Structure and Generation of Crossing-Critical Graphs

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Abstract

We study c-crossing-critical graphs, which are the minimal graphs that require at least c edge-crossings when drawn in the plane. For $c = 1$ there are only two such graphs without degree-2 vertices, K_5 and $K_{3,3}$, but for any fixed $c > 1$ there exist infinitely many c-crossing-critical graphs. It has been previously shown that c-crossing-critical graphs have bounded path-width and contain only a bounded number of internally disjoint paths between any two vertices. We expand on these results, providing a more detailed description of the structure of crossing-critical graphs. On the way towards this description, we prove a new structural characterisation of plane graphs of bounded path-width. Then we show that every c-crossing-critical graph can be obtained from a c-crossing-critical graph of bounded size by replicating bounded-size parts that already appear in narrow "bands" or "fans" in the graph. This also gives an algorithm to generate all the c-crossing-critical graphs of at most given order n in polynomial time per each generated graph.

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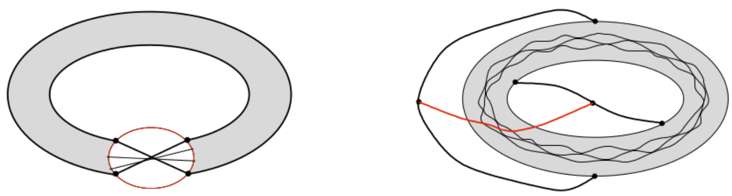

Figure 1 A schematic illustration of two basic methods of constructing crossing-critical graphs.

1 Introduction

Minimizing the number of edge-crossings in a graph drawing in the plane (the *crossing number* of the graph, cf. Definition 2.1) is considered one of the most important attributes of a "nice drawing" of a graph, and this question has found numerous other applications (for example, in VLSI design [12] and in discrete geometry [18]). Consequently, a great deal of research work has been invested into understanding what forces the graph crossing number to be high. There exist strong quantitative lower bounds, such as the famous Crossing Lemma [1, 12]. However, the quantitative bounds show their strength typically in dense graphs, and hence they do not shed much light on the structural properties of graphs of high crossing number.

The reasons for sparse graphs to have many crossings in any drawing are structural – there is a lot of "nonplanarity" in them. These reasons can be understood via corresponding minimal obstructions, the so called *c-crossing-critical* graphs (cf. Section 2 and Definition 2.2), which are the subgraph-minimal graphs that require at least c crossings. There are only two 1-crossing-critical graphs without degree-2 vertices, the Kuratowski graphs K_5 and $K_{3,3}$, but it has been known already since Širáň's [19] and Kochol's [11] constructions that the structure of c-crossing-critical graphs is quite rich and non-trivial for any $c \geq 2$. Already the first nontrivial case of $c = 2$ shows a dramatic increase in complexity of the problem. Yet, Bokal, Oporowski, Richter, and Salazar recently succeeded in obtaining a full description [3] of all the 2-crossing-critical graphs up to finitely many "small" exceptions.

To our current knowledge, there is no hope of extending the explicit description from [3] to any value $c > 2$. We, instead, give for any fixed positive integer c an asymptotic structural description of all sufficiently large c-crossing-critical graphs.

Contribution outline. We refer to subsequent sections for the necessary formal concepts. On a high level of abstraction, our contribution can be summarized as follows:

1. There exist three kinds of local arrangements – a crossed band of uniform width, a twisted band, or a twisted fan – such that any optimal drawing of a sufficiently large c-crossing-critical graph contains at least one of them.
2. There are well-defined local operations (replacements) performed on such bands or fans that can reduce any sufficiently large c-crossing-critical graph to one of finitely many base c-crossing-critical graphs.
3. A converse – a well-defined bounded-size expansion operation – can be used to iteratively construct each c-crossing-critical graph from a c-crossing-critical graph of bounded size. This yields a way to enumerate all the c-crossing-critical graphs of at most given order n in polynomial time per each generated graph. More precisely, the total runtime is $O(n)$ times the output size.

To give a closer (but still informal) explanation of these points, we should review some of the key prior results. First, the infinite 2-crossing-critical family of Kochol [11] explicitly showed one basic method of constructing crossing-critical graphs – take a sequence of suitable small planar graphs (called *tiles*, cf. Section 3), concatenate them naturally into a plane strip and join the ends of this strip with the *Möbius twist*. See Figure 1. Further constructions of this kind can be found, e.g., in [2, 14, 16]. In fact, [3] essentially claims that such a Möbius twist construction is the only possibility for $c = 2$; there, the authors give an explicit list of 42 tiles which build in this way all the 2-crossing-critical graphs up to finitely many exceptions.

The second basic method of building crossing-critical graphs was invented later by Hliněný [9]; it can be roughly described as constructing a suitable planar strip whose ends are now joined without a twist (i.e., making a cylinder), and adding to it a few edges which then have to cross the strip. See again Figure 1 for an illustration. Furthermore, diverse crossing-critical constructions can easily be combined together using so called *zip product* operation of Bokal [2] which preserves criticality. To complete the whole picture, there exists a third, somehow mysterious method of building c-crossing-critical graphs (for sufficiently high values of c), discovered by Dvořák and Mohar in [5]. The latter can be seen as a degenerate case of the Möbius twist construction, such that the whole strip shares a central high-degree vertex, and we skip more details till the technical parts of this paper.

As we will see, the three above sketched construction methods roughly represent the three kinds of local arrangements mentioned in point (1). In a sense, we can thus claim that no other method (than the previous three) of constructing infinite families of c-crossing-critical graphs is possible, for any fixed c. Moving on to point (2), we note that all three mentioned construction methods involve long (and also “thin”) planar strips, or *bands* as subgraphs (which degenerate into *fans* in the third kind of local arrangements; cf. Definition 3.1). We will prove, see Corollary 3.6, that such a long and “thin” planar band or fan must exist in any sufficiently large c-crossing-critical graph, and we analyse its structure to identify elementary connected tiles of bounded size forming the band. We then argue that we can reduce repeated sections of the band while preserving c-crossing-criticality. Regarding point (3), the converse procedure giving a generic bounded-size expansion operation on c-crossing-critical graphs is described in Theorem 4.9 (for a quick illustration, the easiest case of such an expansion operation is edge subdivision, that is replacing an edge with a path, which clearly preserves c-crossing-criticality).

Paper organization. After giving the definitions and preliminary results about crossing-critical graphs in Section 2, we show a new structural characterisation of plane graphs of bounded path-width which forms the cornerstone of our paper in Section 3. Then, in Section 4, we deal with the structure and reductions / expansions of crossing-critical graphs, presenting our main results. In Section 5 we outline the technical steps leading to our cornerstone characterisation from Section 3. Some final remarks are presented in Section 6.

Due to restrictions on the length of the paper, some technical details and proofs of our statements are left for the full paper. Statements whose proofs are in the full paper are marked with **(*)**.

2 Graph drawing and the crossing number

In this paper, we consider multigraphs by default, even though we could always subdivide parallel edges (with a slight adjustment of definitions) in order to make our graphs simple. We follow basic terminology of topological graph theory, see e.g. [13].

A *drawing* of a graph G in the plane is such that the vertices of G are distinct points and the edges are simple curves joining their end vertices. It is required that no edge passes through a vertex, and no three edges cross in a common point. A *crossing* is then an intersection point of two edges other than their common end. A drawing without crossings in the plane is called a *plane drawing* of a graph, or shortly a *plane graph*. A graph having a plane drawing is *planar*.

The following are the core definitions of our research.

▶ **Definition 2.1** (crossing number)**.** The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings of edges in a drawing of G in the plane.

▶ **Definition 2.2** (crossing-critical)**.** Let c be a positive integer. A graph G is *c-crossing-critical* if $cr(G) \geq c$, but every proper subgraph G' of G has $cr(G') < c$.

Furthermore, suppose G is a graph drawn in the plane with crossings. Let G' be the plane graph obtained from this drawing by replacing the crossings with new vertices of degree 4. We say that G' is the plane graph associated with the drawing, shortly the *planarization of G*, and the new vertices are the *crossing vertices* of G'.

Preliminaries. Structural properties of crossing-critical graphs have been studied for more than two decades, and we now briefly review some of the previous important results which we shall use. First, we remark that a c-crossing-critical graph may have no drawing with only c crossings (examples exist already for $c = 2$). Richter and Thomassen [15] proved the following upper bound:

▶ **Theorem 2.3** ([15])**.** *Every c-crossing-critical graph has a drawing with at most $\lceil 5c/2+16 \rceil$ crossings.*

Interestingly, although the bound of Theorem 2.3 sounds rather weak and we do not know any concrete examples requiring more than $c + \mathcal{O}(\sqrt{c})$ crossings, the upper bound has not been improved for more than two decades. We not only use this important upper bound, but also hope to be able to improve it in the future using our results.

Our approach to dealing with "long and thin" subgraphs in crossing-critical graphs relies on the folklore structural notion of *path-width* of a graph, which we recall in Definition 3.4. Hliněný [7] proved that c-crossing-critical graphs have path-width bounded in terms of c, and he and Salazar [8] showed that c-crossing-critical graphs can contain only a bounded number of internally disjoint paths between any two vertices.

▶ **Theorem 2.4** ([7])**.** *Every c-crossing-critical graph has path-width (cf. Definition 3.4) at most* $\lceil 2^{6(72\log_2 c+248)c^3+1} \rceil$.

Another useful concept for this work is that of *nests* in a drawing of a graph (cf. Definition 3.3), implicitly considered already in previous works [7, 8], and explicitly defined by Hernandez-Velez et al. [6] who concluded that no optimal drawing of a c-crossing-critical graph can contain a 0-, 1-, or 2-nest of large depth compared to c.

Lastly, we remark that by trivial additivity of the crossing number over blocks, we may (and will) restrict our attention only to 2-*connected crossing-critical graphs.* We formally argue as follows. For $c, \delta > 0$, let us say a graph is (c, δ)-*crossing-critical* if it has crossing number *exactly* c and all proper subgraphs have crossing number at most $c - \delta$.

▶ **Proposition 2.5** (folklore)**.** *A graph H is c-crossing-critical if and only if there exist positive integers $c_1, \ldots, c_b$ and δ such that $c \leq c_1 + \cdots + c_b \leq c + \delta - 1$, H has exactly b 2-connected blocks $H_1, \ldots, H_b$, and the block H_i is (c_i, δ)-crossing-critical for $i = 1, \ldots, b$.*

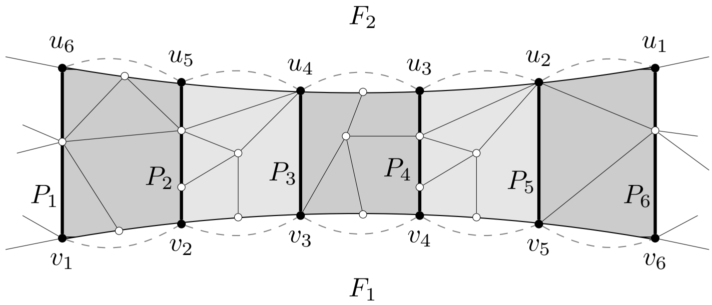

Figure 2 An example of paths $P_1, \ldots, P_6$ (bold lines) forming an (F_1, F_2)-band of length 6, cf. Definition 3.1. The five tiles of this band, as in Definition 3.2, are shaded in grey and the dashed arcs represent α_i and α'_i from that definition.

Hence, strictly respecting Proposition 2.5, we should actually study 2-connected (c, δ)-crossing-critical graphs. To keep the presentation simpler, we stick with c-crossing-critical graphs, but we remark that our results also hold in the more refined setting.

3 Structure of plane tiles

The proof of our structural characterisation of crossing-critical graphs can be roughly divided into two main parts. The first one, presented in this section (leaving technical prerequisites for later Section 5), establishes the existence of specific plane bands (resp. fans) and their tiles in crossing-critical graphs. The second part will then, in Section 4, closely analyse these bands and tiles. Unlike a more traditional "bottom-up" approach to tiles in crossing number research (e.g., [3]), we define tiles and deal with them "top-down", i.e., describing first plane bands or fans and then identifying tiles as their small elementary parts. Our key results are summarized below in Theorem 3.5 and Corollary 3.6.

▶ **Definition 3.1** (band and fan)**.** Let G be a 2-connected plane graph. Let F_1 and F_2 be distinct faces of G and let $v_1, v_2, \ldots, v_m$, and $u_1, u_2, \ldots, u_m$ be some of the vertices incident with F_1 and F_2, respectively, listed in the cyclic order along the faces. If $P_1, \ldots, P_m$ are pairwise vertex-disjoint paths in G such that P_i joins v_i with u_{m+1-i}, for $1 \le i \le m$, then we say that $(P_1, \ldots, P_m)$ forms an (F_1, F_2)-*band of length* m. Note that P_i may consist of only one vertex $v_i = u_{m+1-i}$.

Let F_1 and $v_1, v_2, \ldots, v_m$ be as above. If u is a vertex of G and $P_1, \ldots, P_m$ are paths in G such that P_i joins v_i with u, for $1 \le i \le m$, and the paths are pairwise vertex-disjoint except for their common end u, then we say that $(P_1, \ldots, P_m)$ forms an (F_1, u)-*fan of length* m. The (F_1, u)-fan is *proper* if u is not incident with F_1.

▶ **Definition 3.2** (tiles and support)**.** Let $(P_1, \ldots, P_m)$ be either an (F_1, F_2)-band or an (F_1, u)-fan of length $m \ge 3$. For $1 \le i \le m-1$, let α_i be an arc between v_i and v_{i+1} drawn inside F_1, and let α'_i be an arc drawn between u_i and u_{i+1} in F_2 in the case of the band; α'_i are null when we are considering a fan. Furthermore, choose the arcs to be internally disjoint. Let θ_i be the closed curve consisting of P_i, α_i, P_{i+1}, and α'_{m-i}. Let λ_i be the connected part of the plane minus θ_i that contains none of the paths P_j $(1 \le j \le m)$ in its interior. The subgraphs of G drawn in the closures of $\lambda_1, \ldots, \lambda_{m-1}$ are called *tiles* of the band or fan (and the tile of λ_i includes $P_i \cup P_{i+1}$ by this definition). The union of these tiles is the *support* of the band or fan.

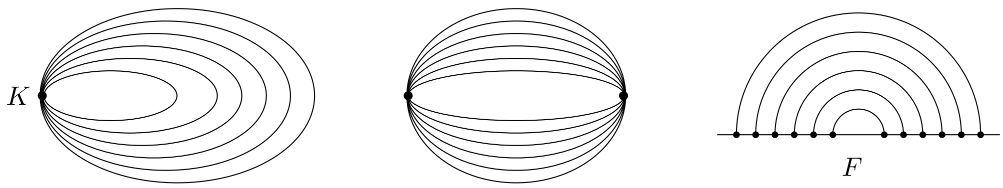

Figure 3 An illustration of Definition 3.3: a 1-nest, a 2-nest, and an F-nest, each of depth 6.

▶ **Definition 3.3** (nests). Let G be a 2-connected plane graph. For an integer $k \geq 0$, a *k-nest* in G of *depth m* is a sequence $(C_1, C_2, \ldots, C_m)$ of pairwise edge-disjoint cycles such that for some set K of k vertices and for every $i < j$, the cycle C_i is drawn in the closed disk bounded by C_j and $V(C_i) \cap V(C_j) = K$.

Let F be a face of G and let $v_1, v_2, \ldots, v_{2m}$ be some of the vertices incident with F listed in the cyclic order along the face. Let $P_1, \ldots, P_m$ be pairwise vertex-disjoint paths in G such that P_i joins v_i with v_{2m+1-i}, for $1 \leq i \leq m$. Then, we say that $(P_1, \ldots, P_m)$ forms an *F-nest of depth m*. Similarly, let $v_1, v_2, \ldots, v_m, u$ be some of the vertices incident with F, let $P_1, \ldots, P_m$ be paths in G such that P_i joins v_i with u, for $1 \leq i \leq m$, and the paths intersect only in u. Then, we say that $(P_1, \ldots, P_m)$ form a *degenerate F-nest of depth m*.

See Figure 3. Note that degenerate F-nests are the same as non-proper (F, u)-fans.

Our cornerstone claim, interesting on its own, is a structure theorem for plane graphs of bounded path-width. Before stating it, we recall the definition of path-width.

▶ **Definition 3.4.** A *path decomposition* of a graph G is a pair (P, β), where P is a path and β is a function that assigns subsets of $V(G)$, called *bags*, to nodes of P such that

- for each edge $uv \in E(G)$, there exists $x \in V(P)$ such that $\{u, v\} \subseteq \beta(x)$, and
- for every $v \in V(G)$, the set $\{x \in V(P) : v \in \beta(x)\}$ induces a non-empty connected subpath of P.

The *width* of the decomposition is the maximum of $|\beta(x)| - 1$ over all vertices x of P, and the *path-width* of G is the minimum width over all path decompositions of G.

▶ **Theorem 3.5** (*). *Let w, m, and k_0 be non-negative integers, and $g : \mathbf{N} \to \mathbf{N}$ be an arbitrary non-decreasing function. There exist integers w_0 and n_0 such that the following holds. Let G be a 2-connected plane graph and let Y be a set of at most k_0 vertices of G of degree at most 4. If G has path-width at most w and $|V(G)| \geq n_0$, then one of the following holds:*

- *G contains a 0-nest, a 1-nest, a 2-nest, an F-nest, or a degenerate F-nest for some face F of G, of depth m, and with all its cycles or paths disjoint from Y, or*
- *for some $w' \leq w_0$, G contains an (F_1, F_2)-band or a proper (F_1, u)-fan (where F_1 and F_2 are distinct faces and u is a vertex) of length at least $g(w')$ and with support disjoint from Y, such that each of its tiles has size at most w'.*

We pay close attention to explaining Theorem 3.5, because of its great importance in this paper. Comparing it to Definition 3.4, one may think that there is not much difference – the bags $\beta(x)$ of a path decomposition of G of width at most w' might perhaps play the role of tiles of the band or fan in the second conclusion. Unfortunately, this simple idea is quite far from the truth. The subgraphs induced by the bags may not be "drawn locally", that is, its edges may be geometrically far apart in the plane graph G. As an example, consider the width 2 path decomposition of a cycle where one of the vertices of the cycle appears in all the bags.

The main message of Theorem 3.5 thus is that in a plane graph of bounded path-width we can find a long band which is "drawn locally" and decomposes into well-defined small and *connected* tiles (cf. Definition 3.2). Otherwise, such a graph must contain some kind of a deep nest or fan. However, as we will see soon in Corollary 3.6, the latter structures are impossible in the planarizations of optimal drawings of crossing-critical graphs.

The proof of Theorem 3.5 requires some preparatory work, and it uses tools of structural graph theory and of semigroup theory in algebra. Since these tools are quite far from the main topic of this paper, we defer their presentation and an outline of their application towards Theorem 3.5 till Section 5. Instead, we now continue with an application of the theorem in the study of crossing-critical graph structure, as a strengthening of Theorem 2.4.

▶ **Corollary 3.6.** *Let c be a positive integer, and let $g : \mathbf{N} \to \mathbf{N}$ be an arbitrary non-decreasing function. There exist integers w_0 and n_0 such that the following holds. Let G be a 2-connected c-crossing-critical graph, and let G' be the plane graph associated with a drawing of G with the minimum number of crossings. Let Y denote the set of crossing vertices of G'. If $|V(G)| \geq n_0$, then for some $w' \leq w_0$, G' contains an (F_1, F_2)-band or a proper (F_1, u)-fan (where F_1 and F_2 are distinct faces and u is a vertex) of length at least $g(w')$ and with support disjoint from Y, such that each of its tiles has size at most w'.*

Proof. Let $k_0 = \lceil 5c/2 + 16 \rceil$, $w = \lceil 2^{6(72 \log_2 c + 248)c^3 + 1} \rceil + k_0$ and $m = 15c^2 + 105c + 16$. Let w_0 and n_0 be the corresponding integers from Theorem 3.5.

By Theorem 2.3, each c-crossing-critical graph has a drawing with at most k_0 crossings, and thus $|Y| \leq k_0$. By Theorem 2.4, G has path-width at most $w - k_0$, and thus G' has path-width at most w. Hliněný and Salazar [8] and Hernandez-Velez et al. [6] proved that the graph G' obtained from a c-crossing-critical graph G as described does not contain a 0-, 1- and 2-nests of depth m with cycles disjoint from Y. Furthermore, arguments analogous to (some of) those used in [7] can prove that no face F of G' has an F-nest or a degenerate F-nest of depth m with paths disjoint from Y. Further details are left for the full paper. ◀

4 Removing and inserting tiles

In the second part of the paper, we study an arrangement of bounded tiles in a long enough plane band or fan (as described by Corollary 3.6), focusing on finding repeated subsequences which then could be shortened. Importantly, this shortening preserves c-crossing-criticality. In the opposite direction we then manage to define the converse operation of "expansion" of a plane band which also preserves c-crossing-criticality. These findings will imply the final outcome – a construction of all c-crossing-critical graphs from an implicit list of base graphs of bounded size. The formal statement can be found in Theorem 4.9.

Again, we start with a few relevant technical terms. Recall Definition 3.1.

▶ **Definition 4.1** (subband, necklace and shelled band)**.** Let $\mathcal{P} = (P_1, \ldots, P_m)$ be an (F_1, F_2)-band or an (F_1, u)-fan in a 2-connected plane graph. A *subband* or *subfan* consists of a contiguous subinterval $(P_i, P_{i+1}, \ldots, P_j)$ of the band or fan (and its *support* is a subset of the support of the original band or fan).

We say that the band $\mathcal{P}$ is a *necklace* if each of its paths consists of exactly one vertex. A tile (cf. Definition 3.2) of the band or fan $\mathcal{P}$ is *shelled* if it is bounded by a cycle, consisting of two consecutive paths P_i and P_{i+1} of $\mathcal{P}$ and parts of the boundary of F_1 and F_2 (respectively, u), and the two paths P_i, P_{i+1} delimiting the tile have at least two vertices each. The band or fan $\mathcal{P}$ is *shelled* if each of its tiles is shelled. See Figure 4.

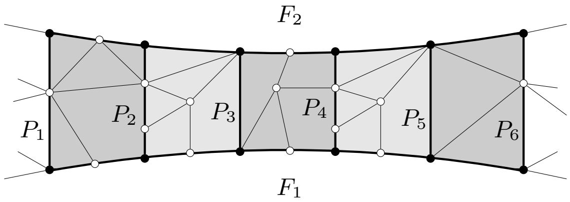

Figure 4 An example of an (F_1, F_2)-band of length 6; this band is shelled (cf. Definition 4.1) and the bounding cycles of the tiles are emphasized in bold lines.

One can easily show that, regarding the outcome of Corollary 3.6, there are only the following two refined subcases that have to be considered in further analysis:

▶ **Lemma 4.2 (*).** *Let w be a positive integer and $f : \mathbf{N} \to \mathbf{N}$ be an arbitrary non-decreasing function. There exist integers n_0 and w' such that the following holds. Let G be a 2-connected plane graph, and let $\mathcal{P} = (P_1, \ldots, P_m)$ be an (F_1, F_2)-band or a proper (F_1, u)-fan in G of length $m \geq n_0$, with all tiles of size at most w. Then either G contains a shelled subband or subfan of $\mathcal{P}$ of length $f(w)$, or G contains a necklace of length $f(w')$ with tiles of size at most w' whose support is contained in the support of $\mathcal{P}$.*

Reducing a necklace. Among the two subcases left by Lemma 4.2, the easier one is that of a necklace which can be reduced simply to a bunch of parallel edges; see also Figure 5.

▶ **Lemma 4.3.** *Let c be a non-negative integer. Let G be a 2-connected c-crossing-critical graph, and let G' be the planarization of a drawing of G with the smallest number of crossings. Let Y denote the set of crossing vertices of G'. Suppose that $\mathcal{P} = (v_1, \ldots, v_m)$, where $m \geq 2$, is a necklace in G' whose support is disjoint from Y. Then for some $p \leq c$, the support of $\mathcal{P}$ consists of p pairwise edge-disjoint paths from v_1 to v_m. Furthermore, the graph G_0 obtained from G by removing the support of $\mathcal{P}$ except for v_1 and v_m and by adding p parallel edges between v_1 and v_m is c-crossing-critical.*

Proof. Let G_1 denote the subgraph of G obtained by removing the support of $\mathcal{P}$ except for v_1 and v_m. Let p be the maximum number of pairwise edge-disjoint paths from v_1 to v_m in the support S of $\mathcal{P}$. Suppose for a contradiction that either $p \geq c + 1$ or some edge e of S is not contained in an edge-cut of size p separating v_1 from v_m. In the former case, let e be an arbitrary edge of S. Let $q = c$ if $p \geq c + 1$ and $q = p$ otherwise.

By criticality of G, the graph $G - e$ can be drawn in the plane with at most $c - 1$ crossings. Consider the drawing of G_1 induced by this drawing, and let a be the minimum number of edges that have to be crossed by any curve in the plane from v_1 to v_m and otherwise disjoint from $V(G_1)$. Note that $a \geq 1$, since otherwise we could draw S without crossings between v_1 and v_m, obtaining a drawing of G with fewer than c crossings. Since $G - e$ contains q pairwise edge-disjoint paths from v_1 to v_m which are not contained in G_1, we conclude that $\mathrm{cr}(G - e) \geq \mathrm{cr}(G_1) + aq \geq q$. Since $\mathrm{cr}(G - e) < c$, we have $q < c$. It follows that $q = p$ and $\mathrm{cr}(G_1) < c - ap$. However, S contains an edge-cut C of order p separating v_1 from v_m by Menger's theorem, and we can add S to the drawing G_1 so that exactly the edges of C are crossed, and each of them exactly a times (by drawing the part of S between v_1 and C close to v_1, and the part of S between v_m and C close to v_m). This way, we obtain a drawing of G with $\mathrm{cr}(G_1) + ap < c$ crossings. This is a contradiction, which shows that $p \leq c$ and that S is the union of p edge-disjoint paths from v_1 to v_m.

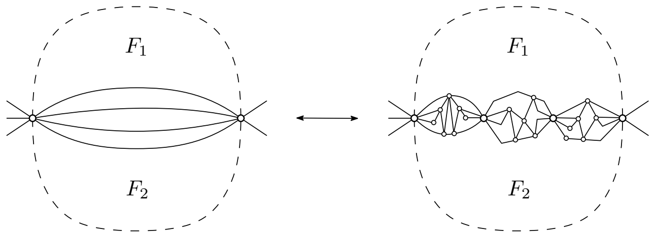

Figure 5 Inserting or removing a necklace (cf. Lemma 4.3 with $p = m = 4$).

Any drawing of G_0 can be transformed into a drawing of G with at most as many crossings in the same way as described in the previous paragraph. Thus $\mathrm{cr}(G_0) \geq c$. Consider now any edge e_0 of G_0. If e_0 is one of the parallel edges between v_1 and v_m, then let e' be any edge of S and $p' = p - 1$, otherwise let $e' = e_0$ and $p' = p$. By the c-crossing-criticality of G, there exists a drawing of $G - e'$ with less than c crossings. Consider the induced drawing of $G_1 - e'$, and let a' denote the minimum number of edges in this drawing that have to be crossed by any curve in the plane from v_1 to v_m and otherwise disjoint from $V(G_1)$. Since $S - e'$ contains p' edge-disjoint paths from v_1 to v_m, we conclude that $\mathrm{cr}(G - e') \geq \mathrm{cr}(G_1 - e') + a'p'$. We can add p' edges between v_1 and v_m to the drawing of $G_1 - e'$ to form a drawing of $G_0 - e_0$ with at most $\mathrm{cr}(G_1 - e') + a'p' \leq \mathrm{cr}(G - e') < c$ crossings. Consequently, G_0 is c-crossing-critical. ◀

Observe that replacing a parallel edge of multiplicity p between vertices u and v in a c-crossing-critical graph with any set of p edge-disjoint plane paths from u to v gives another c-crossing-critical graph. So, the reduction of Lemma 4.3 works in the other direction as well. This two-way process is exhibited by an example with $p = m = 4$ in Figure 5.

Reducing a shelled band or fan. If we could follow the same proof scheme as with necklaces also in the remaining cases of shelled bands and fans, then we would already reach the final goal. Unfortunately, the latter cases are more involved, and require some preparatory work. Compared to the easier case of a necklace, the important difference in the case of a shelled band comes from the fact that the band may be drawn not only in the “straight way” but also in the “twisted way” (recall Figure 1). An indication that this is troublesome comes from the result of Hliněný and Derňár [10], who showed that determining the crossing number of a twisted planar tile is NP-complete (and thus it is not determined by a simple parameter such as the number of edge-disjoint paths between its sides). Consequently, the analysis of shelled bands is significantly more complicated than the relatively straightforward proof of Lemma 4.3. The same remark applies to the shelled fans.

That is why we leave the full details and proofs of the remaining cases for the full paper. Before we dive into technical details needed to at least formulate the final result, Theorem 4.9, we present an informal outline of our approach:

1. Having a very long shelled band $\mathcal{P}$ in our graph G, it is easy to see that the isomorphism types of bounded-size tiles in $\mathcal{P}$ must repeat. Moreover, even bounded-length subbands must have isomorphic repetitions. The first idea is to shorten the band between such repeated isomorphic subbands $\mathcal{P}_1$ and $\mathcal{P}_2$ – by identifying the repeated pieces and discarding what was between (cf. Definition 4.5). If the repeated subband is long enough, we can use some rather easy connectivity properties of $\mathcal{P}$ to show that this yields a smaller graph G_1 of crossing number at least c.

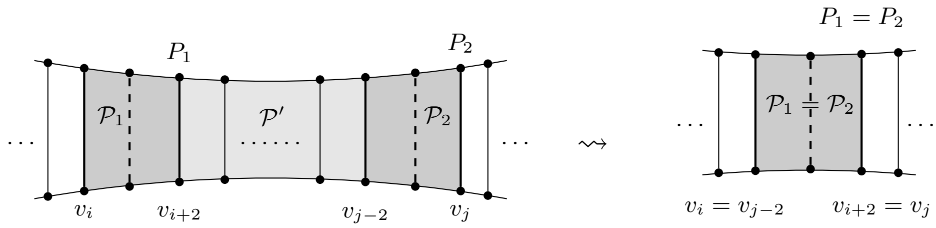

Figure 6 A scheme of a reducible subband $\mathcal{P}'$ (in grey) with repetition $(\mathcal{P}_1, \mathcal{P}_2)$ of order 3 (darker grey), as in Definition 4.5, and the result of the reduction on $\mathcal{P}'$ (on the right).

2. Though, it is not clear that the reduced graph G_1 is c-crossing-critical. Analogously to Lemma 4.3, for any edge $e \in E(G_1)$, we would like to transform a drawing of $G - e$ with less than c crossings to a drawing of $G_1 - e$ with less than c crossings. However, if the drawing of $G - e$ uses some unique properties of the part $\mathcal{P}_{12}$ of the band between $\mathcal{P}_1$ and $\mathcal{P}_2$, we have no way how to mimic this in the drawing of $G_1 - e$ (this is especially troublesome if this part of $G - e$ is drawn in a twisted way, since there is no easy description of what these "unique properties" might be by the NP-completeness result [10]).
   We overcome this difficulty by performing the described reduction only inside longer pieces which repeat elsewhere in the band (cf. Definition 4.6). Hence, in $G_1 - e$ we have many copies of $\mathcal{P}_{12}$, and by appropriate surgery, we can use one of them to mimic the drawing of $\mathcal{P}_{12}$ in $G - e$.
3. A further advantage of reducing within parts that repeat elsewhere is that we can more explicitly describe the converse expansion operation, as duplicating subbands which already exist elsewhere in the (reduced) band.

Let us remark that considering a shelled (F, u)-fan instead of a band is not different, all the arguments simply carry over. The following additional definitions are needed to formalize the outlined claims.

Let $\mathcal{P} = (P_1, \dots, P_m)$ be an (F_1, F_2)-band or an (F_1, u)-fan in a 2-connected plane graph G, and let T_i be the tile of $\mathcal{P}$ delimited by P_i and P_{i+1}. We say that the band $\mathcal{P}$ is *k-edge-linked* if $k \in \mathbf{N}$ and there exist k pairwise edge-disjoint paths from $V(P_1)$ to $V(P_m)$ contained in the support of $\mathcal{P}$, and for each $i = 1, \dots, m-1$, the tile T_i contains an edge-cut of size k separating $V(P_i)$ from $V(P_{i+1})$.

Similarly, the fan $\mathcal{P}$ is *k-edge-linked* if there exist k pairwise edge-disjoint paths from $V(P_1)\setminus\{u\}$ to $V(P_m)\setminus\{u\}$ contained in the support of $\mathcal{P}$ minus u, and for each $i = 1, \dots, m-1$, the sub-tile $T_i - u$ contains an edge-cut of size k separating $V(P_i) \setminus \{u\}$ from $V(P_{i+1}) \setminus \{u\}$. For a closer explanation, one may say that, modulo a trivial adjustment, the fan $\mathcal{P}$ is k-edge-linked iff the corresponding band in $G - u$ is k-edge-linked.

▶ **Definition 4.4** (isomorphic tiles). Two (F_1, F_2)-bands or (F_1, u)-fans $\mathcal{P}_1 = (P_1, \dots, P_m)$ and $\mathcal{P}_2 = (P'_1, \dots, P'_m)$ are *isomorphic* if there exists a homeomorphism mapping the support of $\mathcal{P}_1$ to the support of $\mathcal{P}_2$ and mapping the path P_i to P'_i for $i = 1, \dots, m$, where the paths are taken as directed away from F_1 (i.e., the homeomorphism must map the vertex of P_i incident with F_1 to the vertex of P'_i incident with F_1).

▶ **Definition 4.5** (band or fan reduction). Let G be a graph drawn in the plane with crossings. Let G' be the planarization of G and let Y denote the set of crossing vertices of G'. Let $\mathcal{P}$ be an (F_1, F_2)-band or an (F_1, u)-fan in G' whose support is disjoint from Y. Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are isomorphic subbands or subfans of $\mathcal{P}$, with disjoint supports, except for the vertex u when $\mathcal{P}$ is a fan, and not containing the first and the last path of $\mathcal{P}$. Let $\mathcal{P}'$ be the minimal subband or subfan of $\mathcal{P}$ containing both $\mathcal{P}_1$ and $\mathcal{P}_2$. We then say that $\mathcal{P}'$ is a *reducible subband or subfan with repetition* $(\mathcal{P}_1, \mathcal{P}_2)$. See Figure 6. The *order of* this repetition $(\mathcal{P}_1, \mathcal{P}_2)$ equals the length of $\mathcal{P}_1$ (which is the same as the length of $\mathcal{P}_2$).

Let P_1 and P_2 be the last paths of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Denote by S the support of the subband or subfan between P_1 and P_2, excluding these two paths. Let G'_1 be obtained from G' by removing S and by identifying P_1 with P_2 (stretching the drawing of the support of $\mathcal{P}_1$ within the area originally occupied by S). Let G_1 be obtained from G'_1 by turning the vertices of Y back into crossings. For clarity, note that the support of $\mathcal{P}'$ is disjoint from Y, and so $\mathcal{P}'$ is also a band or fan in a plane subgraph of G. We then say that G_1 is the *reduction of G on $\mathcal{P}'$*.

▶ **Definition 4.6** (t-typical subband or subfan)**.** We say that, in an (F_1, F_2)-band or an (F_1, u)-fan $\mathcal{P}$, a subband $\mathcal{Q}$ is *t-typical* if the following holds: there exist subbands or subfans $\mathcal{P}_1, \ldots, \mathcal{P}_{2t+1}$ of $\mathcal{P}$ appearing in this order, such that they are pairwise isomorphic, with pairwise disjoint supports except for the vertex u when $\mathcal{P}$ is a fan, and $\mathcal{Q} = \mathcal{P}_{t+1}$.

▶ **Lemma 4.7** (*)**.** *Let G be a 2-connected c-crossing-critical graph drawn in the plane with the minimum number of crossings. Let G' be the planarization of G and let Y denote the set of crossing vertices of G'. Let $c_0 = \lceil 5c/2 + 16 \rceil$ and $k \in \mathbf{N}$. Let $\mathcal{P}$ be a k-edge-linked shelled (F_1, F_2)-band or proper (F_1, u)-fan in G' whose support is disjoint from Y. Let $\mathcal{Q}$ be a subband or subfan of $\mathcal{P}$ which is reducible with repetition of order at least $12c_0 + 2k$. If $\mathcal{Q}$ is c-typical in $\mathcal{P}$, then the reduction G_1 of G on $\mathcal{Q}$ is a c-crossing-critical graph again.*

Expanding a band, fan or a necklace. Finally, it is time to formally define what is a generic converse operation of the instances of reduction considered by Lemmas 4.7 and 4.2:

▶ **Definition 4.8** (n-bounded expansion)**.** Let G be a 2-connected c-crossing-critical graph drawn in the plane with the minimum number of crossings. Let G' be the planarization of G and let Y denote the set of crossing vertices of G'. Let $c_0 = \lceil 5c/2 + 16 \rceil$. Assume $\mathcal{P}$ is a k-edge-linked shelled (F_1, F_2)-band or proper (F_1, u)-fan in G' whose support is disjoint from Y. Let $\mathcal{Q}$ be a c-typical subband or subfan of $\mathcal{P}$ which is reducible with repetition of order at least $12c_0 + 2k$. Let the number of vertices of the support of $\mathcal{Q}$ be at most n, and let G_1 denote the reduction of G on $\mathcal{Q}$. In these circumstances, we say that G is an *n-bounded expansion of G_1*.

Assume $\mathcal{P}'$ is a necklace in G' whose support is disjoint from Y, and let $\mathcal{Q}' = (v_1, v_2)$ be a 1-typical subband of $\mathcal{P}'$ of length 2. Let G_2 be obtained from G by replacing the support S of $\mathcal{Q}'$ by a parallel edge of multiplicity equal to the maximum number of pairwise edge-disjoint paths between v_1 and v_2 in S. Let the number of vertices of the support of $\mathcal{Q}'$ be at most n. In these circumstances, we also say that G is an *n-bounded expansion of G_1*.

▶ **Theorem 4.9** (*)**.** *For every integer $c \geq 1$, there exists a positive integer n_0 such that the following holds. If G is a 2-connected c-crossing-critical graph, then there exists a sequence $G_0, G_1, \ldots, G_m$ of 2-connected c-crossing-critical graphs such that $|V(G_0)| \leq n_0$, $G_m = G$, and for $i = 1, \ldots, m$, G_i is an n_0-bounded expansion of G_{i-1}.*

Moreover, the generating sequences claimed by Theorem 4.9 can be turned into an efficient enumeration procedure to generate all 2-connected c-crossing-critical graphs of at most given order n, for each fixed c. The output-sensitive complexity of this procedure has polynomial delay in n. We leave further details for the full paper.

5 Deconstructing plane graphs of bounded path-width

We now return to the topic of Section 3, supplementing the technical prerequisites of Theorem 3.5. We need to add a few terms related to Definition 3.4.

Let (P, β) be a path decomposition of a graph G. Let s denote the first node and t the last node of P. For $x \in V(P) \setminus \{s\}$, let $l(x)$ be the node of P preceding x, and let $L(x) = \beta(l(x)) \cap \beta(x)$. For $x \in V(P) \setminus \{t\}$, let $r(x)$ be the node of P following x, and let $R(x) = \beta(r(x)) \cap \beta(x)$. The path decomposition is *proper* if $\beta(x) \not\subseteq \beta(y)$ for all distinct $x, y \in V(P)$. The *interior width* of the decomposition is the maximum over $|\beta(x)| - 1$ over all nodes x of P distinct from s and t. The path decomposition is *p-linked* if $|L(x)| = p$ for all $x \in V(P) \setminus \{s\}$ and G contains p vertex-disjoint paths from $R(s)$ to $L(t)$. The *order* of the decomposition is $|V(P)|$.

A crucial technical step in the proof of Theorem 3.5 is to analyse a topological structure of the bags of a path decomposition (P, β) of a plane graph G, and to find many consecutive subpaths of P on which the decomposition repeats the same "topological behavior". For this we are going to model the bags of the decomposition (P, β) as letters of a string over a suitable finite semigroup (these letters present an abstraction of the bags), and to apply the following algebraic tool, Lemma 5.1.

Let T be a rooted ordered tree (i.e., the order of children of each vertex is fixed). Let f be a function that to each leaf of T assigns a string of length 1, such that for each non-leaf vertex v of T, $f(v)$ is the concatenation of the strings assigned by f to the children of v in order. We say that (T, f) *yields the string* assigned to the root of T by f. If the letters of the string are elements of a semigroup A, then for each $v \in V(T)$, let $f_A(v)$ denote the product of the letters of $f(v)$ in A. Recall that an element e of A is *idempotent* if $e^2 = e$. A tree (T, f) is an *A-factorization tree* if for every vertex v of T with more than two children, there exists an idempotent element $e \in A$ such that $f_A(x) = e$ for each child x of v (and hence also $f_A(v) = e$). Simon [17] showed existence of bounded-depth A-factorization trees for every string; the improved bound in the following lemma was proved by Colcombet [4]:

▶ **Lemma 5.1** ([4]). *For every finite semigroup A and each string of elements of A, there exists an A-factorization tree of depth at most $3|A|$ yielding this string.*

We further need to formally define what we mean by a "topological behavior" of bags and subpaths of a path decomposition of our G. This will be achieved by the following term of a q-type.

In this context we consider multigraphs (i.e., with parallel edges and loops allowed – each loop contributes 2 to degree of the incident vertex, and not necessarily connected) with some of its vertices labelled by distinct unique labels. A plane multigraph G is *irreducible* if G has no faces of size 1 or 2, and every unlabelled vertex of degree at most 2 is an isolated vertex incident with one loop (this loop, hence, cannot bound a 1-face). Two plane multigraphs G_1 and G_2 with some of the vertices labelled are *homeomorphic* if there exists a homeomorphism φ of the plane mapping G_1 onto G_2 so that for each vertex $v \in V(G_1)$, the vertex $\varphi(v)$ is labelled iff v is, and then v and $\varphi(v)$ have the same label. For G with some of its vertices labelled using the labels from a finite set $\mathcal{L}$, the *q-type* of G is the set of all non-homeomorphic irreducible plane multigraphs labelled from $\mathcal{L}$ and with at most q unlabelled vertices, and whose subdivisions are homeomorphic to subgraphs of G.

Let G be a plane graph and let (P, β) be its p-linked path decomposition. Let s and t be the endpoints of P. Fix pairwise vertex-disjoint paths $Q_1, \ldots, Q_p$ between $R(s)$ and $L(t)$. Consider a subpath P' of $P - \{s, t\}$, and let $G_{P'}$ be the subgraph of G induced by $\bigcup_{x \in V(P')} \beta(x)$. If s' and t' are the (left and right) endpoints of P', we define $L(P') = L(s')$

and $R(P') = R(t')$. Let us label the vertices of $G_{P'}$ using (some of) the labels $\{l_1, \dots, l_p, r_1, \dots, r_p, c_1, \dots, c_p\}$ as follows: For $i = 1, \dots, p$, let u and v be the vertices in which Q_i intersects $L(P')$ and $R(P')$, respectively. If $u \neq v$, we give u the label l_i and v the label r_i. Otherwise, we give $u = v$ the label c_i. For an integer q, the *q-type* of P' is the q-type of $G_{P'}$ with this labelling. If P' contains just one node x, then we speak of the *q-type* of x.

The q-types of subpaths of a linked path decomposition naturally form a semigroup with concatenation of the subpaths, as detailed in the full paper. From Lemma 5.1, specialised to our case, we derive the following structural description which is crucial in the proof of Theorem 3.5. Further technical details are again left for the full paper.

▶ **Theorem 5.2 (*).** *Let w and q be non-negative integers, and let $f : \mathbf{N} \to \mathbf{N}$ be an arbitrary non-decreasing function. There exist integers w_0 and n_0 such that, for any plane graph G that has a proper path decomposition of interior width at most w and order at least n_0, the following holds. For some $w' \leq w_0$ and $p \leq w$, G also has a p-linked proper path decomposition (P, β) of interior width at most w' and order at least $f(w')$, such that for each node x of P distinct from its endpoints, the q-type of x is the same idempotent element.*

In other words, we can find a decomposition in which all topological properties of the drawing that hold in one bag repeat in all the bags. So, for example, if for some node x, the vertices of $L(x)$ are separated in the drawing from vertices of $R(x)$ by a cycle contained in the bag of x, then this holds in every bag, and we conclude that the drawing contains a large 0-nest. Other outcomes of Theorem 3.5 naturally correspond to other possible local properties of the drawings of the bags.

6 Conclusion

To summarize, we have shown a structural characterisation and an enumeration procedure for all 2-connected c-crossing-critical graphs, using bounded-size replication steps over an implicit finite set of base c-crossing-critical graphs. The characterisation can be reused to describe all c-crossing-critical graphs (without the connectivity assumption) since all their proper blocks must be c_i-crossing-critical for some $c_i < c$.

With this characterisation at hand, one can expect significant progress in the crossing number research, both from mathematical and algorithmic perspectives. For example, one can quite easily derive from Theorem 4.9 that, for no c there is an infinite family of 3-regular c-crossing-critical graphs, a claim that has been so far proved only via the Graph minors theorem of Robertson and Seymour. One can similarly expect a progress in some long-time open questions in the area of crossing-critical graphs, such as to improve the bound of Theorem 2.3 or to decide possible existence of an infinite family of 5-regular c-crossing-critical graphs for some c.

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