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LAPLACE FORM OF NAVIER-STOKES EQUATIONS: A SAFE PATH OR A WRONG WAY?

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Abstract. The Navier-Stokes Equations written in Laplace form are often the departure point for the simulation of viscous newtonian flows and some studies of numerical stability. Researchers may not be fully aware that the "physical traction boundary conditions" are not the "natural boundary conditions" of the Laplace form of the Navier-Stokes Equations. This is not a problem per se, as long as one manages to rigurously incorporate the physical boundary conditions into the formulation. However, we have discovered that if some seemenly harmless assumptions are made, like using pseudo-tractions (i.e the natural boundary conditions of the Laplace form) or neglecting viscous terms on the free-surfaces, the resulting formulation violates a basic axiom of continuum mechanics: the principle of objectivity. In the present article we give an accurate account about these topics. We also show that unexpected differences may sometimes arise between Laplace discretizations and Divergence discretizations.

1 INTRODUCTION

Some months ago while working in the development of a monolithic PFEM method for fluid-structure interaction (Marti et al., 2005), we were looking at the simulations obtained for the case of a water dam with an inmersed cube made of a solid hipoelastic material. The cube had the same density than water and seemed to float and move convected by the water streams as it should. However things were not looking quite right. The hipoelastic cube moved but despite of the unsteady flow it never seemed to achieve a lot of rotation, it moved convected by the stream but its initially vertical walls always seemed to remain vertical. Something was wrong in the program, what could it be?. Initially we thought we had made a mistake in the handling of the tensor theory necessary for achieve large deformations and large rotations for the hipoelastic solid or that there had been a mistake at the programming level.

We finally realized that the error was in neither of those things, it was at a more basic level but quietly hidden. And although the error had become visible in the motion of hipoelastic solid it was also present in the fluid. The only difference being that it had become more obvious because of the rigid nature of the solid. Especifically the problem was in the standard formulation we were employing for solving viscous flows and that we had extended to work for the hipoelastic solid.

But what type of formulation were we using?. Nothing strange. On the contrary, we were using the most common approach for solving the viscous flows with free boundaries: a weak form of the Navier-Stokes Equations in *Laplace Form*. On the free surface we had also assumed the standard approach: no external tractions except an external pressure (the atmospheric pressure) from which we could further assume (neglecting viscous efffects) that the internal pressure was equal to the imposed external pressure.

After a careful analysis we discovered the problem. The standard approach we were using was violating one of the most basic principles of continumm mechanics: the principle of objectivity. This was the final cause of the weird motion of the hipoelastic solid in interaction with the fluid. In this paper we will present the mathematical and numerical issues involved.

2 NAVIER-STOKES EQUATIONS IN DIVERGENCE FORM

The *equations of Conservation of Mass and Conservation of Momentum in strong form* of any incompressible fluid material are given by

$$\nabla \cdot \boldsymbol{v} = 0 \tag{1}$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\nabla \boldsymbol{v}) \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{\sigma} \tag{2}$$

In the above equations $\boldsymbol{v}$ is the velocity vector, ρ the density, $\nabla \cdot \boldsymbol{\sigma}$ is the divergence of the Cauchy Stress Tensor $\boldsymbol{\sigma}$ and we have assumed there are not external body forces.

For the particular case of incompressible Newtonian fluids, the Cauchy Stress is given by,

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mu\nabla\boldsymbol{v} + \mu(\nabla\boldsymbol{v})^T \tag{3}$$

where p is the fluid's pressure, $\boldsymbol{I}$ is the Identity Tensor and μ the viscosity. Replacing this relationship into eq. (2) we get the Divergence Form of the Navier-Stokes Equations:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\nabla \boldsymbol{v}) \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{\sigma} = \nabla \cdot (-p\boldsymbol{I} + \mu\nabla\boldsymbol{v} + \mu(\nabla\boldsymbol{v})^T) \tag{4}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{5}$$

3 NAVIER-STOKES EQUATIONS IN LAPLACE FORM

Making use of the following tensor identities,

$$\nabla \cdot (p\boldsymbol{I}) = \nabla p \tag{6}$$

$$\nabla \cdot (\nabla \boldsymbol{V}) = \nabla^2 \boldsymbol{v} \tag{7}$$

$$\nabla \cdot (\nabla \boldsymbol{v})^T = \nabla(\nabla \cdot \boldsymbol{v}) \tag{8}$$

(and if we are dealing with problems where we can neglect the spatial variation of the viscosity), the $\nabla \cdot \boldsymbol{\sigma}$ term of equation (4) simplifies to:

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot (-p\boldsymbol{I} + \mu \nabla \boldsymbol{v}) = -\nabla p + \mu \nabla^2 \boldsymbol{v} \tag{9}$$

Replacing this into the RHS of equation (4), we get an alternative expression for the Navier-Stokes Equations: the *Incompressible Navier-Stokes Equations in Laplace Form*:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla \boldsymbol{v} \cdot \boldsymbol{v} = -\nabla p + \mu \nabla^2 \boldsymbol{v} \tag{10}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{11}$$

People call it the *Laplace Form* of the Navier-Stokes Equations because the viscous contribution is simply the Laplacian of the velocity components:

$$\nabla^2 \boldsymbol{v}$$

The following observations are useful:

- **Observation 1:** Note that the Cauchy Stress contributes to the momentum equation in the form of its divergence.
- **Observation 2**: Note that the $(\nabla \boldsymbol{v})^T$ term does not contribute to the momentum equations because $\nabla \cdot (\nabla \boldsymbol{v})^T = 0$ for incompressible fluids.
- **Observation 3:** Both strong forms of the Navier-Stokes, the Laplace form and the Divergence form should be equivalent (under the constant viscosity approximation).

4 BOUNDARY CONDITIONS OF NAVIER-STOKES EQUATIONS

In general the fluid moves around and between solid surfaces $\Gamma_{\boldsymbol{v}}$ and traction surfaces $\Gamma_{\boldsymbol{t}}$ where suitable boundary conditions have to be prescribed. The set of exact boundary conditions will be presented next.

4.1 Boundary Conditions at Solid Boundaries

At solid boundaries $\Gamma_{\boldsymbol{v}}$ which may be moving with arbitrary velocity $\bar{\boldsymbol{v}}$, the flow velocity is imposed:

$$\boldsymbol{v} = \bar{\boldsymbol{v}} \text{ on } \Gamma_{\boldsymbol{v}} \tag{12}$$

4.2 Boundary Conditions at Traction surfaces

There are many problems of practical importance where flows are not fully bounded and then have traction surfaces. By definition, on traction surfaces external forces are present but the motion is not imposed, so usually these surfaces move, and, part of the problem consists in determining their position and shape. Examples of traction surfaces are all the cases where there exists interaction between two fluids, this includes multi-phase flows.

At traction surfaces Γ_t where arbitrary external traction forces $\bar{\boldsymbol{t}}$ are acting, the condition of continuity of traction forces must hold. So denoting by $\boldsymbol{n}$ the outward pointing surface's unit normals, the following boundary condition must hold:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} \, \text{on} \, \Gamma_{\boldsymbol{t}} \tag{13}$$

4.3 Boundary conditions at Free-Surfaces

Free-surfaces flows are a special case of multi-phase flows. They are essentially two-phase flows, usually a liquid-phase and a gas-phase, where one usually neglegts the density and viscous effects of the lighter phase. For a deeper discussion on the subject refer to Aris (1962) . Examples of free surfaces flows are sea waves, dam breaks, flow over structures, flows around ships, mould filling, flows with bubbles, etc.

The exact free-surface boundary conditions of free-surface flows are obtained as a special case of the general interface condition (13), assuming that external tangential forces are negligible and that only external normal pressures are applied. In these cases we can write the imposed external forces simply as:

$$\bar{\boldsymbol{t}} = -\bar{p}\boldsymbol{n} \tag{14}$$

where $\bar{p}$ is a imposed known external pressure, such the atmospheric pressure. Replacing (14) into the traction boundary condition (13) we get:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = -\bar{p}\boldsymbol{n} \tag{15}$$

5 PRELIMINARY FORMULAS

Nowadays most people solve the Navier-Stokes Equations, not by discretization of the *strong forms* (eqs. (10)-(11) or eqs. (4)-(5)) presented in Section 2 and 3, but by discretization of a *weak form* of the differential equations. The most popular methods to get such weak form are the Finite Element Method (FEM) and the Finite Volume Method (FVM). In the present paper we will assume that we are interested in solving the Navier-Stokes Equations using a FEM formulation. However, the results of this paper are not restricted to FEM. Similar implications are obtained for FVM.

Since weak forms are obtained by volume integration followed by integration by parts, let's recall Green's Theorem for any second order tensor $\boldsymbol{T}$ and scalar function w:

$$\int_\Omega w\nabla \cdot \boldsymbol{T} d\Omega = -\int_\Omega \boldsymbol{T} \cdot \nabla w d\Omega + \int_\Gamma w\boldsymbol{T} \cdot \boldsymbol{n} d\Gamma \tag{16}$$

Applying this theorem to the Pressure Tensor $p\boldsymbol{I}$, to the $\nabla \boldsymbol{v}$ tensor and to the $(\nabla \boldsymbol{v})^T$ tensor we get that:

$$\int_\Omega w\nabla \cdot (p\boldsymbol{I}) d\Omega = -\int_\Omega (p\boldsymbol{I}) \cdot \nabla w d\Omega + \int_\Gamma w(p\boldsymbol{I}) \cdot \boldsymbol{n} d\Gamma \tag{17}$$

$$\int_\Omega w\nabla\cdot(\nabla \boldsymbol{v})d\Omega = -\int_\Omega (\nabla \boldsymbol{v})\cdot\nabla w d\Omega + \int_\Gamma w(\nabla \boldsymbol{v})\cdot \boldsymbol{n} d\Gamma \tag{18}$$

$$\int_\Omega w\nabla\cdot(\nabla \boldsymbol{v})^T d\Omega = -\int_\Omega (\nabla \boldsymbol{v})^T\cdot\nabla w d\Omega + \int_\Gamma w(\nabla \boldsymbol{v})^T\cdot \boldsymbol{n} d\Gamma \tag{19}$$

Combining (8) and (19) we get:

$$\int_\Omega w\nabla(\nabla\cdot \boldsymbol{v})d\Omega = -\int_\Omega (\nabla \boldsymbol{v})^T\cdot\nabla w d\Omega + \int_\Gamma w(\nabla \boldsymbol{v})^T\cdot \boldsymbol{n} d\Gamma \tag{20}$$

Combining (17) and (18) we get:

$$\int_\Omega w\nabla\cdot\left[-p\boldsymbol{I}+\mu\nabla \boldsymbol{v}\right]d\Omega = -\int_\Omega \left[-p\boldsymbol{I}+\mu\nabla \boldsymbol{v}\right]\cdot\nabla w d\Omega + \int_\Gamma w\left[-p\boldsymbol{I}+\mu\nabla \boldsymbol{v}\right]\cdot \boldsymbol{n} d\Gamma \tag{21}$$

Expanding terms:

$$\begin{aligned} -\textstyle\int_\Omega w\nabla p d\Omega + \int_\Omega \mu\nabla^2 \boldsymbol{v} d\Omega &= \textstyle\int_\Omega p\nabla w d\Omega - \int_\Omega \mu\nabla \boldsymbol{v}\cdot\nabla w d\Omega + \\ &\quad + \textstyle\int_\Gamma w\left[-p\boldsymbol{I}+\mu\nabla \boldsymbol{v}\right]\cdot \boldsymbol{n} d\Gamma \end{aligned} \tag{22}$$

Similarly, applying Green's theorem to the Cauchy Stress Tensor we get that

$$\int_\Omega w\nabla\cdot[\boldsymbol{\sigma}]\,d\Omega = -\int_\Omega [\boldsymbol{\sigma}]\cdot\nabla w d\Omega + \int_\Gamma w\,[\boldsymbol{\sigma}]\cdot \boldsymbol{n} d\Gamma \tag{23}$$

6 DIVERGENCE WEAK FORM OF NAVIER-STOKES EQUATIONS

In this Section we will derive the weak form of the Navier-Stokes Equations when they are written in Divergence Form (i.e. eqs. (4)-(5)). For the sake of simplicity we will call it the Divergence weak form of the Navier Stokes Equations or just the Divergence weak form.

6.1 Divergence Weak Form

To find the weak form of the equations (4)-(5) we proceed as it is usual in FEM (Zienkiewicz and Taylor, 1991). We multiply both sides of eq. (4) by a weighting function w and integrate over the whole fluid domain Ω to get:

$$\int_\Omega w\rho\frac{\partial \boldsymbol{v}}{\partial t}d\Omega + \int_\Omega w\rho(\nabla \boldsymbol{v})\cdot \boldsymbol{v} d\Omega = \int_\Omega w\nabla\cdot\boldsymbol{\sigma} d\Omega \tag{24}$$

Using eq. (23) we get:

$$\begin{aligned} \textstyle\int_\Omega w\rho\frac{\partial \boldsymbol{v}}{\partial t}d\Omega + \int_\Omega w\rho(\nabla \boldsymbol{v})\cdot \boldsymbol{v} d\Omega &= -\textstyle\int_\Omega \left[-p\boldsymbol{I}+\mu\nabla \boldsymbol{v}+(\nabla \boldsymbol{v})^T\right]\cdot\nabla w d\Omega + \\ &\quad + \textstyle\int_{\Gamma_t} w\boldsymbol{\sigma}\cdot \boldsymbol{n} d\Gamma \end{aligned} \tag{25}$$

where we have used also that:

$$w = 0 \,\text{on}\, \Gamma_{\boldsymbol{v}} \tag{26}$$

- **Observation 4**: Note that the integrand of the boundary integral in the momentum eq. (25) is exactly the physical traction condition (13). This means that the traction condition is *the natural boundary condition* of the Navier-Stokes Equations written in Divergence Form.

Since we have got the natural boundary condition on the boundary integral, it is straightforward to impose the boundary conditions (13). So for the general traction problem we finally get:

$$\begin{aligned} \int_\Omega w\rho \frac{\partial \boldsymbol{v}}{\partial t} d\Omega + \int_\Omega w\rho (\nabla \boldsymbol{v}) \cdot \boldsymbol{v} d\Omega &= \int_\Omega p \nabla w d\Omega - \int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla w d\Omega + \\ &\quad - \int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla w d\Omega + \int_{\Gamma_t} w \bar{\boldsymbol{t}} d\Gamma \end{aligned} \tag{27}$$

Note that there are not unknowns in the boundary integral. For the particular case of free-surfaces we get:

$$\begin{aligned} \int_\Omega w\rho \frac{\partial \boldsymbol{v}}{\partial t} d\Omega + \int_\Omega w\rho (\nabla \boldsymbol{v}) \cdot \boldsymbol{v} d\Omega &= \int_\Omega p \nabla w d\Omega - \int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla w d\Omega + \\ &\quad - \int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla w d\Omega - \int_{\Gamma_t} w \bar{p} \boldsymbol{n} d\Gamma \end{aligned} \tag{28}$$

We can perform a similar approach for the continuity equation with a weighting function q to get:

$$\int_\Omega q \nabla \cdot \boldsymbol{v} d\Omega = 0 \tag{29}$$

Equations (27) or (28) supplemented with eq. (29) are the Divergence weak form of the the Navier-Stokes Equations.

6.2 Divergence Discretization (FEM)

With the Divergence weak form expressions (28)-(29) for free-surfaces in our hands we can obtain the Divergence discretization using standard FEM procedures (Zienkiewicz and Taylor, 1991). If we identify the following discrete operators in eq. (28):

$$\begin{array}{ccccc} \underbrace{\int_\Omega w\rho \frac{\partial \boldsymbol{v}}{\partial t} d\Omega} & + & \underbrace{\int_\Omega w\rho (\nabla \boldsymbol{v}) \cdot \boldsymbol{v} d\Omega} & = \\ \mathbb{M}\frac{\Delta \boldsymbol{v}}{\Delta t} & + & \mathbb{C}(\boldsymbol{v}) & = \end{array} \tag{30}$$

$$\begin{array}{cccccccc} = \underbrace{\int_\Omega p \nabla w d\Omega} & - & \underbrace{\int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla w d\Omega} & - & \underbrace{\int_\Omega \mu (\nabla \boldsymbol{v})^T \cdot \nabla w d\Omega} & - & \underbrace{\int_{\Gamma_t} w \bar{p} \boldsymbol{n} d\Gamma} \\ = \quad -\mathbb{G}p & + & \mu \mathbb{L} \boldsymbol{v} & + & \mu \mathbb{T} \boldsymbol{v} & + & \mathbb{F} \end{array} \tag{31}$$

and in eq. (29)

$$\begin{array}{ccc} \underbrace{\int_\Omega w \nabla \cdot \boldsymbol{v} d\Omega} & = & 0 \\ \mathbb{D}\boldsymbol{v} & = & 0 \end{array} \tag{32}$$

We can use different forms of the discrete operators to get different numerical schemes.

For example, a fully-implicit (see Ronquist (1996)) scheme at time $(n+1)$ will look like :

$$\begin{bmatrix} \mathbb{M}\boldsymbol{v}^{n+1} + \Delta t \mathbb{C}(\boldsymbol{v}^{n+1}) + \Delta t \mathbb{G} p^{n+1} - \Delta t \mu \left(\mathbb{L} + \mathbb{T} \right) \boldsymbol{v}^{n+1} \\ \mathbb{D}\boldsymbol{v}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbb{M}\boldsymbol{v}^n + \Delta t \mathbb{F} \\ 0 \end{bmatrix} \tag{33}$$

Similarly, if we use (see Perot (1992)) a second-order Crank-Nicholson for the diffusive terms and the convective terms are updated using a second-order Adams-Bashforth Method our discrete system of equations will look something like:

$$\begin{bmatrix} \mathbb{M} - \frac{\Delta t \mu}{2}\left(\mathbb{L}+\mathbb{T}\right) & \Delta t \mathbb{G} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} -\Delta t(\frac{3}{2}\mathbb{C}(\boldsymbol{v}^n) - \frac{1}{2}\mathbb{C}(\boldsymbol{v}^{n-1}) - \frac{\Delta t \mu}{2}\left(\mathbb{L}+\mathbb{T}\right)\boldsymbol{v}^n + \mathbb{M}\boldsymbol{v}^n + \Delta t \mathbb{F} \\ 0 \end{bmatrix} \tag{34}$$

It is worth to make the following observations:

– **Observation 5:** The operator

$$\mathbb{K} = \mathbb{L} + \mathbb{T} \tag{35}$$

is the stiffness matrix of linear elasticity.

– **Observation 6:** The operator

$$\mathbb{T}\boldsymbol{v}$$

produces a full coupling of velocity components

7 LAPLACE WEAK FORMS OF NAVIER-STOKES EQUATIONS

In this Section we will derive the weak form of the Navier-Stokes Equations when written in Laplace Form (i.e. eqs. (10)-(11)). We proceed in a similar way as done for the Divergence form. Since both forms are supposed to be equivalent we expect to get the same results.

7.1 Laplace Weak Form

Multiplying both sides of eq. (10) by the weighting function $w$and integrating over the whole fluid domain Ω we get:

$$\int_\Omega w\rho\frac{\partial \boldsymbol{v}}{\partial t}d\Omega + \int_\Omega w\rho(\nabla \boldsymbol{v})\cdot \boldsymbol{v}d\Omega = -\int_\Omega w\left[\nabla p + \mu\nabla^2\boldsymbol{v}\right]d\Omega \tag{36}$$

Making use of the identity (22) and using equation (26) we get that:

$$\begin{aligned} \int_\Omega w\rho\frac{\partial \boldsymbol{v}}{\partial t}d\Omega + \int_\Omega w\rho(\nabla \boldsymbol{v})\cdot \boldsymbol{v}d\Omega &= \int_\Omega p\nabla w d\Omega - \int_\Omega \mu\nabla\boldsymbol{v}\cdot\nabla w d\Omega + \\ &+ \int_{\Gamma_t} w\left[-p\boldsymbol{I} + \mu\nabla\boldsymbol{v}\right]\cdot\boldsymbol{n}d\Gamma \end{aligned} \tag{37}$$

– **Observation 7**: As opposed to what was done for the Divergence, this time we should be careful in not to replace the integrand in the boundary integral by $\boldsymbol{\sigma}\cdot\boldsymbol{n}$

– **Observation 8**: As opposed to what we might have expected we don't get back in the boundary integral the exact traction condition. This means that the physical boundary condition is not *the natural boundary condition* of the Navier-Stokes Equations written in Laplace Form.

However, we can still make some use of the traction boundary conditions. By definition of the Cauchy Stress (3) we know that:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} - \mu(\nabla \boldsymbol{v})^T \cdot \boldsymbol{n} = [-p\boldsymbol{I} + \mu \nabla \boldsymbol{v}] \cdot \boldsymbol{n} \tag{38}$$

Then at the traction boundary we have:

$$\bar{\boldsymbol{t}} - \mu(\nabla \boldsymbol{v})^T \cdot \boldsymbol{n} = [-p\boldsymbol{I} + \mu \nabla \boldsymbol{v}] \cdot \boldsymbol{n} \, \text{on} \, \Gamma_{\boldsymbol{t}} \tag{39}$$

Replacing eq. (39) in the boundary integral of eq. (37), we see that *the Laplace weak form of the Navies-Stokes Equations* is:

$$\begin{aligned} \int_\Omega w\rho \frac{\partial \boldsymbol{v}}{\partial t} d\Omega + \int_\Omega w\rho(\nabla \boldsymbol{v}) \cdot \boldsymbol{v} d\Omega \quad = & \quad \int_\Omega p \nabla w d\Omega - \int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla w d\Omega + \\ & + \int_{\Gamma_t} w \bar{\boldsymbol{t}} d\Gamma - \int_{\Gamma_t} w \left[\mu(\nabla \boldsymbol{v})^T\right] \cdot \boldsymbol{n} d\Gamma \end{aligned} \tag{40}$$

supplemented with the weak form of the incompressibility condition (29).

In particular for free-surfaces we have:

$$\begin{aligned} \int_\Omega w\rho \frac{\partial \boldsymbol{v}}{\partial t} d\Omega + \int_\Omega w\rho(\nabla \boldsymbol{v}) \cdot \boldsymbol{v} d\Omega \quad = & \quad \int_\Omega p \nabla w d\Omega - \int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla w d\Omega + \\ & - \int_{\Gamma_t} w \bar{p} \boldsymbol{n} d\Gamma - \int_{\Gamma_t} w \left[\mu(\nabla \boldsymbol{v})^T\right] \cdot \boldsymbol{n} d\Gamma \end{aligned} \tag{41}$$

7.2 Laplace Discretization

If we define the discrete operator $\mathbb{V}$ as

$$\underbrace{-\int_{\Gamma_t} w \left[\mu(\nabla \boldsymbol{v})^T\right] \cdot \boldsymbol{n} d\Gamma}_{\mathbb{V}\boldsymbol{v}} \tag{42}$$

we can get the Laplace discretizacions of the Navier-Stokes Equations that are the counterpart the Divergence discretization eqs. (33) and (34).

The exact fully-implicit scheme at time $(n+1)$ replacing (33) is:

$$\begin{bmatrix} \mathbb{M}\boldsymbol{v}^{n+1} \; + \; \Delta t \mathbb{C}(\boldsymbol{v}^{n+1}) \; + \; \Delta t \mathbb{G} p^{n+1} \; - \; \Delta t \mu \left(\mathbb{L} + \mathbb{V}\right) \boldsymbol{v}^{n+1} \\ \mathbb{D}\boldsymbol{v}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbb{M}\boldsymbol{v}^n + \Delta t \mathbb{F} \\ 0 \end{bmatrix} \tag{43}$$

Similarly, the semi-implicit second-order case replacing (34) is:

$$\begin{bmatrix} \mathbb{M} - \frac{\Delta t \mu}{2} \left(\mathbb{L} + \mathbb{V}\right) & \Delta t \mathbb{G} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} -\Delta t(\frac{3}{2}\mathbb{C}(\boldsymbol{v}^n) - \frac{1}{2}\mathbb{C}(\boldsymbol{v}^{n-1}) - \frac{\Delta t \mu}{2} \left(\mathbb{L} + \mathbb{V}\right) \boldsymbol{v}^n + \mathbb{M}\boldsymbol{v}^n + \Delta t \mathbb{F} \\ 0 \end{bmatrix} \tag{44}$$

– **Observation 9:** Note that Laplace Discretizations and the Divergence Discretizations are similar. They would be identical if the discrete operators $\mathbb{T}\boldsymbol{v}$ and $\mathbb{V}\boldsymbol{v}$ would be equal. This is the case in the continuum, and follows from the identity (20) and the incompressibility condition:

$$-\int_\Omega (\nabla \boldsymbol{v})^T \cdot \nabla w d\Omega = -\int_\Gamma w(\nabla \boldsymbol{v})^T \cdot \boldsymbol{n} d\Gamma \tag{45}$$

Then,

$$\mathbb{T}\boldsymbol{v} = \mathbb{V}\boldsymbol{v} \,\text{in the continuum} \tag{46}$$

However, in a real discretization one should be careful to state a *strict* equality because $\mathbb{T}$ involves volume integration while $\mathbb{V}$ involves contributions of boundary elements, and because the incompressibility condition is satisfied weakly. So in general:

$$\mathbb{T}\boldsymbol{v} \neq \mathbb{V}\boldsymbol{v} \,\text{in the discrete} \tag{47}$$

8 LAPLACE-NATURAL WEAK FORMS: NATURAL BOUNDARY CONDITIONS

In our previous derivations we didn't make any approximation (except of course the assumption of negligible spatial variation of the viscosity). Usually the use of approximations are convenient, for example, so as to increase performance without reducing accuracy. Could it be possible to use classical assumptions for the numerical simulations of viscous flows with free surfaces?. As described below this seems to be possible. First, let us recall the exact traction boundary conditions:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} \,\text{on}\, \Gamma_{\boldsymbol{t}} \tag{48}$$

for newtonian fluids:

$$-p\boldsymbol{n} + \mu(\nabla \boldsymbol{v}) \cdot \boldsymbol{n} + \mu(\nabla \boldsymbol{v})^T \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} \,\text{on}\, \Gamma_{\boldsymbol{t}} \tag{49}$$

8.1 Natural boundary conditions: Pseudo-Tractions

In their book Donea and Huerta (2003) mention an alternative approach for boundary conditions. The idea is to replace the correct traction boundary condition (49) with the natural boundary condition of the Laplace form:

$$-p\boldsymbol{n} + \mu(\nabla \boldsymbol{v}) \cdot \boldsymbol{n} = -p\boldsymbol{n} + \mu(\boldsymbol{n} \cdot \nabla)v = \bar{\boldsymbol{t}} \,\text{on}\, \Gamma_{\boldsymbol{t}} \tag{50}$$

Donea and Huerta (2003) call it *pseudo-traction* boundary condition. Comparing eq. (50) with the exact boundary condition (49), we see that the pseudo-traction assumption consists in neglecting the term:

$$\mu(\nabla \boldsymbol{v})^T \cdot \boldsymbol{n} \approx \boldsymbol{0} \,\text{on}\, \Gamma_{\boldsymbol{t}} \tag{51}$$

on the traction boundary or free-surface.

From equation (51) and definition (42), it follows that the pseudo-traction condition implies that

$$\int_{\Gamma_{\boldsymbol{t}}} w \left[\mu(\nabla \boldsymbol{v})^T\right] \cdot \boldsymbol{n} d\Gamma \approx \boldsymbol{0} \tag{52}$$

$$\mathbb{V}\boldsymbol{v} \approx \boldsymbol{0} \tag{53}$$

However, note that the pseudo-traction condition (50) does not necessarily imply that the internal pressure should match the imposed external pressure $\bar{p}$ on the free surface. Effectively for the case of free surfaces (50) becomes:

$$-p\boldsymbol{n} + \mu(\nabla \boldsymbol{v}) \cdot \boldsymbol{n} = -\bar{p}\boldsymbol{n} \,\mathrm{on}\, \Gamma_{\boldsymbol{t}} \tag{54}$$

and then

$$p \neq \bar{p} \,\mathrm{on}\, \Gamma_{\boldsymbol{t}} \tag{55}$$

unless also:

$$\mu(\nabla \boldsymbol{v}) \cdot \boldsymbol{n} \approx \boldsymbol{0} \,\mathrm{on}\, \Gamma_{\boldsymbol{t}} \tag{56}$$

8.2 Inviscid Approximations

It is a common assumption (see Grooss and Hesthaven (2006), pp. 3429 and Peterson (1999) Chapter 1), that for many practical applications (such as a the ones involving water with free surfaces) both viscous terms (51)-(56) become negligible at the free-surfaces:

$$\mu(\nabla \boldsymbol{v}) \cdot \boldsymbol{n} \to 0 \,\mathrm{and}\, \mu(\nabla \boldsymbol{v})^T \cdot \boldsymbol{n} \to 0 \,\mathrm{on}\, \Gamma_{\boldsymbol{t}} \tag{57}$$

This seems reasonable because one expects the velocity gradients in the fluid to be small on the free-surface. Under this *inviscid* assumption, the internal pressure p matches the external pressure $\bar{p}$ (say the atmospheric pressure):

$$p = \bar{p} \,\mathrm{on}\, \Gamma_{\boldsymbol{t}} \tag{58}$$

The matching of internal and external pressures is a classical asumption, see for example (Shao and Lo, 2003; Nallapati and Perot, 2000; Burg et al., 2002; van et al., 2001; Ashgriz et al., 2003; Kyildiz and Unal, 2005; Lohner et al., 2006) , etc.

Note that the invisicid assumption is equivalent to assume the pseudo-traction condition plus the condition of matching external pressure. Under the inviscid assumption equations (52)-(53) apply again.

8.3 Laplace-natural Discretization

If we assume that we can replace the physical boundary conditions with the natural boundary conditions (54) (i.e pseudo-tractions), the Laplace form (41) becomes:

$$\begin{aligned} \int_\Omega w\rho \tfrac{\partial \boldsymbol{v}}{\partial t} d\Omega + \int_\Omega w\rho(\nabla \boldsymbol{v}) \cdot \boldsymbol{v} d\Omega &= \int_\Omega p \nabla w d\Omega - \int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla w d\Omega + \\ &\quad - \int_{\Gamma_{\boldsymbol{t}}} w\bar{p}\boldsymbol{n} d\Gamma \end{aligned} \tag{59}$$

We see then that the viscous contribution of the traction boundary dissapears. Similarly the semi-implicit Laplace Discretization (44) reduces to:

$$\begin{bmatrix} \mathbb{M} - \frac{\Delta t \mu}{2}\mathbb{L} & \Delta t \mathbb{G} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} -\Delta t(\frac{3}{2}\mathbb{C}(\boldsymbol{v}^n) - \frac{1}{2}\mathbb{C}(\boldsymbol{v}^{n-1}) - \frac{\Delta t \mu}{2}\mathbb{L}\boldsymbol{v}^n + \mathbb{M}\boldsymbol{v}^n + \Delta t \mathbb{F} \\ 0 \end{bmatrix} \tag{60}$$

Since we have used the natural boundary conditions of the Laplace Form, we will call this discretization of the weak form a *Laplace-natural* formulation.

It is worth to make the following observations:

– **Observation 10** Note that Laplacian Discrete Operator $\mathbb{L}$ has been defined to be

$$\mathbb{L}\boldsymbol{v} = -\int_{\Omega} \nabla \boldsymbol{v} \cdot \nabla w d\Omega$$

– **Observation 11** The viscous contributions appear only in the form of the Laplacian Discrete Operator $\mathbb{L}$.

– **Observation 12:** The Laplacian Discrete Operator $\mathbb{L}\boldsymbol{v}$ is symmetric and block-diagonal by velocity components. This means that the Laplacian Operator is decoupled in velocity components.

– **Observation 13:** The matrix,

$$\mathbb{A} = \mathbb{M} - \frac{\Delta t \mu}{2} \mathbb{L}$$

Appearing in projection/fractional-step methods is block-diagonal by velocity components. This means that $\mathbb{A}$ could be decoupled in velocity components.

9 THE VIOLATION OF OBJECTIVITY IN LAPLACE-NATURAL FORMULATIONS

The boundary term (42) is the only difference between Laplace-natural formulations and the Laplace weak form derived in Section 7. In the previous section we showed that if we use the natural boundary conditions of the Laplace form such term is not longer part of the formulation. Note that by making use of such approximation one expects to make improvements in the performance of the resulting algorithms (see observation 13).

The boundary term (42) can be written in component form as:

$$\mathbb{V}v = -\mu \int_{\Gamma_t} w \left(\nabla \boldsymbol{v}\right)^T \cdot \boldsymbol{n} d\Gamma = -\mu \int_{\Gamma_t} w \left(\frac{\partial v_i}{\partial x_j} n_i \boldsymbol{e}_j \right) d\Gamma \tag{61}$$

From this we see that the jth-component involves components v_i with $j \neq i$ then this term fully couples all the components of the velocity. The above coupling is a coupling due to viscosity, i.e a *viscous coupling*. The corresponding discrete operator $\mathbb{V}$ produces then a coupling in velocity components. It is not a block-diagonal operator

In this section we will explain why it is a bad idea to simulate viscous flows with Laplace-natural formulations like (60)).

We will show that the lack of the boundary term (42) causes the violation of *the Principle of Objectivity or Invariance under a change of Observer* which is a main axiom of continuum mechanics (see Ogden (1984); Gurtin (1981)).

To demonstrate this, we will prove that the absense of the boundary term (42) is equivalent to write the Cauchy Stress σ (eq. (3)) without the contribution due to $\mu \left(\nabla \boldsymbol{v}\right)^T$:

$$\widetilde{\sigma} = -p\boldsymbol{I} + \widetilde{\sigma}_{\text{viscous}} = -p\boldsymbol{I} + \mu \left(\nabla \boldsymbol{v}\right) \tag{62}$$

and then prove that (62) is an *incorrect or incomplete* constitutive law: it violates the principle of objectivity.

The demostration that the absence of (42) is equivalent to use the constitutive law (62) is as follows. First note that a material whose constitutive law is given by (62) still satisfies the strong form of the Navier-Stokes Equations:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \nabla \boldsymbol{v} \cdot \boldsymbol{v} = \nabla \cdot \tilde{\boldsymbol{\sigma}} = -\nabla p + \mu \nabla^2 \boldsymbol{v} \tag{63}$$

The only change being a slightly different traction boundary condition:

$$\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} \, \text{on} \, \Gamma_{\boldsymbol{t}} \tag{64}$$

$$\Downarrow$$

$$\left[-p\boldsymbol{I} + \mu \left(\nabla \boldsymbol{v}\right)\right] \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} \, \text{on} \, \Gamma_{\boldsymbol{t}} \tag{65}$$

If we insert this boundary condition into the identity (21) we get back the Laplace-natural weak form given in eq. (59):

$$\begin{aligned} \int_\Omega w\rho \tfrac{\partial \boldsymbol{v}}{\partial t} d\Omega + \int_\Omega w\rho(\nabla \boldsymbol{v}) \cdot \boldsymbol{v} d\Omega \;\; &= \;\; \int_\Omega p \nabla w d\Omega - \int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla w d\Omega + \\ &\quad + \int_{\Gamma_{\boldsymbol{t}}} w \bar{\boldsymbol{t}} d\Gamma \end{aligned} \tag{66}$$

This proves that the use of the Laplace-natural weak forms is equivalent to the use of the Cauchy Stress $\tilde{\boldsymbol{\sigma}}$.

Now, let us show that (62) violates *the Principle of Objectivity*. A necessary condition for a constitutive law to satisfy the Principle of Objectivity, is that a rigid motion should not generate viscous stresses and preserve incompressibility on the material. Expressed mathematically this implies that under a velocity field $\boldsymbol{v_R}$ corresponding to a rigid motion we should have:

$$\widetilde{\sigma}_{\text{viscous}} = \mu \nabla \boldsymbol{v_R} \equiv \boldsymbol{0} \tag{67}$$

and

$$\nabla \cdot \boldsymbol{v_R} = 0 \tag{68}$$

Mathematically the velocity field of a rigid motion is given by:

$$\boldsymbol{v_R} = \boldsymbol{\omega} \times \boldsymbol{x}$$

In component form:

$$\boldsymbol{v_R} = \varepsilon_{ijk} \omega_j x_k \boldsymbol{e}_i$$

Then:

$$\nabla \cdot \boldsymbol{v_R} = \frac{\partial}{\partial x_i} \left(\varepsilon_{ijk} \omega_j x_k\right) = \varepsilon_{ijk} \omega_j \frac{\partial}{\partial x_i} x_k = \varepsilon_{ijk} \omega_j \delta_{ki} = \varepsilon_{iji} \omega_j = 0 \tag{69}$$

which proves that the incompressiblity condition is preserved. On the other hand

$$\nabla \boldsymbol{v_R} = \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} \omega_j x_k\right) \boldsymbol{e}_i \boldsymbol{e}_l = \varepsilon_{ijk} \omega_j \frac{\partial}{\partial x_l} \left(x_k\right) \boldsymbol{e}_i \boldsymbol{e}_l = \varepsilon_{ijk} \omega_j \delta_{kl} \boldsymbol{e}_i \boldsymbol{e}_l$$

$$\nabla \boldsymbol{v_R} = \varepsilon_{ijl} \omega_j \boldsymbol{e}_i \boldsymbol{e}_l \neq \boldsymbol{0} \tag{70}$$

In cartesian coordinates we can write the componenst of $\nabla \boldsymbol{v_R}$ in matrix form as:

$$[\nabla \boldsymbol{v_R}] = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \tag{71}$$

Then it follows from (70) that

$$\tilde{\sigma}_{\text{viscous}} \neq \mathbf{0}$$

so the principle of objectivity is violated.

However if we consider the correct expression for the stress:

$$\sigma = -p\boldsymbol{I} + \sigma_{\text{viscous}} = -p\boldsymbol{I} + \mu\left(\nabla \boldsymbol{v}\right) + \mu\left(\nabla \boldsymbol{v}\right)^T \tag{72}$$

we get that:

$$\sigma_{\text{viscous}} = \mu\left(\nabla \boldsymbol{v_R}\right) + \mu\left(\nabla \boldsymbol{v_R}\right)^T \equiv \mathbf{0} \tag{73}$$

Because:

$$\left(\nabla \boldsymbol{v_R}\right)^T = \frac{\partial}{\partial x_l}\left(\varepsilon_{ijk}\omega_j x_k\right)\boldsymbol{e}_l\boldsymbol{e}_i = \varepsilon_{ijk}\omega_j\delta_{kl}\boldsymbol{e}_l\boldsymbol{e}_i = \varepsilon_{ijl}\omega_j\boldsymbol{e}_l\boldsymbol{e}_i$$

$$\left(\nabla \boldsymbol{v_R}\right)^T = \varepsilon_{lji}\omega_j\boldsymbol{e}_i\boldsymbol{e}_l = -\varepsilon_{ijl}\omega_j\boldsymbol{e}_i\boldsymbol{e}_l \tag{74}$$

and then

$$\nabla \boldsymbol{v_R} + \left(\nabla \boldsymbol{v_R}\right)^T = \varepsilon_{ijl}\omega_j\boldsymbol{e}_i\boldsymbol{e}_l - \varepsilon_{ijl}\omega_j\boldsymbol{e}_i\boldsymbol{e}_l \equiv \mathbf{0}$$

This proves then that one must use the entire tensor in order to preserve objectivity and then the term (42) can not be neglegted if the main axioms of continuum mechanics are wanted to be preserved.

9.1 An Error Estimation of Laplace-natural Formulations

Note that:

$$\left(\nabla \boldsymbol{v_R}\right)^T \cdot \boldsymbol{n} = -\varepsilon_{ijl}\omega_j n_l\boldsymbol{e}_i = -\boldsymbol{\omega} \times \boldsymbol{n} \tag{75}$$

Replacing this into the expression of the neglected surface integral, we see that:

$$\text{Neglected term} =$$

$$-\int_{\Gamma_\sigma} w\mu\left(\nabla \boldsymbol{v_R}\right)^T \cdot \boldsymbol{n}d\Gamma = \int_{\Gamma_\sigma} w\mu\boldsymbol{\omega} \times \boldsymbol{n}d\Gamma = \mu\boldsymbol{\omega} \times \int_{\Gamma_\sigma} w\boldsymbol{n}d\Gamma \tag{76}$$

The above expression can be used to provide a good estimation for the magnitude of the error made when using Laplace-natural Navier-Stokes Formulations.

– **Observation 14:** From (76) it is clear that flow simulations where the flow experiences rotations, curvature changes and twisting maybe affected the most. The approximation may be valid for simulations like flow in an straight channel.

– **Observation 15:** Note also from (76) that due to the cross-product with the surface normal (...$\times\boldsymbol{n}$), the neglected term generates a force tangent to the surface of prescribed tractions. This force has the direction $\boldsymbol{\omega} \times \boldsymbol{n}$, and in the case of a two-dimensional flow, because $\boldsymbol{v_R} \cdot \boldsymbol{\omega} \times \boldsymbol{n} \geq 0$, this neglected term generates a force in a direction that helps to keep the rigid motion.

– **Observation 16:** If we consider a two-dimensional flow and decompose the traction forces and the velocity vector in a tangential and a normal component,

$$\boldsymbol{v} = v_t \boldsymbol{e}_t + v_n \boldsymbol{e}_n \,\text{no sum}$$

The exact boundary condition (49) becomes

$$\bar{t}_n \boldsymbol{e}_n + \bar{t}_t \boldsymbol{e}_t = -p\boldsymbol{e}_n + \mu \frac{\partial v_t}{\partial x_n} \boldsymbol{e}_t + \mu \frac{\partial v_n}{\partial x_n} \boldsymbol{e}_n + \mu \frac{\partial v_n}{\partial x_n} \boldsymbol{e}_n + \mu \frac{\partial v_n}{\partial x_t} \boldsymbol{e}_t \,\text{no sum} \tag{77}$$

Separating in components:

$$\bar{t}_n = -p + 2\mu \frac{\partial v_n}{\partial x_n} \,\text{no sum} \tag{78}$$

$$\bar{t}_t = \mu \frac{\partial v_t}{\partial x_n} + \mu \frac{\partial v_n}{\partial x_t} \,\text{no sum} \tag{79}$$

In case, only normal loads are applied:

$$-\bar{p} = -p + 2\mu \frac{\partial v_n}{\partial x_n} \,\text{no sum} \tag{80}$$

$$0 = \mu \frac{\partial v_t}{\partial x_n} + \mu \frac{\partial v_n}{\partial x_t} \,\text{no sum} \tag{81}$$

From (80)-(81) we see that even in 2D pressures do not match.

10 FULLY-BOUNDED DOMAINS AND CAVITY FLOWS

In the previous sections we obtained both Laplace Discretizations and Divergence Discretizations for general cases of flow motion. There we centered the discussion on the handling of traction surfaces and the contribution of traction boundary integrals for flows with free-surfaces.

Now let us consider the case of flows in fully-bounded domains (i.e. no free-surfaces) such as the one of the cavity flow problem (Burggraf (1966), Donea and Huerta (2003) section 6.8).

In such cases the velocity is prescribed in the whole boundary surface Γ of the volume domain Ω so:

$$\boldsymbol{v} = \bar{\boldsymbol{v}} \,\text{on}\, \Gamma = \Gamma_{\boldsymbol{v}} \tag{82}$$

and there are not traction boundaries

$$\Gamma_{\boldsymbol{t}} = 0 \tag{83}$$

As a consequence the following operators become zero:

$$\mathbb{F} = -\int_{\Gamma_{\boldsymbol{t}}} w\bar{p}\boldsymbol{n} d\Gamma = \boldsymbol{0}$$

$$\mathbb{V}\boldsymbol{v} = -\int_{\Gamma_{\boldsymbol{t}}} w \left[\mu(\nabla \boldsymbol{v})^T\right] \cdot \boldsymbol{n} d\Gamma = \boldsymbol{0}$$

Then for the fully-bounded case the Laplace Discretization (i.e equation (60)) reduces to:

$$\begin{bmatrix} \mathbb{M} - \frac{\Delta t \mu}{2}(\mathbb{L}) & \Delta t \mathbb{G} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} -\Delta t(\frac{3}{2}\mathbb{C}(\boldsymbol{v}^n) - \frac{1}{2}\mathbb{C}(\boldsymbol{v}^{n-1}) - \frac{\Delta t \mu}{2}(\mathbb{L})\boldsymbol{v}^n + \mathbb{M}\boldsymbol{v}^n \\ 0 \end{bmatrix} \tag{84}$$

However because of eq. (47):

$$\mathbb{T}\boldsymbol{v} \neq \mathbf{0} \, \text{in the discrete} \tag{85}$$

so for the Divergence discretization (44) we can just say that:

$$\begin{bmatrix} \mathbb{M} - \frac{\Delta t \mu}{2}(\mathbb{L} + \mathbb{T}) & \Delta t \mathbb{G} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} -\Delta t(\frac{3}{2}\mathbb{C}(\boldsymbol{v}^n) - \frac{1}{2}\mathbb{C}(\boldsymbol{v}^{n-1}) - \frac{\Delta t \mu}{2}(\mathbb{L} + \mathbb{T})\boldsymbol{v}^n + \mathbb{M}\boldsymbol{v}^n \\ 0 \end{bmatrix} \tag{86}$$

– **Observation 17:** Note then that in the discrete case, the Laplace discretization and the Laplace-natural discretization match.

– **Observation 18:** However none of the Laplace Discretizations match with the Divergence discretization.

The common approach in publications dealing with projections methods is to start with the Laplace formulation in fully-bounded domains (Johnston and Liu (2002); Claeyssen et al. (199); Minev (2001); Codina et al. (2006); Liu et al. (2005); Guermond and Shen (2003); Perot (1992)). However, the Divergence formulation is a more accurate and general approach. Then such projections methods should be studied from the Divergence discretization perspective. At present we (the authors) don't know if using a Divergence formulation instead of the commonly used Laplace formulation would have a sensible effect on the performance characteristics of such methods, probably it will. It may also be the case that the coupling effect of operator $\mathbb{T}$ will complicate the analysis.

11 DISCUSSION AND CONCLUSIONS

Laplace-natural discretizations are recognized by the fact that the viscous contribution comes only from the Laplacian operator $\mathbb{L}$. Since one can arrive to this type of discretization by using classical assumptions, there is a chance that some people may be using them without noting that they have a extremely serious defect. They violate a main axiom of continuum mechanics: the Principle of Objectivity.

In their book Donea and Huerta (2003) present pseudo-tractions as a convenient form to impose boundary conditions. They make use of them in their description of unsteady Navier-Stokes equations formulations in Section 6.7. In this article we have shown that such proposed formulations violate objectivity.

As a consequence, pseudo-tractions (i.e. natural boundary conditions of the Laplace form) should be avoided. Inviscid assumptions on free surface boundaries should be avoided too.

Laplace-natural discretizations might be being used in combination with fractional-step methods (Chang et al., 2002; Grooss and Hesthaven, 2006). Fractional-step methods allow a decoupling of the pressure variable from the velocity components. The use of an equation like (60) allows to decouple the velocity components too. This is advantageous because instead of solving a system containing $3 * N_V$ unknown velocity components, one would solve 3 smaller systems of size N_V (one system per velocity component). However as explained in Section 9 this approach is incorrect by the violation of objectivity in such discretizations.

Violation of objectivity by Laplace-natural discretizations gives another reason to prefer the Divergence approach (i.e. weak forms derived from the divergence form of the Navier-Stokes equations) over the Laplace approach. Note that the use of Divergence discretizations forces the computation of a discrete operator $\mathbb{T}$ which automatically produces a linkage between velocity components.

One way to remove the linkage would be to evaluate explicitly the terms associated to $\mathbb{T}$. In such case, eq. (44) would change to:

$$\begin{bmatrix} \mathbb{M} - \frac{\Delta t \mu}{2}\mathbb{L} & \Delta t \mathbb{G} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} -\Delta t(\frac{3}{2}\mathbb{C}(\boldsymbol{v}^{n+1}) - \frac{1}{2}\mathbb{C}(\boldsymbol{v}^{n-1}) - \frac{\Delta t \mu}{2}(\mathbb{L} + 2\mathbb{T})\,\boldsymbol{v}^n + \mathbb{M}\boldsymbol{v}^n + \Delta t \mathbb{F} \\ 0 \end{bmatrix} \tag{87}$$

Numerical simulations and convergence and stability analysis is required to see if the linkage can be removed without loss of convergence, accuracy and performance. We doubt that this will be possible at least for low Reynolds numbers.

Divergence discretization is the modern approach chosen by many researchers for the simulation of incompressible viscous flows and free surfaces (see for example Guler et al. (1999); Rabier and Medale (2003); Bonito et al. (2006)). It is also general and compatible with the approach used in solid mechanics. So in case of doubt... choose the Divergence path.

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