Non-simultaneous quenching in a system of heat equations coupled at the boundary

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Abstract

We study the solutions of a parabolic system of heat equations coupled at the boundary through a nonlinear flux. We characterize in terms of the parameters involved when non-simultaneous quenching may appear. Moreover, if quenching is non-simultaneous we find the quenching rate, which surprisingly depends on the flux associated to the other component.

1 Introduction

We study the formation of singularities in finite time for solutions (u, v) of the parabolic system

$$\begin{cases} u_t = u_{xx} \\ v_t = v_{xx} \end{cases} \quad \text{in } (0,1)\times(0,T), \tag{1.1}$$

coupled at the boundary through a nonlinear flux at one border,

$$\begin{cases} u_x(0,t) = v^{-p}(0,t) \\ v_x(0,t) = u^{-q}(0,t) \end{cases} \quad \text{in } (0,T), \tag{1.2}$$

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zero flux at the other border,

$$\begin{cases} u_x(1,t) = 0 \\ v_x(1,t) = 0 \end{cases} \quad \text{in } (0,T), \tag{1.3}$$

and initial data

$$\begin{cases} u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases} \quad \text{in } (0,1). \tag{1.4}$$

Throughout the paper we assume that the initial data u_0, v_0 are positive, smooth and compatible with the boundary data. The constant T denotes the maximal existence time for the solution, which is to be understood in a classical sense. We also assume u_0', $v_0' \geq 0$ and u_0'', $v_0'' < 0$.

We are interested in the *quenching* phenomenon, which has attracted a lot of attention in recent years, see [1], [2], [7] and references therein. There is quenching if $T < \infty$; in this case

$$\liminf_{t \nearrow T} \min\{\min_{0 \leq x \leq 1} u(x,t), \min_{0 \leq x \leq 1} v(x,t)\} = 0,$$

and a singularity appears in the boundary condition. It is easy to see that quenching always happens for our system, see Proposition 2.1. However, a priori there is no reason why both components u and v should reach the level zero simultaneously at the quenching time T. If one of them remains bounded away from zero at $t = T$ we say that quenching is *non-simultaneous*. An example of such a phenomenon was given in [8], where the authors considered a semilinear system. See also [9], [10], [11] for examples of non-simultaneous blow-up.

Our first aim is to characterize when non-simultaneous quenching is possible for problem (1.1)–(1.4).

Theorem 1.1 *If v does not quench then $q < 1$. Moreover, if $q < 1$ then for every v_0 there exist initial data u_0 such that u quenches while v does not.*

The analogous result holds for the component u. This means that if p, $q \geq 1$, then quenching is always simultaneous, while if $p < 1$ or $q < 1$ non-simultaneous quenching indeed occurs. Nevertheless, if p, $q < 1$ simultaneous quenching is also possible.

Theorem 1.2 *If $0 < p$, $q < 1$ then there exist initial data which produce simultaneous quenching.*

If $p \geq 1 > q$ we conjecture that only non-simultaneous quenching is possible. In this direction we prove that this is the case if $q < 1$ and $p \geq p_0 = (1+q)/(1-q) > 1$, see Lemma 3.1.

Next, we concentrate on the non-simultaneous case and we find the quenching rates, the quenching set and the quenching behaviour. The notation $f \sim g$ means that there exist finite positive constants c_1, c_2 such that $c_1\, g \leq f \leq c_2\, g$.

Theorem 1.3 *If quenching is non-simultaneous and, for instance u is the quenching variable, then $u(0,t) \sim (T-t)^{1/(q+1)}$, $u_t(0,t) \sim -(T-t)^{-q/(q+1)}$ and $u(x,T) \sim x$.*

If u does not quench we have analogous estimates for v, replacing q by p. Observe that the quenching rate of the quenching component depends on the exponent appearing in the flux of the other component, something which is rather surprising. For instance, in the system of semilinear heat equations considered in [8], in the case of non-simultaneous quenching the rate exponent is 1.

2 Quenching

In this section we prove some a priori estimates, beginning with the fact that quenching always happens for our problem. To simplify the presentation of the proofs we define the functions

$$U(t) = u(0,t) = \min_{0\le x\le 1} u(x,t), \quad V(t) = v(0,t) = \min_{0\le x\le 1} v(x,t). \tag{2.1}$$

Proposition 2.1 *Quenching happens for system* (1.1)–(1.4) *for every initial data.*

Proof. By the maximum principle we have $u \le M = \|u_0\|_\infty$, $v \le N = \|v_0\|_\infty$. Hence, integration of (1.1) in space gives the following mass estimates,

$$\int_0^1 u(x,t)\,dx \le M - N^{-p}t, \qquad \int_0^1 v(x,t)\,dx \le N - M^{-q}t, \tag{2.2}$$

which yield a contradiction if u and v are positive for all times. □

Some authors understand quenching as the blow-up of the time derivative while the solution itself remains bounded, see for example [5], [6]. This indeed happens for our problem if u and v are strictly decreasing in time, see Corollary 2.1. This will follow for instance if the initial values are strictly concave, something that we assume. We first prove an auxiliary result.

Lemma 2.1 *There exists $\epsilon > 0$ such that*

$$U'(t) \le -\epsilon U^{-q}(t), \qquad V'(t) \le -\epsilon V^{-p}(t). \tag{2.3}$$

Proof. We define the functions $F = u_t + \epsilon v_x$, $G = v_t + \epsilon u_x$. They are solutions to the heat equation. Choosing $\epsilon > 0$ small enough, we have $F(x,0) < 0$ and $G(x,0) < 0$ for $x \in [0,1]$. Also, since u and v are decreasing in time, we have $F(1,t) < 0$ and $G(1,t) < 0$ for $t \in (0,T)$. As to the flux at $x = 0$ we obtain

$$\begin{aligned} F_x(0,t) &= -(pv^{-p-1} - \epsilon)v_t \ge -CG(0,t), \\ G_x(0,t) &= -(qu^{-q-1} - \epsilon)u_t \ge -CF(0,t), \end{aligned}$$

if ϵ is small. The maximum principle implies $F(x,t)$, $G(x,t) \le 0$ for every $x \in [0,1]$ and $t \in [0,T)$. In particular for $x=0$ we obtain (2.3). $\square$

Direct integration of inequalities (2.3) yields the following estimates.

Corollary 2.1

$$\begin{array}{ll} U(t) \ge C(T-t)^{1/(q+1)}, & V(t) \ge C(T-t)^{1/(p+1)}, \\ U'(t) \le -C(T-t)^{-q/(q+1)}, & V'(t) \le -C(T-t)^{-p/(p+1)}. \end{array}$$

Remark. (i) In the case of non-simultaneous quenching, the flux at the boundary of the quenching variable remains bounded. Nevertheless, its time derivative blows up.

(ii) In fact, both time derivatives blow up. Hence quenching is always simultaneous in the sense of [5].

With these estimates we are able to prove the continuity of the quenching time.

Theorem 2.1 *The quenching time is continuous with respect to the initial data.*

Proof. Let (u,v) and $(\widetilde{u},\widetilde{v})$ be the solutions corresponding to the initial values (u_0,v_0) and $(\widetilde{u_0},\widetilde{v_0})$, and let T and $\widetilde{T}$ be their quenching times. Assume for instance that $T<\widetilde{T}$, and also that u quenches. Given $\varepsilon>0$, there is a $\delta>0$ as small as desired such that if $T-\delta<t_0<T$ then $U(t_0)\le\varepsilon/2$. Since up to time t_0 the solutions are classical, by the continuous dependence with respect to the initial data we have that, if $|u_0-\widetilde{u_0}|+|v_0-\widetilde{v_0}|\le\mu$ then $\widetilde{U}(t_0)\le\varepsilon$. Therefore, using the lower estimate given in Corollary 2.1, we get $|T-\widetilde{T}|\le|T-t_0|+|\widetilde{T}-t_0|\le\delta+\varepsilon^{q+1}$. $\square$

Theorem 2.2 *The set of initial data such that one of the components quenches while the other one remains bounded is open.*

Proof. Let (u,v), $(\widetilde{u},\widetilde{v})$ and T, $\widetilde{T}$ be as before. Assume for instance that u quenches and v does not. Let t_0 near T and $(\widetilde{u_0},\widetilde{v_0})$ near (u_0,v_0) be such that $V(t_0)=K$ and $\widetilde{V}(t_0)\ge K/2$. Using the representation formula for $\widetilde{v}$ we have

$$\frac{1}{2}\widetilde{V}(t)\ge C(t_0)-\int_{t_0}^{t}(\widetilde{T}-s)^{-q/(1+q)}(t-s)^{-1/2}\,ds.$$

As $t\to\widetilde{T}$ we obtain

$$\frac{1}{2}\widetilde{V}(\widetilde{T})\ge K-(\widetilde{T}-t_0)^{/1-q)/(2(1+q))}.$$

By the continuity of the quenching time shown above we can choose $\widetilde{T}-t_0$ small enough to make this quantity strictly positive. Therefore $\widetilde{V}$ does not quench. $\square$

We now establish a lower bound for the time derivatives. These estimates will be used in the next section to establish an upper bound for the quenching rates in the case of non-simultaneous quenching.

Lemma 2.2 *There exists a constant* $C > 0$ *such that,*

$$U'(t) \geq -CV^{-p-1}(t)U^{-q}(t), \qquad V'(t) \geq -CU^{-q-1}(t)V^{-p}(t). \tag{2.4}$$

Proof. We adapt an idea from [3]. Let $J = u_x - \phi(x)v^{-p}$, $L = v_x - \phi(x)u^{-q}$, where $\phi : [0,1] \to [0,1]$ is a nonnegative, nonincreasing, convex, C^2 function such that $\phi(0) = 1$, $\phi(1) = 0$, and $\phi(x) \leq u_0'(x)v_0^p(x)$, $\phi(x) \leq v_0'(x)u_0^q(x)$ for $x \in [0,1]$. This implies that J and L are nonnegative at $t = 0$. Differentiating J we get

$$J_t - J_{xx} = \phi'' v^{-p} - 2p\phi' v^{-p-1}v_x + p(p+1)\phi v^{-p-2}(v_x)^2 \geq 0,$$

and the same for L, so that both J and L are supersolutions of the heat equation. In addition, they vanish at the border, $x = 0$ and $x = 1$. Therefore $J(x,t)$, $L(x,t) \geq 0$ for every $x \in [0,1]$, $t \in [0,T)$. In particular this implies $J_x(0,t) \geq 0$ and $L_x(0,t) \geq 0$, i.e.

$$u_t(0,t) = u_{xx}(0,t) \geq \phi'(0)v^{-p}(0,t) - pv^{-p-1}v_x(0,t) \geq -Cv^{-p-1}u^{-q}(0,t),$$

and the analogous estimate for v. □

As a byproduct we obtain the quenching set.

Lemma 2.3 *The only quenching point is the origin* $x = 0$.

Proof. Since $J(x,t) \geq 0$, we have $u_x(x,t) \geq \phi(x)v^{-p}(x,t) \geq N^{-p}/2$ for every $0 \leq x \leq x_0$ such that $\phi(x_0) = 1/2$. Thus $u(x,t) \geq u(0,t) + Cx$. The same happens for v. □

3 Non-simultaneous quenching

This section is devoted to the characterization of non-simultaneous quenching. We first finish the proof of Theorem 1.3. The upper estimate of the non-simultaneous rate is then used to prove Theorem 1.1.

Proof of Theorem 1.3. Lemma 2.1 gives the lower bound of the non-simultaneous rate. The upper bound follows easily by integrating the first estimate in (2.4) using that $V \geq c > 0$:

$$U'(t) \geq -CU^{-q}(t) \Rightarrow U^{q+1}(t) \leq C(T-t).$$

As to the behaviour of the final profile ($t = T$) as $x \sim 0$, we have the lower estimate given by Lemma 2.3. The upper estimate follows directly from the fact that u is concave; therefore $u_x(x,t) \le u_x(0,t) = v^{-p}(0,t) \le C$. □

Proof of Theorem 1.1. We use the representation formula obtained from the heat kernel. Let $\Gamma(x,t) = (4\pi t)^{-1/2} e^{-x^2/4t}$ be the fundamental solution of the heat equation in $\mathbb{R}$. For $x \in (0,1)$ and $t \in (0,T)$, we have

$$\begin{aligned} v(x,t) &= \int_0^1 \Gamma(x-y,t) v_0(y)\,dy - \int_0^t v(0,s)\frac{\partial\Gamma}{\partial x}(x,t-s)\,ds \\ &\quad + \int_0^t v(1,s)\frac{\partial\Gamma}{\partial x}(x-1,t-s)\,ds - \int_0^t u^{-q}(0,s)\Gamma(x,t-s)\,ds. \end{aligned}$$

Taking limits as $x \to 0$, and using the jump relation (see for instance [4]), we get

$$\begin{aligned} \frac{1}{2}V(t) &= \int_0^1 \Gamma(y,t)v_0(y)\,dy + \int_0^t v(1,s)\frac{\partial\Gamma}{\partial x}(-1,t-s)\,ds - \int_0^t U^{-q}(s)\Gamma(0,t-s)\,ds \\ &= I_1 + I_2 + I_3. \end{aligned}$$

The first two integrals are easily bounded by a constant times N. Introducing the upper estimate of the quenching rate in I_3 we get

$$V(t) \le C_1 - C_2 \int_0^T s^{-\frac{q}{q+1}-\frac{1}{2}}\,ds.$$

We conclude by observing that this integral diverges if $q \ge 1$, leading to a contradiction with the fact that v is bounded from below.

Now we use again the above representation formula to show that there exist constants C_1, $C_2 > 0$ such that

$$V(t) \ge C_1 - C_2 \int_0^T (T-s)^{-\frac{q}{q+1}-\frac{1}{2}}\,ds = C_1 - C_2 T^{\frac{1-q}{2(1+q)}}.$$

The integral converges since $q < 1$. On the other hand,

$$T \le \min\{M_v(0)u_0^q(0), M_u(0)v_0^p(0)\}, \tag{3.5}$$

where $M_u(0)$ and $M_v(0)$ are the initial masses of the variables u and v. Thus T can be made as small as we please, just taking $u_0(0)$ small enough. We conclude that $V(t) \ge C_2/2$ for every $0 < t < T$. □

If $0 < p, q < 1$, we have just seen that two situations can appear: u quenches while v does not, or the other way round. We now show that the third possibility, simultaneous quenching, in fact occurs for some initial data. We conjecture that this behaviour is exceptional in this range of parameters, in contrast with what happens for p, $q \geq 1$.

Proof of Theorem 1.2. Fix (u_0, v_0). For any $\lambda > 0$, consider problem (1.1)–(1.4) with initial data $(\lambda u_0, v_0)$, whose solution we denote by (u_λ, v_λ). Arguing as in the proof of Theorem 1.3, we have that

$$\begin{aligned} V_\lambda(t) \geq & \; C_1 - C_2 \int_0^{T_\lambda} (T_\lambda - s)^{-\frac{q}{q+1}-\frac{1}{2}}\, ds = C_1 - C_2 T_\lambda^{\frac{1-q}{2(1+q)}}, \\ U_\lambda(t) \geq & \; C_1\lambda - C_2 \int_0^{T_\lambda} (T_\lambda - s)^{-\frac{p}{p+1}-\frac{1}{2}}\, ds = C_1\lambda - C_2 T_\lambda^{\frac{1-p}{2(1+p)}}, \end{aligned}$$

where the quenching time satisfies $T_\lambda \leq C \min\{\lambda^q, \lambda\}$, see (3.5). Hence, if λ is small we have $V_\lambda(t) \geq C_1 - C_2 \lambda^{\frac{1-q}{2(1+q)}} > 0$ and thus v_λ does not quench. On the other hand, $U_\lambda(t) \geq C_1\lambda - C_2 \lambda^{\frac{q(1-p)}{2(1+p)}} > 0$ if λ is large, and therefore u_λ does not quench.

We define the sets $A^+ = \{\lambda > 0 : U_\lambda(T_\lambda) > 0\}$ and $A^- = \{\lambda > 0 : V_\lambda(T_\lambda) > 0\}$. Theorem 2.2 shows that A^+ and A^- are both open, so there exists a closed subset $A \subset \mathbb{R}^+$ such that if $\lambda \in A$ then quenching is simultaneous. □

Finally, we prove that there exists $p_0 \geq 1$ such that if $q < 1$ and $p \geq p_0$, then quenching is always non-simultaneous. We conjecture that the result is true for $p_0 = 1$.

Lemma 3.1 *If $q < 1$ and $p \geq p_0 = (1 + q)/(1 - q)$ then quenching is always non-simultaneous.*

Proof. We argue by contradiction. Assume that quenching is simultaneous and $q < 1 < p$. Using again the representation formula for v, for $0 < x < 1$ and $0 < t < z < T$, we have

$$\begin{aligned} v(x, z) \; &= \int_0^1 \Gamma(x - y, z - t) v(y, t)\, dy - \int_t^z v(0, s) \frac{\partial \Gamma}{\partial x}(x, z - s)\, ds \\ &+ \int_t^z v(1, s) \frac{\partial \Gamma}{\partial x}(x - 1, z - s)\, ds - \int_t^z u^{-q}(0, s) \Gamma(x, z - s)\, ds, . \end{aligned}$$

Taking limits as $x \to 0$, $z \to T$, we get

$$0 = \int_0^1 \Gamma(y, T - t) v(y, t)\, dy + \int_t^T v(1, s) \frac{\partial \Gamma}{\partial x}(-1, T - s)\, ds - \int_t^T U^{-q}(s) \Gamma(0, T - s)\, ds.$$

Therefore,

$$V(t) \leq C \int_t^T U^{-q}(s)(T - s)^{-1/2}\, ds.$$

On the other hand, the lower bound for $U(t)$ obtained in Corollary 2.1 gives us

$$U(t) \geq C(T-t)^{1/(q+1)} \Rightarrow V(t) \leq C(T-t)^{\frac{1-q}{2(1+q)}}. \tag{3.6}$$

Now, we introduce this upper estimate in the representation formula for u, to obtain

$$0 \leq \int_0^1 \Gamma(y,T)u(y,0)\,dy + \int_0^T u(1,s)\frac{\partial\Gamma}{\partial x}(-1,T-s)\,ds - C\int_0^T (T-s)^{-\frac{1}{2}-\frac{p(1-q)}{2(1+q)}}\,ds.$$

The two first integrals are bounded, while the third one diverges if $p \geq (1+q)/(1-q)$. We arrive to a contradiction. □

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