HOMOTOPY THEORY FOR *CW*-COMPLEXES

by

Archana Tiwari

M.Sc. Student in Mathematics
Roll No.413MA2077
National Institute of Technology, Rourkela
Rourkela, 769 008(India)

A dissertation in conformity with the requirements
for the Degree of Masters of Science in Mathematics at
National Institute of Technology, Rourkela

CERTIFICATE

This is to certify that this review work entitled **"Homotopy theory for CW-Complexes"** which is being submitted by Miss Archana Tiwari, M.Sc. Student in Mathematics, Roll No.-413MA2077, National Institute of Technology, Rourkela-769008 towards the partial fulfillment of the requirement for the award of degree of Master in Science at National Institute of Technology, is carried out under my advice. The matter presented in this dissertation, in the current format has not been submitted anywhere for the award of any other degree or to any other Institute.

To the best of my knowledge Miss Archana Tiwari bears a good moral character and is eligible to get the degree.

Prof. A. Behera

(Advisor)
Depatment of mathematics

.

ACKNOWLEDGEMENT

I am highly indebted to Prof. A. Behera for his advice and for what I know about Algebra, Topology and Algebraic Topology etc., that I learned from him during this work as well as for providing necessary information regarding the work and also for his support in completing the dissertation.

I would like to express my gratitude towards my parents and members of National Institute of Technology, Rourkela for their kind co-operation and encouragement which helped me in completion of this work.

I would like to express my special gratitude and thanks to Miss Snigdha Bharati Choudhury and Miss Mitali Routaray, Ph.D Scholars in Mathematics for giving me their attention towards my work.

Archana Tiwari

M. Sc. Student in Mathematics
Roll Number 413MA2077
National Institute of Technology
Rourkela 769 008

ABSTRACT

In this review work we have studied on homotopy properties of CW-complexes with an emphasis on finite dimensional CW-complexes. We have first given a brief introduction on basic definitions from the general topology and then have discussed the homotopy theory for general topological spaces. Basic definitions and constructions of homotopy and CW-complexes have been discussed exhaustively. Then certain theorems and definitions on homotopy theory of CW-complexes have been discussed briefly. Finally, we have studied Whitehead Theorem.

CONTENTS

Chapter 0

INTRODUCTION

One of the main ideas of algebraic topology is to consider two spaces to be equivalent if they have 'the same shape' in a sense that is much broader than homeomorphism. In this dissertation we review the stages of development.

Chapter 1 is devoted to a general discussion of the most primitive notions of general topology, i.e., topological spaces, Hausdorff spaces, continuous functions, Pasting lemma, homeomorphism and so on.

Chapter 2 deals with the homotopy theory of topological spaces.

Chapter 3 is devoted to the study of CW-complexes. Definitions and examples are presented in a very precise manner.

Chapter 4 deals with homotopy theory for the CW-complex. The notions like fibration and retraction are also recalled in this chapter for the study of Whitehead theorem.

Much of homotopy theory has to do with CW-pairs (X, A). In many standard constructions to work efficiently, it is necessary to make use of homotopy extension property. It was Borsuk who first realised the importance of this notion and many of his earlier papers were devoted to this study. This has been discussed in the final Chapter 5. It also deals with fibration. Most common examples are given and interplay between fibration and cofibration is exploited. Finally Whitehead theorem is proved.

All the results, definition and examples are taken from the textbooks as listed in the references. In almost all such cases references have been stated. In case, in any case, if the reference is missing, the author renders her sincere apology.

Chapter 1

PRELIMINARIES

In this chapter some of the elementary concepts associated with topological spaces have been discussed. Categories, functor and push out have been stated in this chapter which will be used in further chapters.

1.1 Topological preliminaries

In this section we have defined what a topological space is, and different types of topological spaces.

1.1.1 Definition. [5] A *topology* on a set X is a collection $\mathscr{T}$ of subsets of X having the following properties:

(a) φ and X are in $\mathscr{T}$.

(b) The union of the elements of any subcollection of $\mathscr{T}$ is in $\mathscr{T}$.

(c) The intersection of the elements of any finite subcollection of $\mathscr{T}$ is in $\mathscr{T}$.

1.1.2 Definition. [5] A set X for which a topology $\mathscr{T}$ has been defined is called a *topological space.*

1.1.3 Examples. Let X be a non-empty set.

(i) Let $\mathscr{T}$ be the collection of all subsets of X. Then $\mathscr{T}$ is a topology on X, called *disrete topology.*

(ii) Let $\mathscr{T}$ be the collection of φ and X then $\mathscr{T}$ is a topology on X, called as *indiscrete topology* or *trivial topology*.

(iii) Let $X = \{a, b, c\}$.

$\mathscr{T}_1 = \{\varphi, X, \{a, b\}, \{b\}, \{b. c\}\}$

$\mathscr{T}_2 = \{\varphi, X, \{a, b\}, \{a\}, \{b\}\}$

$\mathscr{T}_3 = \{\varphi, X, \{a, b\}, \{c\}\}$

$\mathscr{T}_1$, $\mathscr{T}_2$ and $\mathscr{T}_3$ are the topologies for the set X.

(iv) Let $\mathscr{T}_c$ be the collection of all subsets U of X such that $X - U$ is either countable or is all of X. Then $\mathscr{T}_c$ is a topology on X.

1.2 Hausdorff Spaces

An additional condition which brings the class of spaces under consideration closer to which geometrical intuition applies. This condition was given by the mathematician Felix Hausdorff.

1.2.1 Definition. [5] A topological space X is called a *Hausdorff space* if each pair of distinct points of X, have disjoint neighborhoods.

1.2.2 Example. $\mathbb{R}$ is a Hausdorff space. In general, $\mathbb{R}^n$ is a Hausdorff space.

1.3 Continuous Functions

The concept of continuous function is basic to much of mathematics. In this section, the definition of continuity that will include all the special cases have been formulated.

1.3.1 Definition. [5] A function$f: X \to Y$, where X and Y are topological spaces, is said to be *continuous* if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

1.3.2 Definition. [5] Let X and Y be topological spaces and $f: X \to Y$ be a bijection. If both the functions f and the inverse function $f^{-1}: Y \to X$ are continuous, then f is called a *homeomorphism*.

Pasting lemma is very useful in testing the continuity of a function in algebraic topology.

1.4 Pasting Lemma. [5] *Let*

$$X = A \cup B$$

where A *and* B *are closed in* X. *Let*

$$f: A \to Y \text{ and } g: B \to Y$$

be continuous. If

$$f(x) = g(x)$$

for every $x \in A \cap B$, *then* f *and* g *combine to give a continuous function*

$$h: X \to Y,$$

defined by setting

$$h(x) = f(x) \text{ if } x \in A$$

and

$$h(x) = g(x) \text{ if } x \in B.$$

1.5 Category

A category is an abstract structure which consists of a set of objects and arrows. Basically it exhibits two properties, the first is the ability to compose the arrows associatively and the second one is the existence of an identity arrow for each object. In general, the objects and the arrows can be any abstract entity, and the notion of category provides a fundamental way to describe the mathematical entities and their relationships. Categories can reveal similarities between seemingly different areas of mathematics.

1.5.1 Definition. A *category* $\mathcal{C}$ consists of

(i) A collection $Ob(\mathcal{C})$ of objects, written as $A, B, C, \ldots, X, Y, Z, \ldots$.

(ii) Sets $Mor(X, Y)$ of morphism for each pair $X, Y \in Ob(\mathcal{C})$, including distinguished "*identity*" morphism $1 = 1_X \in Mor(X, X)$ for each X in $\mathcal{C}$. Let $\mathcal{C}(A, B)$ denote the set of morphisms from A to B.

(iii) A composition of morphisms function

$$Mor(X, Y) \times Mor(Y, Z) \to Mor(X, Z)$$

for each triple $X, Y, Z \in Ob(\mathcal{C})$ satisfying

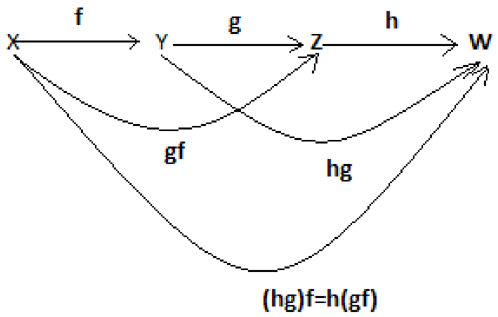
$$f \circ 1_X = f$$

$$X \xrightarrow{1_X} X \xrightarrow{f} Y$$

$$1_Y \circ f = f$$

$$X \xrightarrow{f} Y \xrightarrow{1_Y} Y$$

And $(f \circ g) \circ h = f \circ (g \circ h)$



1.5.1.1 Examples

(a) The collection of sets and functions is a category.

(b) The collection of topological spaces with continuous functions is a category or we could restrict to special classes of spaces such as CW-complexes, keeping continuous maps as the morphism.

(c) The collection of groups and homomorphisms is a category.

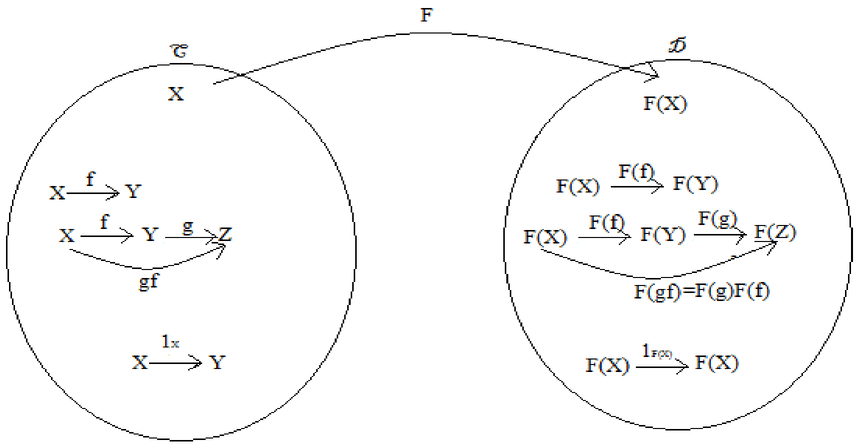
(d) The collection of Banach spaces and bounded linear transformations is a category.

1.6 Definition. [6] A *functor* F from a category $\mathcal{C}$ to another category $\mathcal{D}$ assigns each object X of $\mathcal{C}$ to an object $F(X)$ in $\mathcal{D}$ and to each morphism $f \in \mathcal{C}(X, Y)$ in $\mathcal{C}$ a morphism $F(f) \in \mathcal{D}(F(X), F(Y))$ in $\mathcal{D}$ such that

$$F(1_X) = 1_{F(X)}$$

and

$$F(f \circ g) = F(f) \circ F(g)$$



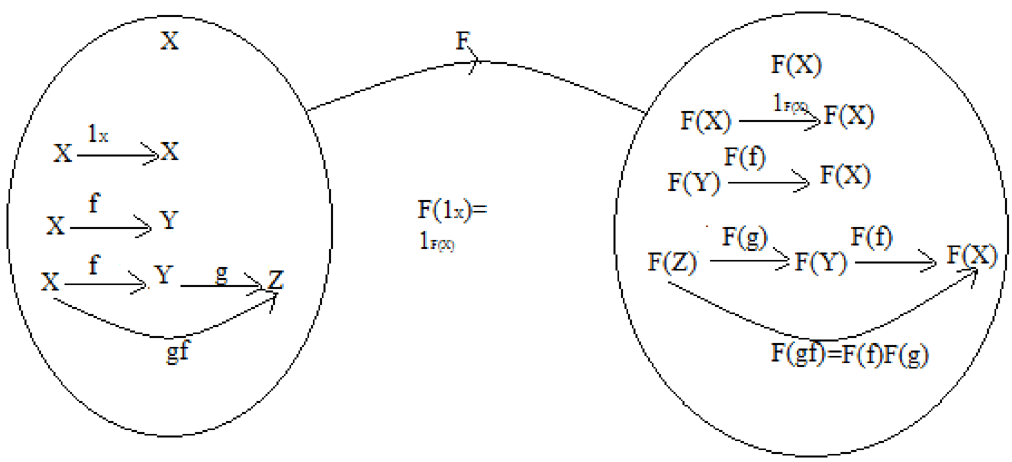
(covariant)

A *contravariant functor* F from a category $\mathcal{C}$ to another category $\mathcal{D}$ assigns to each object X of $\mathcal{C}$ to an object $F(X)$ in $\mathcal{D}$ and each morphism $f \in \mathcal{C}(X, Y)$ in $\mathcal{C}$ to a morphism

$$F(f) \in \mathcal{D}(F(Y), F(X)) \text{ in } \mathcal{D}$$

such that

$$F(1_X) = 1_{F(X)} \text{ and } F(f \circ g) = F(g) \circ F(f).$$

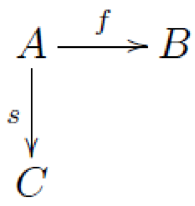


(contravarient)

1.6 Push-out

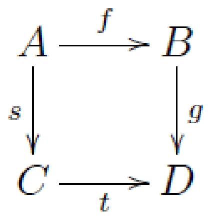
Push-out is an important universal concept in algebraic topology.

1.6.1 Definition. [6] A diagram consisting of two morphisms $f: A \to B$ and $s: A \to C$

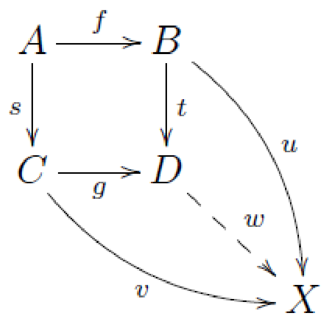


with a common domain is said to be a *push-out* diagram if and only if

(a) the diagram can be completed to a commutative diagram.



(b) for any commutative diagram, i.e., $uf = vs$ there exist a unique morphism $w: D \to X$ such that



$wt = u$ and $wg = v$

The dual of push-out is pull-back (we have not used this concept in our study).

Chapter-2

HOMOTOPY THEORY

The notion of homotopy has fundamental role in algebraic topology. Precisely speaking homotopy theory provides us a machinery to convert topological spaces into algebraic situation. In this chapter we define the concepts of homotopy and homotopy equivalences.

2.1 Definition. Let X and Y be two topological spaces and $I = [0,1] \subset \mathbb{R}$. A *homotopy* is a continuous function $F: X \times I \to Y$. Let $f, g: X \to Y$ be continuous. f is said to be *homotopic* to g ($f \simeq g$) if and only if there exist a homotopy

$$F: X \times I \to Y$$

such that

$$F(x,0) = f(x)$$

and

$$F(x,1) = g(x)$$

for all $x \in X$.

2.1.1 Lemma. *Let (X, Y) be the set of all continuous function from the topological space X to the topological space Y. Then $\simeq$ is an equivalence relation.*

Proof. (i) Reflexive: Let $f \in \mathcal{C}(X,Y)$ be arbitrary. To show: $f \simeq f$. Define $F: X \times I \to Y$ by the rule $F(x,t) = f(x)$ for all $x \in X$ and for all $t \in I$. F is continuous since f is continuous. $F(x,0) = f(x)$ and $F(x,1) = f(x)$ for all $x \in X$. Hence $F: f \simeq f$.

(ii) Symmetry: Let $f, g \in \mathcal{C}(X,Y)$ and $f \simeq g$. To show: $g \simeq f$. Given $g \simeq f: X \to Y$ implies that there exists a homotopy

$$F: X \times I \to Y$$

such that

$$F(x,0) = f(x)$$

and

$$F(x,1) = f(x)$$

for all $x \in X$.

Define

$$G: X \times I \to Y$$

by the rule

$$G(x,t) = F(x, 1-t)$$

for all $(x,t) \in X \times I$. G is continuous since F is continuous.

$$G(x,0) = F(x,1) = g(x)$$

and

$$G(x,1) = F(x,0) = f(x)$$

for all $x \in X$.

Hence $G: g \simeq f$.

(iii) Transitive: Let $f, g, h \in \mathcal{C}(X,Y)$ such that $f \simeq g$ and $g \simeq h$. To show: $f \simeq h$. Given $f \simeq g: X \to Y$ implies that there exist a homotopy

$$F: X \times I \to Y$$

such that

$$F(x,0) = f(x)$$

and

$$F(x,1) = f(x)$$

for all $x \in X$.

Again $g \simeq h: X \to Y$ implies that there exist a homotopy

$$G: X \times I \to Y$$

such that

$$G(x, 0) = g(x)$$

and

$$G(x, 1) = h(x)$$

for all $x \in X$.

Define $H: X \times I \to Y$ by the rule

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \leq t \leq 1/2 \\ G(x, 2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

for all $x \in X$.

$$H(x, 1/2) = \begin{cases} F(x, 1) = g(x) \\ G(x, 0) = h(x) \end{cases}$$

for all $x \in X$. Thus H is continuous by Pasting lemma.

$$H(x, 0) = F(x, 0) = f(x)$$

and

$$H(x, 1) = G(x, 1) = h(x)$$

for all $x \in X$. So $H: f \simeq h$.

Hence $\simeq$ is an equivalence relation. ■

2.1 Homotopy Equivalence

Those spaces which can be deformed continuously into one another or can be transformed into one another by bending, shrinking and expanding operations are homotopically equivalent spaces.

2.2.1 Definition. Two topological spaces X and Y are said to be *homotopically equivalent* if and only if there exist maps $f: X \to Y$ and $g: Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$. If X and Y are homotopically equivalent then it is denoted as $X \simeq Y$.

Here f and g are called homotopy equivalences.

2.2.2 Examples. We present some examples of homotopically equivalent spaces.

(a) Let

$$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

and

$$S^{n-1} = \{n \in \mathbb{R}^n : \|x\| = 1\}.$$

Then

$$D^n - \{0\} \simeq S^{n-1}.$$

(b) For any topological space

$$X \times I \simeq X.$$

(c) Let

$$N = (1,0,0,\cdots,0) \in \mathbb{R}^{n+1}$$

and

$$S = (-1,0,0,\cdots,0) \in \mathbb{R}^{n+1}.$$

Then

$$S - \{N, S\} \simeq S^{n-1}.$$

2.3 Contractible Spaces

A contractible space is a space which can be continuously shrunk to a point.

2.3.1 Definition. [6] A space is *contractible* if it is homotopy equivalent to a one point space. If Y is a point space say $\{*\}$ then X is said to be contractible. Contractible spaces can be characterized in terms of constant maps. Let $x_0 \in X$ and $C_{x_0}: X \to X$ be defined by $C_{x_0}(x) = x_0$ for all $x \in X$, i.e., C_{x_0} is the constant map at $x_0 \in X$.

2.3.2 Lemma. *A space X is contractible iff the identity map 1_X is homotopic to C_{x_0} for some $x_0 \in X$.*

Proof. Assume that X is contractible. So X is homotopically equivalent to a singleton space, i.e., $X \simeq \{*\}$. Hence there exists maps $f: X \to \{*\}$ and $g: \{*\} \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_{\{*\}}$. For any $x \in X$,

$$gf(x) = g(*) = x_0 \text{ for some } x_0 \in X = C_{x_0}(x)$$

implying $gf = C_{x_0}$. Here $gf \simeq 1_X$.

Hence $1_X \simeq C_{x_0}$ (by transitive property) for some $x_0 \in X$.

Conversely, assume that $C_{x_0} \simeq 1_X$ for some $x_0 \in X$. To show: X is contractible, i.e., X is homotopically equivalent to a singleton space. It is enough to show $X \simeq \{x_0\}$.

Define $f: X \to \{x_0\}$ by the rule

$$f(x) = x_0, \text{ for all } x \in X$$

f is continuous being a constant function.

Define $g: \{x_0\} \to X$ by the rule $g(x_0) = x_0$. g is continuous being inclusion map.

$$gf: X \to \{x_0\} \to X$$

and

$$fg: \{x_0\} \to X \to \{x_0\}.$$

$$gf(x) = g(x_0) = x_0 = C_{x_0}(x)$$

and

$$fg(x_0) = f(x_0) = x_0 = 1_{\{x_0\}}(x_0)$$

for all $x \in X$.

Hence

$$gf = C_{x_0}$$

and

$$fg = 1_{\{x_0\}}.$$

But $C_{x_0} \simeq 1_X$ (given). Here $gf \simeq 1_X$ and together with $fg \simeq 1_{\{x_0\}}$ gives $X \simeq \{x_0\}$.

■

2.4 Topological Pair

In algebraic topology, topological pair is used to derive homotopy and homotopy exact sequences.

2.4.1 Definition. [5] Let X be a topological space and A be a subspace of X, i.e., $A \subset X$ and the inclusion function $i: A \hookrightarrow X$ defined by $i(a) = a$, for all $a \in A$ is continuous. (X, A) is called a *topological pair*.

Similarly, (Y, B) is another topological pair i.e., $B \subset Y$ and Y is a topological space. A map from

$$f: (X, A) \to (Y, B),$$

(X, A) to (Y, B) means

$$f: X \to Y \text{ and } f|A: A \to B,$$

i.e., $f(A) \subset B$, in other words, $f(a) \in B$.

f and g are said to be *homotopic* if and only if there exist a homotopy

$F: (X \times I, A \times I) \to (Y, B)$

such that

$$F(x, 0) = f(x)$$

and

$$F(x, 1) = g(x)$$

for all $x \in X$.

$F(A \times I) \subset B$ i.e., $F(a, t) \in B \quad \forall a \in A, \forall t \in I$ and written as $F: f \simeq g$. F is called the homotopy.

2.4.2 Note. We will use the following notations.

$[X, A; Y, B]$ = set of all homotopy classes in the set $(X, A; Y, B)$.
= set of all homotopy classes of maps from (X, A) to (Y, B).

$[X, x_0; Y, y_0]$ = set of all homotopy classes of maps from the based space (X, x_0) to the based space (Y, y_0) that maps x_0 to y_0.

$[S^n, s_0; Y, y_0]$ = set of homotopy classes of based maps from (S^n, s_0) to the based space (Y, y_0).
$= \pi_n(Y, y_0)$.

$\pi_n(Y, y_0)$ is called be called as n^{th} homotopy group of (Y, y_0).

2.4.3 Note. The following are some important results:

(a) $[S^1, s_0; Y, Y, y_0] \cong [I, \dot{I}; Y, y_0]$.

(b) $[I, \dot{I}; Y, y_0]$ is a group for any based topological space (Y, y_0).

(c) Let Y be any path connected. For any $y_0, y_1 \in Y$, $\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$.

(d) If $(X, x_0) \simeq (Y, y_0)$ then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

2.5 Lifting

In homotopy theory the lifting property is a condition on a continuous function from a topological space X to another one, Y. It supports the picture of X "above" Y where it allows a homotopy taking place in Y to be moved "upstairs" to X.

2.5.1 Definition. [5] Let (X, x_0) be a pointed space. A map

$$f: (X, x_0) \to (S^1, s_0)$$

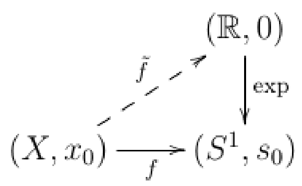
is said to have a lifting w.r.t the map

$$\exp : (\mathbb{R}, 0) \to (S^1, s_0)$$

if and only if there exist a map

$$\tilde{f}: (X, x_0) \to (\mathbb{R}, 0)$$

such that $\exp \circ \tilde{f} = f$.



2.5.2 Lemma. *Any map* $f: (I, 0) \to (S^1, s_0)$ *has a unique lifting* $\tilde{f}: (I, 0) \to (\mathbb{R}, 0)$ *w.r.t* $\exp: (\mathbb{R}, 0) \to (S^1, s_0)$.

2.5.3 Lemma. *The map* $F: (I \times I, (0,0))$ *has a unique lifting* $\widetilde{F}: (I \times I, (0,0)) \to (\mathbb{R}, 0)$ *w.r.t the map* $\exp: (\mathbb{R}, 0) \to (S^1, s_0)$.

Chapter 3

CW-COMPLEXES

In this chapter we recall *CW*-complexes and present some examples. We need the following concept from general topology.

3.1 Quotient Topology

The motivation for quotient topology comes from geometry, where "cut and paste" technique is used to construct geometrical objects as surfaces. The surfaces such as torus can be constructed by taking a rectangle and "pasting" its edges together appropriately.

3.1.1 Definition. [5] Let X be a topological space and let Y be a partition of X into disjoint subsets whose union is X. Let $s: X \to Y^*$ be the surjective map that carries each point of X to the element of Y containing it. In the quotient topology induced by s, the space Y is called *quotient space.*

3.1.2 Definition. [5] If X is a space and A is a space and if $s : X \to A$ is a surjective map, then there exist exactly one topology $\mathscr{T}$ on A relative to which s is a quotient map; it is called *quotient topology* induced by $s.$

3.2 Adjunction Space

An adjunction space (or attaching space) is a common construction in topology where one topological space is attached or "glued" onto another.

3.2.1 Definition. [2] Let X and Y be two topological spaces with A a subspace of Y. Let $f : A \to X$ be a continuous map (called the attaching map). An adjunction space $X \cup_f Y$

is formed by taking the disjoint union of X and Y and identifying X with $f(x)$ for all x in A.

$$X \cup_f Y = (X \sqcup Y)/(f(A) \sim A).$$

Here Y is being glued onto X via the map f.

3.2.2 Examples

(a) Y is a closed n-ball, and let A be the boundary of the ball, the $(n-1)$ sphere. Inductively attaching cells along their spherical boundaries to this space results in an example of a CW-complex.

(b) If A is one point space, then the adjunction is the wedge sum of X and Y.

(c) If X is one point space, then the adjunction is the quotient $Y|A$.

3.3 *CW*-complex

A concept of CW-complex was introduced by J.H.C. Whitehead to meet the needs of homotopy theory. This class of these spaces has some much better categorical properties and still retains a combinatorial nature that allows for computation.

3.3.1 Definition. A CW-complex X consists of

1. X is a Haousdorff topological space.
2. X has a structure of a cell-complex.

(a) A cell complex on X is collection $\{e_\alpha^n \subset X\colon \alpha \in J_n,\ J_n$ is an indexing set of non negative integers$\}$

(b) $X^n = n$-skeleton of X

$= $ collection of all 0-cells, 1-cells, 2-cells,…,$(n-1)$cells and n cells.

(c) $|X^n| = \bigcup_{\alpha\in J_0} e_\alpha^0 \ \cup \bigcup_{\beta\in J_1} e_\beta^1 \cup \cdots \cup \ \bigcup_{\delta\in J_n} e_\delta^n \subset X$

in general, for $0 \le r < \infty$,

$$\bigcup_{\alpha\in J_\alpha} e_\alpha^r \subset X$$

(d) (i) $X = \bigcup_{\alpha\in J_\alpha} e_\alpha^r$, for $0 \le r < \infty$

(ii) $\dot{e}_\alpha^n = e_\alpha^n \cap |X^{n-1}|$

$= $ Boundary of e_α^n

(iii) $\tilde{e}_\alpha^n = e_\alpha^n - \dot{e}_\alpha^n$

$= \text{interior of } e_\alpha^n$

(iv) $\tilde{e}_\alpha^n \cap \tilde{e}_\alpha^m \neq \phi, \ \Rightarrow n = m, \ \alpha = \beta$

(v) $X = \bigcup_{\alpha \in J_r} \tilde{e}_\alpha^r$, for $0 \leq r < \infty$

(vi) The cell e^n compact and hence closed in X.

(vii) $X^0 \subset X^1 \subset X^2 \subset \cdots X^n \subset X$

X^n is obtained from X^{n-1} by attaching n-cells by the characteristic map $f: (D^n, \ S^{n-1}) \to (X^n, \ X^{n-1})$.

C in CW-complex is for closure finite property, which states, for each cell e_α^n its closure $\bar{e}_\alpha^n$ meets (intersects) only at finite number of cells.

W in CW-complex is for the weak topology, where a set B is open in X iff $B \cap e_\alpha^n$ is open in e_α^n for each n, α.

3.3.2 Examples. [4] (a) Spheres as a CW-complexes: Points on the n-sphere $S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ have coordinates of the form (x, u). Let $D_\pm^n$ be the images of the embeddings

$$D^n \to S^n : x \to \left(x, \pm\sqrt{1 - |x|^2}\right).$$

Then

$$S^n = S^{n-1} \cup D_+^n \cup D_-^n = S^{n-1} \cup_{id \coprod id} (D^n \coprod D^n)$$

is obtained from S^{n-1} by attaching two n-cells. The infinite spheres S^∞ is an infinite dimensional CW-complex.

$$S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n \subset \cdots S^\infty = \bigcup_{n=0}^{\infty} S^n$$

with two cells in each dimension. A subspace A of S^∞ is closed iff $A \cap S^n$ is closed in S^n for all n.

(b) Quaternion projective spaces as CW space: Quaternion projective n-space HP^n can be obtained from HP^{n-1} by attaching one $4n$-cell along the canonical quotient map

$$p_{n-1} : S^{4n-1} \to HP^{n-1}.$$

Thus

$$HP^n = {S^{4n-1}}/{S^3}$$

is a CW-complex with one cell in each of the dimension $0,4,\dots,4n$. In particular, HP^0 is a point and

$$HP^1 = HP^0 \cup D^4 = S^4$$

is a sphere.

3.3.3 Definition. [4] If K is a cell complex on X and $L \subset K$, then L is called *a subcomplex* of K if and only if $e_\alpha^n \in L$, which implies every face of e_α^n is in L.

If L is a sub complex, then L is a cell complex on $|L|$, and if K is a CW-complex on X then L is a CW-complex on $|L|$.

3.3.4 Lemma. [4] *Any compact subspace of a CW-complex X is contained in a skeleton.*

Proof: Let X be a CW-complex and C be a compact subspace of X. Choose a point t_n in $e \cap (X^n - X^{n-1})$ for all n where this intersection is non-empty.

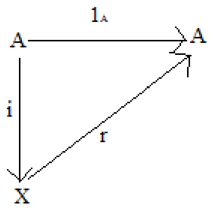
Let $T = \{t_n\}$ be the space of these points. For all n, $T \cap X^n$ is a finite and hence closed in X since points are closed in X. Thus T is closed since X has the coherent topology, (any subspace T is closed). As a closed subspace of the compact space C, T is compact. Thus T is compact and discrete. Hence T is finite. ■

Chapter 4

HOMOTOPY THEORY FOR *CW*-COMPLEXES

4.1 Retraction

Let X be a topological spaces and A a subspace of X and $i : A \hookrightarrow X$ be the inclusion map.



4.1.1 Definition. [4] A is called a *retract* of X if there exist a map $r: X \to A$ such that $ri = 1_A$.

4.1.2 Definition. [4] A is called a *deformation retract* of X if there exist a map $r: X \to A$ such that

$$ri = 1_A \text{ and } ir \simeq 1_X \text{ rel } A.$$

4.2 Mapping cylinder

The mapping cylinder of a continuous map is a fruitful notion. It was introduced by J. H. C. Whitehead in 1939.

4.2.1 Definition.[2] For a map $f: X \to Y$, the *mapping cylinder* M_f is the quotient space of the disjoint union $(X \times I)\coprod Y$ obtained by identifying each $(x.1) \in X \times I$ with $f(x) \in Y$.

A mapping cylinder M_f deformation retracts to the subspace Y by sliding each point (x,t) along the segment $\{x\} \times I \subset M_f$ to the end point $f(x) \in Y$.

The cylinder of f,

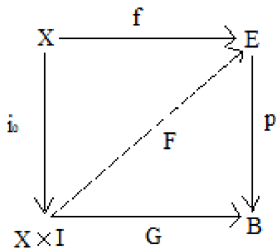
$$M_f = \frac{(X \times I) \coprod Y}{(x,1) \sim f(x)}$$

is obtained by gluing one end of cylinder on X onto Y by means of f.

4.3 Fibration

Fibration is an important concept in algebraic topology. The subsequent chapters deal with many applications of fibrations.

4.3.1 Definition. [3] A map $p: E \to B$ is said to have a homotopy lifting property (HLP) w.r.t a space X if for each map $f: X \to E$ and homotopy $G: X \times I \to B$ of $p \circ f$ there is a homotopy $F: X \times I \to E$ with $f = F_0$ and $p \circ f = G$ (F is said to be a lifting of G)



where $i_0(x) = (x, 0)$, $x \in X$.

p is called a *fibration* it is has the HLP for all spaces X, and a weak fibration if it has the HLP for all disk $D^n, n \geq 0$.

If $b_0 \in B$ is the based point, then the space $F = p^{-1}(b_0)$ is called the fibre of p.

4.3.2 Theorem. *If (X,A) is a relative* CW-complex *then the inclusion* $i: A \to X$ *is a cofibration.*

Proof: For each n, the space $(X,A)^n$ is obtained from $(X,A)^{n-1}$ by attaching n-cells, $(X,A)^{n-1} \subset (X,A)^n$ is a co-fibration.

Given $f: X \to Y$ and $G: A \times I \to Y$ a homotopy of $f|A$, we can construct $F^n: (X,A)^n \times I \to Y$ satisfying

(i) $F^{-1} = G$

(ii) $F_0^n = f|(X,A)^n$

(iii) $F^n|(X,A)^{n-1} \times I = F^{n-1}$

We then define $F: X \times I \to Y$ by the rule

$$F(x,t) = F^n(x,t) \text{ if } x \in (X,A)^n \text{ for all } t \in I.$$

F is well defined because of (iii) and continuous because $F|e_\alpha^n \times I = F^n|e_\alpha^n \times I$. By (ii)

$F_0 = f$. And $F|A \times I = G$ by (i). ■

4.3.3 Theorem. *If $A \to X$ is a cofibration and A is contractible then the projection $p: (X,A) \to (X|A,*)$ is a homotopy equivalence.*

Proof: Let $H: A \times I \to A$ be a contracting homotopy i.e., $H_0 = 1_A$, $H_1 = x_0$ and $H(x_0,t) = x_0$ for all $t \in I$. Since $i: A \to X$ is a cofibration we can extend $i \circ H$ to a homotopy $k: X \times I \to X$ with $k_0 = 1_X$ and $k|A \times I = i \circ H$. Then $k_1(a) = H_1(a) = x_0$ for all $a \in A$. So k_1 induces a map

$$k: (X|A,*) \to (X,x_0)$$

such that $k \circ p = k_1$.

Then k is a homotopy and $1_X \simeq k \circ p$.

Since $p \circ k(a,t) =*$ for all $a \in A$ and $t \in I$, so $p \circ k$ induces a homotopy

$$\overline{k}: X|A \times I \to X \times A$$

such that

$$\overline{k} \circ (p \times 1) = p \circ k$$

Thus for every

$$x \in X, \overline{k}_0\big(p(x)\big) = \overline{k}(p(x),0) = p \circ k(x,0) = p(x)$$

And

$$\overline{k}_1\big(p(x)\big) = \overline{k}(p(x),1) = p \circ k(x,1) = p \circ k_1(x) = p \circ k(p(x))$$

p is surjective, so

$$\bar{k}_0 = 1_{X|A}, \bar{k}_1 = p \circ k.$$

Thus $p \circ k \simeq 1_{X|A}$. Hence k is a homotopy inverse for p.

■

Chapter 5

WHITEHEAD THEOREM

In homotopy theory , the Whitehead theorem was proved by J. H. C. Whitehead in two landmark papers published in 1949 and provides a justification for working with the CW complex concept that he introduced there.

5.1 Cellular map

The notion of cellular map is widely used in proving the Whitehead theorem.

5.1.1 Definition. A map $f: X \to Y$ where, X and Y are CW complexes, satisfying $f(X^n) \subset Y^n$ for all n is called a *cellular map*. Here X^n and Y^n are the n-skeletons.

An exact sequence can either finite or infinite, of objects and morphisms between them such that the image of one morphism equals the kernel of the next.

5.1.2 Definition. [2] A sequence of homomorphisms

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is said to be an *exact sequence* if $ker\alpha_n = Im\alpha_{n+1}$ for each n. The inclusions $Im\alpha_{n+1} \subset ker\alpha_n$ are equivalent to $\alpha_n\alpha_{n+1} = 0$.

5.1.3 Compression Lemma.[2] *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq 0$. For each n, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f: (X, A) \to (Y, B)$ is homotopic $rel\ A$ to a map $X \to B$. When $n = 0$ (Y, B) is 0-connected.*

5.2 Homotopy Extension Property (HEP)

A topological pair (X, A) has the Homotopy extension property (HEP) if for any partial homotopy $A \times I \to Y$ of a map $X \to Y$ into any space Y can be extended to a (full) homotopy of the map.

5.2.1 Theorem. *A pair* (X, A) *has the homotopy extension property if and only if* $X \times \{0\} \cup A \times I$ *is a retract of* $X \times I$.

Proof: For one implication, the homotopy extension property for (X, A) implies that the Identity

$$X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$$

extends to a map

$$X \times I \to X \times \{0\} \cup A \times I.$$

So $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

The converse is easy when A is closed in X. Then any two maps $X \times \{0\} \to Y$ and $A \times I \to Y$ that agree on map $A \times \{0\}$ combine to give a map $X \times \{0\} \cup A \times I \to Y$ which is continuous since it is continuous on the closed sets $X \times \{0\}$ and $A \times I$. By composing this map $X \times \{0\} \cup A \times I \to Y$ with a retraction $X \times I \to X \times \{0\} \cup A \times I$ we get an extension $X \times I \to Y$ so (X, A) has the homotopy extension property.

If $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ and X is Hausdorff, then A must in fact be closed in X. For if $r: X \times I \to X \times I$ is a retraction onto $X \times \{0\} \cup A \times I$, then the image of r is the set of points $z \in X \times I$ with$r(z) = z$, a closed set if X is Hausdorff. So $X \times \{0\} \cup A \times I$ is closed in $X \times I$ and hence A is closed in X. ■

5.2.2 Example. A simple example of a pair (X, A) with A closed for which the homotopy extension property fails is the pair (I, A) where

$$A = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}.$$

It can be shown that there is no retraction

$$I \times I \to I \times \{0\} \cup A \times I$$

which is continuous.

5.3 Whitehead's Theorem.

This theorem was given by J. H. C. Whitehead in 1939.

5.3.1 Basic Construction

(a) Let I^n be n dimensional unit cube, the product of n copies of the interval $[0,1]$ and ∂I^n be boundary of I^n, is the subspace consisting of points with at least one coordinate equal to 0 or 1.

For a space X with base point $x_0 \in X$, define $\pi_n(X, x_0)$, to be the set of homotopy classes of maps $f: (I^n, \partial I^n) \to (X, x_0)$, where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all $t \in I$.

(b) A map $\varphi : (X, A, x_0) \to (Y, B, y_0)$ induces maps $\varphi_*: \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ which are homomorphisms for $n \geq 2$ and have properties

$$(\varphi\psi)_* = \varphi_*\psi_*, \quad 1_* = 1$$

and

$$\varphi_* = \psi_* \text{ if } \varphi \simeq \psi: (X, A, x_0) \to (Y, B, y_0).$$

Here (X, A, x_0) are relative homotopy groups for a pair (X, A) with a base point $x_0 \in A$. These relative groups $\pi_n(X, A, x_0)$ fit into a long exact sequence.

$$\cdots \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0) \cdots$$

Here i and j are the inclusions $i: (A, x_0) \hookrightarrow (X, x_0)$ and $j: (X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, x_0)$ to I^{n-1} or by restricting maps $(D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$ to S^{n-1}. The map, called the boundary map, is a homomorphism when $n > 1$.

The above sequence is exact. The reduced mapping cylinder M_f of f is the space obtained from

$$(I \times X | I \times \{x_0\}) \vee Y$$

by identifying

$$[1, x] \in I \times X | I \times \{x_0\}$$

with $f(x) \in Y$ for all $x \in X$.

$$q: (I \times X | I \times \{x_0\}) \vee Y \to \widetilde{M}_f$$

is the projection. Denote by $[t, x]$ the image $q([t, x])$ for all $[t, x] \in (I \times X | I \times \{x_0\})$ and $[y]$ the image $q(y)$ for all $y \in Y$.

Define

$$j: (Y, y_0) \to (M_f, *)$$

by

$$j(y) = [y]$$

and

$$r: (M_f, *) \to (Y, y_0)$$

by

$$r([y]) = y, r[s, x] = f(x);$$

we have

$$r \circ j(y) = r(j[y]) = r[y] = y = 1_Y(y) \quad \cdots (1)$$

$$j \circ r[y] = j(r[y]) = j(y) = [y] = 1_{M_f}[y] \quad \cdots (2)$$

From (1) and (2) $rj = 1_Y(y)$ and $jr = 1_{M_f}[y]$, which implies $M_f \simeq Y$. i.e., r and j are homotopy equivalences.

5.3.2 Whitehead's Theorem. [2] *If a map $f: X \to Y$ between connected CW-complexes induces isomorphism $f_*: \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence.*

In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X is a deformation retract of Y.

Proof: In the special case that f is the inclusion of a subcomplex, consider the long exact sequence of homotopy groups for the pair (Y, X). Since f induces isomorphisms on all homotopy groups, the relative groups $\pi_n(Y, X)$ are zero. Applying the compression lemma to the identity map $(Y, X) \to (X, Y)$ then yields a deformation retraction of Y onto X.

The general case can be proved using mapping cylinders. Recall that the mapping cylinder M_f of a map $f: X \to Y$ is the quotient space of the disjoint union of $X \times I$ and Y under the identifications $(X, 1) \sim f(X)$. Thus M_f contains both $X = X\{0\}$ and Y as subspaces, and M_f deformation retracts onto. The map f becomes the composition of the inclusion $X \hookrightarrow M_f$ with the retraction $M_f \to Y$. Since this retraction is a homotopy equivalence, it sufficient to show that M_f deformation retracts onto X if f induces isomorphisms on homotopy groups, or also if the relative groups $\pi_n(M_f, X)$ are all zero.

If the map f is cellular, taking the n skeleton of X to the n skeleton of Y for all n, then (M_f, X) is a CW pair. Then there is nothing is proof.

If f is not cellular, then f is homotopic to a cellular map, or using compression lemma, we can obtain a homotopy $rel\, X$ of the inclusion map $(X \cup Y,\ X) \hookrightarrow (M_f, X)$ to a map into $X.(M_f, X \cup Y)$. Obviously this satisfies the homotopy extension property. This homotopy extends to a homotopy from the identity map of M_f to a map $g: M_f \to M_f$ taking $X \cup Y$ into X.

Again applying the compression lemma to the composition

$$\big((X \times I)\coprod Y,\ (X \times \partial I)\coprod Y\big) \to \big(M_f, X \cup Y\big) \to \big(M_f, X\big)$$

we get the construction of a deformation retraction of M_f onto X.

■

REFERENCE

[1] Aguilar M., Gitler S., Carlos P, *Algebraic Topology from a Homotopical viewpoint,* Springer-Verlag, (2002).

[2] Hatcher A, *Algebraic Topology* (2001).

[3] Massey W. S., *A basic course in algebraic topology*, Springer-Verlag New York Inc., (1991).

[4] Moller J. M., *Homotopy theory for beginners.*

[5] Munkers J.R, *Topology*, published by Pearson Education, Inc., publishing as Pearson Prentice Hall, (2000).

[6] Switzer R. M, *Algebraic Topology Homology and Homotopy,* Springer-Verlag Berlin Heidelberg, (2002).

[7] Whitehead G. W*., Elements of Homotopy Theory* 1978 by Springer –Verlag New York Inc.,(1978).

[8] Whitehead J. H. C., *Combinatorial homotopy. I.*, Bull. Amer. Math. Soc., 55 (1949), 213–245.

[9] Whitehead J. H. C., *Combinatorial homotopy. II.*, Bull. Amer. Math. Soc., 55 (1949), 453 - 496.

Archana Tiwari

M.Sc. Student in Mathematics
Roll No. 413MA2077
National Institute of Technology, Rourkela
Rourkela, 769 008(India)