God Save the Queen

Jurek Czyzowicz[1]
Université du Québec en Outaouais, Gatineau, Québec, Canada
jurek.czyzowicz@uqo.ca

Konstantinos Georgiou[2]
Department of Mathematics, Ryerson University, Toronto, Ontario, Canada
konstantinos@ryerson.ca

Ryan Killick[3]
School of Computer Science, Carleton University, Ottawa, Ontario, Canada
ryankillick@cmail.carleton.ca

Evangelos Kranakis[4]
School of Computer Science, Carleton University, Ottawa, Ontario, Canada
kranakis@scs.carleton.ca

Danny Krizanc
Department of Mathematics & Comp. Sci., Wesleyan University, Middletown, CT, USA
dkrizanc@wesleyan.edu

Lata Narayanan[5]
Department of Comp. Sci. and Software Eng., Concordia University, Montreal, Québec, Canada
lata@encs.concordia.ca

Jaroslav Opatrny
Department of Comp. Sci. and Software Eng., Concordia University, Montreal, Québec, Canada
opatrny@cs.concordia.ca

Sunil Shende
Department of Computer Science, Rutgers University, Camden, NJ, USA
shende@camden.rutgers.edu

Abstract

Queen Daniela of Sardinia is asleep at the center of a round room at the top of the tower in her castle. She is accompanied by her faithful servant, Eva. Suddenly, they are awakened by cries of "Fire". The room is pitch black and they are disoriented. There is exactly one exit from the room somewhere along its boundary. They must find it as quickly as possible in order to save the life of the queen. It is known that with two people searching while moving at maximum speed 1 anywhere in the room, the room can be evacuated (i.e., with both people exiting) in $1 + \frac{2\pi}{3} + \sqrt{3} \approx 4.8264$ time units and this is optimal [Czyzowicz et al., DISC'14], assuming that the first person to find the exit can directly guide the other person to the exit using her voice. Somewhat surprisingly, in this paper we show that if the goal is to save the queen (possibly leaving Eva behind to die in the fire) there is a slightly better strategy. We prove that this "priority" version of evacuation can be solved in time at most 4.81854. Furthermore, we show that any strategy for saving the queen requires time at least $3 + \pi/6 + \sqrt{3}/2 \approx 4.3896$ in the worst case. If one or both of the queen's other servants (Biddy and/or Lili) are with her, we show that the time bounds can be improved to 3.8327 for two servants, and 3.3738 for three servants.

[1] NSERC
[2] NSERC
[3] OGS
[4] NSERC
[5] NSERC

9th International Conference on Fun with Algorithms (FUN 2018).
Editors: Hiro Ito, Stefano Leonardi, Linda Pagli, and Giuseppe Prencipe; Article No. 16; pp. 16:1–16:20
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Finally we show lower bounds for these cases of 3.6307 (two servants) and 3.2017 (three servants). The case of $n \geq 4$ is the subject of an independent study by Queen Daniela's Royal Scientific Team.

2012 ACM Subject Classification Computing methodologies → Distributed artificial intelligence,Computing methodologies → Mobile agents

Keywords and phrases Algorithm, Evacuation, Exit, Disk, Wireless Communication, Queen, Priority, Robots, Search, Servants, Trajectory

Digital Object Identifier 10.4230/LIPIcs.FUN.2018.16

Related Version Full version hosted on arXiv `http://arxiv.org/abs/1804.06011`

1 Introduction

In traditional search, a group of searchers (modeled as mobile autonomous agents or robots) may collaboratively search for an exit (or target) placed within a given search domain [1, 2, 20]. Although the searchers may have differing capabilities (communication, perception, mobility, memory) search algorithms, previously employed, generally make no distinction between them as they usually play identical roles throughout the execution of the search algorithm and with respect to the termination time (with the exception of faulty robots, which also do not contribute to searching). In this work we are motivated by real-life safeguarding-type situations where a number of agents have the exclusive role to facilitate the execution of the task by a distinguished entity. More particularly, we introduce and study *Priority Evacuation*, a new form of search , under the wireless communication model, in which the search time of the algorithm is measured by the time it takes a special searcher, called the queen, to reach the exit. The remaining searchers in the group, called servants, are participating in the search but are not required to exit.

1.1 Problem Definition of Priority Evacuation (PE$_n$)

A target (exit) is hidden in an unknown location on the unit circle. The exit can be located by any of the $n+1$ robots (searchers) that walks over it ($n = 1, 2, 3$). Robots share the same coordinate system, start from the center of the circle, and have maximum speed 1. Among them there is a distinguished robot, called the *queen*, and the remaining n robots are referred to as *servants*. All servants are known to the queen by their identities. Robots may run asymmetric algorithms, and can communicate their findings wirelessly and instantaneously (each message is composed by an identity and a location). Only the queen is required to be able to receive messages. Feasible solutions to this problem are *evacuation algorithms*, i.e. robots' movements (trajectories) that guarantee the finding of the hidden exit. The cost of an evacuation algorithm is the *evacuation time* of the queen, i.e., the worst case total time until the queen reaches the exit. None of the n servants needs to evacuate.

1.2 Related work

Related to our work is linear search which refers to search in an infinite line. There have been several interesting studies attempting to optimize the search time which were initiated with the influential works of Bellman [7] and Beck [6]. A long list of results followed for numerous variants of the problem, citing which is outside the scope of this work. For a comprehensive study of seminal search-type problems see [2, 3].

The problem of searching in the plane by one or more searchers, has been considered by [4, 5]. The unit disk model considered in our present paper is a form of two-dimensional search that was initiated in the work of [10]. In this paper the authors obtained evacuation algorithms in the wireless and face-to-face communication models both for a small number of robots as well optimal asymptotic results for a large number of robots. Additional evacuation algorithms in the face-to-face communication model were subsequently analyzed for two robots in [14] and later in [8]. Other variations of the problem include the case of more than one exit, see [9] and [19], triangular and square domains in [15], robots with different moving speeds [18], and evacuation in the presence of crash or byzantine faulty robots [11].

A priority evacuation-type problem has been previously considered in [16, 17] but with different terminology. Using the jargon of the current paper, an immobile queen is hidden somewhere on the unit disk, and a number of robots try to locate her, and fetch (evacuate) her to an exit which is also hidden. The performance of the evacuation algorithm is measured by the time the queen reaches the exit.

Apart from the results in [16, 17], all relevant previous work in search-type problems considered the objective of minimizing the time it takes either by the first or the last agent to reach the hidden target. In contrast, this paper considers an evacuation (search-type) problem where the completion time is defined with respect to a distinguished mobile agent, the *queen*, while the remaining n servants are not required to evacuate. Our current focus is to design efficient algorithms for $n = 1, 2, 3$ servants, as well as give strong lower bounds. Notably, the algorithms we propose significantly improve upon evacuation costs induced by naive trajectories, and in fact the trajectories we propose are non-trivial. Our main contribution concerns priority evacuation for each of the cases of $n = 1, 2, 3$ servants, all of which require special treatment. Moreover, all our algorithms are characterized by the fact that the queen does contribute effectively to the search of the hidden item. In sharp contrast, the independent and concurrent work of [13] studies the same problem for $n \geq 4$ servants where the queen never contributes to the search. More importantly, the proposed algorithms of [13] admit a unified description and analysis that does not intersect with the current work.

1.3 Our Results & Paper Organization

Section 2 introduces necessary notation and terminology and discusses preliminaries. Section 3 is devoted to upper bounds for PE_n for $n = 1, 2, 3$ servants (see Subsections 3.1, 3.2, and 3.3, respectively). All our upper bounds are achieved by fixing optimal parameters for families of parameterized algorithms. In Section 4 we derive lower bounds for PE_n, $n = 1, 2, 3$. An interesting corollary of our positive results is that priority evacuation with $n = 1, 2, 3$ servants (i.e. with $n + 1$ searchers) can be performed strictly faster than ordinary evacuation with $n + 1$ robots where all robots have to evacuate. Indeed, an argument found in [10] can be adjusted to show that the evacuation problem with $n + 1$ robots cannot be solved faster than $1 + \frac{4\pi}{3(n+1)} + \sqrt{3}$. Surprisingly, when one needs to evacuate only one designated robot, the task can provably (due to our upper bounds) be executed faster. All our results, together with the comparison to the lower bounds of [10], are summarized in Table 1. We conclude the paper in Section 5 with a discussion of open problems. Whenever we omit proofs, due to space limitations, we provide an outline of our arguments. The interested reader may consult the full version of our paper [12] for the missing details.

Table 1 Upper and lower bounds for priority evacuation.

# of Servants	Upper Bounds for PE_n	Lower Bounds for PE_n	Lower Bounds for Ordinary Evacuation
$n = 1$	4.8185 (Theorem 8)	4.3896 (Theorem 17)	4.826445 (see [10])
$n = 2$	3.8327 (Theorem 10)	3.6307 (Theorem 19)	4.128314 (see [10])
$n = 3$	3.3738 (Theorem 14)	3.2017 (Theorem 19)	3.779248 (see [10])

2 Notation and Preliminaries

We use n to denote the number of servants, and we set $[n] = \{1, \ldots, n\}$. Queen and servant i will be denoted by $\mathcal{Q}$ and $\mathcal{S}_i$, respectively, where $i \in [n]$. We assume that all robots start from the origin $O = (0, 0)$ of a unit circle in $\mathbb{R}^2$. As usual, points in $A \in \mathbb{R}^2$ will be treated, when it is convenient, as vectors from O to A, and $\|A\|$ will denote the euclidean norm of that vector.

2.1 Problem Reformulation & Solutions' Description

Robots' trajectories will be defined by parametric functions $\mathcal{F}(t) = (f(t), g(t))$, where $f, g : \mathbb{R} \mapsto \mathbb{R}$ are continuous and piecewise differentiable. In particular, search algorithms for all robots will be given by trajectories

$$\mathbb{S}_n := \left\{\mathcal{Q}(t), \{\mathcal{S}_i(t)\}_{i\in[n]}\right\},$$

where $\mathcal{Q}(t), \mathcal{S}_i(t)$ will denote the position of $\mathcal{Q}$ and $\mathcal{S}_i$, respectively, at time $t \geq 0$.

▶ **Definition 1** (Feasible Trajectories). We say that trajectories $\mathbb{S}_n$ are *feasible* for PE_n if:

(a) $\mathcal{Q}(0) = \mathcal{S}_i(0) = O$, for all $i \in [n]$,

(b) $\mathcal{Q}(t), \{\mathcal{S}_i(t)\}_{i\in[n]}$ induce speed-1 trajectories for $\mathcal{Q}, \{\mathcal{S}_i\}_{i\in[n]}$ respectively, and

(c) there is some time $t_0 \geq 1$, such that each point of the unit circle is visited (searched) by at least one robot in the time window $[0, t_0]$. We refer to the smallest such t_0 as the *search time* of the circle.

Note that feasible trajectories do indeed correspond to robots' movements for PE_n in which, eventually the entire circle is searched, and hence the search time is bounded. We will describe all our search/evacuation algorithms as feasible trajectories, and we will assume that once the target is reported, $\mathcal{Q}$ will go directly to the location of the exit.

For feasible trajectories $\mathbb{S}_n$ with search time t_0, and for any trajectory $\mathcal{F}(t)$ (either of the queen or of a servant), we denote by $\mathbb{I}(\mathcal{F})$ the subinterval of $[0, t_0]$ that contains all $x \in [0, t_0]$ such that $\|\mathcal{F}(x)\| = 1$ (i.e. the robot is on the the circle) and no other robot has been to $\mathcal{F}(x)$ before. Since robots start from the origin, it is immediate that $\mathbb{I}(\mathcal{F}) \subseteq [1, t_0]$. With this notation in mind, note that the exit can be discovered by some robot $\mathcal{F}$, say at time x, only if $x \in \mathbb{I}(\mathcal{F})$. In this case, the finding is instantaneously reported, so $\mathcal{Q}$ goes directly to the exit, moving along the corresponding line segment between her current position $\mathcal{Q}(x)$ and the reported position of the exit $\mathcal{F}(x)$. Hence, the total time that $\mathcal{Q}$ needs to evacuate equals

$$x + \|\mathcal{Q}(x) - \mathcal{F}(x)\|.$$

Therefore, the *evacuation time* of feasible trajectories $\mathbb{S}_n$ to PE_n is given by expression

$$\max_{\mathcal{F}\in\mathbb{S}_n} \sup_{x\in\mathbb{I}(\mathcal{F})} \{x + \|\mathcal{Q}(x) - \mathcal{F}(x)\|\}.$$

Notice that for "non-degenerate" search algorithms for which the last point on the circle is not searched by $\mathcal{Q}$ alone, the previous maximum can be simply computed over the servants, i.e the evacuation cost will be

$$\max_{i\in[n]} \sup_{x\in\mathbb{I}(\mathcal{S}_i)} \{x + \|\mathcal{Q}(x) - \mathcal{S}_i(x)\|\}. \tag{1}$$

In other words, we can restate PE_n as the problem of determining feasible trajectories $\mathbb{S}_n$ so as to minimize (1).

2.2 Useful Trajectories' Components

Feasible trajectories induce, by definition, robots that are moving at (maximum) speed 1. The speed restriction will be ensured by the next condition.

▶ **Lemma 2.** *An object following trajectory $\mathcal{F}(t) = (f(t), g(t))$ has unit speed if and only if*

$$(f'(t))^2 + (g'(t))^2 = 1, \quad \forall t \geq 0.$$

Proof. For any $t \geq 0$, the velocity of $\mathcal{F}$ is given by $\mathcal{F}'(t) = (df(t)/dt, dg(t)/dt)$, and its speed is calculated as $\|\mathcal{F}'(t)\|$. ◀

Robots' trajectories will be composed by piecewise smooth parametric functions. In order to describe them, we introduce some further notation. For any $\theta \in \mathbb{R}$, we introduce abbreviation C_θ for point $\{\cos(\theta), \sin(\theta)\}$. Next we introduce parametric equations for moving along the perimeter of a unit circle (Lemma 3), and along a line segment (Lemma 4).

▶ **Lemma 3.** *Let $b \in [0, 2\pi)$ and $\sigma \in \{-1, 1\}$. The trajectory of an object moving at speed 1 on the perimeter of a unit circle with initial location C_b is given by the parametric equation*

$$\mathcal{C}(b, \sigma t) := (\cos(\sigma t + b), \sin(\sigma t + b)).$$

If $\sigma = 1$ the movement is counter-clockwise (ccw), and clockwise (cw) otherwise.

Proof. Clearly, $\mathcal{C}(b, 0) = C_b$. Also, it is easy to see that $\|\mathcal{C}(b, t)\| = 1$, i.e. the object is moving on the perimeter of the unit circle. Lastly,

$$\left(\frac{d}{dt}\cos(\sigma t + b)\right)^2 + \left(\frac{d}{dt}\sin(\sigma t + b)\right)^2 = \sigma^2(-\sin(\sigma t + b))^2 + \sigma^2(\cos(\sigma t + b))^2 = 1,$$

so the claim follows by Lemma 2. ◀

▶ **Lemma 4.** *Consider distinct points $A = (a_1, a_2), B = (b_1, b_2)$ in $\mathbb{R}^2$. The trajectory of a speed 1 object moving along the line passing through A, B and with initial position A is given by the parametric equation*

$$\mathcal{L}(A, B, t) := \left(\frac{b_1 - a_1}{\|A - B\|}t + a_1, \frac{b_2 - a_2}{\|A - B\|}t + a_2\right).$$

Proof. It is immediate that the parametric equation corresponds to a line. Also, it is easy to see that $\mathcal{L}(A, B, 0) = A$ and $\mathcal{L}(A, B, \|A - B\|) = B$, i.e. the object starts from A, and eventually visits B. As for the object's speed, we calculate

$$\left(\frac{d}{dt}\left(\frac{b_1 - a_1}{\|A - B\|}t + a_1\right)\right)^2 + \left(\frac{d}{dt}\left(\frac{b_2 - a_2}{\|A - B\|}t + a_2\right)\right)^2 = \left(\frac{b_1 - a_1}{\|A - B\|}\right)^2 + \left(\frac{b_2 - a_2}{\|A - B\|}\right)^2 = 1$$

so, by Lemma 2, the speed is indeed 1. ◀

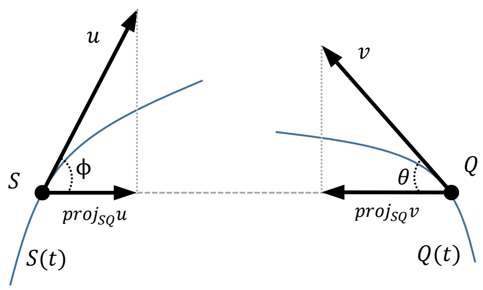

Figure 1 An illustration of trajectories $\mathcal{S}(t)$, $\mathcal{Q}(t)$, and their critical angles at some fixed time τ, with $\mathcal{S}(\tau) = S$, $\mathcal{Q}(\tau) = Q$, $\mathcal{S}'(\tau) = u$, $\mathcal{Q}'(\tau) = v$.

Robots trajectories will be described in phases. In each phase, robot, say $\mathcal{F}$, will be moving between two explicit points, and the corresponding trajectory $\mathcal{F}(t)$ will be implied by the previous description, using most of the times Lemma 3 and Lemma 4. We will summarize the details in tables of the following format.

Robot	#	*Description*	*Trajectory*	*Duration*
$\mathcal{F}$	0		$\mathcal{F}(t)$	t_0
	1		$\mathcal{F}(t)$	t_1
	⋮			⋮

Phase 0 will usually correspond to the deployment of $\mathcal{F}$ from the origin to some point of the circle. Also, for each phase we will summarize it's duration. With that in mind, trajectory $\mathcal{F}(t)$ during phase i, with duration t_i, will be valid for all $t \geq 0$ with $|t-(t_0+t_1+\dots t_{i-1})| \leq t_i$.

Lastly, the following abbreviation will be useful for the exposition of the trajectories. For any $\rho \in [0,1]$ and $\theta \in [0, 2\pi)$, we introduce notation

$$K(\theta, \rho) := (1-\rho)C_{\pi-\theta} + \rho C_{-\theta}.$$

In other words, $K(\theta, \rho)$ is a convex combination of antipodal points $C_{\pi-\theta}, C_{-\theta}$ of the unit circle, i.e. it lies on the diameter of the unit circle passing through these two points. Moreover, it is easy to see that $\|C_{\pi-\theta} - K(\theta,\rho)\| = 2\rho$, and hence

$$\|K(\theta,\rho) - C_{-\theta}\| = 2 - 2\rho.$$

As it will be handy later, we also introduce abbreviation

$$AK(\theta,\rho) := \|C_\pi - K(\theta,\rho)\|.$$

The choice of the abbreviation is clear, if the reader denotes $C_\pi = (-1, 0)$ by A.

2.3 Critical Angles

The following definition introduces a key concept. In what follows, abstract trajectories will be assumed to be continuous and differentiable, which in particular implies that corresponding velocities are continuous.

▶ **Definition 5** (Critical Angle). Let $\mathcal{S}(t) \in \mathbb{R}^2$ denote the trajectory of a speed-1 object, where $t \geq 0$. For some point $Q \in \mathbb{R}^2$, we define the $(\mathcal{S}, Q)$-critical angle at time $t = \tau$ to be the angle between the velocity vector $\mathcal{S}'(\tau)$ and vector $\overrightarrow{\mathcal{S}(\tau)Q}$, i.e. the vector from $\mathcal{S}(\tau)$ to Q.

We make the following critical observation, see also Figure 1.

▶ **Theorem 6.** *Consider trajectories $\mathcal{S}(t), \mathcal{Q}(t)$ of two speed-1 objects $\mathcal{S}, \mathcal{Q}$, where $t \geq 0$. Let also ϕ, θ denote the $(\mathcal{S}, \mathcal{Q}(t))$-critical angle and the $(\mathcal{Q}, \mathcal{S}(t))$-critical angle at time t, respectively. Then $t + \|\mathcal{Q}(t) - \mathcal{S}(t)\|$ is strictly increasing if $\cos(\phi) + \cos(\theta) < 1$, strictly decreasing if $\cos(\phi) + \cos(\theta) > 1$, and constant otherwise.*

Theorem 6 is an immediate corollary of the following lemma.

▶ **Lemma 7.** *Consider trajectories $\mathcal{S}(t), \mathcal{Q}(t)$ and their critical angles π, θ, as in the statement of Theorem 6. Then*

$$\frac{d}{dt}\|\mathcal{Q}(t) - \mathcal{S}(t)\| = \cos(\phi) + \cos(\theta).$$

Proof. For any fixed t, let d denote $D(t)$, and S, Q denote points $\mathcal{S}(t), \mathcal{Q}(t)$, respectively. Denote also by u, v the velocities of $\mathcal{S}, \mathcal{Q}$ at time t, respectively, i.e. $u = \mathcal{S}'(t), v = \mathcal{Q}'(t)$. See also Figure 1.

With that notation, observe that $\left\|\overrightarrow{SQ}\right\| = d$. Since $\|u\| = \|v\| = 1$, we see that

$$\mathrm{proj}_{SQ} u = \frac{\cos(\phi)}{d}\overrightarrow{SQ}$$

and

$$\mathrm{proj}_{SQ} v = \frac{\cos(\theta)}{d}\overrightarrow{QS}.$$

Now consider two imaginary objects $\overline{\mathcal{S}}, \overline{\mathcal{Q}}$, with corresponding velocities $\overline{\mathcal{S}}'(t) = \mathrm{proj}_{SQ} u$ and $\overline{\mathcal{Q}}'(t) = \mathrm{proj}_{SQ} v$. It is immediate that $\|\mathcal{Q}(t) - \mathcal{S}(t)\| = \left\|\overline{\mathcal{Q}}(t) - \overline{\mathcal{S}}(t)\right\|$.

In particular, $\mathrm{proj}_{SQ} u - \mathrm{proj}_{SQ} v$ is the projection of the relative velocities of $\mathcal{S}, \mathcal{Q}$ on the line segment connecting $\mathcal{S}(t), \mathcal{Q}(t)$. As such, the distance between $\mathcal{S}, \mathcal{Q}$ changes at a rate determined by velocity

$$\mathrm{proj}_{SQ} u - \mathrm{proj}_{SQ} v = \frac{\cos(\phi) + \cos(\theta)}{d}\overrightarrow{SQ},$$

where $\left\|\mathrm{proj}_{SQ} u - \mathrm{proj}_{SQ} v\right\| = |\cos(\phi) + \cos(\theta)|$. Moreover, $\mathrm{proj}_{SQ} u, \mathrm{proj}_{SQ} v$ are antiparallel iff and only if $\cos(\phi), \cos(\theta) > 0$, in which case the two objects come closer to each other. ◀

3 Upper Bounds

3.1 Evacuation Algorithm for PE_1

This subsection is devoted in proving the following.

▶ **Theorem 8.** *Consider the real function $f(x) = x + \sin(x)$, and denote by $\alpha_0 > 0$ the solution to equation*

$$f(f(\alpha - \sin(\alpha))) = \sin(\alpha),$$

with $\alpha_0 \approx 1.14193$. Then PE_1 can be solved in time $1 + \pi - \alpha_0 + 2\sin(\alpha_0) \approx 4.81854$.

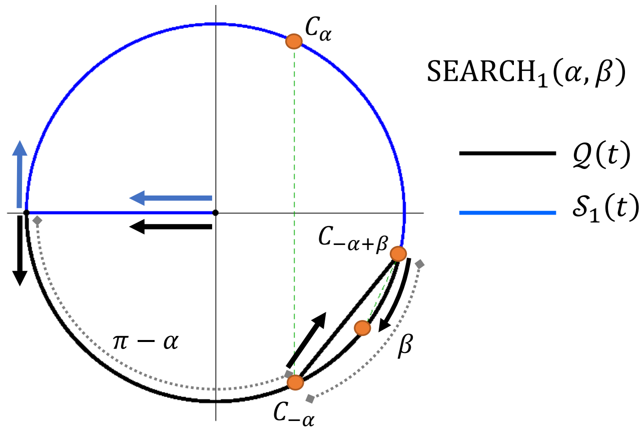

Figure 2 Algorithm $\text{Search}_1(\alpha, \beta)$ depicted for the optimal parameters of the algorithm. In all subsequent figures, as well as here, the orange points on the perimeter of the disc correspond to the worst adversarial placements of the treasure, which due to our optimality conditions induce the same evacuation cost. The orange points in $\mathcal{Q}$'s trajectories correspond to the $\mathcal{Q}$'s positioning when the treasures are reported, in the worst cost induced cases. The green dashed line depict $\mathcal{Q}$'s trajectory after $\mathcal{Q}$ abandons her trajectory and moves toward the reported exit following a straight line.

The value of α_0 is well defined in the statement of Theorem 8. Indeed, by letting $g(x) = f(f(x - \sin(x))) - \sin(x)$, we observe that g is continuous, while $g(1) \approx -0.213934$ and $g(\pi/2) \approx 1.00729$, hence there exists $\alpha_0 \in (1, \pi/2)$ with $g(\alpha_0) = 0$.

In order to prove Theorem 8, and given parameters α, β, we introduce the family of trajectories $\text{Search}_1(\alpha, \beta)$, see also Figure 2.

Algorithm Search$_1(\alpha, \beta)$

Robot	#	*Description*	*Trajectory*	*Duration*
$\mathcal{Q}$	0	Move to point C_π	$\mathcal{L}(O, C_\pi, t)$	1
	1	Search circle ccw till point $C_{-\alpha}$	$\mathcal{C}(\pi, t-1)$	$\pi - \alpha$
	2	Move to point $C_{-\alpha+\beta}$,	$\mathcal{L}(C_{-\alpha}, C_{-\alpha+\beta}, t-(1+\pi-\alpha))$	$2\sin(\beta/2)$
	3	Search circle cw till point $C_{-\alpha}$	$\mathcal{C}(\beta-\alpha, 1+\pi-\alpha+2\sin(\beta/2)-t)$	β
$\mathcal{S}_1$	0	Move to point C_π	$\mathcal{L}(O, C_\pi, t)$	1
	1	Search circle cw till point $C_{\beta-\alpha}$	$\mathcal{C}(\pi, -t+1)$	$\pi+\alpha-\beta$

Partitioning the circle clockwise, we see that the arc with endpoints $C_\pi, C_{\pi+\alpha-\beta}$ is searched by $\mathcal{S}_1$, while the remaining of the circle is searched by $\mathcal{Q}$. Therefore, robots' trajectories in $\text{Search}_1(\alpha, \beta)$ are feasible, and it is also easy to see that they are continuous as well. The search time equals $1 + \pi + \max\{\alpha - \beta, 2\sin(\beta/2) + \beta - \alpha\}$, as well as

$$\mathbb{I}(\mathcal{Q}) = [1, 1+\pi-\alpha] \cup [1+\pi-\alpha+2\sin(\beta/2), 1+\pi-\alpha+2\sin(\beta/2)+\beta], \mathbb{I}(\mathcal{S}_1) = [1, 1+\pi+\alpha-\beta].$$

An illustration of the above trajectories for certain values of α, β can be seen in Figure 2.

First we make some observations pertaining to the monotonicity of the evacuation cost.

▶ **Lemma 9.** *Assuming that $\alpha > \pi/3$ and that $\cos(\alpha) + \cos(\alpha - \beta/2) > 1$, the evacuation cost of $\text{Search}_1(\alpha, \beta)$ is monotonically increasing if the exit is found by $\mathcal{S}_1$ during $\mathcal{Q}$'s phase 1 and monotonically decreasing if the exit is found by $\mathcal{S}_1$ during $\mathcal{Q}$'s phase 2.*

Proof. Suppose that the exit is found by $\mathcal{S}_1$ during $\mathcal{Q}$'s phase 1, i.e. at time x after robots start searching for the first time, where $0 \le x \le \pi - \alpha$. It is easy to see that the critical angles between $\mathcal{Q}, \mathcal{S}_1$ are both equal to $\pi - x$. But then $2\cos(\pi - x) \geq 2\cos(\alpha) > 2\cos(\pi/3) = 1$. Hence, by Theorem 6, the evacuation cost is decreasing in this case.

Now suppose that the exit is found by $\mathcal{S}_1$ during $\mathcal{Q}$'s phase 2, i.e. at time x after $\mathcal{Q}$ starts moving along the chord with endpoints $C_{-\alpha}, C_{-\alpha+\beta}$, where $0 \leq x \leq 2\sin(\beta/2)$. If ϕ_x, θ_x denote the $\mathcal{S}_1, \mathcal{Q}$ critical angles, then it is easy to see that $\phi_0 = \cos(\alpha)$ and that $\theta_0 = \alpha - \beta/2$. Since $\cos(\phi_0) + \cos(\theta_0) > 1$, Theorem 6 implies that the evacuation cost is initially decreasing in this phase. For the remaining of $\mathcal{Q}$'s phase 2, it is easy to see that both ϕ_x, θ_x are decreasing in x, hence $\cos(\phi_x) + \cos(\theta_x)$ is increasing in x, hence, the evacuation cost will remain decreasing in this phase. ◀

Now we can prove Theorem 8 by fixing certain values for parameters α, β of $\textsc{Search}_1(\alpha, \beta)$. In particular, we set α_0 as in the statement of Theorem 8, and $\beta_0 = 2f(\alpha_0 - \sin(\alpha_0)) \approx 0.925793$. The trajectories of the robots, for the exact same values of the parameters, can be seen in Figure 2.

Proof of Theorem 8. Let f, α_0 be as in the statement of Theorem, and set $\beta_0 = 2f(\alpha_0 - \sin(\alpha_0)) \approx 0.925793$. We argue that the worst evacuation time of $\textsc{Search}_1(\alpha_0, \beta_0)$ is $1 + \pi - \alpha_0 + 2\sin(\alpha_0)$. Note that for the given values of the parameters, we have that $\alpha_0 > \pi/3$, that $\alpha_0 - \sin(\beta_0/2) \leq \beta_0$, and that $\cos(\alpha_0) + \cos(\alpha_0 - \beta_0/2) > 1$.

First we observe that if the exit if found by $\mathcal{Q}$, then the worst case evacuation time $E_0(\alpha_0, \beta_0)$ is incurred when the exit is found just before $\mathcal{Q}$ stops searching, that is

$$E_0(\alpha_0, \beta_0) = 1 + \pi - \alpha_0 + 2\sin(\beta_0/2) + \beta_0.$$

Next we examine some cases as to when the exit is found by $\mathcal{S}_1$. If the exit is found by $\mathcal{S}_1$ during the 1st phase of $\mathcal{Q}$, then the evacuation time is, due to Lemma 9, given as

$$E_1(\alpha_0, \beta_0) = \sup_{1 \leq x \leq 1+\pi-\alpha_0} \{x + \|\mathcal{Q}(x) - \mathcal{S}_1(x)\|\} = 1 + \pi - \alpha_0 + 2\sin(\alpha_0).$$

Recall that $\cos(\alpha_0) + \cos(\alpha_0 - \beta_0/2) > 1$, and so, again by Lemma 9 we may omit the case that the exit is found by $\mathcal{S}_1$ while $\mathcal{Q}$ is at phase 2. The end of $\mathcal{Q}$'s phase 2 happens at time $\tau := 1+\pi-\alpha_0+2\sin(\beta_0/2)$, when have that $\mathcal{Q}(\tau) = C_{-\alpha+\beta}$, and $\mathcal{S}_1(\tau) = C_{\alpha-2\sin(\beta_0/2)}$, and both robots are intending to search ccw. Condition $\alpha_0 - \sin(\beta_0/2) \leq \beta_0$ says that $\mathcal{S}_1$ will finish searching prior to $\mathcal{Q}$, and this happens when $\mathcal{S}_1$ reaches point $C_{-\alpha+\beta}$. During this phase, the distance between $\mathcal{Q}, \mathcal{S}_1$ stays invariant and equal to $2\alpha_0 - \beta_0 - 2\sin(\beta_0/2)$. We conclude that the cost in this case would be

$$E_2(\alpha_0, \beta_0) = 1 + \pi + \alpha_0 - \beta_0 + 2\sin(\alpha_0 - \beta_0/2 - \sin(\beta_0/2)).$$

Then, we argue that that the choice of α_0, β_0 guarantees that $E_0(\alpha_0, \beta_0) = E_1(\alpha_0, \beta_0) = E_2(\alpha_0, \beta_0)$, as wanted.

Indeed, $E_0(\alpha_0, \beta_0) = E_1(\alpha_0, \beta_0)$ implies that $\sin(\beta_0/2) + \beta_0/2 = \sin(\alpha_0)$. But then, we can rewrite $E_2(\alpha_0, \beta_0)$ as

$$E_2(\alpha_0, \beta_0) = 1 + \pi + \alpha_0 - \beta_0 + 2\sin(\alpha_0 - \sin(\alpha_0)).$$

Equating the last expression with $E_1(\alpha_0, \beta_0)$ implies that

$$\beta_0/2 = \alpha_0 - \sin(\alpha_0) + \sin(\alpha_0 - \sin(\alpha_0)) = f(\alpha_0 - \sin(\alpha_0)).$$

Substituting twice $\beta_0/2$ in the already derived condition $\sin(\beta_0/2) + \beta_0/2 = \sin(\alpha_0)$ implies that

$$f(f(\alpha - \sin(\alpha_0))) = \sin(\alpha_0).$$

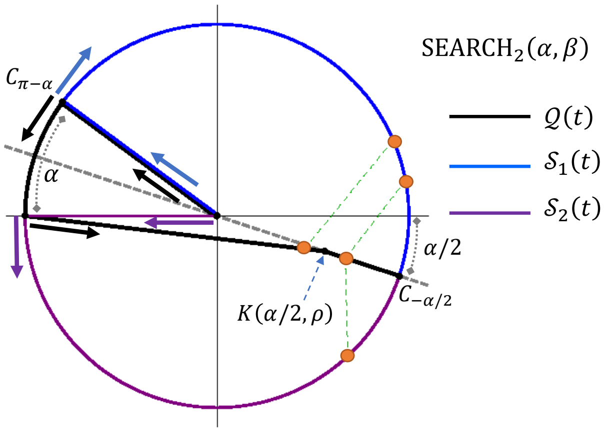

Figure 3 Algorithm SEARCH$_2(\alpha,\beta)$ depicted for the optimal parameters of the algorithm.

Figure 2 depicts the worst placements of the exit, along with the trajectories of the queen (in dashed green lines) after the exit is reported. ◀

It should be stressed that $\mathcal{Q}$'s Phases 2,3 are essential for achieving the promised bound. Indeed, had we chosen $\alpha = \beta = 0$, the worst case evacuation time would have been

$$\sup_{1\leq x\leq 1+\pi} \{x + \|\mathcal{Q}(x) - \mathcal{S}_1(x)\|\} = \sup_{0\leq x\leq \pi} \{1 + x + 2\sin(x)\}.$$

The maximum is attained at $x_0 = 2\pi/3$ (and indeed, both critical angles in this case are $\pi/3$ and in particular $2\cos(\pi/3) = 1$), inducing cost $1 + 2\pi/3 + \sqrt{3} \approx 4.82645$. The latter is the cost of the evacuation algorithm for two robots without priority of [10].

3.2 Evacuation Algorithm for PE$_2$

In this subsection we prove the following theorem.

▶ **Theorem 10.** PE$_2$ *can be solved in time 3.8327.*

Given parameters α, ρ, we introduce the family of trajectories SEARCH$_2(\alpha,\rho)$, see also Figure 3.

Algorithm Search$_2(\alpha,\rho)$

Robot	*#*	*Description*	*Trajectory*	*Duration*
$\mathcal{Q}$	0	Move to point $C_{\pi-\alpha}$	$\mathcal{L}(O, C_{\pi-\alpha}, t)$	1
	1	Search the circle ccw till point C_π	$\mathcal{C}(\pi-\alpha, t-1)$	α
	2	Move to point $K(\alpha/2,\rho)$	$\mathcal{L}(C_\pi, K(\alpha/2,\rho), t-(1+\alpha))$	$AK(\alpha/2,\rho)$
	3	Move to point $C_{-\alpha/2}$	$\mathcal{L}(K(\alpha/2,\rho), C_{-\alpha/2})$	$2-2\rho$
$\mathcal{S}_1$	0	Move to point $C_{\pi-\alpha}$	$\mathcal{L}(O, C_{\pi-\alpha})$	1
	1	Search the circle cw till point $C_{-\alpha/2}$	$\mathcal{C}(\pi-\alpha, -t+1)$	$\pi-\alpha/2$
$\mathcal{S}_2$	0	Move to point C_π	$\mathcal{L}(O, C_\pi)$	1
	1	Search the circle cw till point $C_{-\alpha/2}$	$\mathcal{C}(\pi, t-1)$	$\pi-\alpha/2$

Notice that, by definition of SEARCH$_2(\alpha,\rho)$, robots' trajectories are continuous and feasible, meaning that the entire circle is eventually searched. Indeed, partitioning the circle clockwise, we see that: the arc with endpoints $C_\pi, C_{\pi-\alpha}$ is searched by $\mathcal{Q}$, the arc with endpoints $C_{\pi-\alpha}, C_{-\alpha/2}$ is searched by $\mathcal{S}_1$, and the arc with endpoints $C_{-\alpha/2}, C_\pi$ is searched by $\mathcal{S}_2$.

It is immediate from the description of the trajectories that the search time is $1+\pi-\alpha/2$. Moreover

$$\mathbb{I}(\mathcal{Q}) = [1, 1+\alpha],\ \mathbb{I}(\mathcal{S}_1) = \mathbb{I}(\mathcal{S}_2) = [1, 1+\pi-\alpha/2].$$

An illustration of the above trajectories for certain values of α, ρ can be seen in Figure 3. Now we make some observations, in order to calculate the worst case evacuation time.

▶ **Lemma 11.** *Suppose that $\pi - \alpha/2 \geq \alpha + AK(\alpha/2, \rho) + 2 - 2\rho$. Then $\|\mathcal{Q}(x) - \mathcal{S}_1(t)\|$ is continuous and differentiable in the time intervals I_1, I_2, I_3 of $\mathcal{Q}$'s phases 1,2,3, respectively. Moreover, the worst case evacuation time of* Search$_2(\alpha, \rho)$ *can be computed as*

$$\max \left\{ \begin{array}{l} 1+\alpha+2\sin(\alpha), \\ \sup_{t\in I_2} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ \sup_{t\in I_3} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ 1+\pi-\alpha/2 \end{array} \right\}$$

where

$$I_2 = [1+\alpha, 1+\alpha+AK(\alpha/2,\rho)], I_3 = [1+\alpha+AK(\alpha/2,\rho), 3-2\rho+\alpha+AK(\alpha/2,\rho)].$$

Proof. Note that the line passing through O and $C_{-\alpha/2}$, call it ϵ, has the property that each point of it, including $K(\alpha/2, \rho)$ is equidistant from $\mathcal{S}_1, \mathcal{S}_2$. Moreover, in the time window $[1+\alpha, 1+\alpha+AK(\alpha/2,\rho)]$ that only $\mathcal{S}_1, \mathcal{S}_2$ are searching, $\mathcal{Q}$ stays below line ϵ. At time $1+\alpha+AK(\alpha/2,\rho)$, $\mathcal{Q}$ is, by construction, equidistant from $\mathcal{S}_1, \mathcal{S}_2$, a property that is preserved for the remaining of the execution of the algorithm. As a result, the evacuation time of Search$_2(\alpha, \rho)$ is given by

$$\sup_{1\leq t\leq 1+\pi-\alpha/2} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\}.$$

Now note that condition $\pi - \alpha/2 \geq \alpha + AK(\alpha/2, \rho) + 2 - 2\rho$ guarantees that $\mathcal{Q}$ reaches point $C_{-\alpha/2}$ no later than $\mathcal{S}_1$. Moreover, in each time interval I_1, I_2, I_3, $\mathcal{Q}$'s trajectory is differentiable (and so is $\mathcal{S}_1$'s trajectory). ◀

Now Theorem 10 can be proven by fixing parameters α, ρ for Search$_2(\alpha, \rho)$, in particular, $\alpha = 0.6361, \rho = 0.7944$. Notably, the performance of Search$_2(\alpha, \rho)$ is provably improvable (slightly) using a technique we will describe in the next section.

3.3 Evacuation Algorithm for PE$_3$

3.3.1 A Simple Algorithm

In this section we prove the following preliminary theorem (to be improved in Section 3.3.2).

▶ **Theorem 12.** PE$_3$ *can be solved in time 3.37882.*

Given parameters α, β, ρ, we introduce the family of trajectories Search$_3(\alpha, \beta, \rho)$, corresponding to robots $\mathcal{Q}, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, see also Figure 4.

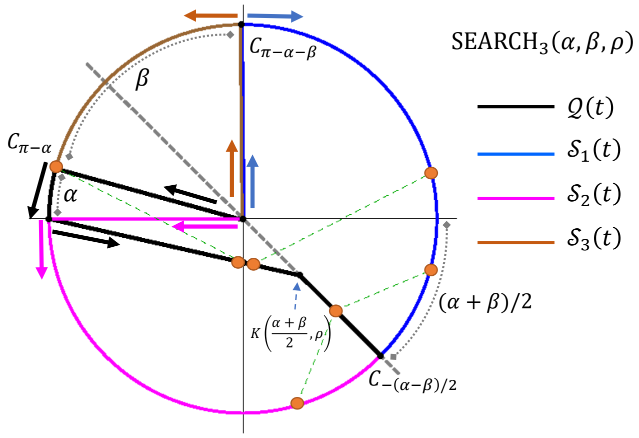

Figure 4 Algorithm SEARCH$_3(\alpha, \beta, \rho)$ depicted for the optimal parameters of the algorithm.

Algorithm Search$_3(\alpha, \beta, \rho)$

Robot	#	*Description*	*Trajectory*	*Duration*
$\mathcal{Q}$	0	Move to point $C_{\pi-\alpha}$	$\mathcal{L}(O, C_{\pi-\alpha}, t)$	1
	1	Search the circle ccw till point C_π	$\mathcal{C}(\pi-\alpha, t-1)$	α
	2	Move to point $K(\frac{\alpha+\beta}{2}, \rho)$	$\mathcal{L}(C_\pi, K(\frac{\alpha+\beta}{2}, \rho), t-(1+\alpha))$	$AK(\frac{\alpha+\beta}{2}, \rho)$
	3	Move to point $C_{-\frac{\alpha+\beta}{2}}$	$\mathcal{L}(K(\frac{\alpha+\beta}{2}, \rho), C_{-\frac{\alpha+\beta}{2}})$	$2-2\rho$
$\mathcal{S}_1$	0	Move to point $C_{\pi-\alpha-\beta}$	$\mathcal{L}(O, C_{\pi-\alpha-\beta})$	1
	1	Search the circle cw till point $C_{-\frac{\alpha+\beta}{2}}$	$\mathcal{C}(\pi-\alpha-\beta, -t+1)$	$\pi-\frac{\alpha+\beta}{2}$
$\mathcal{S}_2$	0	Move to point C_π	$\mathcal{L}(O, C_\pi)$	1
	1	Search the circle ccw till point $C_{-\frac{\alpha+\beta}{2}}$	$\mathcal{C}(\pi, t-1)$	$\pi-\frac{\alpha+\beta}{2}$
$\mathcal{S}_3$	0	Move to point $C_{\pi-\alpha-\beta}$	$\mathcal{L}(O, C_{\pi-\alpha-\beta})$	1
	1	Search the circle ccw till point $C_{-\alpha}$	$\mathcal{C}(\pi-\alpha-\beta, -t+1)$	β

As before, it is immediate that, in SEARCH$_3(\alpha, \beta, \rho)$, robots' trajectories are continuous and feasible, meaning that the entire circle is eventually searched. In particular, the arc with endpoints $C_\pi, C_{\pi-\alpha}$ is searched by $\mathcal{Q}$, the arc with endpoints $C_{\pi-\alpha-\beta}, C_{-\frac{\alpha+\beta}{2}}$ is searched by $\mathcal{S}_1$, the arc with endpoints $C_{-\pi}, C_{-\frac{\alpha+\beta}{2}}$ is searched by $\mathcal{S}_2$, and the arc with endpoints $C_{\pi-\alpha}, C_{\pi-\alpha-\beta}$ is searched by $\mathcal{S}_3$. Also, the search time is $1+\pi-\frac{\alpha+\beta}{2}$, and

$$\mathbb{I}(\mathcal{Q}) = [1, 1+\alpha],\ \mathbb{I}(\mathcal{S}_1) = \mathbb{I}(\mathcal{S}_2) = [1, 1+\pi-\frac{\alpha+\beta}{2}],\ \mathbb{I}(\mathcal{S}_3) = [1, 1+\beta].$$

An illustration of the above trajectories for certain values of α, β, ρ can be seen in Figure 4.

Before we prove Theorem 12, we need to make some observation, in order to calculate the worst case evacuation time.

▶ **Lemma 13.** *Suppose that $\alpha \le \beta$, $\alpha + AK(\frac{\alpha+\beta}{2}, \rho) \ge \beta$, and $\pi - \frac{\alpha+\beta}{2} \ge \alpha + AK(\frac{\alpha+\beta}{2}, \rho) + 2 - 2\rho$. Then the following functions are continuous and differentiable in each associated time intervals: $\|\mathcal{Q}(x) - \mathcal{S}_3(t)\|$ in $I_1 = \{t \ge 0 :\ \alpha \le t-1 \le \beta\}$, $\|\mathcal{Q}(x) - \mathcal{S}_1(t)\|$ in $I_2 = \{t \ge 0 :\ |t-1-\alpha| \le AK(\frac{\alpha+\beta}{2}, \rho)\}$ and in $I_3 = \{t \ge 0 :\ |t-1-\alpha-AK(\frac{\alpha+\beta}{2}, \rho)| \le 2-2\rho\}$. Moreover, the worst case evacuation time of* SEARCH$_3(\alpha, \beta, \rho)$ *can be computed as*

$$\max\left\{\begin{array}{l} \sup_{t\in I_1}\{t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|\} \\ \sup_{t\in I_2}\{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ \sup_{t\in I_3}\{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ 1+\pi-\frac{\alpha+\beta}{2} \end{array}\right\}$$

Proof. Conditions $\alpha \leq \beta$ and $\alpha + AK(\frac{\alpha+\beta}{2}, \rho) \geq \beta$ mean that $\mathcal{Q}$ stops searching no later than $\mathcal{S}_3$, and that when $\mathcal{S}_3$ stops searching $\mathcal{Q}$ is still in her phase 2, respectively.

The line passing through O and $C_{-(\alpha+\beta)/2}$, call it ϵ, has the property that each point of it, including $K(\frac{\alpha+\beta}{2}, \rho)$ is equidistant from $\mathcal{S}_1, \mathcal{S}_2$. Moreover, while $\mathcal{S}_1, \mathcal{S}_2$ are searching, $\mathcal{Q}$ never goes above line ϵ. At time $1 + \alpha + AK(\frac{\alpha+\beta}{2}, \rho)$, $\mathcal{Q}$ is, by construction, equidistant from $\mathcal{S}_1, \mathcal{S}_2$, a property that is preserved for the remaining of the execution of the algorithm. As a result, $\mathcal{S}_2$ can be ignored in the performance analysis, and when it comes to the case that $\mathcal{S}_1$ finds the exit, the evacuation cost is given by the supremum of $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ in the time interval I_2 or in the interval I_3. Note that in both intervals, the evacuation cost is continuous and differentiable, by construction.

If the exit is reported by $\mathcal{S}_3$ then the evacuation cost is $t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|$ for $t \in [1, 1+\beta]$. However, it is easy to see that the cost is strictly increasing for all $t \in [1, 1+\alpha]$ (in fact it is linear). Since the evacuation cost is also continuous, we may restrict the analysis in interval I_1.

Lastly, observe that $\pi - \frac{\alpha+\beta}{2} \geq \alpha + AK(\frac{\alpha+\beta}{2}, \rho) + 2 - 2\rho$ implies that $\mathcal{S}_1, \mathcal{S}_2$ reach point $C_{-(\alpha+\beta)/2}$ no earlier than $\mathcal{Q}$. Hence $\mathcal{Q}$ waits at $C_{-(\alpha+\beta)/2}$ till the search of the circle is over, which can be easily seen to induce the worse evacuation time after $\mathcal{Q}$ reaches $C_{-(\alpha+\beta)/2}$. ◀

We prove Theorem 12 by fixing parameters α, β, ρ for Search$_3(\alpha, \beta, \rho)$, in particular $\alpha = 0.26738, \beta = 1.2949, \rho = 0.70685$.

3.3.2 Improved Search Algorithm

In this section we improve the upper bound of Theorem 12 by 0.00495 additive term.

▶ **Theorem 14.** PE$_3$ *can be solved in time 3.37387.*

The main idea can be described, at a high level, as a cost preservation technique. By the analysis of Algorithm Search$_3(\alpha, \beta, \rho)$ for the value of parameters of α, β, ρ as in the proof of Theorem 12, we know that there are is a critical time window $[\tau_2, \tau_3]$ so that the total evacuation time is the same if the exit is found by $\mathcal{S}_1$ either at time τ_2 or τ_3, and strictly less for time moments strictly in-between. In fact, during time $[\tau_2, 1 + \alpha + AK(\frac{\alpha+\beta}{2}, \rho)]$ $\mathcal{Q}$ is executing phase 2, and in the time window $[1 + \alpha + AK(\frac{\alpha+\beta}{2}, \rho), \tau_3]$ $\mathcal{Q}$ is executing phase 3 of Search$_3(\alpha, \beta, \rho)$.

From the above, it is immediate that we can lower $\mathcal{Q}$'s speed in the time window $[\tau_2, \tau_3]$ so that the evacuation time remains *unchanged* no matter when $\mathcal{S}_1$ finds the exit in the same time interval (notably, $\mathcal{S}_3$ has finished searching prior to τ_2 and $\|\mathcal{Q}(t) - \mathcal{S}_1\| \geq \|\mathcal{Q}(t) - \mathcal{S}_2\|$). But this also implies that we must be able to maintain the evacuation time even if we preserve speed 1 for $\mathcal{Q}$, that will in turn allow us to twist parameters α, β, ρ, hopefully improving the worst case evacuation time. We show this improvement is possible by using the following technical observation

▶ **Theorem 15.** *Consider point $Q = (q_1, q_2) \in \mathbb{R}^2$. Let $\mathcal{S}(t)$ be the trajectory of an object $\mathcal{S}$ moving at speed 1, where $t \geq 0$, and denote by ϕ the $(\mathcal{S}, Q)$-critical angle at time $t = 0$. Assuming that $\cos(\phi) \geq 0$, then there is some $\tau > 0$, and a trajectory $\mathcal{Q}(t) = (f(t), g(t))$ of a speed-1 object, where $t \geq 0$, so that $t + \|\mathcal{Q}(t) - \mathcal{S}(t)\|$ remains constant, for all $t \in [0, \tau]$. Moreover, $\mathcal{Q}(t)$ can be determined by solving the system of differential equations*

$$(f'(t))^2 + (g'(t))^2 = 1 \tag{2}$$

$$t + \|\mathcal{Q}(t) - \mathcal{S}(t)\| = \|\mathcal{S}(0) - Q\| \tag{3}$$

$$(f(0), g(0)) = (q_1, q_2). \tag{4}$$

Proof. An object with trajectory $(f(t), g(t))$ satisfying (2) and (4) has speed 1 (by Lemma 2), and starts from point $Q = (q_1, q_2)$. We need to examine whether we can choose f, g so as to satisfy (3).

By Lemma 7, such a trajectory $\mathcal{Q}(t)$ exists exactly when we can guarantee that $\cos(\phi) + \cos(\theta) = 1$ over time t. When $t = 0$ we are given that $\cos(\phi) > 0$, hence there exists θ satisfying $\cos(\phi) + \cos(\theta) = 1$. This uniquely determines the velocity of $\mathcal{Q}$ at $t = 0$.

By continuity of the velocities, there must exist a $\tau > 0$ such that $\cos(\phi) + \cos(\theta) = 1$ admits a solution for θ also as ϕ changes over time $t \in [0, \tau]$, in which time window the cosine of the $(\mathcal{S}, \mathcal{Q}(t))$-critical angle at time t remains non-negative. ◀

Note that condition $\cos(\phi) \geq 0$ of Theorem 15 translates to that $\|\mathcal{S}(t) - Q\|$ is not increasing at $t = \tau$, i.e. that $\mathcal{S}$ does not move away from point Q.

Now fix parameters α, β, ρ together with the trajectories of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ as in the description of Algorithm $\text{Search}_3(\alpha, \beta, \rho)$. The description of our *new algorithm* N-$\text{Search}_3(\alpha, \beta, \rho)$ will be complete once we fix a new trajectory for $\mathcal{Q}$. Naming specific values for parameters α, β, ρ will eventually prove Theorem 14. In order to do so, we introduce some *further notation and conditions*, denoted below by *(Conditions i-iv)*, that we later make sure are satisfied.

Consider $\mathcal{Q}$'s trajectory as in $\text{Search}_3(\alpha, \beta, \rho)$. Let τ_0 denote a local maximum of

$$t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$$

as it reads for $t \geq 0$ with $|t - 1 - \alpha| \leq AK(\frac{\alpha+\beta}{2}, \rho)$ (recall that in this time window, expression is differentiable by Lemma 13), i.e.

$$|\tau_0 - 1 - \alpha| \leq AK(\frac{\alpha+\beta}{2}, \rho) \qquad \text{(Condition i)}$$

Set $Q = \mathcal{Q}(\tau_0)$, and assume that

"The cosine of the $(\mathcal{S}, Q)$-critical angle at time τ_0 is non-negative." (Condition ii)

Then obtain from Theorem 15 trajectory $(f(t), g(t))$ that has the property that it preserves $\tau_0 + \|\mathcal{Q}(\tau_0) - \mathcal{S}_1(\tau_0)\|$ in the time window $[\tau_0, \tau']$. Assume also that

"There is time $\tau_1 \leq \tau'$ such that point $K_1 := (f(\tau_1), g(\tau_1))$ is equidistant from $\mathcal{S}_1(\tau_1), \mathcal{S}_2(\tau_1)$," (Condition iii)

for the first time after time τ_0, such that

$$\tau_1 \leq 1 + \pi - \frac{\alpha + \beta}{2}. \qquad \text{(Condition iv)}$$

Then consider the following modification of $\text{Search}_3(\alpha, \beta, \rho)$, where the trajectories of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ remain unchanged, see also Figure 5.

Algorithm N-Search$_3(\alpha, \beta, \rho)$

Robot	#	*Description*	*Trajectory*	*Duration*
$\mathcal{Q}$	0	Move to point $C_{\pi-\alpha}$	$\mathcal{L}(O, C_{\pi-\alpha}, t)$	1
	1	Search the circle ccw till point C_π	$\mathcal{C}(\pi - \alpha, t - 1)$	α
	2	Move toward point $K(\frac{\alpha+\beta}{2}, \rho)$	$\mathcal{L}(C_\pi, K(\frac{\alpha+\beta}{2}, \rho), t - (1 + \alpha))$	$\tau_0 - 1 - \alpha$
	3	Preserve $\tau_0 + \|\mathcal{Q}(\tau_0) - \mathcal{S}_1(\tau_0)\|$	$(f(t), g(t))$	$\tau_1 - \tau_0$
	4	Move to point $C_{-\frac{\alpha+\beta}{2}}$	$\mathcal{L}(K_1, C_{-\frac{\alpha+\beta}{2}})$	$\left\|K_1 - C_{-\frac{\alpha+\beta}{2}}\right\|$

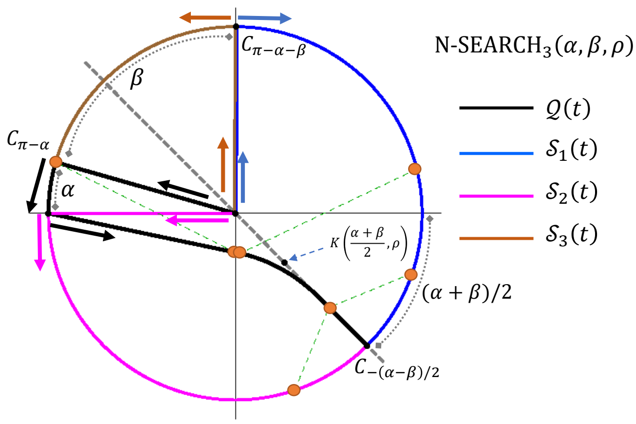

Figure 5 Algorithm SEARCH3(α, β, ρ) depicted for the optimal parameters of the algorithm.

Note that in phase 2, $\mathcal{Q}$ is not reaching (necessarily) point K rather it moves toward it for a certain duration. The search time is still $1 + \pi - \frac{\alpha+\beta}{2}$. Trajectories of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are continuous as before, and

$$\mathbb{I}(\mathcal{S}_1) = \mathbb{I}(\mathcal{S}_2) = [1, 1 + \pi - \frac{\alpha+\beta}{2}],\ \mathbb{I}(\mathcal{S}_3) = [1, 1 + \beta],$$

as well as $\mathbb{I}(\mathcal{Q}) = [1, 1 + \alpha]$.

Condition i makes sure that while $\mathcal{Q}$ is at phase 2, and before it reaches $K(\frac{\alpha+\beta}{2}, \rho)$, there is a time moment τ_0 when the rate of change of $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ is 0. Together with condition ii, this implies that Theorem 15 applies. In fact, for the corresponding critical angles ϕ, θ between $\mathcal{S}_1, \mathcal{Q}$ at time τ_0, we have that $\cos(\phi) + \cos(\theta) = 1$ by construction. Hence trajectory $(f(t), g(t))$ of phase 3 is well defined, and indeed, $\mathcal{Q}$ jumps from phase 2 to phase 3 while $\mathcal{Q}$ is still moving toward point K. Notably, $\mathcal{Q}$'s trajectory is even differentiable at $t = \tau_0$ (but not necessarily at $t = \tau_1$). Then, Condition iii says that $\mathcal{Q}$ eventually will enter phase 4, and that this will happen before $\mathcal{S}_1, \mathcal{S}_2$ finish the exploration of the circle. Overall, we conclude that in N-SEARCH3(α, ρ), robots' trajectories are continuous and feasible. An illustration of the above trajectories for certain values of α, β, ρ can be seen in Figure 5.

Now we make some observations, in order to calculate the worst case evacuation time.

▶ **Lemma 16.** *Suppose that $\alpha \le \beta$, $1 + \beta \le \tau_0$, and $1 + \pi - \frac{\alpha+\beta}{2} \ge \tau_1 + \left\| K_1 - C_{-\frac{\alpha+\beta}{2}} \right\|$ as well as Conditions i-iv are satisfied. Then the following functions are continuous and differentiable in each associated time intervals: $\|\mathcal{Q}(x) - \mathcal{S}_3(t)\|$ in $I_1 = \{t \ge 0 :\ \alpha \le t - 1 \le \beta\}$, $\|\mathcal{Q}(x) - \mathcal{S}_1(t)\|$ in $I_2 = \{t \ge 0 :\ 1 + \alpha \le t \le \tau_0$ and in $I_3 = \left\{t \ge 0 :\ |t - \tau_1| \le \left\| K_1 - C_{-\frac{\alpha+\beta}{2}} \right\| \right\}$. Moreover, the worst case evacuation time of* N-SEARCH3(α, β, ρ) *can be computed as*

$$\max \left\{ \begin{array}{l} \sup_{t \in I_1} \{t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|\} \\ \sup_{t \in I_2} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ \sup_{t \in I_3} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ 1 + \pi - \frac{\alpha+\beta}{2} \end{array} \right\}$$

Proof. Conditions $\alpha \le \beta$ and $1 + \beta \le \tau_0$ mean that $\mathcal{Q}$ stops searching no later than $\mathcal{S}_3$, and that when $\mathcal{Q}$ enters phase 3 after $\mathcal{S}_3$ is done searching, respectively.

The line passing through O and $C_{-(\alpha+\beta)/2}$, call it ϵ, has the property that each point of it, including $K(\frac{\alpha+\beta}{2}, \rho)$ is equidistant from $\mathcal{S}_1, \mathcal{S}_2$. Moreover, while $\mathcal{S}_1, \mathcal{S}_2$ are searching,

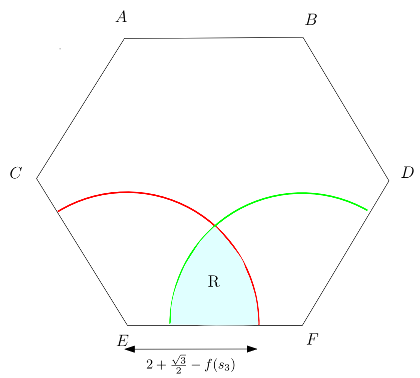

Figure 6 (Left) The queen must be in region R at time $f(s_3)$. Here $s_3 = E$ and $q_3 = F$.

$\mathcal{Q}$ never goes above line ϵ. Also, while $\mathcal{Q}$ is executing phase 3, $\mathcal{Q}$ remains equidistant from $\mathcal{S}_1, \mathcal{S}_2$ and this is preserved for the remainder of the execution of the algorithm. As a result, $\mathcal{S}_2$ can be ignored in the performance analysis, and when it comes to the case that $\mathcal{S}_1$ finds the exit, the evacuation cost is given by the supremum of $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ in the time interval I_2 or in the interval I_3. Note that in both intervals, the evacuation cost is continuous and differentiable, by construction.

If the exit is reported by $\mathcal{S}_3$ then the evacuation cost is $t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|$ for $t \in [1, 1+\beta]$. However, it is easy to see that the cost is strictly increasing for all $t \in [1, 1+\alpha]$ (in fact it is linear). Since the evacuation cost is also continuous, we may restrict the analysis in interval I_1.

Lastly, observe that $1 + \pi - \frac{\alpha+\beta}{2} \geq \tau_1 + \left\| K_1 - C_{-\frac{\alpha+\beta}{2}} \right\|$ implies that $\mathcal{S}_1, \mathcal{S}_2$ reach point $C_{-(\alpha+\beta)/2}$ no earlier than $\mathcal{Q}$. Hence $\mathcal{Q}$ waits at $C_{-(\alpha+\beta)/2}$ till the search of the circle is over, which can be easily seen to induce the worse evacuation time after $\mathcal{Q}$ reaches $C_{-(\alpha+\beta)/2}$. ◀

We can prove now Theorem 14 by fixing parameters α, β, ρ for N-SEARCH$_3(\alpha, \beta, \rho)$, in particular $\alpha = 0.27764, \beta = 1.29839, \rho = 0.68648$.

4 Lower Bounds

In this section we derive lower bounds for evacuation. In Section 4.1 we treat the case of $n = 1$ (see Theorem 17) and in Section 4.2 we treat the case of $n = 2$ and 3 (see Theorem 19).

4.1 Lower Bound for PE$_1$

We will derive the lower bound using an adversarial argument placing the exit at an unknown vertex of a regular hexagon.

▶ **Theorem 17.** *The worst-case evacuation time for* PE$_1$ *is at least* $3 + \pi/6 + \sqrt{3}/2 \approx 4.3896$

Proof. At time $1 + \pi/6$, at most $\pi/3$ of the perimeter of the circle can have been explored by the queen and servant. Thus, there is a regular hexagon, none of whose vertices have been explored. If the exit is at one of these vertices, by Theorem 18, it takes $2 + \sqrt{3}/2$ for the queen to evacuate. The total time is $1 + \pi/6 + 2 + \sqrt{3}/2$. ◀

Next we proceed to provide a lower bound on a unit-side hexagon. Label the vertices of the hexagon V as $A, \ldots, F$ as shown in Figure 6. Fix an evacuation algorithm $\mathcal{A}$. For any vertex v of the hexagon, we call $f(v)$ the time of *first visit* of the vertex v by either the servant or the queen, according to algorithm $\mathcal{A}$. We call $q(v)$ the time that the queen gets to the vertex v. Clearly, $q(v) \geq f(v)$, and if the queen arrives at the vertex no later than the servant, $q(v) = f(v)$.

▶ **Theorem 18.** *For any algorithm $\mathcal{A}$, the evacuation time for the queen when the exit is at one of the vertices of the hexagon is $max_{v \in V}\{q(v)\} \geq 2 + \sqrt{3}/2$.*

Proof. Suppose there is an algorithm in which the queen can always evacuate in time $< 2 + \sqrt{3}/2$. Consider the trajectories of the servant and the queen. If either the queen or the servant are the first to visit 4 vertices, then for the fourth such vertex v, we have $f(v) \geq 3$, a contradiction. Therefore, the queen is the first to visit three vertices, and the servant is the first to visit three vertices. We denote the three vertices visited first by the servant as s_1, s_2, s_3 (in the order they are visited) and the three vertices visited first by the queen as q_1, q_2, q_3, and note that they are all distinct.

Notice that neither s_3 nor q_3 can be visited before time 2, that is, $f(s_3), f(q_3) \geq 2$. If $f(q_3) \leq f(s_3)$, then we place the exit at s_3, and the queen needs time at least 1 to get to s_3, which implies that $T \geq q(s_3) \geq f(q_3) + 1 \geq 3$, a contradiction. We conclude that at time $f(s_3)$, the queen is yet to visit q_3. Since the exit can be at either s_3 or q_3, at time $f(s_3)$, the queen must be at distance $< 2 + \sqrt{3}/2 - f(s_3) \leq \sqrt{3}/2$ from *both* s_3 and q_3.

Assume without loss of generality that $s_3 = E$ (see Figure 6). Since A, B, D are all at distance at least $\sqrt{3}$ from E, we conclude that q_3 is either C or F. Assume without loss of generality that $q_3 = F$. Let R denote the lens-shaped region that is at distance $< 2 + \sqrt{3}/2 - f(s_3)$ from both E and F. Recall that at time $f(s_3)$, the queen must be inside the region R. Notice that if $f(s_3) \geq 1.5 + \sqrt{3}/2$, the region R is empty, yielding a contradiction. So it must be that $2 \leq f(s_3) < 1.5 + \sqrt{3}/2$.

We now work backwards to deduce the trajectories of the servant and the queen. Clearly $s_2 \neq F$ since $q_3 = F$. If $s_2 \neq C$, then $f(s_3) \geq \sqrt{3} + 1 > 1.5 + \sqrt{3}/2$, a contradiction. Therefore, $s_2 = C$. By the same reasoning, $s_1 = A$. Therefore, the queen is the first to visit D and B. If $q_1 = D$ and $q_2 = B$, we place the exit at E; since $f(q_2) \geq 1$ and $dist(B, E) = 2$, we have $T \geq q(E) \geq 3$, a contradiction. Thus, $q_2 = D$ and $q_1 = B$.

Consider the location of the queen at time 1. If she is at distance $\geq 1 + \sqrt{3}/2$ from C at time 1, then if the exit is at C, $q(C) \geq 2 + \sqrt{3}/2$. So at time 1, the queen must be at distance $< 1 + \sqrt{3}/2$ from C and consequently she is at distance $\geq 1 - \sqrt{3}/2$ from vertex D. Therefore $f(q_2) = f(D) \geq 2 - \sqrt{3}/2$. Also, $f(D) < 1.5$ since if the queen reaches D at or after time 1.5, she cannot reach the region R before time $1.5 + \sqrt{3}/2 > f(s_3)$. So $f(D) \leq f(s_3)$. If the exit is at $E = s_3$, the queen cannot reach the exit before time $f(D) + dist(D, E) \geq 2 - \sqrt{3}/2 + \sqrt{3} = 2 + \sqrt{3}$, concluding the proof by contradiction. ◀

We remark that the above bound is optimal, and is achieved by the algorithm depicted in Figure 7.

4.2 Lower Bounds for PE_2 and PE_3

In the case of $n = 2$ and $n = 3$ the proof is rather technical and we will only present a high level outline as to why the lower bounds hold.

▶ **Theorem 19.** *The worst-case evacuation time for* PE_2 *is at least* 3.6307 *and for* PE_3 *at least* 3.2017*.*

Throughout this section we will use $\mathcal{T}$ to refer to the evacuation time of an arbitrary algorithm and use $\mathcal{U}$ to refer to the unit circle which must be evacuated.

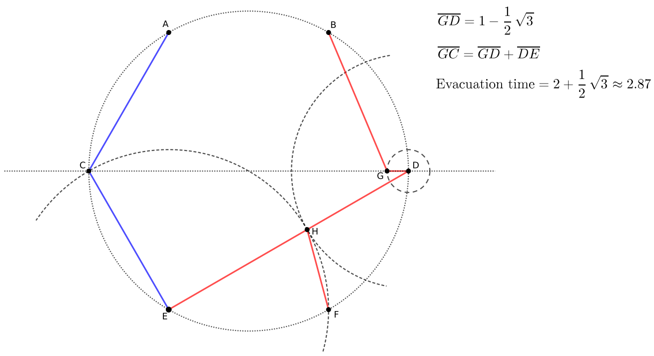

Figure 7 Blue trajectory: servant and red trajectory: queen. At point H, if the queen hears of an exit at E, she goes there, otherwise she goes to F.

The main thrust of the proof relies on a simple idea – the queen should aid in the exploration of $\mathcal{U}$. This is immediately evident for the particular case of $n = 2$ since, if the queen does not explore, it will take time at least $1 + \pi$ for the servants to search all of $\mathcal{U}$ and we already have an upper bound smaller than this (Theorem 10). Thus, a general overview of the proof is as follows: we show that in order to evacuate in time $\mathcal{T}$ the queen must explore some minimum length of the perimeter of $\mathcal{U}$. We will then demonstrate that the queen is not able to explore this minimum amount in any algorithm with evacuation time smaller than what is given in Theorem 19.

To be concrete, consider the case of $n = 2$ and assume that we have an algorithm with evacuation time $\mathcal{T} < 1 + \pi$. Then, in order for the robots to have explored all of $\mathcal{U}$ in time $\mathcal{T}$, the queen must explore a subset of the perimeter of total length at least $2(1 + \pi - \mathcal{T})$. Intuitively, this minimum length of perimeter will increase in size as $\mathcal{T}$ decreases.

Now consider that it is not possible for the queen to always remain on the perimeter (indeed, in each of the algorithms presented, the queen leaves the perimeter). To see why this is consider that, in any algorithm with evacuation time $\mathcal{T}$, it must be the case that all unexplored points of $\mathcal{U}$ are located a distance no more than $\mathcal{T} - t$ from the queen at all times $t \leq \mathcal{T}$. If the queen is on the perimeter at any time t satisfying $\mathcal{T} - t \leq 2$, then, there will be some arc $\theta(t, \mathcal{T}) \subset \mathcal{U}$ such that all points of $\theta(t, \mathcal{T})$ are at a distance at least $\mathcal{T} - t$ from the queen. Thus, if the queen is to be on the perimeter at the time t we can conclude that all of the arc $\theta(t, \mathcal{T})$ must have already been discovered. However, we will find that $\theta(t, \mathcal{T})$ will often grow at a rate much larger than the robots can collectively explore and at some point the queen will have to leave the perimeter. In fact, there will be an interval of time during which it is not possible for the queen to be exploring and this in turn implies that there is a maximum amount of perimeter that can be explored by the queen. Intuitively, the maximum length of perimeter that can be explored by the queen will decrease as $\mathcal{T}$ decreases. The lower bound will result by balancing the minimum amount of perimeter the queen needs to search and the maximum amount of perimeter that the queen is able to search.

The above argument will need a slight modification in the case of $n = 3$. In this case we will show that there is some critical time t_* before which the queen must have explored some minimum amount of perimeter. Again, the lower bound follows by balancing the maximum amount of perimeter the queen can explore by the time t_* and the minimum amount of perimeter the queen needs to explore before the time t_*.

5 Conclusion

We considered an evacuation problem concerning priority searching on the perimeter of a unit disk where only one robot (the queen) needs to find the exit. In addition to the queen, there are $n \leq 3$ other robots (servants) aiding the queen by contributing to the exploration of the disk but which do not need to evacuate. We proposed evacuation algorithms and studied non-trivial tradeoffs on the queen evacuation time depending on the number n of servants. In addition to analyzing tradeoffs and improving the bounds obtained for the wireless communication model, an interesting open problem would be to investigate other models with limited communication range, e.g., face-to-face.

References

1. R. Ahlswede and I. Wegener. *Search problems.* Wiley-Interscience, 1987.
2. S. Alpern and S. Gal. *The theory of search games and rendezvous*, volume 55. Kluwer Academic Publishers, 2002.
3. Steve Alpern, Robbert Fokkink, Leszek Gąsieniec, Roy Lindelauf, and V.S. Subrahmanian, editors. *Ten Open Problems in Rendezvous Search*, pages 223–230. Springer NY, New York, NY, 2013.
4. R. Baeza Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Information and Computation*, 106(2):234–252, 1993.
5. R. Baeza-Yates and R. Schott. Parallel searching in the plane. *Computational Geometry*, 5(3):143–154, 1995.
6. A. Beck. On the linear search problem. *Israel J. of Mathematics*, 2(4):221–228, 1964.
7. R. Bellman. An optimal search. *SIAM Review*, 5(3):274–274, 1963.
8. S. Brandt, F. Laufenberg, Y. Lv, D. Stolz, and R. Wattenhofer. Collaboration without communication: Evacuating two robots from a disk. In *Proceedings of Algorithms and Complexity - 10th International Conference, CIAC 2017, Athens, Greece, May 24-26, 2017*, pages 104–115, 2017.
9. J. Czyzowicz, S. Dobrev, K. Georgiou, E. Kranakis, and F. MacQuarrie. Evacuating two robots from multiple unknown exits in a circle. *Theor. Comput. Sci.*, 709:20–30, 2018.
10. J. Czyzowicz, L. Gasieniec, T. Gorry, E. Kranakis, R. Martin, and D. Pajak. Evacuating robots from an unknown exit located on the perimeter of a disc. In *Proceedings DISC, Austin, Texas*, pages 122–136. Springer, 2014.
11. J. Czyzowicz, K. Georgiou, M. Godon, E. Kranakis, D. Krizanc, W. Rytter, and M. Wlodarczyk. Evacuation from a disc in the presence of a faulty robot. In *Proceedings SIROCCO 2017, 19-22 June 2017, Porquerolles, France*, pages 158–173, 2018.
12. J. Czyzowicz, K. Georgiou, R. Killick, E. Kranakis, D. Krizanc, L. Narayanan, J. Opatrny, and S. Shence. God Save the Queen. *CoRR*, abs/1804.06011, 2018.
13. J. Czyzowicz, K. Georgiou, R. Killick, E. Kranakis, D. Krizanc, L. Narayanan, J. Opatrny, and S. Shende. Priority evacuation from a disk using mobile robots, 2018, Submitted.
14. J. Czyzowicz, K. Georgiou, E. Kranakis, L. Narayanan, J. Opatrny, and B. Vogtenhuber. Evacuating robots from a disk using face-to-face communication (extended abstract). In *Proceedings of Algorithms and Complexity, CIAC 2015, Paris, France, May 20-22, 2015*, pages 140–152, 2015.
15. J. Czyzowicz, E. Kranakis, D. Krizanc, L. Narayanan, J. Opatrny, and S. Shende. Wireless autonomous robot evacuation from equilateral triangles and squares. In *Proceedings of Ad-hoc, Mobile, and Wireless Networks, ADHOC-NOW, Athens, Greece, June 29 - July 1, 2015*, pages 181–194, 2015.

16 Konstantinos Georgiou, George Karakostas, and Evangelos Kranakis. Search-and-fetch with one robot on a disk - (track: Wireless and geometry). In *Algorithms for Sensor Systems - 12th International Symposium on Algorithms and Experiments for Wireless Sensor Networks, ALGOSENSORS 2016, Aarhus, Denmark, August 25-26, 2016, Revised Selected Papers*, pages 80–94, 2016.

17 Konstantinos Georgiou, George Karakostas, and Evangelos Kranakis. Search-and-fetch with 2 robots on a disk - wireless and face-to-face communication models. In Federico Liberatore, Greg H. Parlier, and Marc Demange, editors, *Proceedings of the 6th International Conference on Operations Research and Enterprise Systems, ICORES 2017, Porto, Portugal, February 23-25, 2017*, pages 15–26. SciTePress, 2017.

18 I. Lamprou, R. Martin, and S. Schewe. Fast two-robot disk evacuation with wireless communication. In *Proceedings DISC, Paris, France*, pages 1–15, 2016.

19 D. Pattanayak, H. Ramesh, P.S. Mandal, and S. Schmid. Evacuating two robots from two unknown exits on the perimeter of a disk with wireless communication. In *Proceedings of the 19th International Conference on Distributed Computing and Networking, ICDCN 2018, Varanasi, India, January 4-7, 2018*, pages 20:1–20:4, 2018.

20 L. Stone. *Theory of optimal search*. Academic Press New York, 1975.