Coloring Intersection Hypergraphs of Pseudo-Disks

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Abstract

We prove that the intersection hypergraph of a family of n pseudo-disks with respect to another family of pseudo-disks admits a proper coloring with 4 colors and a conflict-free coloring with $O(\log n)$ colors. Along the way we prove that the respective Delaunay-graph is planar. We also prove that the intersection hypergraph of a family of n regions with linear union complexity with respect to a family of pseudo-disks admits a proper coloring with constantly many colors and a conflict-free coloring with $O(\log n)$ colors. Our results serve as a common generalization and strengthening of many earlier results, including ones about proper and conflict-free coloring points with respect to pseudo-disks, coloring regions of linear union complexity with respect to points and coloring disks with respect to disks.

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1 Introduction

Proper colorings of hypergraphs and graphs defined by geometric regions are studied extensively due to their connections to, among others, conflict-free colorings and to cover-decomposability problems, both of which have real-life motivations in cellular networks, scheduling, etc. For applications and history we refer to the surveys [20, 26]. Here we restrict our attention to the (already long list of) results that directly precede our results. We discuss further directions and connections to other problems in Section 4.

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Our central family of regions is the family of pseudo-disks. Families of pseudo-disks have been regarded in many settings for a long time due to being the natural way of generalizing disks while retaining many of their topological and combinatorial properties. Problems regarded range from classic algorithmic questions like finding maximum size independent (disjoint) subfamilies [5] to classical combinatorial geometric questions like the Erdős-Szekeres problem [9]. Probably the most important pseudo-disk family is the family of homothets of a convex region. Pseudo-disks are also central in the area of geometric hypergraph coloring problems as we will see soon in this section.

Before we state our results and their precursors, we need to introduce some basic definitions.

▶ **Definition 1.** Given a hypergraph $\mathcal{H}$, a **proper coloring** of its vertices is a coloring in which every hyperedge H of size at least 2 of $\mathcal{H}$ contains two differently colored vertices.

Throughout the paper hypergraphs can contain hyperedges of size at least 2 only, in other words we do not allow (or automatically delete, when necessary) hyperedges of size 1.

▶ **Definition 2.** Given a hypergraph $\mathcal{H}$, a **conflict-free coloring** of its vertices is a coloring in which every hyperedge H contains a vertex whose color differs from the color of every other vertex of H.

▶ **Definition 3.** Given a family $\mathcal{B}$ of regions and a family $\mathcal{F}$ of regions, the **intersection hypergraph** of $\mathcal{B}$ **with respect to** (wrt. in short) $\mathcal{F}$ is the simple (that is, it has no multiple edges) hypergraph $\mathcal{I}(\mathcal{B},\mathcal{F})$ which has a vertex v_B corresponding to every $B \in \mathcal{B}$ and for every $F \in \mathcal{F}$ it has a hyperedge $H = \{v_B : B \cap F \neq \emptyset\}$ (if this set has size at least two) which corresponds to F. The **Delaunay-graph** of $\mathcal{B}$ wrt. $\mathcal{F}$ is the graph on the same vertex set containing only the hyperedges of $\mathcal{I}(\mathcal{B},\mathcal{F})$ of size 2.

Note that if $\mathcal{B}$ is finite, then even if $\mathcal{F}$ is infinite, $\mathcal{I}(\mathcal{B},\mathcal{F})$ has finitely many hyperedges. In particular, a hyperedge H can correspond to multiple members of $\mathcal{F}$.

We also define the following subgraph of the Delaunay-graph:

▶ **Definition 4.** The **restricted Delaunay-graph** of $\mathcal{B}$ wrt. $\mathcal{F}$ is the subgraph of the Delaunay-graph containing only those (hyper)edges $H_F = \{v_{B_1}, v_{B_2}\}$ for which the corresponding $F \in \mathcal{F}$ intersects B_1 and B_2 in disjoint regions, that is, $F \cap B_1 \neq \emptyset$, $F \cap B_2 \neq \emptyset$, $F \cap B_1 \cap B_2 = \emptyset$ and $F \cap B = \emptyset$ for every $B \in \mathcal{B} \setminus \{B_1, B_2\}$.

▶ **Observation 5** ([25]). *If $\mathcal{B}$ and $\mathcal{F}$ are families for which every hyperedge of $\mathcal{I}(\mathcal{B},\mathcal{F})$ contains a hyperedge of size 2, then a proper coloring of the Delaunay-graph is also a proper coloring of $\mathcal{I}(\mathcal{B},\mathcal{F})$.*

In the literature hypergraphs that have the property assumed in Observation 5 are sometimes called *rank two* hypergraphs (e.g., in [25]).

▶ **Definition 6.** We say that $\mathcal{B}$ can be properly/conflict-free/etc. colored wrt. $\mathcal{F}$ with $f(n)$ colors if for any $\mathcal{B}'$ subfamily of $\mathcal{B}$ of size n, the hypergraph $\mathcal{I}(\mathcal{B}',\mathcal{F})$ can be properly/conflict-free/etc. colored with $f(n)$ colors.

We note that one could further generalize this notion such that instead of intersection we regard inclusion/reverse-inclusion as the relation that defines the hyperedges, as it is done, e.g., in [16] for hypergraphs defined by intervals. As we do not consider problems in this paper for which this generalization is necessary, for simplicity we do not complicate our notation to contain these cases. For related problems see Section 4.

In the majority of earlier results, coloring points wrt. a family of regions or coloring a family of regions wrt. points were regarded. While they were not defined as intersection hypergraphs, they do fit into our general definition of intersection hypergraphs, choosing either $\mathcal{B}$ or $\mathcal{F}$ to be the family of points of the plane. Notice that whenever every point is or can be added as a member of the family $\mathcal{F}$, e.g., for disks, for homothets of a convex region and for pseudo-disks[1], coloring intersection hypergraphs of $\mathcal{F}$ wrt. to $\mathcal{F}$ is a common generalization both of coloring points wrt. $\mathcal{F}$ and coloring $\mathcal{F}$ wrt. points.

There are a few earlier results regarding this general setting. In [16] intersection (and also inclusion and reverse-inclusion) hypergraphs of intervals of the line were considered. In [11] and [8] they considered intersection hypergraphs (and graphs) of (unit) disks, pseudo-disks, squares and axis-parallel rectangles.

1.1 Results related to pseudo-disks

It is well known that the Delaunay-graph of points wrt. disks is planar and thus proper 4-colorable, and Observation 5 implies that the respective hypergraph is also proper 4-colorable.

In [25] Smorodinsky developed a general framework (based on the framework presented in [7]): using proper colorings of the subhypergraphs of the hypergraph with constantly many colors it can build a conflict-free coloring with $O(\log n)$ colors (where n is the number of vertices of the hypergraph). The results mentioned from now on use this framework to get a conflict-free coloring once there is a proper coloring. First, using the precursor of this framework, it was proved by Even et al. [7] that points wrt. disks admit a conflict-free coloring with $O(\log n)$ colors:

▶ **Theorem 7** ([7]). *Let $\mathcal{D}$ be the family of disks in the plane and S a finite set of points, $\mathcal{I}(S,\mathcal{D})$ admits a conflict-free coloring with $O(\log n)$ colors, where $n = |S|$.*

▶ **Definition 8.** A **Jordan region** is a (simply connected) closed bounded region whose boundary is a closed simple Jordan curve.

A family of Jordan regions is called a family of **pseudo-disks** if the boundaries of every pair of the regions intersect in at most two points.

Although we could not find it in the literature, it is well known that for pseudo-disks the bound of Theorem 7 holds as well, the reason is that the Delaunay-graph of points with respect to pseudo-disks is also a planar graph (implied by, e.g., Lemma 29, see later) and thus proper 4-colorable and then we can apply Observation 5 (we can assume that the hypergraph is rank two by, e.g., Corollary 26) and the same general framework to conclude the $O(\log n)$ upper bound for a conflict-free coloring.

The dual of Theorem 7 was also proved by Even et al. and was generalized to pseudo-disks by Smorodinsky:

▶ **Theorem 9** ([7]). *Let P be the set of all points of the plane and $\mathcal{B}$ a finite family of disks, $\mathcal{I}(\mathcal{B},P)$ admits a proper coloring with 4 colors and a conflict-free coloring with $O(\log n)$ colors, where $n = |\mathcal{B}|$.*

▶ **Theorem 10** ([25]). *Let P be the set of all points of the plane and $\mathcal{B}$ be a finite family of pseudo-disks, $\mathcal{I}(\mathcal{B},P)$ admits a proper coloring with a constant number of colors and a conflict-free coloring with $O(\log n)$ colors, where $n = |\mathcal{B}|$.*

[1] See Corollary 26 for a precise formulation of this property for pseudo-disks.

While for coloring pseudo-disks wrt. points there was no explicit upper bound, for the special case of homothets of a convex region[2], the constant was shown by Cardinal and Korman to be 4, just like for disks:

▶ **Theorem 11** ([4]). *Let P be the set of all points of the plane and $\mathcal{B}$ be a finite family of homothets of a given convex region, $\mathcal{I}(\mathcal{B}, P)$ admits a proper coloring with 4 colors.*

Recently it was proved by Keller and Smorodinsky that disks w.r.t. disks can be also colored in such a way, a common generalization of Theorems 7 and 9:

▶ **Theorem 12** ([11]). *Let $\mathcal{D}$ be the family of all disks in the plane and $\mathcal{B}$ a finite family of disks, $\mathcal{I}(\mathcal{B}, \mathcal{D})$ admits a proper coloring with 6 colors and a conflict-free coloring with $O(\log n)$ colors, where $n = |\mathcal{B}|$.*

While they did not prove the same for pseudo-disks, they solved two special cases (they stated these only as part of the proof of their main result):

▶ **Claim 13** ([11]). *Given a finite family $\mathcal{F}$ of pseudo-disks and a subfamily $\mathcal{B}$ of $\mathcal{F}$, If either $\mathcal{B}$ or $\mathcal{F} \setminus \mathcal{B}$ contains only pairwise disjoint pseudo-disks, then $\mathcal{I}(\mathcal{B}, \mathcal{F})$ admits a proper coloring with a constant number of colors and a conflict-free coloring with $O(\log n)$ colors, where $n = |\mathcal{B}|$.*

In this paper we generalize Theorem 12 to the case of coloring a pseudo-disk family wrt. another pseudo-disk family, which is a common generalization of all the above results (Theorems 7,9,10,11,12 and Claim 13). Moreover we prove the optimal upper bound of 4 colors, which improves the bound of Theorem 10 for coloring pseudo-disks wrt. disks from some constant number of colors to 4 colors and improves the bound of Theorem 12 for coloring disks wrt. disks from 6 colors to 4 colors. Furthermore, it provides an alternative proof for Theorems 9 and 11 (both of these were originally proved using dualization and solving equivalent problems about coloring points in the 3 dimensional space):

▶ **Theorem 14.** *Given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of pseudo-disks, $\mathcal{I}(\mathcal{B}, \mathcal{F})$ admits a proper coloring with 4 colors.*

Along the way we prove that the respective Delaunay-graph is planar:

▶ **Theorem 15.** *Given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of pseudo-disks, the Delaunay-graph of $\mathcal{B}$ wrt. $\mathcal{F}$ is a planar graph.*

This was not known even for the Delaunay-graph of pseudo-disks wrt. points. With standard methods (present also in the proof of Theorem 22) this already implies Theorem 14 with 6 colors instead of 4. To achieve the optimal bound we will need some additional ideas.

We mention that if $\mathcal{B}$ and $\mathcal{F}$ are both families of pairwise disjoint simply connected regions, then $\mathcal{I}(\mathcal{B}, \mathcal{F})$ is a planar hypergraph[3] and all planar hypergraphs can be generated this way. From this perspective Theorem 14 says that even with a much more relaxed definition of planarity of a hypergraph it remains 4-colorable.

Using the usual framework it easily follows from Theorem 14 that:

▶ **Corollary 16.** *Given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of pseudo-disks, $\mathcal{I}(\mathcal{B}, \mathcal{F})$ admits a conflict-free coloring with $O(\log n)$ colors, where $n = |\mathcal{B}|$.*

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[2] For further results about homothets of a convex region see Section 4.
[3] The most common definition of a planar hypergraph is that its bipartite incidence graph is planar.

We note that Claim 13 implies in a straightforward way the main result of [11] about conflict-free coloring the (open/closed) neighborhood hypergraphs of intersection graphs of pseudo-disks (for definitions and details see [11]). Thus, Theorem 14 also implies their main result.

Note that in Theorem 14 $\mathcal{B}$ and $\mathcal{F}$ are not related in any way, thus among others, even though two convex regions can intersect infinitely many times, it implies somewhat surprisingly the following:

▶ **Corollary 17.** *We can proper color with* 4 *colors the family of homothets of a convex region* A *wrt. the family of homothets of another convex region* B*.*

1.2 Results related to families of linear union complexity

In [25] Theorem 10 was shown by proving a more general statement about coloring a family of regions that has linear union complexity wrt. points.

▶ **Definition 18.** Let $\mathcal{B}$ be a family of finitely many Jordan regions in the plane. The **vertices** of the **arrangement** of $\mathcal{B}$ are the intersection points of the boundaries of regions in $\mathcal{B}$, the **edges** are the maximal connected parts of the boundaries of regions in $\mathcal{B}$ that do not contain a vertex and the **faces** are the maximal connected parts of the plane which are disjoint from the edges and the vertices of the arrangement.

▶ **Definition 19.** The **union complexity** $\mathcal{U}(\mathcal{B})$ of a family of Jordan regions $\mathcal{B}$ is the number of edges of the arrangement $\mathcal{B}$ that lie on the boundary of $\cup_{B\in\mathcal{B}}B$.

We say that a family of regions $\mathcal{B}$ has **(c)-linear union complexity** if there exists a constant c such that for any subfamily $\mathcal{B}'$ of $\mathcal{B}$ the union complexity of $\mathcal{B}'$ is at most $c|\mathcal{B}'|$.[4]

▶ **Theorem 20** ([10])**.** *Any finite family of pseudo-disks in the plane has a linear union complexity.*

Theorem 20 shows that the following result of Smorodinsky is indeed more general than Theorem 10.

▶ **Theorem 21** ([25])**.** *Let* P *be the set of all points of the plane and* $\mathcal{B}$ *be a finite family of Jordan regions with linear union complexity. Then* $\mathcal{I}(\mathcal{B},P)$ *admits a proper coloring with a constant number of colors and a conflict-free coloring with* $O(\log n)$ *colors, where* $n=|\mathcal{B}|$*.*[5]

We generalize this in the following way:

▶ **Theorem 22.** *Given a family* $\mathcal{F}$ *of pseudo-disks and a finite family* $\mathcal{B}$ *of Jordan regions with linear union complexity,* $\mathcal{I}(\mathcal{B},\mathcal{F})$ *admits a proper coloring with constant number of colors.*

Note that using Theorem 20 we get that Theorem 22 implies Theorem 14 with a (non-explicit and worse) upper bound.

[4] Sometimes in the literature, in the definition of union complexity they count vertices rather than edges, yet it is easy to see that this does not affect the property of having linear union complexity. Also note that linear union complexity is not always defined hereditarily, we defined it this way in order to simplify our statements.

[5] When a statement is about a family with (c-)linear union complexity, by a constant we mean a constant that depends on c and the O notation similarly hides a dependence on c.

▶ **Corollary 23.** *Given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of Jordan regions with linear union complexity, $\mathcal{I}(\mathcal{B},\mathcal{F})$ admits a conflict-free coloring with $O(\log n)$ colors, where $n = |\mathcal{B}|$.*

The rest of the paper is structured as follows. In Section 2 we prove Theorem 15. In Section 3 we prove Theorem 14. Implications about conflict-free colorings and the results about regions of linear union complexity can be found in the full version of this paper. We also give an example that shows that we cannot always color properly points wrt. a family of linear union complexity with constantly many colors. This shows that Theorem 22 is strongest possible in the sense that we cannot change $\mathcal{F}$ from pseudo-disks to a family with linear union complexity, even if $\mathcal{B}$ would be only a finite family of points in the plane. Finally, in Section 4 we discuss some related (open) problems.

2 The Delaunay-graph of pseudo-disks wrt. pseudo-disks

In this section we prove Theorem 15. First we list some tools we need from the papers of Pinchasi and Buzaglo et al. about pseudo-disks:

▶ **Lemma 24** ([23]). *Let $\mathcal{F}$ be a family of pseudo-disks. Let $D \in \mathcal{F}$ and let $x \in D$ be any point. Then D can continuously be shrunk to the point x so that at each moment $\mathcal{F}$ is a family of pseudo-disks.*

Note that when shrinking this way, we can keep all shrunk copies of D in the family, it remains a pseudo-disk family as their boundaries are pairwise disjoint.

▶ **Definition 25.** We say that a pseudo-disk family $\mathcal{F}$ is **saturated** (for some family of regions $\mathcal{B}$) if we cannot add pseudo-disks to $\mathcal{F}$ in a way that strictly increases the number of hyperedges in $\mathcal{I}(\mathcal{B},\mathcal{F})$.

We can enlarge $\mathcal{F}$ greedily until it becomes saturated (we need to add at most $2^{|\mathcal{B}|}$ pseudo-disks).

▶ **Corollary 26.** *Given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of regions, there exists a saturated (for $\mathcal{B}$) family of pseudo-disks $\mathcal{F}'$ with $\mathcal{F}' \supseteq \mathcal{F}$ and consequently with $\mathcal{I}(\mathcal{B},\mathcal{F}') \supseteq \mathcal{I}(\mathcal{B},\mathcal{F})$. In particular, for every point p, $H_p = \{v_B : p \in B \in \mathcal{B}\}$ is a hyperedge of $\mathcal{I}(\mathcal{B},\mathcal{F}')$ (which we call the hyperedge that corresponds to p) if it is of size at least two.*

The second part of Corollary 26 holds as for a point p not contained in any member of $\mathcal{F}$ we can add to $\mathcal{F}$ a small disk centered at p while for a point p contained in some $F \in \mathcal{F}$ we can apply Lemma 24.

When proving our results, it will be enough to consider the case when $\mathcal{F}$ is saturated as adding hyperedges to a hypergraph cannot remove edges from its Delaunay-graph and cannot decrease the number of colors needed for a proper (or conflict-free) coloring. The following corollary of Lemma 24 will be useful when we deal with a saturated pseudo-disk family:

▶ **Corollary 27** ([23]). *Let $\mathcal{B}$ be a family of pairwise disjoint regions in the plane and let $\mathcal{F}$ be a family of pseudo-disks. Let D be a member of $\mathcal{F}$ and suppose that D intersects exactly k members of $\mathcal{B}$ one of which is the set $B \in \mathcal{B}$. Then for every $2 \le l \le k$ there exists a set $D' \subset D$ such that D' intersects B and exactly $l-1$ other regions from $\mathcal{B}$, and $\mathcal{F} \cup \{D'\}$ is again a family of pseudo-disks.*

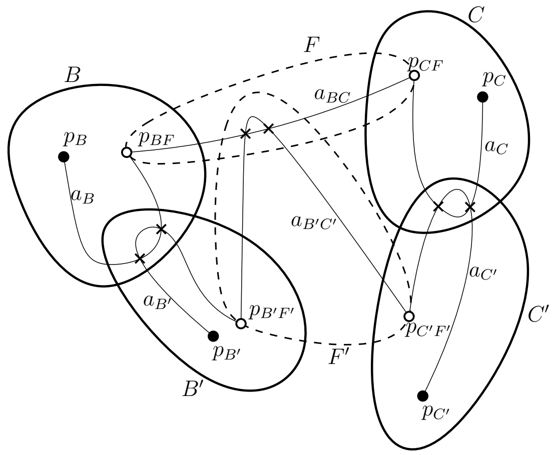


Figure 1 The edges e_f and $e_{f'}$ intersect an even number of times.

▶ **Lemma 28** ([3]). *Let D_1 and D_2 be two pseudo-disks in the plane. Let x and y be two points in $D_1 \setminus D_2$. Let a and b be two points in $D_2 \setminus D_1$. Let e be any Jordan arc connecting x and y that is fully contained in D_1. Let f be any Jordan arc connecting a and b that is fully contained in D_2. Then e and f cross an even number of times.*

▶ **Lemma 29** ([23]). *Given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of pairwise disjoint connected sets in the plane, the Delaunay-graph of $\mathcal{B}$ wrt. $\mathcal{F}$ is a planar graph.*

Note that for a family $\mathcal{B}$ of pairwise disjoint connected sets, the Delaunay-graph and the restricted Delaunay-graph are the same. Assuming also that $\mathcal{B}$ is a family of Jordan regions, Lemma 29 becomes a special case of Theorem 15. Before proving Theorem 15 we prove another special case which for Jordan regions strengthens Lemma 29, while its proof remains relatively simple:

▶ **Lemma 30.** *Given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of pseudo-disks such that each member $B \in B$ contains a point which is in no other $C \in \mathcal{B}$, the Delaunay-graph of $\mathcal{B}$ wrt. $\mathcal{F}$ is a planar graph.*

Proof. We will draw G in the plane in such a way that every pair of edges in G that do not share a common vertex cross an even number of times. The Hanani-Tutte Theorem [6, 29] then implies the planarity of G. In our case the edges may have self-crossings too but the Hanani-Tutte theorem holds in this case as well as we can easily redraw the edges to avoid self-crossings (for further details see, e.g., [24]).

For every $v_B, B \in \mathcal{B}$ choose a point $p_B \in B$ which is in no other $C \in \mathcal{B}$. For an illustration of the rest of the proof see Figure 1.

If v_B and v_C are connected by an edge f in G, then e_f, the drawing of the edge f, connecting p_B and p_C, is as follows. Let $F \in \mathcal{F}$ be a pseudo-disk corresponding to f. Draw an arc a_B inside B from p_B to a point $p_{BF} \in B \cap F$ ($p_{BF} \in C$ is allowed, also p_{BF} may coincide with p_B). Similarly draw an arc a_C inside C from p_C to a point $p_{CF} \in C \cap F$ ($p_{CF} \in B$ is allowed and p_{CF} may coincide with p_C). Finally, draw an arc a_{BC} inside F from p_{BF} to p_{CF} (p_{BF} and p_{CF} may also coincide in which case a_{BC} is of length zero). The concatenation of these three arcs is the drawing e_f of the edge f between the points p_B and p_C. Note that e_f may have self-crossings.

We are left to prove that in this drawing of G every pair of edges that do not share a vertex cross an even number of times.

Let $B, C, B', C' \in \mathcal{B}$ with edges f defined by F between v_B and v_C and f' defined by F' between $v_{B'}$ and $v_{C'}$ Suppose $e_f = a_B \cup a_{BC} \cup a_C$ and $e_{f'} = a_{B'} \cup a_{B'C'} \cup a_{C'}$ are the drawings of the two edges. Notice that the two endpoints of a_B are in $B \setminus B'$ and the two endpoints of $a_{B'}$ are in $B' \setminus B$. Thus using Lemma 28 we get that a_B with $a_{B'}$ intersects an even number of times. The same way we get that a_B with $a_{C'}$, a_C with $a_{B'}$ and a_C with $a_{C'}$ intersect an even number of times. As a_{BC} (resp. $a_{B'C'}$) is disjoint from B' and C' (resp. B and C), there is no intersection between a_{BC} and $a_{B'}, a_{C'}$ nor between $a_{B'C'}$ and a_B, a_C. Finally, the two endpoints of a_{BC} are in $F \setminus F'$ and the two endpoints of $a_{B'C'}$ are in $F' \setminus F$ thus again using Lemma 28 we get that a_{BC} and $a_{B'C'}$ intersect an even number of times. These together imply that indeed e_f and $e_{f'}$ intersect an even number of times, as required. ◀

Pach and Sharir [21] proved that (among others) pseudo-disks have linear union complexity using a similar approach as the proof of Lemma 30, connecting own-points of intersecting regions along their boundaries. More recently, Aronov et al. [2] proved (independently) the case of Lemma 30 when $\mathcal{B} = \mathcal{F}$ (with an almost identical proof) and showed that this implies that for a pseudo-disk family $\mathcal{F}$ and a finite subfamily $\mathcal{B}$ of $\mathcal{F}$, the intersection hypergraph $\mathcal{I}(\mathcal{B}, \mathcal{F})$ has VC-dimension at most 4. In fact, Lemma 30 implies the same way also that $\mathcal{I}(\mathcal{B}, \mathcal{F})$ has VC-dimension at most 4 when $\mathcal{B}$ is any finite pseudo-disk family, not necessarily a subfamily of $\mathcal{F}$, for details see the full version of this paper.

▶ **Corollary 31.** *Given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of pseudo-disks, the restricted Delaunay-graph of $\mathcal{B}$ wrt. $\mathcal{F}$ is a planar graph.*

The proof of Corollary 31 can be found in the full version of this paper.

▶ **Observation 32.** *If $\mathcal{F}$ is saturated (for $\mathcal{B}$), then for every $F \in \mathcal{F}$ the corresponding hyperedge $H = \{v_B : B \cap F \neq \emptyset\}$ of $\mathcal{I}(\mathcal{B}, \mathcal{F})$ either contains an edge of the restricted Delaunay-graph or F contains a point contained in at least 2 members of $\mathcal{B}$.*

Corollary 31 shows that Lemma 30 takes care of the planarity of the restricted Delaunay-graph. From Observation 32 we may think that we are left to take care of hyperedges that contain a point that is contained in at least 2 members of $\mathcal{B}$. This intuition turns out to be good, in all of our main results Lemma 30 will essentially reduce the problem to regarding the intersection hypergraph of $\mathcal{B}$ wrt. points (instead of $\mathcal{B}$ wrt. $\mathcal{F}$).

Before starting the proof of Theorem 15 we make some more preparations:

▶ **Definition 33.** Given a family of regions $\mathcal{B}$, a point is **k-deep** if it is contained in exactly k members of $\mathcal{B}$. We denote by ∂B the boundary of some region B of $\mathcal{B}$. We call a point which is in B but no other $C \in \mathcal{B}$, an **own-point** of B.

▶ **Definition 34.** A hypergraph $\mathcal{H}'$ **supports** another hypergraph $\mathcal{H}$ if they are on the same vertex set and for every hyperedge $H \in \mathcal{H}$ there exists a hyperedge $H' \in \mathcal{H}'$ such that $H' \subseteq H$.

▶ **Observation 35.** *If $\mathcal{H}''$ supports $\mathcal{H}'$ and $\mathcal{H}'$ supports $\mathcal{H}$, then $\mathcal{H}''$ supports $\mathcal{H}$.*

▶ **Observation 36.** *If a hypergraph $\mathcal{H}'$ supports another hypergraph $\mathcal{H}$, then the Delaunay-graph of $\mathcal{H}$ is a subgraph of the Delaunay-graph of $\mathcal{H}'$.*

Our last tool is the following theorem of Snoeyink and Hershberger (we write here a special case of what they called as the Sweeping theorem):

▶ **Theorem 37** ([27]). *Let Γ be a finite set of bi-infinite curves in the plane such that any pair of them intersects at most twice. Let d be a closed curve which intersects at most twice every curve in Γ. We can sweep d such that every member of the sweeping of d intersects at most twice every other curve of Γ.*

A sweeping of d in Theorem 37 is defined as a family $d(t)$, $t \in (-1, 1]$, of pairwise disjoint curves such that $d(0) = d$ and their union contains all points of the plane ($d(-1)$ is a degenerate curve consisting of a singular point).

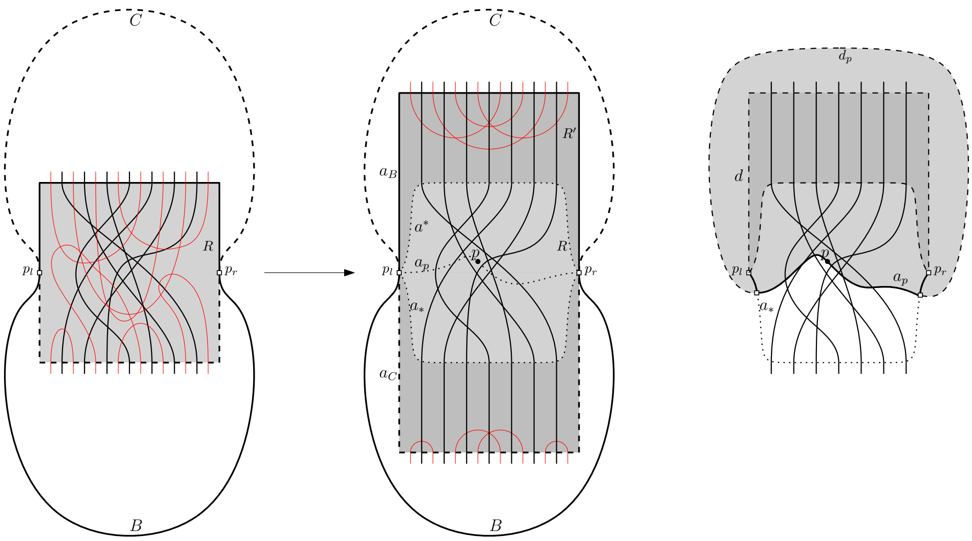
Proof of Theorem 15. Using Corollary 26 we can assume that $\mathcal{F}$ is saturated (for $\mathcal{B}$) and in particular, for every point p, $H_p = \{v_B : p \in B \in \mathcal{B}\}$ is a hyperedge of $\mathcal{I}(\mathcal{B}, \mathcal{F})$ if it is of size at least two.

We can keep deleting members of $\mathcal{B}$ from $\mathcal{B}$ which correspond to a vertex with degree 0 or 1 in the Delaunay-graph, during this process the Delaunay-graph of the new family keeps containing the graph induced by the original Delaunay-graph on the new (reduced) vertex set. If we can prove that the new Delaunay-graph is planar, then adding back the degree 0 and 1 vertices in reverse order we see that the original Delaunay-graph is also planar. Thus we can assume that every vertex of the Delaunay-graph has degree at least two. In this case for every $B \in \mathcal{B}$ we have a point in B which is in at most one other $C \in \mathcal{B}$ (as there are no degree-0 vertices) and there are no two regions $B, C \in \mathcal{B}$ such that $B \subset C$ (as there are no degree-0 or degree-1 vertices).

Given $\mathcal{B}$ and $\mathcal{F}$ we will modify $\mathcal{B}$ such that for the new family $\hat{\mathcal{B}}$ of pseudo-disks every $B \in \hat{\mathcal{B}}$ contains an own-point and furthermore we do this in a way that $\mathcal{I}(\hat{\mathcal{B}}, \mathcal{F})$ supports $\mathcal{I}(\mathcal{B}, \mathcal{F})$. Using Lemma 30 we get that the Delaunay-graph of $\hat{\mathcal{B}}$ wrt. $\mathcal{F}$ is planar and then by Observation 36 we get that the Delaunay graph of $\mathcal{B}$ wrt. $\mathcal{F}$ is also planar.

We do this modification by repeating the below defined operation finitely many times, at each time decreasing by at least one the number of members of $\mathcal{B}$ without an own-point. We now define the three steps of this operation.

- Step 1 - Preparation.
  Take an arbitrary $B \in \mathcal{B}$ that does not contain an own-point. We take a $C \in \mathcal{B}$ for which v_B is connected to v_C in the Delaunay-graph. Thus, there is a point p which is in $B \cap C$ but no other member of $\mathcal{B}$. We morph the plane such that $B \cap C$ becomes a square R such that the two intersection points of the boundary of B and C are on the horizontal halving line of the square (the upper part of ∂R belongs to ∂B and the lower part of ∂R belongs to ∂C) and no member of $\mathcal{B}$ (and $\mathcal{F}$) intersects the vertical sides of the square. This morphing is easily doable and it leaves intact the intersection structure of $\mathcal{B}$ and $\mathcal{F}$. Denote by p_l and p_r the intersection points of ∂B and ∂C, that is, the midpoints of the left and right side of R. This finishes the Preparation Step of our operation. See the left side of Figure 2a for what we get after this step.
- Step 2 - Pulling apart $B \cap C$.
  Now we define the Pulling apart $B \cap C$ Step of the operation which keeps the intersection structure of $\mathcal{B}$ and $\mathcal{F}$ outside $B \cap C$ intact. First we morph the plane such that the square's height is doubled to get the rectangle R' while its horizontal halving line does not change (the part of $\partial R'$ above the halving line is part of ∂B and the part below is part of ∂C) and the drawing inside R remains untouched. We do this such that the intersections of members of $\mathcal{B}$ (and of $\mathcal{F}$) with the horizontal sides of the square are stretched to become vertical lines in $R' \setminus R$. Next, for every $D \in \mathcal{B} \setminus \{B\}$ which intersects exactly one (horizontal) side of R', we redraw the part of ∂D inside B to be a half-circle inside B that has the same endpoints. See the right side of Figure 2a, where these redrawn



(a) Pulling apart $B \cap C$ **(b)** Drawing a_p.

Figure 2 Major steps of the operation used in the proof of Theorem 15.

boundary parts are drawn with thin red strokes. We get a modified family B', while the topology of $\mathcal{F}$ is unmodifed.

It is easy to see that we modified $\mathcal{B}$ in a way that $\mathcal{B}'$ remains a pseudo-disk family. Next we show that $\mathcal{I}(\mathcal{B}', \mathcal{F})$ supports $\mathcal{I}(\mathcal{B}, \mathcal{F})$, that is for every $H \in \mathcal{I}(\mathcal{B}, \mathcal{F})$ there exists a $H' \in \mathcal{I}(\mathcal{B}', \mathcal{F})$ such that $H' \subseteq H$. If $F \in \mathcal{F}$ contains only depth-1 points, then it is disjoint from $B \cap C$ and so the hyperedge H_F remains in the intersection hypergraph after pulling apart $B \cap C$. Otherwise, F contains a depth-2 point q and then H_F contains the hyperedge H_q. So it is enough to prove that for every H_q there is a hyperedge H' (also corresponding to some point) in $\mathcal{I}(\mathcal{B}', \mathcal{F})$ such that $H' \subseteq H_q$. For $q \notin B \cap C$, H_q remains in the intersection hypergraph after pulling apart $B \cap C$. Finally, for $q \in B \cap C$ the hyperedge H_q contains the hyperedge $\{v_B, v_C\}$ which is exactly the hyperedge corresponding to p in $\mathcal{B}'$ (recall that this is the point that was only in B and C and no other member of $\mathcal{B}$, and this property remains true for $\mathcal{B}'$ after the operation).

- Step 3 - Shrinking of C.

  Now we do the final step, the Shrinking of C Step of the operation. Let arc a_B (resp. a_C) be the part of $\partial R'$ above (resp. below) the halving line, which arc is also part of ∂B (resp. ∂C) after the first two steps of the operation. Let arc a^* (resp. arc a_*) be the part of ∂R which is above (resp. below) the halving line. We perturb the vertical parts of a^* and a_* slightly so that they do not overlap (nor intersect) with a_B and a_C. See right side of Figure 2a for an illustration.

  If either a^* or a_* contains a point p which is 2-deep (thus contained by B and C and no other member of $\mathcal{B}'$), then we redraw ∂C such that we change a_C to a^* or a_* (whichever contains the 2-deep point p) and then what we get remains a pseudo-disk family. This way the hyperedge $\{v_B, v_C\}$ is still in $\mathcal{I}(\mathcal{B}'', \mathcal{F})$ as it is corresponding to a point close to p. Thus $\mathcal{I}(\mathcal{B}'', \mathcal{F})$ supports $\mathcal{I}(\mathcal{B}', \mathcal{F})$, and so $\mathcal{I}(\mathcal{B}, \mathcal{F})$ as well. Furthermore, in $\mathcal{B}''$ there is a point close to p which is an own-point of B.

  Unfortunately it can happen that none of a^* and a_* contains a 2-deep point and so they are not suitable for redrawing a_C. Nevertheless, we assumed that there exists a 2-deep point p in B before the operation, which remains 2-deep after the first two steps of the operation. In the following we find an arc a_p connecting p_l and p_r that goes inside R, goes through p and intersects exactly once every maximal part inside R of the boundary of a member of $\mathcal{B}'$, see Figure 2b. Assuming we have this arc a_p we can redraw ∂C such that we change a_C to a_p and what we get remains a pseudo-disk family. Also, the same way as in the previous case, we have that $\mathcal{I}(\mathcal{B}'', \mathcal{F})$ supports $\mathcal{I}(\mathcal{B}, \mathcal{F})$ and in $\mathcal{B}''$ there is a point close to p which is an own-point of B.

- Step 3b - Drawing a_p.
  While the existence of such an a_p is intuitively not surprising, proving its existence is a somewhat technical application of Theorem 37. The proof can be found in the full version of this paper.

Thus, with the above 3-step operation we changed $\mathcal{B}$ such that one more member, B, contains an own-point. Also it is easy to see that no other member could have lost its own-point, as we have redrawn members of $\mathcal{B}$ only inside $B \cap C$, where there were no 1-deep points. So after finitely many times repeating this operation we get a family $\hat{\mathcal{B}}$ in which every member has an own-point and whose intersection hypergraph $\mathcal{I}(\hat{\mathcal{B}}, \mathcal{F})$ supports the intersection hypergraph $\mathcal{I}(\mathcal{B}, \mathcal{F})$ we started with, concluding the proof. ◀

3 Coloring pseudo-disks wrt. pseudo-disks

Note that Observation 5 can be rephrased equivalently such that if the Delaunay-graph of $\mathcal{H}$ supports $\mathcal{H}$, then a proper coloring of the Delaunay-graph is a proper coloring of $\mathcal{H}$. More generally, it is true that:

▶ **Observation 38.** *If a hypergraph $\mathcal{H}'$ supports another hypergraph $\mathcal{H}$, then a proper coloring of $\mathcal{H}'$ is a proper coloring of $\mathcal{H}$.*

Proof of Theorem 14. Given the pseudo-disk family $\mathcal{F}$ and the finite pseudo-disk family $\mathcal{B}$, we want to color the vertices of $\mathcal{I}(\mathcal{B}, \mathcal{F})$ corresponding to the members of $\mathcal{B}$ such that for every $F \in \mathcal{F}$ the hyperedge H_F in $\mathcal{B}$ containing exactly those v_B for which $B \cap F \neq \emptyset$ is not monochromatic (assuming that $|H_F| \geq 2$). Using Corollary 26 and Observation 38 we can assume that $\mathcal{F}$ is saturated (for $\mathcal{B}$).

We prove the existence of a 4-coloring by induction first on the size of $\mathcal{B}$, second on the size of the largest containment-minimal hyperedges and third on the number of largest containment-minimal hyperedges. We say that a hypergraph $\mathcal{H}'$ is *better* than a hypergraph $\mathcal{H}$ (on the same vertex set) if either the size of the largest containment-minimal hyperedges is smaller in $\mathcal{H}'$ than in $\mathcal{H}$ or they are of the same size but there are more of them in $\mathcal{H}$. Notice that if $\mathcal{H}'$ supports $\mathcal{H}$, then $\mathcal{H}'$ is not worse than $\mathcal{H}$.

If $\mathcal{B}$ contains two pseudo-disks B, C such that $B \subset C$, then we can proper color $\mathcal{I}(\mathcal{B} \setminus \{B\}, \mathcal{F})$ by induction and then color v_B with a color different from the color of v_C to get a proper coloring of $\mathcal{I}(\mathcal{B}, \mathcal{F})$ as every hyperedge which contains v_B must contain v_C as well. Thus, we can assume that no two pseudo-disks in $\mathcal{B}$ contain one another.

We can assume that $\mathcal{F}$ is saturated (for $\mathcal{B}$) as adding hyperedges to the intersection hypergraph cannot make it worse. In particular, we can assume that $\mathcal{I}(\mathcal{B}, \mathcal{F})$ contains a hyperedge $H_p = \{v_B : p \in B \in \mathcal{B}\}$ (if it is of size at least two) for every point p in the plane.

Take one of the largest containment-minimal hyperedges, $H \in \mathcal{I}(\mathcal{B}, \mathcal{F})$.

If H is of size 2 that means that the Delaunay-graph supports $\mathcal{I}(\mathcal{B}, \mathcal{F})$ and then by Theorem 15 we can color it with 4 colors, which by Observation 38 is a proper coloring of $\mathcal{I}(\mathcal{B}, \mathcal{F})$, as required. This starts the induction.

Otherwise, H has $l \geq 3$ vertices. Assume by induction that every intersection hypergraph of pseudo-disks wrt. pseudo-disks better than $\mathcal{I}(\mathcal{B},\mathcal{F})$ admits a proper 4-coloring.

If H corresponds to some F that contains only 1-deep points, then using Corollary 27 and that $\mathcal{F}$ is saturated we get that H is not containment-minimal, a contradiction. Thus, H contains a point which is at least 2-deep and then using that H is containment-minimal and $\mathcal{F}$ is saturated there must be a point p (any point inside a pseudo-disk corresponding to H is such) for which actually $H = H_p$. Take two members $B, C \in B$ for which $v_B, v_C \in H_p$. On $B \cap C$ we do the first two steps of the operation used in the proof of Theorem 15 to get a new pseudo-disk family $\mathcal{B}'$. As we have seen, the intersection hypergraph $\mathcal{I}(\mathcal{B}',\mathcal{F})$ supports $\mathcal{I}(\mathcal{B},\mathcal{F})$ and thus $\mathcal{I}(\mathcal{B}',\mathcal{F})$ is not worse than $\mathcal{I}(\mathcal{B},\mathcal{F})$. If $B \cap C$ in $\mathcal{B}'$ contains a 2-deep point, then $\{v_B, v_C\}$ is a hyperedge of $\mathcal{I}(\mathcal{B}',\mathcal{F})$ and the number of size-l hyperedges did actually decrease (as H is not containment minimal anymore) and so $\mathcal{I}(\mathcal{B}',\mathcal{F})$ is better than $\mathcal{I}(\mathcal{B},\mathcal{F})$. Then by induction $\mathcal{I}(\mathcal{B}',\mathcal{F})$ can be colored with 4 colors, which by Observation 38 is a proper coloring of $\mathcal{I}(\mathcal{B},\mathcal{F})$ as well.

The only case left is when $B \cap C$ still does not contain a 2-deep point. In this case similarly to Step 3 of the operation in the proof of Theorem 15, we redraw ∂C such that we change a_C to a^* (see again the right side of Figure 2a). In the new family $\mathcal{B}''$ every point in R is still covered at least twice, and nothing changed outside R, thus $\mathcal{I}(\mathcal{B}'',\mathcal{F})$ supports $\mathcal{I}(\mathcal{B}',F)$ (and then in turn it suppors $\mathcal{I}(\mathcal{B},\mathcal{F})$). Moreover, the hyperedge corresponding to p is a subset of H_p but it does not contain v_C anymore, so it is a proper subset of H and is of size at least 2. This implies that the number of size-l hyperedges did decrease. Then again $\mathcal{I}(\mathcal{B}'',\mathcal{F})$ is better than $\mathcal{I}(\mathcal{B},\mathcal{F})$ and then by induction can be colored with 4 colors, which by Observation 38 is a proper coloring of $\mathcal{I}(\mathcal{B},\mathcal{F})$ as well. ◀

Observe that already for coloring points wrt. disks we may need 4 colors (as there are points whose intersection hypergraph wrt. disks is a complete graph on four vertices), so 4 is also a lower bound for coloring pseudo-disks wrt. pseudo-disks.

Without going into details, we note that from the proofs we can create (low-degree) polynomial time coloring algorithms, supposing that initially we are given reasonable amount of information (e.g., the whole structure of the arrangement (all the vertices, edges and faces) of the pseudo-disks of $\mathcal{B}$ and also for every hyperedge H in $\mathcal{I}(\mathcal{B},\mathcal{F})$ we are given an F corresponding to H).

4 Discussion

As mentioned, the primary motivation behind considering such problems lies in applications to conflict-free colorings and to cover-decomposability problems.

When the main interest is in conflict-free colorings, the $\log n$ factor makes it less interesting to optimize the bound on the proper coloring result. However, when considering the relation to cover-decomposability problems, finding the exact value is of great interest. This makes it important that in Theorem 14 we could prove an exact upper bound for such a wide class of intersection hypergraphs. On the other hand, in Theorem 22 we aimed for keeping the proof as simple as possible, and so we did not optimize the number of colors.

In cover-decomposability problems the typical question asks if for some m we can properly 2-color $\mathcal{I}_m(\mathcal{B},\mathcal{F})$, the subhypergraph of $\mathcal{I}(\mathcal{B},\mathcal{F})$ containing only hyperedges of size at least m. If we properly color $\mathcal{I}_m(\mathcal{B},\mathcal{F})$, then we say that we color $\mathcal{B}$ wrt. $\mathcal{F}$ for m.

Usually either $\mathcal{B}$ or $\mathcal{F}$ is the family of points of the plane. Originally researchers concentrated on the problem when the other family, $\mathcal{F}$ or $\mathcal{B}$, is the family of translates of a convex region (see, e.g., [20] for a history of this research direction).

In the past years researchers started to concentrate more on problems where $\mathcal{B}$ or $\mathcal{F}$ is the family of homothets of a convex region. One of the the most intriguing open problems is whether we can 2-color points wrt. homothets of a given convex polygon for some constant m. The existence of a 2-coloring for some m was proved for triangles [14] and the square [1] and recently for convex polygons the existence of a 3-coloring for some m was proved [17]. On the other hand, for coloring points wrt. disks 2 colors are not enough (for any m) [22]. As we have seen already in the introduction, a 4-coloring of points wrt. disks always exists (for $m = 2$), however the existence of a 3-coloring (for some m) is an open problem (see also the remark after Problem 39).

For the dual problem, about coloring the homothets of a convex polygon wrt. points, while proper 2-coloring exists for the homothets of triangles [14] for some m, for the homothets of other convex polygons there is not always a 2-coloring [18] (for any m). Also, for coloring disks wrt. points 2 colors are not enough (for any m) [19].

In [17] it was conjectured that points wrt. pseudo-disks and pseudo-disks wrt. points can be proper 3-colored (for some m). Encouraged by Theorem 14 we pose the common generalization of these conjectures:

▶ **Problem 39.** *Does there exist a constant m such that given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of pseudo-disks, $\mathcal{I}_m(\mathcal{B}, \mathcal{F})$ always admits a proper coloring with 3 colors?*

We note that Tóth [28] showed an example that choosing $m = 3$ is not enough already when $\mathcal{B}$ is a set of points and $\mathcal{F}$ is a family of pseudo-disks.

The case when $\mathcal{B}$ is a family of points and $\mathcal{F}$ is the family of disks was asked by the author of this paper 10 years ago [12, 13] and is still open (in this special case even $m = 3$ might be true).

As we mentioned in the introduction, one can easily change in the definition of the incidence hypergraph the incidence relation to the containment or reverse-containment relation. Thus we can define the following hypergraphs: $\mathcal{H}^{\subset}(\mathcal{B}, \mathcal{F})$ contains a hyperedge $H = \{v_B : B \subset F\}$ for every $F \in \mathcal{F}$; similarly, $\mathcal{H}^{\supset}(\mathcal{B}, \mathcal{F})$ contains a hyperedge $H = \{v_B : B \supset F\}$ for every $F \in \mathcal{F}$ (if these sets are of size at least two).

Thus we can ask if the respective variants of Theorem 14 and Problem 39 hold for these hypergraphs:

▶ **Problem 40.** *Given a family $\mathcal{F}$ of pseudo-disks and a finite family $\mathcal{B}$ of pseudo-disks, does $\mathcal{H}^{\subset}_m(\mathcal{B}, \mathcal{F})$ (resp. $\mathcal{H}^{\supset}_m(\mathcal{B}, \mathcal{F})$) always admit a proper coloring with 4 colors?*

Does there exist a constant m such that $\mathcal{H}^{\subset}_m(\mathcal{B}, \mathcal{F})$ (resp. $\mathcal{H}^{\supset}_m(\mathcal{B}, \mathcal{F})$) always admits a proper coloring with 3 colors?

The variant of this problem where $\mathcal{F}$ and $\mathcal{B}$ are families of intervals on the line was regarded in [16] where among others it was shown that for intervals these two classes (the class of containment and reverse-containment hypergraphs of intervals) are the same.

Finally we mention that similarly to pseudo-disks, pseudo-halfplanes are natural generalizations of halfplanes. For pseudo-halfplanes it was also possible to reprove the same coloring results that are known for halfplanes [15]. Also, due to the lack of direct relation to pseudo-disks, we did not mention axis-parallel rectangles, whose intersection hypergraph coloring problems are some of the most interesting open problems of the area.

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