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ON WEIGHTED INEQUALITIES FOR FRACTIONAL INTEGRALS OF RADIAL FUNCTIONS

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Abstract. We prove a weighted version of the Hardy–Littlewood–Sobolev inequality for radially symmetric functions, and show that the range of admissible power weights appearing in the classical inequality due to Stein and Weiss can be improved in this particular case.

1. Introduction

Consider the fractional integral operator

$$(T_\gamma v)(x) = \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\gamma}\, dy, \quad 0 < \gamma < n.$$

Weighted estimates for this operator (also called weighted Hardy–Littlewood–Sobolev inequalities) go back to G. H. Hardy and J. E. Littlewood in the 1-dimensional case [5], and were generalized to the space $\mathbb{R}^n, n \geq 1$ by E. M. Stein and G. Weiss in the following celebrated result.

Theorem 1.1 ([10, Theorem B*]). *Let* $n \geq 1$, $0 < \gamma < n, 1 < p < \infty, \alpha < \frac{n}{p'}, \beta < \frac{n}{q}, \alpha+\beta \geq 0$, *and* $\frac{1}{q} = \frac{1}{p} + \frac{\gamma+\alpha+\beta}{n} - 1$. *If* $p \leq q < \infty$, *then the inequality*

$$\left\| |x|^{-\beta} T_\gamma v \right\|_{L^q(\mathbb{R}^n)} \leq C \left\| |x|^\alpha v \right\|_{L^p(\mathbb{R}^n)}$$

holds for any $v \in L^p(\mathbb{R}^n, |x|^{p\alpha}\, dx)$, *where* C *is independent of* v.

Inequalities for the fractional integral with general weights were later studied by several people, see for example [8] and references therein. In particular, it can be deduced from this theory (e.g., [8, Theorem 1]) that if we restrict

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ourselves to power weights, the previous theorem cannot be improved in general.

However, if we reduce ourselves to radially symmetric functions, it is possible to obtain a wider range of exponents for which the fractional integral is continuous with power weights. This is of particular interest for some applications to partial differential equations (see, e.g., [4], [11]). More precisely, our main theorem is the following.

THEOREM 1.2. *Let* $n \geq 1$, $0 < \gamma < n, 1 < p < \infty, \alpha < \frac{n}{p'}, \beta < \frac{n}{q}, \alpha + \beta \geq (n-1)(\frac{1}{q} - \frac{1}{p})$, *and* $\frac{1}{q} = \frac{1}{p} + \frac{\gamma+\alpha+\beta}{n} - 1$. *If* $p \leq q < \infty$, *then the inequality*

$$\left\| |x|^{-\beta} T_\gamma v \right\|_{L^q(\mathbb{R}^n)} \leq C \left\| |x|^{\alpha} v \right\|_{L^p(\mathbb{R}^n)}$$

holds for all radially symmetric $v \in L^p(\mathbb{R}^n, |x|^{p\alpha}\, dx)$, *where* C *is independent of* v.

REMARK 1.1. If $p = 1$, then the result of Theorem 1.2 holds for $\alpha + \beta > (n-1)(\frac{1}{q} - 1)$ as may be seen from the proof of the theorem.

REMARK 1.2. When $\gamma \leq n - 1$, the condition $\frac{1}{q} = \frac{1}{p} + \frac{\gamma+\alpha+\beta}{n} - 1$ automatically implies $\alpha + \beta \geq (n-1)(\frac{1}{q} - \frac{1}{p})$.

REMARK 1.3. It is worth noting that if $n = 1$ or $p = q$, Theorem 1.2 gives the same range of exponents as Theorem 1.1.

Our method of proof can also be used, with slight modifications, to reobtain the general case of Stein and Weiss' Theorem. Since this result is not new and the main ideas needed will appear in our proof of the radial case, we leave the details to the reader.

Previous results in the direction of Theorem 1.2 include the works of M. C. Vilela, who made a proof for the case $p < q$ and $\beta = 0$ in [11, Lemma 4]; the work of G. Gasper, K. Stempak and W. Trebels [2, Theorem 3.1], who proved a fractional integration theorem in the context of Laguerre expansions which in the particular case of radial functions in $\mathbb{R}^n$ gives Theorem 1.2 for $\alpha + \beta \geq 0$; and the work of K. Hidano and Y. Kurokawa [4, Theorem 2.1], who proved Theorem 1.2 for $p < q$ under the stronger condition $\beta < \frac{1}{q}$. Notice that this restriction, together with the additional assumptions on α and β, implies $n - 1 < \gamma < n$, whereas our conditions on α and β allow for $0 < \gamma < n$, which is the natural range of γ's for the fractional integral. This is because the proof of Hidano and Kurokawa reduces to the 1-dimensional case of the Stein–Weiss theorem, while our method of proof is completely different. Moreover, our proof is simpler than that in [4], particularly when $n = 2$.

The rest of the paper is organized as follows: in Section 2, we recall some definitions and preliminary results that will be needed in this paper. In Section 3, we prove Theorem 1.2 in the case $n = 1$. As we have already pointed out, in this case our range of weights coincides with that of Stein and Weiss (and Hardy and Littlewood) and, therefore, the assumption that v be radially symmetric (i.e., even) is unnecessary. Although this result is not new, we have included it because the proof we provide is very simple and uses some of the ideas that we will use to prove the general theorem. Section 4 is devoted to the proof of Theorem 1.2 in the general case, and we show, by means of an example when $n = 3$, that the condition on $\alpha + \beta$ is sharp. Finally, in Section 5, we use Theorem 1.2 to obtain a weighted imbedding theorem for radially symmetric functions.

2. Preliminaries

Let X be a measure space and μ be a positive measure on X. Recall that if f is a measurable function, its distribution function d_f on $[0,\infty)$ is defined as

$$d_f(\alpha) = \mu\big(\{x \in X : |f(x)| > \alpha\}\big).$$

For $0 < p < \infty$, the space weak-$L^p(X,\mu)$ is defined as the set of all μ-measurable functions f such that $\|f\|_{L^{p,\infty}}$ is finite, where

$$\|f\|_{L^{p,\infty}} = \inf\big\{C > 0 : d_f(\alpha) \le \big(C\alpha^{-1}\big)^p \text{ for all } \alpha > 0\big\}.$$

If G is a locally compact group, then G posseses a Haar measure, that is, a positive Borel measure μ which is left invariant (i.e., $\mu(At) = \mu(A)$ whenever $t \in G$ and $A \subseteq G$ is measurable) and nonzero on nonempty open sets. In particular, if $G = \mathbb{R}^* := \mathbb{R} - \{0\}$ (with multiplicative structure), then $\mu = \frac{dx}{|x|}$, and if $G = \mathbb{R}^+$, then $\mu = \frac{dx}{x}$.

The convolution of two functions $f, g \in L^1(G)$ is defined as:

$$(f * g)(x) = \int_G f(y) g\big(y^{-1}x\big)\, d\mu(y),$$

where y^{-1} denotes the inverse of y in the group G.

With these definitions in mind, we are ready to recall the following improved version of Young's inequality that will be needed in what follows.

THEOREM 2.1 ([3, Theorem 1.4.24]). *Let G be a locally compact group with left Haar measure μ that satisfies $\mu(A) = \mu(A^{-1})$ for all measurable $A \subseteq G$. Let $1 < p, q, s < \infty$ satisfy*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{s}.$$

Then, there exists a constant $B_{pqs} > 0$ such that for all $f \in L^p(G,\mu)$ and $g \in L^{s,\infty}(G,\mu)$ we have

$$\|f * g\|_{L^q(G,\mu)} \le B_{pqs} \|g\|_{L^{s,\infty}(G,\mu)} \|f\|_{L^p(G,\mu)}. \tag{2.1}$$

REMARK 2.1. Notice that if $p=1$, and $g\in L^s(G,\mu)$, then we can replace (2.1) by the classical Young's inequality, to obtain

$$\|f*g\|_{L^s(G,\mu)}\le B_{pqs}\|g\|_{L^s(G,\mu)}\|f\|_{L^1(G,\mu)}.$$

This can be used to prove the extension to the case $p=1$ of Theorem 1.2 (see Remark 1.1).

3. The 1-dimensional case

Recall that we want to prove

$$\big\||x|^{-\beta}T_\gamma f\big\|_{L^q(\mathbb{R})}\le C\big\|f|x|^\alpha\big\|_{L^p(\mathbb{R})}. \tag{3.1}$$

The key point in our proof is to write the above inequality as a convolution inequality in the group $\mathbb{R}^*$ with the corresponding Haar measure $\mu=\frac{dx}{|x|}$. Indeed, inequality (3.1) can be rewritten as

$$\big\||x|^{-\beta+\frac{1}{q}}T_\gamma f\big\|_{L^q(\mu)}\le C\big\||x|^{\alpha+\frac{1}{p}}f\big\|_{L^p(\mu)}.$$

Now,

$$|x|^{-\beta+\frac{1}{q}}T_\gamma f(x)=\int_{-\infty}^{\infty}\frac{|x|^{-\beta+\frac{1}{q}}f(y)|y|^{\alpha+\frac{1}{p}}}{|y|^{\gamma-1+\alpha+\frac{1}{p}}|1-\frac{x}{y}|^\gamma}\,\frac{dy}{|y|}=(h*g)(x),$$

where $h(x)=f(x)|x|^{\alpha+\frac{1}{p}}, g(x)=\frac{|x|^{-\beta+\frac{1}{q}}}{|1-x|^\gamma}$, and we have used that $\gamma-1+\alpha+\frac{1}{p}=-\beta+\frac{1}{q}$. Using Theorem 2.1, we obtain

$$\big\||x|^{-\beta+\frac{1}{q}}T_\gamma f\big\|_{L^q(\mu)}\le C\big\||x|^{\alpha+\frac{1}{p}}f\big\|_{L^p(\mu)}\big\|g(x)\big\|_{L^{s,\infty}(\mu)},$$

where

$$\frac{1}{q}=\frac{1}{p}+\frac{1}{s}-1.$$

Therefore, it suffices to check that $\|g(x)\|_{L^{s,\infty}(\mu)}<\infty$. For this purpose, consider $\varphi\in C^\infty(\mathbb{R})$, supported in $[\frac{1}{2},\frac{3}{2}]$ and such that $0\le\varphi\le 1$ and $\varphi\equiv 1$ in $(\frac{3}{4},\frac{5}{4})$. We split $g=\varphi g+(1-\varphi)g:=g_1+g_2$.

Clearly, $g_2\in L^s(\mu)$, since the integrability condition at the origin for $|g_2|^s$ (with respect to the measure μ) is $\beta<\frac{1}{q}$, and the integrability condition when $x\to\infty$ is $\frac{1}{q}-\beta-\gamma<0$, which, under our assumptions on the exponents, is equivalent to $\alpha<\frac{1}{p'}$.

Therefore,

$$\begin{aligned}\mu\big(\{g_1+g_2>\lambda\}\big)&\le\mu\left(\left\{g_1>\frac{\lambda}{2}\right\}\right)+\left(\frac{\|g_2\|_{L^s(\mu)}}{\lambda}\right)^s\\&\le\mu\left(\left\{g_1>\frac{\lambda}{2}\right\}\right)+\frac{C}{\lambda^s}\end{aligned}$$

but,

$$\begin{aligned}\mu\Big(\Big\{g_1>\frac{\lambda}{2}\Big\}\Big)&\le\mu\Big(\Big\{\frac{C}{|1-x|^\gamma}>\lambda\Big\}\Big)\\&=\mu\Big(\Big\{\frac{C}{\lambda^{\frac{1}{\gamma}}}>|x-1|\Big\}\Big)\\&\le\frac{C}{\lambda^{\frac{1}{\gamma}}}\le\frac{C}{\lambda^s}\end{aligned}$$

as long as $s\gamma\le 1$, that is, $\gamma\le 1+\frac{1}{q}-\frac{1}{p}$, which is equivalent to $\alpha+\beta\ge 0$. Hence, $g\in L^{s,\infty}(\mu)$ and this concludes the proof.

4. Proof of the weighted HLS theorem for radial functions

In this section, we prove Theorem 1.2. The main idea, as in the one-dimensional case, will be to write the fractional integral operator acting on a radial function as a convolution in the multiplicative group $\mathbb{R}^+$ with Haar measure $\mu=\frac{dx}{x}$. For this purpose, we shall need the following lemma.

LEMMA 4.1. *Let $x\in S^{n-1}=\{x\in\mathbb{R}^n:|x|=1\}$ and consider an integral of the form*:

$$I(x)=\int_{S^{n-1}}f(x\cdot y)\,dy$$

(*the integral is taken with respect to the surface measure on the sphere*), *where $f:[-1,1]\to\mathbb{R}, f\in L^1([-1,1],(1-t^2)^{(n-3)/2})$. Then, $I(x)$ is a constant independent of x and moreover*

$$I(x)=\omega_{n-2}\int_{-1}^{1}f(t)\big(1-t^2\big)^{\frac{n-3}{2}}\,dt,$$

where ω_{n-2} denotes the area of S^{n-2}.

Proof. First, observe that $I(x)$ is constant for all $x\in S^{n-1}$. Indeed, given $\tilde{x}\in S^{n-1}$, there exists a rotation $R\in O(n)$ such that $\tilde{x}=Rx$ and, therefore,

$$I(\tilde{x})=\int_{S^{n-1}}f(\tilde{x}\cdot y)\,dy=\int_{S^{n-1}}f(Rx\cdot y)\,dy=\int_{S^{n-1}}f\big(x\cdot R^{-1}y\big)\,dy=I(x).$$

So, taking $x=e_n$, it suffices to compute $I(e_n)=\int_{S^{n-1}}f(y_n)\,dy$. To this end, we split the integral in two and consider first the integral on the upper-half sphere $(S^{n-1})^+$. Since $(S^{n-1})^+$ is the graph of the function $g:\{x\in\mathbb{R}^{n-1}:|x|<1\}\to(S^{n-1})^+, g(x)=(x,\sqrt{1-|x|^2})$, we obtain

$$\int_{(S^{n-1})^+}f(y_n)\,dy=\int_{\{|x|<1\}}f\big(\sqrt{1-|x|^2}\big)\frac{1}{\sqrt{1-|x|^2}}\,dx$$

using polar coordinates, this is

$$\int_{S^{n-2}}\int_0^1 f\big(\sqrt{1-r^2}\big)\frac{1}{\sqrt{1-r^2}}r^{n-2}\,dr\,dy=\omega_{n-2}\int_0^1 f(t)\big(1-t^2\big)^{\frac{n-3}{2}}\,dt.$$

Analogously, one obtains

$$\int_{(S^{n-1})^-} f(y_n)\,dy = \omega_{n-2}\int_{-1}^{0} f(t)\big(1-t^2\big)^{\frac{n-3}{2}}\,dt.$$

This completes the proof. $\square$

Now we can proceed to the proof of our main theorem.

Using polar coordinates,

$$\begin{aligned} y &= ry', \qquad & r &= |y|, \qquad & y' &\in S^{n-1}, \\ x &= \rho x', \qquad & \rho &= |x|, \qquad & x' &\in S^{n-1} \end{aligned}$$

and the identity

$$|x-y|^2 = |x|^2 - 2|x||y|x'\cdot y' + |y|^2$$

we write the fractional integral of a radial function $v(x) = v_0(|x|)$ as

$$T_\gamma v(x) = \int_0^\infty \int_{S^{n-1}} \frac{v_0(r) r^{n-1}\,dr\,dy'}{(r^2 - 2r\rho x'\cdot y' + \rho^2)^{\gamma/2}}.$$

Using Lemma 4.1, we have that:

$$T_\gamma v(x) = \omega_{n-2}\int_0^\infty v_0(r) r^{n-1}\left\{\int_{-1}^{1} \frac{(1-t^2)^{(n-3)/2}}{(\rho^2 - 2\rho r t + r^2)^{\gamma/2}}\,dt\right\} dr.$$

Now, we may write the inner integral as:

$$\int_{-1}^{1} \frac{(1-t^2)^{(n-3)/2}}{(\rho^2 - 2\rho r t + r^2)^{\gamma/2}}\,dt = \int_{-1}^{1} \frac{(1-t^2)^{(n-3)/2}}{r^\gamma[1-2(\frac{\rho}{r})t + (\frac{\rho}{r})^2]^{\gamma/2}}\,dt.$$

Therefore,

$$T_\gamma v(x) = \omega_{n-2}\int_0^\infty v_0(r) r^{n-\gamma} I_{\gamma,k}\left(\frac{\rho}{r}\right)\frac{dr}{r},$$

where $k = \frac{n-3}{2}$, and, for $a \geq 0$,

$$I_{\gamma,k}(a) = \int_{-1}^{1} \frac{(1-t^2)^k}{(1-2at+a^2)^{\gamma/2}}\,dt.$$

Notice that the denominator of this integral vanishes if $a = 1$ and $t = 1$ only. Therefore, $I_{\gamma,k}(a)$ is well defined and is a continuous function for $a \neq 1$.

This formula shows in a explicit way that $T_\gamma v$ is a radial function, and can be therefore thought of as a function of ρ. Furthermore, we observe that as consequence of this formula, $\rho^{\frac{n}{q}-\beta} T_\gamma v$ has the structure of a convolution on the multiplicative group $\mathbb{R}^+$:

$$\begin{aligned} \rho^{\frac{n}{q}-\beta} T_\gamma v(x) &= \omega_{n-2}\int_0^\infty v_0(r) r^{n-\gamma+\frac{n}{q}-\beta} \frac{\rho^{\frac{n}{q}-\beta}}{r^{\frac{n}{q}-\beta}} I_{\gamma,k}\left(\frac{\rho}{r}\right)\frac{dr}{r} \\ &= \omega_{n-2}\big(v_0 r^{n-\gamma+\frac{n}{q}-\beta}\big) * \big(r^{\frac{n}{q}-\beta} I_{\gamma,k}(r)\big). \end{aligned}$$

Hence, using Theorem 2.1 we get that

$$\begin{aligned}\big\||x|^{-\beta}T_\gamma v\big\|_{L^q(\mathbb{R}^n)} &= \left(\omega_{n-1}\int_0^\infty \big|T_\gamma v(\rho)\big|^q \rho^{n-\beta q}\frac{d\rho}{\rho}\right)^{1/q}\\ &= \omega_{n-1}^{1/q}\big\|T_\gamma v(\rho)\rho^{\frac{n}{q}-\beta}\big\|_{L^q(\mu)}\\ &\le \omega_{n-1}^{1/q}\omega_{n-2}\big\|v_0(r)r^{n-\gamma+\frac{n}{q}-\beta}\big\|_{L^p(\mu)}\big\|r^{\frac{n}{q}-\beta}I_{\gamma,k}(r)\big\|_{L^{s,\infty}(\mu)}\end{aligned}$$

provided that:

$$\frac{1}{p}+\frac{1}{s}-1=\frac{1}{q}. \tag{4.1}$$

Using polar coordinates once again:

$$\begin{aligned}\omega_{n-1}^{1/p}\big\|v_0(r)r^{n-\gamma+\frac{n}{q}-\beta}\big\|_{L^p(\mu)} &= \omega_{n-1}^{1/p}\left(\int_0^\infty \big|v_0(r)\big|^p r^{(n-\gamma+\frac{n}{q}-\beta)p-n}r^n\frac{dr}{r}\right)^{1/p}\\ &= \big\|v_0|x|^{n-\gamma+\frac{n}{q}-\beta-\frac{n}{p}}\big\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

But, by the conditions of our theorem,

$$n-\gamma+\frac{n}{q}-\beta-\frac{n}{p}=\alpha.$$

Therefore, it suffices to prove that

$$\big\|r^{\frac{n}{q}-\beta}I_{\gamma,k}(r)\big\|_{L^{s,\infty}(\mu)}<+\infty. \tag{4.2}$$

For this purpose, consider $\varphi\in C^\infty(\mathbb{R})$, supported in $[\frac{1}{2},\frac{3}{2}]$ and such that $0\le\varphi\le 1$ and $\varphi\equiv 1$ in $(\frac{3}{4},\frac{5}{4})$. We split $r^{\frac{n}{q}-\beta}I_{\gamma,k}=\varphi r^{\frac{n}{q}-\beta}I_{\gamma,k}+(1-\varphi)r^{\frac{n}{q}-\beta}\times I_{\gamma,k}:=g_1+g_2$.

We claim that $g_2\in L^s(\mu)$. Indeed, since $I_{\gamma,k}(r)$ is a continuous function for $r\ne 1$, to analyze the behavior (concerning integrability) of g_2 it suffices to consider the behavior of $r^{(\frac{n}{q}-\beta)s}|I_{\gamma,k}(r)|^s$ at $r=0$, and when $r\to+\infty$.

Since $I_{\gamma,k}(r)$ has no singularity at $r=0$ ($I_{\gamma,k}(0)$ is finite) the local integrability condition at $r=0$ is $\beta<\frac{n}{q}$.

When $r\to+\infty$, we observe that

$$I_{\gamma,k}(r)=\frac{1}{r^\gamma}\int_{-1}^1\frac{(1-t^2)^k}{(r^{-2}-2r^{-1}t+1)^{\gamma/2}}\,dt$$

and using the bounded convergence theorem, we deduce that

$$I_{\gamma,k}(r)\sim\frac{C_k}{r^\gamma}\quad\text{as } r\to+\infty\quad\left(\text{with } C_k=\int_{-1}^1\big(1-t^2\big)^k\,dt\right).$$

It follows that the integrability condition at infinity is $\frac{n}{q}-\beta-\gamma<0$, which, under our conditions on the exponents, is equivalent to $\alpha<\frac{n}{p'}$.

We proceed now to g_1. To analyze its behavior near $r=1$, we shall need the following lemma.

LEMMA 4.2. *For $a\sim 1$ and $k\in\mathbb{N}_0$ or $k=m-\frac{1}{2}$ with $m\in\mathbb{N}_0$, we have that*

$$\left|I_{\gamma,k}(a)\right| \le \begin{cases} C_{k,\gamma}, & \text{if } \gamma<2k+2,\\ C_{k,\gamma}\log\frac{1}{|1-a|}, & \text{if } \gamma=2k+2,\\ C_{k,\gamma}|1-a|^{-\gamma+2k+2}, & \text{if } \gamma>2k+2.\end{cases}$$

REMARK 4.1. Notice that since in the proof of our theorem $k=\frac{n-3}{2}$, the conditions relating γ and k above correspond to conditions on γ and n which cover all the range $0<\gamma<n$.

Proof of Lemma 4.2. Assume first that $k\in\mathbb{N}_0$ and $-\frac{\gamma}{2}+k>-1$ (that is, $0<\gamma<n-1$). Then,

$$I_{\gamma,k}(1)\sim\int_{-1}^{1}\frac{(1-t^2)^k}{(2-2t)^{\frac{\gamma}{2}}}\,dt\sim C\int_{-1}^{1}\frac{(1-t)^k}{(1-t)^{\frac{\gamma}{2}}}\,dt.$$

Therefore, $I_{\gamma,k}$ is bounded.

If $-\frac{\gamma}{2}+k=-1$ (that is, $\gamma=n-1$), then

$$I_{\gamma,k}(a)\sim\int_{-1}^{1}\left(1-t^2\right)^k\frac{d^k}{dt^k}\left\{\left(1-2at+a^2\right)^{-\frac{\gamma}{2}+k}\right\}dt.$$

Integrating by parts k times (the boundary terms vanish),

$$I_{\gamma,k}(a)\sim\left|\int_{-1}^{1}\frac{d^k}{dt^k}\left\{\left(1-t^2\right)^k\right\}\left(1-2at+a^2\right)^{-\frac{\gamma}{2}+k}dt\right|.$$

But $\frac{d^k}{dt^k}\{(1-t^2)^k\}$ is a polynomial of degree k and therefore is bounded in $[-1,1]$ (in fact, it is up to a constant the classical Legendre polynomial). Therefore,

$$I_{\gamma,k}(a)\sim\frac{1}{2a}\log\left(\frac{1+a}{1-a}\right)^2\le C\log\frac{1}{|1-a|}.$$

Finally, if $-\frac{\gamma}{2}+k<-1$ (that is, $n-1<\gamma<n$), then integrating by parts as before,

$$I_{\gamma,k}(a)\le C_k\int_{-1}^{1}\left(1-2at+a^2\right)^{-\frac{\gamma}{2}+k}dt.$$

Thus,

$$I_{\gamma,k}(a)\sim\left(1-2at+a^2\right)^{-\frac{\gamma}{2}+k+1}\Big|_{t=-1}^{t=1}\le C_{k,\gamma}|1-a|^{-\gamma+2k+2}.$$

This finishes the proof if $k\in\mathbb{N}_0$ (i.e., if n is odd).

We proceed now to the case $k = m + \frac{1}{2}$, $m \in \mathbb{N}_0$. For $-\frac{\gamma}{2} + k > -1$ the proof is exactly as in the case $k \in \mathbb{N}_0$. If $-\frac{\gamma}{2} + k < -1$, assume first that $-\frac{\gamma}{2} + m + 1 < -1$. Then,

$$\begin{aligned} I_{\gamma,k}(a) &= \int_{-1}^{1} \left(1-t^2\right)^k \left(1-2at+a^2\right)^{-\frac{\gamma}{2}} dt \\ &= \int_{-1}^{1} \left(1-t^2\right)^{\frac{1}{2}m} \left(1-2at+a^2\right)^{-\frac{\gamma}{4}} \left(1-t^2\right)^{\frac{1}{2}(m+1)} \left(1-2at+a^2\right)^{-\frac{\gamma}{4}} dt \end{aligned}$$

and applying the Cauchy–Schwarz inequality we get that

$$I_{\gamma,k}(a) \le I_{\gamma,m}(a)^{\frac{1}{2}} I_{\gamma,m+1}(a)^{\frac{1}{2}}.$$

Using the bound for the case in which k is an integer for $I_{\gamma,m}(a)$ and $I_{\gamma,m+1}(a)$, we conclude that, $I_{k,\gamma}(a) \le C|1-a|^{-\gamma+2m+3} = C|1-a|^{-\gamma+2k+2}$.

If, on the contrary, $-\frac{\gamma}{2} + m + 1 \ge -1$, we proceed as follows: notice that we can always assume $a < 1$, since $I_{\gamma,k}(a) = a^{-\gamma} I_{\gamma,k}(a^{-1})$, then

$$\begin{aligned} I'_{\gamma,k}(a) &= \gamma \int_{-1}^{1} \frac{(1-t^2)^k (t-a)}{(1-2at+a^2)^{\frac{\gamma}{2}+1}}\, dt \le \gamma \int_{a}^{1} \frac{(1-t^2)^k (t-a)}{(1-2at+a^2)^{\frac{\gamma}{2}+1}}\, dt \\ &\le \gamma (1-a) I_{\gamma+2,k}(a). \end{aligned}$$

But, $-\frac{\gamma+2}{2} + k + \frac{1}{2} = -\frac{\gamma}{2} + k - \frac{1}{2} < -1$, therefore, $I_{\gamma+2,k}$ can be bounded as in the previous case to obtain $I_{\gamma+2,k}(a) \le C|1-a|^{-\gamma+2k}$. Using this bound, when $-\frac{\gamma}{2} + k < -1$, we obtain

$$I_{\gamma,k}(a) = \int_0^a I'_{\gamma,k}(s)\, ds \le C \int_0^a (1-s)^{-\gamma+2k+1}\, ds \le C|1-a|^{-\gamma+2k+2},$$

and when $-\frac{\gamma}{2} + k = -1$,

$$I_{\gamma,k}(a) \le C \int_0^a \frac{1}{1-s}\, ds = C \log \frac{1}{|1-a|}.$$

It remains to check the case $k = -\frac{1}{2}$

$$\begin{aligned} I_{\gamma,-\frac{1}{2}}(a) &= \int_{-1}^{0} \frac{(1-t^2)^{-\frac{1}{2}}}{(1-2at+a^2)^{\frac{\gamma}{2}}}\, dt + \int_{0}^{1} \frac{(1-t^2)^{-\frac{1}{2}}}{(1-2at+a^2)^{\frac{\gamma}{2}}}\, dt \\ &= I + II. \end{aligned}$$

Since $\gamma > 0$,

$$\begin{aligned} I &\le \int_{-1}^{0} \frac{dt}{(1+t)^{\frac{1}{2}}} = 2, \\ II &\le \int_0^1 \frac{(1-t)^{-\frac{1}{2}}}{(1-2at+a^2)^{\frac{\gamma}{2}}}\, dt = -2 \int_0^1 \frac{\frac{d}{dt}[(1-t)^{\frac{1}{2}}]}{(1-2at+a^2)^{\frac{\gamma}{2}}}\, dt \\ &\le 2a\gamma \int_0^1 \frac{(1-t^2)^{\frac{1}{2}}}{(1-2at+a^2)^{\frac{\gamma}{2}+1}}\, dt \le C I_{\gamma+2,\frac{1}{2}}. \end{aligned}$$

□

Now we can go back to the study of g_1. We shall split the proof into three cases, depending on whether γ is less than, equal to or greater than $n-1$.

i. Assume first that $0<\gamma<n-1$. Then $|r|^{(-\beta+\frac{n}{q})s}|I_{\gamma,k}(r)|^s$ is bounded when $r\sim 1$, and, therefore, $\|g_1\|_{L^s(\mu)}<+\infty$.

ii. Consider now the case $\gamma=n-1$. Since in this case
$$\left|I_{\gamma,k}(r)\right|\leq C\log\frac{1}{|1-r|},$$
we conclude, as before, that $\|g_1\|_{L^s(\mu)}<+\infty$.

iii. Finally, we have to consider the case $n-1<\gamma<n$. In this case,
$$\left|I_{\gamma,k}(r)\right|\leq C|1-r|^{-\gamma+2k+2}=C|1-r|^{-\gamma+n-1}.$$
Therefore,
$$\begin{aligned}\mu\left(\left\{g_1>\frac{\lambda}{2}\right\}\right)&\leq\mu\left(\left\{\frac{C}{|1-x|^{\gamma-n+1}}>\lambda\right\}\right)=\mu\left(\left\{\frac{C}{\lambda^{\frac{1}{\gamma-n+1}}}>|1-x|\right\}\right)\\&\leq\frac{C}{\lambda^{\frac{1}{\gamma-n+1}}}\leq\frac{C}{\lambda^s}\end{aligned}$$
as long as $s(\gamma-n+1)\leq 1$, which is equivalent to $\alpha+\beta\geq(n-1)(\frac{1}{p}-\frac{1}{q})$. Therefore, $\|g_1\|_{L^{s,\infty}(\mu)}<+\infty$.

Remark 4.2. The following example shows that for $n=3$ the condition $\alpha+\beta\geq(n-1)(\frac{1}{q}-\frac{1}{p})$ is necessary.

Assume that $\alpha+\beta<(n-1)(\frac{1}{q}-\frac{1}{p})$. Then, by Remark 1.2, $\gamma>n-1$.

Since $\frac{1}{q}=\frac{1}{p}+\frac{1}{s}-1$, we obtain $\gamma-n+1>\frac{1}{s}$ and, therefore, by Lemma 4.2, for $n=3$ and $r\sim 1$, $I_{\gamma,k}(r)\sim\frac{1}{|1-r|^{\frac{1}{s}+\varepsilon}}$ for some $\varepsilon>0$.

Fix η such that $\eta p>1$ and let
$$f(r)=\frac{\chi_{[\frac{1}{2},\frac{3}{2}]}(r)}{|1-r|^{\frac{1}{p}}\log(\frac{1}{|1-r|})^{\eta}}.$$

Then $f\in L^p(\mu)$ and, for $r>1$,
$$\begin{aligned}(I_{\gamma,k}*f)(r)&\geq\int_r^{\frac{3}{2}}\frac{t^{\frac{1}{s}+\varepsilon}}{t^{\frac{1}{s}+\varepsilon}|1-\frac{r}{t}|^{\frac{1}{s}+\varepsilon}|1-t|^{\frac{1}{p}}\log(\frac{1}{|1-t|})^{\eta}}\,dt\\&\geq\int_r^{\frac{3}{2}}\frac{1}{(t-r)^{\frac{1}{s}+\varepsilon}(t-1)^{\frac{1}{p}}(\log\frac{1}{|1-r|})^{\eta}}\,dy\\&\geq\frac{1}{(\log\frac{1}{|1-r|})^{\eta}}\int_r^{\frac{3}{2}}\frac{dy}{(t-1)^{\frac{1}{s}+\frac{1}{p}+\varepsilon}}\\&\sim\frac{1}{(\log\frac{1}{|1-r|})^{\eta}|1-r|^{\frac{1}{q}+\varepsilon}}\notin L^q.\end{aligned}$$

Recall now that for a radial function,

$$\rho^{\frac{n}{q}-\beta} T_\gamma f_0(\rho) = f_0 r^{\frac{n}{p}+\alpha} * r^{\frac{n}{q}-\beta} I_{\gamma,k}(r).$$

Therefore, defining $f_0 = f(|x|)|x|^{-\frac{n}{p}-\alpha}$ we have, $\|f_0|x|^\alpha\|_{L^p} < \infty$ but $T_\gamma f|x|^{-\beta} \notin L^q$.

5. An application to weighted imbedding theorems

Consider the fractional order Sobolev space

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : (-\Delta)^{s/2} u \in L^2(\mathbb{R}^n)\} \quad (s \geq 0).$$

As an application of our main theorem, we will prove a weighted imbedding theorem for $H^s_{\mathrm{rad}}(\mathbb{R}^n)$, the subspace of radially symmetric functions of $H^s(\mathbb{R}^n)$.

THEOREM 5.1. *Let* $0 < s < \frac{n}{2}$, $2 < q < 2^*_c := \frac{2(n+c)}{n-2s}$. *Then, we have the compact imbedding*

$$H^s_{\mathrm{rad}}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c\, dx)$$

provided that $-2s < c < \frac{(n-1)(q-2)}{2}$.

REMARK 5.1. The case $s = 1$ of this lemma was already proved in the work of W. Rother [7], while the general case was already proved in a completely different way in our work [1].

REMARK 5.2. The unweighted case $c = 0$ gives the classical Sobolev imbedding (in the case of radially symmetric functions). In that case, the compactness of the imbedding $H^s_{\mathrm{rad}}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ under the conditions $0 < s < \frac{n}{2}$ and $2 < q < \frac{2n}{n-2s}$ was proved by P. L. Lions [6].

Proof of Theorem 5.1. Let $u \in H^s_{\mathrm{rad}}(\mathbb{R}^n)$. Then, $f := (-\Delta)^{s/2} u \in L^2$, and, recalling the relation between the negative powers of the Laplacian and the fractional integral (see, e.g., [9, Chapter V]), we obtain

$$T_{n-s} f = C(-\Delta)^{-s/2} f = Cu.$$

Then, it follows from Theorem 1.2 that

$$\||x|^{\frac{c}{q}} u\|_{L^{2^*_c}(\mathbb{R}^n)} = C\||x|^{\frac{c}{q}} T_{n-s} f\|_{L^{2^*_c}(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}.$$

Therefore, writing $q = 2\nu + (1-\nu)2^*_c$, and using Hölder's inequality, we obtain

$$\||x|^{\frac{c}{q}} u\|_{L^q(\mathbb{R}^n)} \leq \||x|^{\frac{c}{q}} u\|^{\nu}_{L^{2^*_c}(\mathbb{R}^n)} \|u\|^{1-\nu}_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}.$$

It remains to prove that the imbedding $H^s_{\mathrm{rad}}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c\, dx)$ is compact. The proof can be made in the same way as that in [1, Theorem 2.1].

Indeed, it suffices to show that if $u_n \to 0$ weakly in $H^s_{\rm rad}(\mathbb{R}^n)$, then $u_n \to 0$ strongly in $L^q(\mathbb{R}^n, |x|^c\, dx)$. Since

$$2 < q < 2^*_c = \frac{2(n+c)}{n-2s}$$

by hypothesis, it is possible to choose r and $\tilde{q}$ so that $2 < r < q < \tilde{q} < 2^*_c$. We write $q = \theta r + (1-\theta)\tilde{q}$ with $\theta \in (0,1)$ and, using Hölder's inequality, we have that

$$\int_{\mathbb{R}^n} |x|^c |u_n|^q\, dx \le \left(\int_{\mathbb{R}^n} |u_n|^r\, dx \right)^{\theta} \left(\int_{\mathbb{R}^n} |x|^{\tilde{c}} |u_n|^{\tilde{q}}\, dx \right)^{1-\theta}, \tag{5.1}$$

where $\tilde{c} = \frac{c}{1-\theta}$. By choosing r close enough to 2 (hence making θ small), we can fulfill the conditions

$$\tilde{q} < \frac{2(n+\tilde{c})}{n-2s}, \qquad -2s < \tilde{c} < \frac{(n-1)(\tilde{q}-2)}{2}.$$

Therefore, by the imbedding that we have already established:

$$\left(\int_{\mathbb{R}^n} |x|^{\tilde{c}} |u_n|^{\tilde{q}}\, dx \right)^{1/\tilde{q}} \le C \|u_n\|_{H^s} \le C'.$$

Since the imbedding $H^s_{\rm rad}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ is compact by Lions theorem [6], we have that $u_n \to 0$ in $L^r(\mathbb{R}^n)$. From (5.1), we conclude that $u_n \to 0$ strongly in $L^q(\mathbb{R}^n, |x|^c\, dx)$, which shows that the imbedding in our theorem is also compact. This concludes the proof. □

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