New bases for a general definition for the moving preferred basis.

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Abstract: One of the challenges of the Environment-Induced Decoherence (EID) approach is to provide a simple general definition of the moving pointer basis or moving preferred basis. In this letter we prove that the study of the poles that produce the decaying modes in non-unitary evolution, could yield a general definition of the relaxation, the decoherence times, and the moving preferred basis. These probably are the most important concepts in the theory of decoherence, one of the most relevant chapters of theoretical (and also practical) quantum mechanics. As an example we solved the Omnès (or Lee-Friedrich) model using our theory.

Decoherence is extremely important both for theoretical and applied physics. In fact, decoherence is the main element of the quantum to classical limit and this limit is one of the essential features of any successful interpretation of quantum mechanics. But it is also essential for technological subjects as for quantum computation. Nevertheless nowadays we have not a generic definition of moving preferred basis, an essential element for this theory.

In order to prove this assertion let us take into account the state of the art on the subject: the book of Maximilian Schlosshauer "Decoherence and the quantum to classical transition"[1], where it is said that the fundamental equation $\langle E_i(t)|E_j(t)\rangle \sim e^{-\frac{t}{\tau_d}}$ is valid for *many environment models* (not for all), that the characteristic decoherence timescale *can be evaluated numerically* (i. e. not by a general mathematical formula), and that *frequently an exponential-decay behavior* is found (not always). It is also said that a *simple and intuitive criterion will be introduced* to define the preferred set of pointer states while a rigorous criterion would be necessary given the importance of the subject. In fact the book presents a set of different solved models but not a rigorous and general rule to define the moving preferred basis. This also is the common lore according to the literature on the subject.

Of course this is not a criticism to the immense scientific value of the research already done on decoherence, but the justification in trying to do a further step. And this step is also justified for both the theoretical and practical importance on the subject and for the urgent need of a more rigorous treatment.

This letter contains a tentative proposal for a general definition of the *moving preferred basis* where the states of the (open) system become diagonal in a certain decoherence time.

a.- Let us begin introducing the fundamental concepts:

- Relaxation is the (non-unitary) evolution of a quantum state $\rho(t)$ towards an equilibrium state ρ_* at a relaxation time t_R. At this time ρ_* is obviously diagonal in its own eigen-basis $\{|i*\rangle\}$, the relaxation basis.

- Decoherence is the (non-unitary) evolution of a quantum state $\rho(t)$ towards a state where $\rho(t)$ is diagonal in some well defined moving preferred basis $\{|i(t)\rangle\}$ at a certain decoherence time t_D.

The problem is that the last definitions are circular since these three notions: decoherence, moving decoherence basis, and t_D depend among each other and none of them has a general independent definition. Nevertheless there is a large number of convincing examples [2] where these notions are defined, case by case. Often the moving decoherence basis is related to some (macroscopic) collective variables which are experimentally accessible [3], and in this case we can call it a *moving "pointer" basis*. Using these accessible variables an almost general definition can be found in [4], but only in systems that satisfy a certain number of (reasonable) conditions.

Obviously this state of affairs is not theoretically satisfactory so we need a general definition of the three concepts: decoherence, moving preferred basis, and t_D. In this letter all these notions are defined independently and unambiguously but only for systems that satisfy the usual properties of continuity and quasi-analiticity, i. e. the existence of poles (but not more complex objects like branch-cuts, etc.). The poles theory is widely used in quantum mechanics [5], QFT [6], and especially in scattering theory [7]. Essentially, the states associated with the poles or resonances are similar to plane waves, both are non normalizable states. The plane waves would be the eigenstates of the free Hamiltonian while the eigenstates corresponding to the perturbed Hamiltonian would be the Gamov states [8] [9].

b.- Thus we will try to introduce a new general definition that would encompass the various definitions of "pointer" basis in a general one that we will simply call the moving preferred basis. Based on this definition we will define decoherence and decoherence time t_D. The main idea will be the following:

- Given a mechanical system we can find its typical oscillation modes. Let us consider the Hamiltonian equations and find the constant of motion. Then, the corresponding conjugated variables will evolve as $\varphi_i = \varphi_i(0) + \omega_i t$ and these variables will define the typical oscillation modes.

- The quantum version of this mechanical example is obtained if we quantize the system [10]. Then the typical modes would be related to the complete set of commuting observables of the quantum Hamiltonian and they would be $\exp(-\frac{i}{\hbar}\omega_i t)$ where $H|i\rangle = \omega_i |i\rangle$. Of course in this case the evolution of the system is unitary.

To obtain non-unitary evolutions that would reach equilibrium and decoherence we must consider some (relevant) observables $O_R = O_S \otimes I_E$ where O_S is any observable of the Hilbert space $\mathcal{H}_S$ of the (proper or relevant) system $\mathcal{S}$ and I_E is the unit operator of the Hilbert space $\mathcal{H}_E$ of the environment $\mathcal{E}$. Then if $(\rho|O) = Tr(\rho O)$ we have

$$(\rho(t)|O_R) = (\rho(t)|O_S \otimes I_E) = (\rho_S(t)|O_S) \tag{1}$$

where the last member is computed in the Hilbert space of system $\mathcal{S}$ and $\rho_S(t) = Tr_E \rho(t)$, i. e. we have "traced away" the environment. $O_S \in \mathcal{O}_S$ is the vector space of the observables of the system and $\rho_S(t) \in \mathcal{O}'_S$, the dual space of O_S. The $\rho_S(t)$ evolves non-unitarily.

Then to obtain the typical non-unitary modes we must extend the range of the exponents of the evolution of the $(\rho_S(t)|O_S)$ from the real semiaxis to the complex plane obtaining the complex eigenvalues [11]

$$z_i = \omega_i - \frac{i}{2}\gamma_i \tag{2}$$

which also are the complex poles of the resolvent or those of the complex extension of the S-matrix [5]. Then the characteristic decaying times are $t_i = \frac{\hbar}{\gamma_i}$ and also a "long time" or *Khalfin* decaying mode [5]. Usually the corresponding time of this last mode is so long [12] that it can be neglected for practical reasons, as we will always do below.

c.- Then it can be proved that the γ_i define the decaying modes [13], since essentially

$$(\rho_S(t)|O_S) = (\rho_{S*}|O_S)+$$

$$a_0 \exp(-\frac{\gamma_0}{\hbar}t) + \sum_{i=1}^{N} a_i(t) \exp(-\frac{\gamma_i}{\hbar}t) \tag{3}$$

where ρ_{S*} is the equilibrium state of the proper system. Thus it is quite obvious that the minimum of the γ_i, let us say γ_0, related to the pole z_0 closest to the real axis, corresponds to the slowest decaying mode and therefore the *relaxation time* is

$$t_R = \frac{\hbar}{\gamma_0} \tag{4}$$

Let us now go to the simplest example: a model with only two poles, z_0 and z_1, such that $\gamma_0 \ll \gamma_1$. Then we have

$$(\rho_S(t)|O_S) = (\rho_{S*}|O_S) + a_0(t) \exp(-\frac{\gamma_0}{\hbar}t)$$

$$+a_1(t) \exp(-\frac{\gamma_1}{\hbar}t) \tag{5}$$

where $a_0(t)$ and $a_1(t)$ the are real oscillating functions and, in this case, since we only have two modes, necessarily γ_1 corresponds to the decoherence mode so

$$t_D = \frac{\hbar}{\gamma_1} \tag{6}$$

is a reasonable candidate for decoherence time. In fact, for $t > t_D$ the third term of the l.h.s. of (5) is negligible and we have

$$(\rho_S(t)|O_S) = (\rho_{S*}|O_S) + a_0(t) \exp(-\frac{\gamma_0}{\hbar}t) \tag{7}$$

Thus let us define a *preferred state* $\rho_P(t)$, for all times, such that

$$(\rho_P(t)|O_S) = (\rho_{S*}|O_S) + a_0(t)\exp(-\frac{\gamma_0}{\hbar}t) \tag{8}$$

According to the Riezs theorem [14] the inner product $(\rho_P(t)|O_S)$ defines the functional $(\rho_P(t)| \in O'_S$. $\rho_P(t)$ would be defined *for all times* and would be self adjoint since the rhs of the last equation is real. Then we can find the eigen-decomposition of $\rho_P(t)$:

$$\rho_P(t) = \sum_{i=1}^{D_S} \rho_i(t)|i(t)\rangle\langle i(t)| \tag{9}$$

where D_S is the dimension of the space $\mathcal{H}_S$. Comparing eqs. (5) to (8) it is quite evident that if we define $t_D = \frac{\hbar}{\gamma_1}$, $\rho_S(t)$ becomes $\rho_P(t)$ for $t > t_D$. Then in this period $\rho_S(t)$ becomes diagonal in the basis $\{|i(t)\rangle\}$ which is the only possible *moving preferred basis* and $t_D = \frac{\hbar}{\gamma_1}$ is the *decoherence time*.

   d.- Going now to a the general case and therefore with an arbitrary number N of poles and therefore N decaying modes $\gamma_0 < \gamma_1 < \gamma_2... < \gamma_N$ and again we would be forced to say that the relaxation time corresponds to the pole placed nearest to the real semiaxis (then we obtain eq. (4)). Now we know that the main achievement of the EID program is to obtain a decoherence time such that $t_D \ll t_R$ and as

$$\sum_{i=1}^N a_i \exp(-\frac{\gamma_i}{\hbar}t) = \exp g(t) = \exp(g(0) + g'(0)t + \frac{1}{2}g''(0)t^2 + ...$$

and since we are only interested in small times we can only consider the two first terms: i.e. $g(0) + g'(0)t$ where

$$g(0) = \log\sum_{i=1}^N a_i(0), \; g'(0) = -\hbar^{-1}\frac{\sum_{i=1}^N a_i(0)\gamma_i}{\sum_{i=1}^N a_i(0)}$$

Then we have

$$\begin{aligned}(\rho_S(t)|O_{RS}) &= (\rho_{S*}|O_S) + a_0\exp(-\frac{\gamma_0}{\hbar}t) \\ &\quad + a_{eff}\exp(-\frac{\gamma_{eff}}{\hbar}t)\end{aligned} \tag{10}$$

and expression similar to eq. (5) but where

$$\gamma_{eff} = \frac{\sum_{i=1}^N a_i(0)\gamma_i}{\sum_{i=1}^N a_i(0)}$$

then in this general case we can define

$$t_D = \frac{\hbar}{\gamma_{eff}} \tag{11}$$

and

$$(\rho_P(t)|O_S) = (\rho_{S*}|O_S) + a_0(t)\exp(-\frac{\gamma_0}{\hbar}t)+$$

$$\sum_{i=1}^M a_i(t)\exp(-\frac{\gamma_i}{\hbar}t) \tag{12}$$

where in the last sum the term labelled by $i \leq M \leq N$ are those such that $\gamma_i < \gamma_{eff}$. Repeating the reasoning of point c in this general case we can say that the moving preferred basis $\{|i(t)\rangle\}$ is the one that diagonalizes the new $\rho_P(t)$.

   This would be our candidate for the general definition of moving pointer basis, that must be compared to those of the different models in the literature [2]. The characteristic properties of this candidate preferred basis are:

    i. $\rho_S(t)$ do decohere in this basis at time t_D.

    ii. If we classify the decaying modes in slow modes (such that $\gamma_i < \gamma_{eff}$) and fast modes (such that $\gamma_i > \gamma_{eff}$), the evolution of our basis could be considered as "adiabatic" since it only contains the slow modes.

e.- Moreover, up to this point the reader may consider that all the presented structure is completely alien to the usual literature on the subject. It is not so, and at the end this letter we will use our technique in an important example: the Omnès (or Lee-Friedrich) model with Hamiltonian [3]

$$H \; = \; \omega a^\dagger a + \int \omega_k b_k^\dagger b_k dk$$
$$+ \int (\lambda_k a^\dagger b_k + \lambda_k^* a b_k^\dagger) dk \tag{13}$$

where $a^\dagger$ $(b_k^\dagger)$ is the "creation" operator of the system (environment) (an interesting study of this model can be found in [16]). The initial condition for the system corresponds to the sum of two Gaussian functions and the environment initial condition is the vacuum state. Moreover the Wigner transform of this initial condition must correspond to two mountain-like density functions separated by a distance L (which will allow us to consider another important ingredient: *macroscopicity*). One of the properties of Hamiltonian (13) is that if $|\nu\rangle$ is an ν-mode state it is easy to see that $H|\nu\rangle$ is also a ν-mode state so the evolution of the system conserves the *number of modes sectors* (or number of "particles" sectors). Thus, we can decompose the problem in number of modes sectors, and we can study the one mode sector and find the corresponding pole z_0, which is produced by the interaction of the Hamiltonian (13). Then z_0, up to the second order in λ, is [15]:

$$z_0 = \omega + \int \frac{\lambda_k^2 dk}{\omega - \omega_k + i0} \tag{14}$$

where

$$z_0 = \omega_0' - \frac{i}{2}\gamma_0 \tag{15}$$

and

$$\gamma_0 = \pi \int n(\omega')|\lambda_{\omega'}|^2 \delta(\omega - \omega')d\omega' \tag{16}$$

and where $dk = n(\omega')d\omega'$. So we conclude that in this sector $t_R = \frac{\hbar}{\gamma_0}$.

    Now, going to the many sectors case (and always neglecting the Khalfin term), it is easy to prove that the poles of the Omnès system are

$$z_n = nz_0, \qquad n = 1, 2, 3, ... \tag{17}$$

so z_0 is the pole closest to the real axis, for the whole system, and, in fact, $t_R = \frac{\hbar}{\gamma_0}$ in this system. Then we have our first coincidence since this result coincides with the one obtained by Omnès in page 288 of [3], and a first evidence that the definition of t_R can be used.

    Also using this result we can define an effective Hamiltonian, that produces the non-unitary evolution of the system, and if we neglect the Khalfin term, we obtain

$$H_{eff} = z_0 a_0^\dagger a_0 = z_0 N \tag{18}$$

where $a_0^\dagger$ (a_0) is the creation (annihilation) operator of the poles and N is the number of poles operator. As we will see, the Omnès model contains only the poles terms given by eq. (17) but not the Khalfin term (very slow motion term). This, completely justified approximation, forces us to introduce the non Hermitian Hamiltonian of eq. (18), as it is usual in many cases. The presence of a non Hermitian Hamiltonian is a good indicator since we are looking for a non unitary evolution. Then $\{|n\rangle\}$ is the common eigenbasis of H_{eff} and N. Therefore if we define the amplitude $A(t) = \langle \psi | \varphi(t) \rangle$, and if $|\psi\rangle = \sum_n a_n |n\rangle$, $|\varphi(t)\rangle = \sum_n b_n |n(t)\rangle$ we have

$$A(t) = \sum_n b_n a_n^* e^{-i\frac{nz_0}{\hbar}t} \tag{19}$$

Now if we consider that the initial state of the system is $|\Phi(0)\rangle = a|\alpha_1(0)\rangle + b|\alpha_2(0)\rangle$ and $\rho_S(0) = |\Phi(0)\rangle\langle\Phi(0)\rangle|$ where $|\alpha_1(0)$ and $|\alpha_2(0)\rangle$ are to two *coherent states* placed at $\alpha_1(0) = \frac{m\omega_0'}{\sqrt{2m\hbar^2\omega_0'}}x_1(0)$ and $\alpha_2(0) = \frac{m\omega_0'}{\sqrt{2m\hbar^2\omega_0'}}x_1(0)$, $p_1(0) = p_2(0) = 0$ of phase space the non diagonal part of $\rho_S(t)$ is

$$\rho_S^{(ND)}(t) = ab^*|\alpha_1(t)\rangle\langle\alpha_2(t)|$$
$$+a^*b|\alpha_2(t)\rangle\langle\alpha_1(t)| \tag{20}$$

Then using eq. (19), the following inner products can be computed:

$$\langle\alpha_1(0)|\alpha_1(t)\rangle = e^{-|\alpha_1(0)|^2}e^{|\alpha_1(0)|^2e^{-i\frac{z_0}{\hbar}t}} \tag{21}$$

$$\langle\alpha_1(0)|\alpha_2(t)\rangle = e^{-\frac{|\alpha_1(0)|^2+|\alpha_2(0)|^2}{2}}e^{\alpha_1(0)\alpha_2(0)e^{-i\frac{z_0}{\hbar}t}}$$
$$= \langle\alpha_2(0)|\alpha_1(t)\rangle \tag{22}$$

$$\langle\alpha_2(o)|\alpha_2(t)\rangle = e^{-|\alpha_2(0)|^2}e^{(\alpha_2(0))^2e^{-i\frac{z_0}{\hbar}t}} \tag{23}$$

So we can also compute the time evolution of $\rho_S^{(ND)}(t)$. Then, with no lost of generality, we can choose

$$\alpha_1(0) = 0, \quad \alpha_2(0)\frac{m\omega_0'}{\sqrt{2m\hbar^2\omega_0'}}L \tag{24}$$

where L is the distance between the centers of the initial positions of the two coherent states.

Then it turns out that

$$\rho_S^{(ND)}(t) = (ab^*|\alpha_1(0)\rangle\langle\alpha_2(0)| +$$
$$+a^*b|\alpha_2(0)\rangle\langle\alpha_1(0)|)e^{-\frac{1}{2}L^2\left(1-e^{-\frac{\gamma}{\hbar}t}\right)} \tag{25}$$

Thus, using now the recipe $g(0) + g'(0)t$ we obtain that the , $\rho^{(ND)}(t)$ vanishes when $t \to \infty$ as

$$\rho_S^{(ND)}(t) \sim \exp\left(-\frac{m\omega_0'}{2\hbar^2}\gamma_0 L^2 t\right) \tag{26}$$

(see [17], from details). Therefore, in this case, the moving preferred basis is $\{|\alpha_1(t)\rangle, |\alpha_2(t)\rangle\}$ for large L and from our previous definition of relaxation time $t_R = \frac{\hbar}{\gamma_0}$ we find that the decoherence time is

$$t_D = \frac{2\hbar^2}{m\omega}\frac{t_R}{L^2} \tag{27}$$

namely the result of Omnès in page 291 [3]. This is our second coincidence. So the Omnès result can be obtained with our pole technique. In this case macroscopicity appears for large L. Moreover, it can be proved (see [17]) that the Omnès moving preferred basis coincides with ours for large L.

The coincidence of our proposal with the Omnès model leads us to consider that our moving preferred basis (that diagonalizes ρ_P) is a good candidate for the moving preferred basis.

f.- Of course we are aware that, to improve our proposals, more examples must be added, as we will try to do elsewhere, because we also believe that we have a good point of depart. In fact, our group is now studying our proposal in other models as the Brownian motion and spin systems. The Brownian motion models are so similar to the Omnès one that we believe that we will find a similar result. For the spin model, in order to use the analytical continuation theory it is necessary to approximate the quasi-continuous spectrum to a continuous one. We have already studied the conditions for this approximation in [18]. Probably the coincidences that we have found in the Omnès model could be a general feature of the decoherence phenomenon and would allow us to obtain complete general definitions. Essentially because, being the pole catalogue the one that contains *all the possible decaying modes* of the non unitary evolutions, relaxation and decoherence must be included in this catalogue, since they are non-unitary evolutions.

In conclusion, we have given a quite general definition of a moving preferred basis, decoherence time, and of the relaxation time. The Omnès formalism, of references [3] and [4], contains the most general definition of moving

preferred basis of the literature on the subject. Our basis have another conceptual frame: the catalogue of decaying modes in the non-unitary evolution of a quantum system. Finally we hope that our result would open a new way to obtain a general and rigorous formalism for one of the most important chapters of quantum physics.

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