The Trisection Genus of Standard Simply Connected PL 4–Manifolds

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Abstract

Gay and Kirby recently introduced the concept of a trisection for arbitrary smooth, oriented closed 4–manifolds, and with it a new topological invariant, called the trisection genus. In this note we show that the $K3$ surface has trisection genus 22. This implies that the trisection genus of all standard simply connected PL 4–manifolds is known. We show that the trisection genus of each of these manifolds is realised by a trisection that is supported by a singular triangulation. Moreover, we explicitly give the building blocks to construct these triangulations.

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1 Introduction

Gay and Kirby's construction of a *trisection* for arbitrary smooth, oriented closed 4–manifolds [13] defines a decomposition of the 4–manifold into three 4-dimensional handlebodies[3] glued along their boundaries in the following way: Each handlebody is a boundary connected sum of copies of $S^1 \times B^3$, and has boundary a connected sum of copies of $S^1 \times S^2$ (here, B^i denotes the i-dimensional ball and S^j denotes the j-dimensional sphere). The triple intersection of the 4-dimensional handlebodies is a closed orientable surface Σ, called the

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3 A d-dimensional handlebody (or, more precisely, 1–handlebody) is the regular neighbourhood of a graph embedded into Euclidean d-space.

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central surface, which divides each of their boundaries into two 3–dimensional handlebodies (and hence is a Heegaard surface). These 3–dimensional handlebodies are precisely the intersections of pairs of the 4–dimensional handlebodies.

A trisection naturally gives rise to a quadruple of non-negative integers $(g; g_0, g_1, g_2)$, encoding the genus g of the central surface Σ and the genera g_0, g_1, and g_2 of the three 4–dimensional handlebodies. The *trisection genus* of M, denoted $g(M)$, is the minimal genus of a central surface in any trisection of M. A trisection with $g(\Sigma) = g(M)$ is called a *minimal genus trisection*.

For example, the standard 4–sphere has trisection genus equal to zero, the complex projective plane equal to one, and $S^2 \times S^2$ equal to two. Meier and Zupan [19] showed that there are only six 4–manifolds of trisection genus at most two. An extended set of examples of trisections of 4–manifolds can be found in [13]. The recent works of Gay [14], and Meier, Schirmer and Zupan [16, 19, 18] give some applications and constructions arising from trisections of 4–manifolds and relate them to other structures on 4–manifolds.

A key feature is the so-called *trisection diagram* of the 4–manifold, comprising three sets of simple closed curves on the 2–dimensional central surface from which the 4–manifold can be reconstructed and various invariants of the 4–manifold can be computed. This is particularly interesting in the case where the central surface is of minimal genus, giving a minimal representation of the 4–manifold.

An approach to trisections using triangulations is given by Rubinstein and the second author in [23], and used by Bell, Hass, Rubinstein and the second author in [2] to give an algorithm to compute trisection diagrams of 4–manifolds, using a modified version of the input triangulation. The approach uses certain partitions of the vertex set (called *tricolourings*), and the resulting trisections are said to be *supported* by the triangulation. We combine this framework with a greedy-type algorithm for collapsibility to explicitly calculate trisections from existing triangulations of 4–manifolds. In doing so we are able to prove the following statement about the *K3 surface*, a 4–manifold that can be described as a quartic in $\mathbb{C}P^3$ given by the equation

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0.$$

► **Theorem 1.** *The trisection genus of the $K3$ surface is 22, that is, it is equal to its second Betti number.*

The algorithms of [2] can in particular be applied to a minimal trisection of the $K3$ surface to compute a minimal trisection diagram. This straightforward application is not done in this paper. Instead we focus on the following application of our result:

We already know that the trisection genera of $\mathbb{C}P^2$ and $S^2 \times S^2$ are equal to their respective second Betti numbers. Moreover, the second Betti number is additive, and the trisection genus is subadditive under taking connected sums. It thus follows from Theorem 1 and Lemma 6 that the trisection genus of every 4–manifold which is a connected sum of arbitrarily many (PL standard) copies of $\mathbb{C}P^2$, $S^2 \times S^2$ and the $K3$ surface must be equal to its second Betti number. This family includes the standard 4–sphere as the empty connected sum.

We refer to each member of this family of manifolds as a *standard simply connected PL 4–manifold*. Due to work by Freedman [11], Milnor and Husemoller [20], Donaldson [8], Rohlin [22], and Furuta [12], each of these manifolds must be homeomorphic to one of

$$k(\mathbb{C}P^2)\#m(\overline{\mathbb{C}P^2}),\ k(K3)\#m(S^2 \times S^2),\ \text{or}\ k(\overline{K3})\#m(S^2 \times S^2), \quad k, l, m, r \geq 0,$$

where $\overline{X}$ denotes X with the opposite orientation. Furthermore, modulo the 11/8-conjecture, standard simply connected PL 4–manifolds comprise all topological types of PL simply connected 4–manifolds. See [24, Section 5] for a more detailed discussion of the above statements, and see Section 2.1 for more details about 4–manifolds.

By analysing a particular family of singular triangulations of $\mathbb{C}P^2$, $S^2 \times S^2$ and the $K3$ surface due to Basak and the first author [1], we are able to prove the following even stronger statement.

▶ **Theorem 2.** *Let M be a standard simply connected PL 4–manifold. Then M has a trisection of minimal genus supported by a singular triangulation of M.*

The singular triangulations used as building blocks (connected summands) for this construction are all highly regular and come from what is called *simple crystallisations* (see Section 2.5 for details). They are explicitly listed in Appendix A.

In addition, we run a variant of the algorithm from [2] on the 440 495 triangulations of the 6-pentachoron census of closed 4–manifolds [4]. Surprisingly, while only 445 of these triangulations admit at least one trisection supported by its simplicial structure, some triangulations from the census admit as many as 48 of them.

The paper is organised as follows. In Section 2 we briefly go over some basic concepts used in the article, some elementary facts on trisections, as well as how to construct trisections from triangulations of 4–manifolds. In Section 3 we give details on the algorithm to compute trisections from a given triangulation and we present data obtained from running the algorithm on the 6–pentachora census of closed 4–manifolds. In Section 4 we then prove Theorems 1 and 2.

2 Preliminaries

2.1 Manifolds and triangulations

We assume basic knowledge of geometric topology and in particular manifolds. For a definition of homology groups and Betti numbers of a manifold see [15], for an introduction into simply connected 4–manifolds and their intersection forms, see [24].

In dimensions ≤ 4, there is a bijective correspondence between isotopy classes of smooth and piecewise linear structures [6, 7]. In this paper, all manifolds and maps are assumed to be piecewise linear (PL) unless stated otherwise. Our results apply to any compact smooth manifold by passing to its unique piecewise linear structure [26].

Recall that a manifold M is *simply connected* if every closed loop in M can be contracted to a single point (in other words, the fundamental group of M is trivial). While in dimensions 2 and 3 the only simply connected closed manifolds are the 2– and 3–sphere respectively, in dimension 4 there are infinitely many such manifolds.

Given two oriented simply connected 4–manifolds M and N, their *connected sum*, written $M \# N$, is defined as follows. First remove a 4-ball from both M and N and then glue the resulting manifolds along their boundaries via an orientation reversing homeomorphism. The resulting manifold is again oriented. The second Betti number is additive and the property of being simply connected is preserved under taking connected sums.

In this article, simply connected 4–manifolds are presented in the form of *singular triangulations*. A singular triangulation $\mathcal{T}$ of a (simply connected) 4–manifold M is a collection of $2n$ abstract pentachora together with $5n$ *gluing maps* identifying their $10n$ tetrahedral faces in pairs such that the underlying set of the quotient space $|\mathcal{T}|$ is PL-homeomorphic to M. The gluing maps generate an equivalence relation on the faces of the pentachora, and an equivalence class of faces is referred to as a single face of the triangulation $\mathcal{T}$. Faces of dimension zero are called *vertices* and faces of dimension one are referred to as *edges* or the triangulation. The triangulation is *simplicial* if each equivalence class of faces has at most one member from each pentachoron; i.e. no two faces of a pentachoron are identified in $|\mathcal{T}|$. Every simplicial triangulation is also singular, and the second barycentric subdivision of a singular triangulation is always simplicial. The triangulation is *PL* if, in addition, the boundary of a small neighbourhood of every vertex is PL homeomorphic to a standard sphere. The number $2n$ of pentachora of $\mathcal{T}$ is referred to as the size of $\mathcal{T}$.

2.2 Trisections

We begin with a formal definition of a trisection of a 4–manifold.

▶ **Definition 3** (Trisection of closed manifold)**.** Let M be a closed, connected, piecewise linear 4–manifold. A *trisection* of M is a collection of three piecewise linear codimension zero submanifolds $H_0, H_1, H_2 \subset M$, subject to the following four conditions:

1. Each H_i is PL homeomorphic to a standard piecewise linear 4–dimensional 1–handlebody of genus g_i.
2. The handlebodies H_i have pairwise disjoint interior, and $M = \bigcup_i H_i$.
3. The intersection $H_i \cap H_j$ of any two of the handlebodies is a 3–dimensional 1–handlebody.
4. The common intersection $\Sigma = H_0 \cap H_1 \cap H_2$ of all three handlebodies is a closed, connected surface, the *central surface.*

The submanifolds $H_{ij} = H_i \cap H_j$ and Σ are referred to as the *trisection submanifolds.* In our illustrations, we use the colours blue, red, and green instead of the labels 0, 1, and 2 and we refer to $H_{\text{blue red}} = H_{br}$ as the *green* submanifold and so on.

The trisection in the above definition is also termed a $(g; g_0, g_1, g_2)$–trisection, where $g = g(\Sigma)$ is the genus of the central surface Σ. This definition is somewhat more general than the one originally given by Gay and Kirby [13] in that they ask for the trisection to be *balanced* in the sense that each handlebody H_i has the same genus. It is noted in [17, 23] that any unbalanced trisection can be stabilised to a balanced one.[4]

▶ **Definition 4** (Trisection genus)**.** The *trisection genus* of M, denoted $g(M)$, is the minimal genus of a central surface in any trisection of M. A trisection with $g(\Sigma) = g(M)$ is called a *minimal genus trisection.*

2.3 Three elementary facts

▶ **Lemma 5.** *If M has a trisection in which all 4–dimensional handlebodies are 4–balls, then the trisection is a minimal genus trisection.*

Proof. Suppose M has a $(g'; 0, 0, 0)$–trisection and a $(g; g_0, g_1, g_2)$–trisection. Moreover, suppose that $g = g(M)$. By [13], these have a common stabilisation. Suppose this is a $(g''; k_0, k_1, k_2)$–trisection. Each elementary stabilisation increases the genus of one handlebody and the genus of the central surface by one. Hence $g'' = g' + k_0 + k_1 + k_2$ and $g'' =$

[4] A stabilisation of a trisection with central surface of genus g is obtained by attaching a 1-handle to a 4-dimensional handlebody along a properly embedded boundary parallel arc in the 3-dimensional handlebody that is the intersection of the other two 4-dimensional handlebodies of the trisection. The result is a new trisection with central surface of genus g + 1, and exactly one of the 4-dimensional handlebodies has its genus increased by one.

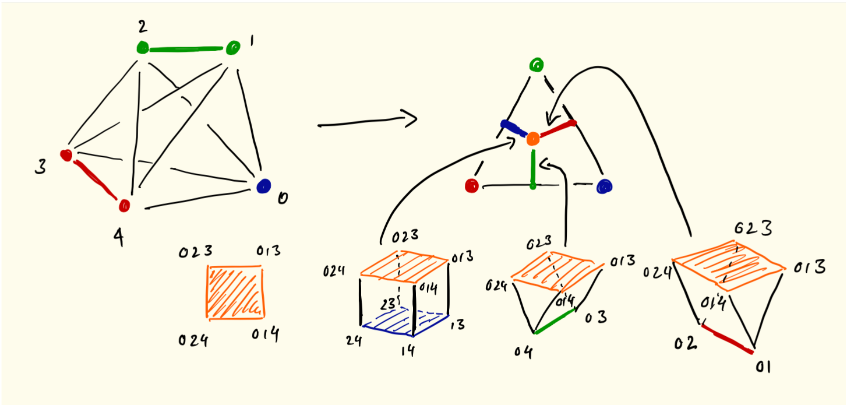

Figure 1 Pieces of the trisection submanifolds. The vertices of the pieces are barycenters of faces and labelled with the corresponding vertex labels. The central surface meets the pentachoron in a square. Two of the 3–dimensional trisection submanifolds meet the pentachoron in triangular prisms and the third (corresponding to the singleton) meets it in a cube. Moreover, any two of these meet in the square of the central surface.

$g + (k_0 - g_0) + (k_1 - g_1) + (k_2 - g_2)$. This gives $g' \geq g(M) = g = g' + g_0 + g_1 + g_2$. This forces $g_0 = g_1 = g_2 = 0$ and $g = g'$. ◀

▶ **Lemma 6.** *Suppose M has a $(g; g_0, g_1, g_2)$–trisection. Then $g \geq \dim H_2(M)$.*

Proof. In [9] it is shown how to compute the homology of a 4–manifold from a trisection diagram. Equation (3.9) in [9] in particular implies that $g \geq \dim H_2(M)$. ◀

▶ **Lemma 7.** *If M_0 has a trisection of genus g_0 and M_1 has a trisection of genus g_1, then the connected sum of M_0 and M_1 has a trisection of genus $g_0 + g_1$. In particular, $g(M_0 \# M_1) \leq g(M_0) + g(M_1)$.*

Proof. As explained in [13, §2], this follows by choosing, for the connect sum operation, two standardly trisected 4–balls in both 4–manifolds. ◀

2.4 Algorithms to compute trisections

Our set-up for algorithms to compute trisections follows [2], where a trisection on M is induced by *tricolourings* of the triangulation. This is now summarised; the reader is referred to [2] for a more detailed discussion.

Let M be a closed, connected 4–manifold with (not necessarily simplicial, but possibly singular) triangulation $\mathcal{T}$. A partition $\{P_0, P_1, P_2\}$ of the set of all vertices of $\mathcal{T}$ is a *tricolouring* if every 4–simplex meets two of the partition sets in two vertices and the remaining partition set in a single vertex. In this case, we also say that the triangulation is *tricoloured*.

Denote the vertices of the standard 2–simplex Δ^2 by v_0, v_1, and v_2. A tricolouring determines a natural map $\mu\colon M \to \Delta^2$ by sending the vertices in P_k to v_k and extending this map linearly over each simplex. Note that the pre-image of v_k is a graph Γ_k in the 1–skeleton of M spanned by the vertices in P_k.

The strategy in [2, 23] is to use μ to pull back the *cubical structure* of the simplex to a trisection of M. The *dual spine* Π^n in an n–simplex Δ^n is the $(n-1)$-dimensional subcomplex of the first barycentric subdivision of Δ^n spanned by those vertices of the first barycentric subdivision which are not vertices of Δ^n itself. This is shown for $n = 2$ and $n = 3$ in Figure 2.

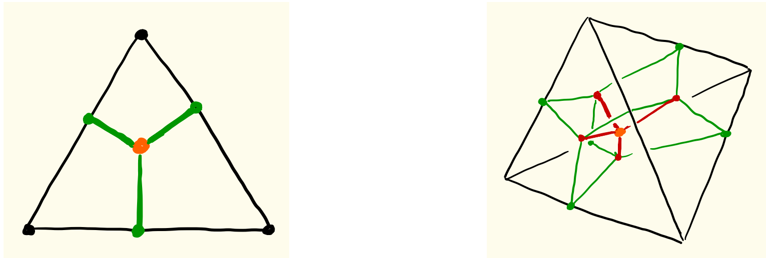

Figure 2 Dual cubes. Left: $\Pi^2 \subset \Delta^2$. Right: $\Pi^3 \subset \Delta^3$.

Decomposing along Π^n gives Δ^n a natural *cubical structure* with $n+1$ cubes of dimension n, and the lower-dimensional cubes that we focus on are the intersections of non-empty collections of these top-dimensional cubes.

Recall that a compact subpolyhedron P in the interior of a manifold M is called a *(PL) spine of* M if M collapses onto P. If P is a spine of M, then $M \setminus P$ is PL homeomorphic with $\partial M \times [0, 1)$.

The pre-images under μ of the dual cubes of $\Pi^2 \subset \Delta^2$ have very simple combinatorics. The barycenter of Δ^2 pulls back to exactly one 2–cube in each pentachoron of M, and these glue together to form a surface Σ in M. This surface is the common boundary of each of the three 3–manifolds obtained as pre-images of an interior 1–cube (edge) of Δ^2. Each such 3–manifold is made up of cubes and triangular prisms, as in Figure 1. Each interior 1–cube c has boundary the union of the barycenter of Δ^2 and the barycenter b of an edge of Δ^2. Since the map $\mu\colon M \to \Delta^2$ is linear on each simplex, the pre-image $\mu^{-1}(c)$ collapses to the pre-image $\mu^{-1}(b)$. In particular, each 3–manifold has a spine made up of 1–cubes and 2–cubes.

It is shown in [2] that the above construction gives a trisection if:

1. the graph Γ_k is connected for each k; and
2. the pre-image of an interior 1–cube of Δ^2 has a 1–dimensional spine.

A tricolouring is a *c-tricolouring* if Γ_k is connected for each k. A c-tricolouring is a *ts-tricolouring* if the pre-image of each interior 1–cube collapses onto a 1–dimensional spine. In this case, the dual cubical structure of Δ^2 pulls back to a trisection of M.

▶ **Definition 8** (Trisection supported by triangulation)**.** We say that a trisection of M is *supported* by the triangulation $\mathcal{T}$ of M, if $\mathcal{T}$ is ts-tricolourable and the trisection is isotopic to the pull-back of the dual cubical structure of Δ^2. In this case, the trisection is said to be *dual* to the corresponding ts-tricolouring.

An algorithm to construct a ts-tricolouring from an arbitrary initial triangulation is given in [2]. It uses subdivisions and bistellar moves in order to justify that the resulting triangulation has a ts-tricolouring. As a result, the number of pentachora of the final triangulation is larger by a factor of 120 compared to the number of pentachora of the initial one (see [2, Theorem 4].)

The new algorithmic contribution of this paper is the development and implementation of an algorithm to determine whether a given triangulation directly admits a ts-tricolouring. This is given in §3.

2.5 Crystallisations of simply connected 4–manifolds

Let $G = (V, E)$ be a multigraph without loops. An *edge colouring* of G is a surjective map $\gamma\colon E \to C = \{0, 1, \dots, d\}$ such that $\gamma(e) \neq \gamma(f)$ whenever e and f are adjacent. The tuple (G, γ) is said to be a $(d + 1)$-*coloured graph*. For the remainder of this section we fix d to be 4.

For a 5-coloured graph (G, γ), and $B \subseteq C$, $|B| = k$, the graph $G_B = (V(G), \gamma^{-1}(B))$ together with the colouring γ restricted to $\gamma^{-1}(B)$ is a k-coloured graph. If for all $j \in C$, $G_{C\setminus\{j\}}$ is connected, then (G, γ) is called *contracted*.

A 5-coloured graph (G, γ) defines a 4-dimensional simplicial cell complex $K(G)$: For each $v \in V(G)$ take a pentachoron Δ_v and label its vertices by $0, 1, 2, 3, 4$. If $u, v \in V(G)$ are joined by an edge e in G and $\gamma(e) = j$, we identify Δ_u and Δ_v along the tetrahedra opposite to vertex j with equally labelled vertices glued together. This way, no face of $K(G)$ can have self-identifications and $K(G)$ is a regular simplicial cell complex. We say that (G, γ) *represents* the simplicial cell complex $K(G)$. Since, in addition, the number of j-labelled vertices of $K(G)$ is equal to the number of components of $G_{C\setminus\{j\}}$ for each $j \in C$, the simplicial cell complex $K(G)$ contains exactly 5 vertices if and only if G is contracted [10].

For a 4–manifold M we call a 5-coloured contracted graph (G, γ) a *crystallisation* of M if the simplicial cell complex $K(G)$ is a singular triangulation of M (with no self-identifications of any of its faces). Every PL 4–manifold admits such a crystallisation due to work by Pezzana [21].

We refer to (G, γ) as simple, if $K(G)$ has exactly 10 edges. That is, if the one-skeleton of $K(G)$ is equal to the 1-skeleton of any of its pentachora. While not every simply connected PL 4–manifold can admit a simple crystallisation[5], this is true for the standard ones, see the article by Basak and the first author [1] for an explicit construction.

3 Implementation and data

In this section we describe the algorithm to check whether a singular triangulation admits a trisection and present some experimental data for the Budney-Burton census of singular triangulations of 4–manifolds up to six pentachora [4].

3.1 Implementation

Given a singular triangulation $\mathcal{T}$ with vertex set $V = V(\mathcal{T})$, perform the following three basic and preliminary checks.

1. Check if $\mathcal{T}$ has at least three vertices.
2. For all triangles $t \in \mathcal{T}$, check if t contains at least two vertices.
3. For all partitions of the vertex set into three non-empty parts $P_0 \sqcup P_1 \sqcup P_2 = V$, for all triangles $t \in \mathcal{T}$, check that no triangle is monochromatic with respect to (P_0, P_1, P_2) (i.e., every triangle contains vertices from at least two of the P_i, $i = 0, 1, 2$).

If any of these checks fail we conclude that $\mathcal{T}$ does not admit a tricolouring. Otherwise, for each tricolouring of $\mathcal{T}$ we proceed in the following way.

Compute the spines of the 4-dimensional handlebodies, that is, the graphs Γ_i in the 1-skeleton of $\mathcal{T}$ spanned by the vertices in P_i, $i = 0, 1, 2$. If all of the Γ_i, $i = 0, 1, 2$, are

[5] Note that, on the one hand there exist simply connected topological 4–manifolds with an infinite number of PL structures, and on the other hand, every topological type of simply connected PL 4–manifold can only admit a finite number of simple crystallisations.

connected, (P_0, P_1, P_2) defines a c-tricolouring of $\mathcal{T}$. Otherwise we conclude that the partition (P_0, P_1, P_2) is not a c-tricolouring of $\mathcal{T}$.

A spine of the 3-dimensional trisection submanifolds is given by the 1- and 2-cubes sitting in bi-coloured triangles and tetrahedra. Denote these three complexes γ_i, $i = 0, 1, 2$ (with γ_i being disjoint of all faces containing vertices in P_i). We compute the Hasse diagram of each of the γ_i and check their connectedness. In case all three complexes are connected, we perform a greedy-type collapse (as defined in [3]) to check whether γ_i collapses onto a 1-dimensional complex (and thus the 3-dimensional trisection submanifolds are handlebodies). Note that this is a deterministic procedure, see, for instance, [25].

If all of these checks are successful, (P_0, P_1, P_2) defines a ts-tricolouring and thus a trisection of M dual to this ts-tricolouring of $\mathcal{T}$. Otherwise we conclude that (P_0, P_1, P_2) does not define a ts-tricolouring.

Finally we compute the central surface, its genus, the genera of the handlebodies Γ_i and γ_i, $i = 0, 1, 2$, and, as a plausibility check, compare the genus of the central surface to the genera of the 3-dimensional handlebodies.

For $\mathcal{T}$ a v-vertex, n-pentachora triangulation the running time of this procedure is roughly the partition number $p(v)$ times a small polynomial in n. Since the triangulations we need to consider usually come with a very small number of vertices, the running time of our algorithm is sufficiently feasible for our purposes.

3.2 Experimental data

The census of singular triangulations of orientable, closed 4–manifolds [4] contains 8 triangulations with two pentachora, 784 triangulations with four pentachora and 440 495 triangulations with 6 pentachora.

In the 2-pentachora case, exactly 6 of the 8 triangulations have at least 3 vertices, and in exactly 3 of them all triangles contain at least two vertices. Overall, these three triangulations admit 19 tricolourings, 18 of which are c-tricolourings covering 2 triangulations. Of these 18 c-tricolourings all are dual to trisections. Overall, exactly 2, that is, 25% of all 2-pentachora triangulations admit a trisection.

Of the 784 triangulations with 4-pentachora, exactly 324 have at least three vertices, and exactly 24 only have triangles containing at least two vertices. Each of these 24 triangulations admits at least one tricolouring. Altogether they admit a total of 88 tricolourings. In 15 triangulations, at least one tricolouring defines connected graphs Γ_i, $0, 1, 2$, and is thus a c-tricolouring. These 15 triangulations admit a total of 72 c-tricolourings. All 72 of them are dual to trisections giving rise to a total of 15 triangulations (1.9% of the census) in the 4-pentachora census admitting a trisection.

There are 440 495 triangulations in the 6-pentachoron census, exactly 116 336 of which have at least three vertices, and exactly 837 of which have no triangles with only one vertex. 824 of these 837 triangulations admit at least one tricolouring, summing up to a total of 1 689 tricolourings in the census. Amongst these, 450 triangulations have at least one *c*-tricolouring, summing up to a total of 1 100 c-tricolourings. In 8 of these 1 100 c-tricolourings, the 2-complexes γ_i, $i = 0, 1, 2$, are not all connected, and in a further 5 of them not all γ_i, $i = 0, 1, 2$, collapse to a 1-dimensional complex. The remaining $1,087$ ts-tricolourings occur in 445 distinct triangulations. It follows that a total of 445 (or 0.1%) of all 6-pentachora triangulations admit a trisection.

Table 1 gives an overview of how many tricolourings, c-tricolourings, and ts-tricolourings can be expected from a triangulation admitting at least one tricolouring. As can be observed

Table 1 Number of triangulations in the 6-pentachora census having k tricolourings ($\mathfrak{n}_{tc}$ in second column), c-tricolourings ($\mathfrak{n}_c$ in third column), and ts-tricolourings ($\mathfrak{n}_{ts}$ in fourth column).

k	$\mathfrak{n}_{tc}$	$\mathfrak{n}_c$	$\mathfrak{n}_{ts}$
1	518	248	243
2	155	76	79
3	67	56	55
4	24	36	34
5	22	0	0
6	15	16	16
8	4	6	6
9	4	0	0
10	3	1	1
11	1	0	0
12	2	2	2
15	4	4	5
18	1	1	0
24	0	2	2
27	2	0	0
36	0	1	1
48	1	1	1
51	1	0	0
Σ	824	450	445

from the table, there exist triangulations in the census with as many as 51 distinct tricolourings, and 48 distinct ts-tricolourings.

For many of the triangulations with multiple ts-tricolourings, all trisections dual to them have central surfaces of the same genus and 4-dimensional handlebodies of the same genera. For some, however, a smallest genus trisection occurs amongst a number of trisections of higher genera. For instance, one of the largest ranges of genera of central surfaces is exhibited by a triangulation with 15 ts-tricolourings, supporting trisections of type $(0;0,0,0)$ $(\times 10)$, $(1;1,0,0)$ $(\times 4)$, and $(2;1,1,0)$ $(\times 1)$. The triangulation is necessarily homeomorphic to S^4. Its isomorphism signature is given in Appendix A.

4 Trisection genus for all known standard simply connected PL 4–manifolds

This section contains the proofs of Theorems 1 and 2. We start with a purely theoretical observation.

▶ **Lemma 9.** *Let M be a simply connected 4-manifold with second Betti number β_2, and let $\mathcal{T}$ be a triangulation of M coming from a simple crystallisation. Then $\mathcal{T}$ admits 15 c-tricolourings.*

Moreover, if some of these c-tricolourings are in fact ts-tricolourings, then their dual trisections must be of type $(\beta_2;0,0,0)$. In particular they must all be minimal genus trisections.

Proof. A triangulation of a 4–manifold M coming from a simple crystallisation is a simplicial cell complex $\mathcal{T}$ such that a) $M \cong_{PL} |\mathcal{T}|$, b) no face has self-identifications, and c) every

4-simplex shares the same 5 vertices and the same 10 edges.

Given such a triangulation $\mathcal{T}$, property c) ensures, that there are 15 ways to properly tricolour the 5 vertices of $\mathcal{T}$ (one colour colours a single vertex and the remaining two colours colour the remaining four vertices in pairs), and each one of them colours every 4-simplex equally and thus in a valid way. Moreover, also because of property c), every colour either spans a single edge or a single vertex of $\mathcal{T}$, and because of property b) a neighbourhood of this edge or vertex is diffeomeomorphic to a 4-ball. In particular, M decomposes into three 4-balls (i.e., handlebodies of genus zero) and all possible 15 tricolourings from above are in fact c-tricolourings.

Hence, let us assume that the 3-dimensional trisection submanifolds dual to some of the c-tricolourings are handlebodies – and thus some of the c-tricolourings are, in fact, ts-tricolourings. It remains to determine the genus of the central surface of the trisection dual to these ts-tricolourings.

The f-vector of $\mathcal{T}$ is given by

$$f(\mathcal{T}) = (5, 10, 10\beta_2 + 10, 15\beta_2 + 5, 6\beta_2 + 2),\ [6]$$

where $\beta_2 = \dim(H_2(M,\mathbb{Z}))$. Every pair of the 5 vertices spans exactly one of the 10 edges, and each of the 10 boundaries of triangles spanned by the 10 edges of $\mathcal{T}$ bounds exactly $(\beta_2 + 1)$ parallel copies of triangles. It follows that there are exactly $4\beta_2 + 4$ tricoloured triangles. The central surface is thus spanned by $6\beta_2 + 2$ quadrilaterals, $4\beta_2 + 4$ vertices, and is (orientable) of Euler characteristic $2 - 2\beta_2$. Hence, its genus is equal to the second Betti number of M.

The trisection must be a minimal genus trisection for (at least) two reasons: (i) all of its 4-dimensional handlebodies are of genus zero (Lemma 5), and (ii) the genus of its central surface equals the second Betti number of M (Lemma 6). ◀

Proof of Theorem 1. In [1], Basak and the first author constructively show that there exists a simple crystallisation of the $K3$ surface. The result now follows from Lemma 9 together with a check of the collapsibility of the 3-dimensional trisection submanifolds for a particular example triangulation coming from a simple crystallisation.

The isomorphism signature given in Appendix A belongs to such a singular triangulation of the $K3$ surface supporting a trisection. ◀

Proof of Theorem 2. Given an arbitrary standard simply connected 4-manifold M and a simple crystallisation of M, Lemma 9 tells us that its associated triangulation $\mathcal{T}$ admits 15 c-tricolourings. It remains to show for a particular such triangulation of M and its 15 c-tricolourings that every 3-dimensional trisection submanifold is a handlebody, that is, it retracts to a 1-dimensional complex.

For this, w.l.o.g. assume that for every choice of c-tricolouring, the colour colouring only one vertex is blue b, and the colours colouring two vertices are red r and green g. By construction, the two 3-dimensional trisection submanifolds defined by b and r (b and g) retract to the (multi-)graph whose vertices are the mid-points of the two edges with endpoints coloured by b and r (b and g), and whose edges are normal arcs parallel to the

[6] Property c) in the definition of a simple crystallisation states that $\mathcal{T}$ must have 5 vertices and 10 edges. Since all ten edges of $\mathcal{T}$ are contained in a single pentachoron, $M \cong_{PL} |\mathcal{T}|$ must be simply connected and thus its Euler characteristic is given by $\chi(M) = 2 + \beta_2(M,\mathbb{Z})$. The other entries of the f-vector are then determined by the Dehn–Sommerville equations for 4-manifolds.

monochromatic edge in the β_2+1 triangles coloured rrb (ggb respectively). In particular, these two 3-dimensional trisection submanifolds retract to a 1-dimensional complex of genus β_2.

The third 3-dimensional trisection submanifold defined by r and g initially retracts to a 2-dimensional subcomplex Q_{rg} with 4 vertices, one for every rg-coloured edge, $4\beta_2 + 4$ edges, one for each rrg- and each rgg-coloured triangle, and $3\beta_2 + 1$ quadrilaterals, one for each $rrgg$-coloured tetrahedron. This 2-dimensional complex might or might not continue to collapse to a 1-dimensional complex depending on the combinatorial properties of $\mathcal{T}$. If Q_{rg} collapses, however, it must collapse to a 1-dimensional complex of genus β_2.

The unique simple crystallisation of $\mathbb{C}P^2$ as well as 266 of the 267 simple crystallisations of $S^2 \times S^2$ translate to triangulations where all of the 15 c-tricolourings (guaranteed by Lemma 9) are in fact ts-tricolourings. Moreover, there exist various simple triangulations of the $K3$ surface with this property. See [1] for pictures of representative simple crystallisations of $\mathbb{C}P^2$, $S^2 \times S^2$ and $K3$. These particular simple crystallisations turn out to have associated triangulations with 15 ts-tricolourings. See Appendix A for their isomorphism signatures.

Assume that for all connected sums N of these three prime simply connected 4–manifolds with arbitrary orientation and second Betti number $\leq k$, there always exists a triangulation coming from a simple crystallisation with all 15 c-tricolourings being also ts-tricolourings.

It remains to show that for two such manifolds N_1 and N_2, there exists a triangulation coming from a simple crystallisation of $N_1 \# N_2$ with this property. By the induction hypothesis we can assume that there exist such triangulations $\mathcal{T}_i$ of N_i, $i = 1, 2$, with all 15 c-tricolourings producing 2-complexes $Q_{rg}(\mathcal{T}_i)$, $i = 1, 2$, collapsing to a 1-dimensional complex.

Fix a ts-tricolouring on $\mathcal{T}_1$ and a collapsing sequence of the quadrilaterals of $Q_{rg}(\mathcal{T}_1)$. Remove one of the two 4-simplices containing the quadrilateral which is collapsed last. Similarly, fix a ts-tricolouring and collapsing sequence on $\mathcal{T}_2$ and remove one of the two 4-simplices containing the quadrilateral removed first. Glue together both triangulations along their boundaries such that the edges through which the last quadrilateral of $Q_{rg}(\mathcal{T}_1)$ and the first quadrilateral of $Q_{rg}(\mathcal{T}_2)$ are collapsed, are aligned. For this, colours r and g of $\mathcal{T}_2$ might need to be swapped (note that such a swap in colours does not change the ts-tricolouring class as such). The collapsing sequence can now be concatenated, yielding a collapsing sequence for $Q_{rg}(\mathcal{T}_1 \# \mathcal{T}_2)$.

Moreover, the property of a triangulation of coming from a simple crystallisation is preserved under taking connected sums of the type as described above [1]. Hence, repeating this process for all 15 tricolourings finishes the proof. ◀

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A Isomorphism signatures of triangulations of S^4, $\mathbb{C}P^2$, $S^2 \times S^2$ and the $K3$ surface

Isomorphism signatures of the triangulations associated to three simple crystallisations of $\mathbb{C}P^2$, $S^2 \times S^2$ and the $K3$ surface, and a triangulation homeomorphic to S^4. Each of them admits 15 ts-tricolourings (see Section 4 for details, see [1] for pictures).

To construct the triangulations download *Regina* [5], and produce a new 4-manifold triangulation by selecting type of triangulation "From isomorphism signature" and pasting in one of the strings given below (please note that you need to remove newline characters before pasting in the isomorphism signature of the $K3$ surface).

Unique 8-pentachora simple crystallisation of $\mathbb{C}P^2$

```
iLvLQQQkbghhghhfffggfaaaaaaaaaaaaaaaaaaaaaaaaaa
```

14-pentachora simple crystallisation of $S^2 \times S^2$

```
oLvMPLQAPMQPkbfgghjihhiilkkmllmnnnnaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa
```

134-pentachora simple crystallisation of the $K3$ surface

```
-cgcLvLALLvvwwzvzMvwLAwvwvvwvwAAPvPPwzQQwMvPAQLQzQzPwLvwvPzvPwQLAPAPQMQwMQQQQAQ
zQQQQQAQPQQQQQQQQQwzvQQMMMMQQQQQQQQPkcahafafakaiasauaxapaBaDaGaKaDaFaFaQaWa3a6a
1aTa7aYaSaTa-aVa4a9aVaab9abbabPaPaZaibPagbfb5a3a5a1aZa0a2a0a-aVa-a8ahbQaubvbubz
bAbGbIbHbHbNbObQbQbKbLbRbHbvbLbwbMbTbTbububWbGbXbHbObRbDbYbNbYbKbwbObSbGbIbVbMb
MbZbZbGbVb1bWbxbTbPbWbXb0b3bPbPb3bCbCb3bYb0bJbybybJb2bIbvbIbvb3bzbxbzb5bWb1b5bz
bAbDbRb4bFbDbSb2bUbUb2bUbUb4bKbCbybybCb8bac-b+b+b+b9b9b8b8bbc6bdc6becacbc9b9bdc
6bec-b7bdc-b-bdcecccaccc7b+b8bbc6b7b7bfcfcfcaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa
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```

Triangulation homeomorphic to S^4 supporting trisections of multiple types

Triangulation homeomorphic to S^4 with 15 ts-tricolourings, supporting trisections of type $(0; 0, 0, 0)$ $(\times 10)$, $(1; 1, 0, 0)$ $(\times 4)$, and $(2; 1, 1, 0)$ $(\times 1)$. See Section 3.2 for details.

```
gLAAMQbbcddeffffaaaaaaaaaaaaaaaaaaaa
```