Herugolf and Makaro are NP-complete

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Abstract

Herugolf and Makaro are Nikoli's pencil puzzles. We study the computational complexity of Herugolf and Makaro puzzles. It is shown that deciding whether a given instance of each puzzle has a solution is NP-complete.

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1 Introduction

The Herugolf puzzle is played on a rectangular grid of cells (see Fig. 1(a)). Initially, there are circles (balls) and holes (H) on the grid, where an integer is in each circle. The purpose of the puzzle is to move (hit) all balls one or more times, and bring them to a cell with an H in the following rules [1]: (1) One ball must be brought to every hole H. (2) The movement of a ball is shown by an arrow, with the tip of the arrow in the cell where it stops. The arrows cannot cross other balls, holes, or lines of other arrows. (3) A ball moves across as many cells as the number in it in the first move, vertically or horizontally. The next move becomes one shorter; it decreases one by one. (4) The direction of movement may change after a move. When the next movement becomes 0, or the ball stops at an H, the ball cannot move any further. (5) A ball cannot leave the grid (OB), and cannot stop in a grey area (water hazard).

Figure 1(a) is the initial configuration of a Herugolf puzzle. In this figure, there are eight balls and eight holes in the 6×6 cells. From Figs. 1(b)–(f), the reader can understand the basic technique for finding a solution. (b) The bottom right ball ③ must be moved 3 cells to the left, since there is a hole H in the blue cell. Then the ball is moved 2 cells to the upper direction, since there is a water hazard in the bottom left cell. (c) There is exactly one ball which can be brought to the hole H in the red cell. On the other hand, there is exactly one hole which the ball in the blue cell can reach. (d) Balls ③ and ① are moved to holes H in the red and blue cells, respectively. (e) If the ball ② in the blue cell is brought to the hole H in the red cell, then one of the two balls ② in the green cells cannot reach any hole. (f) is a solution.

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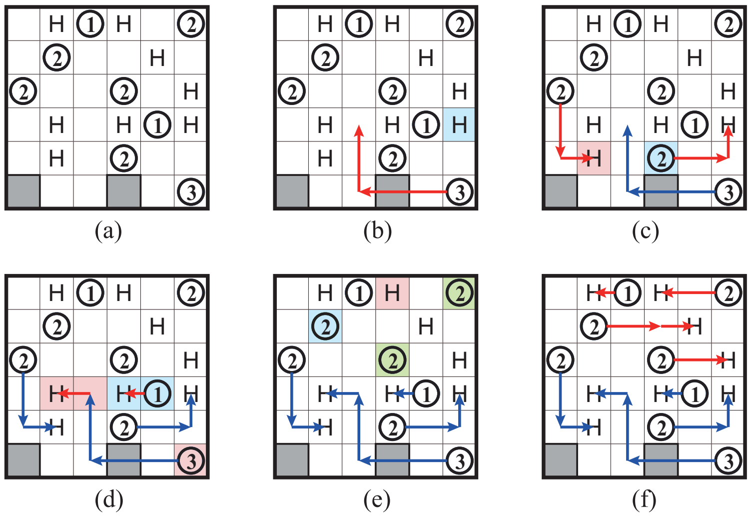

Figure 1 (a) Initial configuration of a Herugolf puzzle. (b)–(f) are the progress from the initial configuration to a solution.

The Makaro puzzle is also played on a rectangular grid of cells (see Fig. 2(a)). Initially, some of the cells are colored black and contain an arrow, and the remaining cells are divided into *rooms* surrounded by bold lines. The purpose of the puzzle is to fill in all white cells with numbers under the following rules [2]: (1) Each room contains all the natural numbers up to the number of cells in it, starting from 1 (see Fig. 2(h)). (2) Every arrow in black cells must point at the biggest number among the numbers in the adjacent cells. (3) A number must not be next to the same number in another room.

Figure 2(a) is the initial configuration of a Makaro puzzle. (b) Since an arrow exists between two yellow 2-cell rooms, numbers 2 and 1 are placed so that the arrow is between them and points at the number 2, which is bigger than the other number 1. (c) Four numbers are placed in the blue cells. (d) Since number 2 in a green cell is pointed by an arrow, another green cell must contain 1. (e) Since the yellow cell contains number 2, one of the two red cells must contain 2; therefore the green and blue cells must contain 3 and 4, respectively. (f) Since the two red cells can contain numbers less than 3, the number 3 is in the blue cell. (g) Two and seven numbers are placed in the yellow and red cells, respectively. (h) is a solution.

In this paper, we study the computational complexity of the decision version of the Herugolf and Makaro puzzles. The instance of the *Herugolf puzzle problem* is defined as a rectangular grid of cells, on which there are circled integers in $\{①, ②, ③, \ldots\}$ and holes H. The instance of the *Makaro puzzle problem* is a rectangular grid of cells, where some of the cells are colored black and contain an arrow, and the remaining cells are divided into rooms. The problem is to decide whether there is a solution to the instance. It is shown that the Herugolf and Makaro puzzle problems are NP-complete. It is clear that both problems belong to NP.

There has been a huge amount of literature on the computational complexities of games and puzzles. In 2009, a survey of games, puzzles, and their complexities was reported by Hearn and Demaine [9]. After the publication of this book, the following Nikoli's pencil puzzles were shown to be NP-complete: Fillmat [16], Hashiwokakero [4], Hebi, Satogaeri, and Suraromu [12], Kurodoko [13], LITS and Norinori [5], Numberlink [3], Pipe link [17], Shakashaka [7], Shikaku and Ripple Effect [15], Yajilin and Country Road [10], and Yosenabe [11].

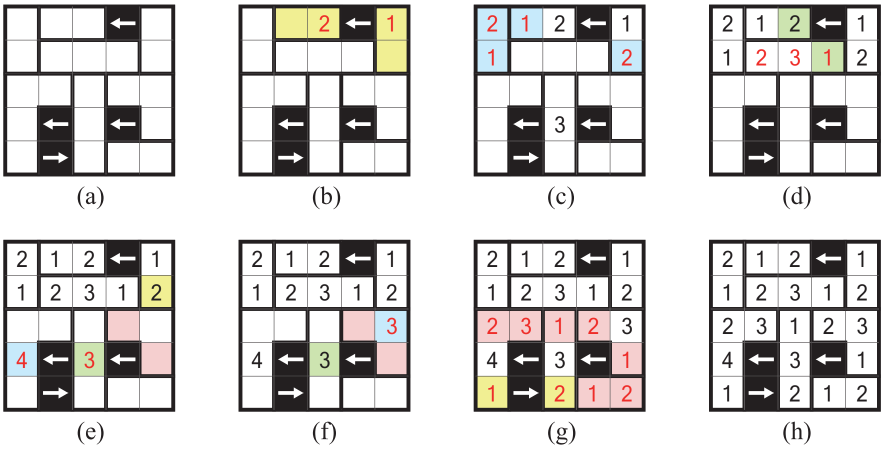

Figure 2 (a) Initial configuration of a Makaro puzzle. (b)–(h) are the progress from the initial configuration to a solution.

2 NP-completeness of Herugolf and Makaro

2.1 3SAT Problem

The definition of 3SAT is mostly from [8]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean *variables*. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U, then x and $\overline{x}$ are *literals* over U. The value of $\overline{x}$ is 1 (true) if and only if x is 0 (false). A *clause* over U is a set of literals over U, such as $\{\overline{x_1}, x_3, x_4\}$. It represents the disjunction of those literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment.

An instance of PLANAR 3SAT is a collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over U such that (i) $|c_j| \leq 3$ for each $c_j \in C$ and (ii) the bipartite graph $G = (V, E)$, where $V = U \cup C$ and E contains exactly those pairs $\{x, c\}$ such that either literal x or $\overline{x}$ belongs to the clause c, is planar.

The PLANAR 3SAT problem asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C. This problem is known to be NP-complete. For example, $U = \{x_1, x_2, x_3, x_4\}$, $C = \{c_1, c_2, c_3, c_4\}$, and $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, $c_3 = \{\overline{x_1}, \overline{x_3}, x_4\}$, $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$ provide an instance of PLANAR 3SAT. For this instance, the answer is "yes," since there is a truth assignment $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfying all clauses. It is known that PLANAR 3SAT is NP-complete even if each variable occurs exactly once positively and exactly twice negatively in C [6].

2.2 Transformation from an Instance of 3SAT to a Herugolf Puzzle

We present a polynomial-time transformation from an arbitrary instance C of 3SAT to a Herugolf puzzle such that C is satisfiable if and only if the puzzle has a solution.

Each variable $x_i \in \{x_1, x_2, \ldots, x_n\}$ is transformed into the variable gadget as illustrated in Fig. 3(a), which is composed of six holes and eight balls. (Each gadget constructed in this section uses no water hazards, although some of the cells in Fig. 3(a) are colored grey.) Note that the instances of 3SAT considered in this section have the restriction explained at the end of Sect. 2.1.

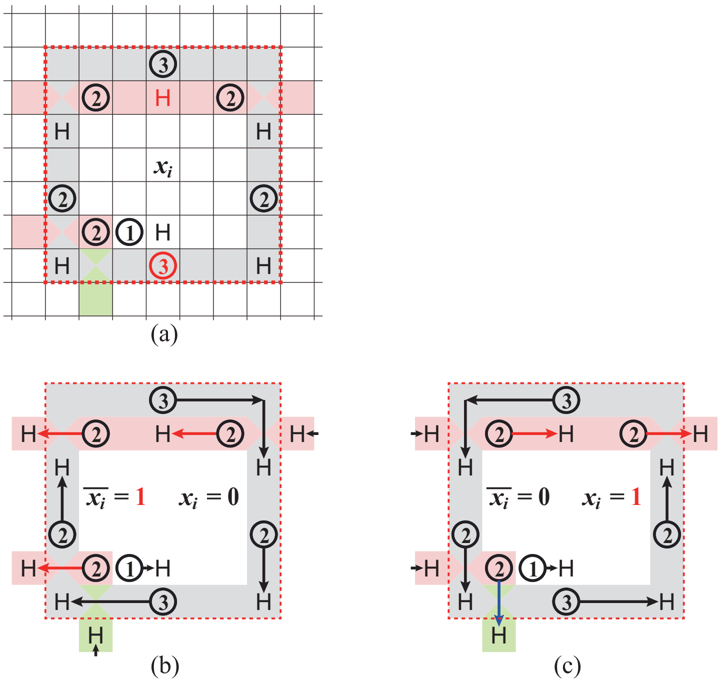

Figure 3 (a) Variable gadget of Herugolf transformed from x_i. (b) Assignment $\overline{x_i} = 1$. (c) Assignment $x_i = 1$.

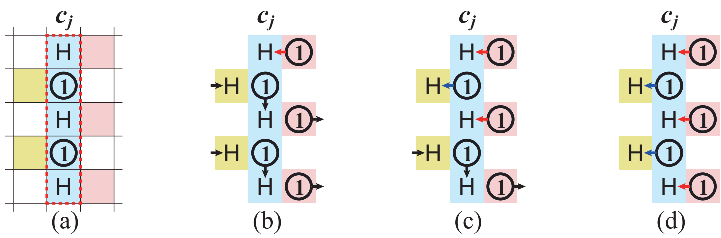

Figure 4 (a) Clause gadget of Herugolf transformed from c_j. (b)–(d) If at least one of the three balls ① is moved from a red cell to a blue cell, then the three holes H in the blue cells receive three balls.

Consider the four balls and four holes in the grey area of Fig. 3(b). There are two possible solutions to those balls. Suppose that the top ball ③ is moved 3 cells to the right and then 2 cells to the downward direction. Then the left ball ② (resp. right ball ②) in the grey area must be moved upward (resp. downward), and the bottom ball ③ must be moved 3 cells to the left. In this case, three balls ② in the red cells can be moved 2 cells to the left. This configuration corresponds to $\overline{x_i} = 1$. (Note that four holes H outside the red dotted square belong to the connection gadget, which will be explained later.)

On the other hand, if the top ball ③ is moved 3 cells to the left and then 2 cells to the downward direction (see Fig. 3(c)), then two of the three balls ② in the red cells can be moved 2 cells to the right, and the remaining one ball ② can be moved downward. This corresponds to $x_i = 1$. In Fig. 3(a), the red ball ③ cannot reach the red hole H, since there is a hole H just above the red ball ③.

Clause $c_j \in \{c_1, c_2, \ldots, c_m\}$ is transformed into the clause gadget as illustrated in the blue cells of Fig. 4(a), which is composed of three holes H and two balls ①. If either literal x_i

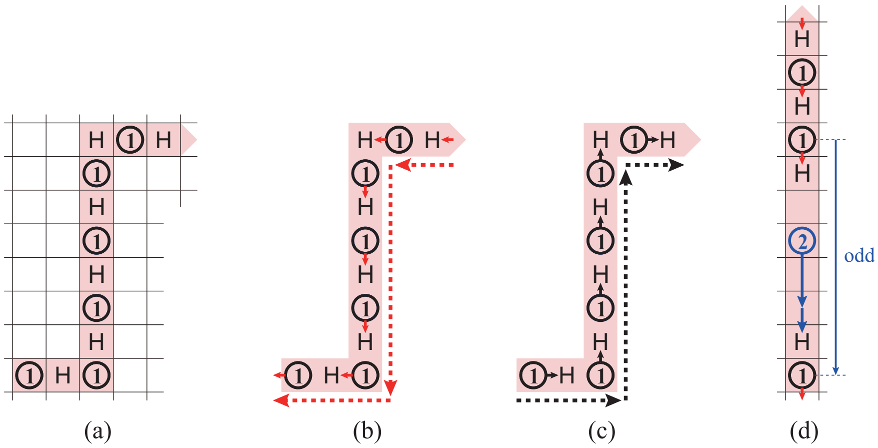

Figure 5 (a) Connection gadget of Herugolf. (b) and (c) are two possible movements. (d) The distance between two balls ① is odd.

or $\overline{x_i}$ belongs to the clause c_j, then clause gadget c_j is connected to a variable gadget x_i by using the connection gadget as illustrated in Fig. 5(a) (see also Fig. 8). The three holes of a clause gadget can each receive balls if and only if at least one of the three balls are moved into one of these holes from a neighboring red cell. The cases are illustrated in Figs. 4(b)–(d). If clause c_j contains only two literals, Fig. 4(a) is replaced with 3×1 blue cells containing "H,①,H." (The clauses of two literals are essential, since it is known that 3SAT with exactly three occurrences per variable is polynomial-time solvable if every clause has three literals [14].)

Figure 5(a) is a connection gadget connecting between variable and clause gadgets. In Figs. 5(b) and 5(c), a "signal" is transmitted from the top right hole H to the bottom left ball ① and vice versa. Namely, if the top right hole of Fig. 5(b) receives a ball from the right side, then the bottom left ball can be moved to the left. If you want the distance between two balls ① to be odd (see Fig. 5(d)), then ball ② is used in a connection gadget. Figure 6 is a crossover gadget.

In each variable gadget (see Fig. 3(a)), the number of balls is two larger than the number of holes. In each clause gadget (see Fig. 4(a)), the number of balls is one smaller than the number of holes. Therefore, the number of balls is $2n - m$ larger than the number of holes in total, where n and m are the numbers of variables and clauses, respectively. Finally, we add a terminator gadget as illustrated in the red dotted rectangle of size $(2n-m) \times (2n+4m-1)$ of Fig. 7. The top row of the terminator gadget is an alternating sequence of "H ① H ① $\cdots$ ① H" of length $2n + 4m - 1$. The second row is an alternating sequence of length $2n + 4m - 3$, and so on.

Every pair of "H,①" in green cells of Fig. 7 is connected to a hole H in the green cell of Fig. 3(a) or a hole H in the yellow cells of Fig. 4(a) by a connection gadget (see Fig. 8). In the terminator gadget, the number of holes is $2n - m$ larger than the number of balls, so $2n - m$ signals are terminated in the terminator gadget. (In Fig. 8, $2n - m$ $(= 4)$ signals are terminated at the leftmost $2n - m$ holes in the red dotted rectangle.)

Figure 8 is a Herugolf puzzle transformed from $C = \{c_1, c_2, c_3, c_4\}$ and $U = \{x_1, x_2, x_3, x_4\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, $c_3 = \{\overline{x_1}, \overline{x_3}, x_4\}$, $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. In this figure, several pairs of a black ball and a hole are placed in the white areas so that balls in grey, red, green, and yellow areas do not move to unintended directions. (If variable and clause gadgets

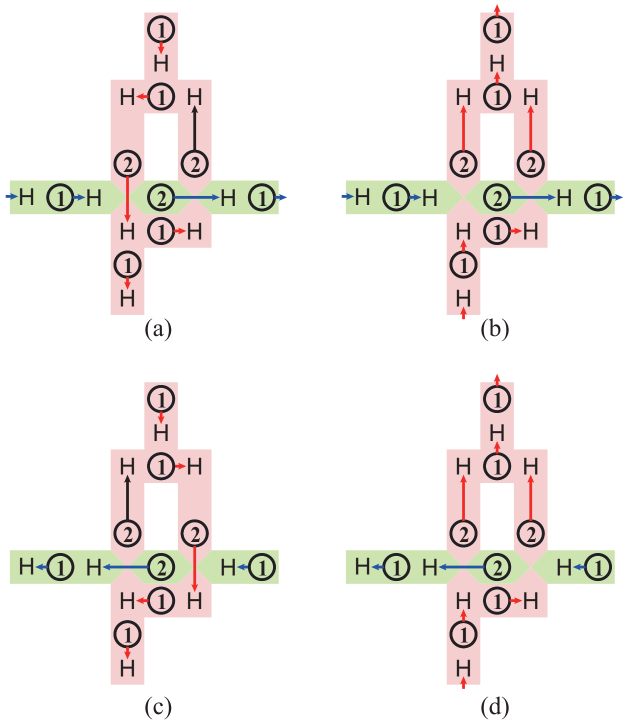

Figure 6 Crossover gadget of Herugolf.

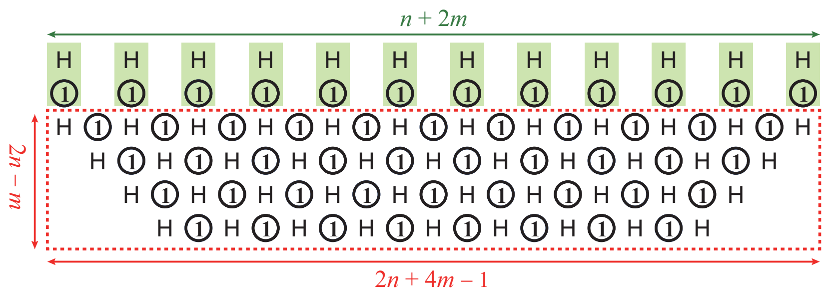

Figure 7 Terminator gadget of Herugolf.

are embedded on a sufficiently large space, no such pair is required.) From this construction, the instance C of 3SAT is satisfiable if and only if the corresponding Herugolf puzzle has a solution.

2.3 Transformation from an Instance of 3SAT to a Makaro Puzzle

We present a polynomial-time transformation from an arbitrary instance C of PLANAR 3SAT to a Makaro puzzle such that C is satisfiable if and only if the puzzle has a solution.

Each variable $x_i \in \{x_1, x_2, \ldots, x_n\}$ is transformed into the variable gadget as illustrated in Fig. 9(a), which is composed of three grey 2-cell rooms. Figures 9(b) and 9(c) correspond to $\overline{x_i} = 1$ and $x_i = 1$, respectively. Note that the instances of PLANAR 3SAT considered in this section have the restriction explained at the end of Sect. 2.1.

Clause $c_j \in \{c_1, c_2, \ldots, c_m\}$ is transformed into the clause gadget as illustrated in Fig. 10(a), which is composed of one blue 4-cell room, three black cells, two grey 4-cell rooms, and four 1-cell rooms. If all of the three black cells are adjacent to number 2 in red 2-cell rooms (see Fig. 10(b)), then there is no solution to the blue 4-cell room. On the other hand, if at least one of the three black cells is adjacent to number 1 in a red 2-cell room (see Figs. 10(c) and 10(d), and Fig. 12), then there is a solution to the blue 4-cell room. If clause c_j contains only two literals, the gadget is composed of one blue 3-cell room, two black cells, and four grey 1-cell rooms (see Figs. 10(e)–(g)).

Fig. 11(a) is a connection gadget connecting between variable and clause gadgets. In Figs. 11(b) and 11(c), a "signal" is transmitted from the top right room to the bottom left room and vice versa. If you want the distance between two 2-cell rooms to be odd, you can use a gadget of Fig. 11(d).

Figure 12 is a Makaro puzzle transformed from $C = \{c_1, c_2, c_3, c_4\}$ and $U = \{x_1, x_2, x_3, x_4\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, $c_3 = \{\overline{x_1}, \overline{x_3}, x_4\}$, $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. In this figure, there are six large white rooms separated by connection, variable, and clause gadgets. Those white rooms can easily be filled with numbers $1, 2, 3, \cdots$. From this construction, the instance C of PLANAR 3SAT is satisfiable if and only if the corresponding Makaro puzzle has a solution.

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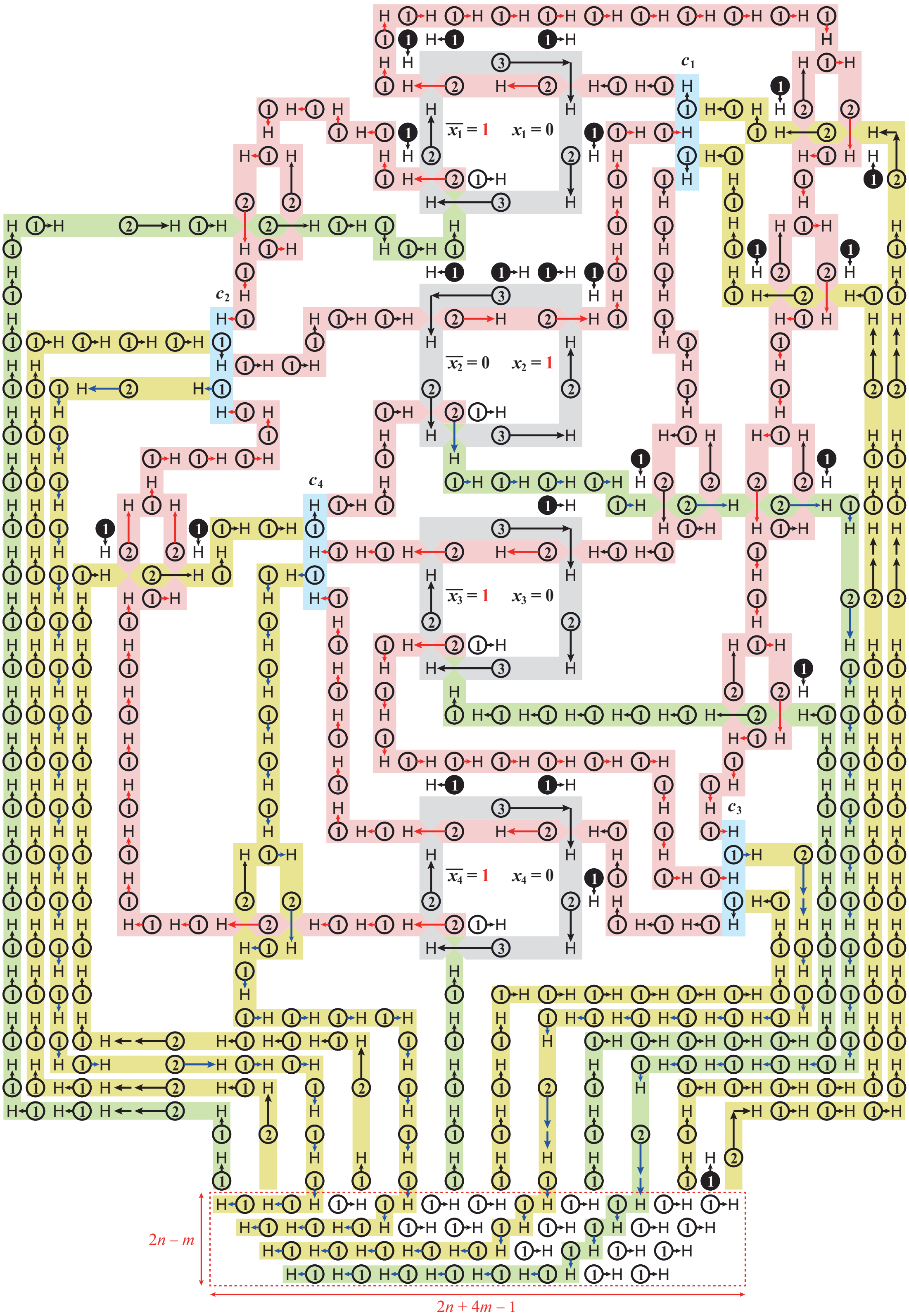

Figure 8 A Herugolf puzzle transformed from $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, $c_3 = \{\overline{x_1}, \overline{x_3}, x_4\}$, $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. From the solution of the puzzle, one can see that the assignment $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfies all clauses of C.

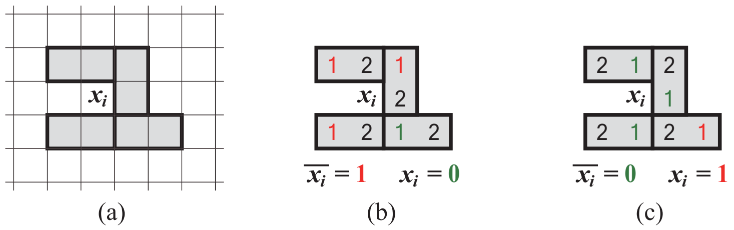

Figure 9 (a) Variable gadget of Makaro transformed from x_i. (b) Assignment $\overline{x_i} = 1$. (c) Assignment $x_i = 1$.

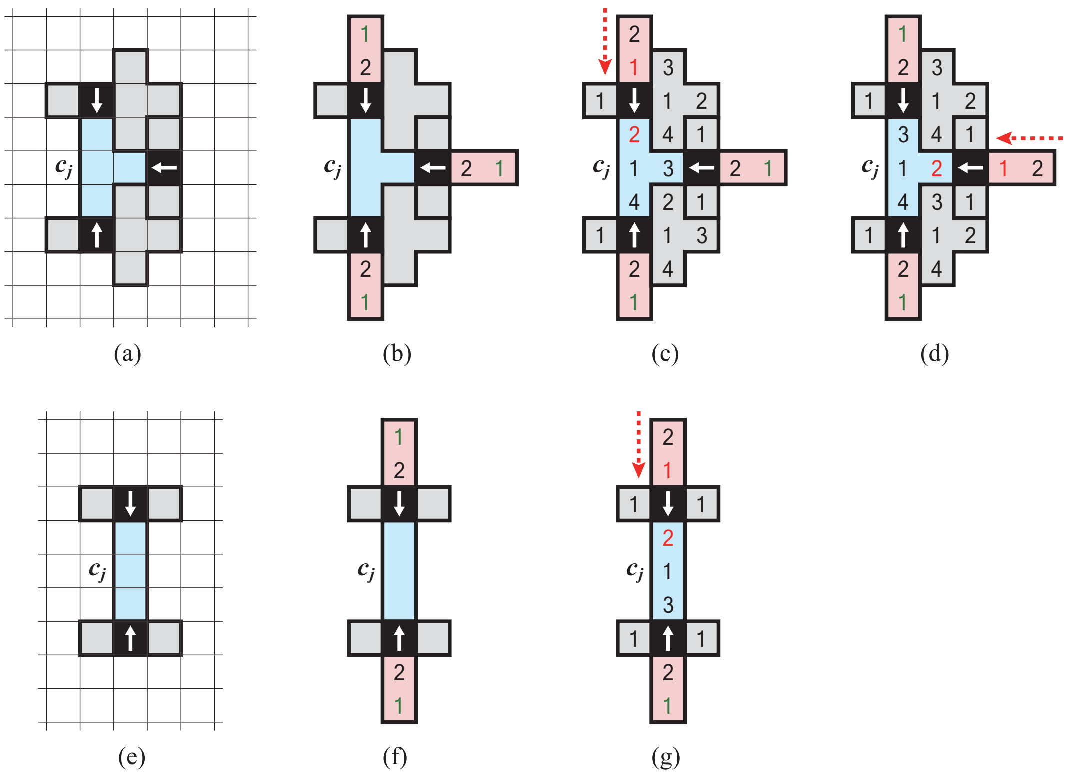

Figure 10 (a) Clause gadget of Makaro transformed from c_j. (b) If all of the three black cells are adjacent to number 2 in red 2-cell rooms, then there is no solution to the blue 4-cell room. (c),(d) If at least one of the three black cells is adjacent to number 1 in a red 2-cell room, then there is a solution to the blue room. (e)–(g) If clause c_j contains only two literals, the gadget is composed of one blue 3-cell room, two black cells, and four grey 1-cell rooms.

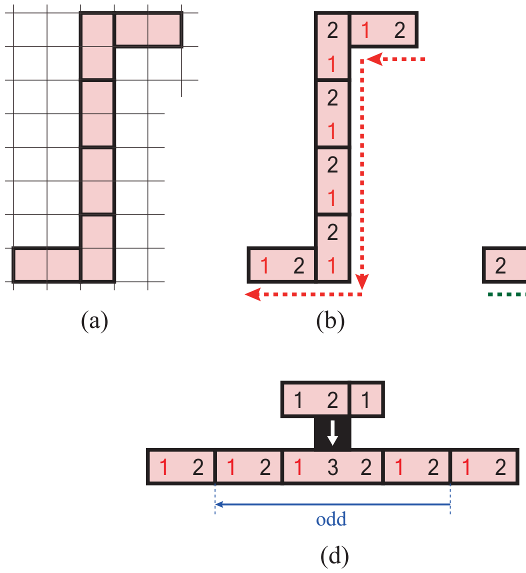

Figure 11 (a) Connection gadget of Makaro. (b) and (c) are two possible solutions. (d) The interval between two 2-cell rooms is odd.

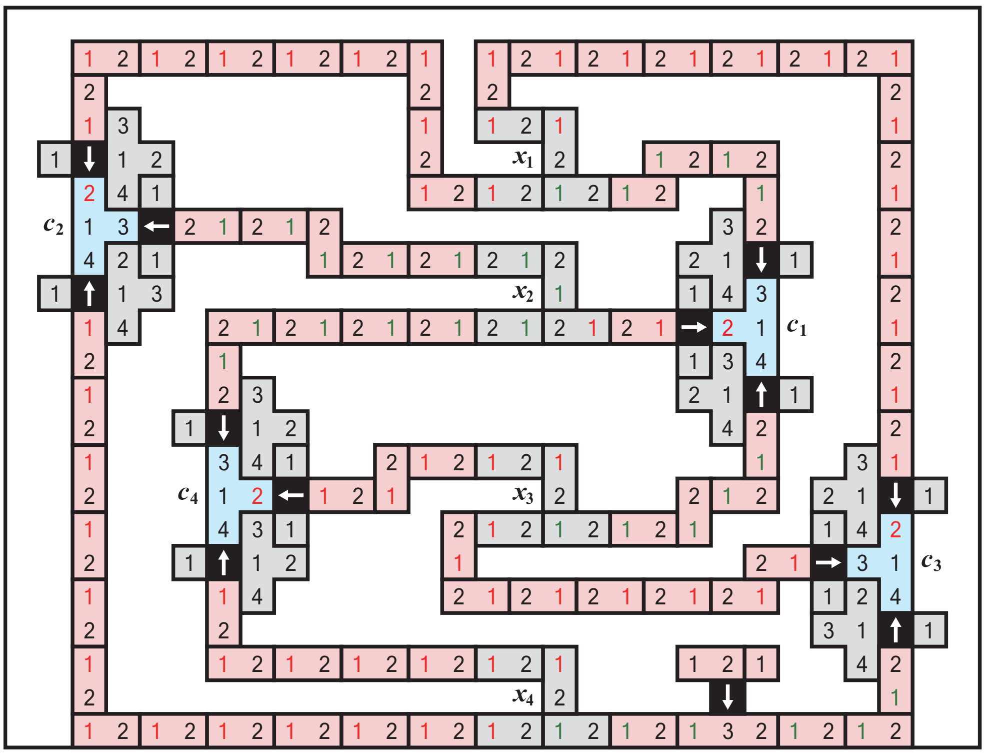

Figure 12 A Makaro puzzle transformed from $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, $c_3 = \{\overline{x_1}, \overline{x_3}, x_4\}$, $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. From the solution of the puzzle, one can see that the assignment $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfies all clauses of C.