# Vietoris–Rips and Čech Complexes of Metric Gluings


**Michał Adamaszek**
MOSEK ApS
Copenhagen, Denmark
aszek@mimuw.edu.pl
https://orcid.org/0000-0003-3551-192X

**Henry Adams**
Department of Mathematics, Colorado State University
Fort Collins, CO, USA
adams@math.colostate.edu
https://orcid.org/0000-0003-0914-6316

**Ellen Gasparovic**
Department of Mathematics, Union College
Schenectady, NY, USA
gasparoe@union.edu
https://orcid.org/0000-0003-3775-9785

**Maria Gommel**
Department of Mathematics, University of Iowa
Iowa City, IA, USA
maria-gommel@uiowa.edu
https://orcid.org/0000-0003-2714-9326

**Emilie Purvine**
Computing and Analytics Division, Pacific Northwest National Laboratory
Seattle, WA, USA
emilie.purvine@pnnl.gov
https://orcid.org/0000-0003-2069-5594

**Radmila Sazdanovic**[1]
Department of Mathematics, North Carolina State University
Raleigh, NC, USA
rsazdanovic@math.ncsu.edu
https://orcid.org/0000-0003-1321-1651

**Bei Wang**[2]
School of Computing, University of Utah
Salt Lake City, UT, USA
beiwang@sci.utah.edu
https://orcid.org/0000-0002-9240-0700

**Yusu Wang**[3]
Department of Computer Science, The Ohio State University
Columbus, OH, USA
yusu@cse.ohio-state.edu
https://orcid.org/0000-0001-7950-4348



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**Lori Ziegelmeier**
Department of Mathematics, Statistics, and Computer Science, Macalester College
Saint Paul, MN, USA
lziegel1@macalester.edu
https://orcid.org/0000-0002-1544-4937

## Abstract

We study Vietoris–Rips and Čech complexes of metric wedge sums and metric gluings. We show that the Vietoris–Rips (resp. Čech) complex of a wedge sum, equipped with a natural metric, is homotopy equivalent to the wedge sum of the Vietoris–Rips (resp. Čech) complexes. We also provide generalizations for certain metric gluings, i.e. when two metric spaces are glued together along a common isometric subset. As our main example, we deduce the homotopy type of the Vietoris–Rips complex of two metric graphs glued together along a sufficiently short path. As a result, we can describe the persistent homology, in all homological dimensions, of the Vietoris–Rips complexes of a wide class of metric graphs.

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## 1 Introduction

When equipped with a notion of similarity or distance, data can be thought of as living in a metric space. Our goal is to characterize the homotopy types of geometric thickenings of a wide class of metric spaces. In particular, we consider metric spaces formed by gluing smaller metric spaces together along certain nice intersections; our results then characterize the persistent homology of these spaces. Persistent homology is a central tool in topological data analysis that captures complex interactions within a system at multiple scales [13, 19].

The geometric complexes of interest are Vietoris–Rips and Čech complexes, which build a simplicial complex on top of a metric space according to the choice of a scale parameter $r$. We first study Vietoris–Rips and Čech complexes of metric wedge sums: given two metric spaces $X$ and $Y$ with specified basepoints, the metric wedge sum $X \vee Y$ is obtained by gluing $X$ and $Y$ together at the specified points. We show that the Vietoris–Rips (resp. Čech) complex of the metric wedge sum is homotopy equivalent to the wedge sum of the Vietoris–Rips (resp. Čech) complexes. We also provide generalizations for certain more general metric gluings, namely, when two metric spaces are glued together along a common isometric subset.

One common metric space that appears in applications such as road networks [1], brain functional networks [8], and the cosmic web [29] is a *metric graph*, a structure where any two points of the graph (not only vertices) are assigned a distance equal to the minimum length of a path from one point to the other. In this way, a metric graph encodes the proximity data of a network into the structure of a metric space. As a special case of our results, we show that the Vietoris–Rips complex of two metric graphs glued together along a sufficiently short common path is homotopy equivalent to the union of the Vietoris–Rips complexes. This enables us to determine the homotopy types of geometric thickenings of a large class of metric graphs, namely those that can be constructed iteratively via simple gluings.

The motivation for using Vietoris–Rips and Čech complexes in the context of data analysis is that these complexes can recover topological features of an unknown sample space underlying the data. Indeed, in [22, 25], it is shown that if the underlying space $M$ is a closed Riemannian manifold, if scale parameter $r$ is sufficiently small compared to the injectivity radius of $M$, and if a sample $X$ is sufficiently close to $M$ in the Gromov-Hausdorff distance, then the Vietoris–Rips complex of the sample $X$ at scale $r$ is homotopy equivalent to $M$. Analogously, the Nerve Theorem implies that the Čech complex (the nerve of all metric balls of radius $r$) of a similarly nice sample is homotopy equivalent to $M$ for small $r$ ([10] or [21, Corollary 4G.3]). In this paper, we identify the homotopy types of Vietoris–Rips and Čech complexes of certain metric graphs at all scale parameters $r$, not just at small scales.

Our paper builds on the authors' prior work characterizing the 1-dimensional intrinsic Čech persistence module associated to an arbitrary metric graph. Indeed, [20] shows that the 1-dimensional intrinsic Čech persistence diagram associated to a metric graph of genus $g$ (i.e., the rank of the 1-dimensional homology of the graph) consists of the points $\left\{\left(0, \frac{\ell_i}{4}\right) : 1 \leq i \leq g\right\}$, where $\ell_i$ corresponds to the length of the $i^{th}$ loop. In the case of the Vietoris–Rips complex, the results hold with the minor change that the persistence points are $\left\{\left(0, \frac{\ell_i}{6}\right) : 1 \leq i \leq g\right\}$. An extension of this work is [31], which studies the 1-dimensional persistence of geodesic spaces. In [3, 4], the authors show that the Vietoris–Rips or Čech complex of the circle obtains the homotopy types of the circle, the 3-sphere, the 5-sphere, ..., as the scale $r$ increases, giving the persistent homology in all dimensions of a metric graph consisting of a single cycle. In this paper, we extend to a larger class of graphs: our results characterize the persistence profile, in any homological dimension, of Vietoris–Rips complexes of metric graphs that can be iteratively built by gluing trees and cycles together along short paths.

Our results on Vietoris–Rips and Čech complexes of metric gluings have implications for future algorithm development along the line of "topological decompositions". The computation time of homotopy, homology, and persistent homology depend on the size of the simplicial complex. It would be interesting to investigate if our Theorem 10 means that one can break a large metric graph into smaller pieces, perform computations on the simplicial complex of each piece, and then subsequently reassemble the results together. This has the potential to use less time and memory.

**Outline.** Section 2 introduces the necessary background and notation. Our main results on the Vietoris–Rips and Čech complexes of metric wedge sums and metric gluings are established in Section 3. In addition to proving homotopy equivalence in the wedge sum case, we show that the persistence module (for both Vietoris–Rips and Čech) of the wedge sum of the complexes is isomorphic to the persistence module of the complex for the wedge sum. We develop the necessary machinery to prove that the Vietoris–Rips complex of metric spaces glued together along a sufficiently short path is homotopy equivalent to the union of the Vietoris–Rips complexes. The machinery behind this proof technique does not hold in the analogous case for the Čech complex, and we provide an example illustrating why not. In Section 4, we describe families of metric graphs to which our results apply and furthermore discuss those that we cannot yet characterize. In Section 5, we conclude with our overall goal of characterizing the persistent homology profiles of families of metric graphs.

## 2 Background

In this section, we recall the relevant background in the settings of simplicial complexes and metric spaces, including metric graphs. For a more comprehensive introduction of related concepts from algebraic topology, we refer the reader to [21], and to [23] and [19] for a combinatorial and computational treatment, respectively.

**Simplicial complexes.** An *abstract simplicial complex* $K$ is a collection of finite subsets of some (possibly infinite) vertex set $V = V(K)$, such that if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$. In this paper, we use the same symbol $K$ to denote both the abstract simplicial complex and its geometric realization. For $V' \subseteq V$, we let $K[V']$ denote the induced simplicial complex on the vertex subset $V'$. If $K$ and $L$ are simplicial complexes with disjoint vertex sets $V(K)$ and $V(L)$, then their *join* $K * L$ is the simplicial complex whose vertex set is $V(K) \cup V(L)$, and whose set of simplices is $K * L = \{\sigma_K \cup \sigma_L \mid \sigma_K \in K \text{ and } \sigma_L \in L\}$ [23, Definition 2.16]. The *join* of two disjoint simplices $\sigma = \{x_0, \cdots, x_n\}$ and $\tau = \{y_0, \cdots, y_m\}$ is the simplex $\sigma \cup \tau := \{x_0, \cdots, x_n, y_0, \cdots, y_m\}$.

By an abuse of notation, a simplex $S \in K$ can be considered as either a single simplex, or as a simplicial complex $\{S' \mid S' \subseteq S\}$ with all subsets as faces. When taking joins, we use $\cup$ to denote that the result is a simplex, and we use $*$ to denote that the result is a simplicial complex. For example, for $a \in V(K)$ a vertex and $S \in K$ a simplex, we use the notation $a \cup S := \{a\} \cup S$ to denote the simplex formed by adding vertex $a$ to $S$. We instead use $a * S := \{a \cup S' \mid S' \subseteq S\}$ to denote the associated simplicial complex. Similarly, for two simplices $\sigma, S \in K$, we use $\sigma \cup S$ to denote a simplex, and we instead use $\sigma * S := \{\sigma' \cup S' \mid \sigma' \subseteq \sigma,\ S' \subseteq S\}$ to denote the associated simplicial complex. We let $\dot{S}$ be the boundary simplicial complex $\dot{S} = \{S' \mid S' \subsetneq S\}$, and therefore $a * \dot{S} := \{a \cup S' \mid S' \subsetneq S\}$ and $\sigma * \dot{S} := \{\sigma' \cup S' \mid \sigma' \subseteq \sigma,\ S' \subsetneq S\}$ are simplicial complexes.

A simplicial complex $K$ is equipped with the topology of a CW-complex [21]: a subset of the geometric realization of $K$ is closed if and only if its intersection with each finite-dimensional skeleton is closed.

**Simplicial collapse.** Recall that if $\tau$ is a face of a simplex $\sigma$, then $\sigma$ is said to be a *coface* of $\tau$. Given a simplicial complex $K$ and a maximal simplex $\sigma \in K$, we say that a face $\tau \subseteq \sigma$ is a *free face of* $\sigma$ if $\sigma$ is the unique maximal coface of $\tau$ in $K$. A *simplicial collapse* of $K$ with respect to a pair $(\tau, \sigma)$, where $\tau$ is a free face of $\sigma$, is the removal of all simplices $\rho$ such that $\tau \subseteq \rho \subseteq \sigma$. If $\dim(\sigma) = \dim(\tau) + 1$ then this is an *elementary simplicial collapse*. If $L$ is obtained from a finite simplicial complex $K$ via a sequence of simplicial collapses, then $L$ is homotopy equivalent to $K$, denoted $L \simeq K$ [23, Proposition 6.14].

**Metric spaces.** Let $(X, d)$ be a metric space, where $X$ is a set equipped with a distance function $d$. Let $B(x, r) := \{y \in X \mid d(x, y) \leq r\}$ denote the closed ball with center $x \in X$ and radius $r \geq 0$. The *diameter* of $X$ is $\operatorname{diam}(X) = \sup\{d(x, x') \mid x, x' \in X\}$. A *submetric space* of $X$ is any set $X' \subseteq X$ with a distance function defined by restricting $d$ to $X' \times X'$.

**Vietoris–Rips and Čech complexes.** We consider two types of simplicial complexes constructed from a metric space $(X, d)$. These constructions depend on the choice of a scale parameter $r \geq 0$. First, the *Vietoris–Rips complex* of $X$ at scale $r \geq 0$ consists of all finite subsets of diameter at most $r$, that is, $\mathrm{VR}(X;r) = \{\text{finite } \sigma \subseteq X \mid \mathrm{diam}(\sigma) \leq r\}$. Second, for $X$ a submetric space of $X'$, we define the *ambient Čech complex* with vertex set $X$ as $\check{\mathrm{C}}\mathrm{ech}(X, X'; r) := \{\text{finite } \sigma \subseteq X \mid \exists\, x' \in X' \text{ with } d(x, x') \leq r \ \forall\, x \in \sigma\}$. The set $X$ is often called the set of "landmarks", and $X'$ is called the set of "witnesses" [17]. This complex can equivalently be defined as the nerve of the balls $B_{X'}(x, r)$ in $X'$ that are centered at points $x \in X$, that is, $\check{\mathrm{C}}\mathrm{ech}(X, X'; r) = \{\text{finite } \sigma \subseteq X \mid \bigcap_{x \in \sigma} B_{X'}(x, r) \neq \emptyset\}$. When $X = X'$, we denote the *(intrinsic) Čech complex* of $X$ as $\check{\mathrm{C}}\mathrm{ech}(X; r) = \check{\mathrm{C}}\mathrm{ech}(X, X; r)$. Alternatively, the Čech complex can be defined with an open ball convention, and the Vietoris–Rips complex can be defined as $\mathrm{VR}(X;r) = \{\sigma \subseteq X \mid \mathrm{diam}(\sigma) < r\}$. Unless otherwise stated, all of our results hold for both the open and closed convention for Čech complexes, as well as for both the $<$ and $\leq$ convention on Vietoris–Rips complexes.

**Persistent homology.** For $k$ a field, for $i \geq 0$ a homological dimension, and for $Y$ a filtered topological space, we denote the *persistent homology* (or *persistence*) *module* of $Y$ by $\mathrm{PH}_i(Y;k)$. Persistence modules form a category [16, Section 2.3], where morphisms are given by commutative diagrams.

**Gluings of topological spaces.** Let $X$ and $Y$ be two topological spaces that share a common subset $A = X \cap Y$. The *gluing* space $X \cup_A Y$ is formed by gluing $X$ to $Y$ along their common subspace $A$. More formally, let $\iota_X \colon A \to X$ and $\iota_Y \colon A \to Y$ denote the inclusion maps. Then the gluing space $X \cup_A Y$ is the quotient space of the disjoint union $X \coprod Y$ under the identification $\iota_X(a) \sim \iota_Y(a)$ for all $a \in A$. The gluing of two simplicial complexes along a common subcomplex is itself a simplicial complex.

**Gluings of metric spaces.** Following Definition 5.23 in [11], we define a way to glue two metric spaces along a closed subspace. Let $X$ and $Y$ be arbitrary metric spaces with closed subspaces $A_X \subset X$ and $A_Y \subset Y$. Let $A$ be a metric space with isometries $\iota_X : A \to A_X$ and $\iota_Y : A \to A_Y$. Let $X \cup_A Y$ denote the quotient of the disjoint union of $X$ and $Y$ by the equivalence between $A_X$ and $A_Y$, i.e., $X \cup_A Y = X \sqcup Y / \{\iota_X(a) \sim \iota_Y(a) \mid a \in A\}$. Then $X \cup_A Y$ is the *gluing of $X$ and $Y$ along $A$.* We define a metric on $X \cup_A Y$, which extends the metrics on $X$ and $Y$:

$$d_{X \cup_A Y}(s,t) = \begin{cases} d_X(s,t) & \text{if } s,t \in X \\ d_Y(s,t) & \text{if } s,t \in Y \\ \inf_{a \in A} d_X(s, \iota_X(a)) + d_Y(\iota_Y(a), t) & \text{if } s \in X, t \in Y. \end{cases}$$

Lemma 5.24 of [11] shows that the gluing $(X \cup_A Y, d_{X \cup_A Y})$ of two metric spaces along common isometric closed subsets is itself a metric space. In this paper all of our metric gluings will be done in the case where $X \cap Y = A$ and the $\iota_X$ and $\iota_Y$ are identity maps. This definition of gluing metric spaces agrees with that of gluing their respective topological spaces, with the standard metric ball topology.

**Pointed metric space and wedge sum.** A *pointed metric space* is a metric space $(X, d_X)$ with a distinguished basepoint $b_X \in X$. In the notation of metric gluings, given two pointed metric spaces $(X, d_X)$ and $(Y, d_Y)$, let $X \vee Y = X \cup_A Y$ where $A_X = \{b_X\}$ and $A_Y = \{b_Y\}$; we also refer to $X \vee Y$ as the *wedge sum* of $X$ and $Y$. The gluing metric on $X \vee Y$ is the same as in the gluing of metric spaces above with $|A| = 1$.

**Metric graphs.** A graph $G$ consists of set $V = \{v_i\}$ of vertices and a set $E = \{e_j\}$ of edges connecting the vertices. A graph $G$ is a *metric graph* if each edge $e_j$ is assigned a positive finite length $l_j$ [11, 12, 24]. Under mild hypotheses[4], the graph $G$ can be equipped with a natural metric $d_G$: the distance between any two points $x$ and $y$ (not necessarily vertices) in the metric graph is the infimum of the length of all paths between them.

**Loops of a metric graph.** A *loop* of a metric graph $G$ is a continuous map $c : \mathbb{S}^1 \to G$. We also use the word *loop* to refer to the image of this map. Intuitively, elements of the singular 1-dimensional homology of $G$ may be represented by collections of loops in $G$ [21]. The *length* of a loop is the length of the image of the map $c$.

# 3 Homotopy equivalences for metric gluings

## 3.1 Homotopy lemmas for simplicial complexes

In this section, we present three lemmas that will be vital to our study of homotopy equivalences of simplicial complexes. We begin with a lemma proved by Barmak and Minian [7] regarding a sequence of elementary simplicial collapses between two simplicial complexes (Lemma 1). We then generalize this lemma in order to use it in the case where the simplicial collapses need not be elementary (Lemma 2). While these first two lemmas are relevant in the context of finite metric spaces, the third lemma will be useful when passing to arbitrary metric spaces. These three lemmas will later allow us to show that a complex built on a gluing is homotopy equivalent to the gluing of the complexes.

▶ **Lemma 1** (Lemma 3.9 from [7]). *Let $L$ be a subcomplex of a finite simplicial complex $K$. Let $T$ be a set of simplices of $K$ which are not in $L$, and let $a$ be a vertex of $L$ which is contained in no simplex of $T$, but such that $a \cup S$ is a simplex of $K$ for every $S \in T$. Finally, suppose that $K = L \cup \bigcup_{S \in T}\{S, a \cup S\}$. Then $K$ is homotopy equivalent to $L$ via a sequence of elementary simplicial collapses.*

In [7], Barmak and Mimian observe that there is an elementary simplicial collapse from $K$ to $L$ if there is a simplex $S$ of $K$ and a vertex $a$ of $K$ not in $S$ such that $K = L \cup \{S, a \cup S\}$ and $L \cap (a * S) = a * \dot{S}$, where $\dot{S}$ denotes the boundary of $S$. Indeed, $S$ is the free face of the elementary simplicial collapse, and the fact that $a \cup S$ is the unique maximal coface of $S$ follows from $L \cap (a * S) = a * \dot{S}$ (which implies the intersection of $L$ with $S$ is the boundary of $S$). See Figure 1 (left) for an illustration.

It's not difficult to show that Barmak and Minian's observation can be made more general. In fact, there is a simplicial collapse from $K$ to $L$ if there is a simplex $S$ of $K$ and another simplex $\sigma$ of $K$, disjoint from $S$, such that $K = L \cup \{\tau : S \subseteq \tau \subseteq \sigma \cup S\}$ and $L \cap (\sigma * S) = \sigma * \dot{S}$. Indeed, $S$ is again the free face of the simplicial collapse, and the fact that $\sigma \cup S$ is the unique maximal coface of $S$ in $K$ follows from $L \cap (\sigma * S) = \sigma * \dot{S}$ (which implies the intersection of $L$ with $S$ is the boundary of $S$). See Figure 1 (right) for an illustration.

Our more general Lemma 2 will be used in the proof of Theorem 8 when we consider gluings along sets that are larger than just a single point.

[4] For every vertex, the lengths of the edges incident to that vertex are bounded away from zero [11, Section 1.9].

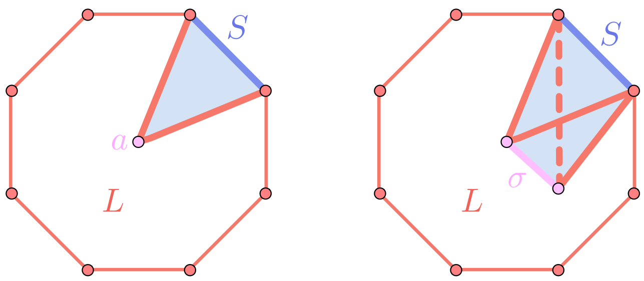


**Figure 1** Examples of simplicial collapses when $T = \{S\}$ for Lemma 1 (left) and Lemma 2 (right).

▶ **Lemma 2** (Generalization of Lemma 1). *Let $L$ be a subcomplex of a finite simplicial complex $K$, and let $\sigma$ be a simplex in $L$. Suppose $T$ is a set of simplices of $K$ which are not in $L$ and which are disjoint from $\sigma$, but such that $\sigma \cup S$ is a simplex of $K$ for every $S \in T$. Finally, suppose $K = L \cup \bigcup_{\{S \in T\}} \{\tau \mid S \subseteq \tau \subseteq \sigma \cup S\}$. Then $K$ is homotopy equivalent to $L$ via a sequence of simplicial collapses.*

**Proof.** We mimic the proof of Lemma 3.9 in [7], except that we perform a sequence of simplicial collapses rather than elementary simplicial collapses. Order the elements $S_1$, $S_2$, $\ldots$, $S_n$ of $T$ in such a way that for every $i$, $j$ with $i \leq j$, we have $|S_i| \leq |S_j|$. Define $K_i = L \cup \bigcup_{j=1}^{i} \{\tau \mid S_j \subseteq \tau \subseteq \sigma \cup S_j\}$ for $0 \leq i \leq n$. Let $S \subsetneq S_i$. If $S \in T$, then $\sigma \cup S \in K_{i-1}$ since $|S| < |S_i|$. If $S \notin T$, then $\sigma \cup S$ is in $L \subseteq K_{i-1}$. This proves that $K_{i-1} \cap (\sigma * S_i) = \sigma * \dot{S}_i$, and so $S_i$ is the free face of a simplicial collapse from $K_i$ to $K_{i-1}$. Then we are done since $K = K_n$ and $L = K_0$. ◀

The next lemma will be useful when passing from wedge sums or gluings of finite metric spaces to wedge sums or gluings of arbitrary metric spaces.

▶ **Lemma 3.** *Let $K$ be a (possibly infinite) simplicial complex with vertex set $V$, and let $L$ be a subcomplex also with vertex set $V$. Suppose that for every finite $V_0 \subseteq V$, there exists a finite subset $V_1$ with $V_0 \subseteq V_1 \subseteq V$ such that the inclusion $L[V_1] \hookrightarrow K[V_1]$ is a homotopy equivalence. It then follows that the inclusion map $\iota \colon L \hookrightarrow K$ is a homotopy equivalence.*

**Proof.** We will use a compactness argument to show that the induced mapping on homotopy groups $\iota_* \colon \pi_k(L, b) \to \pi_k(K, b)$ is an isomorphism for all $k$ and for any basepoint $b$ in the geometric realization of $L$. The conclusion then follows from Whitehead's theorem [21, Theorem 4.5].

First, suppose we have a based map $f \colon \mathbb{S}^k \to K$ where $\mathbb{S}^k$ is the $k$-dimensional sphere. Since $f$ is continuous and $\mathbb{S}^k$ is compact, it follows that $f(\mathbb{S}^k)$ is compact in $K$. Then by [21, Proposition A.1] we know that $f(\mathbb{S}^k)$ is contained in a finite subcomplex of $K$. Therefore, there exists a finite subset $V_0 \subseteq V$ so that $f$ factors through $K[V_0] \subseteq K$. By assumption, there exists a finite subset $V_1$ with $V_0 \subseteq V_1 \subseteq V$ such that the inclusion $\iota_1 \colon L[V_1] \hookrightarrow K[V_1]$ is a homotopy equivalence. Thus, we can find a based map $\widetilde{f} \colon \mathbb{S}^k \to L[V_1]$ such that $[\iota_1 \widetilde{f}] = [f] \in \pi_k(K[V_1], b)$ and hence $[\iota \widetilde{f}] = [f] \in \pi_k(K, b)$. This proves that $\iota_*$ is surjective.

Next, suppose that $f \colon \mathbb{S}^k \to L$ is a based map such that $\iota f \colon \mathbb{S}^k \to K$ is null-homotopic. Let $F \colon \mathbb{B}^{k+1} \to K$ be a null-homotopy between $\iota f$ and the constant map, where $\mathbb{B}^{k+1}$ is the $(k+1)$-dimensional ball. By compactness of $S^k$ and $B^{k+1}$, we can find a finite subset $V_0 \subseteq V$ such that $f$ factors through $L[V_0]$ and $F$ factors through $K[V_0]$. By assumption, there exists a finite subset $V_1$ with $V_0 \subseteq V_1 \subseteq V$ such that the inclusion $\iota_1 \colon L[V_1] \hookrightarrow K[V_1]$ is a homotopy equivalence. Note that $\iota_1 f \colon \mathbb{S}^k \to K[V_1]$ is null-homotopic via $F$, and since the inclusion $\iota_1$ is a homotopy equivalence, it follows that $f$ is null-homotopic, and thus $\iota_*$ is injective. ◀

## 3.2 Vietoris–Rips and Čech complexes of wedge sums

As a warm-up, we first show in this subsection that the Vietoris–Rips complex of a metric wedge sum (i.e, gluing along a single point) is homotopy equivalent to the wedge sum of the Vietoris–Rips complexes, and similarly for Čech complexes. In the next subsection, Proposition 4 will be extended in Corollary 9 and Theorem 10 to gluings of metric spaces and to gluings of metric graphs along short paths, respectively. Intuitively, such results allow us to characterize the topology of a bigger space via the topology of smaller individual pieces.

Given pointed metric spaces $X$ and $Y$, we use the symbol $b \in X \vee Y$ to denote the point corresponding to the identified distinguished basepoints $b_X \in X$ and $b_Y \in Y$.

▶ **Proposition 4.** *For $X$ and $Y$ pointed metric spaces and $r > 0$, we have the homotopy equivalence* $\mathrm{VR}(X;r) \vee \mathrm{VR}(Y;r) \xhookrightarrow{\simeq} \mathrm{VR}(X \vee Y;r)$.

**Proof.** We first consider the case where $X$ and $Y$ are finite. We apply Lemma 1 with $L = \mathrm{VR}(X;r) \vee \mathrm{VR}(Y;r)$, with $K = \mathrm{VR}(X \vee Y;r)$, with $T = \{\sigma \in K \setminus L \mid b \notin \sigma\}$, and with basepoint $b \in X \vee Y$ serving the role as $a$. It is easy to check the conditions on $K, L$ and $T$ required by Lemma 1 are satisfied. Furthermore, if $\sigma \in T$, then at least one vertex of $X \setminus \{b_X\}$ and one vertex of $Y \setminus \{b_Y\}$ are in $\sigma$. Hence $\mathrm{diam}(\sigma \cup b) \leq r$ and $\sigma \cup b$ is a simplex of $K$. Since $K = L \cup \bigcup_{\sigma \in T} \{\sigma, \sigma \cup b\}$, Lemma 1 implies $L \simeq K$.

Now let $X$ and $Y$ be arbitrary (possibly infinite) pointed metric spaces. For finite subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $b_X \in X_0$ and $b_Y \in Y_0$, the finite case guarantees that $\mathrm{VR}(X_0;r) \vee \mathrm{VR}(Y_0;r) \simeq \mathrm{VR}(X_0 \vee Y_0;r)$. Therefore, we can apply Lemma 3 with $L = \mathrm{VR}(X;r) \vee \mathrm{VR}(Y;r)$ and $K = \mathrm{VR}(X \vee Y;r)$. ◀

Proposition 4, in the case of finite metric spaces, is also obtained in [26].

▶ **Corollary 5.** *Let $X$ and $Y$ be pointed metric spaces. For any homological dimension $i \geq 0$ and field $k$, the persistence modules* $\mathrm{PH}_i(\mathrm{VR}(X;r) \vee \mathrm{VR}(Y;r);k)$ *and* $\mathrm{PH}_i(\mathrm{VR}(X \vee Y;r);k)$ *are isomorphic.*

See the full version of this paper [5] for the proof of Corollary 5.

For a submetric space $X \subseteq X'$, let $\check{\mathrm{C}}\mathrm{ech}(X, X';r)$ be the ambient Čech complex with landmark set $X$ and witness set $X'$. Note that if $X \subseteq X'$ and $Y \subseteq Y'$ are pointed with $b_X = b_{X'}$ and $b_Y = b_{Y'}$, then $X \vee Y$ is a submetric space of $X' \vee Y'$.

▶ **Proposition 6.** *For $X \subseteq X'$ and $Y \subseteq Y'$ pointed metric spaces and $r > 0$, we have the homotopy equivalence* $\check{\mathrm{C}}\mathrm{ech}(X, X';r) \vee \check{\mathrm{C}}\mathrm{ech}(Y, Y';r) \xhookrightarrow{\simeq} \check{\mathrm{C}}\mathrm{ech}(X \vee Y, X' \vee Y';r)$.

The proof of Proposition 6 is in the full version of this paper. It proceeds similarly to the proof of Proposition 4, except applying Lemma 1 with $L = \check{\mathrm{C}}\mathrm{ech}(X, X';r) \vee \check{\mathrm{C}}\mathrm{ech}(Y, Y';r)$, $K = \check{\mathrm{C}}\mathrm{ech}(X \vee Y, X' \vee Y';r)$, and $T = \{\sigma \in K \setminus L \mid b \notin \sigma\}$.

▶ **Corollary 7.** *Let $X \subseteq X'$ and $Y \subseteq Y'$ be pointed metric spaces. For any homological dimension $i \geq 0$ and field $k$, the persistence modules* $\mathrm{PH}_i(\check{\mathrm{C}}\mathrm{ech}(X, X';r) \vee \check{\mathrm{C}}\mathrm{ech}(Y, Y';r))$ *and* $\mathrm{PH}_i(\check{\mathrm{C}}\mathrm{ech}(X \vee Y, X' \vee Y';r))$ *are isomorphic.*

## 3.3 Vietoris–Rips complexes of set-wise gluings

We now develop the machinery necessary to prove, in Theorem 10, that the Vietoris–Rips complex of two metric graphs glued together along a sufficiently short path is homotopy equivalent to the union of the Vietoris–Rips complexes. First, we prove a more general result for arbitrary metric spaces that intersect in a sufficiently small space.

▶ **Theorem 8.** *Let $X$ and $Y$ be metric spaces with $X \cap Y = A$, where $A$ is a closed subspace of $X$ and $Y$, and let $r > 0$. Consider $X \cup_A Y$, the metric gluing of $X$ and $Y$ along the intersection $A$. Suppose that if* $\mathrm{diam}(S_X \cup S_Y) \le r$ *for some $\emptyset \ne S_X \subseteq X \setminus A$ and $\emptyset \ne S_Y \subseteq Y \setminus A$, then there is a unique maximal nonempty subset $\sigma \subseteq A$ such that* $\mathrm{diam}(S_X \cup S_Y \cup \sigma) \le r$. *Then* $\mathrm{VR}(X \cup_A Y; r) \simeq \mathrm{VR}(X; r) \cup_{\mathrm{VR}(A;r)} \mathrm{VR}(Y; r)$. *Hence if* $\mathrm{VR}(A; r)$ *is contractible, then* $\mathrm{VR}(X \cup_A Y; r) \simeq \mathrm{VR}(X; r) \vee \mathrm{VR}(Y; r)$.

**Proof.** We first restrict our attention to the case when $X$ and $Y$ (and hence $A$) are finite. Let $n = |A|$. Order the nonempty subsets $\sigma_1, \sigma_2, \ldots, \sigma_{2^n-1}$ of $A$ so that for every $i, j$ with $i \le j$, we have $|\sigma_i| \ge |\sigma_j|$. For $i = 1, 2, \ldots, 2^n - 1$, let $T_i$ be the set of all simplices of the form $S_X \cup S_Y$ such that

- $\emptyset \ne S_X \subseteq X \setminus A$ and $\emptyset \ne S_Y \subseteq Y \setminus A$,
- $\mathrm{diam}(S_X \cup S_Y) \le r$, and
- $\sigma_i$ is the maximal nonempty subset of $A$ satisfying $\mathrm{diam}(S_X \cup S_Y \cup \sigma_i) \le r$.

Let $L_0 = \mathrm{VR}(X; r) \cup_{\mathrm{VR}(A;r)} \mathrm{VR}(Y; r)$. We apply Lemma 2 repeatedly to obtain

$$\begin{aligned}
L_0 \simeq & L_0 \cup_{\{S \in T_1\}} \{\tau \mid S \subseteq \tau \subseteq \sigma_1 \cup S\} =: L_1 \\
\simeq & L_1 \cup_{\{S \in T_2\}} \{\tau \mid S \subseteq \tau \subseteq \sigma_2 \cup S\} =: L_2 \\
& \vdots \\
\simeq & L_{2^n-3} \cup_{\{S \in T_{2^n-2}\}} \{\tau \mid S \subseteq \tau \subseteq \sigma_{2^n-2} \cup S\} =: L_{2^n-2} \\
\simeq & L_{2^n-2} \cup_{\{S \in T_{2^n-1}\}} \{\tau \mid S \subseteq \tau \subseteq \sigma_{2^n-1} \cup S\} =: L_{2^n-1}.
\end{aligned}$$

The fact that each $L_j$ is a simplicial complex follows since if $S_X \cup S_Y \in T_j$ and $\emptyset \ne S'_X \subseteq S_X$ and $\emptyset \ne S'_Y \subseteq S_Y$, then we have $S'_X \cup S'_Y \in T_i$ for some $i \le j$ (meaning $\sigma_j \subseteq \sigma_i$). For each $j = 1, \ldots, 2^n - 1$, we set $K = L_j$, $L = L_{j-1}$, $T = T_j$, and $\sigma = \sigma_j$ and apply Lemma 2 to get that $L_j \simeq L_{j-1}$ (This works even when $T_j = \emptyset$, in which case $L_j = L_{j-1}$).

To complete the proof of the finite case it suffices to show $L_{2^n-1} = \mathrm{VR}(X \cup_A Y; r)$. This is because any simplex $\tau \in \mathrm{VR}(X \cup_A Y; r) \setminus L_0$ is necessarily of the form $\tau = S_X \cup S_Y \cup \sigma$, with $\emptyset \ne S_X \subseteq X \setminus A$, with $\emptyset \ne S_Y \subseteq Y \setminus A$, with $\sigma \subseteq A$, and with $\mathrm{diam}(S_X \cup S_Y \cup \sigma) \le r$. By assumption, there exists a unique maximal non-empty set $\sigma_j \subseteq A$ such that $\mathrm{diam}(S_X \cup S_Y \cup \sigma_j) \le r$. Since $\sigma_j$ is unique, we have that $\sigma \subseteq \sigma_j$. Therefore $S_X \cup S_Y \in T_j$, and $\tau$ will be added to $L_j$ since $S_X \cup S_Y \subseteq \tau \subseteq S_X \cup S_Y \cup \sigma_j$. Hence $\tau \in L_{2^n-1}$ and so $L_{2^n-1} = \mathrm{VR}(X \cup_A Y; r)$.

Now let $X$ and $Y$ be arbitrary metric spaces. Note that for any finite subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $A_0 = X_0 \cap Y_0 \ne \emptyset$, we have $\mathrm{VR}(X_0; r) \cup_{\mathrm{VR}(A_0;r)} \mathrm{VR}(Y_0; r) \simeq \mathrm{VR}(X_0 \cup_{A_0} Y_0; r)$ by the finite case. Hence we can apply Lemma 3 with $L = \mathrm{VR}(X; r) \cup_{\mathrm{VR}(A;r)} \mathrm{VR}(Y; r)$ and $K = \mathrm{VR}(X \cup_A Y; r)$ to complete the proof. ◀

▶ **Corollary 9.** *Let $X$ and $Y$ be metric spaces with $X \cap Y = A$, where $A$ is a closed subspace of $X$ and $Y$, and $X \cup_A Y$ is their metric gluing along $A$. Let $r > 0$, and suppose* $\mathrm{diam}(A) \le r$. *Then* $\mathrm{VR}(X \cup_A Y; r) \simeq \mathrm{VR}(X; r) \vee \mathrm{VR}(Y; r)$.

**Proof.** The fact that $\mathrm{diam}(A) \le r$ implies that if $\mathrm{diam}(S_X \cup S_Y) \le r$ for some $S_X \subseteq X \setminus A$ and $S_Y \subseteq Y \setminus A$, there is a unique maximal nonempty subset $\sigma \subseteq A$ such that $\mathrm{diam}(S_X \cup S_Y \cup \sigma) \le r$. Indeed, the set of all such $\sigma \subseteq A$ satisfying $\mathrm{diam}(S_X \cup S_Y \cup \sigma) \le r$ is closed under unions since $\mathrm{diam}(A) \le r$, and hence there will be a unique maximal $\sigma$. The definition of the metric on $X \cup_A Y$ implies that $\sigma \ne \emptyset$. The claim now follows from Theorem 8 and from the fact that $\mathrm{VR}(A; r)$ is contractible. ◀

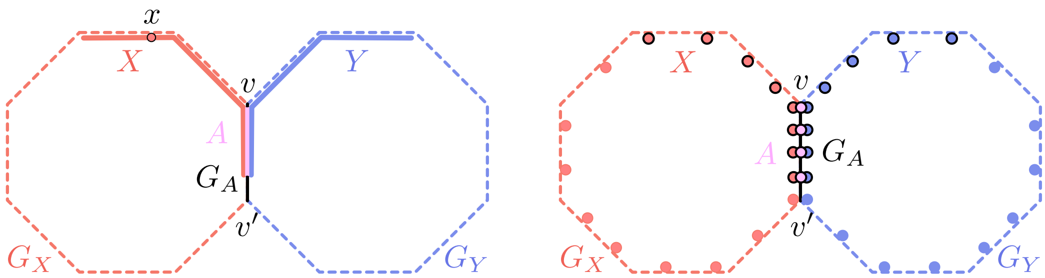


**Figure 2** Illustration of Theorem 10 on metric graph gluings and both infinite (left) and finite (right) subsets thereof. The metric graphs $G_X$ and $G_Y$ are shown with thin, dotted red and blue lines respectively; $X$ and $Y$ are shown in the infinite case with thick, solid red and blue lines respectively; $G_A$ corresponds to the black solid line while $A$ corresponds to the pink solid line. The finite subset case uses the same color scheme.

Note that if $A$ is a single point (i.e. $|A| = 1$), then $X \cup_A Y$ is the same as $X \vee Y$. Therefore, Proposition 4 is a special case of Corollary 9.

The setup of the following theorem regarding metric graph gluings is illustrated in Figure 2. In a graph, the *degree* of a vertex without self-loops is the number of incident edges to that vertex. A *path graph* (or simply a path) is one in which the $n$ vertices can be ordered, and in which the $n-1$ edges connect pairs of successive vertices.

▶ **Theorem 10.** *Let $G = G_X \cup_{G_A} G_Y$ be a metric graph, where $G_A = G_X \cap G_Y$ is a closed metric subgraph of the metric graphs $G_X$ and $G_Y$. Suppose furthermore that $G_A$ is a path, and that each vertex of $G_A$ besides the two endpoints has degree 2 not only as a vertex in $G_A$, but also as a vertex in $G$. Suppose the length of $G_A$ is at most $\frac{\ell}{3}$, where any simple loop in $G$ that goes through $G_A$ has length at least $\ell$. Let $X \subseteq G_X$ and $Y \subseteq G_Y$ be arbitrary subsets such that $X \cap G_Y = Y \cap G_X = X \cap Y := A$. Then $\mathrm{VR}(X \cup_A Y; r) \simeq \mathrm{VR}(X; r) \cup_{\mathrm{VR}(A;r)} \mathrm{VR}(Y; r)$ for all $r > 0$. Hence if $\mathrm{VR}(A; r)$ is contractible, then $\mathrm{VR}(X \cup_A Y; r) \simeq \mathrm{VR}(X; r) \vee \mathrm{VR}(Y; r)$.*

**Proof.** Let the length of $G_A$ be $\alpha \leq \frac{\ell}{3}$. If $r \geq \alpha$, then the conclusion follows from Corollary 9.

For the case $r < \alpha \leq \frac{\ell}{3}$, let $v$ and $v'$ be the endpoints of the path $G_A$. We claim that no point $z \in (X \cup_A Y) \setminus A$ is within distance $r$ of both $v$ and $v'$. Indeed, if there were such a point $z \in (X \cup_A Y) \setminus A$ satisfying $d(z, v) \leq r$ and $d(z, v') \leq r$, then we could produce a homologically non-trivial loop through the gluing path $G_A$ that is shorter than $\ell$, giving a contradiction. It follows that if $\mathrm{diam}(S_X \cup S_Y) \leq r$ for some $S_X \subseteq X \setminus A$ and $S_Y \subseteq Y \setminus A$, then there is a unique maximal nonempty set $\sigma \subseteq A$ such that $\mathrm{diam}(S_X \cup S_Y \cup \sigma) \leq r$. To be more explicit, one can show that this maximal set is $\sigma = A \cap \bigcap_{z \in S_X \cup S_Y} B(z, r)$; it is nonempty since it contains either $v$ or $v'$. Hence, Theorem 8 implies that $\mathrm{VR}(X \cup_A Y; r) \simeq \mathrm{VR}(X; r) \cup_{\mathrm{VR}(A;r)} \mathrm{VR}(Y; r)$. ◀

▶ **Corollary 11.** *Let $G$, $G_X$, $G_Y$, $G_A$, $X$, $Y$, and $A$ satisfy the same hypotheses as in the statement of Theorem 10. Suppose furthermore that $\mathrm{VR}(A; r)$ is contractible for all $r > 0$. Then for any homological dimension $i \geq 0$ and field $k$, the persistence modules $\mathrm{PH}_i(\mathrm{VR}(X; r) \vee \mathrm{VR}(Y; r); k)$ and $\mathrm{PH}_i(\mathrm{VR}(X \cup_A Y; r))$ are isomorphic.*

**Remarks on set-wise gluings in the Čech case.** It seems natural to ask if our results in Section 3.3 extend to Čech complexes. In other words, is it necessarily the case that $\check{\mathrm{C}}\mathrm{ech}(X; r) \cup_{\check{\mathrm{C}}\mathrm{ech}(A;r)} \check{\mathrm{C}}\mathrm{ech}(Y; r) \simeq \check{\mathrm{C}}\mathrm{ech}(X \cup_A Y; r)$, where $X, Y$ and $A$ are as described in Theorem 10? Interestingly, while the desired result may hold true, the arguments of Section 3.3 do not all directly transfer to the Čech case. In particular, Theorem 8 can be extended to the Čech case by replacing the condition $\mathrm{diam}(S_X \cup S_Y \cup \sigma) \leq r$ with $\bigcap_{z \in S_X \cup S_Y \cup \sigma} B(z;r) \neq \emptyset$. However, the arguments for Corollary 9 do not transfer to the Čech case no matter how small the size of the gluing portion $A$ is, which subsequently makes it hard to adapt the strategy behind Theorem 10 to the Čech case. An example to illustrate this is given in the full version. This suggests that a different technique needs to be developed in order to show an analog of Theorem 10 for the Čech setting (if such an analog holds).

## 4 Applicability to certain families of graphs

The results in Section 3 provide a mechanism to compute the homotopy types and persistent homology of Vietoris–Rips complexes of metric spaces built from gluing together simpler ones. For the sake of brevity, if the results of Section 3 can be used to completely describe the homotopy types and persistence module of the Vietoris–Rips complex of metric space $X$, then we will simply say that space $X$ can be *characterized*. Figure 3 shows examples of two metric graphs that can be characterized (a and b) and two that cannot (c and d). In Section 4.1, we describe some families of metric spaces that can be characterized, and in Section 4.2, we discuss graphs (c) and (d).

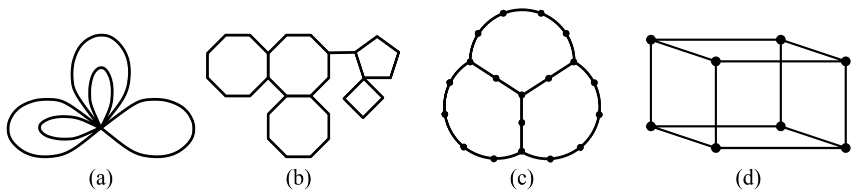


**Figure 3** Graphs (a) and (b) can be characterized while (c) and (d) cannot.

### 4.1 Families of graphs

In this section we consider finite metric spaces and metric graphs that can be understood using the results in this paper. Examples of finite metric spaces whose Vietoris–Rips complexes are well-understood include the vertex sets of dismantlable graphs (defined below) and vertex sets of single cycles [2]. Examples of metric graphs whose Vietoris–Rips complexes are well-understood include trees and single cycles [3].

Let $G$ be a graph with vertex set $V$ and with all edges of length one.[5] The vertex set $V$ is a metric space equipped with the shortest path metric. We say that a vertex $v \in V$ is *dominated* by $u \in V$ if $v$ is connected to $u$, and if each neighbor of $v$ is also a neighbor of $u$. We say that a graph is *dismantlable* if we can iteratively remove dominated vertices from $G$ in order to obtain the graph with a single vertex. Note that if $v$ is dominated by $u$, then $v$ is dominated by $u$ in the 1-skeleton of $\mathrm{VR}(V;r)$ for all $r \geq 1$. It follows from the theory of folds, elementary simplicial collapses, or LC reductions [6, 9, 27] that if $G$ is dismantlable, then $\mathrm{VR}(V;r)$ is contractible for all $r \geq 1$. Examples of dismantlable graphs include trees, chordal graphs, and unit disk graphs of sufficiently dense samplings of convex sets in the plane [25, Lemma 2.1]. We will also need the notion of a *$k$-cycle* graph, a simple cycle with $k$ vertices and $k$ edges. The following proposition specifies a family of finite metric spaces that can be characterized using the results in this paper.

[5] We make this assumption for simplicity's sake, even though it can be relaxed.

▶ **Proposition 12.** *Let $G$ be a finite graph, with each edge of length one, that can be obtained from a vertex by iteratively attaching (i) a dismantlable graph or (ii) a $k$-cycle graph along a vertex or along a single edge. Let $V$ be the vertex set of the graph $G$. Then we have $\mathrm{VR}(V;r) \simeq \bigvee_{i=1}^{n} \mathrm{VR}(V(C_{k_i});r)$ for $r \geq 1$, where $n$ is the number of times operation (ii) is performed, $k_i$ are the corresponding cycle lengths, and $V(C_{k_i})$ is the vertex set of a $k_i$-cycle.*

**Proof.** It suffices to show that an operation of type (i) does not change the homotopy type of the Vietoris–Rips complex of the vertex set, and that an operation of type (ii) has the effect up to homotopy of taking a wedge sum with $\mathrm{VR}(V(C_k);r)$. The former follows from applying Corollary 9 (or alternatively Theorem 10), as the Vietoris–Rips complex of the vertex set of a dismantlable graph is contractible for all $r \geq 1$, and the latter also follows from Corollary 9 (or alternatively Theorem 10). ◀

The iterative procedure outlined in Proposition 12 can be used to obtain some recognizable families of graphs. Examples include trees and wedge sums of cycles (Figure 3(a)). More complicated are *polygon trees* [18] in which cycles are iteratively attached along a single edge. Graph (b) in Figure 3 is an example that is built by using both (i) and (ii).

A similar procedure is possible for metric graphs, except that we must replace arbitrary dismantlable graphs with the specific case of trees.[6] Note that any tree, $T$, is obtained by starting with a single vertex and iteratively taking the wedge sum with a single edge. This implies that $\mathrm{VR}(T;r)$ is contractible; thus, the persistent homology filtration of any tree is trivial, a result that is also established in [14, 28].

▶ **Proposition 13.** *Let $G$ be a metric graph, with each edge of length one, that can be obtained from a vertex by iteratively attaching (i) an edge along a vertex or (ii) a $k$-cycle graph along a vertex or a single edge. Then we have $\mathrm{VR}(G;r) \simeq \bigvee_{i=1}^{n} \mathrm{VR}(C_{k_i};r)$ for $r \geq 1$, where $n$ is the number of times operation (ii) is performed, $k_i$ are the corresponding cycle lengths, and $C_{k_i}$ is a loop of length $k_i$.*

**Proof.** The proof is analogous to that of Proposition 12. ◀

Proposition 13 can be generalized to allow for arbitrary edge lengths, as long as the conditions of Theorem 10 hold (see the full version for further discussion and an example).

## 4.2 Obstructions to using our results

When gluing two metric graphs, $G_1$ and $G_2$, the most restrictive requirement is that the gluing path must be a simple path with all vertices except the endpoints having degree 2. This requirement is what disallows the configuration (c) in Figure 3. Notice that every pair of shortest cycles shares only a simple path of length 2. However, once one pair is glued, the third must be glued along a path of length 4 which traverses both of the two other cycles. This path includes a vertex of degree 3 in its interior, meaning that Theorem 10 is not applicable. Moreover, when $r < 4$, (where 4 is the diameter of the gluing set), then Corollary 9 is also not applicable. Nevertheless, we can computationally verify that for (c), the homology of the Vietoris–Rips complex is still the direct sum of the homology groups of the component cycles (where $\mathrm{VR}(V(C_9);r) \simeq S^1$ for $r = 1$ or 2, and where $\mathrm{VR}(V(C_9);3) \simeq S^2 \vee S^2$ [4]). In

[6] The case of $C_3$, a cyclic graph with three unit-length edges, is instructive. Since $C_3$ is dismantlable we have that $\mathrm{VR}(V(C_3);r)$ is contractible for any $r \geq 1$. But since the metric graph $C_3$ is isometric to a circle of circumference 3, it follows from [3] that $\mathrm{VR}(C_3;r)$ is not contractible for $0 < r < \frac{3}{2}$.

light of this example, in future work, we hope to extend the results in this paper to gluing metric graphs along admissible isometric trees (a generalization of isometric simple paths).

The final graph (d) in Figure 3, the cube graph, is another case for which Theorem 10 is not applicable. However, unlike example (c) above, we cannot compute the homology of the vertex set of the cube as the direct sum of the homology groups of smaller component pieces. Indeed, if $V$ is the vertex set of the cube with each edge of length one, then $\dim(H_3(\mathrm{VR}(V;2))) = 1$ since $\mathrm{VR}(V;2)$ is homotopy equivalent to the 3-sphere. However, this graph is the union of five cycles of length four, and the Vietoris–Rips complex of the vertex set of a cycle of length four never has any 3-dimensional homology.

## 5 Discussion

We have shown that the wedge sum of Vietoris–Rips (resp. Čech) complexes is homotopy equivalent to the corresponding complex for the metric wedge sum, and generalized this result in the case of Vietoris–Rips complexes for certain metric space gluings. Our ultimate goal is to understand to the greatest extent possible the topological structure of large classes of metric graphs via persistent homology. Building on previous work in [3] and [20], the results in this paper constitute another important step toward this goal by providing a characterization of the persistence profiles of metric graphs obtainable via certain types of metric gluing. Many interesting questions remain for future research.

**Gluing beyond a single path.** We are interested in studying metric graphs obtainable via metric gluings other than along single paths of degree 2, such as gluing along a tree or self-gluing. For the case of gluings along a tree, the gluing graph may have vertices of degree greater than 2. Examples include gluing four square graphs together into a larger square with a degree 4 node in the center. Moreover, the techniques of our paper do not allow one to analyze self-gluings such as forming an $n$-cycle $C_n$ from a path of length $n$.

**Generative models for metric graphs.** Our results can be considered as providing a generative model for metric graphs, where we specify a particular metric gluing rule for which we have a clear understanding of its effects on persistent homology. Expanding the list of metric gluing rules would in turn lead to a larger collection of generative models.

**Approximations of persistent homology profiles.** A particular metric graph that arises from data in practice may not directly correspond to an existing generative model. However, we may still be able to approximate its persistent homology profile via stability results (e.g. [15, 30]) by demonstrating close proximity between its metric and a known one.

### References

1 Mridul Aanjaneya, Frederic Chazal, Daniel Chen, Marc Glisse, Leonidas Guibas, and Dmitry Morozon. Metric graph reconstruction from noisy data. *International Journal of Computational Geometry and Applications*, 22(04):305–325, 2012.

2 Michał Adamaszek. Clique complexes and graph powers. *Israel Journal of Mathematics*, 196(1):295–319, 2013.

3 Michał Adamaszek and Henry Adams. The Vietoris–Rips complexes of a circle. *Pacific Journal of Mathematics*, 290:1–40, 2017.

4 Michał Adamaszek, Henry Adams, Florian Frick, Chris Peterson, and Corrine Previte-Johnson. Nerve complexes of circular arcs. *Discrete & Computational Geometry*, 56(2):251–273, 2016.

5 Michał Adamaszek, Henry Adams, Ellen Gasparovic, Marix Gommel, Emilie Purvine, Radmila Sazdanovic, Bei Wang, Yusu Wang, and Lori Ziegelmeier. Vietoris-Rips and Čech complexes of metric gluings. *arXiv:1712.06224*, 2018.

6 Eric Babson and Dmitry N. Kozlov. Complexes of graph homomorphisms. *Israel Journal of Mathematics*, 152(1):285–312, 2006.

7 Jonathan Ariel Barmak and Elias Gabriel Minian. Simple homotopy types and finite spaces. *Advances in Mathematics*, 218:87–104, 2008.

8 Bharat Biswal, F. Zerrin Yetkin, Victor M. Haughton, and James S. Hyde. Functional connectivity in the motor cortex of resting human brain using echo-planar MRI. *Magnetic Resonance in Medicine*, 34:537–541, 1995.

9 Anders Björner. Topological methods. *Handbook of Combinatorics*, 2:1819–1872, 1995.

10 Karol Borsuk. On the imbedding of systems of compacta in simplicial complexes. *Fundamenta Mathematicae*, 35(1):217–234, 1948.

11 Martin R. Bridson and André Haefliger. *Metric Spaces of Non-Positive Curvature*. Springer-Verlag, 1999.

12 Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33. American Mathematical Society Providence, RI, 2001.

13 Gunnar Carlsson. Topology and data. *Bulletin of the American Mathematical Society*, 46(2):255–308, 2009.

14 Joseph Minhow Chan, Gunnar Carlsson, and Raul Rabadan. Topology of viral evolution. *Proceedings of the National Academy of Sciences*, 110(46):18566–18571, 2013.

15 Frédéric Chazal, David Cohen-Steiner, Leonidas J. Guibas, Facundo Mémoli, and Steve Y. Oudot. Gromov-Hausdorff stable signatures for shapes using persistence. *Computer Graphics Forum*, 28(5):1393–1403, 2009.

16 Frédéric Chazal, Vin De Silva, Marc Glisse, and Steve Oudot. *The structure and stability of persistence modules.* Springer, 2016.

17 Frédéric Chazal, Vin de Silva, and Steve Oudot. Persistence stability for geometric complexes. *Geometriae Dedicata*, pages 1–22, 2013.

18 Fengming Dong, Khee-Meng Koh, and Kee L Teo. *Chromatic Polynomials and Chromaticity of Graphs.* World Scientific, 2005.

19 Herbert Edelsbrunner and John L Harer. *Computational Topology: An Introduction.* American Mathematical Society, 2010.

20 Ellen Gasparovic, Maria Gommel, Emilie Purvine, Radmila Sazdanovic, Bei Wang, Yusu Wang, and Lori Ziegelmeier. A complete characterization of the one-dimensional intrinsic Čech persistence diagrams for metric graphs. In *Research in Computational Topology*, 2018.

21 Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

22 Jean-Claude Hausmann. On the Vietoris–Rips complexes and a cohomology theory for metric spaces. *Annals of Mathematics Studies*, 138:175–188, 1995.

23 Dmitry N. Kozlov. *Combinatorial Algebraic Topology*, volume 21 of *Algorithms and Computation in Mathematics.* Springer-Verlag, 2008.

24 Peter Kuchment. Quantum graphs I. Some basic structures. *Waves in Random Media*, 14(1):S107–S128, 2004.

25 Janko Latschev. Vietoris–Rips complexes of metric spaces near a closed Riemannian manifold. *Archiv der Mathematik*, 77(6):522–528, 2001.

26 Michael Lesnick, Raul Rabadan, and Daniel Rosenbloom. Quantifying genetic innovation: Mathematical foundations for the topological study of reticulate evolution. In preparation, 2018.

**27** Jiří Matoušek. LC reductions yield isomorphic simplicial complexes. *Contributions to Discrete Mathematics*, 3(2), 2008.

**28** Steve Oudot and Elchana Solomon. Barcode embeddings, persistence distortion, and inverse problems for metric graphs. *arXiv: 1712.03630*, 2017.

**29** Thierry Sousbie, Christophe Pichon, and Hajime Kawahara. The persistent cosmic web and its filamentary structure – II. Illustrations. *Monthly Notices of the Royal Astronomical Society*, 414(1):384–403, 2011.

**30** Katharine Turner. Generalizations of the Rips filtration for quasi-metric spaces with persistent homology stability results. *arXiv:1608.00365*, 2016.

**31** Žiga Virk. 1-dimensional intrinsic persistence of geodesic spaces. *arXiv:1709.05164*, 2017.