partial_{a_j} I_f = i a_j \, I_f.$$

Such a function is given by

$$I_f = \exp\left(\frac{i}{2} \sum_{j=1}^{n} a_j^2\right),$$

which clearly is a null solution of the operator $\grave{\mathfrak{f}}_j^\dagger$, i.e.

$$\grave{\mathfrak{f}}_j^\dagger I_f = -\frac{1}{2}\left(\grave{e}_{2j-1} + i\grave{e}_{2j}\right) I_f = -\frac{e_{2m+1}}{2}\left(\partial_{a_j} - i a_j\right)[I_f] = 0.$$

Proposition 7.5. *A superfunction* $F \in C^\infty(\Omega) \otimes \mathfrak{G}_{2n}$ *is holomorphic in the variables* $z_1, \ldots, z_m, \grave{z}_1, \ldots, \grave{z}_n$ *if and only if the spinor-valued function* $F I_b I_f$ *is sh-monogenic.*

Proof.

We first observe that

$$\partial_{\mathbf{Z}}[F I_b I_f] = \sum_{j=1}^m \partial_{z_j}[F]\, \mathfrak{f}_j^\dagger\, I_b I_f + 2i \sum_{j=1}^n \partial_{\grave{z}_j}[F]\, \grave{\mathfrak{f}}_j^\dagger\, I_b I_f = 0,$$

while on the other hand,

$$\partial_{\mathbf{Z}^\dagger}[F I_b I_f] = \sum_{j=1}^m \partial_{z_j^c}[F]\, \mathfrak{f}_j\, I_b I_f - 2i \sum_{j=1}^n \partial_{\grave{z}_j^c}[F]\, \grave{\mathfrak{f}}_j\, I_b I_f, \tag{7.31}$$

whence it is clear that holomorphicity for F implies sh-monogenicity for $F I_b I_f$.

Assume now that $F I_b I_f$ is sh-monogenic. In order to prove that F is holomorphic, it suffices to show that all the elements $\mathfrak{f}_j\, I_b I_f$, $\grave{\mathfrak{f}}_k I_b I_f$ are linearly independent when considering coefficients in $C^\infty(\Omega) \otimes \mathfrak{G}_{2n}$, see (7.31).

We have,

$$\begin{aligned}
\mathfrak{f}_j\, I_b &= \mathfrak{f}_1^\dagger \mathfrak{f}_1 \cdots \left(\mathfrak{f}_j \mathfrak{f}_j^\dagger\right) \mathfrak{f}_j \cdots \mathfrak{f}_m^\dagger \mathfrak{f}_m \\
&= \mathfrak{f}_1^\dagger \mathfrak{f}_1 \cdots \left(1 - \mathfrak{f}_j^\dagger \mathfrak{f}_j\right) \mathfrak{f}_j \cdots \mathfrak{f}_m^\dagger \mathfrak{f}_m \\
&= \mathfrak{f}_1^\dagger \mathfrak{f}_1 \cdots \mathfrak{f}_{j-1}^\dagger \mathfrak{f}_{j-1}\, \mathfrak{f}_j\, \mathfrak{f}_{j+1}^\dagger \mathfrak{f}_{j+1} \cdots \mathfrak{f}_m^\dagger \mathfrak{f}_m,
\end{aligned}$$

yielding,

$$\mathfrak{f}_k^\dagger \mathfrak{f}_j I_b = \delta_{k,j} I_b. \tag{7.32}$$

Moreover,

$$\grave{\mathfrak{f}}_j I_b I_f = I_b \grave{\mathfrak{f}}_j I_f = I_b \frac{e_{2m+1}}{2}(\partial_{a_j} + i a_j)\, \exp\left(\frac{i}{2}\sum_{j=1}^n a_j^2\right) = I_b e_{2m+1} i a_j \exp\left(\frac{i}{2}\sum_{j=1}^n a_j^2\right).$$

Hence, taking into account (7.31), the sh-monogenicity of $F I_b I_f$ reduces to

$$\left(\sum_{j=1}^m \partial_{z_j^c}[F]\, \mathfrak{f}_j\, I_b - 2i \sum_{j=1}^n \partial_{\grave{z}_j^c}[F]\, I_b e_{2m+1} i a_j\right) \exp\left(\frac{i}{2}\sum_{j=1}^n a_j^2\right) = 0,$$

implying

$$\sum_{j=1}^{m} \partial_{z_j^c}[F]\, \mathfrak{f}_j\, I_b - 2i \sum_{j=1}^{n} \partial_{\grave{z}_j^c}[F]\, I_b e_{2m+1} i a_j = 0.$$

Multiplying the above expression from the left by $\mathfrak{f}_k^\dagger$ we get, on account of (7.32), that $\partial_{z_k^c}[F]\, I_b = 0$. This directly implies that $\partial_{z_k^c}[F] = 0$ for every $k = 1, \ldots, m$, whence the previous equality reduces to

$$\sum_{j=1}^{n} \partial_{\grave{z}_j^c}[F]\, I_b e_{2m+1} i a_j = 0.$$

Finally, by taking $a_k = 1$ and $a_j = 0$ $(j \neq k)$ we obtain $\partial_{\grave{z}_k^c}[F] = 0$ for every $k = 1, \ldots, n$. □

7.4.2 Bochner-Martinelli theorem for holomorphic superfunctions

The above result shows that considering functions of the form $F I_b I_f$ establishes a connection between Hermitian monogenicity and holomorphic functions in superspace. In this section, we investigate the nature of the Hermitian Cauchy integral formula obtained in Corollary 7.9 for this type of functions.

To this end, we will explicitly compute the left-hand side of (7.29), taking $G = F I_b I_f$ where $F(\mathbf{Z}) = \sum_A F_A(\underline{Z})\, \underline{\grave{Z}}_A$ is a holomorphic function. We first obtain,

$$\begin{aligned}\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g(\mathbf{x})]\, \boldsymbol{G}_0(\mathbf{x}) &= \begin{pmatrix} \partial_{\mathbf{Z}}[g(\mathbf{x})] & * \\ \partial_{\mathbf{Z}^\dagger}[g(\mathbf{x})] & * \end{pmatrix} \begin{pmatrix} F(\mathbf{Z}) I_b I_f & * \\ 0 & * \end{pmatrix} \\ &= \begin{pmatrix} \partial_{\mathbf{Z}}[g(\mathbf{x})] F(\mathbf{Z}) I_b I_f & * \\ \partial_{\mathbf{Z}^\dagger}[g(\mathbf{x})] F(\mathbf{Z}) I_b I_f & * \end{pmatrix},\end{aligned}$$

where the second columns have not been written, since they only duplicate the first ones (in reversed order) on account of the circulant structure of the involved matrices. Further calculation yields,

$$\partial_{\mathbf{Z}}[g(\mathbf{x})] F(\mathbf{Z}) I_b I_f = \sum_{j=1}^{m} \partial_{z_j}[g(\mathbf{x})]\, F(\mathbf{Z})\, (\mathfrak{f}_j^\dagger\, I_b) I_f + 2i \sum_{j=1}^{n} \partial_{\grave{z}_j}[g(\mathbf{x})]\, F(\mathbf{Z})\, I_b\, (\grave{\mathfrak{f}}_j^\dagger\, I_f) = 0,$$

and as a consequence,

$$\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g]\, \boldsymbol{G}_0(\mathbf{x}) = \begin{pmatrix} 0 & * \\ \partial_{\mathbf{Z}^\dagger}[g] F I_b I_f & * \end{pmatrix}.$$

Hence

$$
\begin{aligned}
&\boldsymbol{\Psi}^{m|n}_{2\times 2}(\mathbf{Z}-\mathbf{U})\,\delta(g(\mathbf{x}))\,\boldsymbol{\mathcal{D}}_{(\mathbf{Z},\mathbf{Z}^\dagger)}[g(\mathbf{x})]\,\boldsymbol{G}_0(\mathbf{x})\\
&\quad=\begin{pmatrix} \Psi^{m|n}_1(\mathbf{Z}-\mathbf{U}) & * \\ {\Psi^{m|n}_1}^\dagger(\mathbf{Z}-\mathbf{U}) & * \end{pmatrix}\begin{pmatrix} 0 & * \\ \delta(g(\mathbf{x}))\,\partial_{\mathbf{Z}^\dagger}[g(\mathbf{x})]\;F(\mathbf{Z})I_bI_f & * \end{pmatrix}\\
&\quad=\begin{pmatrix} {\Psi^{m|n}_1}^\dagger(\mathbf{Z}-\mathbf{U})\delta(g(\mathbf{x}))\,\partial_{\mathbf{Z}^\dagger}[g(\mathbf{x})]\;F(\mathbf{Z})I_bI_f & * \\ \Psi^{m|n}_1(\mathbf{Z}-\mathbf{U})\,\delta(g(\mathbf{x}))\,\partial_{\mathbf{Z}^\dagger}[g(\mathbf{x})]\;F(\mathbf{Z})I_bI_f & * \end{pmatrix}.
\end{aligned}
$$

Denoting the even element $|\mathbf{Z}-\mathbf{U}|=\boldsymbol{\rho}$ we compute

$$
\begin{aligned}
&{\Psi^{m|n}_1}^\dagger(\mathbf{Z}-\mathbf{U})\delta(g(\mathbf{x}))\,\partial_{\mathbf{Z}^\dagger}[g(\mathbf{x})]\\
&\quad=\frac{2\delta(g(\mathbf{x}))}{|\mathbb{S}^{2m-1|2n}|\boldsymbol{\rho}^M}\Big[(\underline{Z}-\underline{U})^\dagger\,\partial_{\underline{Z}^\dagger}[g(\mathbf{x})]+(\underline{Z}-\underline{U})^\dagger\,\partial_{\underline{\grave{Z}}^\dagger}[g(\mathbf{x})]\Big]\\
&\qquad+\frac{2\delta(g(\mathbf{x}))}{|\mathbb{S}^{2m-1|2n}|\boldsymbol{\rho}^M}\Big[\Big(\underline{\grave{Z}}-\underline{\grave{U}}\Big)^\dagger\partial_{\underline{\grave{Z}}^\dagger}[g(\mathbf{x})]+\Big(\underline{\grave{Z}}-\underline{\grave{U}}\Big)^\dagger\partial_{\underline{Z}^\dagger}[g(\mathbf{x})]\Big].
\end{aligned}
$$

We now consider each term in the previous sum separately, obtaining

$$
\begin{aligned}
(\underline{Z}-\underline{U})^\dagger\,\partial_{\underline{Z}^\dagger}[g(\mathbf{x})]&=\left(\sum_{j=1}^m (z_j-u_j)^c\,\mathfrak{f}_j^\dagger\right)\left(\sum_{k=1}^m \mathfrak{f}_k\,\partial_{z_k^c}[g(\mathbf{x})]\right)\\
&=-\sum_{j\neq k}(z_j-u_j)^c\,\partial_{z_k^c}[g(\mathbf{x})]\,\mathfrak{f}_k\mathfrak{f}_j^\dagger+\sum_{j=1}^m (z_j-u_j)^c\,\partial_{z_j^c}[g(\mathbf{x})]\,(1-\mathfrak{f}_j\mathfrak{f}_j^\dagger),\\
(\underline{Z}-\underline{U})^\dagger\,\partial_{\underline{\grave{Z}}^\dagger}[g(\mathbf{x})]&=-2i\left(\sum_{j=1}^m (z_j-u_j)^c\,\mathfrak{f}_j^\dagger\right)\left(\sum_{k=1}^n \grave{\mathfrak{f}}_k\partial_{\grave{z}_k^c}[g(\mathbf{x})]\right)\\
&=2i\sum_{\substack{1\leq j\leq m\\1\leq k\leq n}}(z_j-u_j)^c\,\partial_{\grave{z}_k^c}[g(\mathbf{x})]\,\grave{\mathfrak{f}}_k\mathfrak{f}_j^\dagger,\\
\Big(\underline{\grave{Z}}-\underline{\grave{U}}\Big)^\dagger\partial_{\underline{\grave{Z}}^\dagger}[g(\mathbf{x})]&=-2i\left(\sum_{j=1}^n (\grave{z}_j-\grave{u}_j)^c\,\grave{\mathfrak{f}}_j^\dagger\right)\left(\sum_{k=1}^n \grave{\mathfrak{f}}_k\partial_{\grave{z}_k^c}[g(\mathbf{x})]\right)\\
&=-2i\sum_{j\neq k}(\grave{z}_j-\grave{u}_j)^c\,\partial_{\grave{z}_k^c}[g(\mathbf{x})]\,\grave{\mathfrak{f}}_k\grave{\mathfrak{f}}_j^\dagger-2i\sum_{j=1}^n(\grave{z}_j-\grave{u}_j)^c\,\partial_{\grave{z}_j^c}[g(\mathbf{x})]\left(\frac{i}{2}+\grave{\mathfrak{f}}_j\grave{\mathfrak{f}}_j^\dagger\right),\\
\Big(\underline{\grave{Z}}-\underline{\grave{U}}\Big)^\dagger\partial_{\underline{Z}^\dagger}[g(\mathbf{x})]&=\left(\sum_{j=1}^n (\grave{z}_j-\grave{u}_j)^c\,\grave{\mathfrak{f}}_j^\dagger\right)\left(\sum_{k=1}^m \mathfrak{f}_k\,\partial_{z_k^c}[g(\mathbf{x})]\right)\\
&=-\sum_{\substack{1\leq j\leq n\\1\leq k\leq m}}(\grave{z}_j-\grave{u}_j)^c\,\partial_{z_k^c}[g(\mathbf{x})]\,\mathfrak{f}_k\grave{\mathfrak{f}}_j^\dagger.
\end{aligned}
$$

This yields,

$$\begin{aligned}
&(\underline{Z}-\underline{U})^{\dagger}\,\partial_{\underline{Z}^{\dagger}}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f=\sum_{j=1}^{m}(z_j-u_j)^c\,\partial_{z_j^c}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f,\\
&(\underline{Z}-\underline{U})^{\dagger}\,\partial_{\underline{\grave{Z}}^{\dagger}}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f=0,\\
&\left(\underline{\grave{Z}}-\underline{\grave{U}}\right)^{\dagger}\,\partial_{\underline{\grave{Z}}^{\dagger}}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f=\sum_{j=1}^{n}(\grave{z}_j-\grave{u}_j)^c\,\partial_{\grave{z}_j^c}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f,\\
&\left(\underline{\grave{Z}}-\underline{\grave{U}}\right)^{\dagger}\,\partial_{\underline{Z}^{\dagger}}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f=0.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\Psi_1^{m|n\,\dagger}(\mathbf{Z}-\mathbf{U})\delta(g(\mathbf{x}))\,\partial_{\mathbf{Z}^{\dagger}}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f\\
&\quad=\frac{2\delta(g(\mathbf{x}))}{|\mathbb{S}^{2m-1|2n}|\boldsymbol{\rho}^M}\left[\sum_{j=1}^{m}(z_j-u_j)^c\,\partial_{z_j^c}[g(\mathbf{x})]+\sum_{j=1}^{n}(\grave{z}_j-\grave{u}_j)^c\,\partial_{\grave{z}_j^c}[g(\mathbf{x})]\right]\,F(\mathbf{Z})I_bI_f\\
&\quad=\frac{2\delta(g(\mathbf{x}))}{|\mathbb{S}^{2m-1|2n}|\boldsymbol{\rho}^M}\,D^{\dagger}_{\mathbf{Z}-\mathbf{U},\mathbf{Z}}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f,
\end{aligned}$$

where the differential operator

$$D^{\dagger}_{\mathbf{Z}-\mathbf{U},\mathbf{Z}}=\sum_{j=1}^{m}(z_j-u_j)^c\,\partial_{z_j^c}+\sum_{j=1}^{n}(\grave{z}_j-\grave{u}_j)^c\,\partial_{\grave{z}_j^c}=\left\{\partial^{\dagger}_{\mathbf{Z}},(\mathbf{Z}-\mathbf{U})^{\dagger}\right\}-\frac{1}{2}\left(\frac{M}{2}-i\mathbf{B}\right)$$

is the Hermitian directional derivative with respect to $\mathbf{Z}^{\dagger}$ in the direction $(\mathbf{Z}-\mathbf{U})^{\dagger}$ introduced in Chapter 3.

Thus, the Hermitian Cauchy integral formula (7.29) for $g_0(\underline{y})<0$ yields the following two statements:

$$\begin{aligned}
\frac{2}{|\mathbb{S}^{2m-1|2n}|}\int_{\mathbb{R}^{2m|2n}}\frac{\delta(g(\mathbf{x}))}{|\mathbf{Z}-\mathbf{U}|^M}\,D^{\dagger}_{\mathbf{Z}-\mathbf{U},\mathbf{Z}}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f&=F(\mathbf{U})I_bI_f,\\
\frac{2}{|\mathbb{S}^{2m-1|2n}|}\int_{\mathbb{R}^{2m|2n}}\frac{\delta(g(\mathbf{x}))}{|\mathbf{Z}-\mathbf{U}|^M}\,(\mathbf{Z}-\mathbf{U})\,\partial_{\mathbf{Z}^{\dagger}}[g(\mathbf{x})]\,F(\mathbf{Z})I_bI_f&=0.
\end{aligned}\tag{7.33}$$

The first one leads to the following integral representation of holomorphic superfunctions.

Theorem 7.6 (Bochner-Martinelli formula in superspace). *Let $F(\mathbf{Z})\in C^{\infty}(\Omega)\otimes\mathfrak{G}_{2n}$ be a holomorphic function in the variables $z_1\ldots,z_m,\grave{z}_1,\ldots,\grave{z}_n$, $(m>n)$, and let $g=g_0+\mathbf{g}\in C^{\infty}(\mathbb{R}^{2m})\otimes\mathfrak{G}_{2n}^{(ev)}$ be a phase function such that $\{g_0\leq 0\}\subset\Omega$ is compact and $\partial_{\underline{x}}[g_0]\neq 0$ on $g_0^{-1}(0)$. It then follows for $g_0(\underline{y})<0$ that*

$$\frac{2}{|\mathbb{S}^{2m-1|2n}|}\int_{\mathbb{R}^{2m|2n}}\frac{\delta(g(\mathbf{x}))}{|\mathbf{Z}-\mathbf{U}|^M}\,D^{\dagger}_{\mathbf{Z}-\mathbf{U},\mathbf{Z}}[g(\mathbf{x})]\,F(\mathbf{Z})=F(\mathbf{U}).\tag{7.34}$$

Remark 7.3. *The above theorem indeed constitutes an extension of the classical Bochner-Martinelli formula to superspace. In the next section it will be shown that (7.34) reduces to (7.1) when* $n = 0$.

On the other hand, the second statement yields the following result.

Theorem 7.7. *Under the same conditions as in Theorem 7.6 one has*

$$\begin{aligned}
\int_{\mathbb{R}^{2m|2n}} \delta(g(\mathbf{x}))\, \frac{z_j - u_j}{|\mathbf{Z}-\mathbf{U}|^M}\, \partial_{z_k^c}[g(\mathbf{x})]\, F(\mathbf{Z}) &= \int_{\mathbb{R}^{2m|2n}} \delta(g(\mathbf{x}))\, \frac{z_k - u_k}{|\mathbf{Z}-\mathbf{U}|^M}\, \partial_{z_j^c}[g(\mathbf{x})]\, F(\mathbf{Z}),\\
-2i \int_{\mathbb{R}^{2m|2n}} \delta(g(\mathbf{x}))\, \frac{z_j - u_j}{|\mathbf{Z}-\mathbf{U}|^M}\, \partial_{\grave{z}_k^c}[g(\mathbf{x})]\, F(\mathbf{Z}) &= \int_{\mathbb{R}^{2m|2n}} \delta(g(\mathbf{x}))\, \frac{\grave{z}_k - \grave{u}_k}{|\mathbf{Z}-\mathbf{U}|^M}\, \partial_{z_j^c}[g(\mathbf{x})]\, F(\mathbf{Z}),\\
\int_{\mathbb{R}^{2m|2n}} \delta(g(\mathbf{x}))\, \frac{\grave{z}_j - \grave{u}_j}{|\mathbf{Z}-\mathbf{U}|^M}\, \partial_{\grave{z}_k^c}[g(\mathbf{x})]\, F(\mathbf{Z}) &= -\int_{\mathbb{R}^{2m|2n}} \delta(g(\mathbf{x}))\, \frac{\grave{z}_k - \grave{u}_k}{|\mathbf{Z}-\mathbf{U}|^M}\, \partial_{\grave{z}_j^c}[g(\mathbf{x})]\, F(\mathbf{Z}).
\end{aligned}$$

Proof.

The proof directly follows from expanding expression (7.33). We have,

$$\begin{aligned}
(\mathbf{Z}-\mathbf{U})\, \partial_{\mathbf{Z}^\dagger}[g(\mathbf{x})] = (\underline{Z}-\underline{U})\, \partial_{\underline{Z}^\dagger}[g(\mathbf{x})] + (\underline{Z}-\underline{U})\, \partial_{\underline{\grave{Z}}^\dagger}[g(\mathbf{x})]&\\
+ \left(\underline{\grave{Z}}-\underline{\grave{U}}\right) \partial_{\underline{Z}^\dagger}[g(\mathbf{x})] + \left(\underline{\grave{Z}}-\underline{\grave{U}}\right) \partial_{\underline{\grave{Z}}^\dagger}[g(\mathbf{x})]&,
\end{aligned}$$

where

$$\begin{aligned}
(\underline{Z}-\underline{U})\, \partial_{\underline{Z}^\dagger}[g(\mathbf{x})] &= \left(\sum_{j=1}^m (z_j-u_j)\, \mathfrak{f}_j\right)\left(\sum_{k=1}^m \mathfrak{f}_k\, \partial_{z_k^c}[g(\mathbf{x})]\right) = \sum_{j\neq k} (z_j-u_j)\, \partial_{z_k^c}[g(\mathbf{x})]\, \mathfrak{f}_j \mathfrak{f}_k,\\
(\underline{Z}-\underline{U})\, \partial_{\underline{\grave{Z}}^\dagger}[g(\mathbf{x})] &= -2i\left(\sum_{j=1}^m (z_j-u_j)\, \mathfrak{f}_j\right)\left(\sum_{k=1}^n \grave{\mathfrak{f}}_k\, \partial_{\grave{z}_k^c}[g(\mathbf{x})]\right) = -2i \sum_{\substack{1\leq j\leq m\\ 1\leq k\leq n}} (z_j-u_j)\, \partial_{\grave{z}_k^c}[g(\mathbf{x})]\, \mathfrak{f}_j \grave{\mathfrak{f}}_k,\\
\left(\underline{\grave{Z}}-\underline{\grave{U}}\right) \partial_{\underline{Z}^\dagger}[g(\mathbf{x})] &= \left(\sum_{k=1}^n (\grave{z}_k-\grave{u}_k)\, \grave{\mathfrak{f}}_k\right)\left(\sum_{j=1}^m \mathfrak{f}_j\, \partial_{z_j^c}[g(\mathbf{x})]\right) = \sum_{\substack{1\leq j\leq m\\ 1\leq k\leq n}} (\grave{z}_k-\grave{u}_k)\, \partial_{z_j^c}[g(\mathbf{x})]\, \grave{\mathfrak{f}}_k \mathfrak{f}_j,\\
\left(\underline{\grave{Z}}-\underline{\grave{U}}\right) \partial_{\underline{\grave{Z}}^\dagger}[g(\mathbf{x})] &= -2i\left(\sum_{j=1}^n (\grave{z}_j-\grave{u}_j)\, \grave{\mathfrak{f}}_j\right)\left(\sum_{k=1}^n \grave{\mathfrak{f}}_k\, \partial_{\grave{z}_k^c}[g(\mathbf{x})]\right) = -2i \sum_{1\leq j,k\leq n} (\grave{z}_j-\grave{u}_j)\, \partial_{\grave{z}_k^c}[g(\mathbf{x})]\, \grave{\mathfrak{f}}_j \grave{\mathfrak{f}}_k.
\end{aligned}$$

Hence (7.33) reads

$$
\begin{aligned}
&\frac{2}{|\mathbb{S}^{2m-1|2n}|}\int_{\mathbb{R}^{2m|2n}} \frac{\delta(g(\mathbf{x}))}{|\mathbf{Z}-\mathbf{U}|^M} \sum_{1\leq j<k\leq m} \Big[(z_j-u_j)\,\partial_{z_k^c}[g(\mathbf{x})]-(z_k-u_k)\,\partial_{z_j^c}[g(\mathbf{x})]\Big]\, F(\mathbf{Z})\,\mathfrak{f}_j\mathfrak{f}_k\, I_bI_f\\
&+\frac{2}{|\mathbb{S}^{2m-1|2n}|}\int_{\mathbb{R}^{2m|2n}} \frac{\delta(g(\mathbf{x}))}{|\mathbf{Z}-\mathbf{U}|^M} \sum_{\substack{1\leq j\leq m\\ 1\leq k\leq n}} \Big[-2i(z_j-u_j)\,\partial_{z_{\grave{k}}^c}[g(\mathbf{x})]-(z_{\grave{k}}-u_{\grave{k}})\,\partial_{z_j^c}[g(\mathbf{x})]\Big]\, F(\mathbf{Z})\,\mathfrak{f}_j\mathfrak{f}_{\grave{k}}\, I_bI_f\\
&-\frac{4i}{|\mathbb{S}^{2m-1|2n}|}\int_{\mathbb{R}^{2m|2n}} \frac{\delta(g(\mathbf{x}))}{|\mathbf{Z}-\mathbf{U}|^M} \sum_{1\leq j<k\leq n} \Big[(z_{\grave{j}}-u_{\grave{j}})\,\partial_{z_{\grave{k}}^c}[g(\mathbf{x})]+(z_{\grave{k}}-u_{\grave{k}})\,\partial_{z_{\grave{j}}^c}[g(\mathbf{x})]\Big]\, F(\mathbf{Z})\,\mathfrak{f}_{\grave{j}}\mathfrak{f}_{\grave{k}}\, I_bI_f\\
&\qquad -\frac{4i}{|\mathbb{S}^{2m-1|2n}|}\int_{\mathbb{R}^{2m|2n}} \frac{\delta(g(\mathbf{x}))}{|\mathbf{Z}-\mathbf{U}|^M} \sum_{1\leq j\leq n} (z_{\grave{j}}-u_{\grave{j}})\,\partial_{z_{\grave{j}}^c}[g(\mathbf{x})]\, F(\mathbf{Z})\,\mathfrak{f}_{\grave{j}}^{2}\, I_bI_f=0.
\end{aligned}
$$

Thus, it suffices to prove that all the elements

$$
\begin{aligned}
&\mathfrak{f}_j\mathfrak{f}_k\, I_bI_f, && 1\leq j<k\leq m,\\
&\mathfrak{f}_j\mathfrak{f}_{\grave{k}}\, I_bI_f, && 1\leq j\leq m,\ \ 1\leq k\leq n,\\
&\mathfrak{f}_{\grave{j}}\mathfrak{f}_{\grave{k}}\, I_bI_f && 1\leq j\leq k\leq n,
\end{aligned}
\tag{7.35}
$$

are linearly independent when considering coefficients in $C^\infty(\Omega)\otimes\mathfrak{G}_{2n}$. So take $A_{j,k}$, $B_{j,k}$, $C_{j,k}\in C^\infty(\Omega)\otimes\mathfrak{G}_{2n}$ such that

$$
\sum_{1\leq j<k\leq m} A_{j,k}\,\mathfrak{f}_j\mathfrak{f}_k\, I_bI_f+\sum_{\substack{1\leq j\leq m\\ 1\leq k\leq n}} B_{j,k}\,\mathfrak{f}_j\mathfrak{f}_{\grave{k}}\, I_bI_f+\sum_{1\leq j\leq k\leq n} C_{j,k}\,\mathfrak{f}_{\grave{j}}\mathfrak{f}_{\grave{k}}\, I_bI_f=0. \tag{7.36}
$$

We now observe that

$$
\begin{aligned}
&\mathfrak{f}_\ell^\dagger\,\mathfrak{f}_s^\dagger\,(\mathfrak{f}_j\mathfrak{f}_k\, I_bI_f)=(\delta_{s,j}\delta_{\ell,k}-\delta_{\ell,j}\delta_{s,k})I_bI_f,\\
&\mathfrak{f}_\ell^\dagger\,\mathfrak{f}_s^\dagger\,(\mathfrak{f}_j\mathfrak{f}_{\grave{k}}\, I_bI_f)=0,\\
&\mathfrak{f}_\ell^\dagger\,\mathfrak{f}_s^\dagger\,(\mathfrak{f}_{\grave{j}}\mathfrak{f}_{\grave{k}}\, I_bI_f)=0.
\end{aligned}
$$

Multiplying (7.36) from the left by $\mathfrak{f}_\ell^\dagger\,\mathfrak{f}_s^\dagger$ $(1\leq s<\ell\leq m)$ we obtain $A_{s,\ell}\, I_bI_f=0$ which implies $A_{s,\ell}=0$, whence we are left with

$$
\sum_{\substack{1\leq j\leq m\\ 1\leq k\leq n}} B_{j,k}\,\mathfrak{f}_j\mathfrak{f}_{\grave{k}}\, I_bI_f+\sum_{1\leq j\leq k\leq n} C_{j,k}\,\mathfrak{f}_{\grave{j}}\mathfrak{f}_{\grave{k}}\, I_bI_f=0.
$$

In the same order of ideas we get,

$$
\mathfrak{f}_{\grave{\ell}}^\dagger\,\mathfrak{f}_s^\dagger\,(\mathfrak{f}_j\mathfrak{f}_{\grave{k}}\, I_bI_f)=\frac{i}{2}\delta_{\ell,k}\delta_{s,j}I_bI_f, \qquad\qquad \mathfrak{f}_{\grave{\ell}}^\dagger\,\mathfrak{f}_s^\dagger\,(\mathfrak{f}_{\grave{j}}\mathfrak{f}_{\grave{k}}\, I_bI_f)=0,
$$

whence multiplying now the remainder of (7.36) from the left by $\mathfrak{f}_{\grave{\ell}}^\dagger\,\mathfrak{f}_s^\dagger$ $(1\leq s\leq m$, $1\leq \ell\leq n)$ yields $B_{s,\ell}=0$, thus further reducing the equality to

$$
\sum_{1\leq j\leq k\leq n} C_{j,k}\,\mathfrak{f}_{\grave{j}}\mathfrak{f}_{\grave{k}}\, I_bI_f=0.
$$

Finally, we compute

$$\mathfrak{f}_\ell^{\grave{}\dagger}\,\mathfrak{f}_s^{\grave{}\dagger}\,(\mathfrak{f}_j^{\grave{}}\mathfrak{f}_k^{\grave{}} I_b I_f) = -\frac{1}{4}(\delta_{\ell,j}\delta_{s,k} + \delta_{s,j}\delta_{\ell,k}) I_b I_f,$$

which allows to conclude that $C_{\ell,s} = 0$ after multiplying (7.36) by $\mathfrak{f}_\ell^{\grave{}\dagger}\,\mathfrak{f}_s^{\grave{}\dagger}$ $(1 \leq \ell \leq s \leq n)$. In this way, we have proven that all coefficients in (7.36) are zero, meaning that all elements (7.35) indeed are linearly independent. □

7.4.3 Some examples

In this section we study some particular but important applications of the Bochner-Martinelli formula in superspace.

Case 1.
We first consider the case of a purely bosonic phase function $g(\mathbf{x}) = g_0(\underline{x}) \in C^\infty(\mathbb{R}^{2m})$, i.e. $\mathbf{g} = 0$, which satisfies the same conditions as in Theorem 7.6. In this case, formula (7.34) reads

$$\frac{2}{|\mathbb{S}^{2m-1|2n}|}\int_{\mathbb{R}^{2m|2n}} \frac{\delta(g_0(\underline{x}))}{|\mathbf{Z}-\mathbf{U}|^M}\left(\sum_{j=1}^m (z_j-u_j)^c \partial_{z_j^c}[g_0(\underline{x})]\right) F(\mathbf{Z}) = F(\mathbf{U}), \quad g_0(\underline{y}) < 0. \tag{7.37}$$

This formula reduces to the classical Bochner-Martinelli formula (7.1) as we will show next. We begin by recalling the following classical result for surface integration over $\Gamma := g_0^{-1}(0)$, see [53, p. 136],

$$\int_{\mathbb{R}^{2m}} \delta(g_0(\underline{x}))|\partial_{\underline{x}}[g_0(\underline{x})]| f(\underline{x})\, dV_{\underline{x}} = \int_\Gamma f(\underline{x})\, dS_{\underline{x}}.$$

The j-th coordinate $n_j(\underline{x})$ of the exterior normal vector $n(\underline{x})$ to the surface Γ at the point $\underline{x} \in \Gamma$ is given by

$$n_j(\underline{x}) = \frac{\partial_{x_j}[g](\underline{x})}{|\partial_{\underline{x}}[g_0(\underline{x})]|}.$$

Hence, from the above formula one easily obtains

$$\int_{\mathbb{R}^{2m}} \delta(g_0(\underline{x}))\partial_{x_j}[g_0(\underline{x})] f(\underline{x})\, dV_{\underline{x}} = \int_\Gamma n_j(\underline{x}) f(\underline{x})\, dS_{\underline{x}}.$$

Moreover, since Γ is a smooth surface in $\mathbb{R}^{2m}$, we can write

$$\widehat{dx_j} = (-1)^{j-1} n_j(\underline{x})\, dS_{\underline{x}}$$

where $\widehat{dx_j}$ is the differential $(2m-1)$-form

$$\widehat{dx_j} = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_{2m}.$$

This allows to change the above distributional approach to classical surface integration by differential forms. In particular, we have that

$$\begin{aligned}\int_{\mathbb{R}^{2m}} \delta(g_0(\underline{x}))\, \partial_{z_j^c}[g_0(\underline{x})] f(\underline{x})\, dV_{\underline{x}} &= \frac{1}{2}\int_{\mathbb{R}^{2m}} \delta(g_0(\underline{x}))\left(\partial_{x_j} + i\partial_{x_{m+j}}\right)[g_0(\underline{x})]\, f(\underline{x})\, dV_{\underline{x}}\\ &= \frac{1}{2}\int_{\Gamma}\left(n_j(\underline{x}) + i n_{m+j}(\underline{x})\right) f(\underline{x})\, dS_{\underline{x}}\\ &= \frac{1}{2}\int_{\Gamma}\left((-1)^{j-1}\widehat{dx_j} + i(-1)^{m+j-1}\widehat{dx_{m+j}}\right) f(\underline{x}).\end{aligned}$$

We now write,

$$(-1)^{j-1}\widehat{dx_j} = (-1)^{\frac{m(m-1)}{2}}\widetilde{\widehat{dx_j}}, \qquad -(-1)^{m+j-1}\widehat{dx_{m+j}} = (-1)^{\frac{m(m-1)}{2}}\widetilde{\widehat{dx_{m+j}}},$$

with

$$\begin{aligned}\widetilde{\widehat{dx_j}} &= (dx_1 \wedge dx_{m+1}) \wedge \ldots \wedge ([dx_j] \wedge dx_{m+j}) \wedge \ldots \wedge (dx_m \wedge dx_{2m}), \quad j = 1, \ldots, m,\\ \widetilde{\widehat{dx_{m+j}}} &= (dx_1 \wedge dx_{m+1}) \wedge \ldots \wedge (dx_j \wedge [dx_{m+j}]) \wedge \ldots \wedge (dx_m \wedge dx_{2m}), \quad j = 1, \ldots, m,\end{aligned}$$

where $[\cdot]$ denotes omitting that particular differential. We then obtain,

$$\int_{\mathbb{R}^{2m}} \delta(g_0(\underline{x}))\, \partial_{z_j^c}[g_0(\underline{x})] f(\underline{x})\, dV_{\underline{x}} = \frac{(-1)^{\frac{m(m-1)}{2}}}{2}\int_{\Gamma}\left(\widetilde{\widehat{dx_j}} - i\widetilde{\widehat{dx_{m+j}}}\right) f(\underline{x})$$

On the other hand,

$$\widetilde{\widehat{dx_j}} - i\widetilde{\widehat{dx_{m+j}}} = -2\left(\frac{i}{2}\right)^m \widehat{dz_j^c} = (-1)^{\frac{m(m+1)}{2}-j}(-2)\left(\frac{i}{2}\right)^m \widetilde{\widehat{dz_j^c}}$$

where we have introduced the complex differential forms

$$\begin{aligned}\widehat{dz_j^c} &= (dz_1 \wedge dz_1^c) \wedge \ldots \wedge \left(dz_j \wedge [dz_j^c]\right) \wedge \ldots \wedge (dz_m \wedge dz_m^c),\\ \widetilde{\widehat{dz_j^c}} &= dz_1^c \wedge \ldots \wedge [dz_j^c] \wedge \ldots \wedge dz_m^c \wedge dz_1 \wedge \ldots \wedge dz_m,\end{aligned}$$

which are clearly connected by $\widehat{dz_j^c} = (-1)^{\frac{m(m+1)}{2}-j}\, \widetilde{\widehat{dz_j^c}}$.

Hence,

$$\int_{\mathbb{R}^{2m}} \delta(g_0(\underline{x}))\, \partial_{z_j^c}[g_0(\underline{x})] f(\underline{x})\, dV_{\underline{x}} = (-1)^{m-j+1}\left(\frac{i}{2}\right)^m \int_{\Gamma} f(\underline{x})\, \widetilde{\widehat{dz_j^c}}.$$

Applying this last result to the left side of formula (7.37) yields

$$\begin{aligned}
&\frac{2}{|\mathbb{S}^{2m-1|2n}|}\int_B \sum_{j=1}^{m}\left(\int_{\mathbb{R}^{2m}} \delta(g_0(\underline{x}))\,\partial_{z_j^c}[g_0(\underline{x})]\frac{(z_j-u_j)^c}{|\mathbf{Z}-\mathbf{U}|^M}\,F(\mathbf{Z})\,dV_{\underline{x}}\right)\\
&\quad=\frac{2}{|\mathbb{S}^{2m-1|2n}|}\int_B \sum_{j=1}^{m}(-1)^{m-j+1}\left(\frac{i}{2}\right)^m\int_\Gamma \widetilde{\widehat{dz_j^c}}\,\frac{(z_j-u_j)^c}{|\mathbf{Z}-\mathbf{U}|^M}\,F(\mathbf{Z})\\
&\quad=\frac{2}{|\mathbb{S}^{2m-1|2n}|(2i)^m}\int_\Gamma\int_B \sum_{j=1}^{m}(-1)^{j-1}\widetilde{\widehat{dz_j^c}}\,\frac{(z_j-u_j)^c}{|\mathbf{Z}-\mathbf{U}|^M}\,F(\mathbf{Z}).
\end{aligned}$$

Then (7.37) can be rewritten as

$$\frac{(m-n-1)!\,\pi^n}{(2i\pi)^m}\int_\Gamma\int_B\left(\sum_{j=1}^{m}(-1)^{j-1}\widetilde{\widehat{dz_j^c}}\,\frac{(z_j-u_j)^c}{|\mathbf{Z}-\mathbf{U}|^M}\right)F(\mathbf{Z})=F(\mathbf{U}),\qquad g_0(\underline{y})<0,$$

which exactly coincides with formula (7.1) for $n=0$.

Case 2.
We now examine the form which (7.34) takes on the supersphere of radius $R>0$ defined by means of the phase function $g(\mathbf{x})=-\mathbf{x}^2-R^2$. Observe that

$$g(\mathbf{x})=|\mathbf{x}|^2-R^2=|\mathbf{Z}|^2-R^2=\{\mathbf{Z},\mathbf{Z}^\dagger\}-R^2=\sum_{j=1}^{m} z_j z_j^c-\frac{i}{2}\sum_{j=1}^{n}\grave{z}_j\grave{z}_j^c-R^2.$$

Then, $\partial_{z_j^c}[g(\mathbf{x})]=z_j$ and $\partial_{\grave{z}_j^c}[g(\mathbf{x})]=\frac{i}{2}\grave{z}_j$, leading to

$$D^\dagger_{\mathbf{Z}-\mathbf{U},\mathbf{Z}}[g(\mathbf{x})]=\sum_{j=1}^{m}(z_j-u_j)^c\,z_j+\frac{i}{2}\sum_{j=1}^{n}(\grave{z}_j-\grave{u}_j)^c\,\grave{z}_j=\{\mathbf{Z},(\mathbf{Z}-\mathbf{U})^\dagger\}.$$

Hence, the Bochner-Martinelli formula on the supersphere of radius $R>0$ takes the form

$$\frac{2}{|\mathbb{S}^{2m-1|2n}|}\int_{\mathbb{R}^{2m|2n}}\frac{\delta(\{\mathbf{Z},\mathbf{Z}^\dagger\}-R^2)}{|\mathbf{Z}-\mathbf{U}|^M}\,\{\mathbf{Z},(\mathbf{Z}-\mathbf{U})^\dagger\}\,F(\mathbf{Z})=F(\mathbf{U}),\qquad |\underline{y}|<R.$$

Summary

Clifford analysis nowadays is a well-established discipline within mathematical analysis that may be seen as both a generalization of the theory of holomorphic functions in the complex plane and a refinement of classical harmonic analysis. In its most simple setting, the Clifford function theory focusses on the notion of monogenic function, i.e. a null solution of the Dirac operator $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$. This setting is known as the Euclidean case (also the term orthogonal Clifford analysis can be found in the literature). The fundamental group leaving the Dirac operator invariant is $\mathrm{Spin}(m)$ which is a double covering of $\mathrm{SO}(m)$.

More recently further refinements and generalizations of Euclidean Clifford analysis have been studied. Amongst the most important and well-studied refinements of Euclidean Clifford analysis we find so-called Hermitian Clifford analysis, which focusses on h-monogenic functions, h-monogenicity being expressed by means of two mutually adjoint Dirac operators which are invariant under the realization of the unitary group $\mathrm{U}(m)$ in $\mathrm{Spin}(2m)$.

Harmonic analysis and Euclidean Clifford analysis have been extended to superspace by means of a representation of the so-called radial algebra. This theory introduces some important differential operators (such as Dirac and Laplace operators) on the flat supermanifold $\mathbb{R}^{m|2n}$, and uses them in the study of special functions, orthogonal polynomials, integration, etc.

The main goal of this thesis is to further develop the extension of Clifford analysis to superspace initiated in the works of Hendrik De Bie and Kevin Coulembier, [29, 22]. In particular our purpose is threefold. In first place, we aim at extending Hermitian Clifford analysis to superspace (Chapters 2, 3). The second goal is to provide a suitable definition for the (super) spin group and to study the underlying group actions in both Euclidean and Hermitian settings in superspace (Chapters 4, 5). Finally, the third objective is to further develop integration theory by introducing and studying integration over general domains and surfaces in superspace depending on bosonic and fermionic variables on equal footing (Chapters 6, 7).

We now give a detailed overview of the contents of the chapters in this thesis.

Chapter 1: Introduction

In this introductory chapter we describe the mathematical framework for this thesis. First we give a brief overview of both Euclidean and Hermitian Clifford analysis, together with a description of the underlying abstract radial algebra.Then an explanation of the meaning of superanalysis (or analysis in superspace) is given. Here, we provide an account of some important historical approaches that have been used for superanalysis and comment on the extension of Clifford analysis towards super-symmetry. Finally, a detailed account of the goals and the content of this thesis is provided.

Chapter 2: Radial algebra

This chapter is devoted to the study of the abstract framework for Clifford analysis offered by the radial algebra. The radial algebra has been proven to be an important tool for deriving a theory of Dirac operators in superspace from the standard Euclidean one and for giving a meaning to spaces with negative integer dimension. In this chapter, we study the Hermitian radial algebra in order to extend the theory of Hermitian Dirac operators to superspace. We first provide a detailed account of the algebra of endomorphisms of the radial algebra and on the notion of radial algebra representation. Then, the Hermitian radial algebra is introduced by means of the notion of abstract complex structure. At the end, an important representation of the radial algebra with a complex structure is presented at the level of endomorphisms.

Chapter 3: Hermitian Clifford analysis on superspace

The main goal of this chapter is to introduce the building blocks of a Hermitian monogenic function theory in superspace by means of a representation of the Hermitian radial algebra. To that end, we first provide an overview of the superanalysis framework. We mainly discuss two approaches to superanalysis: the one based on differential geometry where variables are represented as co-ordinates taking values in some commutative Banach superalgebra, and the one based on modern algebraic geometry where variables are defined in a purely symbolic way giving rise to a supermanifold with a structural sheaf of superfunctions. Using this last approach, we then recall the main aspects of the extension of Euclidean Clifford analysis to superspace. In particular, the vector multipliers give rise to a natural way of introducing a complex structure on superspace which immediately leads to the corresponding extensions of all basic objects such as Hermitian Dirac operators, complex Euler operators, etc. The definition of all these objects is validated by checking that they satisfy the abstract relations provided in Chapter 2 for the Hermitian radial algebra.

Chapter 4: The Spin group in superspace

There are two well-known ways of describing elements of the rotation group $\mathrm{SO}(m)$. First, according to the Cartan-Dieudonné theorem, every rotation matrix can be written as an even number of reflections. And second, they can also be expressed as the exponential of some anti-symmetric matrix. In this chapter, we study similar descriptions of the corresponding extension of $\mathrm{SO}(m)$ to superspace. To that end, we consider linear actions on supervector variables using both commuting and anti-commuting coefficients in a Grassmann algebra. This allows to study the invariance of the inner product in superspace from a classical group theoretic approach which contains all information on the underlying symmetry superalgebras obtained in [22, 23].

We first provide some basics on Grassmann algebras, Grassmann envelopes and supermatrices. Next, we further develop the Clifford setting in superspace by introducing the Lie algebra of superbivectors. An extension of this algebra is crucial in the description of the super spin group. While studying the invariance of the bilinear form that extends the Euclidean inner product to superspace, we obtain the group of so-called superrotations SO_0 whose Lie algebra $\mathfrak{so}_0$ turns out to be a Grassmann envelope of $\mathfrak{osp}(m|2n)$. The group SO_0 is also an extension of the symplectic group. While still being connected, it is thus no longer compact. As a consequence, it cannot be fully described by just one action of the exponential map on its Lie algebra. Instead, we obtain an Iwasawa-type decomposition for this group in terms of three exponentials acting on three direct summands of the corresponding Lie algebra of supermatrices. At the same time, SO_0 strictly contains the group generated by supervector reflections. Therefore, its Lie algebra is isomorphic to a certain extension of the algebra of superbivectors. This means that the Spin group in superspace has to be seen as the group generated by the exponentials of the so-called extended superbivectors in order to cover SO_0. We also study the actions of this Spin group on supervectors and provide a proper subset of it that is a double cover of SO_0. Finally, we show that every fractional Fourier transform in n bosonic dimensions can be seen as an element of the spin group in superspace.

Chapter 5: Spin action in Euclidean and Hermitian Clifford analysis in superspace

In this chapter we study the action of the spin group on superfunctions. In the first place, we prove the invariance of the super Dirac operator $\partial_{\mathbf{x}}$ under the classical H and L actions. This follows from the $\mathfrak{so}_0$-invariance of $\partial_{\mathbf{x}}$, i.e. the commutation of $\partial_{\mathbf{x}}$ with the infinitesimal representation dL of L acting on the Lie algebra of superbivectors. These actions are also studied in the Hermitian setting, where we study the group invariance of the Hermitian inner product of supervectors introduced in [42]. The resulting group of complex supermatrices leaving this inner product invariant constitutes an extension of $\mathrm{U}(m) \times \mathrm{U}(n)$ and is isomorphic to the subset $\mathrm{SO}_0^{\mathbf{J}} \subset \mathrm{SO}_0$ of elements that commute with the complex structure $\mathbf{J}$. The realization of $\mathrm{SO}_0^{\mathbf{J}}$ within the spin group is studied simultaneously with the invariance under its actions of the super Hermitian Dirac system.

Chapter 6: Distributions and integration in superspace

Distributions have been proven to be an important tool in the development of an integration theory in superspace. In particular, distributions were used in [24] to obtain a suitable extension of the Cauchy formula to superspace and to define integration over the superball and the supersphere through the Heaviside and Dirac distributions respectively. The goal of this chapter is to extend the distributional approach to integration over more general domains and surfaces in superspace.

We first introduce domains and surfaces in superspace by means of smooth commuting phase functions g. In this way, one can define domain integrals and oriented (and non-oriented) surface integrals in terms of the Heaviside and Dirac distributions of the superfunction g. Then it is shown that the presented definition for the integrals does not depend on the choice of the phase function g defining the corresponding domain or surface. Moreover, some applications of these approaches are shown by computing the volume and surface area of a super-paraboloid and a super-hyperboloid of revolution. At the end of the chapter, a new distributional Cauchy-Pompeiu formula is obtained generalizing and unifying the previously known approaches.

Chapter 7: Bochner-Martinelli formula in superspace

In this final chapter the Bochner-Martinelli integral representation for holomorphic functions of several complex variables is extended to the superspace setting. This is done by exploiting the intrinsic connection existing between the Hermitian monogenic function theory and the theory of holomorphic functions.

We start by addressing the problem of establishing a Cauchy integral formula in the framework of Hermitian Clifford analysis in superspace. We then establish the connection between Hermitian monogenicity and super holomorphicity by considering a specific class of spinor-valued superfunctions. It turns out that after a certain projection of the obtained (super) Hermitian Cauchy integral formula one obtains a new extension of the Bochner-Martinelli formula for holomorphic superfunctions. It is indeed proven, at the end of the chapter, that such a projection coincides with the classical Bochner-Martinelli representation when considering only m complex bosonic dimensions.

Nederlandse samenvatting

Cliffordanalyse is heden ten dage een gevestigde discipline binnen de klassieke wiskundige analyse, die terzelfder tijd kan worden beschouwd als een veralgemening van de theorie van holomorfe functies in het complexe vlak en als een verfijning van klassieke harmonische analyse. In haar meest eenvoudige vorm kunnen we de cliffordanalyse zien als de studie van monogene functies, dit zijn nuloplossingen van de diracoperator $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$; in deze context spreken we van euclidische cliffordanalyse (al wordt in sommige bronnen ook de term orthogonale cliffordanalyse gebruikt). De fundamentele groep die de diracoperator invariant laat, is de zogenaamde spingroep $\mathrm{Spin}(m)$, die een dubbele bedekking vormt van de speciaal orthogonale groep $\mathrm{SO}(m)$.

Meer recent werden verdere verfijningen en veralgemeningen van euclidische cliffordanalyse bestudeerd, zoals bijvoorbeeld hermitische cliffordanalyse. Deze theorie draait rond de studie van h-monogene functies, oplossingen van twee hermitisch toegevoegde diracoperatoren, welke invariant zijn onder de actie van een realisatie van de unitaire groep $\mathrm{U}(m)$ in de spingroep $\mathrm{Spin}(2m)$.

Zowel harmonische analyse als euclidische cliffordanalyse werden ook ingevoerd in de superruimte, door gebruik te maken van een representatie van de zogenaamde radiale algebra. Binnen deze theorie worden een aantal belangrijke differentiaaloperatoren (zoals de diracoperator en de laplace-operator) geïntroduceerd op de vlakke supervariëteit $\mathbb{R}^{m|n}$, teneinde ook daar speciale functies, orthogonale polynomen, een integratietheorie, en dies meer, in te voeren en te bestuderen.

Het doel van deze thesis is de verdere ontwikkeling van cliffordanalyse in de superruimte, als vervolg op het werk van Hendrik De Bie en Kevin Coulembier, [29, 22]. In het bijzonder kunnen drie specifieke doelstellingen worden onderscheiden. Eerst en vooral zullen we hermitische cliffordanalyse invoeren in de superruimte (Hoofdstukken 2 en 3). Vervolgens gaan we op zoek naar een geschikte definitie voor de super-spingroep en bestuderen we de corresponderende groep-acties in zowel de euclidische als de hermitische context (Hoofdstukken 4 en 5). Tot slot ontwikkelen we een theorie van integratie over algemene domeinen en oppervlakken in de superruimte, die de fermionische en bosonische variabelen op een gelijkwaardige manier incorporeert (Hoofdstukken 6 en 7).

We geven nu een gedetailleerd overzicht van de inhoud van de respectieve hoofdstukken.

Hoofdstuk 1: Inleiding

In dit inleidend hoofdstuk beschrijven we het wiskundig kader voor deze thesis. Eerst wordt een kort overzicht gegeven van euclidische en hermitische cliffordanalyse, alsook een beschrijving van de corresponderende radiale algebra. Vervolgens wordt de betekenis uitgelegd van superanalyse (of: analyse op de superruimte). Hierbij geven we een overzicht van de verschillende historische benaderingen van superanalyse en wordt er ook ingegaan op de uitbreiding van cliffordanalyse naar dit kader. Tot slot geven we een gedetailleerd overzicht van doelstellingen en inhoud van deze thesis.

Hoofdstuk 2: De radiale algebra

Dit hoofdstuk is gewijd aan de studie van de radiale algebra, die het abstracte kader vormt voor cliffordanalyse. De radiale algebra is een belangrijk instrument gebleken om van de theorie van diracoperatoren in de euclidische ruimte over te gaan naar de superruimte en hierbij een betekenis te geven aan ruimten met een negatieve (gehele) dimensie. Daarnaast bestuderen we in dit hoofdstuk ook de hermitische radiale algebra, teneinde ook de theorie van hermitische diracoperatoren uit te breiden tot de superruimte. Eerst bespreken we in detail de algebra der endomorfismen in de radiale algebra en het concept van een representatie van de radiale algebra. Vervolgens wordt de hermitische radiale algebra ingevoerd middels het concept van een abstracte complexe structuur. Tot slot stellen we een belangrijke representatie voor van de radiale algebra met complexe structuur op het niveau van de endomorfismen.

Hoofdstuk 3: Hermitische cliffordanalyse in de superruimte

Het belangrijkste doel van dit hoofdstuk is de bouwstenen in te voeren voor een hermitisch monogene functietheorie in de superruimte, door gebruik te maken van een representatie van de hermitische radiale algebra. Daartoe geven we eerst een overzicht van het kader waarbinnen we in de superanalyse werken, en we beschouwen hierbij twee verschillende mogelijke benaderingen. De eerste is gebaseerd op differentiaalmeetkunde, waarbij de variabelen voorgesteld worden als coördinaten die waarden aannemen in een commutatieve Banach superalgebra. De tweede is gebaseerd op moderne algebraïsche meetkunde, waarbij de variabelen op een puur symbolische manier gedefinieerd worden, wat aanleiding geeft tot een supervariëteit uitgerust met een structurele schoof van superfuncties. We werken verder binnen het tweede kader, waarbij we vectormultiplicatoren gebruiken om op een natuurlijke manier een complexe structuur te definiëren op de superruimte, wat ons in staat stelt onmiddellijk alle fundamentele objecten van hermitische cliffordanalyse in te voeren, zoals hermitische diracoperatoren, complexe Euleroperatoren, etc. De respectieve definities worden hierbij gevalideerd door te checken dat ze aan alle abstracte voorwaarden van de hermitische radiale algebra voldoen (zie Hoofdstuk 2).

Hoofdstuk 4: The super-spingroep

Er zijn essentieel twee verschillende manieren om de elementen van de rotatiegroep $\mathrm{SO}(m)$ te beschrijven. Vooreerst zegt de stelling van Cartan-Dieudonné dat elke rotatiematrix kan worden ontbonden in een even aantal spiegelingen. Daarnaast kan een rotatiematrix ook worden geschreven als de exponentiële van een anti-symmetrische matrix. In dit hoofdstuk bestuderen we gelijkaardige beschrijvingen van de tegenhanger van $\mathrm{SO}(m)$ in de superruimte, en we beschouwen daartoe lineaire acties op de supervector variabelen, gebruik makend van zowel commuterende als anti-commuterende coëfficiënten in een Grassmann algebra. Dit stelt ons in staat de invariantie van het inproduct in de superruimte te bestuderen via een groeptheoretische aanpak, zoals in [22, 23].

We beschrijven eerst de belangrijkste aspecten van Grassmann algebra's and supermatrices, en we voeren in de Clifford setting de Lie algebra van de superbivectoren in. Een uitbreiding van deze algebra zal cruciaal blijken voor de karakterisatie van de super spingroep. Als we de invariantie bestuderen van de bilineaire vorm die de tegenhanger is in de superruimte van het euclidisch inproduct, bekomen we de groep van zogenaamde superrotaties SO_0, een uitbreiding van de symplectische groep waarvan de Lie algebra $\mathfrak{so}_0$ een Grassmann omhullende van $\mathfrak{osp}(m|2n)$ blijkt te zijn. Deze groep is wel samenhangend maar niet langer compact, en kan bijgevolg niet meer volledig worden gekarakteriseerd door de actie van één enkele exponentiële afbeelding op zijn Lie algebra. In de plaats daarvan verkrijgen we een Iwasawa-decompositie in drie exponentiële afbeeldingen die inwerken op de drie termen in de directe somontbinding van de corresponderende Lie algebra van supermatrices. We merken ook op dat SO_0 de groep omvat die gegenereerd wordt door spiegelingen t.o.v. supervectoren, waardoor de corresponderende Lie algebra isomorf moet zijn met een welbepaalde extensie van de Lie algebra der superbivectoren. Dit betekent dat, om een bedekking te verkrijgen van SO_0, we de super-spingroep dus moeten zien als de groep gegenereerd door de exponentiëlen van de zogenaamde uitgebreide superbivectoren. We bestuderen vervolgens de actie van deze spingroep op de supervectoren en identificeren de deelgroep die een dubbele bedekking is van SO_0. Tot slot tonen we aan dat elke fractionele fouriertransformatie in n bosonische dimensies kan worden gezien als een element van deze super-spingroep.

Hoofdstuk 5: De actie van de super-spingroep in euclidische en hermitische cliffordanalyse

In dit hoofdstuk bestuderen we de actie van de super-spingroep op superfuncties. Vooreerst tonen we de invariantie aan van de super-diracoperator $\partial_{\mathbf{x}}$ onder de traditionele H en L acties, steunend op de $\mathfrak{so}_0$-invariantie van $\partial_{\mathbf{x}}$, i.e. het commuteren ervan met de infinitesimale representatie dL van L inwerkend op de Lie algebra der superbivectoren. Deze acties worden ook bestudeerd in het hermitisch kader, waar we de groepinvariantie onderzoeken van het hermitisch inproduct van supervectoren ingevoerd in [42]. De groep van complexe supermatrices die dit inproduct invariant laten, vormt een uitbreiding van $\mathrm{U}(m) \times \mathrm{U}(n)$ en is isomorf met de deelgroep $\mathrm{SO}_0^{\mathbf{J}} \subset \mathrm{SO}_0$ van elementen die commuteren met de complexe structuur $\mathbf{J}$. De realisatie van $\mathrm{SO}_0^{\mathbf{J}}$ in de spingroep wordt terzelfder tijd bestudeerd als de invariantie van het hermitisch systeem onder de acties ervan.

Hoofdstuk 6: Distributies en integratie in de superruimte

Distributies zijn een belangrijk instrument gebleken bij de ontwikkeling van een integratietheorie in de superruimte. In het bijzonder werden distributies gebruikt in [24] om een geschikte uitbreiding te bekomen van de Cauchy formule en om integratie te definiëren over de superbal en de supersfeer, door gebruik te maken van de Heaviside en de Dirac distributie. Het doel van dit hoofdstuk is om de distributionele benadering van integratie uit te breiden naar meer algemene domeinen en oppervlakken in de superruimte.

We voeren domeinen en oppervlakken in de superruimte in door middel van een commuterende, gladde fasefunctie g. Op die manier kunnen domein- zowel als georiënteerde (en niet georiënteerde) oppervlakintegralen worden gedefinieerd in functie van de Heaviside en de Dirac distributie geassocieerd met de superfunctie g. We tonen dan aan dat onze correponderende definitie van integraal niet afhangt van de keuze van g als fasefunctie voor het beschouwde domein of oppervlak. We passen onze aanpak toe op de berekening van het volume en de oppervlakte van een super-omwentelingsparaboloïde en een super-omwentelingshyperboloïde. Op het einde van het hoofdstuk stellen we nog een nieuwe en unificerende distributionele Cauchy-Pompeiu formule op.

Hoofdstuk 7: De formule van Bochner-Martinelli in de superruimte

In dit laatste hoofdstuk bekomen we een tegenhanger in de superruimte van de integraalrepresentatie van Bochner-Martinelli voor holomorfe functies van meerdere complexe variabelen. Dit gebruikt door voluit gebruik te maken van het intrinsieke verband dat bestaat tussen de theorie van hermitisch monogene functies en deze van holomorfe functies.

We starten hierbij met het opstellen van een Cauchy integraalformule in de context van hermitische cliffordanalyse in de superruimte. Daarna leggen we het verband tussen hermitisch monogeniteit en superholomorfie door een klasse van superfuncties te beschouwen met waarden in de spinorruimte. De bekomen (super) hermitische Cauchy formule leidt dan, door projectie, tot een nieuwe uitbreiding van de Bochner-Martinelli formule voor holomorfe superfuncties. Op het einde van het hoofdstuk wordt dan bewezen dat de gebruikte projectie samenvalt met de klassieke Bochner-Martinelli representatie als we enkel de m complexe bosonische dimensies beschouwen.

List of Tables

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