Improvements in I.F.S. Formulation for its Use in Still Image Coding

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ABSTRACT — **In this paper, we report on a study on still image coding using I.F.S. In particular, we focus on two aspects of the algorithm. The first one concerns the definition of the contractive constraint during the coding stage, in order to ensure the convergence of the iterative decoding process. The second one concerns the choice of the initial image for starting the decoding stage.**

1 Introduction

The I.F.S. (Iterated Functions Systems) technique was invented by the mathematician J. Hutchinson in early eighties [1]. It defines iterative processes which converge towards a fixed point independent of their starting point. This fixed point is called the attractor of the I.F.S.. The notion of I.F.S. is part of a more general theory developed by the mathematician B. Mandelbrot known as fractal theory. The I.F.S. technique is not covered in this paper, we only deal with image compression based on iterated transforms.

The basic theorem called the "Collage Theorem" and the algorithm due to A. Jacquin are reviewed in section 2.

In section 3, we focus on the algorithm convergence and discuss the link between geometric and photometric transformations playing a role in the definition of the contractivity constraint. This aspect is important in the estimation of the optimum I.F.S. code of an image.

In section 4, the optimal choice for the initial image in the decoding process is considered in order to increase decoding speed.

2 A review of fractal image coding

2.1 Notations

- x, y designate two generic images.
- x_c designates the image to be encoded.
- x_0 designates the initial image of the iterative process.
- $\mathcal{W}$ designates the image transform.
- x_a designates the attractor of $\mathcal{W}$.
- d designates a metric defined on the image space.

2.2 Collage theorem

This theorem says the following [2, 3]:

$$\begin{aligned}
&\text{if } \exists \mathcal{W} \;/\; d(x_c, \mathcal{W}(x_c)) \le \epsilon \\
&\qquad \text{and } d(\mathcal{W}(x), \mathcal{W}(y)) \le \sigma . d(x,y) \\
&\qquad \text{where } 0 < \sigma < 1 \; (\mathcal{W} \text{ contractive}) \\
&\text{then } d(x_c, x_a) \le \frac{\epsilon}{1-\sigma} \\
&\qquad \text{with } x_a = \lim_{n\to\infty} \mathcal{W}^{0,n}(x_0) \\
&\qquad \text{and } \mathcal{W}^{0,n}(x_0) = \underbrace{\mathcal{W}(\mathcal{W}(\ldots(\mathcal{W}(x_0))\ldots))}_{nterms}
\end{aligned}$$

The proof is based by repeated application of triangular inequality.

2.3 Jacquin's algorithm

The basic algorithm for still image coding using I.F.S. was proposed by A. Jacquin, who introduced the idea of Local-IFS [4, 5]:
x_c is partitioned twice at two levels of resolution. For instance, this may be into squared-blocks of size B×B and 2B×2B (typically, B is fixed at 8). The former are called range blocks and the latter are called domain blocks. For each range block, the algorithm searches for the best matching domain block according to the local quadratic criterion.

$$\text{err}_k = \sum_{(i,j)\in \mathrm{R}_k} \{\mathcal{W}_k(D)(i,j) - R_k(i,j)\}^2 \qquad (1)$$

where

- $R_k(i,j)$ designates the grey-value at pixel (i,j) in the range block k.

- $\mathcal{W}_k(D)(i,j)$, designates the grey-value at pixel (i,j) in the transformed domain block associated with R_k.

Before the matching stage, domain blocks are transformed as follows

- sub-sampling by a factor two —in each direction—;
- geometric transformations —eight isometries are considered—;
- scale and shift of luminance value.
  Different strategies exist for computing these values. In this study, they are computed according to a minimum mean squared error criterion, as described in [6].

Finally, each area of the image —i.e. each range block— has an associated affine transformation

$$\mathcal{W}_k \begin{bmatrix} u \\ v \\ z \end{bmatrix} = \begin{bmatrix} a_k & b_k & 0 \\ c_k & d_k & 0 \\ 0 & 0 & s_k \end{bmatrix} . \begin{bmatrix} u \\ v \\ z \end{bmatrix} + \begin{bmatrix} e_k \\ f_k \\ o_k \end{bmatrix} \quad (2)$$

Where $a_k, b_k, c_k, d_k, e_k, f_k$ represent the geometric transformation and s_k, o_k the grey level transformation. u, v are the pixel coordinates and z the grey level.

For decoding image x_c from its I.F.S. code and image x_0, the algorithm proceeds as follows: Image x_0 is partitioned into set of square-blocks. Each area of the image is computed by taking the associated block in image x_0 and applying an associated contractive transformation defined during the coding stage. Then, image x_1 is obtained. The algorithm iterates this process to obtain x_2 from x_1 ..., until it reaches x_a. In practice, less than ten iterations are needed.

3 Contractivity constraint and error reconstruction

3.1 Contractivity constraint

According to the Collage theorem, the contraction condition $\sigma < 1$ on $\mathcal{W}$ is necessary and sufficient to provide the attractor existence x_a. In Jacquin's algorithm, the contractive condition appears locally at two different levels. One is geometric (i.e. subsampling equal to 0.5) and the other is photometric (i.e. each local scale luminance factors s_k less than 1).
These local contraction conditions are sufficient but not necessary. As a result, the reconstruction error is not optimal because the set of possibilities for $\mathcal{W}_k$ is more restricted than necessary.

In order to solve this problem, a global condition which links the two components of the contraction condition could be used. For this, we introduce a matrix form to represent the overall transformation $\mathcal{W}$ applied on the entire image. So, if each image is written as a column vector,

$$\mathcal{W}(x) = [A] . [x] + [b] \quad (3)$$

where

- Column vector b contains the brightness shifts o_k.
- Matrix A contains the contrast scaling factors s_k and zeros. The distribution of s_k among the zeros represents the geometric transformation. Note that matrix A is a sparse matrix (see figures 1 and 2 on next page). Furthermore, the possibilities of localization of s_k can be limited, for each range block, by restricting the research space of domain blocks (e.g.: the search of domain block can be limited in a neighbourhood of the range block).

Let σ_A be the spectral radius of matrix A (i.e. the largest eigenvalue of A) [7, 8] .
if

$$\sigma_A < 1 \quad (4)$$

then

$$\lim_{n \to \infty} [A]^n = [0] \, and \sum_{k=0}^{n} [A]^k = (I - A)^{-1} \quad (5)$$

therefore

$$x_a = \lim_{n \to \infty} ([A]^n . [x_0] + (\sum_{k=0}^{n} [A]^k) . [b]). \quad (6)$$

Note that this necessary and sufficient condition can be respected although several local scale luminance factors should be greater than one [9], or no geometric contraction applied [10]. The main problem of this criterion is that σ_A can only be computed *a posteriori*. Nevertheless, a way to overcome this problem can be proposed by indexing the obtained matrix A with Jacquin's algorithm A_1 in order to access, by the construction of a series A_k, to the matrix A_{opt} (i.e. : which respects the necessary and sufficient constraint). At each iteration k, one or severals range blocks are selected. The selection criterion is based on a large reconstruction error due to the restriction on the luminance scale factor to be less than one. They are then recomputed (see next subsection).

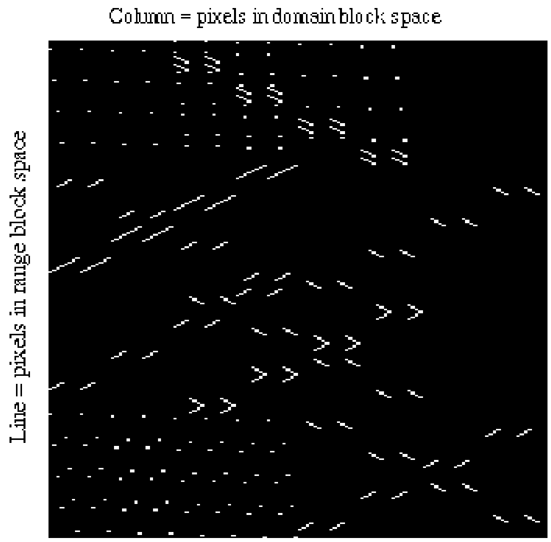

Figure 1: *The size of the considered image, in this example, is 16x16 in which 16 range blocks of size 4x4 and 9 domain blocks of size 8x8 are defined. Zeros are represented in black and scale values in white. Each isometry has an associated motif. This structure can be permuted without modification of the spectral radius of A (see figure 2).*

3.2 Error reconstruction

According to the Collage Theorem, and with the condition of contractivity defined in the previous subsection, the upper bound on the reconstruction error (i.e. the maximum of $d(x_c, x_a)$) depends on two parameters, ϵ and σ_A.

$$e_r \leq \frac{\epsilon}{1-\sigma_A}. \quad (7)$$

Yet, the basic algorithm which computes the transformation $\mathcal{W}$ is based only on the minimization of local errors err_k, and then ϵ (i.e. error obtained during the coding stage).
To design the iterative algorithm to determine the optimal matrix $[A_{opt}]$, we propose to take into account the duality between ϵ and $\frac{1}{1-\sigma_A}$ in order to minimize $\frac{\epsilon}{1-\sigma_A}$ during the coding stage.

Such an algorithm would consist in:

- Selecting one or several range blocks according to criterion defined in the previous paragraph, and computing new I.F.S. code for them without a constraint on grey scale value or without subsampling.
- Computing $\delta\epsilon = \epsilon_n - \epsilon_{n-1}$, the improvement of the coding stage error between two iteration denoted n and $n-1$. Note that $\delta\epsilon$ satisfies $\delta\epsilon \geq 0$.
- Computing $\delta\sigma_A = \sigma_{A_n} - \sigma_{A_{n-1}}$,
- if $(\delta\sigma_A \leq 0)$ or
$(\frac{1}{1-\sigma_{A_n}} - \frac{1}{1-\sigma_{A_{n-1}}} < |\delta\epsilon|$, with $\sigma_{A_n} < 1)$
then iteration is accepted,
else iteration is rejected.
- Selecting other local modifications to decrease ϵ_n if possible (step one) else stop iteration.

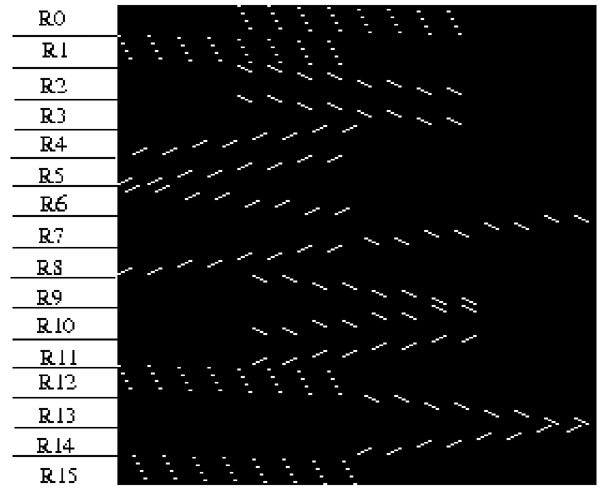

Figure 2: *This matrix is obtained from the previous one by line permutations. It is now composed of horizontal bands which corresponds to range block (from R_0 to R_{15})*
.

4 Initial image

According to the Collage Theorem, the attractor x_a is independent of x_0 so that the discussion about the choice of an initial image for the decoding stage could seem inappropriate. Actually, this conclusion is absolutely correct only if the number of interations is infinite. In pratice, this is not the case because algorithms try to minimize the number of iterations in order to minimize computing time. So only an approximate attractor is computed. The optimal initial image must lead to a better reconstruction image with a minimum of iterations. Furthermore, the requirement of this initial image must not decrease the compression ratio. To solve this problem, let the attractor approximation at iteration k be defined by

$$x_k = [A]^k x_0 + (\sum_{n=0}^{k-1} [A]^n)b. \quad (8)$$

The choice $x_0 = b$ is appropriate because if we consider now the attractor approximation at iteration $k+1$

$$x_{k+1} = [A]^{k+1}\, x_0 + [A]^k\, b + (\sum_{n=0}^{k-1} [A]^n) b, \qquad (9)$$

it is clear that only $[A]^k\, b + (\sum_{n=0}^{k-1} [A]^n) b$ contains useful information which can be identified in expression (9) of x_k, with $x_0 = b$. $[A]^{k+1}\, x_0$ represents an negligible term.
We notice that the choice $x_0 = b$ does not require the transmission of more information. Furthermore, in this case, it is then possible to reorganize the order of the summation operations included in the decoding stage.

5 Conclusion and Perspectives

5.1 Conclusion

In this paper, an optimal condition is defined for the contractivity constrainst in using I.F.S. for still image coding. This constraint, which is less restrictive than Jacquin's constraint, could lead to better results. Nevertheless, its use in practice is not immediate and we are continuing research in this direction.
The use of shift values as initial image for the decoding stage is an optimal choice because one iteration is gained and no additional information need be transmitted.

5.2 Perspectives

Coding using I.F.S. is based on geometric and photometric similarities in an image itself. This technique could be used, in addition to compression for some image processing, such as zoom: I.F.S. allows for possibility of definition or resolution manipulation on images. This is due to the property of invariance by change of scale included in fractals. Other applications could also be defined from I.F.S. code of an image such as grey scale to half-tone conversion.

6 Acknowlegments

This work has been partially supported by CANON, Research Centre France. The authors thank Dr. C. Dierieck for helpful discussions.

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